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Nuclear operators on the space $C_{\mathrm{rc}}(X, E)$ of vector-valued continuous functions

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Abstract. Let X be a completely regular Hausdorff space and E and F be Banach spaces. Let $C_{\mathrm{rc}}(X, E)$ denote the Banach space of all continuous functions $f : X \to E$ such that $f(X)$ is a relatively compact set in E. Let β_σ be the strict topology on $C_{\mathrm{rc}}(X, E)$. We characterize the nuclearity of a $(\beta_\sigma, \|\cdot\|_F)$-continuous operator $T : C_{\mathrm{rc}}(X, E) \to F$ in terms of its representing operator-valued Baire measure. As an application, we establish the relationship between the nuclearity of a $(\beta_\sigma, \|\cdot\|_F)$-continuous operator $T : C_{\mathrm{rc}}(X, E) \to F$ and the nuclearity of its conjugate operator T'.

1. Introduction and terminology

Throughout the paper, let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be real Banach spaces and E' and F' denote the Banach duals of E and F, respectively. By $\mathcal{L}(E, F)$ we denote the Banach space of all bounded linear operators $U : E \to F$, equipped with the operator norm $\|\cdot\|$. Given a locally convex space (L, ξ), by $(L, \xi)'$ we denote its topological dual.

Now, we recall terminology concerning operator-valued measures (see [5,6]). Assume that $\mathcal{F}$ is an algebra of subsets of a set X and $m : \mathcal{F} \to \mathcal{L}(E, F)$ is a finitely additive measure. By $\tilde{m}(A)$ we denote the *semivariation* of m on $A \in \mathcal{F}$; that is,

$$\tilde{m}(A) := \sup \Big\| \sum m(A_i)(x_i) \Big\|_F,$$

where the supremum is taken over all finite $\mathcal{F}$-partitions (A_i) of A and $x_i \in E$, $\|x_i\|_E \leq 1$, for each i. By $|m|(A)$ we denote the *variation* of m on $A \in \mathcal{F}$; that is, $|m|(A) := \sup \Sigma \|m(A_i)\|$, where the supremum is taken over all finite $\mathcal{F}$-partitions (A_i) of A. For $y' \in F'$, let $m_{y'} : \mathcal{F} \to E'$ be the measure defined by

$$m_{y'}(A)(x) := y'(m(A)(x)) \quad \text{for all } A \in \mathcal{F},\ x \in E.$$

Note that, for a finitely additive measure $\mu : \mathcal{F} \to E'$, we have $\widetilde{\mu}(A) = |\mu|(A)$ for $A \in \mathcal{F}$ (see [5, Section 4, Proposition 4, p. 54]). For $x \in E$, let

$$\mu_x(A) := \mu(A)(x) \quad \text{for } A \in \mathcal{F}.$$

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The concept of a nuclear operator between Banach spaces is due to Ruston [29] (see also [40, p. 279], [4, 7, 21, 22, 30] for more details). Grothendieck [8, 9] carried over the concept of a nuclear operator to locally convex spaces (see also [40, p. 289], [21], [33, Chapter 3, Section 7], [11, 38]).

In particular, for X a compact Hausdorff space, nuclear operators $T : C(X) \to F$ have been studied intensively (see [4, 10, 21, 22, 30, 37]). According to Tong [37, Theorem 1.2], a linear operator $T : C(X) \to F$ is nuclear if and only if T is Bochner representable (see also [4, Theorem 4, pp. 173–174], [30, Proposition 5.30]). Nuclear operators $T : C(X, E) \to F$ have been studied intensively by Alexander [1], Bilyeu and Lewis [3], Saab and Smith [32], Smith [34], Saab [31], and Popa [24, 25, 27, 28]. The study of nuclear operators $T : C(X, E) \to F$ was initiated by Alexander [1], where some of the known results in scalar case, we extended. Bilyeu and Lewis [3] showed that if $T : C(X, E) \to F$ is nuclear, then its representing measure m takes values in the Banach space $\mathcal{N}(E, F)$ of all nuclear operators from E to F. Saab and Smith [32] and Popa [24] established the relationship between nuclear operators $T : C(X, E) \to F$ and their representing operator-valued Borel measures.

From now on, we assume that X is a completely regular Hausdorff space. Let $C_{\mathrm{rc}}(X, E)$ stand for the Banach space of all continuous functions $f : X \to E$ such that $f(X)$ is a relatively compact set in E, equipped with the topology τ_u of the supremum norm $\|\cdot\|$. By $C_{\mathrm{rc}}(X, E)'$ and $C_{\mathrm{rc}}(X, E)''$ we denote the Banach dual and the Banach bidual of $C_{\mathrm{rc}}(X, E)$, respectively. We write $C_b(X)$ instead of $C_{\mathrm{rc}}(X, \mathbb{R})$.

A subset H of $C_{\mathrm{rc}}(X, E)$ is said to be *solid* whenever $\|f_1(t)\|_E \le \|f_2(t)\|_E$ for all $t \in X$, $f_1 \in C_{\mathrm{rc}}(X, E)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $C_{\mathrm{rc}}(X, E)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets (see [18, Definition 2.1], [17, Section 8]).

The *strict topology* β_σ (denoted also by β_1) on the space $C_{\mathrm{rc}}(X, E)$ plays an important role in the topological measure theory (see [13–18] for definitions and more details).

Now, we recall a definition of the strict topology β_σ on $C_{\mathrm{rc}}(X, E)$. Let βX stand for the Stone-Čech compactification of X and $\mathcal{C}_\sigma$ denote the family of all the zero sets of continuous functions on $\beta X \setminus X$. For a set $Q \in \mathcal{C}_\sigma$, let $C_Q(X) := \{v \in C_b(X) : \bar{v}|_Q \equiv 0\}$, where $\bar{v}$ denotes the unique extension of $v \in C_b(X)$ on βX. For each $v \in C_Q(X)$, let us define

$$\rho_v(f) := \sup_{t \in X} |v(t)| \|f(t)\|_E \quad \text{for } f \in C_{\mathrm{rc}}(X, E).$$

By β_Q we denote the locally convex Hausdorff topology on $C_{\mathrm{rc}}(X, E)$ defined by the family of seminorms $\{\rho_v : v \in C_Q(X)\}$. The strict topology β_σ on $C_{\mathrm{rc}}(X, E)$, defined by $\mathcal{C}_\sigma$, is the greatest lower bound (in the class of all locally convex Hausdorff topologies on $C_{\mathrm{rc}}(X, E)$) of the topologies β_Q as Q runs over $\mathcal{C}_\sigma$ (see [18, p. 181], [13, p. 322]). Then, β_σ is a locally convex-solid topology on $C_{\mathrm{rc}}(X, E)$ (see [18, Proposition 2.7]).

It is known that $\beta_\sigma \subset \tau_u$ and $\beta_\sigma = \tau_u$ if X is pseudocompact (see [13, Theorem 4.3]). The strict topology β_σ is a σ*-Dini topology*; that is, $f_n \to 0$ in β_σ whenever (f_n) is a sequence $C_{\mathrm{rc}}(X, E)$ such that $\|f_n(t)\|_E \downarrow_n 0$ for all $t \in X$; and β_σ is the finest locally convex-solid topology on $C_{\rm rc}(X, E)$ with this property (see [18, Corollary 2.9], [39, Corollary 11.16]).

Recall that a linear operator $T : C_{\rm rc}(X, E) \to F$ is said to be *nuclear* between the Banach spaces $C_{\rm rc}(X, E)$ and F if there exist a bounded sequence (Φ_n) in $C_{\rm rc}(X, E)'$, a bounded sequence (y_n) in F, and a sequence $(\alpha_n) \in \ell^1$ so that

$$T(f) = \sum_{n=1}^{\infty} \alpha_n \Phi_n(f) y_n \quad \text{for } f \in C_{\rm rc}(X, E) \tag{1.1}$$

(see [40, p. 279] and [35]). The nuclear norm $\|T\|_{\rm nuc}$ of T is defined by

$$\|T\|_{\rm nuc} := \inf\left\{ \sum_{n=1}^{\infty} |\alpha_n| \|\Phi_n\| \|y_n\|_F \right\},$$

where the infimum is taken over all sequences (Φ_n) in $C_{\rm rc}(X, E)'$ and (y_n) in F and $(\alpha_n) \in \ell^1$ such that T admits a representation (1.1). Every nuclear operator

$$T : C_{\rm rc}(X, E) \to F$$

is compact.

Let $\mathcal{N}(E, F)$ denote the Banach space of all nuclear operators $U : E \to F$, equipped with the nuclear norm $\|\cdot\|_{\rm nuc}$ (see [21, Proposition, p. 51]). Then, we have $\|U\| \le \|U\|_{\rm nuc}$ for $U \in \mathcal{N}(E, F)$.

A linear operator $T : C_{\rm rc}(X, E) \to F$ is called a *nuclear operator* between the locally convex space $(C_{\rm rc}(X, E), \beta_\sigma)$ and the Banach space F if there exist a β_σ-equicontinuous sequence (Φ_n) in $(C_{\rm rc}(X, E), \beta_\sigma)'$, a bounded sequence (y_n) in F, and a sequence $(\alpha_n) \in \ell^1$ so that

$$T(f) = \sum_{n=1}^{\infty} \alpha_n \Phi_n(f) y_n \quad \text{for } f \in C_{\rm rc}(X, E)$$

(see [40, p. 289] and [35, 38] for more details). Every $(\beta_\sigma, \|\cdot\|_F)$-nuclear operator is $(\beta_\sigma, \|\cdot\|_F)$-compact and hence T is $(\beta_\sigma, \|\cdot\|_F)$-continuous.

