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Positive solutions for a p-Laplacian equation with sub-critical singular parametric reaction term

Pasquale Candito, Giuseppe Failla, and Roberto Livrea

Abstract. The existence of at least two smooth positive solutions for a parametric quasilinear elliptic problem driven by a p-Laplacian operator involving a mildly singular non-linearity perturbed with a sub-critical term is established. Although, to get our conclusions, we combine variational and truncation techniques, we do not use the usual trick of C^1 versus Sobolev minimizers. An explicit quantitative estimate from below of the best theoretical parameters considered is furnished.

1. Introduction

In this paper, the following p-Laplacian problem involving a singular non-linearity perturbed with a sub-critical term is studied, namely,

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$ is a bounded domain with a smooth boundary, $\partial\Omega \in C^2$, the driven operator is the usual p-Laplacian,

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

with $p \in (1, N)$, λ and μ are two positive parameters. Furthermore, we assume that $f : \Omega \times (0, +\infty) \to [0, +\infty)$ and $g : \Omega \times [0, +\infty) \to [0 + \infty)$ are two Carathéodory functions fulfilling the following conditions:

- (Q_1) $\lim_{s\to 0^+} f(x,s) = +\infty$ uniformly w.r.t. $x \in \Omega$;
- (Q_2) there exist positive constants c_i for $i \in \{1, \dots, 4\}$ and some $\gamma \in (0,1)$ such that
  - (Q_{21}) $f(x,s) \leq c_1 s^{-\gamma} + c_2$ for a.a. $x \in \Omega$ $\forall s > 0$;
  - (Q_{22}) $g(x,s) \leq c_3 s^{q-1} + c_4$ for a.a. $x \in \Omega$ $\forall s > 0$, $q \in (1, p^*)$,

with $p^* = \frac{Np}{N-p}$ being the critical Sobolev exponent.

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A wide literature on parametric problems has been produced since the pioneering and seminal papers due to Ambrosetti, Brézis, Cerami [1] and García Azorero, Peral, Manfredi [16], where among the others results, in this latter the fundamental result of Brézis, Nirenberg [8] concerning H^1 versus C^1 local minimizers is extended to the p-Laplacian with $p \neq 2$, which is a well-known key point to allow merging sub-super-solution methods with variational ones.

Subsequently, these ideas have been developed in singular settings, overcoming, in an original way, many technical difficulties arising from the presence of the singularity on the non-linearity, as, for instance, the lack of regularity of the energy functional associated with the problem investigated; see Giacomoni, Saoudi [18] and Giacomoni, Schindler, and Takáč [19].

For a general look on p-Laplacian singular problems, we refer the interested reader to the recent survey [20], the papers [28–31] ($\gamma < 1$), [11, 15] for strongly singular problems ($\gamma \geq 1$) and references therein.

The study of this type of singular problems with a p-Laplacian operator is also largely encouraged by a wide range of physical and engineering applications, particularly the so-called non-Newtonian fluids, see [13, Remark 2.2] and references therein for more details.

Our main results, Theorems 3.1 and 3.4, establish the existence of at least one or two smooth positive solutions for ($P_{\lambda,\mu}$), respectively.

Adapting here some arguments introduced and developed in [9, 10], the first solution for ($P_{\lambda,\mu}$) is obtained by applying a local minimum theorem (Theorem 2.12) due to Bonanno [3], which beyond the existence of a solution, allows us to get two additional features:

(B_1) the first solution is a local minimum of the functional J_λ, regardless of the asymptotic behavior at infinity of the perturbation g;

(B_2) a quantitative estimate of the parameters λ, μ, for which ($P_{\lambda,\mu}$) admits a positive solution, i.e.,

$$\lambda\Big(\frac{c_1}{1-\gamma} + c_2\Big) + \mu(c_3 + c_4)$$
$$< \begin{cases} +\infty & \text{if } 1 < q < p, \\ \frac{N}{p}\pi^{\frac{p}{2}}|\Omega|^{-\frac{p}{N}}\left(\frac{N-p}{p-1}\right)^{p-1}\left[\frac{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}{\Gamma(1+\frac{N}{2})\Gamma(N)}\right]^p & \text{if } q = p, \\ \frac{N}{p}\pi^{\frac{p}{2}}|\Omega|^{-\frac{p}{N}}\left(\frac{N-p}{p-1}\right)^{p-1}\left(\frac{q-p}{2p}\right)^{\frac{q-p}{q}}\left[\frac{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}{\Gamma(1+\frac{N}{2})\Gamma(N)}\right]^p & \text{if } p < q < p^*, \end{cases}$$

where Γ is the gamma function.

In a few words, we combine variational and truncation techniques, but we do not use the usual trick of C^1 versus Sobolev minimizers, recalled above, as well as we do not need a priori estimates or to prove the existence of a super-solution. More precisely, we can use the point (B_1) together with the request on g to fulfill the classical unilateral Ambrosetti–Rabinowitz condition (in the short (AR)-condition), see [2, 26], i.e., there exists $\eta > p$

such that

$$\operatorname*{essinf}_{x\in\Omega} \int_0^s g(x,t)dt > 0, \quad 0 < \eta \int_{s_1}^s g(x,t)dt \le g(x,s)s \tag{AR}$$

for a.a. $x \in \Omega$ and for all $s \ge s_1 > 0$. Here, we realize that the functional J_λ satisfies two key ingredients of the mountain pass theorem [2], that are, the so-called mountain pass geometry, see [4, Theorem 2.1], and the Palais–Smale condition.

Roughly speaking, to obtain the second solution in the super-linear case, $p < q < p^*$, as in the above-mentioned papers, we apply the powerful mountain pass theorem of Ambrosetti and Rabinowitz [2] to a C^1 energy functional J_λ associated to an auxiliary problem, whose positive smooth solutions give back solutions of ($P_{\lambda,\mu}$).

