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# The variable exponent compound boundary value problem for Liapunov open curve

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**Abstract.** In this paper, we study the compound boundary value problem for a class of Cauchy-type integrals with density in variable exponent space. Based on the Smirnov theorem for multi-connected domains and the elimination method, we transform the compound boundary value problem of open curve into that of closed curve. The solvable condition and the explicit solution are obtained by the elimination method.

## 1. Introduction

The Riemann–Hilbert boundary value problem (BVP), abbreviated as the Riemann–Hilbert problem, is a problem that seeks an analytical function within a region to satisfy certain boundary conditions at the boundary of the region [10, 32]. Actually, this theory was researched intensively by many mathematicians, including N. I. Muskhelishvili, F. D. Gakhov, I. I. Privalov, I. N. Vekua, and L. Bers; these boundary value problems also have been systematically investigated by many authors; see, e.g., [22, 23, 25, 26, 33]. Recently, the Riemann–Hilbert problem received a renewed research interest within the function spaces theory, and with respect to applications in different kinds of partial differential equations, see, e.g., [1, 4, 21, 28]. In the meanwhile, Riemann–Hilbert problem in high-dimensional Euclidean space has been widely studied by Clifford analysis [7, 8, 19, 20].

The variable exponent function spaces were proposed in the study of elasticity and fluid mechanics problems with non-standard local growth conditions; due to their importance in the mechanical background and their essential extension as classical function spaces, the research on variable exponent Lebesgue space and Sobolev space has achieved fruitful results. In recent years, due to the rapid development of variable exponent theory and its widespread application in harmonic analysis and partial differential equations, the study of variable exponent function spaces has received more attention.

The origin of variable metric spaces can be traced back to the idea of Orlicz [29] in 1931; However, its modern development was achieved by the work of O. Kováčik and J. Ráksoník in 1991 and X. Fan and D. Zhao in 2001 using the method of Musielak–Orlicz spaces [6, 18], and the variable exponent Hardy space was considered by E. Nakai

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and Y. Sawano [27]. The further development of variable exponent space theory is more favorable for us to describe the relationship between boundary curves and solvable conditions of boundary value problems of analytic functions; for example, the literature [14] in the class of functions represented by the Cauchy-type integral of the kernel density function $f(t) \in L^{p(\cdot)}(\gamma)$ for the boundary $\gamma$ is a piecewise Liapunov curve, and the solvability of the Dirichlet problem depends largely on the value at the corner of $\gamma$.

For monographs on variable exponent Lebesgue space and variable exponent Sobolev space, you can refer to [3, 5]. With the development of variable exponent spaces, many researchers such as [9, 13, 16, 17, 24, 31], studied Riemann and Hilbert BVP for non-standard Banach function spaces, such as variable exponent Lebesgue and Smirnov space. In 2021, the authors of [35] investigated the compound Riemann–Hilbert BVP in the class of Cauchy-type integrals with density in variable exponent Lebesgue spaces and obtained the solvable conditions and explicit solutions of the compound Riemann–Hilbert BVP in variable exponent space [34].

Many scholars, such as Kokilashvili, Paatashvili, and so on, discussed Dirichlet BVP, Riemann BVP, and Hilbert BVP of analytic function by introducing variable exponent space [14, 16, 17]. Kokilashvili and other scholars mainly discussed two kinds of boundary value problems in variable exponent spaces, one is the space which is composed of Cauchy-type integrals of kernel density function $f(t)\in L^{p(\cdot)}(\Gamma)$ and denoted by $\mathcal{K}^{p(\cdot)}(\Gamma)$. The other is $f(t) \in E^{p(\cdot)}(D^{\pm})$, where $E^{p(\cdot)}(D^{\pm})$ is a variable exponent Smirnov space [12]. By means of the Tumarkin theorem, we obtain the properties of the variable exponent function $p(\cdot)$ defined on the boundary of the piecewise smooth curve $\Gamma$, the equivalent relation, and equivalent condition between $L^{p(\cdot)}(\Gamma)$ space and Smirnov space $E^{p(\cdot)}(D^{\pm})$ are established, where $D^{\pm}$ is a simply connected region surrounded by $\Gamma$, and the relation between $L^{p(\cdot)}(\Gamma)$ space and Smirnov space $E^{p(\cdot)}(D^{\pm})$ is obtained; then, the Riemann boundary value problem on the variable exponent space can be transformed into the constant exponent space to be discussed, and the solvable conditions and general solutions of the problem are obtained.

In this paper, we consider compound Riemann–Hilbert BVP for Liapunov open curve in the variable exponent space; the idea is transforming the compound Riemann–Hilbert BVP into the Hilbert BVP. We will discuss the equivalence of the transformation. As the curve is open curve, we also carefully discuss the singularities of the ends for the open curve; we try to get the solvable conditions and explicit solutions of the compound BVP [34]. These results are an extension of the classical theory of boundary value problems in complex analysis and have profound implications in solving the various mathematical models of fluid mechanics, nonlinear elasticity theory, variational problems in mathematical physics, differential equation with non-standard growth conditions, nonlinear partial differential equation, etc.

The paper is arranged as follows. In Section 2, some basic definitions of curves and variable exponent spaces are introduced; we will explain the meaning of the defined curve in the variable exponent space. In Section 3, we will briefly describe the compound boundary value problem in the variable exponent space for Liapunov open curve; we will give the solvable conditions and explicit solutions to the compound boundary value problem for Liapunov open curve in the variable exponent space.

## 2. Preliminaries

Firstly, we introduce some basic definitions of curve and variable exponent spaces.