The problem of integral representation of different classes of linear operators $T : C_{\rm rc}(X, E) \to F$ has been studied by Katsaras and Liu [15] and Nowak [18, 19]. The aim of this paper is to establish the relationship between the nuclearity of $(\beta_\sigma, \|\cdot\|_F)$-continuous nuclear operators $T : C_{\rm rc}(X, E) \to F$ and their representing operator-valued Baire measures (see Theorem 4.3, Theorem 4.5, and Corollary 4.6 below). Moreover, as an application, we establish the relationship between the nuclearity of a $(\beta_\sigma, \|\cdot\|_F)$-continuous operator $T : C_{\rm rc}(X, E) \to F$ and the nuclearity of its conjugate operator T' (see Corollary 4.11 below).

Remark 1.1. Let $C_b(X, E)$ be the space of all bounded continuous functions $f : X \to E$, equipped with the tight strict topology β. Then, the nuclear operators

$$T : C_b(X, E) \to F$$

between the locally convex space $(C_b(X,E),\beta)$ and the Banach space F have been studied by Nowak and Stochmal [20] and Stochmal [36].

2. Integration in the space $C_{rc}(X,E)$

We recall the terminology concerning spaces of Baire measures and study the problem of integration of functions in $C_{rc}(X,E)$ with respect to Baire vector measures (see [12, 14–16] for more details).

Recall that a zero set in X is of the form $Z=\{t\in X: u(t)=0\}$, where $u\in C_b(X)$. Let $\mathcal{B}$ (resp., $\mathcal{B}a$) be the algebra (resp., σ-algebra) of Baire sets in X, which is the algebra (resp., σ-algebra) generated by the class $\mathcal{Z}$ of all zero sets in X.

Let $M(X)$ stand for the space of all finitely additive real-valued zero-set regular measures on $\mathcal{B}$, that is, $\nu\in M(X)$, if for every $A\in\mathcal{B}$ and $\varepsilon>0$ there exists $Z\in\mathcal{Z}$ with $Z\subset A$ such that $|\nu|(A\setminus Z)\le\varepsilon$. Then, $M(X)$ with the norm $\|\nu\|:=|\nu|(X)$ is a Dedekind complete Banach lattice (see [39, p. 114]).

Following [13], by $M(X,E')$ we denote the set of all finitely additive measures $\mu:\mathcal{B}\to E'$ with $|\mu|(X)<\infty$ and such that $\mu_x\in M(X)$ for each $x\in E$.

Note that if $\mu\in M(X,E')$, then $|\mu|\in M(X)$ (see [13, p. 314]).

Let $A\in\mathcal{B}$ and $\mu\in M(X,E')$. Following [12], for $f\in C_{rc}(X,E)$, one can define a *Riemann–Stieltjes-type integral* on $A\in\mathcal{B}$ with respect to μ by

$$(RS)\int_A f(t)\,d\mu := \lim\sum\mu(A_i)f(t_i),$$

where the limit is taken over the directed set of all finite $\mathcal{B}$-partitions (A_i) of A and $t_i\in A_i$ (see also [13–16] for more details).

According to Katsaras [12, Theorem 2.5], $C_{rc}(X,E)'$ can be identified with $M(X,E')$ through the linear mapping $M(X,E')\ni\mu\mapsto\Phi_\mu\in C_{rc}(X,E)'$, where

$$\Phi_\mu(f)=(RS)\int_X f(t)\,d\mu\quad\text{for } f\in C_{rc}(X,E) \text{ and } \|\Phi_\mu\|=|\mu|(X).$$

It follows that $M(X,E')$, equipped with the norm $\|\mu\|:=|\mu|(X)$, is a Banach space.

By $B(\mathcal{B},E)$ we denote the Banach space of totally $\mathcal{B}$-measurable functions $g:X\to E$ (= the uniform limits of sequences of E-valued $\mathcal{B}$-simple functions on X), equipped with the uniform norm $\|\cdot\|$ (see [5, 6]). Then, we have (see [17, p. 196])

$$C_b(X)\otimes E\subset C_{rc}(X,E)\subset B(\mathcal{B},E). \tag{2.1}$$

Recall that $C_b(X)\otimes E$ is the linear span of all functions $u\otimes x$, where $u\in C_b(X)$ and $x\in E$ and $(u\otimes x)(t)=u(t)x$ for all $t\in X$.

Since $C_b(X)\otimes E$ is a dense subset of the Banach space $C_{rc}(X,E)$ (see [12, Lemma 2.2]), one can easily show that, for each $\mu\in M(X,E')$, we have

$$(RS)\int_X f(t)\,d\mu=\int_X f(t)\,d\mu\quad\text{for } f\in C_{rc}(X,E),$$

where $\int_X f(t)d\mu$ denotes the so-called *immediate integral* of f with respect to μ (see [5, Section 9] for more details).

Let $M_\sigma(X)$ denote the subspace of $M(X)$ of all σ-additive Baire measures ν, that is, $\nu(Z_n) \to 0$ if $Z_n \downarrow \emptyset$, $Z_n \in \mathcal{Z}$. It is known that, for $\nu \in M(X)$, $\nu \in M_\sigma(X)$ if and only if ν is countably additive on the algebra $\mathcal{B}$ (see [39, Section 6.2, pp. 117–118]).

Let

$$M_\sigma(X, E') := \{\mu \in M(X, E') : \mu_x \in M_\sigma(X) \text{ for each } x \in E\}.$$

According to [13, Theorem 4.7], we have the following result.

Theorem 2.1. *For $\mu \in M(X, E')$, the following statements are equivalent.*

(i) $\mu \in M_\sigma(X, E')$.

(ii) $\Phi_\mu \in (C_{rc}(X, E), \beta_\sigma)'$.

It follows that $M_\sigma(X, E')$ is a Banach space because $(C_{rc}(X, E), \beta_\sigma)'$ is a closed subspace of the Banach space $C_{rc}(X, E)'$ (see [14, Corollary 2.5]).

In view of [18, Corollary 3.2 and Proposition 3.5] and [39, Theorem 11.14, p. 142], we get the following result.

Theorem 2.2. *Let $\mathcal{M}$ be a subset of $M_\sigma(X, E')$. Then, the following statements are equivalent.*

(i) $\{\Phi_\mu : \mu \in \mathcal{M}\}$ *is β_σ-equicontinuous.*

(ii) $\sup_{\mu\in\mathcal{M}} |\mu|(X) < \infty$ *and* $\sup_{\mu\in\mathcal{M}} |\mu|(Z_n) \to 0$ *if $Z_n \downarrow \emptyset$, $Z_n \in \mathcal{Z}$.*

By $M_\sigma(X, \mathcal{L}(E, F))$ we denote the space of all finitely additive measures $m : \mathcal{B} \to \mathcal{L}(E, F)$ with $\tilde{m}(X) < \infty$ such that $m_{y'} \in M_\sigma(X, E')$ for each $y' \in F'$.

Following [13, 15], by $M_\sigma(\mathcal{B}a)$ we denote the space of all countably additive real-valued zero-set regular measures on $\mathcal{B}a$.

Remark 2.3. Note that every real-valued countably additive measure ν on $\mathcal{B}a$ must be zero-set regular; that is, $\nu \in M_\sigma(\mathcal{B}a)$ (see [39, p. 118]).

By $M_\sigma(\mathcal{B}a, E')$ we denote the space of all finitely additive measures $\mu : \mathcal{B}a \to E'$ with $|\mu|(X) < \infty$ such that $\mu_x \in M_\sigma(\mathcal{B}a)$ for each $x \in E$.

The following result will be needed.

Proposition 2.4. *The following statements hold.*

(i) *If $\mu \in M_\sigma(\mathcal{B}a, E')$, then $|\mu| \in M_\sigma(\mathcal{B}a)^+$.*

(ii) *If $\mu \in M_\sigma(X, E')$, then μ possesses a unique extension $\overline{\mu} \in M_\sigma(\mathcal{B}a, E')$ and $|\overline{\mu}|(A) = |\mu|(A)$ for $A \in \mathcal{B}$.*

Proof. (i) See [13, Lemma 2.1].

(ii) In view of [13, Theorem 2.5], μ possesses a unique extension $\overline{\mu} \in M_\sigma(\mathcal{B}a, E')$ and $|\overline{\mu}|(X) = |\mu|(X)$. According to [4, Corollary 10, p. 4], we have $|\overline{\mu}|(A) = |\mu|(A)$ for $A \in \mathcal{B}$. ∎

By $M_\sigma(\mathcal{B}a, \mathcal{L}(E,F))$ we denote the space of all measures $m : \mathcal{B}a \to \mathcal{L}(E,F)$ with $\tilde{m}(X) < \infty$ such that $m_{y'} \in M_\sigma(\mathcal{B}a, E')$ for each $y' \in F'$.

3. Integral representation of operators on $C_{\mathrm{rc}}(X, E)$

In this section, we collect basic results concerning integral representation of weakly compact operators $T : C_{\mathrm{rc}}(X,E) \to F$ (see [15, 18, 19]).

Since $C_{\mathrm{rc}}(X,E) \subset B(\mathcal{B}, E)$, one can embed $B(\mathcal{B}, E)$ into $C_{\mathrm{rc}}(X,E)''$ by the mapping $\pi : B(\mathcal{B}, E) \to C_{\mathrm{rc}}(X,E)''$, where, for $g \in B(\mathcal{B}, E)$,

$$\pi(g)(\Phi_\mu) = \int_X g(t)\, d\mu \quad \text{for } \mu \in M(X, E').$$

Then, for $g \in B(\mathcal{B}, E)$ and $\mu \in M(X, E')$, we have

$$|\pi(g)(\Phi_\mu)| = \left| \int_X g(t)\, d\mu \right| \leq \|g\| |\mu|(X) = \|g\| \|\Phi_\mu\|,$$

and hence, π is bounded and $\|\pi(g)\| \leq \|g\|$; that is, $\|g\| \leq 1$.