We also highlight that the estimates achieved in (B_2) are not the best theoretical ones, as, for example, in [18] or [19], but are completely computable since they involve only the constants c_i given in (Q_2) and an upper bound of the Sobolev embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ as in (2.1) below.

To summarize, Section 2 shows some preliminary results and describes the variational setting adopted to solve ($P_{\lambda,\mu}$). Section 3 is devoted to the main results.

Finally, we observe that our results guarantee the existence of at least two solutions for the following class of elliptic problems:

$$\begin{cases} -\Delta_4 u = \frac{\lambda}{\sqrt{u}} + \mu u^8 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subseteq \mathbb{R}^N$ $(5 \le N \le 7)$.

2. Preliminaries

In this paper, we will use the symbol $\|\cdot\|_p$ to denote the norm in the Lebesgue space $L^p(\Omega)$, i.e.,

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \quad \forall u \in L^p(\Omega);$$

in addition, the symbol $\|\cdot\|$ refers to the norm in Sobolev space $W_0^{1,p}(\Omega)$, i.e.,

$$\|u\| = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

We recall that by compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, p^*)$ we obtain

$$\|u\|_q \le C_q \|u\| \quad \forall u \in W_0^{1,p}(\Omega),$$

where according to Talenti [34] and by Hölder's inequality, we have

$$C_q \le S_q := \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} |\Omega|^{\frac{p^*-q}{p^*q}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}\right], \tag{2.1}$$

with Γ being the gamma function. Since our problem involves a singular term, we also need the inequality [27, Theorem 21.3] in the following particular theorem.

Theorem 2.1 (Hardy–Sobolev's inequality). *Let $p \in (1, N)$, $\tau \in [0, 1]$ and $\frac{1}{r} = \frac{1}{p} - \frac{1-\tau}{N}$. Then, for any $u \in W_0^{1,p}(\Omega)$, we have $ud^{-\tau} \in L^r(\Omega)$ and*

$$\|ud^{-\tau}\|_r \le D_\tau \|u\|,$$

where D_τ is a positive constant and d denotes the distance function, i.e.,

$$d(x) := \operatorname{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y| \quad \forall x \in \Omega.$$

As in [12, Definitions 3.14–3.15], we recall the definitions of sub-solution and weak solution for the problem $(P_{\lambda,\mu})$.

Definition 2.2. The function $\underline{u} \in W_0^{1,p}(\Omega)$ is called a (weak) sub-solution of $(P_{\lambda,\mu})$, if

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla v dx \le \lambda \int_\Omega f(x, \underline{u}) v dx + \mu \int_\Omega g(x, \underline{u}) v dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega); \tag{2.2}$$

likewise, the function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of problem $(P_{\lambda,\mu})$, if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_\Omega f(x, u) v dx + \mu \int_\Omega g(x, u) v dx \quad \forall v \in W_0^{1,p}(\Omega); \tag{2.3}$$

in both cases, it is understood that the right-hand side is well posed.

Now, adopting some reasoning as in [9, 10], we prove the existence of a sub-solution for $(P_{\lambda,\mu})$.

Lemma 2.3. *Suppose (Q_1) holds, then, for all $\lambda, \mu > 0$, there exist $0 < \alpha < 1$, $\delta > 0$, $l > 0$ and a sub-solution $\underline{u}$ of problem $(P_{\lambda,\mu})$ such that $\underline{u} \in C^{1,\alpha}(\overline{\Omega})$ and $ld(x) \le \underline{u} \le \delta$ for a.a. $x \in \Omega$.*

Proof. Notice that, from (Q_1), there exists $\delta > 0$ small enough such that

$$f(x, s) \ge 1 \quad \forall (x, s) \in \Omega \times (0, \delta). \tag{2.4}$$

For $\lambda > 0$, we study the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Since $\lambda \in W^{-1,p'}(\Omega)$ and negative p-Laplacian is strictly monotone for $1 < p < 2$, strongly monotone for $p = 2$ and uniformly monotone for $p > 2$ (see (2.18) below and [25, Example 2.27 (c)]), by Minty–Browder's theorem [6, Theorem 5.16] and combining [35, Theorem 5], [24, Theorem 1.1] (see also [23]) and [21, Lemma 3.1], there exists

$e_\lambda \in C^{1,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$ (see also [22, Theorem 4.8]), a unique positive solution of (2.5). Let $M := \max\{\delta, \max_{\overline{\Omega}} e_\lambda\}$ and choose c such that $cM \leq \delta$. We put $\underline{u} = ce_\lambda$, then, by (2.4),

$$-\Delta_p \underline{u} = c^{p-1}(-\Delta_p e_\lambda) = c^{p-1}\lambda \leq \lambda \leq \lambda f(x, \underline{u}) \leq \lambda f(x, \underline{u}) + \mu g(x, \underline{u}).$$

Therefore, $\underline{u}$ is a sub-solution of the problem $(P_{\lambda,\mu})$. Moreover, from [32, Theorem 5.3.1], there exists a positive constant l such that

$$l d(x) \leq \underline{u} \leq c \max_{\Omega} e_\lambda \leq \delta \tag{2.6}$$

and this achieves the proof. ∎

To apply variational tools and to avoid blow up phenomena, we truncate the reaction term as follows, set $f^* : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g^* : \Omega \times \mathbb{R} \to \mathbb{R}$

$$f^*(x,s) = \begin{cases} f(x,s) & \text{if } s \geq \underline{u}; \\ f(x,\underline{u}) & \text{if } |s| < \underline{u}; \\ f(x,-s) & \text{if } s \leq -\underline{u}; \end{cases} \tag{2.7}$$

and

$$g^*(x,s) = \begin{cases} g(x,s) & \text{if } s \geq \underline{u}; \\ g(x,\underline{u}) & \text{if } |s| < \underline{u}; \\ g(x,-s) & \text{if } s \leq -\underline{u}. \end{cases} \tag{2.8}$$

Notice that f^* and g^* are Carathéodory functions. In particular, by (Q_2) and (2.6), we obtain

$$f^*(x,s) \leq c_1 \underline{u}^{-\gamma}(x) + c_2 \leq \tilde{c_1} d^{-\gamma}(x) + c_2 \quad \text{for a.a. } x \in \Omega, \forall s \in \mathbb{R}, \tag{2.9}$$

where $\tilde{c_1} = c_1 l^{-\gamma}$. Analogously, we get that by (2.6),

$$g^*(x,s) \leq c_3 |s|^{q-1} + c_5 \quad \text{for a.a. } x \in \Omega, \forall s \in \mathbb{R}, \tag{2.10}$$

where $c_5 = c_3 \delta^{q-1} + c_4$.

