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Enlargeable foliations and the monodromy groupoid

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Abstract. Let M be a closed spin manifold, the Dirac operator with coefficient in the universal flat Hilbert $C^*\pi_1$-module determines a Rosenberg index element which, according to B. Hanke and T. Schick, subsumes the enlargeability obstruction of positive scalar curvature on M. In this paper, we generalize this result to the case of spin foliation. More precisely, given a foliation (M, F) with F spin, we will define a foliation version of Rosenberg index element and prove that it is nonzero in the presence of enlargeability of (M, F). As a consequence, the foliation version of Rosenberg index element subsumes the enlargeability obstruction to the existence of leafwise positive scalar curvature metric.

1. Introduction

1.1. Enlargeable manifold

Enlargeability [11] is an important notion in studying which manifold admits positive scalar curvature metric. A famous theorem of Gromov and Lawson states that closed spin enlargeable manifolds cannot carry positive scalar curvature metric.

If M is an even dimensional spin manifold with the fundamental group π_1 and the Dirac operator

$$D : C_c^\infty(M, S^+) \to C_c^\infty(M, S^-),$$

according to [18], the Dirac-type operator twisted by the canonical flat $C^*\pi_1$-bundle

$$\widetilde{M} \times_{\pi_1} C^*\pi_1 \tag{1.1}$$

determines an element $[\alpha(M)]$ (we will simply write $[\alpha]$ when there is no confusion) in $K_0(C^*\pi_1)$ which is usually called the Rosenberg index element. In fact, if M is of odd dimensional, by replacing M with $M \times S^1$, $[\alpha] \in K_1(C^*\pi_1)$ can also be defined. The main result of [12, 13] is the following theorem.

Theorem 1.1 ([12, Proposition 4.2] and [13, Theorem 1.2]). *If M is an enlargeable spin manifold, then $[\alpha] \neq 0$ in $K_n(C^*\pi_1)$, where n is the dimension of M.*

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1.2. Previous approach

The main idea of [12] is as follows: if M is compactly enlargeable, let $E \to S^n$ be a vector bundle whose top degree Chern class is not zero while all other Chern classes are zero. The pullback $f_\varepsilon^* E \to \widetilde{M}_\varepsilon$ can be "wrapped up" to a finite dimensional vector bundle $E_\varepsilon \to M$. As ε range over $1, 1/2, \ldots, 1/n, \ldots$, we get a sequence of leafwise increasingly flat vector bundles $\{E_i\}_{i \in \mathbb{N}}$ over M whose Chern classes vanish except the top degree one. Denote by d_i the dimension of E_i. Let P_i be the principal frame bundle of E_i, $\mathcal{K}$ be the C^*-algebra of compact operators. Unitary matrices act on $\mathcal{K}$ by the inclusion $U(d_i) \hookrightarrow \mathcal{K}$. Denote by q_i the image of $1 \in U(d_i)$ inside $\mathcal{K}$. The associated product

$$P_i \times_{U(d_i)} \mathcal{K} \tag{1.2}$$

is a Hilbert $\mathcal{K}$-module bundle.

Definition 1.2. Let A be the C^*-algebra of bounded sequence of compact operators. Namely,

$$A = \left\{ (a_i) \in \prod_{\mathbb{N}} \mathcal{K} : \sup_{i \in \mathbb{N}} \|a_i\| < \infty \right\}.$$

Let $A_i \subset A$ be the subalgebra of sequences such that all but the ith component vanish. It is clear that $A_i \cong \mathcal{K}$ for all $i \in \mathbb{N}$. Let $A' \subset A$ be the subalgebra consisting of sequences that converge to zero. In other word, A' is the closure of $\bigoplus \mathcal{K} \subset A$. Let Q be the quotient C^*-algebra A/A'.

Thanks to the boundedness of the curvatures of E_i's, the sequence of Hilbert module bundles $P_i \times_{U(d_i)} \mathcal{K}$ can be assembled into a Hilbert A-module bundle V. The almost flatness of E_i is reflected in the fact that the curvature of V is endomorphism of A which take value in $\hom(A, A')$. Therefore, V can be reduced into a genuinely flat Hilbert Q-module bundle $W = V/A'$. As a consequence of the flatness of W, there is a holonomy representation of the fundamental group π_1 and correspondingly, a C^*-algebras homomorphism $C^*\pi_1 \to Q$. To detect the non-vanishing of $[\alpha]$, it is enough to show the non-vanishing of its image under the map

$$K_0(C^*\pi_1) \to K_0(Q). \tag{1.3}$$

It is known (see [12, Proposition 3.6]) that the K-theory of Q is explicitly computable as a quotient of $\prod \mathbb{Z}$, and the ith argument of $[\alpha]$ in $K_0(Q)$ is computed as the index of the Dirac-type operator twisted by E_i. The non-vanishing of $[\alpha]$ then follows from the non-vanishing of the top degree Chern class and the Atiyah–Singer index theorem.

1.3. Difficulty with noncompactness

The main difficulty with non-compactly enlargeable foliation is that the covering spaces $\widetilde{M}_\varepsilon$ are non-compact so that the "wrapped up" bundles $E_\varepsilon \to M$ are infinite dimensional. To get some useful information out of E_ε's, Hanke and Schick [13] organize these bundles into some Hilbert module bundles in the following novel way.

Let G and H_ε be the fundamental groups for M and $\widetilde{M}_\varepsilon$, respectively. Then, each fiber of the covering $\widetilde{M}_\varepsilon \to M$ can be parameterized by G/H_ε. We fix such a parameterization. The C^*-algebras C_T, C_S and $C_{S,T}$ are as in [13] except that we will add a superscript ε to indicate its dependence on ε.

Definition 1.3. Let $C_T^\varepsilon \subset \mathcal{B}(\ell^2(G/H_\varepsilon) \otimes \mathbb{C}^d)$ be the C^*-algebra generated by G/H_ε families of matrices $M_d(\mathbb{C})$ all but finitely many of which are zero. Let us assume that $C_S^\varepsilon \subset \mathcal{B}(\ell^2(G/H_\varepsilon) \otimes \mathbb{C}^d)$ be the C^*-algebra generated by permutations of G/H_ε and let $C_{S,T}^\varepsilon$ be the C^*-algebra generated by C_S^ε and C_T^ε. Notice here C_T^ε is isomorphic to the C^*-algebra of compact operators and is a two-sided ideal in $C_{S,T}^\varepsilon$.

Fix ε and let $\pi_\varepsilon : \widetilde{M}_\varepsilon \to M$ be the covering map, let $\{U_\alpha\}$ be a finite open cover of M such that $f_\varepsilon^* E$ is trivial on $\pi_\varepsilon^{-1}(U_\alpha)$ for all α. Fix these trivializations:

$$f_\varepsilon^* E|_{\pi_\varepsilon^{-1}(U_\alpha)} \xrightarrow{\varphi_\alpha} \pi_\varepsilon^{-1}(U_\alpha) \times \mathbb{C}^d. \tag{1.4}$$

The transition functions

$$\pi_\varepsilon^{-1}(U_\alpha \cap U_\beta) \times \mathbb{C}^d \xrightarrow{\varphi_{\alpha\beta}} \pi_\varepsilon^{-1}(U_\alpha \cap U_\beta) \times \mathbb{C}^d \tag{1.5}$$

can be viewed as a map from $U_\alpha \cap U_\beta$ to $C_{S,T}^\varepsilon$, which, used as new set of transition functions, build the Hilbert $C_{S,T}^\varepsilon$-module bundle $E_\varepsilon \to M$.

Apart from Hilbert module bundle structure, one more ingredient is needed to tackle the noncompactness of $\widetilde{M}_\varepsilon$, that is *relative index*. Let $R(C_{S,T}^\varepsilon)$ be the C^*-algebra defined by $R(C_{S,T}^\varepsilon) = \{(c_1, c_2) \in C_{S,T}^\varepsilon \times C_{S,T}^\varepsilon \mid c_1 - c_2 \in C_T^\varepsilon\}$. Inspired by Roe's approach to relative index [22], Hanke and Schick organize the virtual bundle $E_\varepsilon - C_{S,T}^\varepsilon$ into a Hilbert $R(C_{S,T}^\varepsilon)$-module bundle. Then, the index of the twisted Dirac operator $D_{E_\varepsilon - C_{S,T}^\varepsilon}$ belongs to $K_0(R(C_{S,T}^\varepsilon))$.

Definition 1.4. Let $A \subset \prod_i R(C_{S,T}^{1/i})$ be the C^*-algebra of bounded sequences, $A' \subset A$ be the subalgebra consisting of sequences that converge to zero and $Q = A/A'$.

As in the compact case, we may assemble the sequence of bundles $\{E_i\}$ into a Hilbert A-module bundle V. Its quotient V/A' is a flat Hilbert Q-module bundle which determines a homomorphism $\pi_1(M) \to Q$. The advantage of $R(C_{S,T}^\varepsilon)$ is that its K-theory splits out a $\mathbb{Z}$-component which can be computed by a generalization of Mischenko–Fomenko index theorem [23] plus a trace calculation. Then, an analogous map as (1.3) detects the nonvanishing of $[\alpha]$.

1.4. Present work

In this paper, we will first define a foliation version of Rosenberg index element (Definition 4.3) and then generalize Theorem 1.1 to the case of enlargeable foliations. The following is the main theorem of this paper.

Theorem 1.5. *If (M, F) is an enlargeable foliation (the precise definition is given in Definition 1.6) with F spin, then the foliation version of the Rosenberg index element is not zero.*

On the other hand, we will see that the existence of leafwise positive scalar curvature metric forces the Rosenberg index to be zero (Proposition 4.4). Together with the above main theorem, the foliation version of the Rosenberg index element subsumes the enlargeability obstruction to the existence of the leafwise positive scalar curvature metric.

We will first verify the main theorem in the compactly enlargeable case and then reduce the general case back to the compact case.

In fact, there are several notions of enlargeability for foliation in the literatures. We will use the following definition.

Definition 1.6. Let M be a compact Riemannian metric. A foliation (M, F) is compactly enlargeable if there is $C > 0$ such that for any $\varepsilon > 0$, there is a compact covering $\widetilde{M}_\varepsilon$ of M and a smooth map

$$f_\varepsilon : \widetilde{M}_\varepsilon \to S^n$$

with

- $|f_{\varepsilon,*}X| \leq \varepsilon |X|$ for all $X \in C^\infty(\widetilde{M}_\varepsilon, \widetilde{F}_\varepsilon)$, where $\widetilde{F}_\varepsilon$ is the lifting of F to $\widetilde{M}_\varepsilon$;
- $|f_{\varepsilon,*}X| \leq C \cdot |X|$ for all $X \in C^\infty(\widetilde{M}_\varepsilon, T\widetilde{M}_\varepsilon)$;
- f_ε has nonzero degree.

A foliation (M, F) is enlargeable if the above condition holds with possibly non-compact coverings $\widetilde{M}_\varepsilon$ and f_ε is constant outside some compact subset $K_\varepsilon \subset \widetilde{M}_\varepsilon$.

Remarks 1.7. The notion of enlargeability given in the above definition is, in some sense, in between that of [25, Definition 0.1] and [3, Definition 1.5]. More precisely, the enlargeability used in [3, Definition 1.5] requires the ε-contracting condition in all tangent directions while the one used in [25, Definition 0.1] only needs the ε-contracting condition in the leafwise direction. In the above definition, the second bullet point replaces the ε-contracting condition in the transverse direction by a uniformly bounded condition.

The definition above of enlargeability explicitly uses the Riemannian metric on M. However, as shown in [17, Chapter IV, Theorem 5.3], the enlargeability is independent of the Riemannian metric. More precisely, any compact foliated manifold (N, F') which admits a map of nonzero degree onto an enlargeable (M, F) that sends F' into F is enlargeable. The proof given in [17, Chapter IV, Theorem 5.3] can be applied verbatim.

Let (M, F) be a compactly enlargeable foliation with F spin, let G_M and G_H be the monodromy groupoid and the holonomy groupoid of (M, F), respectively. The leafwise Dirac operator

$$D : C^\infty(M, S^+(F)) \to C^\infty(M, S^-(F))$$

defines a K-theory element $[\alpha(M, F)]$ (we will simply write $[\alpha]$ when there is no confusion) in $K_0(C^*G_M)$.

Definition 1.8. Recall that $q_i \in \mathcal{K}$ is the image of $1 \in U(d_i)$ inside $\mathcal{K}$. Let $q = (q_1, q_2, \ldots) \in A$, then qAq is an unital C^*-algebra. We will write $qA'q = A' \cap qAq$ which is an ideal in qAq. Finally,

$$qQq = qAq/qA'q.$$

As in the unfoliated case, the enlargeability condition gives a sequence of leafwise almost flat vector bundles $\{E_i\}$ of dimension d_i whose Chern classes vanish except the top degree part. The sequence of principal frame bundles and their associated product with the truncated compact operators $q_i \mathcal{K} q_i$ can be defined in the same way as in (1.2). The sequence $\{E_i\}$ can be assembled into a Hilbert qAq-module bundle V, and the almost leafwise flatness will be reflected in a genuinely leafwise flat Hilbert qQq-module bundle $W = V/qA'q$. However, to the best of the authors' knowledge, there is no characterization of leafwise flat vector bundle in the form of (1.1). To find the counterpart of (1.3), we will make use of basic KK-theory.