Let $i_F : F \to F''$ stand for the canonical embedding, i.e., $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ denote the left inverse of i_F; that is,

$$j_F \circ i_F = id_F.$$

Now, assume that $T : C_{\mathrm{rc}}(X,E) \to F$ is a weakly compact linear operator. Let

$$T' : F' \to C_{\mathrm{rc}}(X,E)' \quad \text{and} \quad T'' : C_{\mathrm{rc}}(X,E)'' \to F''$$

stand for the conjugate and biconjugate operators of T, respectively. Then, $T'(y') := y' \circ T$ for $y' \in F'$ and $T''(\varphi)(y') := \varphi(y' \circ T)$ for $\varphi \in C_{\mathrm{rc}}(X,E)''$. Due to the Gantmacher theorem (see [2, Theorem 17.2]), we have that $T''(C_{\mathrm{rc}}(X,E)'') \subset i_F(F)$. Let us define

$$\widehat{T} := j_E \circ T'' \circ \pi : B(\mathcal{B}, E) \to F.$$

Then, $\widehat{T}$ is a weakly compact operator, and we define its *representing measure* $m : \mathcal{B} \to \mathcal{L}(E,F)$ by

$$m(A)(x) := \widehat{T}(\mathbb{1}_A \otimes x) \quad \text{for } A \in \mathcal{B} \text{ and } x \in E. \tag{3.1}$$

Thus, it follows that, for $g \in B(\mathcal{B}, E)$, we have

$$\widehat{T}(g) = \int_X g(t)\, dm \quad \text{and} \quad \|\widehat{T}\| = \tilde{m}(X),$$

where $\int_X g(t)\, dm$ denotes the so-called *immediate integral* of g with respect to m (see [5, Section 9], [6, Section 1, pp. 10–11]). Then, for $f \in C_{\mathrm{rc}}(X,E)$, we have

$$T(f) = \int_X f(t)\, dm \quad \text{and} \quad \|T\| = \tilde{m}(X),$$

and, for each $y' \in F'$, $y'(T(f)) = \int_X f(t) dm_{y'}$.

The following result will be of importance.

Theorem 3.1. *Assume that* $T : C_{\mathrm{rc}}(X, E) \to F$ *is a weakly compact* $(\beta_\sigma, \|\cdot\|_F)$*-continuous operator and* $m : \mathcal{B} \to \mathcal{L}(E, F)$ *is its representing measure. Then, the following statements hold.*

(i) $m \in M_\sigma(X, \mathcal{L}(E, F))$.

(ii) *m has a unique extension* $\bar{m} \in M_\sigma(\mathcal{B}a, \mathcal{L}(E, F))$ *such that* $\tilde{\bar{m}}(X) = \tilde{m}(X) = \|T\|$.

Proof. (i) It follows from Theorem 2.1 because $y' \circ T \in (C_{\mathrm{rc}}(X, E), \beta_\sigma)'$.

(ii) Since $\widehat{T} : B(\mathcal{B}, E) \to F$ is weakly compact, we have that, for every $x \in E$, the set $\{m(A)(x) : A \in \mathcal{B}\}$ is weakly compact in F. Hence, in view of [15, Theorem 7], m has a unique extension $\bar{m} \in M_\sigma(\mathcal{B}a, \mathcal{L}(E, F))$ with $\tilde{\bar{m}}(X) = \tilde{m}(X) = \|T\|$. ∎

4. Nuclear operators on $C_{\mathrm{rc}}(X, E)$

In this section, we establish the relationship between the nuclearity of weakly compact $(\beta_\sigma, \|\cdot\|_F)$-continuous operators $T : C_{\mathrm{rc}}(X, E) \to F$ and the properties of their representing measures $m : \mathcal{B} \to \mathcal{L}(E, F)$.

For $\nu \in ca(\mathcal{B}a)^+$ and a Banach space $(G, \|\cdot\|_G)$, let $L^1(\nu, G)$ be the Banach space of ν-equivalence classes of all ν-Bochner integrable functions $g : X \to G$, equipped with the norm

$$\|g\|_1 := \int_X \|g(t)\|_G \, d\nu.$$

We will need the following lemma.

Lemma 4.1. *For* $\nu \in M_\sigma(\mathcal{B}a)^+$ *and* $g \in L^1(\nu, E')$, *let* $\mu(B) = \int_B g(t)\, d\nu$ *for* $B \in \mathcal{B}a$. *Then,* $\mu \in M_\sigma(\mathcal{B}a, E')$ *and* $\int_X f(t)\, d\mu = \int_X g(t)(f(t))\, d\nu$ *for* $f \in C_{\mathrm{rc}}(X, E)$.

Proof. Note that $\mu : \mathcal{B}a \to E'$ is a countably additive measure and

$$|\mu|(X) = \int_X \|g(t)\|_{E'} d\nu < \infty$$

(see [4, Theorem 4, p. 46]). Hence, $|\mu|$ is countably additive and $|\mu| \in M_\sigma(\mathcal{B}a)^+$ (see Remark 2.3). It follows that $\mu \in M_\sigma(\mathcal{B}a, E')$.

Assume that $f \in C_{\mathrm{rc}}(X, E)$. Since $C_{\mathrm{rc}}(X, E) \subset B(\mathcal{B}, E)$ (see (2.1)), we can choose a sequence (s_n) of E-valued $\mathcal{B}$-simple functions on X such that $\|f - s_n\| \overset{n}{\to} 0$. Note that

$$\int_X s_n(t)\, d\mu = \int_X g(t)(s_n(t))\, d\nu.$$

Then, we have

$$\int_X f(t)\, d\mu = \lim_n \int_X s_n(t)\, d\mu = \lim_n \int_X g(t)(s_n(t))\, d\nu.$$

On the other hand, we have

$$\left| \int_X g(t)(f(t))\, d\nu - \int_X g(t)(s_n(t)) d\nu \right| \leq \int_X |g(t)(f(t) - s_n(t))|\, d\nu$$
$$\leq \int_X \|g(t)\|_{E'} \|f(t) - s_n(t)\|_E\, d\nu$$
$$\leq \|f - s_n\| \int_X \|g(t)\|_{E'}\, d\nu \xrightarrow{n} 0.$$

Thus, it follows that $\int_X f(t)\, d\mu = \int_X g(t)(f(t))\, d\nu$. ■

Remark 4.2. A similar result as in Lemma 4.1 appears in the proof of Theorem 2 in [26] and in [23].

Now, we can state our main result.

Theorem 4.3. *Let $T : C_{\mathrm{rc}}(X, E) \to F$ be a weakly compact $(\beta_\sigma, \|\cdot\|_F)$-continuous operator, and let $m : \mathcal{B} \to \mathcal{L}(E, F)$ be its representing measure. Assume that $\bar{m}(B) \in \mathcal{N}(E, F)$ for each $B \in \mathcal{B}a$ and $\bar{m} : \mathcal{B}a \to \mathcal{N}(E, F)$ is a countably additive measure and $|\bar{m}|_{\mathrm{nuc}}(X) < \infty$ (with respect to the norm $\|\cdot\|_{\mathrm{nuc}}$).*

If there exists $H \in L^1(|\bar{m}|_{\mathrm{nuc}}, \mathcal{N}(E, F))$ so that

$$\bar{m}(B) = \int_B H(t)\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{for } B \in \mathcal{B}a,$$

then the following statements hold.

(i) $T(f) = \int_X H(t)(f(t))\, d|\bar{m}|_{\mathrm{nuc}}$ *for* $f \in C_{\mathrm{rc}}(X, E)$.

(ii) *T is a nuclear operator between the locally convex space $(C_{\mathrm{rc}}(X, E), \beta_\sigma)$ and the Banach space F.*

(iii) *T is a nuclear operator between the Banach spaces $C_{\mathrm{rc}}(X, E)$ and F, and*

$$\|T\|_{\mathrm{nuc}} \leq |m|_{\mathrm{nuc}}(X) = |\bar{m}|_{\mathrm{nuc}}(X) = \int_X \|H(t)\|_{\mathrm{nuc}}\, d|\bar{m}|_{\mathrm{nuc}}.$$

Proof. (i) Let $f \in C_{\mathrm{rc}}(X, F) \subset B(\mathcal{B}, E)$. Then, there exists a sequence (s_n) of E-valued $\mathcal{B}$-simple functions on X such that $\|f - s_n\| \xrightarrow{n} 0$. Note that, for $n \in \mathbb{N}$, we have

$$\int_X s_n(t)\, d\bar{m} = \int_X H(t)(s_n(t))\, d|\bar{m}|_{\mathrm{nuc}}.$$

Hence,

$$\left\| \int_X H(t)(f(t))\, d|\bar{m}|_{\mathrm{nuc}} - \int_X H(t)(s_n(t))\, d|\bar{m}|_{\mathrm{nuc}} \right\|_F$$
$$\leq \int_X \|H(t)(f(t)) - H(t)(s_n(t))\|_F\, d|\bar{m}|_{\mathrm{nuc}} \leq \int_X \|H(t)\|_{\mathrm{nuc}} \|f(t) - s_n(t)\|_E\, d|\bar{m}|_{\mathrm{nuc}}$$
$$\leq \|f - s_n\| \int_X \|H(t)\|_{\mathrm{nuc}}\, d|\bar{m}|_{\mathrm{nuc}} \xrightarrow{n} 0.$$

On the other hand, we get

$$T(f) = \int_X f(t)\, dm = \lim_n \int_X s_n(t)\, dm = \lim_n \int_X s_n(t)\, d\bar{m} = \lim_n \int_X H(t)(s_n(t))\, d|\bar{m}|_{\mathrm{nuc}}.$$

Thus, it follows that

$$T(f) = \int_X H(t)(f(t))\, d|\bar{m}|_{\mathrm{nuc}}.$$

(ii) Let $L^1(|\bar{m}|_{\mathrm{nuc}})\widehat{\otimes}_\gamma \mathcal{N}(E, F)$ denote the projective tensor product of the Banach spaces $L^1(|\bar{m}|_{\mathrm{nuc}})$ and $\mathcal{N}(E, F)$, equipped with the complete projective norm γ (see [4, p. 227]). Note that, for $z \in L^1(|\bar{m}|_{\mathrm{nuc}})\widehat{\otimes}_\gamma \mathcal{N}(E, F)$, we have

$$\gamma(z) = \inf\left\{ \sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_1 \|U_n\|_{\mathrm{nuc}} \right\},$$

where the infimum is taken over all sequences (v_n) in $L^1(|\bar{m}|_{\mathrm{nuc}})$ and (U_n) in $\mathcal{N}(E, F)$ with $\lim \|v_n\|_1 = 0 = \lim \|U_n\|_{\mathrm{nuc}}$ and $(\alpha_n) \in \ell^1$ such that

$$z = \sum_{n=1}^{\infty} \alpha_n v_n \otimes U_n$$

in γ-norm (see [30, Proposition 2.8, pp. 21–22]).