Now, with the aim of applying variational methods for $(P_{\lambda,\mu})$, we consider the following problem:

$$\begin{cases} -\Delta_p u = \lambda h^*_{\lambda,\mu}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P*}$$

where $h_{\lambda,\mu} : \Omega \times \mathbb{R} \to \mathbb{R}$ is given, for a.a. $x \in \Omega$, by

$$h^*_{\lambda,\mu}(x,u) := \begin{cases} f^*(x,u) + \frac{\mu}{\lambda} g^*(x,u) & \text{if } |u| \geq \underline{u}; \\ f(x,\underline{u}) + \frac{\mu}{\lambda} g(x,\underline{u}) & \text{if } |u| < \underline{u}. \end{cases}$$

Remark 2.4. We explicitly observe that the weak solution u of (P*) is well-defined, i.e., $u \in W_0^{1,p}(\Omega)$ is such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega f^*(x,u)v dx + \mu \int_\Omega g^*(x,u)v dx \quad \forall v \in W_0^{1,p}(\Omega).$$

Indeed, if $u, v \in W_0^{1,p}(\Omega)$, by (2.9), (2.10), Hölder, Sobolev, and Hardy–Sobolev inequalities, it follows that

$$\begin{aligned}\left|\int_\Omega h^*_{\lambda,\mu}(x,u)v dx\right| &\leq \int_\Omega f^*(x,u)|v| dx + \frac{\mu}{\lambda}\int_\Omega g^*(x,u)|v| dx \\ &\leq \int_\Omega (\tilde{c_1} d^{-\gamma}(x) + c_2)|v| dx + \frac{\mu}{\lambda}\int_\Omega (c_3|u|^{q-1} + c_5)|v| dx \\ &\leq \tilde{c_1} D_\gamma \|v\| + c_2\|v\|_1 + \frac{\mu}{\lambda}(c_3\|u\|_q^{q-1}\|v\|_q + c_5\|v\|_1) < \infty.\end{aligned}$$

Now, we set

$$H_{\lambda,\mu}(x,s) := \int_0^s h^*_{\lambda,\mu}(x,t) dt \quad \text{for a.a. } x \in \Omega, \forall s \in \mathbb{R}.$$

Since $h^*_{\lambda,\mu}(x,s)$ is an even function and by (2.6) (we recall that one can choose $\underline{u} \leq \delta < 1$), we point out that for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$,

$$|H_{\lambda,\mu}(x,s)| \leq \int_0^{|s|} h^*_{\lambda,\mu}(x,t) dt = \int_0^{\underline{u}} h^*_{\lambda,\mu}(x,\underline{u}) dt + \int_{\underline{u}}^{|s|} h^*_{\lambda,\mu}(x,t) dt. \tag{2.11}$$

Moreover, for a.a. $x \in \Omega$, one has

$$\int_0^{\underline{u}} f(x,\underline{u}) dt + \frac{\mu}{\lambda}\int_0^{\underline{u}} g(x,\underline{u}) dt \leq \frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4)$$

and

$$\int_{\underline{u}}^{|s|} f^*(x,t) dt + \frac{\mu}{\lambda}\int_{\underline{u}}^{|s|} g^*(x,t) dt \leq \frac{c_1}{1-\gamma}|s|^{1-\gamma} + c_2|s| + \frac{\mu}{\lambda}(c_3|s|^q + c_4|s|),$$

which implies that

$$\int_0^{|s|} h^*(x,t) dt \leq \frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4) + \frac{c_1}{1-\gamma}|s|^{1-\gamma} + c_2|s| + \frac{\mu}{\lambda}(c_3|s|^q + c_4|s|).$$

At this point, it follows that

$$H_{\lambda,\mu}(x,|s|) \leq \begin{cases} 2\left[\frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4)\right] & \text{if } |s| \leq 1; \\ \frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4) + \left[\frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4)\right]|s|^q & \text{if } |s| > 1. \end{cases}$$

So, for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$, we have

$$H(x,|s|) \leq 2\left[\frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4)\right] + \left[\frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4)\right]|s|^q. \tag{2.12}$$

To simplify the notation, for our convenience, let us put

$$A_{\lambda,\mu} := \frac{c_1}{1-\gamma} + c_2 + \frac{\mu}{\lambda}(c_3 + c_4) \tag{2.13}$$

from (2.12), we get

$$|H_{\lambda,\mu}(x,s)| \leq H_{\lambda,\mu}(x,|s|) \leq A_{\lambda,\mu}(2 + |s|^q) \quad \text{for a.a. } x \in \Omega, \forall s \in \mathbb{R}. \tag{2.14}$$

Lemma 2.5. *For all $\lambda, \mu > 0$, every weak solution of* (P*) *is a weak solution of* ($P_{\lambda,\mu}$).