The foliation counterpart of universal cover and fundamental group is the monodromy groupoid G_M. The role of C^*-algebras A, A', Q will be played by three crossed product C^*-algebras $C^*(G_M, qAq)$, $C^*(G_M, qA'q)$ and $C^*(G_M, qQq)$ which are constructed by taking the completion of the algebra of compactly supported smooth functions on G_M with values in the corresponding C^*-algebras. The fact that qAq is an unital C^*-algebras is crucial in the corresponding pseudodifferential calculus that we will need. If W is a leafwise flat Hilbert qQq-module bundle over M, the space of smooth compactly supported sections of the pullback bundle $r^*W \to G_M$ can be completed into a Hilbert $C^*(G_M, qQq)$-module $\mathcal{E}_W$. It can be shown, due to the leafwise flatness, this module also has a left C^*G_M-action which, together with the zero operator, determines a KK-theory element in

$$KK(C^*G_M, C^*(G_M, qQq)). \tag{1.6}$$

This KK-element will play the role of the map (1.3). The sequence of C^*-algebras

$$0 \to C^*(G_M, qA'q) \to C^*(G_M, qAq) \to C^*(G_M, qQq) \to 0$$

is exact (see Proposition 3.8) and induces the following exact sequence at the level of K-theory:

$$K_0(C^*(G_M, qA'q)) \to K_0(C^*(G_M, qAq)) \to K_0(C^*(G_M, qQq)). \tag{1.7}$$

The image of $[\alpha]$ under the map

$$K_0(C^*G_M) \to K_0(C^*(G_M, qQq)),$$

which is induced by Kasparov product with the KK-element (1.6), is given by the twisted leafwise Dirac operator $[D_{W-qQq}] \in K_0(C^*(G_M, qQq))$ twisted by the virtual bundle $W - qQq$. It can be lifted, through the second map of (1.7), to the element in $[D_{V-qAq}] \in K_0(C^*(G_M, qAq))$ given by the leafwise Dirac operator twisted by the virtual bundle

$V - qAq$. To see the nonvanishing of $[\alpha]$, it suffices to show that $[D_{V-qAq}]$ is not in the image of the first map of (1.7). Consider the following diagram:

$$\begin{array}{ccccc} K_0(C^*(G_M, qA'q)) & \longrightarrow & \bigoplus K_0(C^*G_M) & \xrightarrow{\mu} & \bigoplus \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ K_0(C^*(G_M, qAq)) & \xrightarrow{\prod p_i} & \prod K_0(C^*G_M) & \xrightarrow{\mu} & \prod \mathbb{C}, \end{array} \tag{1.8}$$

where the first vertical arrow is precisely the first map of (1.7), the horizontal arrows from the first column to the second column are induced from the projections $A \to \mathcal{K}$ into the each component and the μ maps are given by the Connes' transversal fundamental class. Then, the image of $[D_{V-qAq}]$ under the lower horizontal line is given by

$$(\langle \widehat{A}(F)\,\mathrm{ch}(E_i - \mathbb{C}^{d_i}), [M]\rangle)_{i\in\mathbb{N}},$$

which, according to the assumption on Chern classes, has infinitely many nonzero terms. Thus, it cannot be in the image of the first horizontal line.

1.5. Reduction to compact case

For general enlargeable foliation, the situation can be reduced back to the compactly enlargeable case by the following observation.

Observation 1.9. As $C_T^\varepsilon \subset C_{S,T}^\varepsilon$ is an ideal, the C^*-algebra $C_{S,T}^\varepsilon$ can be mapped to the multiplier algebra of C_T^ε,

$$C_{S,T}^\varepsilon \to M(C_T^\varepsilon), \tag{1.9}$$

the compositions of the set of transition functions (1.5) and the map (1.9) can be taken as a new set of transition functions. Together with trivializations $U_\alpha \times C_T^\varepsilon$, it builds a Hilbert C_T^ε-module bundle E_ε.

Now, the difficulty of non-compactness is reflected in the fact that $C_T^\varepsilon \cong \mathcal{K}$ is not unital so that the KK-theory element construction (1.6) cannot be applied directly. We need the relative index theorem to overcome the difficulty. The virtual bundle

$$E_\varepsilon - C_T^\varepsilon \tag{1.10}$$

can be organized into an element in the group $K_0(C(M) \otimes C_T^\varepsilon)$. Under the light of KK-equivalence between $\mathcal{K}$ and $\mathbb{C}$ the difference bundle (1.10) can be reduced to a finite dimensional virtual bundle $E_\varepsilon^0 - \mathbb{C}^{d_\varepsilon}$ with $d_\varepsilon = \dim(E_\varepsilon^0)$. The advantage of E_ε^0 over E_ε is that, apart from being asymptotic flat as $\varepsilon \to 0$, it is finite dimensional vector bundle so that the construction of (1.6) can be applied. The Chern characters of E_ε^0's can be calculated to satisfy

$$\langle \widehat{A}(F)\,\mathrm{ch}(E_\varepsilon^0 - \mathbb{C}^{d_\varepsilon}), [M]\rangle = \langle \widehat{A}(\widetilde{F}_\varepsilon)\,\mathrm{ch}(f_\varepsilon^* E - \mathbb{C}^d), [\widetilde{M}_\varepsilon]\rangle. \tag{1.11}$$

From here on, exactly the same methods in the compactly enlargeable case can be applied, and the result follows.

1.6. Organization

This paper is organized as follows. In Section 2, we briefly recall the definition of monodromy groupoids and holonomy groupoids of a foliated manifold. In Section 3, we review the notion of Haar system on Lie groupoids, the construction of full and reduced groupoid C^*-algebras and introduce $C^*(G, A)$ groupoid C^*-algebras with coefficient in another C^*-algebra. In Section 4, under the assumption that (M, F) is a foliation with F spin and even dimensional, we define the foliation counterpart of Rosenberg index $[\alpha] \in K_0(C^*G_M)$ and relate it to the longitudinal index element. In Section 5, we define the Rosenberg index twisted by a Hilbert C^*-module bundle. In Section 6, we construct a Hilbert module in the presence of a leafwise flat Hilbert Q-module bundle. This Hilbert module will later determines a KK-theory element which play the role of (1.3). In Section 7, we write down the definition of the genuinely leafwise flat Hilbert Q-module bundle out of the compactly enlargeability of (M, F) and prove the non-vanishing of $[\alpha]$. In Section 8, we construct the finite dimensional vector bundles E_ε^0, calculate their curvatures and Chern characters and explain how to reduce the general enlargeable case to the compactly enlargeable case. In Section 9, we deal with odd dimensional F. We define $[\alpha] \in K_1(C^*G_M)$ and show how to reduce the non-vanishing problem to the even dimensional case.

2. Monodromy groupoids and holonomy groupoids

Let (M, F) be a compact foliation, we will denote the monodromy groupoid by G_M and the holonomy groupoid by G_H. The unit space of G_H is the compact manifold M, the morphism space is the set of holonomy classes of curves along leaves of (M, F). The range map and the source map $r, s : G_H \to M$ are given by sending curves to their terminal and initial points, respectively. Groupoid multiplications are given by concatenation of curves.

Proposition 2.1 ([19, Proposition 5.6]). *The morphism space of holonomy groupoid G_H has a manifold structure.*

Proof. Let $\gamma \in G_H$ be some curves in a leaf of the foliation (M, F). We will construct an open neighborhood of γ which is homeomorphic to some Euclidean space.

Assume that $r(\gamma) = x$ and $s(\gamma) = y$. Pick local foliation charts $x \in U = T_1 \times L_1 \to \mathbb{R}^p \times \mathbb{R}^q$ with $x = (x_T, x_L) \in U$ and $y \in V = T_2 \times L_2 \to \mathbb{R}^p \times \mathbb{R}^q$ with $y = (y_T, y_L) \in V$. If we pick two foliation charts small enough, there is a smooth map

$$H : T_1 \times [0, 1] \to T_2 \tag{2.1}$$

such that $H(x_T, t) = \gamma(t)$ and $H(*, t)$ is a curve within some leaves connecting $H(*, 0) \in T_1$ and $H(*, 1) \in T_2$. Now, we can define a map

$$T_1 \times L_1 \times L_2 \to \mathrm{Hol}(M, F), \tag{2.2}$$

which assign $(a,b,c) \in T_1 \times L_1 \times L_2$ the curve $\tau \circ H(a,t) \circ \eta$ where τ is any curve connecting (a,b) to (a,x_L) in U, H is the smooth map described in (2.1) with $H(a,0) = (a,x_L)$ and η is any curve connecting $H(a,1)$ with (y_T,c) in V. The map (2.2) is well-defined since the holonomy class of $\tau \circ H(a,t) \circ \eta$ is independent of the choice of τ, H, η. It is also clear that (2.2) is injective, and form a topological basis. In this way, we define the local Euclidean structure, and hence, a manifold structure of G_H. ∎

Remark 2.2. As for the monodromy groupoid G_M, the unit space is given by M, the morphism space is the set of homotopy classes of curves along leaves of (M, F), the manifold structure and the source and range maps are given in a similar way.

The source fibers of G_H and G_M over $x \in M$ is the holonomy cover and universal cover of the leaf passing through $x \in M$, respectively.

Example 2.3. If $F = TM$, the holonomy groupoid degenerates into the pair groupoid, namely, $G_H = M \times M$. Under the same assumption $F = TM$, the morphism space of the monodromy groupoid is given by the space of homotopy classes of all curves in M. In this particular case, G_M is usually called fundamental groupoid and denoted by $\Pi(M) \rightrightarrows M$. It can be shown that the fundamental groupoid is Morita equivalent to fundamental group taken as groupoid over a single point. Hence, their corresponding groupoid C^*-algebras are Morita equivalent.

3. Groupoid C^*-algebras

Parallel to the notion of Haar measures on locally compact topological groups, there is a notion of Haar systems on Lie groupoids.

Definition 3.1. Let $G \rightrightarrows G^{(0)}$ be a Lie groupoid, a family of measures $\{\mu_x\}_{x \in G^0}$ is called a Haar system on G if the following statements hold.

(1) The measure μ_x is supported on the source fiber G_x.

(2) For any smooth compactly supported function f on G, the function on the unit space $G^{(0)}$ given by the assignment

$$x \mapsto \int_{G_x} f(\gamma) d\mu_x(\gamma)$$

is smooth.

(3) Let $\eta \in G$, f be any smooth compactly supported function on G; then,

$$\int_{G_{s(\eta)}} f(\gamma) d\mu_{s(\eta)}(\gamma) = \int_{G_{r(\eta)}} f(\gamma \circ \eta) d\mu_{r(\eta)}(\gamma).$$

The above family of measures is sometimes referred to as right invariant Haar system. Left invariant Haar system $\{\mu^x\}_{x \in G^0}$ can be defined in a similar way where we replace the source fibers G_x with the range fibers G^x.

Example 3.2. Fix a metric on F; then, there is an induced measure $\{\mu_x\}_{x\in M}$ on each leaf $L_x \subset M$. The leafwise measures, in turn, determine measures $\{\mu_x^H\}$ on their holonomy covers $G_{H,x}$ and measures $\{\mu_x^M\}$ on universal covers $G_{M,x}$ accordingly. One can check that these measures form Haar systems on G_H and G_M, respectively.

In the presence of a Haar system $\{\mu_x\}_{x\in G^{(0)}}$, the space of compactly supported smooth functions on groupoid G can be made into an algebra. Let $f, g \in C_c^\infty(G)$, the multiplication $f * g$ is given by

$$f * g(\gamma) = \int_{\gamma_1 \in G_{s(\gamma)}} f(\gamma \circ \gamma_1^{-1}) g(\gamma_1) d\mu_{s(\gamma)}(\gamma_1), \tag{3.1}$$

and the adjoint is given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Remark 3.3. In general, the monodromy groupoid G_M and the holonomy groupoid G_H may not be Hausdorff. We need to be careful with the definition of $C_c^\infty(G)$. Since G_M and G_H all have smooth manifold structure, every point in the groupoid have Hausdorff local coordinate chart. According to [6], the space $C_c^\infty(G)$ is defined to be the span of functions each of which is smooth on a Hausdorff chart of G and vanishes outside a compact subset of the Hausdorff chart. More precisely, a typical function in $C_c^\infty(G)$ can be written as finite sum

$$f = \sum_i f_i,$$

where f_i is smooth function on a Hausdorff chart $U_i \subset G$ that vanishes outside a compact subset of U_i. If G is indeed Hausdorff, then so defined $C_c^\infty(G)$ has its usual meaning (see [20] for more details).

Definition 3.4. Let $f \in C_c^\infty(G)$ and define

$$\|f\|_I = \sup_{x\in G^{(0)}} \left\{ \int_{G_x} |f(\gamma)| d\mu_x(\gamma), \int_{G_x} |f(\gamma^{-1})| d\mu_x(\gamma) \right\}.$$

It is easy to check that $\|\cdot\|_I$ is a norm. We will say a representation $\varphi : C_c^\infty(G) \to \mathcal{B}(\mathcal{H}_\varphi)$ is bounded if it satisfies

$$\|\varphi(f)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \le \|f\|_I$$

for all $f \in C_c^\infty(G)$. The full groupoid C^*-algebra is the completion of $C_c^\infty(G)$ with respect to the norm

$$\sup_\varphi \|\varphi(f)\|_{\mathcal{B}(\mathcal{H})},$$

where φ ranges over all bounded representations of $C_c^\infty(G)$. The full groupoid C^*-algebra is usually denoted by C^*G.

Analogous to the fact that the holonomy group at a fixed point is a quotient of the fundamental group of the leaf passing through the fixed point, there is a canonical quotient map $\pi : G_M \to G_H$ which sends the universal cover of a leaf to its holonomy cover. There is a homomorphism of algebras $\Phi : C_c^\infty(G_M) \to C_c^\infty(G_H)$ which is given by

$$\Phi(f)(\eta) = \sum_{\pi(\gamma)=\eta} f(\gamma).$$

Proposition 3.5. *The map Φ extends to a C^*-algebras homomorphism $C^*G_M \to C^*G_H$.*

Proof. It is straightforward to check that $\|\Phi(f)\|_{C^*G_H} \leq \|\Phi(f)\|_I \leq \|f\|_I$. ■

Within the set of bounded representations of groupoid algebra $C_c^\infty(G)$ there is a distinguished one called regular representation which is described as follows. Let $\{\mu_x\}$ be a right Haar system on the groupoid G. For any $x \in G^{(0)}$, the groupoid algebra $C_c^\infty(G)$ acts on the Hilbert space $L^2(G_x, \mu_x)$ as follows:

$$\pi_x(f)\xi(\gamma) = \int_{\eta \in G_x} f(\gamma \circ \eta^{-1})\xi(\eta) d\mu_x(\eta).$$

It is easy to check that this is a bounded representation. The completion of $C_c^\infty(G)$ with respect to the norm

$$\|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|$$

is denoted by C_r^*G and called the reduced groupoid C^*-algebra. By definition $\|\cdot\|_{C_r^*G} \leq \|\cdot\|_{C^*G}$, so there is a canonical map $C^*G \to C_r^*G$.

Following the construction of groupoid C^*-algebra, we will consider a construction of crossed product C^*-algebra. This algebra will be useful in the following sections. Let B be a C^*-algebra, notice that the $C_c(G, B)$ has a $*$-algebra structure whose multiplication is given in the same way as in (3.1) and the adjoint is given by $f^*(\gamma) = f(\gamma^{-1})^*$.

Definition 3.6. Let B be a C^*-algebra, let $f \in C_c(G, B)$, define a norm $\|\cdot\|_I$ on $C_c(G, B)$:

$$\|f\|_I = \sup_{x \in G^{(0)}} \left\{ \int_{G_x} \|f(\gamma)\|_B d\mu_x(\gamma), \int_{G_x} \|f(\gamma^{-1})\|_B d\mu_x(\gamma) \right\}.$$

A representation $\varphi : C_c(G, B) \to \mathcal{B}(\mathcal{H}_\varphi)$ is called bounded if

$$\|\varphi(f)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \leq \|f\|_I$$

for all $f \in C_c(G, B)$. The C^*-algebra $C^*(G, B)$ is defined to be the completion of $C_c(G, B)$ with respect to the norm

$$\|f\|_{C^*(G,B)} = \sup_\varphi \|\varphi(f)\|_{\mathcal{B}(\mathcal{H}_\varphi)},$$

where φ ranges over all bounded representations.