It is known that $L^1(|\bar{m}|_{\mathrm{nuc}})\widehat{\otimes}_\gamma \mathcal{N}(E, F)$ is isometrically isomorphic to the Banach space $L^1(|\bar{m}|_{\mathrm{nuc}}, \mathcal{N}(E, F))$ by the isometry

$$J : L^1(|\bar{m}|_{\mathrm{nuc}})\widehat{\otimes}_\gamma \mathcal{N}(E, F) \to L^1(|\bar{m}|_{\mathrm{nuc}}, \mathcal{N}(E, F)),$$

defined by $J(v \otimes U) := v(\cdot)U$ for $v \in L^1(|\bar{m}|_{\mathrm{nuc}})$ and $U \in \mathcal{N}(E, F)$ (see [4, Example 10, p. 228], [30, Example 2.19, p. 29]).

Let $\varepsilon > 0$ be given. Then, there exist sequences (v_n) in $L^1(|\bar{m}|_{\mathrm{nuc}})$ and (U_n) in $\mathcal{N}(E,F)$ with $\lim_n \|v_n\|_1 = 0 = \lim_n \|U_n\|_{\mathrm{nuc}}$ and $(\alpha_n) \in \ell^1$ so that

$$J^{-1}(H) = \sum_{n=1}^{\infty} \alpha_n v_n \otimes U_n \quad \text{in } L^1(|m|_{\mathrm{nuc}})\widehat{\otimes}_\gamma \mathcal{N}(X, Y)$$

and

$$\sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_1 \|U_n\|_{\mathrm{nuc}} \le \gamma\big(J^{-1}(H)\big) + \frac{\varepsilon}{2} = \|H\|_1 + \frac{\varepsilon}{2}. \tag{4.1}$$

Thus, it follows that

$$H = J\left(\sum_{n=1}^{\infty} \alpha_n v_n \otimes U_n \right) = \sum_{n=1}^{\infty} \alpha_n v_n(\cdot) U_n \quad \text{in } L^1\big(|\bar{m}|_{\mathrm{nuc}}, \mathcal{N}(X, Y)\big).$$

Note that for $f \in C_{\mathrm{rc}}(X,E)$, $T(f) = \sum_{n=1}^{\infty} \alpha_n \int_X v_n(t) U_n(f(t))\, d|\bar{m}|_{\mathrm{nuc}}$. Indeed, using (i), we have

$$\begin{aligned}
&\left\| T(f) - \sum_{i=1}^{n} \alpha_i \int_X v_i(t) U_i(f(t))\, d|\bar{m}|_{\mathrm{nuc}} \right\|_F \\
&\quad \le \int_X \left\| H(t)(f(t)) - \left(\sum_{i=1}^{n} \alpha_i v_i(t) U_i \right)(f(t)) \right\|_F d|\bar{m}|_{\mathrm{nuc}} \\
&\quad \le \int_X \left\| H(t) - \sum_{i=1}^{n} \alpha_i v_i(t) U_i \right\|_{\mathrm{nuc}} \|f(t)\|_E \, d|\bar{m}|_{\mathrm{nuc}} \\
&\quad \le \|f\| \int_X \left\| H(t) - \sum_{i=1}^{n} \alpha_i v_i(t) U_i \right\|_{\mathrm{nuc}} d|\bar{m}|_{\mathrm{nuc}} \xrightarrow{n} 0.
\end{aligned}$$

For every $n \in \mathbb{N}$, we can choose bounded sequences $(x'_{n,k})$ in E' and $(y_{n,k})$ in F and a sequence $(\alpha_{n,k}) \in \ell^1$ so that $U_n(x) = \sum_{k=1}^{\infty} \alpha_{n,k} x'_{n,k}(x) y_{n,k}$ for $x \in E$ and

$$\sum_{k=1}^{\infty} |\alpha_{n,k}| \|x'_{n,k}\|_{E'} \|y_{n,k}\|_F \le \|U_n\|_{\mathrm{nuc}} + \frac{\varepsilon}{2\left(\sum_{j=1}^{\infty} |\alpha_j| \|v_j\|_1\right)}. \tag{4.2}$$

Then, we have

$$\begin{aligned}
T(f) &= \sum_{n=1}^{\infty} \alpha_n \int_X v_n(t) U_n(f(t))\, d|\bar{m}|_{\mathrm{nuc}} \\
&= \sum_{n=1}^{\infty} \alpha_n \int_X v_n(t) \left(\sum_{k=1}^{\infty} \alpha_{n,k} x'_{n,k}(f(t)) y_{n,k} \right) d|\bar{m}|_{\mathrm{nuc}} \\
&= \sum_{n=1}^{\infty} \alpha_n \sum_{k=1}^{\infty} \alpha_{n,k} \|x'_{n,k}\|_{E'} \|y_{n,k}\|_F \left(\int_X v_n(t) \frac{x'_{n,k}(f(t))}{\|x'_{n,k}\|_{E'}}\, d|\bar{m}|_{\mathrm{nuc}} \right) \frac{y_{n,k}}{\|y_{n,k}\|_F}.
\end{aligned}$$

For $n, k \in \mathbb{N}$, let us define

$$g_{n,k}(t) := v_n(t) \frac{x'_{n,k}}{\|x'_{n,k}\|_{E'}} \quad \text{for } t \in X.$$

Then, $g_{n,k} \in L^1(|\bar{m}|_{\mathrm{nuc}}, E')$ and $\|g_{n,k}(t)\|_{E'} = |v_n(t)|$ for $t \in X$. Let

$$\mu_{n,k}(B) := \int_B g_{n,k}(t)\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{for } B \in \mathcal{B}a.$$

Then,

$$|\mu_{n,k}|(B) = \int_B \|g_{n,k}(t)\|_{E'}\, d|\bar{m}|_{\mathrm{nuc}} = \int_B |v_n(t)|\, d|\bar{m}|_{\mathrm{nuc}}.$$

Since $|\bar{m}|_{\text{nuc}} \in M_\sigma(\mathcal{B}a)^+$ (see Remark 2.3), in view of Lemma 4.1, we get

$$\mu_{n,k} \in M_\sigma(\mathcal{B}a, E').$$

For $n, k \in \mathbb{N}$, let us define

$$\Phi_{n,k}(f) := \int_X f(t)\, d\mu_{n,k} \quad \text{for } f \in C_{\text{rc}}(X, E).$$

Then, $\Phi_{n,k} \in (C_{\text{rc}}(X, E), \beta_\sigma)'$, and in view of Lemma 4.1, we have

$$\Phi_{n,k}(f) = \int_X g_{n,k}(t)(f(t))\, d|\bar{m}|_{\text{nuc}} \quad \text{for } f \in C_{\text{rc}}(X, E).$$

Note that $\sup\{|\mu_{n,k}|(X) : n, k \in \mathbb{N}\} = \sup_n \|v_n\|_1 < \infty$, and since

$$\lim_n \|v_n\|_1 = \lim_n \int_X |v_n(t)|\, d|\bar{m}|_{\text{nuc}} = 0,$$

the set $\{v_n : n \in \mathbb{N}\}$ is uniformly integrable in $L^1(|\bar{m}|_{\text{nuc}})$.

Assume that $\eta > 0$ and $Z_i \downarrow \emptyset$, $Z_i \in \mathcal{Z}$. Then, there exists $\delta > 0$ such that

$$\sup_n \int_B |v_n(t)|\, d|\bar{m}|_{\text{nuc}} \leq \eta$$

for all $B \in \mathcal{B}a$ with $|\bar{m}|_{\text{nuc}}(B) \leq \delta$. Choose $i_0 \in \mathbb{N}$ such that $|\bar{m}|_{\text{nuc}}(Z_i) \leq \delta$ for $i \geq i_0$. Hence, for $i \geq i_0$, we get

$$\sup_{n,k\in\mathbb{N}} |\mu_{n,k}|(Z_i) = \sup_n \int_{Z_i} |v_n(t)|\, d|\bar{m}|_{\text{nuc}} \leq \eta.$$

Hence, in view of Theorem 2.2, the set $\{\Phi_{n,k} : n, k \in \mathbb{N}\}$ is β_σ-equicontinuous. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_n||\alpha_{n,k}|\|x'_{n,k}\|_{E'}\|y_{n,k}\|_F &= \sum_{n=1}^{\infty}|\alpha_n|\Bigg(\sum_{k=1}^{\infty}|\alpha_{n,k}|\|x'_{n,k}\|_{E'}\|y_{n,k}\|_F\Bigg)\\
&\leq \sum_{n=1}^{\infty}|\alpha_n|\Bigg(\|U_n\|_{\text{nuc}} + \frac{\varepsilon}{2\big(\sum_{j=1}^{\infty}|\alpha_j|\|v_j\|_1\big)}\Bigg)\\
&\leq \sum_{n=1}^{\infty}|\alpha_n|\Bigg(\sup_{j\in\mathbb{N}}\|U_j\|_{\text{nuc}} + \frac{\varepsilon}{2\big(\sum_{j=1}^{\infty}|\alpha_j|\|v_j\|_1\big)}\Bigg)\\
&< \infty.
\end{aligned}$$

This means that (ii) holds.