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P*). Following the idea in [12, Lemma 3.50], it is enough to show that $u \geq \underline{u}$. Indeed, by (2.7)–(2.8), if $u \geq \underline{u}$ then $h^*_{\lambda,\mu}(x,s) = f(x,s) + \frac{\mu}{\lambda} g(x,s)$ for a.a. $x \in \Omega$, for all $s \in \mathbb{R}$. Since u is a weak solution of (P*), then by (2.3), one has

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx = \lambda \int_\Omega f^*(x,u) v\, dx + \mu \int_\Omega g^*(x,u) v\, dx, \quad v \in W_0^{1,p}(\Omega). \tag{2.15}$$

Therefore, $\underline{u}$ is a sub-solution of (P*), then by (2.2), we have

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla v\, dx \leq \lambda \int_\Omega f^*(x,\underline{u}) v\, dx + \mu \int_\Omega g^*(x,\underline{u}) v\, dx, \quad v \in W_0^{1,p}(\Omega) \cap L^p_+(\Omega). \tag{2.16}$$

Subtracting (2.16) and (2.15) and choosing $v = (\underline{u} - u)^+$,

$$\begin{aligned}
&\int_\Omega (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u|^{p-2}\nabla u)\nabla(\underline{u} - u)^+ dx \\
&\quad \leq \lambda \int_\Omega (f^*(x,\underline{u}) - f^*(x,u))(\underline{u} - u)^+ dx + \mu \int_\Omega (g^*(x,\underline{u}) - g^*(x,u))(\underline{u} - u)^+ dx.
\end{aligned}$$

Notice that in $\{\underline{u} > u\}$, from (2.7)–(2.8), we have that $f \equiv f^*$, $g \equiv g^*$, then

$$\int_\Omega (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u|^{p-2}\nabla u)\nabla(\underline{u} - u)^+ dx \leq 0. \tag{2.17}$$

On the other hand, by monotonicity of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see, [25, Example 2.27 (c)] and [12, Example 2.110]), it follows that

$$\langle -\Delta_p u + \Delta_p v, u - v\rangle \geq \begin{cases} c_1(p)\big((\|u\| + \|v\|)^{p-2}\|u - v\|^2\big) & \text{if } 1 < p < 2, \\ c_2(p)\|u - v\|^p & \text{if } p \geq 2, \end{cases} \tag{2.18}$$

for suitable positive constants $c_1(p), c_2(p)$. Moreover, by (2.17)–(2.18), there exists a positive constant c such that

$$\begin{aligned}
0 \leq \|(\underline{u} - u)^+\|^p &= \int_{\{\underline{u} > u\}} |\nabla(\underline{u} - u)|^p dx \\
&\leq c \int_{\{\underline{u} > u\}} (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u|^{p-2}\nabla u)|\nabla(\underline{u} - u)| dx \\
&= c \int_\Omega (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u|^{p-2}\nabla u)\nabla(\underline{u} - u)^+ dx \leq 0.
\end{aligned}$$

Therefore, $\|(\underline{u} - u)^+\|^p = 0$ and $u \geq \underline{u}$. ∎

The energy functional associated to (P*) is defined by setting

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u), \tag{2.19}$$

where

$$\Phi(u) = \frac{1}{p}\|u\|^p, \quad \Psi(u) = \int_\Omega H_{\lambda,\mu}(x,u)dx \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.20}$$

Lemma 2.6. *Suppose* (Q_1) *and* (Q_2) *hold. Then,* Ψ *is well-posed, continuously Gâteaux differentiable and* Ψ' *is completely continuous (i.e., if* $x_n \rightharpoonup x$ *then* $\Psi'(x_n) \to \Psi'(x)$*, see* [12, Definition 2.95]*).*

Proof. By (2.14) Ψ is well-posed. Now, we compute and show that

$$\langle \Psi'(u), v\rangle = \int_\Omega h^*_{\lambda,\mu}(x,u)v\,dx \quad \forall u, v \in W_0^{1,p}(\Omega).$$

Indeed, fixing $u, v \in W_0^{1,p}(\Omega)$, we get

$$\langle \Psi'(u), v\rangle = \lim_{t\to 0^+} \frac{\Psi(u+tv)-\Psi(u)}{t} = \lim_{t\to 0^+}\int_\Omega \frac{H_{\lambda,\mu}(x,u+tv)-H_{\lambda,\mu}(x,u)}{t}dx,$$

and Torricelli–Barrow's theorem ensures that, for each $t \in [0,1]$ and for a.a. $x \in \Omega$, we get

$$\begin{aligned} H_{\lambda,\mu}(x,u+tv) - H_{\lambda,\mu}(x,u) &= tv\int_0^1 \frac{d}{ds}H_{\lambda,\mu}(x,u+stv)ds \\ &= tv\int_0^1 h^*_{\lambda,\mu}(x,u+stv)ds. \end{aligned}$$

Furthermore, fixed $x \in \Omega$,

$$h^*_{\lambda,\mu}(x,y+z) \le \tilde{c_1}d^{-\gamma}(x) + c_2 + \frac{\mu}{\lambda}c_3|y+z|^{q-1} + \frac{\mu}{\lambda}c_5 \quad \forall y,z \in \Omega.$$

Then, by Fubini's theorem and Lebesgue's dominate convergence theorem (see, respectively, [6, Theorems 4.5 and 4.2]), it follows that

$$\begin{aligned} &\lim_{t\to 0^+}\int_\Omega \frac{H_{\lambda,\mu}(x,u+tv)-H_{\lambda,\mu}(x,u)}{t}dx \\ &\quad = \lim_{t\to 0^+}\int_\Omega\left(\int_0^1 \frac{tvh^*_{\lambda,\mu}(x,u+stv)}{t}ds\right)dx \\ &\quad = \lim_{t\to 0^+}\int_0^1\left(\int_\Omega h^*_{\lambda,\mu}(x,u+stv)v\,dx\right)ds \\ &\quad = \int_\Omega h^*_{\lambda,\mu}(x,u)v\,dx. \end{aligned}$$