**Definition 2.1** (Hölder condition [11, 22, 34, 35]). Let $f(t)$ be a function defined on a given curve $\Gamma$; if for all $t_1, t_2 \in \Gamma$,

$$|f(t_1) - f(t_2)| \leq C|t_1 - t_2|^{\alpha}, \quad 0 < \alpha \leq 1,$$

where $C$ is definite constant, then $f(t)$ is said to satisfy Hölder condition of $\alpha$ order on $\Gamma$, denoted as $f(t) \in H^{\alpha}(\Gamma)$. We define

$$H(\Gamma) = \bigcup_{0<\alpha\leq 1} H^{\alpha}(\Gamma).$$

**Definition 2.2** ([30, 34, 35]). Let $p(\cdot)$ be a Lebesgue measurable function, mapping the curve $\Gamma$ to the interval $[1, +\infty]$. If $p(\cdot)$ satisfies the following two conditions:

(i) $\underline{p} = \operatorname{ess\,inf}_{t\in\Gamma} p(t) > 1$, $\bar{p} = \operatorname{ess\,sup}_{t\in\Gamma} p(t) < \infty$;

(ii) There is a constant $C$ so that for any $t_1, t_2 \in \Gamma$ and $|t_1 - t_2| < \frac{1}{2}$ satisfy

$$|p(t_1) - p(t_2)| \leq \frac{C}{|\ln|t_1 - t_2||},$$

then the two conditions are called log-Hölder condition and we claim that $p(\cdot)$ belongs to $\mathcal{P}(\Gamma)$ on $\Gamma$.

**Definition 2.3** ([13, 34, 35]). Let $\Gamma$ be a curve and $p(\cdot) \in \mathcal{P}(\Gamma)$. If $f(t)$ is an almost everywhere measurable function on $\Gamma$ satisfying the condition

$$\|f\|_{L^{p(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0 : \int_{\Gamma}\left|\frac{f(t)}{\lambda}\right|^{p(\cdot)}|\mathrm{d}t| \leq 1\right\},$$

then the space composed of function $f(z)$ is called the variable exponent Lebesgue space, denoted as

$$L^{p(\cdot)}(\Gamma) = \{f(t) : \|f(t)\|_{L^{p(\cdot)}(\Gamma)} < \infty\},$$

and $L^{p(\cdot)}(\Gamma)$ becomes a Banach space.

**Definition 2.4** ([13, 34, 35]). Let $D^+$ be a simply connected domain. If analytic function $f(z)$ in $D^+$ satisfies

$$\sup_{0<r<1}\int_0^{2\pi} |f(z(re^{i\theta}))|^{p(z(e^{i\theta}))}|z'(re^{i\theta})||\mathrm{d}t| < \infty,$$

where $z = z(w)$ is a conformal mapping that maps the unit disk $S^+$ to a simply connected region $D^+$, then $f(z) \in E^{p(\cdot)}(D^+)$. It is easy to obtain that $E^p(D^+) \subseteq E^q(D^+)$ when $p \geq q$.

**Definition 2.5** ([16, 30, 34, 35]). Let $\Gamma$ be a curve; $f(t)$ is the complex function defined on $\Gamma$.

(i) By $S_\Gamma$ we denote the Cauchy singular operator

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} \mathrm{d}\tau, \quad t \in \Gamma.$$

(ii) By $K_\Gamma$ we denote the Cauchy-type integral

$$(K_\Gamma f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z} \mathrm{d}t, \quad z \notin \Gamma.$$

(iii) By $\mathcal{K}^{p(\cdot)}(\Gamma)$ we denote the Cauchy-type integral with density in $L^{p(\cdot)}(\Gamma)$

$$\mathcal{K}^{p(\cdot)}(\Gamma) = \{\Phi(z) : \Phi(z) = (K_\Gamma f)(z) \text{ with } f(t) \in L^{p(\cdot)}(\Gamma)\},$$

where $p(\cdot) \in \mathcal{P}(\Gamma)$.

(iv) By $W^{p(\cdot)}(\Gamma)$ we denote the class of weight functions

$$W^{p(\cdot)}(\Gamma) = \{\rho(t) : S_\Gamma \text{ is a continuous linear operator on } L^{p(\cdot)}(\Gamma, \rho)\},$$

where

$$L^{p(\cdot)}(\Gamma, \rho) = \{f(z) : f(z)\rho(t) \in L^{p(\cdot)}(\Gamma)\},$$

and $\rho(t)$ is the measurable function on $\Gamma$.

**Definition 2.6** ([22, 34, 35]). Let $t = t(s)$, $0 \leq s \leq l$, be the equation for a curve $\Gamma$ of arc length $l$ on the complex plane $\mathbb{C}$

(i) If $t = t'(s)$ with $0 \leq s \leq l$ is a function of the Hölder class, then $\Gamma$ is called a Liapunov curve.

(ii) If $t = t'(s)$ is a piecewise Hölder class function, the discontinuity points of the first type are $A_1, A_2, \ldots, A_n$, and the angle between two one-sided tangents of $\Gamma$ at point $A_i$ about bounded inner domain is $\pi v_k$ $(i = 1, \ldots, n)$ with $0 < v_k \leq 2$, then $\Gamma$ is called piecewise Liapunov curve, denoted as

$$\Gamma \in C^\Gamma(A_1, \ldots, A_n; v_1, \ldots, v_n).$$

(iii) If a closed rectifiable curve $\Gamma$ satisfies the condition

$$\sup_{t \in \Gamma, r > 0} \frac{|\Gamma \cap B(t, r)|}{r} < \infty,$$

where $B(t, r) = \{\tau \in \mathbb{C} : |z - t| \leqslant r\}$, then the curve $\Gamma$ is called a Carleson curve.