(iii) Since $\beta_\sigma \subset \tau_u$, by (ii), we have that T is a nuclear operator between the Banach spaces $C_{\text{rc}}(X, E)$ and F. Moreover, in view of (ii), for $f \in C_{\text{rc}}(X, E)$, we have

$$\begin{aligned}
&T(f)\\
&= \sum_{n=1}^{\infty}\alpha_n\|v_n\|_1\sum_{k=1}^{\infty}\alpha_{n,k}\|x'_{n,k}\|_{E'}\|y_{n,k}\|_F\Bigg(\int_X \frac{v_n(t)}{\|v_n\|_1}\frac{x'_{n,k}(f(t))}{\|x'_{n,k}\|_{E'}}\, d|\bar{m}|_{\text{nuc}}\Bigg)\frac{y_{n,k}}{\|y_{n,k}\|_F}.
\end{aligned}$$

Note that by (4.1) and (4.2) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_n|\|v_n\|_1|\alpha_{n,k}|\|x'_{n,k}\|_{E'}\|y_{n,k}\|_F \\
&\quad = \sum_{n=1}^{\infty} |\alpha_n|\|v_n\|_1\left(\sum_{k=1}^{\infty}|\alpha_{n,k}|\|x'_{n,k}\|_{E'}\|y_{n,k}\|_F\right) \\
&\quad \le \sum_{n=1}^{\infty} |\alpha_n|\|v_n\|_1\left(\|U_n\|_{\mathrm{nuc}} + \frac{\varepsilon}{2\left(\sum_{j=1}^{\infty}|\alpha_j|\|v_j\|_1\right)}\right) \\
&\quad \le \|H\|_1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \|H\|_1 + \varepsilon.
\end{aligned} \tag{4.3}$$

For $n, k \in \mathbb{N}$, let us define

$$F_{n,k}(f) := \int_X \frac{v_n(t)}{\|v_n\|_1}\frac{x'_{n,k}(f(t))}{\|x'_{n,k}\|_{E'}}\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{for } f \in C_{\mathrm{rc}}(X, E).$$

Then, $|F_{n,k}(f)| \le \|f\|$; that is, $F_{n,k} \in C_{\mathrm{rc}}(X, E)'$ and $\|F_{n,k}\| \le 1$ for all $n, k \in \mathbb{N}$.

In view of (4.3), we get $\|T\|_{\mathrm{nuc}} \le \|H\|_1 + \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we have

$$\|T\|_{\mathrm{nuc}} \le \|H\|_1 = \int_X \|H(t)\|_{\mathrm{nuc}}\, d|\bar{m}|_{\mathrm{nuc}} = |\bar{m}|_{\mathrm{nuc}}(X).$$

Since $|m|_{\mathrm{nuc}}(X) = |\bar{m}|_{\mathrm{nuc}}(X)$ (see [4, Corollary 10, p. 4]), the proof is complete. ∎

We will need the following lemma.

Lemma 4.4. *Assume that (λ_n) is a bounded sequence in $M(X, E')$, (y_n) is a bounded sequence in F, and $(\alpha_n) \in \ell^1$. Then, for $y' \in F'$, we have that*

$$\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n \in M(X, E')$$

and

$$\left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n\right)(A)(x) = \sum_{n=1}^{\infty} \alpha_n \lambda_n(A)(x) y'(y_n) \quad \textit{for } A \in \mathcal{B}, x \in E.$$

Proof. Since for $y' \in F'$ we have

$$\sum_{n=1}^{\infty} |\alpha_n||y'(y_n)||\lambda_n|(X) < \infty$$

and $M(X, E')$ is a Banach space, we get

$$\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n \in M(X, E').$$

Then, for $A \in \mathcal{B}$, $x \in E$, we have

$$\begin{aligned}
&\left|\left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n\right)(A)(x) - \sum_{i=1}^{n} \alpha_i \lambda_i(A)(x) y'(y_i)\right| \\
&\quad = \left|\left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n - \sum_{i=1}^{n} \alpha_i y'(y_i)\lambda_i\right)(A)(x)\right| \\
&\quad \leq \left\|\left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n - \sum_{i=1}^{n} \alpha_i y'(y_i)\lambda_i\right)(A)\right\|_{E'} \|x\|_E \\
&\quad \leq \left|\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n - \sum_{i=1}^{n} \alpha_i y'(y_i)\lambda_i\right|(X)\|x\|_E \xrightarrow{n} 0.
\end{aligned}$$

■

Now, we can derive the properties of the representing measure m of a nuclear $(\beta_\sigma, \|\cdot\|_F)$-continuous operator

$$T : C_{\mathrm{rc}}(X, E) \to F.$$

Theorem 4.5. *Assume that $T : C_{\mathrm{rc}}(X, E) \to F$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous nuclear operator between the Banach spaces $C_{\mathrm{rc}}(X, E)$ and F and $m : \mathcal{B} \to \mathcal{L}(E, F)$ is its representing measure. Then, the following statements hold.*

(i) *$\bar{m}(B) \in \mathcal{N}(E, F)$ for $B \in \mathcal{B}a$ and $\bar{m} : \mathcal{B}a \to \mathcal{N}(E, F)$ is a countably additive measure with*

$$|\bar{m}|_{\mathrm{nuc}}(X) \leq \|T\|_{\mathrm{nuc}}.$$

(ii) *If, in particular, E' has the Radon–Nikodym property, then there exists*

$$H \in L^1(|\bar{m}|_{\mathrm{nuc}}, \mathcal{N}(E, F))$$

so that

$$\bar{m}(B) = \int_B H(t)\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{for } B \in \mathcal{B}a$$

and

$$T(f) = \int_X H(t)(f(t))\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{for } f \in C_{\mathrm{rc}}(X, E).$$

Proof. (i) Let $\varepsilon > 0$ be given. There exist a bounded sequence (λ_n) in $M(X, E')$, a bounded sequence (y_n) in F, and a sequence $(\alpha_n) \in \ell^1$ so that

$$T(f) = \sum_{n=1}^{\infty} \alpha_n \left(\int_X f(t)\, d\lambda_n\right) y_n \quad \text{for } f \in C_{\mathrm{rc}}(X, E)$$

and

$$\sum_{n=1}^{\infty} |\alpha_n| |\lambda_n|(X) \|y_n\|_F \leq \|T\|_{\mathrm{nuc}} + \varepsilon.$$

Let $A \in \mathcal{B}$. Then, using (3.1), for each $x \in E$ and $y' \in F'$, we have

$$\begin{aligned}
y'(m(A)(x)) &= ((T'' \circ \pi)(\mathbb{1}_A \otimes x))(y') = \pi(\mathbb{1}_A \otimes x)(y' \circ T) \\
&= \sum_{n=1}^{\infty} \alpha_n y'(y_n)\pi(\mathbb{1}_A \otimes x)(\Phi_{\lambda_n}) = \sum_{n=1}^{\infty} \alpha_n y'(y_n) \int_X (\mathbb{1}_A \otimes x)(t)\, d\lambda_n \\
&= \sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_n(A)(x) = y'\left(\sum_{n=1}^{\infty} \alpha_n \lambda_n(A)(x) y_n\right).
\end{aligned}$$

Hence, $m(A)(x) = \sum_{n=1}^{\infty} \alpha_n \lambda_n(A)(x) y_n$. This means that $m(A) \in \mathcal{N}(E, F)$.

Note that $M(X, E') \subset \mathrm{bva}(\mathcal{B}, E')$, where $\mathrm{bva}(\mathcal{B}, E')$ denotes the Banach space of all finitely additive measures $\lambda : \mathcal{B} \to E'$ of finite variation, equipped with the norm $\|\lambda\| = |\lambda|(X)$. Let $\mathrm{bvca}(\mathcal{B}, E')$ (resp., $\mathrm{bvpfa}(\mathcal{B}, E')$) denote the linear subspaces of $\mathrm{bva}(\mathcal{B}, E')$ consisting of all countably additive measures (resp., purely finitely additive measures). Note that $M_\sigma(X, E') \subset \mathrm{bva}(\mathcal{B}, E')$.

Due to the Yosida–Hewitt decomposition theorem (see [4, Theorem 8, p. 30]) for every $n \in \mathbb{N}$, there exist uniquely $\lambda_{n,c} \in \mathrm{bvca}(\mathcal{B}, E')$ and $\lambda_{n,p} \in \mathrm{bvpfa}(\mathcal{B}, E')$ so that

$$\lambda_n = \lambda_{n,c} + \lambda_{n,p}, \quad |\lambda_n| = |\lambda_{n,c}| + |\lambda_{n,p}|,$$

where $\lambda_{n,c}$ and $\lambda_{n,p}$ are mutually singular.

We will show that $\lambda_{n,c} \in M_\sigma(X, E')$ for $n \in \mathbb{N}$. Indeed, since $|\lambda_{n,c}| \le |\lambda_n|$ and $|\lambda_n| \in M(X)$ (see [13, p. 314]), we get $|\lambda_{n,c}| \in M(X)$, and it follows that $\lambda_{n,c} \in M(X, E')$. Note that $|\lambda_{n,c}| \in ca(\mathcal{B})$ because $\lambda_{n,c} \in \mathrm{bvca}(\mathcal{B}, E')$, and hence, $|\lambda_{n,c}| \in M_\sigma(X)$ (see [39, pp. 117–118]). This means that $\lambda_{n,c} \in M_\sigma(X, E')$, as desired.