Finally, we prove that Ψ' is completely continuous and $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ is a sequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and let $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}
&\lim_{n\to\infty} \left|\langle \Psi'(u_n) - \Psi(u), v\rangle\right| \\
&= \lim_{n\to\infty} \left| \int_\Omega (h^*_{\lambda,\mu}(x,u_n) - h^*_{\lambda,\mu}(x,u))v dx \right| \\
&\leq \lim_{n\to\infty} \int_\Omega |h^*_{\lambda,\mu}(x,u_n) - h^*_{\lambda,\mu}(x,u)||v| dx \\
&\leq \lim_{n\to\infty} \left(\int_\Omega |f^*(x,u_n) - f^*(x,u)||v| dx + \frac{\mu}{\lambda} \int_\Omega |g^*(x,u_n) - g^*(x,u)||v| dx \right).
\end{aligned}$$

By Rellich–Kondrachov's embedding theorem [6, Theorem 9.16], $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q \in [1, p^*)$. So, in $L^q(\Omega)$, there exists $w \in L^q(\Omega)$ such that $u_n \to u$ in $L^q(\Omega)$, $u_n \to u$ a.e. in Ω and $|u_n| \leq w$ [6, Theorem 4.9]. Moreover, from (Q_2), (2.9)–(2.10) and the Lebesgue's dominate convergence theorem, we get that Ψ' is completely continuous. ∎

Definition 2.7. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. We say that J satisfies the Palais–Smale condition (briefly, (PS)-condition), if any sequence $\{u_n\} \subseteq X$ such that

(1) $J(u_n)$ is bounded,

(2) $\|J'(u_n)\|_{X^*} \to 0$ as $n \to +\infty$,

admitting a convergent subsequence.

A more general condition can be found in [3].

Definition 2.8 ([3, Section 2]). Let Φ, Ψ be two continuously Gâteaux differentiable functions; put

$$J = \Phi - \Psi,$$

and fix $r \in [-\infty, +\infty]$; we say that the function J fulfills the Palais–Smale condition cut off upper at r (in short $(PS)^r$-condition), if any sequence $\{u_n\}$, in addition to (1) and (2) in the previous definition, accomplishing also

(3) $\Phi(u_n) < r \ \forall n \in \mathbb{N}$,

possesses a convergent subsequence.

Lemma 2.9. *Suppose that (Q_1), (Q_2), and (AR) hold. Then, for all $p < q < p^*$, the functional J_λ associated to (P*) satisfies the (PS)-condition.*

Proof. Fix $\lambda, \mu > 0$, let J_λ be as in (2.19) and $\{u_n\}$ a (PS)-sequence in $W_0^{1,p}(\Omega)$, i.e., it fullfills (1) and (2) introduced in the Definition 2.7, that is, for all $n \in \mathbb{N}$, there exists a suitable constant c_b such that

$$J_\lambda(u_n) = \frac{1}{p}\|u_n\|^p - \lambda \int_\Omega H_{\lambda,\mu}(x,u_n) dx \leq c_b \tag{2.21}$$

and

$$|\langle J'_\lambda(u_n), v\rangle| = \left| \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx - \lambda \int_\Omega h^*_{\lambda,\mu}(x, u_n) v dx \right| \le \|v\| \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.22}$$

In order to prove that $\{u_n\}$ is bounded first we verify it on $\{u_n^-\}$. From (2.22), with $v = -u_n^-$, we obtain

$$\|u_n^-\|^p \le \|u_n^-\|^p + \lambda \int_\Omega h^*_{\lambda,\mu}(x, u_n) u_n^- dx \le \|u_n^-\|,$$

then $\|u_n^-\| \le 1$. Now, we prove that also $\{u_n^+\}$ is bounded. Arguing as in (2.14), putting $A_1 := (\frac{c_1}{1-\gamma} + c_2)$, we have

$$\int_0^{|s|} f^*(x,t)dt \le A_1(2 + |s|) \quad \forall s \in \mathbb{R}.$$

At this point, we use (AR). Let $n \in \mathbb{N}$ for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$, we compute

$$H_{\lambda,\mu}(x, u_n^+) \le \int_0^{u_n^+} f^*(x,t)dt + \frac{\mu}{\lambda}\int_0^{u_n^+} g^*(x,t)dt \le A_1(2 + u_n^+) + \frac{\mu}{\lambda}\int_0^{u_n^+} g^*(x,t)dt.$$

Define

$$\Omega_R := \{x \in \Omega : u_n^+(x) \ge R\}$$

and

$$\Omega'_R = \Omega \setminus \Omega_R.$$

We remember that we can choose $\underline{u} < 1$ and $R \ge \max\{s_1, 1\}$. So, from the previous inequalities, we point out

$$\begin{aligned} &\int_\Omega H_{\lambda,\mu}(x, u_n^+)dx \\ &\quad \le \int_\Omega \int_0^{u_n^+} f^*(x,t)dtdx + \frac{\mu}{\lambda}\int_\Omega \int_0^{u_n^+} g^*(x,t)dtdx \\ &\quad \le A_1\big(2|\Omega| + S_1\|u_n^+\|\big) + \frac{\mu}{\lambda}\int_{\Omega_R}\int_0^{u_n^+} g^*(x,t)dtdx + \frac{\mu}{\lambda}\int_{\Omega'_R}\int_0^{u_n^+} g^*(x,t)dtdx. \end{aligned} \tag{2.23}$$

In particular, one has

$$\int_{\Omega'_R}\int_0^{u_n^+} g^*(x,t)dtdx \tag{2.24}$$

$$\begin{aligned} &\le \int_{\Omega'_R}\int_0^{R} g^*(x,t)dtdx \\ &= \int_{\Omega'_R}\int_0^{\underline{u}} g(x,\underline{u})dtdx + \int_{\Omega'_R}\int_{\underline{u}}^{R} g(x,t)dtdx \le (c_3 + c_4)|\Omega|(1 + R^q). \end{aligned} \tag{2.25}$$