(iv) If $\Gamma$ is a closed curve and the Cauchy singular integral $S_\Gamma$ is bound in $L^{p(\cdot)}(\Gamma)$, then $\Gamma \in R_{p(\cdot)}$.

According to the definition of the above curve, we get the following conclusion.

(a) Liapunov curve and piecewise Liapunov closed curve are Carleson curve [15].

(b) If $p(\cdot) \in \mathscr{P}(\Gamma)$ with $\Gamma \in R_{p(\cdot)}$ if and only if $\Gamma$ is Carleson curve and

$$R_{p_0} = \bigcap_{p>1} R_p, \quad \forall p_0 > 1,$$

and if $f(t) \in L^{p(\cdot)}(\Gamma)$, then $S_\Gamma f(t)$ exist (see, e.g., [11, 13]).

(c) If $f(t) \in L^{p(\cdot)}(\Gamma)$ with $p(\cdot) \in \mathscr{P}(\Gamma)$ and $\Gamma \in R_{p(\cdot)}$, then the Cauchy-type integral $K_\Gamma f(t) \in L^{p(\cdot)}(\Gamma)$ has angular boundary values that are almost everywhere finite and the Plemelj formula holds (see, e.g., [30]).

## 3. The compound problem boundary value for Liapunov open curve in the variable exponent space

In this section, we will introduce the compound boundary value problem and use the elimination method [2, 35] to transform the compound boundary value problem into a Hilbert boundary value problem to deal with.

### 3.1. Introducing the compound boundary value problem for Liapunov open curve in the variable exponent space

Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ be composed of $n$ non-intersecting Liapunov open curves where $\Gamma_i = \overset{\frown}{a_i b_i}$ cut the whole complex plane with endpoints $a_i$ and $b_i$ and it is a positive direction from the end point $b_i$ to the starting point $a_i$, where $i = 1, 2, \ldots, n$. Assume that $L \in C^L(A_0, \ldots, A_m; v_0, \ldots, v_m)$ is a closed curve and it is positive in the counter-clockwise direction, the internally bounded region by this curve is denoted as $D^+$ and the outer region containing $z = \infty$ point is denoted as $D^-$. $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ is contained in the bounded region $D^+$, we find a function $\Phi(z) \in \mathcal{K}^{p(\cdot)}(\Gamma \cup L)$, where $p(\cdot) \in \mathscr{P}(\Gamma \cup L)$, the angular boundary value exists and satisfies the following relationship:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{3.1}$$

where $G(t)$ belongs to the piecewise Hölder continuous class on $\Gamma$, $g(t) \in L^{p(\cdot)}(\Gamma)$, and satisfies the following relationship:

$$\operatorname{Re}[a(t) + ib(t)]\Phi^+(t) = c(t), \quad t \in L, \tag{3.2}$$

where $a(t)$, $b(t)$ are real functions belong to the piecewise Hölder continuous class on $L$, $a^2(t) + b^2(t) \neq 0$. $c(t) \in L^{p(\cdot)}(L)$ is the real function on $L$ and bounded near the discontinuous points $\{t_k \in \Gamma : k = 0, 1, \ldots, m_1\}$.

Note, we generally required that $\Phi^{\pm}(t)$ are permitted to have integrable singularities, i.e.,

$$|\Phi^{\pm}(t)| \leq \frac{K}{|z-c|^{\alpha}}, \quad 0 \leq \alpha < 1, \tag{3.3}$$

where $c = a, b$. So, we can classify $\Phi(z)$ as follows.

(i) If $\Phi(z)$ is bounded near $z = a, b$, then it is defined by $h_2 = h(a, b)$.

(ii) If $\Phi(z)$ has integrable singularities near $z = a, b$, then it is defined by $h_0$.

(iii) If $\Phi(z)$ is bounded near $z = a$ or $z = b$, then it is defined by $h(a)$ or $h(b)$.

According to [11], this Liapunov open curve $\Gamma_k$ can be completed as a closed curve through the smooth curve $\Gamma'_k = \Gamma_k \cup \Upsilon_k \in R_{p(\cdot)}$; the simply connected and bounded region enclosed by $\Gamma'_k$ is $D_k^-$. We assume $G(t) = 1$ and $g(t) = 0$ on $\Upsilon_k$ with $k = 1, 2, \ldots, n$, then the boundary value problem for Liapunov open curve is transformed into the closed curve boundary value problem.

We first consider the boundary value problem (3.1) and temporarily ignore the boundary value condition (3.2). Notice that $c_k = a_k$ or $b_k$ are discontinuity points of the first kind; thus, $G(c^{\pm})$ exists. Assume that

$$\frac{G(c_k^-)}{G(c_k^+)} = e^{2\pi i \lambda_{c_k}}, \quad \text{with } c_k = a_k, b_k, \tag{3.4}$$

where $\lambda_{c_k} = \alpha_{c_k} + i\beta_{c_k}$ and $\alpha_{c_k}, \beta_{c_k} \in \mathbb{R}$. Now, we demand

$$\alpha_{c_k} \neq \frac{1}{q(c_k)} \mod (1), \quad k = 1, \ldots, n, \tag{3.5}$$

where $c_k = a_k, b_k$ and $1/p(c_k) + 1/q(c_k) = 1$. Writing

$$G_k(t) = (t - z_k)^{\lambda_{a_k}} (t - z_k)^{\lambda_{b_k}} G(t), \tag{3.6}$$

where $z_k$ belongs to the bounded region $D_k^-$ bounded by the smooth curve $\Gamma'_k$, where $D_k^+ = \mathbb{C}/D_k^-$, we write $D_0^+ = D^+ - \sum_{k=1}^n D_k^-$.