Note that $\mathrm{bvca}(\mathcal{B}, E')$ and $\mathrm{bvpfa}(\mathcal{B}, E')$ are closed subspaces of the Banach space $\mathrm{bva}(\mathcal{B}, E')$. It follows that, for $y' \in F'$, we have

$$\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_{n,c} \in \mathrm{bvca}(\mathcal{B}, E') \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_{n,p} \in \mathrm{bvpfa}(\mathcal{B}, E').$$

For $A \in \mathcal{B}$ and $x \in E$, let us put

$$m_c(A)(x) := \sum_{n=1}^{\infty} \alpha_n \lambda_{n,c}(A)(x) y_n \quad \text{and} \quad m_p(A)(x) := \sum_{n=1}^{\infty} \alpha_n \lambda_{n,p}(A)(x) y_n.$$

Then, for every $y' \in F'$, in view of Lemma 4.4, we have

$$\begin{aligned}
y'(m_c(A)(x)) &:= \sum_{n=1}^{\infty} \alpha_n \lambda_{n,c}(A)(x) y'(y_n) = \left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_{n,c}\right)(A)(x), \\
y'(m_p(A)(x)) &:= \sum_{n=1}^{\infty} \alpha_n \lambda_{n,p}(A)(x) y'(y_n) = \left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_{n,p}\right)(A)(x).
\end{aligned}$$

Hence,

$$m_{c,y'} = \sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_{n,c} \quad \text{and} \quad m_{p,y'} = \sum_{n=1}^{\infty} \alpha_n y'(y_n)\lambda_{n,p},$$

where $m_{c,y'} \in \mathrm{bvca}(\mathcal{B}, E')$, $m_{p,y'} \in \mathrm{bvpfa}(\mathcal{B}, E')$, and $m_{y'} = m_{c,y'} + m_{p,y'}$. Since $m_{y'} \in M_\sigma(X, E')$ (see Theorem 3.1) and $M_\sigma(X, E') \subset \mathrm{bvca}(\mathcal{B}, E')$, we get $m_{y'} - m_{c,y'} \in \mathrm{bvca}(\mathcal{B}, E') \cap \mathrm{bvpfa}(\mathcal{B}, E') = \{0\}$, and hence, $m = m_c$; that is,

$$m(A)(x) = \sum_{n=1}^{\infty} \alpha_n \lambda_{n,c}(A)(x) y_n.$$

Let $\overline{\lambda_{n,c}} \in M_\sigma(\mathcal{B}a, E')$ stand for the unique extension of $\lambda_{n,c} \in M_\sigma(X, E')$, where $|\overline{\lambda_{n,c}}|(A) = |\lambda_{n,c}|(A)$ for $A \in \mathcal{B}$ (see Proposition 2.4). Let us set

$$m_0(B)(x) := \sum_{n=1}^{\infty} \alpha_n \overline{\lambda_{n,c}}(B)(x) y_n \quad \text{for } B \in \mathcal{B}a,\ x \in E.$$

Note that $m_0(B) \in \mathcal{N}(E, F)$ for $B \in \mathcal{B}a$. We will show that $m_0 : \mathcal{B}a \to \mathcal{N}(E, F)$ is a countably additive measure. Indeed, let (B_k) be a pairwise disjoint sequence in $\mathcal{B}a$, and let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} |\alpha_n||\overline{\lambda_{n,c}}|(X) < \infty$, for $a = \sup_n \|y_n\|_F$, we can choose $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} |\alpha_n||\overline{\lambda_{n,c}}|(X) \leq \frac{\varepsilon}{2(a+1)}.$$

Since $|\overline{\lambda_{n,c}}| \in M_\sigma(\mathcal{B}a)^+$ (see Proposition 2.4), there exists $k_\varepsilon \in \mathbb{N}$ such that

$$|\alpha_n||\overline{\lambda_{n,c}}|\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right) \leq \frac{\varepsilon}{2n_\varepsilon(a+1)} \quad \text{for } n = 1, \ldots, n_\varepsilon.$$

Hence, we get

$$\begin{aligned}
&\left\| m_0\left(\bigcup_{k=1}^{\infty} B_k\right) - \sum_{k=1}^{k_\varepsilon - 1} m_0(B_k) \right\|_{\mathrm{nuc}} = \left\| m_0\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right)\right\|_{\mathrm{nuc}} \\
&\leq \sum_{n=1}^{\infty} |\alpha_n||\overline{\lambda_{n,c}}|\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right)\|y_n\|_F \\
&\leq \sum_{n=1}^{n_\varepsilon} |\alpha_n||\overline{\lambda_{n,c}}|\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right)\|y_n\|_F + \sum_{n=n_\varepsilon+1}^{\infty} |\alpha_n||\overline{\lambda_{n,c}}|\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right)\|y_n\|_F \\
&\leq a\sum_{n=1}^{n_\varepsilon} |\alpha_n||\overline{\lambda_{n,c}}|\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right) + a\sum_{n=n_\varepsilon+1}^{\infty} |\alpha_n||\overline{\lambda_{n,c}}|\left(\bigcup_{k=k_\varepsilon}^{\infty} B_k\right) \\
&\leq a\frac{\varepsilon}{2(a+1)} + a\frac{\varepsilon}{2(a+1)} \leq \varepsilon.
\end{aligned}$$

This means that $m_0 : \mathcal{B}a \to \mathcal{N}(E, F)$ is countably additive.

Now, we will show that $|m_0|_{\mathrm{nuc}}(X) \le \|T\|_{\mathrm{nuc}}$. Indeed, let $(B_i)_{i=1}^k$ be a $\mathcal{B}a$-partition of X. Then, we get

$$\begin{aligned}\sum_{i=1}^{k} \|m_0(B_i)\|_{\mathrm{nuc}} &\le \sum_{i=1}^{k} \left(\sum_{n=1}^{\infty} |\alpha_n| \|\overline{\lambda_{n,c}}(B_i)\|_{E'} \|y_n\|_F \right) \\ &\le \sum_{n=1}^{\infty} |\alpha_n| \|y_n\|_F \left(\sum_{i=1}^{k} |\overline{\lambda_{n,c}}|(B_i) \right) = \sum_{n=1}^{\infty} |\alpha_n| \|y_n\|_F |\overline{\lambda_{n,c}}|(X) \\ &\le \sum_{n=1}^{\infty} |\alpha_n| |\lambda_n|(X) \|y_n\|_F \le \|T\|_{\mathrm{nuc}} + \varepsilon.\end{aligned}$$

Hence, $|m_0|_{\mathrm{nuc}}(X) \le \|T\|_{\mathrm{nuc}} + \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we get $|m_0|_{\mathrm{nuc}}(X) \le \|T\|_{\mathrm{nuc}}$, as desired. Since $|m_0|(X) \le |m_0|_{\mathrm{nuc}}(X) < \infty$, in view of Theorem 3.1, we have $m_0(B) = \bar{m}(B)$ for all $B \in \mathcal{B}a$. It follows that the measure $\bar{m} : \mathcal{B}a \to \mathcal{N}(E, F)$ is countably additive with $|\bar{m}|_{\mathrm{nuc}}(X) < \infty$ and

$$\bar{m}(B)(x) = \sum_{n=1}^{\infty} \alpha_n \overline{\lambda_{n,c}}(B)(x) y_n \quad \text{for } B \in \mathcal{B}a,\ x \in E.$$

(ii) Let $\mathrm{bvca}(\mathcal{B}a, E')$ denote the Banach space of all countably additive measures $\lambda : \mathcal{B}a \to E'$ of finite variation, equipped with the norm $\|\lambda\| := |\lambda|(X)$.

Since $|\overline{\lambda_{n,c}}| \in M_\sigma(\mathcal{B}a)$ for $n \in \mathbb{N}$, we get $\overline{\lambda_{n,c}} \in \mathrm{bvca}(\mathcal{B}a, E')$. In view of the Lebesgue decomposition theorem, for every $n \in \mathbb{N}$, we have

$$\overline{\lambda_{n,c}} = \mu_{n,a} + \mu_{n,s}, \quad |\overline{\lambda_{n,c}}| = |\mu_{n,a}| + |\mu_{n,s}|,$$

where $\mu_{n,a} \in \mathrm{bvca}(\mathcal{B}a, E')$ and $\mu_{n,a}$ is $|\bar{m}|_{\mathrm{nuc}}$-absolutely continuous ($\mu_{n,a} \ll |\bar{m}|_{\mathrm{nuc}}$) and $\mu_{n,s} \in \mathrm{bvca}(\mathcal{B}a, E')$ and $\mu_{n,s}$ and $|\bar{m}|_{\mathrm{nuc}}$ are mutually singular (see [4, Theorem 9, p. 31]).

Since E' is supposed to have the Radon–Nikodym property, for each $n \in \mathbb{N}$, there exists $\psi_n \in L^1(|\bar{m}|_{\mathrm{nuc}}, E')$ such that, for each $B \in \mathcal{B}a$, we have

$$\mu_{n,a}(B) = \int_B \psi_n(t)\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{and} \quad |\mu_{n,a}|(B) = \int_B \|\psi_n(t)\|_{E'}\, d|\bar{m}|_{\mathrm{nuc}}. \tag{4.4}$$

Moreover, note that, for each $n \in \mathbb{N}$, there exist sets $B_n \in \mathcal{B}a$ and $C_n \in \mathcal{B}a$ with $B_n \cap C_n = \emptyset$ such that $|\bar{m}|_{\mathrm{nuc}}$ is concentrated on B_n and $|\mu_{n,s}|$ is concentrated on C_n; that is, for each $B \in \mathcal{B}a$, $|\bar{m}|_{\mathrm{nuc}}(B) = |\bar{m}|_{\mathrm{nuc}}(B \cap B_n)$ and $|\mu_{n,s}|(B) = |\mu_{n,s}|(B \cap C_n)$. Hence, for each $n \in \mathbb{N}$,

$$|\mu_{n,s}|(B_n) = 0 \quad \text{and} \quad |\bar{m}|_{\mathrm{nuc}}(X \setminus B_n) = 0.$$

Let $D_0 = \bigcap_{n=1}^{\infty} B_n$ and $B \in \mathcal{B}a$ be given. Then, we have

$$\begin{aligned}\|\bar{m}(B \cap (X \setminus D_0))\|_{\mathrm{nuc}} &\le |\bar{m}|_{\mathrm{nuc}}(B \cap (X \setminus D_0)) \le |\bar{m}|_{\mathrm{nuc}}(X \setminus D_0) \\ &= |\bar{m}|_{\mathrm{nuc}}\left(\bigcup_{n=1}^{\infty} (X \setminus B_n) \right) \le \sum_{n=1}^{\infty} |\bar{m}|_{\mathrm{nuc}}(X \setminus B_n) = 0.\end{aligned} \tag{4.5}$$