Proposition 3.7. *Let $B \to C$ be a homomorphism between C^*-algebras , then it induces a homomorphism $C^*(G, B) \to C^*(G, C)$.*

Proof. It is clear that the homomorphism $B \to C$ induces a $*$-homomorphism at the level of continuous maps $\pi : C_c(G, B) \to C_c(G, C)$. Let $\varphi : C_c(G, C) \to \mathcal{B}(\mathcal{H}_\varphi)$ be any bounded representation. Then, the composition

$$C_c(G, B) \to C_c(G, C) \to \mathcal{B}(\mathcal{H}_\varphi) \tag{3.2}$$

is also a representation. For any $f \in C_c(G, B)$, we have the following estimate:

$$\|\varphi(\pi(f))\|_{\mathcal{B}(\mathcal{H}_\varphi)} \leq \|\pi(f)\|_I \leq \|f\|_I,$$

where the first inequality is a consequence of the boundedness assumption on φ and the second inequality is implied by the fact that homomorphism between C^*-algebra is contractive. Therefore, the composition (3.2) is still a bounded representation of $C_c(G, B)$ and $\|f\|_{C^*(G,B)} \geq \|\pi(f)\|_{C^*(G,C)}$ for all $f \in C_c(G.B)$. This completes the proof. ∎

Proposition 3.8. *Let B be a C^*-algebra, $J \subset B$ an ideal. If $G^{(0)}$ is compact, the exact sequence of C^*-algebras $0 \to J \to B \to B/J \to 0$ induces an exact sequence*

$$0 \to C^*(G, J) \to C^*(G, B) \to C^*(G, B/J) \to 0. \tag{3.3}$$

Proof. We first notice that the sequence at the level of continuous maps

$$0 \to C_c(G, J) \to C_c(G, B) \to C_c(G, B/J) \to 0$$

is exact. Indeed, the injectivity of the second arrow and the exactness in the middle term is clear. We only need to show the surjectivity of the third arrow. Pick a $f \in C_c(G, B/J)$ whose support is a compact subset K of a coordinate chart $U \subset G$. There is an open neighborhood U' of K such that the closure of U' is contained in U and is compact. Then, $f \in C_0(U') \otimes B/J$. Since $C_0(U') \otimes B \to C_0(U') \otimes B/J$ is surjective (see [4, Section 3.7], for example), there is a preimage in $C_0(U') \otimes B \subset C_c(U, B) \subset C_c(G, B)$. General elements in $C_c(G, B)$ are spanned by those f's. This proves the surjectivity of $C_c(G, B) \to C_c(G, B/J)$.

It is clear that any bounded representation of $C_c(G, B)$ restricts to a bounded representation of $C_c(G, J)$. The C^*-norm $\|\cdot\|_{C^*(G,J)}$ on $C_c(G, J)$ is greater than or equal to the restriction of $\|\cdot\|_{C^*(G,B)}$ to $C_c(G, J)$. To show the injectivity of the second arrow in (3.3), it suffices to show that any bounded representation of $C_c(G, J)$ extends to a bounded representation of $C_c(G, B)$. Indeed, let $\varphi : C_c(G, J) \to \mathcal{B}(\mathcal{H}_\varphi)$ be a bounded representation of $C_c(G, J)$, let

$$\mathcal{H}' = \text{closure of span}\left\{\varphi(f)h \mid f \in C_c(G, J), h \in \mathcal{H}_\varphi\right\} \subset \mathcal{H}_\varphi$$

be the Hilbert subspace of $\mathcal{H}_\varphi$. The algebra $C_c(G, B)$ acts on $\mathcal{H}'$ in the following way:

$$g \cdot \varphi(f)h = \varphi(gf)h \tag{3.4}$$

for all $g \in C_c(G, B)$ and $f \in C_c(G, J)$. To proceed, we need the following lemma.

Lemma 3.9. *The representation* (3.4) *is bounded.*

Proof of Lemma 3.9. Let $g \in C_c(G, B)$ and $f \in C_c(G, J)$. Since φ is a bounded representation, we have $\|\varphi(gf)\| \le \|g\|_I \cdot \|f\|_I$. Moreover, $\|\varphi(gf)\|^2 = \|\varphi(f^*g^*)\varphi(gf)\| \le \|\varphi(f)\| \cdot \|g\|_I^2 \cdot \|f\|_I$. By induction, we have $\|\varphi(gf)\|^{2^k} \le \|\varphi(f)\|^{2^k-1} \cdot \|g\|_I^{2^k} \cdot \|f\|_I$ for all integers k. Taking the 2^kth root, we have $\|\varphi(gf)\| \le \|\varphi(f)\|^{1-2^{-k}} \cdot \|g\|_I \cdot \|f\|_I^{2^{-k}}$ for all $k \in \mathbb{N}$. Let $k \to \infty$, we get

$$\|\varphi(gf)\| \le \|g\|_I \cdot \|\varphi(f)\|. \tag{3.5}$$

Let $\{e_i\}$ be norm 1 approximate identity of $C^*(G, J)$. Choose a sequence $\{v_i\}$ from $C_c(G, J)$ such that $\|v_i - e_i\|_{C^*(G,J)} \le 1/i$. Then, according to (3.5), we have

$$\varphi(gf)v = \lim_{i\to\infty} \varphi(gv_i f)v.$$

Moreover,

$$\|\varphi(gv_i f)v\| \le \|\varphi(gv_i)\| \cdot \|\varphi(f)v\|_{\mathcal{H}_\varphi} \le \|g\|_I \cdot \|\varphi(v_i)\| \cdot \|\varphi(f)v\|_{\mathcal{H}_\varphi}.$$

Taking the limit $i \to \infty$, we have

$$\|\varphi(gf)v\|_{\mathcal{H}_\varphi} \le \|g\|_I \cdot \|\varphi(f)v\|_{\mathcal{H}_\varphi}.$$

Since elements of the form $\varphi(f)v$ form a dense subspace of $\mathcal{H}'$, the above estimate completes the proof of the lemma. ∎

This shows that for any $f \in C_c(G, J)$, we have $\|f\|_{C^*(G,J)} = \|f\|_{C^*(G,B)}$. Hence, the second arrow of (3.3) is injective. Since the range of homomorphism between C^*-algebras is closed, the third map of (3.3) is surjective. It remains to show the exactness in the middle of (3.3).

A priori, the sequence (3.3) is only a complex, namely, the composition of the second arrow and the third arrow is zero in (3.3). There is a quotient map

$$C^*(G, B)/C^*(G, J) \to C^*(G, B/J). \tag{3.6}$$

On the other hand, $C_c(G, B)/C_c(G, J) \cong C_c(G, B/J)$ sits inside $C^*(G, B/J)$. So, there is a dense embedding

$$C_c(G, B/J) \hookrightarrow C^*(G, B)/C^*(G, J) \tag{3.7}$$

of algebras. Pick any faithful representation $\Psi : C^*(G, B)/C^*(G, J) \to \mathcal{B}(\mathcal{H}_\Psi)$, then the composition with (3.7) gives a representation π of $C_c(G, B/J)$. We will now prove that π is a bounded representation. Let $f \in C_c(G, B/J)$ and $\bar{f} \in C_c(G, B)$ be a lift of f. Then, we have

$$\begin{aligned}\|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\Psi)} &= \|f\|_{C^*(G,B)/C^*(G,J)} \\ &= \inf_{h\in C^*(G,J)} \|\bar{f} + h\|_{C^*(G,B)}.\end{aligned}$$

Let $\{v_i\}$ be approximate identity of J such that $0 < v_i \leq v_j < 1$ in the unitalization of J if $i \leq j$. Then, for any $h \in C_c(G, J)$, we have $h^*(x)(1 - v_i)h(x) \geq h^*(x)(1 - v_j)h(x)$ if $i \leq j$ which implies $\|(1 - v_i)^{1/2}h(x)\|_J \geq \|(1 - v_j)^{1/2}h(x)\|_J$ if $i \leq j$. Therefore, the function

$$g_i(u) = \int_{G^u} \|(1 - v_i)^{1/2}h(x)\|_J dx$$

is continuous in u and decreasing in i. According to the monotone convergence theorem, the function $g_i(u)$ pointwise converge to zero function. On the other hand, since $G^{(0)}$ is compact, according to the Dini theorem, $g_i(u)$ converges uniformly to zero function. This implies that $(1 - v_i)^{1/2} f \to 0$ in I-norm and also in the norm of $C^*(G, J)$. To proceed, we need the following lemma.

Lemma 3.10. *We have* $\|f\|_{C^*(G,B)/C^*(G,J)} = \lim_{i\to\infty} \|(1 - v_i)^{1/2} \bar{f}\|_{C^*(G,B)}$.

Proof of Lemma 3.10. Fix $\varepsilon > 0$, there is $h \in C_c(G, J)$ such that

$$\|\bar{f} - h\|_{C^*(G,B)} \leq \|f\|_{C^*(G,B)/C^*(G,J)} + \varepsilon.$$

Then,

$$\|(1 - v_i)^{1/2} \bar{f}\|_{C^*(G,B)} \leq \|(1 - v_i)^{1/2}(\bar{f} - h)\|_{C^*(G,B)} + \|(1 - v_i)^{1/2}h\|_{C^*(G,B)}.$$

Using the method of Lemma 3.9, we can show that

$$\|(1 - v_i)^{1/2}(\bar{f} - h)\|_{C^*(G,B)} \leq \|(1 - v_i)^{1/2}\| \cdot \|\bar{f} - h\|_{C^*(G,B)}.$$

Therefore, choosing i sufficiently large, we can arrange that

$$\|(1 - v_i)^{1/2} \bar{f}\|_{C^*(G,B)} \leq \|f\|_{C^*(G,B)/C^*(G,J)} + 2\varepsilon.$$

This completes the proof of the lemma. ■

Thanks to the above lemma, we have $\|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\Psi)} \leq \lim_{i\to\infty} \|(1 - v_i)^{1/2} \bar{f}\|_I$. Again, let

$$g_i(u) = \int_{G^u} \|(1 - v_i)^{1/2} \bar{f}(x)\|_B dx$$

and apply the Dini theorem once again, we have $g_i(u)$ is uniformly convergent in u and

$$\lim_{i\to\infty} \sup_u g_i(u) = \sup_u \int_{G^u} \|f\|_{B/J}.$$

Overall, we have $\|\pi(f)\|_{\mathcal{B}(\mathcal{H}_\Psi)} \leq \|f\|_I$. So, π is a bounded representation. Therefore, the norm on $C^*(G, B)/C^*(G, J)$ is less than or equal to the norm on $C^*(G, B/J)$. Together with the fact that homomorphism between C^*-algebra is contractive, we have (3.6) is an isomorphism. This completes the proof. ■

4. Rosenberg index

From this section on, except the last section, we will assume that $F \to M$ is a spin vector bundle of even dimension with spinor given by $S = S^+(F) \oplus S^-(F)$. Let D_+ be the positive part of the leafwise Dirac operator acting on S, which means the following:

- $D_+ : C^\infty(M, S^+) \to C^\infty(M, S^-)$ is a usual differential operator.
- For any smooth section ξ of $S^+ \to M$ and any leaf L of (M, F), the restriction $D_+\xi|_L$ only depends on the restriction $\xi|_L$.
- For any leaf L of M, $D_+|_L : C_c^\infty(L, S^+) \to C_c^\infty(L, S^-)$ is the classical Dirac operator on L.

The notion of leafwise Dirac type operator or more generally, the notion of leafwise elliptic differential operator can be defined in a similar way (see [8, Chapter 2, Section 9]).

The Dirac operator $D_+|_L$ can be lifted to universal covers $D_{+,\widetilde{L}} : C_c^\infty(\widetilde{L}, \pi^* S^+) \to C_c^\infty(\widetilde{L}, \pi^* S^-)$ where $\pi : \widetilde{L} \to L$ is the covering map. All those $D_{+,\widetilde{L}}$'s can be assembled to an operator $D_+ : C_c^\infty(G_M, r^* S^+) \to C_c^\infty(G_M, r^* S^-)$ such that

$$D_+ f(\gamma) = (D_{+,\widetilde{L}_{s(\gamma)}} f)(\gamma),$$

where $f \in C_c^\infty(G_M, r^* S^+)$ and $\widetilde{L}_{s(\gamma)}$ is the universal cover of the leaf passing through $s(\gamma) \in M$. Similarly, the operator $D_- : C_c^\infty(G_M, r^* S^-) \to C_c^\infty(G_M, r^* S^+)$ can be defined.

Proposition 4.1. *The space $C_c^\infty(G_M, r^* S)$ can be completed into a Hilbert $C^* G_M$-module which will be denoted by $\mathcal{E}$.*

Proof. Let $\varphi, \psi \in C_c^\infty(G_M, r^* S)$ and $\gamma \in G_M$, the formula

$$\langle \varphi, \psi \rangle(\gamma) = \int_{\gamma_1 \in G_{M,s(\gamma)}} \langle \varphi(\gamma_1 \circ \gamma^{-1}), \psi(\gamma_1) \rangle d\mu_{s(\gamma)}(\gamma_1)$$

defines a $C^* G_M$-valued inner product on $C_c^\infty(G_M, r^* S)$. Let $f \in C_c^\infty(G_M)$, it acts on $C_c^\infty(G_M, r^* S)$ by

$$\varphi \cdot f(\gamma) = \int_{\gamma_1 \in G_{M,s(\gamma)}} \varphi(\gamma \circ \gamma_1^{-1}) f(\gamma_1) d\mu_{s(\gamma)}(\gamma_1).$$

It is easy to check that this inner product satisfies the pre-Hilbert module condition and $\langle \varphi, \varphi \rangle = 0$ implies $\varphi = 0$. The completion of $C_c^\infty(G_M, r^* S)$ under the norm

$$\|\varphi\|_{\mathcal{E}}^2 = \|\langle \varphi, \varphi \rangle\|_{C^* G_M}$$

is a Hilbert $C^* G_M$-module. ∎

The same constructions can be done for S^+ and S^- and the corresponding Hilbert $C^* G_M$-modules will be denoted by $\mathcal{E}_+$ and $\mathcal{E}_-$, respectively. Clearly, $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$.