Considering to the function

$$\rho(z) = \prod_{k=1}^{n} \rho_{a_k}(z)\rho_{b_k}(z), \tag{3.7}$$

where

$$\rho_{c_k}(z) = \begin{cases} (\frac{z-c_k}{z-z_k})^{\lambda_{c_k}}, & z \in D_k^+, \\ (z-c_k)^{\lambda_{c_k}}, & z \in D_k^-, \end{cases} \quad c_k = a_k, b_k,$$

when $z \to \infty$, $[(z-c_k)/(z-z_k)]^{\lambda_{c_k}} \to 1$ with $z_k \in D_k^-$, so $\rho(z)$ is analytic in $D^+$, according to Definition 2.4, there exists $\varepsilon > 0$ such that

$$\rho(z)^{\pm} \in E^{1+\varepsilon}(D^{\pm}), \quad \frac{1}{\rho^{\pm}(z)} - 1 \in E^{1+\varepsilon}(D^{\pm}). \tag{3.8}$$

Let

$$\Phi_1(z) = \frac{\Phi(z)}{\rho(z)}; \tag{3.9}$$

then, the boundary value condition (3.1) is converted to

$$\Phi_1^+(t) = G^*(t)\Phi_1^-(t) + g_1(t), \quad t \in \Gamma, \tag{3.10}$$

where $g_1(t) = g(t)/\rho^+(t)$, according to [13, 17],

$$G^*(t) = \left[\prod_{k=1}^{n} (t - z_k)^{\lambda_{a_k}} (t - z_k)^{\lambda_{b_k}}\right] G(t) \in H\left(\bigcup_{k=1}^{n} \Gamma_k'\right).$$

Now, we consider

$$\psi(t) = \frac{1}{2\pi i} \int_\Gamma \frac{\log G^*(t)}{\tau - t} \mathrm{d}\tau; \tag{3.11}$$

according to [22], we obtain that near $z = a_k$

$$\psi(t) = \gamma_{a_k} \log(a_k - z) + \Phi_{a_k}(z), \quad \text{with } \gamma_{a_k} = \alpha'_{a_k} + i\beta'_{a_k} = -\frac{\log G^*(a_k)}{2\pi i}, \tag{3.12}$$

where $\Phi_{a_k}(z)$ is holomorphic near $z = a_k$. Similarly, near $z = b_k$

$$\psi(t) = \gamma_{b_k} \log(z - b_k) + \Phi_{b_k}(z), \quad \text{with } \gamma_{b_k} = \alpha'_{b_k} + i\beta'_{b_k} = \frac{\log G^*(b_k)}{2\pi i}, \tag{3.13}$$

where $\Phi_{b_k}(z)$ is holomorphic near $z = b_k$. So, we define an integer $w_{a_k}$ [22]:

**(a) Case $\Phi(z) \in h_2$:** (i) If $\alpha_{a_k} + \alpha'_{a_k}$ is an integer, then $w_{a_k} = -\alpha_{a_k} + \alpha'_{a_k}$.
(ii) If $\alpha_{a_k} + \alpha'_{a_k}$ is not integer, then $0 < w_{a_k} + \alpha_{a_k} + \alpha'_{a_k} < 1$.

**(b) Case $\Phi(z) \in h_0$:** (i) If $\alpha_{a_k} + \alpha'_{a_k}$ is an integer, then $w_{a_k} = -\alpha_{a_k} + \alpha'_{a_k}$.
(ii) If $\alpha_{a_k} + \alpha'_{a_k}$ is not integer, then $-1 < w_{a_k} + \alpha_{a_k} + \alpha'_{a_k} < 0$.
As for $w_{b_k}$, it is similarly defined by $\alpha_{b_k} + \alpha'_{b_k}$.
If

$$\mathrm{Ind}_{\Gamma_k'} G_k(t) = \frac{1}{2\pi}[\arg G_k(t)]_{\Gamma_k} = \kappa_k. \tag{3.14}$$

According to [22, 35], we get its canonical function

$$X(z) = \left[\prod_{k=1}^{n} (z - a_k)^{\omega_{a_k}} (z - b_k)^{\omega_{b_k}}\right] X^*(z), \tag{3.15}$$

where

$$X^*(z) = \begin{cases} \prod^{-1} e^{\Gamma_1(z)}, & z \in D_0^+, \\ e^{\Gamma_1(z)}, & z \in \sum_{k=1}^{n} D_k^-, \end{cases} \tag{3.16}$$

where

$$\Gamma_1(z) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \frac{\log(\prod G^*)(t)}{t-z_k}\mathrm{d}t\right\}, \quad \prod(z) = \prod_{k=1}^{n}(z-z_k)^{\kappa_k}.$$

We define the index of boundary value problem (3.1)

$$\kappa = \sum_{k=1}^{n}\kappa_k - \sum_{k=1}^{n}(w_{a_k}+w_{b_k}), \tag{3.17}$$

for $X(z)$ can be called the canonical function of problem (3.1) and has the following properties (see, e.g., [22]):

- $X(z) \neq 0$ on the whole complex plane;
- $X(\infty)$ has finite order;
- $X^+(t) = G(t)X^-(t)$;
- $X^+(t)$ is bounded in $D^\pm$;
- $X^+(t)$ has singularity at the ends $z = a_k, b_k$ of order less than 1, $k = 1, \dots, n$.