Moreover, we have

$$\begin{aligned}\int_{\Omega_R}\int_0^{u_n^+} & g^*(x,t)dtdx \\ &= \int_{\Omega_R}\int_0^{\underline{u}} g(x,\underline{u})dtdx + \int_{\Omega_R}\int_{\underline{u}}^{R} g(x,t)dtdx + \int_{\Omega_R}\int_R^{u_n^+} g(x,t)dtdx, \quad (2.26)\end{aligned}$$

where

$$\int_{\Omega_R}\int_0^{\underline{u}} g(x,\underline{u})dtdx \le (c_3+c_4)|\Omega|, \tag{2.27}$$

$$\int_{\Omega_R}\int_{\underline{u}}^{R} g(x,t)dtdx \le (c_3+c_4)|\Omega|R^q, \tag{2.28}$$

since $\Omega = \Omega_R \cup \Omega_R'$, we have

$$\int_{\Omega_R}\int_R^{u_n^+} g(x,t)dtdx \le \int_{\Omega}\int_R^{u_n^+} g(x,t)dtdx. \tag{2.29}$$

So, combining (2.23)–(2.29), we get

$$\begin{aligned}\int_\Omega & H_{\lambda,\mu}(x,u_n^+)dx \\ &\le A_1(2|\Omega| + S_1\|u_n^+\|) + \frac{\mu}{\lambda}(2(c_3+c_4)|\Omega|(1+R^q)) + \frac{\mu}{\lambda}\int_\Omega\int_R^{u_n^+} g^*(x,t)dtdx.\end{aligned}$$

Furthermore, by (AR)-condition,

$$\begin{aligned}\int_\Omega & H_{\lambda,\mu}(x,u_n^+)dx \\ &\le A_1(2|\Omega| + S_1\|u_n^+\|) + \frac{\mu}{\lambda}(2(c_3+c_4)|\Omega|(1+R^q)) + \frac{\mu}{\lambda\eta}\int_\Omega g^*(x,u_n)u_n^+dx.\end{aligned}$$

By (2.22), with $v = u_n^+$, it follows that

$$\begin{aligned}-\|u_n^+\|^p - \|u_n^+\| &\le -\lambda\int_\Omega f^*(x,u_n^+)u_n^+dx - \mu\int_\Omega g^*(x,u_n^+)u_n^+dx \\ &\le -\mu\int_\Omega g^*(x,u_n)u_n^+dx. \end{aligned} \tag{2.30}$$

By (2.21), we arrive at

$$\begin{aligned}\frac{1}{p}\|u_n^+\|^p \le c_b + \lambda\int_\Omega H_{\lambda,\mu}(x,u_n^+)dx &\le c_b + \lambda A_1(2|\Omega| + S_1\|u_n^+\|) \\ &+ \mu(2(c_3+c_4)|\Omega|(1+R^q)) + \frac{\mu}{\eta}\int_\Omega g^*(x,u_n)u_n^+dx,\end{aligned} \tag{2.31}$$

then, dividing (2.30) by η and adding to (2.31), we derive

$$\left(\frac{1}{p}-\frac{1}{\eta}\right)\|u_n^+\|^p - \left(\frac{1}{\eta}+\lambda S_1 A_1\right)\|u_n^+\| \le c_b + 2|\Omega|[\lambda A_1 + \mu(c_3+c_4)(1+R^q)].$$

So, $\{u_n^+\}$ is also bounded and our claim is shown, i.e., $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By Lemma 2.6, Ψ' is completely continuous, then

$$\lim_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle = \lim_{n\to\infty}\langle J_\lambda'(u_n), u_n - u\rangle + \lambda \lim_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle = 0,$$

that is,

$$\limsup_{n\to\infty}\langle -\Delta_p(u_n), u_n - u\rangle = 0.$$

Thus, by (S_+) property of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see [12, Definition 2.96 and Lemma 2.111]), we have that if $u_n \rightharpoonup u$ then $u_n \to u$ and J_λ satisfies the (PS)-condition. ■

The following is folklore, but for completeness we prove it also in our setting.

Lemma 2.10. *Suppose* (Q_1), (Q_2), *and (AR) hold. Then, for all* $p < q < p^*$, *the functional* J_λ *associated to* (P*) *is unbounded from below.*

Proof. Fix $\lambda, \mu > 0$. For all $M \ge 1$, we put $\Omega_M = \{x \in \Omega : M\varphi_1(x) \ge R\}$ and we consider $\Omega_M' = \Omega \setminus \Omega_M$, where $R \ge \max\{s_1, 1\}$ and φ_1 is the first positive eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$, normalized in $L^p(\Omega)$. From (AR), for all $\omega \ge R$,

$$\eta\int_{s_1}^{\omega} g(x,t)dt \le g(x,\omega)\omega,$$

$$\frac{\eta}{\omega} \le \frac{g(x,\omega)}{\int_{s_1}^{\omega} g(x,t)dt} = \frac{\frac{\partial}{\partial\omega}\left(\int_{s_1}^{\omega} g(x,t)dt\right)}{\int_{s_1}^{\omega} g(x,t)dt}.$$

By integrating both sides for $s \ge R$ in $[R,s]$ w.r.t. ω, we obtain

$$\eta \ln\left(\frac{s}{R}\right) \le \ln\left(\frac{\int_{s_1}^{s} g(x,t)dt}{\int_{s_1}^{R} g(x,t)dt}\right);$$

then,

$$\int_{s_1}^{s} g(x,t)dt \ge s^\eta R^{-\eta}\int_{s_1}^{R} g(x,t)dt. \tag{2.32}$$