Proposition 4.2. *The operators D_+ and D_- are formal adjoint to each other. Namely, for any $f \in C_c^\infty(G_M, r^*S^+)$ and $g \in C_c^\infty(G_M, r^*S^-)$, we have*

$$\langle D_+ f, g\rangle = \langle f, D_- g\rangle.$$

Proof. Indeed,

$$\begin{aligned}\langle D_+ f, g\rangle(\gamma) &= \int_{\gamma_1 \in G_{M,s(\gamma)}} \langle D_+ f(\gamma_1 \circ \gamma^{-1}), g(\gamma_1)\rangle d\mu_{s(\gamma)}(\gamma_1)\\ &= \int_{\gamma_1 \in G_{M,s(\gamma)}} \langle (D_{+,\widetilde{L}_{r(\gamma)}} f)(\gamma_1 \circ \gamma^{-1}), g(\gamma_1)\rangle d\mu_{s(\gamma)}(\gamma_1)\\ &= \langle D_{+,\widetilde{L}_{s(\gamma)}}(U_{\gamma^{-1}} f), g\rangle = \langle U_{\gamma^{-1}} f, D_{-,\widetilde{L}_{s(\gamma)}} g\rangle = \langle f, D_- g\rangle(\gamma),\end{aligned}$$

where U_γ is the translation operator $U_\gamma f(\gamma_1) = f(\gamma_1 \circ \gamma)$. In the last line, the first two inner products are given by the L^2 inner product of the space $C_c^\infty(\widetilde{L}_{s(\gamma)}, \pi^* S)$. The first two terms in the last line are the same because D_+ is formal adjoint to D_- on the universal cover of leaves. ∎

In the following discussion, we will use D_+, D_- for their closure. According to [24, Proposition 21, Lemma 22], D_+ and D_- can be taken as unbounded regular operators and $D_-^* = D_+$. So,

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix} : \mathcal{E} \to \mathcal{E}$$

is self-adjoint and regular. Since $D \pm iI$ is a first order elliptic operator, there is a smoothing operator R and pseudodifferential operator Q of order negative one such that

$$(D \pm iI)Q = I + R.$$

Multiply $(D \pm iI)^{-1}$ on both sides, we get $(D \pm iI)^{-1}$ is compact. So, the functional calculus $f(D)$ (see [16], for example) is compact for all $f \in C_0(\mathbb{R})$.

Recall that in [14, Chapter 10], a continuous function $f : \mathbb{R} \to [-1, 1]$ is called normalizing if

- f is odd;
- $f(c) \geq 0$ if $c \geq 0$;
- $\lim_{c\to\pm\infty} f(c) \to \pm 1$.

Definition 4.3. The Rosenberg index of D is an element $[\alpha]$ in $K_0(C^*G_M)$ which is given by the Kasparov module $(\mathcal{E}, f(D))$ for any normalizing function f.

Proposition 4.4. *If F is spin and (M, F) admits leafwise positive scalar curvature, then $[\alpha] = 0$ as a K-theory element in $K_0(C^*G_M)$.*

Proof. By Lichnerowicz formula, if (M, F) has leafwise positive scalar curvature, the leafwise Dirac D is invertible. It has a spectrum gap around $0 \in \mathbb{R}$. We can choose the normalizing function f such that $f^2 = 1$ on spectrum of D. Under this circumstances, the Kasparov module $(\mathcal{E}, f(D))$ is degenerate. ■

Remark 4.5. In [9], the authors define the longitudinal index as an element in $K_0(C_r^* G_H)$. Following their method, we set the Rosenberg index to live in $K_0(C^* G_M)$. In fact, under the map

$$C^* G_M \to C^* G_H \to C_r^* G_H, \tag{4.1}$$

the Rosenberg index defined above is mapped to the longitudinal index.

5. Twisted Rosenberg index

In this section, we assume B to be a C^*-algebra with unit. The theory of pseudodifferential operators over unital C^*-algebras can be found in [18]. In [18] the author define pseudodifferential operators over unital C^*-algebras for compact smooth manifolds, the method there also works for paracompact manifold. One can choose locally finite partition of unity in the formula (3.12) in [18].

Let $S^m(A^*G, B)$ be the set of all $a \in C^\infty(A^*G, B)$ such that for every compact subset $K \subset G^0$ and every multi-indices α, β there is constant $C_{\alpha,\beta,K} > 0$ with the following inequality:

$$\|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\|_B \leq C_{\alpha,\beta,K} \cdot (1 + |\xi|)^{m-|\beta|}$$

for all $x \in K$. Let $S^m_{\mathrm{phg}}(A^*G, B)$ be the set of all $a \in S^m(A^*G, B)$ such that for every $j \in \mathbb{N}$ one can find $a_{m-j} \in C^\infty(A^*G, B)$ with the property $a_{m-j}(x, t\xi) = t^{m-j} a(x, \xi)$ for all $t > 0$, $\|\xi\| \geq 1$ and

$$a - \sum_{j=0}^{N-1} a_{m-j} \in S^{m-N}(A^*G, B)$$

for all $N \in \mathbb{N}$.

Definition 5.1. A pseudodifferential operator of order m on Lie groupoid G with values in B is a compactly supported G-operator $\{P_x\}_{x \in G^{(0)}}$ in the sense of [24, Section 3.3] such that

- each P_x is a pseudodifferential operator on source fiber $s^{-1}(x)$ of order m over a C^*-algebra B;
- for each trivializing open subset $U \times V \cong \Omega \subset G$ to which the source map restricts to the projection onto the first factor, and for all $\phi, \psi \in C_c(\Omega)$ the operator $\phi P_x \psi$ is given by a symbol $a(x, y, \xi) \in S^m_{\mathrm{phg}}(U \times V \times \mathbb{R}^n, B)$.

If, in addition, the distributional kernel of $\{P_x\}$ is compactly supported, the pseudodifferential operator is called compactly supported. The principal symbol $\sigma_P \in S^*_{\mathrm{phg}}(A^*G, B)$ of a pseudodifferential operator P is defined by

$$\sigma_P(x, \xi) = \sigma(P_x)(x, \xi),$$

where $\sigma(P_x) \in S^*(T^*G_x, B)$ is the principal symbol of P_x as pseudodifferential operator on the source fiber G_x. From the definition, it is clear that if P, Q are compactly supported pseudodifferential operators, then PQ is still a pseudodifferential operator and $\sigma_{PQ} = \sigma_P \cdot \sigma_Q$.

Proposition 5.2. *Pseudodifferential operators on G with compact support of order less than or equal to zero extend to morphisms between $C^*(G, B)$ and pseudodifferential operators with compact support of order strictly less than zero extend to elements of $C^*(G, B)$.*

Proof. See [10, Proposition 3.4]. We first assume that the pseudodifferential operator P has order less than or equal to $p = \dim G^{(0)} - \dim G$. Then, for any trivializing open subset $U \times V \cong \Omega \subset G$ and any $\phi, \psi \in C_c(\Omega)$ the operator $\phi P_x \psi$ has smooth integral kernel. Therefore, P has compactly supported smooth kernel which clearly extends to an element of $C^*(G, B)$.

If P has order $\leq p/2$, then $\|Pf\|^2_{C^*(G,B)} \leq \|\langle Pf, Pf\rangle\|_{C^*(G,B)} \leq \|P^*P\| \cdot \|f\|^2_{C^*(G,B)}$ which implies that P is a multiplier of $C^*(G, B)$. Since P^*P extends to an element of $C^*(G, B)$, it follows that $P \in C^*(G, B)$. By induction, if P has order $\leq p/2^k$ for some integer k, then $P \in C^*(G, B)$. This proves compactly supported pseudodifferential operators of negative order extend to an element of $C^*(G, B)$.

Now, assume that P is of order 0 with principal symbol $\sigma_P \in S^0_{\text{phg}}(A^*G, B)$. Let $c \in \mathbb{R}_+$ such that $c > \sigma_p(x, \xi)$ for all $(x, \xi) \in A^*G$. Put $b(x, \xi) = (c^2 + 1 - |\sigma_p(x, \xi)|^2)^{1/2}$ and let Q be pseudodifferential operator with principal symbol $b(x, \xi)$. Then, $P^*P + Q^*Q$ has principal symbol $1 + c^2$ and is bounded. A direct calculation

$$\|Pf\|^2_{C^*(G,B)} \leq \|\langle Pf, Pf\rangle\|_{C^*(G,B)} \leq \|\langle (P^*P + Q^*Q)f, f\rangle\|_{C^*(G,B)}$$

shows that P is bounded. This completes the proof. ∎

Given $a \in S^m_{\text{phg}}(A^*G, B)$, a pseudodifferential operator $P_a : C^\infty_c(G, B) \to C^\infty_c(G, B)$ can be defined by the formula in [24, Proposition 14]. Namely, we fix a diffeomorphism ϕ from a tubular neighborhood of $G^{(0)} \subset G$ to an open neighborhood W of the zero section of AG. Let χ be a function with values in $[0, 1]$, whose restriction to $G^{(0)}$ equals 1 and its support is contained in W. Let $\xi \in A^*G, \gamma \in G$, and $e_\xi(\gamma) = \chi(\gamma)\exp(i\langle\phi(\gamma), \xi\rangle)$, then P_a is given by the distributional kernel

$$k(\gamma) = \frac{1}{(2\pi)^n}\int_{A^*_{r(\gamma)}G} e_{-\xi}(\gamma^{-1})a(r(\gamma), \xi)d\xi.$$

Then, P_a is a pseudodifferential operator on G of order m whose principal symbol is a.

Remark 5.3. The above discussion also applies to pseudodifferential operators between finitely generated projective Hilbert B-module bundles.

Let $G^{(0)}$ be compact. A pseudodifferential operator is called elliptic if its principal symbol $a \in S^m_{\text{phg}}(A^*G, B)$ is invertible outside a compact neighborhood of the zero section

$G^{(0)} \subset A^*G$. Then, there is $a' \in S^{-m}_{\text{phg}}(A^*G, B)$ which agrees with the inverse of a outside a compact neighborhood of $G^{(0)} \subset A^*G$.

Now, let E be a finitely generated projective Hilbert B-module bundle over M. The space $C^\infty_c(G_M, r^*E)$ can be completed into a Hilbert $C^*(G_M, B)$-module in the same way as Proposition 4.1. Notice that $S^+ \otimes E$ ($S^- \otimes E$, $S \otimes E$, respectively) is still a finitely generated projective Hilbert B-module bundle over M, the corresponding Hilbert module will be denoted by $\mathcal{E}_{+,B}$ ($\mathcal{E}_{-,B}, \mathcal{E}_B$, respectively). Notice that $\mathcal{E}_B = \mathcal{E}_{+,B} \oplus \mathcal{E}_{-,B}$. Let $D_{+,E} : C^\infty_c(G_M, r^*S^+ \otimes r^*E) \to C^\infty_c(G_M, r^*S^- \otimes r^*E)$ denote the leafwise Dirac type operator twisted by E which is a first order elliptic differential operator. The operator $D_{+,E}$ can be taken as an unbounded operator from $\mathcal{E}_{+,B}$ to $\mathcal{E}_{-,B}$. We will use the same notation for its closure.

Proposition 5.4. *The operator $D_{+,E} : \mathcal{E}_{+,B} \to \mathcal{E}_{-,B}$ is regular and $D^*_{+,E} = D_{-,E}$.*

Proof. Since $D_{+,E}$ is elliptic, there is a pseudodifferential operator Q of order -1 such that $D_{+,E}Q - I = R$ and $QD_{+,E} - I = S$ are smoothing operators. Then, the proof in [24, Proposition 21] works verbatim. ■

Let $D_E = \begin{bmatrix} 0 & D_{-,E} \\ D_{+,E} & 0 \end{bmatrix} : \mathcal{E}_B \to \mathcal{E}_B$. It is a self-adjoint regular operator. It can be checked that the operator $(D_E \pm i)^{-1} : \mathcal{E}_B \to \mathcal{E}_B$ is compact.

Proposition 5.5. *Let f be a normalizing function, then the pair $(\mathcal{E}_B, f(D_E))$ forms a Kasparov module and determines an element in $K_0(C^*(G_M, B))$. This element will be called twisted Rosenberg index and denoted by $[D_E]$.*

Proof. It suffices to show that if g vanishes at infinity, $g(D_E) : \mathcal{E}_B \to \mathcal{E}_B$ is a compact operator. The result follows from the fact that $C_0(\mathbb{R})$ is generated by $(x \pm i)^{-1}$ as C^*-algebra. ■

6. The Hilbert module out of leafwise flat bundles

The basic theory of Hilbert C^*-module and Hilbert C^*-module bundle can be found in [23]. Let B be an unital C^*-algebra, let W be a leafwise flat, finitely generated projective Hilbert B-module bundle over M.

The space $C^\infty_c(G_M, r^*W)$ has a $C^\infty_c(G_M, B) \subset C^*(G_M, B)$-valued inner product given by

$$\langle \varphi, \psi \rangle(\gamma) = \int_{\gamma_1 \in G_{M,s(\gamma)}} \langle \varphi(\gamma_1 \circ \gamma^{-1}), \psi(\gamma_1) \rangle d\mu_{s(\gamma)}(\gamma_1), \tag{6.1}$$

where $\varphi, \psi \in C^\infty_c(G_M, r^*W)$. Assume $\mathcal{E}_W$ be the completion of $C^\infty_c(G_M, r^*W)$ under $\|\varphi\|_{\mathcal{E}_W} = \|\langle \varphi, \varphi \rangle\|^{1/2}_{C^*(G_M,B)}$. The space $C^\infty_c(G_M, r^*W)$ has an obvious right $C^\infty_c(G_M, B)$ action which is given by

$$\varphi \cdot f(\gamma) = \int_{\gamma_1 \in G_{M,s(\gamma)}} \varphi(\gamma \circ \gamma_1^{-1}) f(\gamma_1) d\mu_{s(\gamma)}(\gamma_1), \tag{6.2}$$

where $\varphi \in C_c^\infty(G_M, r^*W)$ and $f \in C_c^\infty(G_M, B)$. The action (6.2) extends to a right $C^*(G_M, B)$ action on $\mathcal{E}_W$. As a consequence, $\mathcal{E}_W$ has a Hilbert $C^*(G_M, B)$-module structure.

There is a left $C_0(M)$-action on $C_c^\infty(G_M, r^*W)$ which is given by

$$h \cdot \varphi(\gamma) = h(r(\gamma)) \cdot \varphi(\gamma) \tag{6.3}$$

for all $h \in C_0(M)$ and $\varphi \in C_c^\infty(G_M, r^*W)$.