According to Smirnov theorem for multi-connected domain [35], assume condition (3.5) is satisfied; we obtain all solutions satisfying the boundary condition (3.1).

**Theorem 3.1** ([13, 15, 22]). *If* (3.5) *holds and* $\kappa$ *is defined by* (3.17)*, the solution to the boundary value problem* (3.1) *can be obtained:*

(i) *If* $\kappa \geq 0$*, all solutions to problem* (3.1) *can be derived:*

$$\Phi_1(z) = \frac{\rho(z)X(z)}{2\pi i}\int_\Gamma \frac{g(t)}{\rho^+(t)X^+(t)(t-z)}\mathrm{d}t + F(z)\rho(z)X(z), \tag{3.18}$$

*where* $\Phi_1(z) \in \mathcal{K}^{p(\cdot)}(\Gamma)$*, whereas* $F(z)$ *is holomorphic on* $D^+$ *and continuous as an arbitrary function on* $\overline{D^+}$*.* $\Phi_1(z)$ *has less than first-order singularity at* $a_k$*,* $b_k$*, with* $k = 1, \dots, n$*.*

(ii) *If* $\kappa < 0$*, problem* (3.1) *is solvable if and only if the condition is satisfied:*

$$\int_\Gamma \frac{t^j g(t)}{\rho^+(t)X^+(t)}\mathrm{d}t = 0, \quad j = 0, \dots, -\kappa-1, \tag{3.19}$$

*and problem* (3.1) *has a unique solution:*

$$\Phi_1(z) = \rho(z)X(z)\frac{1}{2\pi i}\int_L \frac{g(t)}{\rho^+(t)X^+(t)(t-z)}\mathrm{d}t. \tag{3.20}$$

After we get $\Phi_1(z)$, we transform the unknown function $\Phi(z)$ into a new function $\Phi_0(z)$ by

$$\Phi(z) = X(z)\Phi_0(z) + \Phi_1(z), \tag{3.21}$$

which is sectionally holomorphic in $D^+$ and continuous to $L$. Since $\Phi_1(z)$ fulfills (3.1) and $X(z)$ is the canonical function of problem (3.1), when $t \in \Gamma$,

$$\begin{aligned}\Phi^+(t) &= \Phi_1^+(t) + X^+(t)\Phi_0^+(t)\\ &= G(t)\Phi_1^-(t) + g(t) + G(t)X^-(t)\Phi_0^+(t)\\ &= G(t)[\Phi_1^-(t) + X^-(t)\Phi_0^+(t)] + g(t).\end{aligned}$$

By (3.1) again,

$$\Phi^+(t) = G(t)[\Phi_1^-(t) + X^-(t)\Phi_0^-(t)] + g(t). \tag{3.22}$$

Comparing these two equations (note $X^-(t) \neq 0$), we have

$$\Phi_0^+(t) = \Phi_0^-(t), \quad t \in \Gamma. \tag{3.23}$$

Hence, $\Phi_0(z)$ is in fact a function holomorphic in $D^+$ and continuous on $\overline{D^+}$.

Conversely, if $\Phi_0(z)$ is such a function, then it is easy to verify that the sectionally holomorphic function $\Phi(z)$ determined by (3.21) ought to be continuous on $L$ and fulfills (3.1). This method is called the elimination method [22].

According to [35], if $\Phi(t) \in \mathcal{K}^{p(\cdot)}(\Gamma + L)$ is the solution to boundary value problem (3.1), then $\Phi_0(z)$ defined by (3.21) belongs to $\mathcal{K}^{p(\cdot)}(L)$. On the contrary, if $\Phi_0(z)$ belongs to $\mathcal{K}^{p(\cdot)}(L)$, then $\Phi(z)$ defined by (3.21) belongs to $\mathcal{K}^{p(\cdot)}(\Gamma + L)$ and satisfies condition (3.1).

Thus, the compound boundary value problem on the Liapunov open curve can be transformed into a function $\Phi_0(z)$ which finds holomorphic function in $D^+$ and continues to $\overline{D^+}$ so that it satisfies the corresponding condition transformed from (3.2). Now, substitute (3.21) into (3.2) to obtain the following condition:

$$\operatorname{Re}[a(t) + ib(t)]X(t)\Phi_0^+(t) = c^*(t), \quad t \in L, \tag{3.24}$$

where

$$c^*(t) = c(t) - \operatorname{Re}[a(t) + ib(t)]\Phi_1^+(t), \tag{3.25}$$

because $X(t)$, $\Phi_1^+(t)$ belongs to the piecewise Hölder continuous class on $L$, so $c^*(t) \in L^{p(\cdot)}(L)$, $[a(t) + ib(t)]X(t)$ belongs to the piecewise Hölder continuous class on $L$, then condition (3.1) and the closed curve $\Gamma_k'$ are eliminated, where $\Gamma_k' = \Gamma_k \cup \Upsilon_k, k = 1, \dots, n$.