Since $\|\mu_{n,s}(B \cap D_0)\|_{E'} \le |\mu_{n,s}|(B \cap D_0) \le |\mu_{n,s}|(B_n) = 0$, we get $\mu_{n,s}(B \cap D_0) = 0$ for $n \in \mathbb{N}$. Hence, in view of (4.5), for each $x \in E$, we have

$$\begin{aligned}\bar{m}(B)(x) &= \bar{m}(B \cap D_0)(x) + \bar{m}(B \cap (X \setminus D_0))(x) = \bar{m}(B \cap D_0)(x) \\ &= \sum_{n=1}^{\infty} \alpha_n \mu_{n,a}(B \cap D_0)(x) y_n.\end{aligned}$$

But $|\bar{m}|_{\text{nuc}}(B \cap (X \setminus D_0)) = 0$ and $\mu_{n,a} \ll |\bar{m}|_{\text{nuc}}$, so $\mu_{n,a}(B \cap (X \setminus D_0)) = 0$, and hence,

$$\begin{aligned}\mu_{n,a}(B) &= \mu_{n,a}(B \cap (D_0 \cup (X \setminus D_0))) \\ &= \mu_{n,a}(B \cap D_0) + \mu_{n,a}(B \cap (X \setminus D_0)) \\ &= \mu_{n,a}(B \cap D_0)(x).\end{aligned}$$

Thus, we have $\bar{m}(B)(x) = \sum_{n=1}^{\infty} \alpha_n \mu_{n,a}(B)(x) y_n$, and using (4.4), we get

$$\bar{m}(B)(x) = \sum_{n=1}^{\infty} \alpha_n \int_B \psi_n(t)\, d|\bar{m}|_{\text{nuc}}(x) y_n. \tag{4.6}$$

For $n \in \mathbb{N}$, let us put $H_n(t) := \sum_{i=1}^{n} \alpha_i \psi_i(t) \otimes y_i$ for $t \in X$, where $(\alpha_i \psi_i(t) \otimes y_i)(x) := \alpha_i \psi_i(t)(x) y_i$ for $x \in E$. Then, $H_n(t) \in \mathcal{N}(E, F)$ for $t \in X$. For $n_1, n_2 \in \mathbb{N}$ with $n_2 > n_1$, we have

$$\begin{aligned}\int_X \|H_{n_2}(t) - H_{n_1}(t)\|_{\text{nuc}}\, d|\bar{m}|_{\text{nuc}} &\le \int_X \left(\sum_{i=n_1+1}^{n_2} |\alpha_i| \|\psi_i(t)\|_{E'} \|y_i\|_F \right) d|\bar{m}|_{\text{nuc}} \\ &= \sum_{i=n_1+1}^{n_2} |\alpha_i| \left(\int_X \|\psi_i(t)\|_{E'}\, d|\bar{m}|_{\text{nuc}} \right) \|y_i\|_F \\ &\le \sum_{i=n_1+1}^{n_2} |\alpha_i| |\mu_{i,a}|(X) \|y_i\|_F \\ &\le \sup_{j \in \mathbb{N}} |\mu_{j,a}|(X) \sup_{j \in \mathbb{N}} \|y_j\|_F \sum_{i=n_1+1}^{n_2} |\alpha_i|.\end{aligned}$$

It follows that (H_n) is a Cauchy sequence in $L^1(|\bar{m}|_{\text{nuc}}, \mathcal{N}(E, F))$, so there exists $H \in L^1(|\bar{m}|_{\text{nuc}}, \mathcal{N}(E, F))$ so that $\int_X \|H_n(t) - H(t)\|_{\text{nuc}}\, d|\bar{m}|_{\text{nuc}} \xrightarrow{n} 0$.

One can show that, for each $x \in E$, a linear operator $S_x : \mathcal{N}(E, F) \to F$ defined by $S_x(U) := U(x)$ for $U \in \mathcal{N}(E, F)$ is $(\|\cdot\|_{\text{nuc}}, \|\cdot\|_F)$-bounded. Hence, using Hille's theorem (see [6, Section 1, Theorem 36, p. 16]), for $B \in \mathcal{B}a$, we have

$$\int_B H(t)\, d|\bar{m}|_{\text{nuc}}(x) = \int_B S_x(H(t))\, d|\bar{m}|_{\text{nuc}} = \int_B H(t)(x)\, d|\bar{m}|_{\text{nuc}}.$$

Then, we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} \alpha_i \int_B \psi_i(t)\, d|\bar{m}|_{\text{nuc}}(x) y_i - \int_B H(t)(x)\, d|\bar{m}|_{\text{nuc}} \right\|_F \\
&\quad = \left\| \int_B \left(\left(\sum_{i=1}^{n} \alpha_i \psi_i(t) \otimes y_i \right) - H(t) \right)(x)\, d|\bar{m}|_{\text{nuc}} \right\|_F \\
&\quad \le \|x\|_E \int_B \|H_n(t) - H(t)\|_{\text{nuc}}\, d|\bar{m}|_{\text{nuc}} \xrightarrow{n} 0.
\end{aligned}$$

In view of (4.6), we get $\bar{m}(B)(x) = \int_B H(t)(x)\, d|\bar{m}|_{\text{nuc}} = \int_B H(t)\, d|\bar{m}|_{\text{nuc}}(x)$, and hence, $\bar{m}(B) = \int_B H(t)\, d|\bar{m}|_{\text{nuc}}$, and using Theorem 4.3, we get

$$T(f) = \int_X H(t)(f(t))\, d|\bar{m}|_{\text{nuc}} \quad \text{for } f \in C_{\text{rc}}(X,E). \qquad \blacksquare$$

As a consequence of Theorems 4.3 and 4.5, we have the following characterization of $(\beta_\sigma, \|\cdot\|_F)$-continuous nuclear operators $T : C_{\text{rc}}(X,E) \to F$.

Corollary 4.6. *Let $T : C_{\text{rc}}(X,E) \to F$ be a weakly compact $(\beta_\sigma, \|\cdot\|_F)$-continuous operator and $m : \mathcal{B} \to \mathcal{L}(E,F)$ its representing measure. If E' has the Radon–Nikodym property, then the following statements are equivalent.*

(i) *$\bar{m}(B) \in \mathcal{N}(E,F)$ for $B \in \mathcal{B}a$ and $\bar{m} : \mathcal{B}a \to \mathcal{N}(E,F)$ is a countably additive measure with $|\bar{m}|_{\text{nuc}}(X) < \infty$, and there exists $H \in L^1(|\bar{m}|_{\text{nuc}}, \mathcal{N}(E,F))$ so that*

$$\bar{m}(B) = \int_B H(t)\, d|\bar{m}|_{\text{nuc}} \quad \text{for } B \in \mathcal{B}a.$$

(ii) *T is a nuclear operator between the locally convex space $(C_{\text{rc}}(X,E), \beta_\sigma)$ and the Banach space F.*

(iii) *T is a nuclear operator between the Banach spaces $C_{\text{rc}}(X,E)$ and F.*

(iv) *$\bar{m}(B) \in \mathcal{N}(E,F)$ for $B \in \mathcal{B}a$ and $\bar{m} : \mathcal{B}a \to \mathcal{N}(E,F)$ is a countably additive measure with $|\bar{m}|_{\text{nuc}}(X) < \infty$, and there exists $H \in L^1(|\bar{m}|_{\text{nuc}}, \mathcal{N}(E,F))$ so that*

$$T(f) = \int_X H(t)(f(t))\, d|\bar{m}|_{\text{nuc}} \quad \text{for } f \in C_{\text{rc}}(X,E).$$

In this case, $\|T\|_{\text{nuc}} = |\bar{m}|_{\text{nuc}}(X) = |m|_{\text{nuc}}(X) = \int_X \|H(t)\|_{\text{nuc}}\, d|\bar{m}|_{\text{nuc}}$.

Proof. (i)⇒(ii). It follows from Theorem 4.3. (ii)⇒(iii). It is obvious because $\beta_\sigma \subset \tau_u$. (iii)⇒(iv). It follows from Theorem 4.5. (iv)⇒(i). Assume that (iv) holds.

For each $y' \in F'$, the linear operator $R_{y'} : \mathcal{N}(E,F) \to E'$ defined by

$$R_{y'}(U) := y' \circ U$$

for $U \in \mathcal{N}(E,F)$ is $(\|\cdot\|_{\mathrm{nuc}}, \|\cdot\|_{E'})$-bounded. Hence, according to Hille's theorem (see [6, Section 1, Theorem 36, p. 16]), we have

$$y' \circ \int_X H(t)\, d|\bar{m}|_{\mathrm{nuc}} = R_{y'}\Big(\int_X H(t)\, d|\bar{m}|_{\mathrm{nuc}}\Big) = \int_X R_{y'}(H(t))\, d|\bar{m}|_{\mathrm{nuc}}$$
$$= \int_X y' \circ H(t)\, d|\bar{m}|_{\mathrm{nuc}},$$

and the function $X \ni t \mapsto y' \circ H(t) \in E'$ belongs to $L^1(|\bar{m}|_{\mathrm{nuc}}, E')$.

Let

$$\mu_{y'}(B) := \int_B y' \circ H(t)\, d|\bar{m}|_{\mathrm{nuc}}$$

for $B \in \mathcal{B}a$. Then, by Lemma 4.1 $\mu_{y'} \in M_\sigma(\mathcal{B}a, E')$ and, for $f \in C_{\mathrm{rc}}(X,E)$, we have

$$\int_X f(t)\, d\mu_{y'} = \int_X y'(H(t)(f(t)))\, d|\bar{m}|_{\mathrm{nuc}} = y'(T(f)).$$

Assume now that $A \in \mathcal{B}$. Then, for $y' \in F'$, $x \in E$, we have

$$y'(m(A)(x)) = ((T'' \circ \pi)(\mathbb{1}_A \otimes x))(y') = \pi(\mathbb{1}_A \otimes x)(y' \circ T)$$
$$= \int_X (\mathbb{1}_A \otimes x)(t)\, d\mu_{y'} = \mu_{y'}(A)(x) = y'\Big(\int_A H(t)\, d|\bar{m}|_{\mathrm{nuc}}(x)\Big),$$

and it follows that $m(A) = \int_A H(t)\, d|\bar{m}|_{\mathrm{nuc}}$.