Proposition 6.1. *For any $h \in C_0(M)$ and $\varphi \in C_c^\infty(G_M, r^*W)$, we have*

$$\|h \cdot \varphi\| \leq \|h\| \cdot \|\varphi\|_{\mathcal{E}_W},$$

where $\|h\|$ is the sup-norm of h in $C_0(M)$.

Proof. We will use the estimate in [21, Lemma 1.1.13]. Let $k \in C_0(M)$ be the function defined by

$$k(m) = (\|h\|^2 - |h(m)|^2)^{1/2}.$$

Then, it is easy to check the following:

$$\begin{aligned}\|h.\varphi\|^2 &= \|\langle h \cdot \varphi, h \cdot \varphi\rangle\| \\ &= \|\,\|h\|^2\langle \varphi, \varphi\rangle - \langle k \cdot \varphi, k \cdot \varphi\rangle\| \\ &\leq \|h\|^2\|\langle \varphi, \varphi\rangle\|,\end{aligned}$$

which completes the proof. ∎

Corollary 6.2 ([21, Proposition 2.1.14]). *The action* (6.3) *extends to a $*$-homomorphism $C_0(M) \to \mathcal{L}(\mathcal{E}_W)$.*

Proof. Thanks to the above proposition, the action extends to the Hilbert module $\mathcal{E}_W$. It is a matter of direct calculation to check that it preserves the $*$-operation. ∎

We will show, in the rest of this section, the Hilbert module $\mathcal{E}_W$ determines a KK-theory element in $KK(C^*G_M, C^*(G_M, B))$. There is a left $C_c^\infty(G_M)$ action on the space $C_c^\infty(G_M, r^*W)$ which is given by

$$f \cdot \varphi(\gamma) = \int_{\gamma_1 \in G_{M,s(\gamma)}} f(\gamma \circ \gamma_1^{-1})(\gamma \circ \gamma_1^{-1}) \cdot \varphi(\gamma_1) d\mu_{s(\gamma)}(\gamma_1), \tag{6.4}$$

where $(\gamma \circ \gamma_1^{-1}) \cdot \varphi(\gamma_1)$ is the image of $\varphi(\gamma_1)$ under the parallel translation along the curve $\gamma \circ \gamma_1^{-1}$. Thanks to the leafwise flatness of W, this parallel translation is well defined. It is also convenient to have an alternative description of the action (6.4).

Let $\mu = \{\mu_x\}$ be a right invariant Haar system on G_M. The inverse map $\iota : G_M \to G_M$ induces a left invariant Haar system which we denote by $\tilde{\mu}$. The space $C_c^\infty(G_M)$ can be completed into Hilbert $C_0(M)$-modules in two ways given by two inner products

$$\langle f, g\rangle_s(m) = \int_{s(\gamma)=m} \overline{f(\gamma)} \cdot g(\gamma) d\mu(\gamma)$$

and

$$\langle f, g\rangle_r(m) = \int_{r(\gamma)=m} \overline{f(\gamma)} \cdot g(\gamma) d\widetilde{\mu}(\gamma),$$

where $f, g \in C_c^\infty(G_M)$. Following [5], we will denote the completions by $L^2(G_M, s, \mu)$ and $L^2(G_M, r, \widetilde{\mu})$, respectively. According to Corollary 6.2, we can form the inner tensor product $L^2(G_M, s, \mu) \otimes_{C_0(M)} \mathcal{E}_W$ and $L^2(G_M, r, \widetilde{\mu}) \otimes_{C_0(M)} \mathcal{E}_W$. We denote by $C_c^\infty(G_M) \otimes_{\mathrm{alg}} C_c^\infty(G_M, r^*W)$ the dense subset of $L^2(G_M, s, \mu) \otimes_{C_0(M)} \mathcal{E}_W$ consists of linear span of elements of the form $f \otimes \varphi$ with $f \in C_c^\infty(G_M)$ and $\varphi \in C_c^\infty(G_M, r^*W)$.

Proposition 6.3. *There is $U : C_c^\infty(G_M) \otimes_{\mathrm{alg}} C_c^\infty(G_M, r^*W) \to L^2(G_M, r, \widetilde{\mu}) \otimes_{C_0(M)} \mathcal{E}_W$ given by*

$$U(F)(\gamma_1, \gamma_2) = \gamma_1 \cdot F(\gamma_1, \gamma_1^{-1} \circ \gamma_2), \tag{6.5}$$

where $\gamma_1 \cdot$ is the parallel translation of W along γ_1.

Proof. We have to show that $U(F) \in L^2(G_M, r, \widetilde{\mu}) \otimes_{C_0(M)} \mathcal{E}_W$. It is enough to verify the case where $F = f \otimes \varphi$. For $\gamma_1, \gamma_2 \in G_M$ with $r(\gamma_1) = r(\gamma_2)$, we have

$$U(F)(\gamma_1, \gamma_2) = f(\gamma_1) \cdot \gamma_1 \cdot \varphi(\gamma_1^{-1} \circ \gamma_2).$$

By using the fact that $C_c^\infty(G_M, r^*W)$ is a finitely generated projective module over $C_c^\infty(G_M, B)$, the above equation can be written as a finite sum,

$$U(F)(\gamma_1, \gamma_2) = \sum_i F_i(\gamma_1, \gamma_2)\varphi_i(r(\gamma_2)),$$

where F_i are compactly supported smooth functions on

$$H = \{(\gamma_1, \gamma_2) \in G_M \times G_M \mid r(\gamma_1) = r(\gamma_2)\}$$

with values in B and φ_i are smooth compactly supported sections of $W \to M$. Since the image of $C_c^\infty(G_M) \otimes_{\mathrm{alg}} C_c^\infty(G_M, B) \to C_c^\infty(H, B)$ is dense in the inductive topology, there is a sequence $F_i^k \in C_c^\infty(G_M) \otimes_{\mathrm{alg}} C_c^\infty(G_M, B)$ such that $F_i^k \to F_i$ in the inductive topology of $C_c^\infty(H, B)$ for all i. Therefore, for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\left\| \sum_i (F_i^k - F_i^{k'}) \cdot r^*\varphi_i \right\|_{L^2(G_M, s, \mu) \otimes_{C_0(M)} \mathcal{E}_W} \leq \varepsilon,$$

whenever $k, k' > N$. As a consequence,

$$\sum_i F_i^k \cdot r^*\varphi_i \in C_c^\infty(G_M) \otimes_{\mathrm{alg}} C_c^\infty(G_M, r^*W)$$

is a Cauchy sequence parametrized by k and converging to $U(F)$ in the topology of $L^2(G_M, s, \mu) \otimes_{C_0(M)} \mathcal{E}_W$. ■

Proposition 6.4. *If F, G belong either to $C_c^\infty(G_M) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W)$ or the image of $C_c^\infty(G_M) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W)$ under U, then*

$$\langle F, G\rangle_{L^2(G_M,r,\tilde{\mu})\otimes_{C_0(M)}\mathcal{E}_W} \in C_c^\infty(G_M, B)$$

and

$$\begin{aligned}\langle F, G\rangle_{L^2(G_M,r,\tilde{\mu})\otimes_{C_0(M)}\mathcal{E}_W}(\gamma) \\ = \int_{s(\gamma_1)=s(\gamma), r(\gamma_2)=r(\gamma_1)} \langle F(\gamma_2, \gamma_1 \circ \gamma^{-1}), G(\gamma_2, \gamma_1)\rangle d\mu(\gamma_1) d\mu(\gamma_2).\end{aligned} \tag{6.6}$$

Proof. If F, G both belong to $C_c^\infty(G_M) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W)$ equation (6.6) is obvious. If at least one of them belongs to the image of U, then F, G can be approximated by $F_i, G_i \in C_c^\infty(G_M) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W)$ as constructed in Proposition 6.3. Moreover, $\langle F_i, G_i\rangle \to \langle F, G\rangle$ in the inductive topology of $C_c^\infty(G_M, B)$. This completes the proof. ∎

Proposition 6.5. *We have*

$$\langle U(F), U(F)\rangle_{L^2(G_M,r,\tilde{\mu})\otimes_{C_0(M)}\mathcal{E}_W} = \langle F, F\rangle_{L^2(G_M,s,\mu)\otimes_{C_0(M)}\mathcal{E}_W}$$

for all $F \in C_c^\infty(G_M) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W)$.

Proof. According to (6.6), we have

$$\begin{aligned}\langle U(F), U(F)\rangle(\gamma) &= \int \langle \gamma_2 \cdot F(\gamma_2, \gamma_2^{-1} \circ \gamma_1 \circ \gamma^{-1}), \gamma_2 \cdot F(\gamma_2, \gamma_2^{-1} \circ \gamma_1)\rangle d\mu(\gamma_1) d\mu(\gamma_2) \\ &= \int \langle F(\gamma_2, \gamma_2^{-1} \circ \gamma_1 \circ \gamma^{-1}), F(\gamma_2, \gamma_2^{-1} \circ \gamma_1)\rangle d\mu(\gamma_1) d\mu(\gamma_2) \\ &= \langle F, F\rangle(\gamma),\end{aligned}$$

where from the first line to the second line we use the fact that parallel translation is unitary and from the second line to the third line we use the right invariance of Haar system (see Definition 3.1 (iii)). ∎

Therefore, the map U can be extended to an isometry

$$U : L^2(G_M, s, \mu) \otimes_{C_0(M)} \mathcal{E}_W \to L^2(G_M, r, \tilde{\mu}) \otimes_{C_0(M)} \mathcal{E}_W.$$

Let $f \in C_c^\infty(G_M)$, let $T_f : \mathcal{E}_W \to L^2(G_M, s, \mu) \otimes_{C_0(M)} \mathcal{E}_W$ denote the Hilbert module map $x \mapsto f \otimes x$ and $T_f^* : L^2(G_M, r, \tilde{\mu}) \otimes_{C_0(M)} \mathcal{E}_W \to \mathcal{E}_W$ be the operator which sends $g \otimes x$ to $\langle f, g\rangle_r \cdot x$. Choose $f_1, f_2 \in C_c^\infty(G_M)$ such that $f = \bar{f}_1 \cdot f_2$, where $\cdot$ is the point-wise multiplication.

Proposition 6.6. *The action (6.4) can be realized as $T_{f_1}^* U T_{f_2}$.*

Proof. Let $\varphi, \psi \in C_c^\infty(G_M, r^*W)$. Then,

$$\begin{aligned}\langle T_{f_1}^* U T_{f_2}\varphi, \psi\rangle(\gamma) &= \langle U(f_2 \otimes \varphi), f_1 \otimes \psi\rangle(\gamma)\\ &= \int \langle U(f_2 \otimes \varphi)(\gamma_2, \gamma_1 \circ \gamma^{-1}), f_1 \otimes \psi(\gamma_2, \gamma_1)\rangle d\mu(\gamma_1) d\mu(\gamma_2)\\ &= \int \langle f_2(\gamma_2) \cdot \gamma_2 \cdot \varphi(\gamma_2^{-1} \circ \gamma_1 \circ \gamma^{-1}), f_1(\gamma_2)\psi(\gamma_1)\rangle d\mu(\gamma_1) d\mu(\gamma_2)\\ &= \int \langle f(\gamma_2)\gamma_2 \cdot \varphi(\gamma_2^{-1} \circ \gamma_1 \circ \gamma^{-1}), \psi(\gamma_1)\rangle d\mu(\gamma_1) d\mu(\gamma_2)\\ &= \langle f \cdot \varphi, \psi\rangle(\gamma),\end{aligned}$$

here from the first line to the second line we use (6.6). This completes the proof. ■

Let E be a finitely generated projective Hilbert B-module and let E^* be the space of adjointable operators between E and B. It has naturally a left B-action which is given by $(b \cdot \varphi)(e) = b \cdot \varphi(e)$ for $b \in B, e \in E$ and $\varphi \in E^*$.

Lemma 6.7. *E^* can be given a Hilbert B-module structure and E^* and E are isomorphic. The isomorphism $E \to E^*$ is given by sending $e \in E$ to the adjointable operator*

$$E \ni e' \mapsto \langle e, e'\rangle_E \in B.$$

Moreover, $E \otimes_B E^* \cong \mathcal{K}_B(E)$.

Proof. If $E = B^n$ for some integer n, an adjointable map $E \to B$ is determined by the images of $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0 \ldots, 0, 1)$ in B which we will denote by $b_1, b_2 \ldots, b_n$. In this case, $E^* = B^n$, and the isomorphism is given by sending $(b_1, b_2, \ldots, b_n) \in E$ to $v \mapsto \langle v, (b_1, b_2, \ldots, b_n)\rangle$. In general, E is finitely generated and projective, there is an orthogonal complemented Hilbert B-module bundle $E^\perp$ with $E \oplus E^\perp = B^n$. An adjointable operator $E \to B$ can be complemented to an adjointable operator $B^n \to B$ and is given by taking the inner product with some element $w \in B^n$. Let p be the projection from B^n to E, then the restriction of the adjointable operator $B^n \to B$ to E is given by sending $v \in E$ to $\langle v, pw\rangle$. Therefore, there is an isomorphism $E \cong E^*$. The space E^* is a left B-module and right Hilbert $\mathcal{K}_B(E)$-module, and the inner tensor product $E \otimes_B E^* \cong \mathcal{K}_B(E)$. ■

Lemma 6.8. *If E_1 and E_2 are two finitely generated projective Hilbert modules over some unital C^*-algebra, then the set of compact operators between E_1 and E_2 equals the set of adjointable operators between E_1 and E_2.*

Proof. Let E_1, E_2 be finitely generated projective Hilbert modules over unital C^*-algebra B. Then, there are complemented Hilbert modules $E_1^\perp, E_2^\perp$ with $E_i \oplus E_i^\perp = B^n$ for some $n \in \mathbb{N}$ and $i = 1, 2$. Then,

$$\mathcal{K}_B(E_1 \oplus E_1^\perp, E_2 \oplus E_2^\perp) = \begin{bmatrix} \mathcal{K}_B(E_1, E_2) & \mathcal{K}_B(E_1, E_2^\perp) \\ \mathcal{K}_B(E_1^\perp, E_2) & \mathcal{K}_B(E_1^\perp, E_2^\perp) \end{bmatrix}. \tag{6.7}$$

On the other hand, since B is unital, $\mathcal{K}_B(E_1 \oplus E_1^{\perp}, E_2 \oplus E_2^{\perp}) = \mathcal{L}_B(E_1 \oplus E_1^{\perp}, E_2 \oplus E_2^{\perp})$ and

$$\mathcal{L}_B(E_1 \oplus E_1^{\perp}, E_2 \oplus E_2^{\perp}) = \begin{bmatrix} \mathcal{L}_B(E_1, E_2) & \mathcal{L}_B(E_1, E_2^{\perp}) \\ \mathcal{L}_B(E_1^{\perp}, E_2) & \mathcal{L}_B(E_1^{\perp}, E_2^{\perp}) \end{bmatrix}. \tag{6.8}$$

By comparing (6.7) with (6.8), we have $\mathcal{K}_B(E_1, E_2) = \mathcal{L}_B(E_1, E_2)$. ∎

Proposition 6.9. *The action* (6.4) *extends to a* $*$*-homomorphism* $C^*G_M \to \mathcal{L}(\mathcal{E}_W)$ *whose image is contained in the algebra of compact operators on* $\mathcal{E}_W$.