Thus, the compound boundary value problem is transformed into the following boundary value problem in $D^+$; i.e., we find a function $\Phi(z) \in \mathcal{K}^{p(\cdot)}(L)$ such that the angular boundary value of $\Phi(z)$ satisfies

$$\operatorname{Re}[a(t) + ib(t)]X(t)\Phi_0^+(t) = c^*(t), \quad t \in L, \tag{3.26}$$

where $L \in C^L\{A_0, \dots, A_m; \nu_0, \dots, \nu_m\}$ and

$$c^*(t) = c(t) - \operatorname{Re}[a(t) + ib(t)]\Phi_1^+(t). \tag{3.27}$$

### 3.2. The solution to the compound boundary value problem

In the previous section, we have transformed the compound boundary value problem into finding $\Phi_0(z) \in \mathcal{K}^{p(\cdot)}(L)$, whose boundary value condition satisfies

$$\operatorname{Re}[A(t) + iB(t)]\Phi_0^+(t) = c^*(t), \quad t \in L, \tag{3.28}$$

where

$$A(t) + iB(t) = [a(t) + ib(t)]X(t), \quad c^*(t) = c(t) - \operatorname{Re}[a(t) + ib(t)]\Phi_1^+(t),$$

where $c^*(t) \in L^{p(\cdot)}(L)$, $[A(t) + iB(t)]$ belongs to the piecewise Hölder continuous class on $L$ and $A^2(t) + B^2(t) \neq 0$.

Suppose $l = \{\tau : |\tau| = 1\}$, $z = z(w)$ is a conformal mapping from the unit disk $U^+$ to the bounded region $D^+$, and its inverse mapping is

$$w = w(z).$$

Assume $A^*(\tau) = A(z(\tau))$, $B^*(\tau) = B(z(\tau))$, $C^*(\tau) = c^*(z(\tau))$. By [22],

$$A^*(\tau) = A(z(\tau)), \quad B^*(\tau) = B(z(\tau))$$

belong to the piecewise Hölder continuous class on $l$, whose the discontinuous points are $\{\tau_k = z(t_k) \in \Gamma : k = 0, 1, \ldots, m_1\}$. $A^{*2}(t) + B^{*2}(t) \neq 0$, $C^*(\tau) \in L^{\ell(\cdot)}(l)$ is function which is bounded near $\{\tau_k = z(t_k);\ k = 0, 1, \ldots, m_1\}$.

The $\Phi_0(z)$ function is symmetrically extended to partition holomorphic functions on the complex plane

$$\Omega(w) = \begin{cases} \Phi_0(w), & |w| < 1, \\ \overline{\Phi_0(\frac{1}{\bar{w}})}, & |w| > 1. \end{cases} \tag{3.29}$$

Since $\Omega^-(\tau) = \overline{\Phi_0^+(\tau)}$, then the boundary value problem (3.28) can be converted to seek the function $\Omega(w) \in \mathcal{K}^{p(\cdot)}(l)$ such that the angular boundary value of $\Omega(w)$ satisfies

$$\Omega^+(\tau) = G_*(\tau)\Omega^-(\tau) + g(\tau), \tag{3.30}$$

where

$$G_*(\tau) = -\frac{A^*(\tau) - iB^*(\tau)}{A^*(\tau) + iB^*(\tau)}, \tag{3.31}$$

and the solution to problem (3.28) satisfies the condition

$$\Omega(w) = \overline{\Omega\left(\frac{1}{\bar{w}}\right)}. \tag{3.32}$$

Since $\{\tau_k = z(t_k);\ k = 0, 1, \ldots, m_1\}$ are discontinuous points for $G_*(\tau)$ and

$$|G_*(\tau_k^{\pm})| = 1,$$

then we have

$$\frac{G_*(\tau_k^-)}{G_*(\tau_k^+)} = e^{2\pi i \mu_k}, \quad \text{with } k = 0, 1, \ldots, m_1, \tag{3.33}$$

where $\mu_k$ is a real number.

Remember that

$$r_k(w) = \begin{cases} (w - \tau_k)^{\mu_k}, & |w| < 1, \\ \overline{(\frac{1}{\overline{w}} - \tau_k)^{\mu_k}}, & |w| > 1, \end{cases} \tag{3.34}$$

and

$$r(w) = \prod_{k=0}^{m} r_k^{-1}(w). \tag{3.35}$$

We get

$$R_k(\tau) = \frac{r_k^+(\tau)}{r_k^-(\tau)} = e^{2i \arg(\tau - b_k)\mu_k}. \tag{3.36}$$

Then,

$$\frac{R_k(\tau_k^-)}{R_k(\tau_k^+)} = e^{-2i\pi\mu_k}, \quad \text{with } k = 0, 1, \ldots, m_1; \tag{3.37}$$

therefore,

$$G_{**}(\tau) = G_*(\tau) \prod_{k=0}^{m_1} R_k(\tau),$$

where $G_{**}(\tau) \in H(l)$ and $G_{**}(\tau) \neq 0$ for $\tau \in l$ (see [35]).

We obtain the canonical function of problem (3.30) by

$$X_*(w) = \begin{cases} r(w)e^{\Gamma(w)}, & |w| < 1, \\ r(w)(w - c)^{-\kappa_1} e^{\Gamma(w)}, & |w| > 1, \end{cases} \tag{3.38}$$

where

$$\Gamma(w) = \frac{1}{2\pi i} \int_l \frac{\log(\tau - w_0)^{-\kappa_1} G_{**}(\tau)}{\tau - w} d\tau,$$

and $c \neq \tau_k \in l$, $w_0 \in U^+$, where $U^+$ is the unit disk enclosed by curve $l$. According to [34], we define $\mathrm{Ind}_l G_*(\tau) = \kappa_1 = -\omega_c$,