For $B \in \mathcal{B}a$, let us put $m_0(B) := \int_B H(t)\, d|\bar{m}|_{\mathrm{nuc}}$. One can observe that $m_0 \in M_\sigma(\mathcal{B}a, \mathcal{L}(E,F))$, and hence, in view of Theorem 3.1, $\bar{m}(B) = m_0(B) = \int_B H(t)\, d|\bar{m}|_{\mathrm{nuc}}$ for $B \in \mathcal{B}a$. Moreover, in view of Theorems 4.3 and 4.5, we have

$$\|T\|_{\mathrm{nuc}} = |\bar{m}|_{\mathrm{nuc}}(X) = |m|_{\mathrm{nuc}}(X) = \int_X \|H(t)\|_{\mathrm{nuc}}\, d|\bar{m}|_{\mathrm{nuc}}. \qquad \blacksquare$$

Remark 4.7. The result of Corollary 4.6 extends to the setting of completely regular Hausdorff spaces, the classical results of Diestel and Uhl (see [4, p. 173]) and Popa (see [24, Theorem 1]), where X is a compact Hausdorff space.

We will need the following lemma. (The proof is similar to the proof of Lemma 4.4 and will be omitted.)

Lemma 4.8. *Assume that (μ_n) is a bounded sequence in $M_\sigma(\mathcal{B}a, E')$, (y_n) is a bounded sequence in F, and $(\alpha_n) \in \ell^1$. Then, for $y' \in F'$, we have that*

$$\sum_{n=1}^{\infty} \alpha_n y'(y_n)\mu_n \in M_\sigma(\mathcal{B}a, E')$$

and

$$\Big(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\mu_n\Big)(B)(x) = \sum_{n=1}^{\infty} \alpha_n \mu_n(B)(x) y'(y_n) \quad \text{for } B \in \mathcal{B}a,\ x \in E.$$

Assume that $T : C_{\mathrm{rc}}(X, E) \to F$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator. Then, we can consider the conjugate mapping

$$T' : F' \ni y' \mapsto \bar{m}_{y'} \in M_\sigma(\mathcal{B}a, E').$$

Now, as a consequence of Theorems 4.3 and 4.5, we establish the relationship between the nuclearity of a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator $T : C_{\mathrm{rc}}(X, E) \to F$ and the nuclearity of its conjugate operator $T' : F' \to M_\sigma(\mathcal{B}a, E')$.

Corollary 4.9. *Assume that $T : C_{\mathrm{rc}}(X, E) \to F$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous nuclear operator between the Banach spaces $C_{\mathrm{rc}}(X, F)$ and F. Then, the mapping*

$$T' : F' \to M_\sigma(\mathcal{B}a, E')$$

is a nuclear operator and $\|T'\|_{\mathrm{nuc}} \le \|T\|_{\mathrm{nuc}}$.

Proof. Let $\varepsilon > 0$ be given. Then, in view of the proof of Theorem 4.5, there exist a bounded sequence (μ_n) in $M_\sigma(\mathcal{B}a, E')$, a bounded sequence (y_n) in F, and a sequence $(\alpha_n) \in \ell^1$ so that

$$\bar{m}(B)(x) = \sum_{n=1}^{\infty} \alpha_n \mu_n(B)(x) y_n \quad \text{for } B \in \mathcal{B}a,\ x \in E$$

and

$$\sum_{n=1}^{\infty} |\alpha_n||\mu_n|(X)\|y_n\|_F \le \|T\|_{\mathrm{nuc}} + \varepsilon.$$

Then, according to Lemma 4.8, for $y' \in F'$, we have $\sum_{n=1}^{\infty} \alpha_n y'(y_n)\mu_n \in M_\sigma(\mathcal{B}a, E')$, and for any $B \in \mathcal{B}a$, $x \in E$, we get

$$T'(y')(B)(x) = \bar{m}_{y'}(B)(x) = \sum_{n=1}^{\infty} \alpha_n \mu_n(B)(x) y'(y_n) = \left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\mu_n \right)(B)(x).$$

Thus, it follows that

$$T'(y') = \sum_{n=1}^{\infty} \alpha_n y'(y_n)\mu_n = \sum_{n=1}^{\infty} \alpha_n i_F(y_n)(y')\mu_n \quad \text{for } y' \in F'.$$

This means that T' is a nuclear operator and

$$\|T'\|_{\mathrm{nuc}} \le \sum_{n=1}^{\infty} |\alpha_n| \|y_n\|_F |\mu_n|(X) \le \|T\|_{\mathrm{nuc}} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\|T'\|_{\mathrm{nuc}} \le \|T\|_{\mathrm{nuc}}$. ∎

Corollary 4.10. *Let $T : C_{\mathrm{rc}}(X, E) \to F$ be a weakly compact $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator. Assume that E' has the Radon–Nikodym property and F is reflexive. If the mapping $T' : F' \to M_\sigma(\mathcal{B}a, E')$ is a nuclear operator, then T is a nuclear operator between the Banach spaces $C_{\mathrm{rc}}(X, E)$ and F and $\|T\|_{\mathrm{nuc}} \le \|T'\|_{\mathrm{nuc}}$.*

Proof. Let $\varepsilon > 0$ be given. Then, there exist a bounded sequence (y_n'') in F'', a bounded sequence (μ_n) in $M_\sigma(\mathcal{B}a, E')$, and $(\alpha_n) \in \ell^1$ so that

$$T'(y') = \sum_{n=1}^{\infty} \alpha_n y_n''(y')\mu_n \quad \text{for } y' \in F'$$

and

$$\sum_{n=1}^{\infty} |\alpha_n| \|y_n''\|_{F''} |\mu_n|(X) \leq \|T'\|_{\mathrm{nuc}} + \varepsilon.$$

Since F is supposed to be reflexive, we can choose a sequence (y_n) in F such that $y_n'' = i_F(y_n)$ for $n \in \mathbb{N}$. Then, for each $y' \in F'$ and $B \in \mathcal{B}a$, $x \in E$, we get (see Lemma 4.8)

$$\begin{aligned} y'(\bar{m}(B)(x)) = T'(y')(B)(x) &= \left(\sum_{n=1}^{\infty} \alpha_n i_F(y_n)(y')\mu_n\right)(B)(x) \\ &= \left(\sum_{n=1}^{\infty} \alpha_n y'(y_n)\mu_n\right)(B)(x) = \sum_{n=1}^{\infty} \alpha_n \mu_n(B)(x) y'(y_n) \\ &= y'\left(\sum_{n=1}^{\infty} \alpha_n \mu_n(B)(x) y_n\right). \end{aligned}$$

Hence, $\bar{m}(B)(x) = \sum_{n=1}^{\infty} \alpha_n \mu_n(B)(x) y_n$, and this means that $\bar{m}(B) : E \to F$ is a nuclear operator.

To show that $|\bar{m}|(X) \leq \|T'\|_{\mathrm{nuc}}$, assume that $(B_i)_{i=1}^{k}$ is a $\mathcal{B}a$-partition of X. Then, we have

$$\begin{aligned} \sum_{i=1}^{k} \|\bar{m}(B_i)\|_{\mathrm{nuc}} &\leq \sum_{i=1}^{k} \left(\sum_{n=1}^{\infty} |\alpha_n| \|\mu_n(B_i)\|_{E'} \|y_n''\|_{F''}\right) \\ &= \sum_{n=1}^{\infty} |\alpha_n| \|y_n''\|_{F''} \left(\sum_{n=1}^{\infty} \|\mu_n(B_i)\|_{E'}\right) \\ &\leq \sum_{n=1}^{\infty} |\alpha_n| \|y_n''\|_{F''} |\mu_n|(X) \leq \|T'\|_{\mathrm{nuc}} + \varepsilon. \end{aligned}$$

Hence, $|\bar{m}|_{\mathrm{nuc}}(X) \leq \|T'\|_{\mathrm{nuc}} + \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we get $|\bar{m}|_{\mathrm{nuc}}(X) \leq \|T'\|_{\mathrm{nuc}}$.

Arguing as in the proof of (i) of Theorem 4.5, we can show that the measure $\bar{m} : \mathcal{B}a \to \mathcal{N}(E, F)$ is countably additive. Since E' is supposed to have the Radon–Nikodym property, arguing as in the proof of (ii) of Theorem 4.5, we obtain that there exists $H \in L^1(|\bar{m}|_{\mathrm{nuc}}, \mathcal{N}(E, F))$ so that

$$\bar{m}(B) = \int_B H(t)\, d|\bar{m}|_{\mathrm{nuc}} \quad \text{for } B \in \mathcal{B}a.$$

Hence, by Theorem 4.3, T is nuclear, and we get $\|T\|_{\mathrm{nuc}} = |\bar{m}|_{\mathrm{nuc}}(X) \leq \|T'\|_{\mathrm{nuc}}$. ∎

As a consequence of Corollaries 4.9 and 4.10, we have the following result.

Corollary 4.11. *Let $T : C_{rc}(X, E) \to F$ be a weakly compact $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator. Assume that E' has the Radon–Nikodym property and F is reflexive. Then, the following statements are equivalent.*

(i) *T is a nuclear operator between the Banach spaces $C_{rc}(X, E)$ and F.*

(ii) *The mapping $T' : F' \to M_\sigma(\mathcal{B}a, E')$ is a nuclear operator.*

In this case, $\|T\|_{nuc} = \|T'\|_{nuc}$.

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