Proof. According to the above discussion, we have

$$\|f \cdot \varphi\| \le \|T_{f_1}^*\| \cdot \|T_{f_2}\| \cdot \|\varphi\|, \tag{6.9}$$

where we omit the norm of U since it is an isometry. It is easy to check that

$$\|T_f\| = \|f\|_{L^2(G_M, s, \mu)} = \sup_{x \in G_M^{(0)}} \left| \int_{s(\gamma)=x} |f(\gamma)|^2 d\mu(\gamma) \right|^{1/2}$$

and

$$\|T_f^*\| = \|f\|_{L^2(G_M, r, \tilde{\mu})} = \sup_{x \in G_M^{(0)}} \left| \int_{r(\gamma)=x} |f(\gamma)|^2 d\tilde{\mu}(\gamma) \right|^{1/2}.$$

Let $f_1(\gamma) = |f(\gamma)|^{1/2}$, $f_2(\gamma) = f(\gamma)/f_1(\gamma)$ if $f(\gamma) \neq 0$ and $f_2(\gamma) = 0$ if $f(\gamma) = 0$. In this way, we have $f = f_1 \cdot f_2$ and $|f_1|^2 = |f_2|^2 = |f|$. According to the definition $\|f\|_I = \max\{\|T_{f_2}\|^2, \|T_{f_1}^*\|^2\}$. Therefore, the inequality (6.9) becomes

$$\|f . \varphi\| \le \|f\|_I \cdot \|\varphi\|,$$

which completes the extension part of the proof.

Since B is unital, according to Lemma 6.8 the parallel translation along a curve $\gamma \in G_M$ is an element of the space of compact operators $\mathcal{K}(W_{s(\gamma)}, W_{r(\gamma)})$. Therefore, the action of $C_c^\infty(G_M)$ is given by convolution multiplication with an element in $C_c^\infty(G_M, r^*W \otimes_B s^*W^*)$.

It suffices to show that the operator given by convolution multiplication with element in $C_c^\infty(G_M, r^*W \otimes_B s^*W^*)$ is a compact operator. Let φ_0, ψ_0 be sections of $W \to M$, $f \in C_c^\infty(G_M, B)$, and denote by ψ_0^* the section of $W^* \to M$ which is given by $\psi_0^*(\varphi_0) = \langle \psi_0, \varphi_0 \rangle_W$. Since W is a finitely generated projective Hilbert module bundle over M, $C_c^\infty(M, W)$ is a finitely generated projective module over $C_c^\infty(M, B)$. Hence, $C_c^\infty(G_M, r^*W), C_c^\infty(G_M, s^*W^*)$ are finitely generated projective modules over the space $C_c^\infty(G_M, B)$. More precisely, let $\{\varphi_i\}$ be a finite sequence of smooth sections of $W \to M$ such that the span of $\{\varphi_i(m)\}$ is W_m for all $m \in M$. Then, $C_c^\infty(G_M, r^*W)$ can be obtained as span of $f_i \cdot r^*\varphi_i$ where $f_i \in C_c^\infty(G_M, B)$. Similar result holds for $C_c^\infty(G_M, s^*W^*)$.

Accordingly, $\{\varphi_i(m) \otimes \varphi_j^*(n)\}$ span the vector space $W_m \otimes W_n^*$ for all $m \in M$ and $n \in M$. Therefore, elements in $C_c^\infty(G_M, r^*W \otimes_B s^*W^*)$ can be written as span of

$$r^*\varphi \otimes f \otimes s^*\psi^*, \tag{6.10}$$

where $\varphi, \psi \in C_c^\infty(M, W)$, $\psi^* \in C_c^\infty(M, W^*)$ and $f \in C_c^\infty(G_M, B)$. It suffices to show the operator $T_{\varphi,f,\psi}$ which is given by convolution multiplication with elements of the form (6.10) is a compact operator.

If there are $f_1, f_2 \in C_c^\infty(G_M, B)$ such that $f_1 * f_2 = f$, we pick $\varphi_1 \in C_c^\infty(G_M, r^*W)$ which is given by $\varphi_1(\gamma) = \varphi(r(\gamma)) f_1(\gamma)$ and $\psi_1 \in C_c^\infty(G_M, r^*W)$ which is given by $\psi_1(\gamma^{-1}) = \psi(s(\gamma)) f_2(\gamma)$. Then, $\theta_{\varphi_1,\psi_1} h(\gamma)$ equals

$$\begin{aligned}
&\varphi_1 \cdot \langle \psi_1, h\rangle(\gamma)\\
&= \int_{\gamma_1 \in G_{M,s(\gamma)}} \varphi_1(\gamma\gamma_1^{-1}) \langle \psi_1, h\rangle(\gamma_1) d\mu_{s(\gamma)}(\gamma_1)\\
&= \int_{\gamma_1 \in G_{M,s(\gamma)}} \varphi_1(\gamma\gamma_1^{-1}) d\mu_{s(\gamma)}(\gamma_1) \int_{\gamma_2 \in G_{M,s(\gamma_1)}} \langle \psi_1(\gamma_2\gamma_1^{-1}), h(\gamma_2)\rangle d\mu_{s(\gamma_1)}(\gamma_2)\\
&= \int \varphi(r(\gamma)) f_1(\gamma\gamma_1^{-1}) \langle \psi(r(\gamma_2)) f_2(\gamma_1\gamma_2^{-1}), h(\gamma_2)\rangle d\mu_{s(\gamma)}(\gamma_1) d\mu_{s(\gamma_1)}(\gamma_2)\\
&= \int \varphi(r(\gamma)) f(\gamma\gamma_2^{-1}) \psi^*(r(\gamma_2)) h(\gamma_2) d\mu_{s(\gamma)}(\gamma_2)\\
&= \int_{\gamma_2 \in G_{M,s(\gamma)}} (r^*\varphi \otimes f \otimes s^*\psi^*)(\gamma\gamma_2^{-1}) h(\gamma_2) d\mu_{s(\gamma)}(\gamma_2),
\end{aligned}$$

where from the first line to the second line we use equation (6.2), from the second line to the third line we use equation (6.1), from the third line to the fourth line we plug-in the definition of φ_1 and ψ_1 and from the fourth line to the fifth line we use the assumption that $f = f_1 * f_2$. Therefore, the operator $T_{\varphi,f,\psi}$ which is the convolution with the element of the form (6.10) is a compact operator.

In general, since C^*-algebras have approximate identity and $C_c^\infty(G_M, B)$ is dense in $C^*(G_M, B)$, any $f \in C_c^\infty(G_M, B)$ can be approximated by elements of the form $f_1 * f_2$ in norm $\|\cdot\|_{C^*(G_M,B)}$. It can be checked that the operator norm of $r^*\varphi \otimes f \otimes s^*\varphi^*$ is less than or equal to

$$\|\varphi\|_\infty \cdot \|f\|_{C^*(G_M,B)} \cdot \|\psi\|_\infty,$$

where $\|\varphi\|_\infty = \sup_{m \in M} \|\varphi(m)\|_W$ and $\|\psi\| = \sup_{m \in M} \|\psi(m)\|_W$. Then, for any $\varepsilon > 0$ there is $f_1, f_2 \in C_c^\infty(G_M, B)$ such that

$$\|f - f_1 * f_2\|_{C^*(G_M,B)} < \varepsilon/(\|\varphi\|_\infty \cdot \|\psi\|_\infty),$$

and there is a compact operator θ_ε such that $\|\theta_\varepsilon - T_{\varphi,f,\psi}\| \leq \varepsilon$. This completes the proof. ■

Corollary 6.10. *The Hilbert module $\mathcal{E}_W$ together with the zero operator $(\mathcal{E}_W, 0)$ form a Kasparov module which defines an element in $KK(C^*G_M, C^*(G_M, B))$.* ■

7. Compactly enlargeable foliation

Pick a complex vector bundle E over sphere S^n such that all its Chern classes vanish except the top-degree one $c_n(E) \neq 0$. Let $\widetilde{M}_\varepsilon$ be the compact cover with fundamental group H_ε , the pull back bundle f^*E can be extended to a G/H_ε-equivariant bundle

$$\bigoplus_{g \in G/H_\varepsilon} g^*(f^*E) \to \widetilde{M}_\varepsilon,$$

which can be reduced to a vector bundle E_ε over M. As a result all Chern classes of E_ε vanish except the top degree one $c_n(E_\varepsilon) \neq 0$. As ε ranges over $1, 1/2, 1/3, \ldots$, we get a sequence of bundles E_i. We will denote by $P_i \to M$ the frame bundle of E_i which are by themselves principal $U(d_i)$ bundles. They are equipped with natural connections whose leafwise curvatures tend to zero as $i \to \infty$.

In the following discussion, we will make use of several C^*-algebras A, A', Q and their variations. The definitions are given in Definition 1.2 and Definition 1.8. Let q_i denote the image of $1 \in U(d_i)$ on $\mathcal{K}$. We will consider the family of Hilbert $q_i \mathcal{K} q_i \cong M_{d_i}(\mathbb{C})$-module bundles

$$V_i = P_i \times_{U(d_i)} q_i \mathcal{K} q_i, \tag{7.1}$$

where $U(d_i)$ acts on $\mathcal{K}$ by matrix multiplications. We will briefly explain how they can be assembled into a leafwise flat Hilbert qQq-module (see [12, Section 2] for a detailed construction). Indeed, let $\{U_\alpha\}$ be an open cover of M over which each V_i is trivializable and each U_α is homeomorphic to an unit open disc $(0, 1)^n$. We can choose local trivializations

$$\psi_{\alpha,i} : V_i|_{U_\alpha} \to U_\alpha \times q_i \mathcal{K} q_i \tag{7.2}$$

as in [12, Section 2] such that

$$\nabla^i_{\frac{\partial}{\partial x_k}} s = 0 \tag{7.3}$$

if s is a smooth section which is constant, under the trivialization, in $[0, 1]^k \times \{0\} \times \cdots \times \{0\}$ the first k variable of U_α. Here, ∇^i is the connection on V_i.

The corresponding transition functions is denoted by

$$\varphi_{\alpha,\beta,i} : U_\alpha \cap U_\beta \to \mathrm{End}(q_i \mathcal{K} q_i) \cong q_i \mathcal{K} q_i.$$

Since the norm of curvature of V_i is universally bounded with respect to $i \in \mathbb{N}$. According to [12, Lemma 2.3, Lemma 2.5, and Proposition 2.6], $\varphi_{\alpha,\beta,i}$ is a Lipschitz function with Lipschitz constant independent of i. Therefore, the transition functions can be assembled into

$$\varphi_{\alpha,\beta} = (\varphi_{\alpha,\beta,1}, \varphi_{\alpha,\beta,2}, \ldots, \varphi_{\alpha,\beta,i}, \ldots), \tag{7.4}$$

which is a Lipschitz map from $U_\alpha \cap U_\beta$ to qAq. It determines a Lipschitz Hilbert qAq-module bundle over M which can be approximated by a smooth Hilbert qAq-module bundle V over M.

The properties of bundle V are summarized in the following.

Proposition 7.1. *There is a Hilbert qAq-module bundle V over M such that*

- V_i*, defined in* (7.1)*, is isomorphic to* $V \cdot q_i A_i q_i$ *as Hilbert* $\mathcal{K}$*-module bundle;*
- *the connection of* V *preserves subbundle* V_i*;*
- *the leafwise curvature takes values in* $\hom(qAq, qA'q)$.

Therefore, the bundle $W = V/qA'q$ is a leafwise flat Hilbert qQq-module bundle which, according to Corollary 6.10, determines an element in $KK(C^*G_M, C^*(G_M, qQq))$. The KK-element induces $(\phi_1)_* : K_0(C^*G_M) \to K_0(C^*(G_M, qQq))$. The above procedure can be replicated if we start with a sequence of trivial principal bundles $\{P_i'\}$ with $P_i' = M \times M_{d_i}(\mathbb{C})$. We will get a new KK-theory element in $KK(C^*G_M, C^*(G_M, qQq))$ and corresponding $(\phi_2)_* : K_0(C^*G_M) \to K_0(C^*(G_M, qQq))$. Let

$$\phi_* = (\phi_1)_* - (\phi_2)_*. \tag{7.5}$$

Recall that the Rosenberg index $[\alpha] \in K_0(C^*G_M)$ is given in Definition 4.3.

Proposition 7.2. *Let* $[D_W] \in K_0(C^*(G_M, qQq))$ *denote the image of* $[\alpha] \in K_0(C^*G_M)$ *under the map* $(\phi_1)_* : K_0(C^*G_M) \to K_0(C^*(G_M, qQq))$*. Then,* $[D_W]$ *coincides with the Rosenberg index twisted by the leafwise flat Hilbert* qQq*-module bundle* W*.*

Proof. $[\alpha]$ is given by the Kasparov module $(\mathcal{E}, f(D))$, while the KK-theory element is given by the Kasparov module $(\mathcal{E}_W, 0)$. Their Kasparov product is given by the pair $(\mathcal{E} \otimes_{C^*G_M} \mathcal{E}_W, f(D) \otimes 1)$. According to the definition, the inner tensor product is completion of $\mathcal{E} \otimes_{\text{alg}} \mathcal{E}_W / N$, where N is the span of elements of the form

$$\varphi \cdot a \otimes \psi - \varphi \otimes \Theta(a)\psi$$

with $\varphi \in \mathcal{E}, a \in C^*G_M, \psi \in \mathcal{E}_W$ and $\Theta : C^*G_M \to \mathcal{L}(\mathcal{E}_W)$ being the map defined in Proposition 6.9. Consider the following map $\pi : C_c^\infty(G_M, r^*S) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W) \to C_c^\infty(G_M, r^*S \otimes r^*W)$ given by

$$\pi(\varphi \otimes \psi)(\gamma) = \int_{G_{M,s(\gamma)}} \varphi(\gamma \circ \gamma_1^{-1}) \otimes (\gamma \circ \gamma_1^{-1}) \cdot \psi(\gamma_1) d\mu(\gamma_1), \tag{7.6}$$

where $\varphi \in C_c^\infty(G_M, r^*S)$, $\psi \in C_c^\infty(G_M, r^*W)$ and $(\gamma \circ \gamma_1^{-1}).\psi(\gamma_1)$ is the parallel translation of $\psi(\gamma_1)$ along the curve $\gamma \circ \gamma_1^{-1}$. It is a matter of direct calculation to check that π vanishes on $C_c^\infty(G_M, r^*S) \otimes_{\text{alg}} C_c^\infty(G_M, r^*W) \cap N$ and preserves the inner product if taken as a map from $\mathcal{E} \otimes_{C^*G_M} \mathcal{E}_W$ to the completion of $C_c^\infty(G_M, r^*S \otimes r^*W)$.