$$\kappa_1 = \frac{1}{2\pi} \arg \frac{G_*(c - 0)}{G_*(c + 0)}.$$

Since $\{\tau_k = w(t_k), k = 0, 1, \ldots, m_1\}$ are discontinuous points on $l$, $\{d_k = w(A_k), k = 0, 1, \ldots, m\}$ are angular points on $l$, then the points $\{\tau_k;\ k = 0, \ldots, m\}$ and $\{d_k; k = 0, \ldots, n\}$ are denoted as follows:

$$\begin{cases} w_0 = \tau_0 = d_0, w_1 = \tau_1 = d_1, \ldots, w_s = \tau_s = d_s, \\ w_{s+1} = \tau_{s+1}, w_{s+2} = \tau_{s+2}, \ldots, w_{s+u} = \tau_{s+u}, \\ w_{s+u+1} = d_{s+u+1}, w_{s+u+2} = d_{s+u+2}, \ldots, w_{s+u+p} = d_{s+u+p}, \end{cases} \tag{3.39}$$

where $1 + s + u + p = h$; according to [17], we get

$$0 < m \leq \frac{|\tau - \xi|^{\frac{\alpha}{\ell(\tau)}}}{|\tau - \xi|^{\frac{\alpha}{\ell(\xi)}}} \leq M < \infty, \tag{3.40}$$

where $\zeta \in l$, $\ell(\tau) = p(z(\tau))$, and $\alpha \in \mathbb{R}$, and we have

$$\delta_k = \begin{cases} \frac{\nu_k - 1}{\ell(b_k)} - \mu_k, & k = 0, 1, \ldots, s, \\ -\mu_k, & k = s+1, \ldots, s+u, \\ \frac{\nu_{k-u} - 1}{\ell(b_{k-u})}, & k = s+u+1, \ldots, 1+s+u+p. \end{cases} \tag{3.41}$$

As for the real number $x$ can be decomposed as

$$x = [x] + \{x\},$$

where $[x]$ is the integer part for $x$ and $0 \leq \{x\} < 1$, we require

$$\{\delta_k\} \neq \frac{1}{\ell'(w_k)}, \quad \text{with } \frac{1}{\ell(w_k)} + \frac{1}{\ell'(w_k)} = 1, \tag{3.42}$$

for all $k = 0, 1, \ldots, h$.

Assume

$$\eta_k = \begin{cases} [\delta_k], & \{\delta_k\} < \frac{1}{\ell'(w_k)}, \\ [\delta_k] + 1, & \{\delta_k\} > \frac{1}{\ell'(w_k)}. \end{cases} \tag{3.43}$$

Therefore, we obtain

$$-\frac{1}{\ell(w_k)} < \delta_k - \eta_k < \frac{1}{\ell'(w_k)}. \tag{3.44}$$

Assume

$$Q(w) = \prod_{k=0}^{h} (w - w_k)^{\eta_k}, \quad \rho(w) = \prod_{k=0}^{h} (w - w_k)^{\delta_k - \eta_k}. \tag{3.45}$$

By [30, Theorem A], we obtain

$$\rho(\tau) \in W^{\ell(\cdot)}(l). \tag{3.46}$$

We define the index of this problem (3.30)

$$\kappa_0 = \kappa_1' + \kappa_2', \tag{3.47}$$

where $\kappa_1'$ is the order of the function $X_*^{-1}(w)$ and $\kappa_2'$ is the order of the function $Q(w)$.

If $\kappa_0 \geq 0$, $P_{\kappa_0}(w)$ is the arbitrary polynomial of order $\kappa_0$, i.e.,

$$P_{\kappa_0}(w) = a_{\kappa_0} w^{\kappa_0} + a_{\kappa_0 - 1} w^{\kappa_0 - 1} + \cdots + a_1 w + a_0. \tag{3.48}$$

Let

$$\Omega_1(w) = \frac{\Omega_1'(w) + (\Omega_1')_*(w)}{2}, \tag{3.49}$$

where

$$\Omega_1'(w) = H(w) X_*(w) Q^{-1}(w) \quad \text{with } H(w) = K_l(Q(X_*^+)^{-1}g)(w);$$

however, $(\Omega_1')_*(w)$ is the symmetric function of $\Omega_1'(w)$; it is defined by

$$(\Omega_1')_*(w) = \overline{\Omega_1'(\frac{1}{\bar{w}})}. \tag{3.50}$$

Assume

$$\Omega_2(w) = \frac{X_*(w) P_{\kappa_0}(w)}{Q(w)}, \tag{3.51}$$

where $Q(w)$ is defined by (3.45). According to [13], combining all the above results, we obtain the following theorem about the solutions of problem (3.28).

**Theorem 3.2.** *Assume that* (3.42) *holds and $\kappa_0$ is defined by* (3.47)*; the solution to the boundary value problem* (3.28) *can be obtained:*

(i) *If $\kappa_0 \geq 0$, the boundary value problem* (3.28) *is solvable if the polynomial $P_{\kappa_0}(w)$ satisfies the conditions*

$$\overline{a_k} = (-1)^{\kappa_2'} \prod_{k=0}^{h} w_k^{-\eta_k} a_{\kappa_0 - k}, \quad k = 0, \ldots, \kappa_0, \tag{3.52}$$

*where $h = 1 + s + u + p$, $\kappa_2'$ is the order of $Q(w)$ and $\eta_k$ is defined by* (3.43)*, then the solution to the boundary value problem* (3.28) *is given by*

$$\Phi(z) = \Omega_1(w(z)) + \Omega_2(w(z)), \tag{3.53}$$

*where $\Omega_1(w(z))$, $\Omega_2(w(z))$ are defined by* (3.49), (3.51).