Compute

$$\begin{aligned} J_\lambda(M\varphi_1) &= \frac{1}{p}\|M\varphi_1\|^p - \lambda\int_\Omega\int_0^{M\varphi_1} h_{\lambda,\mu}^*(x,t)dtdx \\ &= \frac{\lambda_1}{p}M^p - \lambda\int_\Omega\int_0^{M\varphi_1} h_{\lambda,\mu}^*(x,t)dtdx, \end{aligned}$$

we focus on the right-hand side term, in particular,

$$\int_\Omega \int_0^{M\varphi_1} h^*_{\lambda,\mu}(x,t)dtdx = \int_\Omega \int_0^{M\varphi_1} f^*(x,t)dtdx + \frac{\mu}{\lambda} \int_\Omega \int_0^{M\varphi_1} g^*(x,t)dtdx.$$

As in (2.14), we get

$$\int_\Omega \int_0^{M\varphi_1} f(x,t)dtdx \le A_1(2|\Omega| + MS_1\|\varphi_1\|) = A_1(2|\Omega| + \lambda_1 S_1 M).$$

While, on the integral of g, we have

$$\int_\Omega \int_0^{M\varphi_1} g(x,t)dtdx = \int_{\Omega_M} \int_0^{M\varphi_1} g(x,t)dtdx + \int_{\Omega'_M} \int_0^{M\varphi_1} g(x,t)dtdx,$$

where

$$\int_{\Omega'_M} \int_0^{M\varphi_1} g(x,t)dtdx \le \int_{\Omega'_M} \int_0^{R} g(x,t)dtdx \le (c_3 + c_4)R^q|\Omega|$$

and

$$\int_{\Omega_M} \int_0^{M\varphi_1} g(x,t)dtdx = \int_{\Omega_M} \int_0^{R} g(x,t)dtdx + \int_{\Omega_M} \int_R^{M\varphi_1} g(x,t)dtdx$$

with

$$\int_{\Omega_M} \int_0^{R} g(x,t)dtdx \le (c_3 + c_4)R^q|\Omega|.$$

Thanks to these estimates, we have

$$\begin{aligned} J_\lambda(M\varphi_1) \le{}& \frac{\lambda_1 M^p}{p} + \lambda A_1(2|\Omega| + S_1\lambda_1 M) \\ &+ 2\mu(c_3 + c_4)R^q|\Omega| - \mu \int_{\Omega_M} \int_R^{M\varphi_1} g(x,t)dtdx. \end{aligned}$$

By (2.32), we get

$$\int_R^s g(x,t)dt \ge \Big(\frac{s^\eta}{R^\eta} - 1\Big) \int_{s_1}^R g(x,t)dt.$$

We use this inequality to see that

$$\begin{aligned} J_\lambda(M\varphi_1) \le{}& \frac{\lambda_1 M^p}{p} + \lambda A_1(2|\Omega| + S_1\lambda_1 M) \\ &+ 3\mu(c_3 + c_4)R^q|\Omega| - \mu \frac{M^\eta}{R^\eta} \int_{\Omega_M} \int_{s_1}^R g(x,t)\varphi_1^\eta(x)dtdx; \end{aligned}$$

then, $J_\lambda(M\varphi_1) \to -\infty$ as $M \to +\infty$. ∎

For the reader's convenience, let us recall our two fundamental tools. The first one is the celebrated Ambrosetti–Rabinowitz's mountain pass theorem (see [2, 7]).

Theorem 2.11 (Mountain pass theorem). *Let X be a Banach space. Let $J : X \to \mathbb{R}$ be a functional such that $J \in C^1(X, \mathbb{R})$. Let x_0, $x_1 \in X$, $r > 0$ such that $\|x_1 - x_0\| > r$ and*

$$\inf_{\|u-x_0\|=r} J(u) > \max \{J(x_0), J(x_1)\}.$$

Let

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$. Suppose that J satisfies the Palais–Smale condition. Then, c is a critical value for J.

For the second one, fixing $r > 0$, let $\varphi(r)$ be as follows:

$$\varphi(r) = \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)}. \tag{2.33}$$

Theorem 2.12 ([3, Theorem 5.2]). *Let X be a Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functions with Φ bounded from below. Fix $r > \inf_X \Phi$ and assume that for each $\lambda \in]0, \frac{1}{\varphi(r)}[$, where φ is given in (2.33), the function $J_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^r$-condition. Then, for each $\lambda \in]0, \frac{1}{\varphi(r)}[$ there is $u^* \in \Phi^{-1}(]-\infty, r[)$ such that $J_\lambda(u^*) \leq J_\lambda(u)$ $\forall u \in \Phi^{-1}(]-\infty, r[)$ and $J'_\lambda(u^*) = 0$.*

Remark 2.13. We point out that this type of local minima theorems follows from the ideas introduced by Ricceri in [33], in which the author used the weakly closure of suitable sublevels. In [5], Bonanno and Candito got the result starting from direct methods of calculus of variations and, in [3], Bonanno himself obtained the local minimum via Ekeland variational principle.

3. Main results

In this section, we prove our main results. In particular, we obtain the existence of at least one solution for $1 < q \leq p$ and the existence of at least two solutions for $p < q < p^*$ for problem $(P_{\lambda,\mu})$. In addition, we derive a computable estimate of the parameters λ, μ.