The covariant derivative on $S \otimes W$ is given by $\nabla^{S \otimes W} = \nabla^S \otimes 1 + 1 \otimes \nabla^W$. We have

$$\nabla_{e_i}^{S \otimes W} \pi(\varphi \otimes \psi)(\gamma) = \int_{G_{M,s(\gamma)}} \nabla_{e_i}^S \varphi(\gamma \circ \gamma_1^{-1}) \otimes (\gamma \circ \gamma_1^{-1}) \cdot \psi(\gamma_1) d\mu(\gamma_1),$$

where $1 \otimes \nabla^W$ does not appear because $(\gamma \circ \gamma_1^{-1}).\psi(\gamma_1)$ is, by definition, parallel with respect to the curve γ and the connection ∇^W. So, the operator $f(D) \otimes 1$ is precisely $f(D_W)$ under the identification (7.6). ∎

By the same reason, the image of $[\alpha]$ under the map $(\phi_2)_*$ is the Rosenberg index $[D_{qQq}]$ twisted by the trivial bundle $M \times qQq$. Let $\pi : A \to Q$ be the canonical projection, it induces $\pi_* : K_0(C^*(G_M, qAq)) \to K_0(C^*(G_M, qQq))$. Let $[D_V], [D_{qAq}] \in K_0(C^*(G_M, qAq))$ be the elements defined by the leafwise Dirac-type operators twisted by the non-flat bundle V and the trivial bundle $M \times qAq$, respectively. Then, it is straightforward to verify that we have $\pi_*[D_V] = [D_W]$ and $\pi_*[D_{qAq}] = [D_{qQq}]$ (see also [12, Lemma 3.1]).

Consider the following composition:

$$K_0(C^*(G_M, qAq)) \to K_0(C^* G_M) \to K_0(C_r^* G_H), \tag{7.7}$$

where the first arrow is given by the homomorphism sending A to its ith component $\mathcal{K}$, and the second arrow is given by (4.1).

Proposition 7.3. *The image of* $[D_V]$ *under the map* (7.7) *is computed by the longitudinal index element corresponding to* D_{E_i}.

Proof. It is a consequence of Remark 4.5. ∎

Proposition 7.4. *We have* $K_0(C^*(G_M, qA'q)) = \bigoplus K_0(C^* G_M)$.

Proof. By the Dini theorem, the subspace

$$\bigoplus C_c^\infty(G_M, q_i \mathcal{K} q_i) \subset C_c^\infty(G_M, qA'q)$$

is dense in the I-norm. It is clear that $\bigoplus_{i=1}^k q_i \mathcal{K} q_i$ is an ideal in $qA'q$ for all $k \in \mathbb{N}$. According to Proposition 3.8, we have $\bigoplus_{i=1}^k C^*(G_M, q_i \mathcal{K} q_i) \subset C^*(G_M, qA'q)$ for all $k \in \mathbb{N}$. Therefore, the C^*-algebra $C^*(G_M, qA'q)$ can be realized as direct limit of $\bigoplus C^*(G_M, q_i \mathcal{K} q_i)$. ∎

Proposition 7.5. *Let* $\phi_* : K_0(C^* G_M) \to K_0(C^*(G_M, qQq))$ *be defined as in* (7.5). *Then,* $\phi_*[\alpha] \neq 0$ *in* $K_0(C^*(G_M, qQq))$.

Proof. By Proposition 7.2, the image of $[D_V] - [D_{qAq}] \in K_0(C^*(G_M, qAq))$ under the map

$$\pi_* : K_0(C^*(G_M, qAq)) \to K_0(C^*(G_M, qQq))$$

is precisely $\phi_*[\alpha]$. By the exact sequence (3.3), it suffices to show that $[D_V] - [D_{qAq}] \in K_0(C^*(G_M, qAq))$ does not come from the image of $K_0(C^*(G_M, qA'q))$.

Consider the following commutative diagram:

$$\begin{array}{ccc} K_0(C^*(G_M, qA'q)) & \longrightarrow & K_0(C^*(G_M, qAq)) \\ & \searrow & \downarrow \\ & & \prod K_0(C^* G_M), \end{array}$$

where the downward arrows are given by sending A and A' to A_i's. By the Proposition 7.4, it then suffices to show the image of $[D_V] - [D_{qAq}]$ under the vertical downward arrow has infinitely many nonzero terms.

Indeed, according to Proposition 7.3, the image of the ith component of $[D_V] - [D_{qAq}]$ under the map $K_0(C^*G_M) \to K_0(C_r^*G_H)$ is given by the longitudinal index of the Dirac type operator twisted by the virtual bundle $E_i - \mathbb{C}^{d_i}$. And according to Connes [7], there is a transverse fundamental class μ such that

$$\mu([D_{E_i - \mathbb{C}^{d_i}}]) = \langle \widehat{A}(F)\,\mathrm{ch}(E_i - \mathbb{C}^{d_i}), [M] \rangle,$$

where $[M]$ is a fundamental class of M. Our non-vanishing assumption on top Chern classes ensure that the sequence $\mu([D_{E_i - \mathbb{C}^{d_i}}])$ is nonzero for all i. ∎

The above proposition directly implies the following theorem.

Theorem 7.6. *If (M, F) is a compactly enlargeable foliation in the sense of Definition 1.6 with F spin and even dimensional, then $[\alpha] \neq 0$ in $K_0(C^*G_M)$.*

8. Reduction to compact case

By the definition of enlargeability, the pullback $f_\varepsilon^* E \to \widetilde{M}_\varepsilon$ and the trivial bundle $\mathbb{C}^d \to \widetilde{M}_\varepsilon$ are isomorphic outside the compact subset $K_\varepsilon \subset \widetilde{M}_\varepsilon$. Since $\widetilde{M}_\varepsilon$ is locally compact, there is an open neighborhood $K_\varepsilon \subset K_\varepsilon'$ whose closure is compact. Let $\theta_\varepsilon : f_\varepsilon^* E \to \mathbb{C}^d$ be the unitary outside K_ε' which, according to the Tietze extension theorem, admits an extension to $\widetilde{M}_\varepsilon$. We will use the same notation to denote the extension.

$$\begin{array}{ccc} f_\varepsilon^* E & \xrightarrow{\theta_\varepsilon} & \mathbb{C}^d \\ \downarrow & & \downarrow \\ \widetilde{M}_\varepsilon & \xrightarrow{\mathrm{id}} & \widetilde{M}_\varepsilon \end{array} \qquad\qquad \begin{array}{ccc} E_\varepsilon & \xrightarrow{\theta_\varepsilon^{\mathcal{K}}} & C_T^\varepsilon \\ \downarrow & & \downarrow \\ M & \xrightarrow{\mathrm{id}} & M \end{array}$$

Proposition 8.1. *There is a bundle map*

$$\theta_\varepsilon^{\mathcal{K}} : E_\varepsilon \to C_T^\varepsilon$$

such that the restriction $\theta_\varepsilon^{\mathcal{K}}|_{U_\alpha}$ is given by left multiplication of element in $C(U_\alpha, C_T^{\varepsilon,+})$ for all α. Here, we denote by $C_T^{\varepsilon,+}$ the one-point unitlization.

Proof. Under the trivializations (1.4), the bundle map $\theta_\varepsilon : f_\varepsilon^* E \to \mathbb{C}^d$ is given by

$$\theta_\varepsilon|_{\pi_\varepsilon^{-1}(U_\alpha)} \circ \varphi_\alpha^{-1},$$

which can be taken as a G/H_ε family of $M(\mathbb{C}^d)$ all but finitely many are 1. In other words, it can be taken as a map from U_α to $C_T^{\varepsilon,+}$. Define $\theta_\varepsilon^{\mathcal{K}}$ to be the left multiplication with $\theta_\varepsilon|_{\pi_\varepsilon^{-1}(U_\alpha)} \circ \varphi_\alpha^{-1}$ over U_α for all α. It is clear that this definition of $\theta_\varepsilon^{\mathcal{K}}$ is invariant under the transition functions of E_ε. ∎

Proposition 8.2. *Denote by $C(M, E_\varepsilon \oplus C_T^\varepsilon)$ the graded Hilbert module over $C(M) \otimes C_T^\varepsilon$ whose even part is given by E_ε and the odd part is given by the trivial bundle C_T^ε. Then, the triple*

$$\left(C(M, E_\varepsilon \oplus C_T^\varepsilon), \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \theta_\varepsilon^{\mathcal{K},*} \\ \theta_\varepsilon^{\mathcal{K}} & 0 \end{bmatrix} \right) \tag{8.1}$$

is a Kasparov module in $KK(\mathbb{C}, C(M) \otimes C_T^\varepsilon)$.

Proof. It suffices to check that $\begin{bmatrix} 0 & \theta_\varepsilon^{\mathcal{K},*} \\ \theta_\varepsilon^{\mathcal{K}} & 0 \end{bmatrix}^2 - 1$ is a compact operator. Indeed, according to Proposition 8.1, $\theta_\varepsilon^{\mathcal{K}} \theta_\varepsilon^{\mathcal{K},*} - 1$ and $\theta_\varepsilon^{\mathcal{K},*} \theta_\varepsilon^{\mathcal{K}} - 1$ are given by left multiplication with elements in $C^\infty(M, C_T^\varepsilon)$ and $C(M, \mathcal{K}(E_\varepsilon))$, respectively. ■

Let $\mathcal{H}$ be the standard separable Hilbert space. Analogous to the observation 1.9, the same set of transition functions (1.5) together with the trivializations $U_\alpha \times \mathcal{H}$ build a bundle of Hilbert spaces $\mathcal{H}_\varepsilon$, and parallel to Proposition 8.1, there is a bundle map $\theta_\varepsilon^{\mathcal{H}} : \mathcal{H}_\varepsilon \to \mathcal{H} \times M$ such that $\theta_\varepsilon^{\mathcal{H}} \theta_\varepsilon^{\mathcal{H},*} - 1$ and $\theta_\varepsilon^{\mathcal{H},*} \theta_\varepsilon^{\mathcal{H}} - 1$ are given by compact operators.

The KK-equivalence between $\mathcal{K}$ and $\mathbb{C}$ is implemented by the elements $x = (\mathcal{H}, 1, 0) \in KK(\mathcal{K}, \mathbb{C})$ and $y = (\mathcal{K}, p_1, 0) \in KK(\mathbb{C}, \mathcal{K})$, where $p_1 \in \mathcal{K}$ is some rank one projection. Under the Kasparov product $KK(\mathbb{C}, C(M) \otimes \mathcal{K}) \otimes KK(\mathcal{K}, \mathbb{C}) \to K_0(C(M))$ the Kasparov module (8.1) becomes

$$\left(C(M, \mathcal{H}_\varepsilon \oplus \mathcal{H}), \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \theta_\varepsilon^{\mathcal{H},*} \\ \theta_\varepsilon^{\mathcal{H}} & 0 \end{bmatrix} \right). \tag{8.2}$$

We recall a trick used in [2], to find an equivalent finite dimensional virtual bundle to (8.2). This trick has its root in Atiyah–Jänich theorem [1, 15].

Proposition 8.3. *There is a finite set of sections $\{s_1, s_2, \dots, s_q\}$ of $\mathcal{H} \times M \to M$ such that the map $\overline{\theta}_\varepsilon : C(M, \mathcal{H}_\varepsilon \oplus \mathbb{C}^q) \to C(M, \mathcal{H})$ given by*

$$(u, \lambda_1, \lambda_2, \dots, \lambda_q) \mapsto \theta_\varepsilon^{\mathcal{H}}(u) + \sum_{i=1}^{q} \lambda_i s_i$$

is surjective and whose kernel is a sub-bundle of $\mathcal{H}_\varepsilon \oplus \mathbb{C}^q$. Moreover, the K-theory element (8.2) is equivalent to the virtual bundle $[\mathrm{Ker}(\overline{\theta}_\varepsilon)] - [M \times \mathbb{C}^q]$ in $K_0(C(M))$.

Proof. For any $m_0 \in M$, there is an open neighborhood U_{m_0} where the bundle $\mathcal{H}_\varepsilon$ is trivial. Then, the restriction $\theta_\varepsilon^{\mathcal{H}}|_{U_{m_0}}$ can be identified with a map

$$U_{m_0} \xrightarrow{\theta_\varepsilon^{\mathcal{H}}} \mathcal{F}(\mathcal{H}),$$

here, $\mathcal{F}(\mathcal{H})$ denotes the set of Fredholm operators on Hilbert space $\mathcal{H}$. Let us assume that $V_0 = \ker(\theta_\varepsilon^{\mathcal{H},*}(m_0))$, and define $T_m^0 : \mathcal{H}_\varepsilon \oplus V_0 \to \mathcal{H}$ to be

$$T_m^0(u \oplus v) = \theta_\varepsilon^{\mathcal{H}} u(m) + v.$$

It is surjective at $m = m_0$, and therefore, surjective in an open neighborhood W_0 of m_0. Since M is compact, it can be covered by finitely many such open sets W_i where the maps

$$T_m^i : \mathcal{H}_\varepsilon \oplus V_i \to \mathcal{H}$$

are surjective. Let ρ_i be the partition of unity associated to the cover $\{W_i\}$. Define

$$\overline{\theta}_\varepsilon : C\left(M, \mathcal{H}_\varepsilon \bigoplus \oplus_i V_i\right) \to C(M, \mathcal{H})$$

to be

$$\overline{\theta}_\varepsilon(u, v_i)(m) = \sum_i \rho_i(m) T_m^i(u, v_i),$$

which is clearly surjective.

Over the open subset U_{m_0}, $\overline{\theta}_\varepsilon$ can be identified with $U_{m_0} \xrightarrow{\overline{\theta}_\varepsilon} \mathcal{B}(\mathcal{H} \oplus \mathbb{C}^q, \mathcal{H})$. Its composition with $\mathcal{B}(\mathcal{H} \oplus \mathbb{C}^q, \mathcal{H}) \to \mathcal{B}(\ker(\overline{\theta}_\varepsilon(m_0))^\perp, \mathcal{H})$ is invertible at m_0, and therefore, invertible on an open neighborhood of m_0 where the kernel of $\overline{\theta}_\varepsilon$ is trivial.