(ii) *If $\kappa_0 < 0$, then $P_{\kappa_0}(w) = 0$, and the boundary value problem* (3.28) *is solvable if and only if the following conditions are satisfied:*

$$\int_\Gamma \frac{c^*(t) Q(w(t)) w^k(t) w'(t)}{X_*^+(w(t))(a(t) + ib(t))} \mathrm{d}t = 0, \quad k = 0, 1, \ldots, |\kappa_0| - 2, \tag{3.54}$$

*where $X_*(t)$ is defined by* (3.38) *and $Q(w) = \prod_{k=0}^{h} (w - w_k)^{\eta_k}$. Then, the solution to the boundary value problem* (3.28) *is given by*

$$\Phi(z) = \Omega_1(w(z)). \tag{3.55}$$

Combining Theorem 3.1, Theorem 3.2, and (3.21), we obtain the solutions of the compound boundary value problem as follows.

**Theorem 3.3.** *Assume $\kappa$, $\kappa_0$ are defined as (3.17) and (3.47), respectively, $Q(z)$ is defined (3.45), $X_*(z)$ is obtained by (3.38), $X(z)$ is obtained by (3.15), $(\Omega_1')_*(w)$ are defined by (3.50) and $P_{\kappa_0}(z)$ is a polynomial of order $\kappa_0$, $\Phi_1(z)$ is the solution to problem (3.1). If the following conditions are met*

$$\alpha_{c_k} \neq \frac{1}{q(c_k)} \mod (1), \quad k = 1, \dots, n,$$

*where $c_k = a_k, b_k$ and $1/p(c_k) + 1/q(c_k) = 1$,*

$$\{\delta_j\} \neq \frac{1}{\ell'(w_j)}, \quad \frac{1}{\ell(w_j)} + \frac{1}{\ell'(w_j)} = 1, \quad j = 0, \dots, h,$$

*where $\ell(\tau) = p(z(\tau))$ and $w_j$ is defined in (3.39), then the solution to the compound boundary value problem exists and is given as follows.*

**Case 1.** *When $g(t) \equiv 0, c(t) \equiv 0$ and $\kappa < 0$.*

(a) *If $\kappa_0 \geq 0$, if and only if the condition can be met (3.52), the Hilbert boundary value problem (3.28) has the solution*

$$\Phi_0(z) = \frac{X_*(w)P_{\kappa_0}(w)}{Q(w)},$$

*so the solution to the compound boundary value problem is*

$$\Phi(z) = X(z)\Phi_0(z). \tag{3.56}$$

(b) *If $\kappa_0 < 0$, then the Hilbert boundary value problem (3.28) has only zero solution, so the compound boundary value problem has zero solution.*

**Case 2.** *When $g(t) \equiv 0$, $c(t)$ almost everywhere is not $0$ on $L$ and $\kappa < 0$.*

(a) *If $\kappa_0 \geq 0$ and condition (3.52) is satisfied and the Hilbert boundary value problem (3.28) has a solution*

$$\Phi_0(z) = \frac{\Omega_1'(w) + (\Omega_1')_*(w)}{2} + \frac{X_*(w)P_{\kappa_0}(w)}{Q(w)}, \tag{3.57}$$

*then the solution to the compound boundary value problem is*

$$\Phi(z) = X(z)\Phi_0(z). \tag{3.58}$$

(b) *If $\kappa_0 < 0$, the Hilbert boundary value problem (3.28) has a solution*

$$\Phi_0(z) = \frac{\Omega_1'(w) + (\Omega_1')_*(w)}{2}$$

*if and only if condition (3.54) is satisfied, then the solution to the compound boundary value problem is*

$$\Phi(z) = X(z)\Phi_0(z). \tag{3.59}$$

**Case 3.** *Other situations include the following.*

(i) *When* $c(t) = \mathrm{Re}[a(t) + ib(t)]\Phi_1^+(t)$.

(a) *If* $\kappa_0 \geq 0$, *the Hilbert boundary value problem* (3.28) *has a solution*

$$\Phi_0(z) = \frac{X_*(w)P_{\kappa_0}(w)}{Q(w)},$$

*if and only if condition* (3.52), *so the solution to the compound boundary value problem is*

$$\Phi(z) = X(z)\Phi_0(z) + \Phi_1(z). \tag{3.60}$$

(b) *If* $\kappa_0 < 0$, *then the Hilbert boundary value problem* (3.28) *has only zero solution, so the compound boundary value problem has a solution.*

$$\Phi(z) = \Phi_1(z). \tag{3.61}$$

(ii) *When* $c(t) \neq \mathrm{Re}[a(t) + ib(t)]\Phi_1^+(t)$.

(a) *If* $\kappa_0 \geq 0$, *the Hilbert boundary value problem* (3.28) *has a solution*

$$\Phi_0(z) = \frac{\Omega_1'(w) + (\Omega_1')_*(w)}{2} + \frac{X(w)P_{\kappa_0}(w)}{Q(w)},$$

*if and only if condition* (3.52) *is satisfied, so the solution to the compound boundary value problem is*

$$\Phi(z) = X(z)\Phi_0(z) + \Phi_1(z). \tag{3.62}$$

(b) *If* $\kappa_0 < 0$, *the Hilbert boundary value problem* (3.28) *has a solution*

$$\Phi_0(z) = \frac{\Omega_1'(w) + (\Omega_1')_*(w)}{2}$$

*if and only if condition* (3.54), *so the solution of the compound boundary value problem is*

$$\Phi(z) = X(z)\Phi_0(z(w)) + \Phi_1(z). \tag{3.63}$$

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