Now, we will verify the equality

$$\text{The Kasparov module (8.2)} = (\ker(\overline{\theta}_\varepsilon) \oplus C(M, \mathbb{C}^q), 1, 0)$$

in $KK(\mathbb{C}, C(M))$. Indeed, by adding degenerate Kasparov module, we have

$$\begin{aligned}\text{The Kasparov module (8.2)} &= \left(C(M, \mathcal{H}_\varepsilon \oplus \mathbb{C}^q \oplus \mathcal{H} \oplus \mathbb{C}^q), 1, \begin{bmatrix} 0 & \theta_\varepsilon^{\mathcal{H},*} \\ \theta_\varepsilon^{\mathcal{H}} & 0 \end{bmatrix}\right) \\ &= \left(C(M, \mathcal{H}_\varepsilon \oplus \mathbb{C}^q \oplus \mathcal{H} \oplus \mathbb{C}^q), 1, \begin{bmatrix} 0 & \overline{\theta}_\varepsilon^* \\ \overline{\theta}_\varepsilon & 0 \end{bmatrix}\right),\end{aligned}$$

here from the first line to the second line is a compact perturbation. Decomposing according to $\overline{\theta}_\varepsilon$, the above equation continues

$$\begin{aligned}&= \left(\ker(\overline{\theta}_\varepsilon) \oplus \ker(\overline{\theta}_\varepsilon)^\perp \oplus C(M, \mathcal{H}) \oplus C(M, \mathbb{C}^q), 1, \begin{bmatrix} 0 & \overline{\theta}_\varepsilon^* \\ \overline{\theta}_\varepsilon & 0 \end{bmatrix}\right) \\ &= (\ker(\overline{\theta}_\varepsilon) \oplus C(M, \mathbb{C}^q), 1, 0) \bigoplus \left(\ker(\overline{\theta}_\varepsilon)^\perp \oplus C(M, \mathcal{H}), 1, \begin{bmatrix} 0 & \overline{\theta}_\varepsilon^* \\ \overline{\theta}_\varepsilon & 0 \end{bmatrix}\right).\end{aligned}$$

It suffices to show that the second summand is a degenerate Kasparov module. Indeed, let $U_\varepsilon = \overline{\theta}_\varepsilon(\overline{\theta}_\varepsilon^* \overline{\theta}_\varepsilon)^{-1/2}$ be the unitary, and according to the polar decomposition

$$\overline{\theta}_\varepsilon = U_\varepsilon(\overline{\theta}_\varepsilon^* \overline{\theta}_\varepsilon)^{1/2},$$

and the fact that $(\overline{\theta}_\varepsilon^* \overline{\theta}_\varepsilon)^{1/2} - 1$ take value in compact operators, the operator $\left[\begin{smallmatrix} 0 & \overline{\theta}_\varepsilon^* \\ \overline{\theta}_\varepsilon & 0 \end{smallmatrix}\right]$ is a compact perturbation of $\left[\begin{smallmatrix} 0 & U_\varepsilon^* \\ U_\varepsilon & 0 \end{smallmatrix}\right]$. This completes the proof. ■

Definition 8.4. Let E^0_ε be the finite dimensional vector bundle $\ker(\overline{\theta}_\varepsilon)$.

The Hilbert bundle $\mathcal{H}_\varepsilon$ has connection induced from that of $f_\varepsilon^* E$ in the following way: the trivializations (1.4) can be viewed as G/H_ε-families of $U_\alpha \times \mathbb{C}^d$. The connection has local form $d + \omega$ on each connected component. So, these connection 1-forms ω's can be assembled into a single 1-form with value in G/H_ε-families of $M_d(\mathbb{C})$. This can be used as a connection $\nabla^{\mathcal{H}_\varepsilon}$ on $\mathcal{H}_\varepsilon$ whose curvature converges to zero as $\varepsilon \to 0$.

Notice that U_ε extends to a map from $C(M, \mathcal{H}_\varepsilon \oplus \mathbb{C}^q)$ to $C(M, \mathcal{H})$ whose kernel is $\ker(\overline{\theta}_\varepsilon)$. Let U_ε^{-1} be the inverse map from $C(M, \mathcal{H}) \to C(M, \mathcal{H}_\varepsilon \oplus \mathbb{C}^q)$ whose range is $\ker(\overline{\theta}_\varepsilon)^\perp$. Then, $U_\varepsilon U_\varepsilon^{-1} = \mathrm{id}_{C(M,\mathcal{H})}$. Let s be a section of $\ker(U_\varepsilon)$, then $U_\varepsilon(s) = 0$ and

$$0 = \nabla^{\mathcal{H}}(U_\varepsilon(s)) = \nabla^{\mathcal{B}(\mathcal{H}_\varepsilon \oplus \mathbb{C}^q, \mathcal{H})}(U_\varepsilon)(s) + U_\varepsilon(\nabla^{\mathcal{H}_\varepsilon \oplus \mathbb{C}^q} s).$$

Therefore, the subbundle $E^0_\varepsilon = \ker(\overline{\theta}_\varepsilon) = \ker(U_\varepsilon)$ can be equipped with the following connection:

$$s \mapsto \nabla^{\mathcal{H}_\varepsilon \oplus \mathbb{C}^q} s + U_\varepsilon^{-1}(\nabla^{\mathcal{B}(\mathcal{H}_\varepsilon \oplus \mathbb{C}^q, \mathcal{H})} U_\varepsilon) s. \tag{8.3}$$

It has curvature

$$\nabla^{\mathcal{H}_\varepsilon \oplus \mathbb{C}^q, 2} + U_\varepsilon^{-1}(\nabla^{\mathcal{B}(\mathcal{H}_\varepsilon \oplus \mathbb{C}^q, \mathcal{H}), 2} U_\varepsilon),$$

which clearly converges to zero as $\varepsilon \to 0$.

The natural connection on $\mathcal{H}_\varepsilon$ induced from that of $E \to S^n$ have curvature $\Omega^{\mathcal{H}_\varepsilon}$ with value in G/H_ε-families of $M_d(\mathbb{C})$ all but finitely many are zero. Therefore, $\exp(-\Omega^{\mathcal{H}_\varepsilon})$-1 is of trace class. Define the Chern form $\mathrm{ch}(\nabla^{\mathcal{H}_\varepsilon}, \nabla^{\mathcal{H}})$ to be $\mathrm{tr}(\exp(-\Omega^{\mathcal{H}_\varepsilon}) - \exp(-\Omega^{\mathcal{H}}))$.

Proposition 8.5. *The cohomology class determined by* $\mathrm{ch}(\nabla^{\mathcal{H}_\varepsilon}, \nabla^{\mathcal{H}})$ *is the same as that of* $\mathrm{ch}(E^0_\varepsilon - \mathbb{C}^q)$.

Proof. This follows from the standard transgression argument. Let $\nabla^{E^0_\varepsilon}$ be the connection on E^0_ε defined by the equation (8.3), $\nabla^{E^{0,\perp}_\varepsilon}$ be the trivial connection on $E^{0,\perp}_\varepsilon$ that is pulled back from that of $\mathcal{H}$ by U_ε and $\nabla^{E^0_\varepsilon \oplus E^{0,\perp}_\varepsilon}$ be the direct sum.

Let $A_t, 0 \le t \le 1$ be a family of connections on $\mathcal{H}_\varepsilon \oplus \mathbb{C}^q$ that is defined by $A_t = t\nabla^{\mathcal{H}_\varepsilon \oplus \mathbb{C}^q} + (1-t)\nabla^{E^0_\varepsilon \oplus E^{0,\perp}_\varepsilon}, 0 \le t \le 1$. Notice that the connection 1-forms of $\nabla^{\mathcal{H}_\varepsilon \oplus \mathbb{C}^q}$, $\nabla^{E^0_\varepsilon \oplus E^{0,\perp}_\varepsilon}$ and the curvatures of A_t are finite rank operators, therefore, $\exp(-A_t^2) - 1$ are of trace class. Then,

$$\begin{aligned}\frac{d}{dt}\,\mathrm{tr}(\exp(-A_t^2) - \exp(-\Omega^{\mathcal{H}})) &= \mathrm{tr}\Big(-\frac{dA_t^2}{dt}\exp(-A_t^2)\Big)\\ &= \mathrm{tr}\Big(-\Big[A_t, \frac{dA_t}{dt}\exp(-A_t^2)\Big]\Big)\\ &= d\,\mathrm{tr}\Big(-\frac{dA_t}{dt}\exp(-A_t^2)\Big).\end{aligned}$$

Here, dA_t/dt is a 1-form $\nabla^{\mathcal{H}_\varepsilon \oplus \mathbb{C}^q} - \nabla^{E^0_\varepsilon \oplus E^{0,\perp}_\varepsilon}$ with value in finite rank operators, so the expressions on the left-hand side of the equations are of finite rank and, in particular, trace class. As a consequence,

$$\mathrm{ch}(\nabla^{\mathcal{H}_\varepsilon}, \nabla^{\mathcal{H}}) - \mathrm{tr}(\exp(-(\nabla^{E_\varepsilon^0 \oplus E_\varepsilon^{0,\perp}})^2) - \exp(-\Omega^{\mathcal{H}})) = d\int_0^1 \mathrm{tr}\Big(-\frac{dA_t}{dt}\exp(-A_t^2)\Big).$$

Therefore, the Chern form $\mathrm{ch}(\nabla^{\mathcal{H}_\varepsilon}, \nabla^{\mathcal{H}})$ determines the same class as

$$\mathrm{tr}(\exp(-(\nabla^{E_\varepsilon^0 \oplus E_\varepsilon^{0,\perp}})^2) - \exp(-\Omega^{\mathcal{H}})),$$

which is easily computed to be equal to $\mathrm{ch}(E_\varepsilon^0 - \mathbb{C}^q)$. ∎

As a consequence of the above proposition,

$$\langle \widehat{A}(F)\,\mathrm{ch}(E_\varepsilon^0 - \mathbb{C}^q), [M]\rangle = \langle \widehat{A}(F)\,\mathrm{tr}(\exp(-\Omega^{\mathcal{H}_\varepsilon}) - \exp(-\Omega^{\mathcal{H}})), [M]\rangle.$$

Over each local trivialization U_α, the above integral is equal to

$$\int_{U_\alpha} \widehat{A}(F)\,\mathrm{tr}(\exp(-\Omega^{\mathcal{H}_\varepsilon}) - \exp(-\Omega^{\mathcal{H}}))$$
$$= \int_{\pi_\varepsilon^{-1}(U_\alpha)} \widehat{A}(\widetilde{F}_\varepsilon)\,\mathrm{tr}(\exp(-\Omega^{f_\varepsilon^* E}) - \exp(-\Omega^{\mathcal{H}})).$$

A partition of unity argument proves equation (1.11). This proves the following theorem.

Theorem 8.6. *If (M, F) is an enlargeable foliation in the sense of Definition 1.6 with F spin and even dimensional, then $[\alpha] \neq 0$ in $K_0(C^*G_M)$.*

9. Odd dimensional case

If the foliation $F \to M$ is of odd dimensional, the exterior product of vector bundles $F \boxtimes TS^1 \to M \times S^1$ defines an even dimensional foliation. Moreover, the monodromy groupoid of $(M \times S^1, F \boxtimes TS^1)$ is the direct product of monodromy groupoid of (M, F) and the fundamental groupoid of S^1. Accordingly, the corresponding maximal groupoid C^*-algebra is $C^*G_M \otimes \mathcal{K} \otimes C^*\mathbb{Z}$ whose K-theory is computed by the universal coefficient theorem

$$K_*(C^*G_M \otimes \mathcal{K} \otimes C^*\mathbb{Z}) = K_*(C^*G_M) \otimes K_*(C^*\mathbb{Z}).$$

Here, $\mathcal{K}$ is the C^*-algebra of compact operators and $C^*\mathbb{Z}$ is the group C^* algebra of $\mathbb{Z}$. In particular, we have

$$K_0(C^*G_M \otimes \mathcal{K} \otimes C^*\mathbb{Z}) = K_0(C^*G_M) \otimes 1 \oplus K_1(C^*G_M) \otimes e,$$

where 1 is the generator of $K_0(C^*\mathbb{Z})$ and e is the generator of $K_1(C^*\mathbb{Z})$. As in [12], $[\alpha(M, F)] \in K_1(C^*G_M)$ is defined by requiring

$$[\alpha(M, F)] \otimes e = [\alpha(M \times S^1, F \boxtimes TS^1)] \in K_0(C^*G_M \otimes \mathcal{K} \otimes C^*\mathbb{Z}).$$

Proposition 9.1. *The foliation* $(M \times S^1, F \boxtimes TS^1)$ *is enlargeable if* (M, F) *is enlargeable.*

Proof. Assume that (M, F) is enlargeable. Then, for any $\varepsilon > 0$ there is a covering space $\widetilde{M}_\varepsilon \to M$ and map $f_\varepsilon : \widetilde{M}_\varepsilon \to S^n$ with the properties given in Definition 1.6. Since S^1 is also enlargeable, there is $g_\varepsilon : S^1_\varepsilon \to S^1$ with the properties of Definition 1.6. Fix a degree one map $\varphi : S^n \times S^1 \to S^{n+1}$ and let $C_1 = \max |\varphi_*|$, we claim that the following composition:

$$\widetilde{M}_\varepsilon \times S^1_\varepsilon \xrightarrow{(f_\varepsilon, g_\varepsilon)} S^n \times S^1 \xrightarrow{\varphi} S^{n+1}$$

has the wanted property. Indeed, let $\widetilde{F}$ be the lifting of F to $\widetilde{M}_\varepsilon$ and $\Phi = \varphi \circ (f_\varepsilon, g_\varepsilon)$, then

$$|\Phi_*(X, Y)| = |\varphi_*(f_{\varepsilon,*}X, g_{\varepsilon,*}Y)| \le \varepsilon C_1 |(X, Y)|$$

for $(X, Y) \in C^\infty(\widetilde{M}_\varepsilon \times S^1_\varepsilon, \widetilde{F} \boxtimes TS^1_\varepsilon)$ and

$$|\Phi_*(Z, Y)| \le C C_1 |(Z, Y)|$$

for any tangent vector (Z, Y) of $\widetilde{M}_\varepsilon \times S^1_\varepsilon$. This completes the proof. ∎

According to Theorem 8.6 and Proposition 9.1, $[\alpha(M \times S^1, F \boxtimes TS^1)]$ is nonzero, so $[\alpha(M, F)]$ is also nonzero.

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