The following positive constants are central in the main results:

$$Z_1 := \frac{N}{p}\pi^{\frac{p}{2}}|\Omega|^{-\frac{p}{N}}\left(\frac{N-p}{p-1}\right)^{p-1}\left[\frac{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}{\Gamma(1+\frac{N}{2})\Gamma(N)}\right]^p, \tag{3.1}$$

$$Z_2 := \frac{N}{p}\pi^{\frac{p}{2}}|\Omega|^{-\frac{p}{N}}\left(\frac{N-p}{p-1}\right)^{p-1}\left(\frac{q-p}{2p}\right)^{\frac{q-p}{q}}\left[\frac{\Gamma(\frac{N}{p})\Gamma(1+N-\frac{N}{p})}{\Gamma(1+\frac{N}{2})\Gamma(N)}\right]^p. \tag{3.2}$$

Theorem 3.1. *Assume that* (Q_1) *and* (Q_2) *hold. In addition, we suppose that*

$$\lambda\Big(\frac{c_1}{1-\gamma}+c_2\Big)+\mu(c_3+c_4)<\begin{cases}+\infty & \text{if } 1<q<p;\\ Z_1 & \text{if } q=p,\\ Z_2 & \text{if } p<q<p^*,\end{cases}$$

where Z_1, Z_2 *are as in* (3.1)–(3.2). *Then, problem* $(P_{\lambda,\mu})$ *admits at least one solution* $u \in C_0^{1,\alpha}(\bar{\Omega})$, *with* $\alpha \in (0,1)$.

Proof. Let J_λ, Φ, and Ψ be as in (2.19)–(2.20). By Lemma 2.6, J_λ is a C^1-functional. Fix $r>0$, let $\{u_n\}$ be a $(PS)^r$-sequence. Since $\Phi(u_n)<r$ and reasoning is as in Lemma 2.9, we have that J_λ satisfies the $(PS)^r$-condition as in Definition 2.8. In this setting, Theorem 2.12 furnishes a local minimum $u^*=u^*(\lambda,\mu)$ of J_λ for all $\lambda\in\,]0,\bar{\lambda}[$. So, u^* is a weak solution of (P*) and then, by Lemma 2.5, a weak solution of $(P_{\lambda,\mu})$. To give an approximation of $\bar{\lambda}$ we need to estimate (2.33). Since $r>\inf_X\Phi=\Phi(0)=\Psi(0)=0$ and (2.14), we detect that

$$\begin{aligned}\varphi(r)&\le\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)\\ &\le\frac{1}{r}\Big(\sup_{\|u\|\le(pr)^{\frac{1}{p}}}\int_\Omega|H_{\lambda,\mu}(x,u)|dx\Big)\\ &\le\frac{1}{r}\sup_{\|u\|\le(pr)^{\frac{1}{p}}}\int_\Omega A_{\lambda,\mu}(2+|u|^q)dx\\ &\le\frac{1}{r}A_{\lambda,\mu}\big(2|\Omega|+S_q^q p^{\frac{q}{p}}r^{\frac{q}{p}}\big)\quad\forall r>0.\end{aligned}\tag{3.3}$$

where $A_{\lambda,\mu}$ is defined in (2.13). We put

$$k(r):=\frac{1}{r}\big(2|\Omega|+S_q^q p^{\frac{q}{p}}r^{\frac{q}{p}}\big)\quad\text{for all } r>0.$$

A straightforward calculation shows that

$$\lim_{r\to0^+}k(r)=\lim_{r\to0^+}\frac{1}{r}\big(2|\Omega|+S_q^q p^{\frac{q}{p}}r^{\frac{q}{p}}\big)=+\infty\quad\text{for all } 1<q<p^*,\tag{3.4}$$

and, for $r\to+\infty$, three cases arise:

$$\lim_{r\to+\infty}k(r)=\begin{cases}0 & \text{if } 1<q<p;\\ S_p^p p & \text{if } q=p;\\ +\infty & \text{if } p<q<p^*.\end{cases}\tag{3.5}$$

To guarantee that $\varphi(r)<\frac{1}{\lambda}$ for some $r>0$, by (3.3), it follows that

$$\varphi(r)\le A_{\lambda,\mu}k(r)<\frac{1}{\lambda}.$$

Considering that $p > 1, q > 1$, then by (3.4)–(3.5), we have the following cases:

(1) if $1 < q < p$, since $k(r)$ is a continuous function, there exists $\bar{r}_{\lambda,\mu} > 0$ such that

$$k(\bar{r}_{\lambda,\mu}) = \frac{1}{\lambda A_{\lambda,\mu}}.$$

Moreover, from the strictly decreasing of $k(r)$, our claim holds for all $r > \bar{r}_{\lambda,\mu}$;

(2) if $q = p$, there exists $\bar{r}_{\lambda,\mu} > 0$ such that $\lambda A_{\lambda,\mu} = \frac{1}{k(r)} < \frac{1}{S_p^p p} =: Z_1$, then since $k(r)$ is strictly decreasing, the result holds for all $r > \bar{r}_{\lambda,\mu}$;

(3) if $p < q < p^*$, then $k(r)$ has a global minimum point for $r > 0$. We compute the external point of $k(r)$,

$$\bar{r} = \left(\frac{2p|\Omega|}{S_q^q p^{\frac{q}{p}}(q-p)}\right)^{\frac{p}{q}}, \quad \inf_{r>0} k(\bar{r}) = S_q^p p\left(\frac{2p|\Omega|}{q-p}\right)^{1-\frac{p}{q}}.$$

Thus, by (3.3), it follows that

$$\frac{1}{Z_2} := S_q^p p\left(\frac{2p|\Omega|}{q-p}\right)^{1-\frac{p}{q}} < \frac{1}{\lambda A_{\lambda,\mu}}.$$

So, also, in this case, our statement holds by keeping in mind (2.1).

Notice that our solution belongs to $C_0^{1,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$. In particular, from the estimates in (2.9) and (2.10), it is sufficient first to apply [17, Theorem 1.5.5] to obtain $u^* \in L^\infty(\Omega)$; finally, from [17, Theorem 1.5.6], we derive u belongs to a Hölder space. ■

Remark 3.2. A careful reading of the proof of Theorem 3.1, gives us, for $p < q < p^*$, an estimate on the Sobolev norm of the solution, i.e.,

$$\|u\| \leq \bar{r} = \left(\frac{2p|\Omega|}{S_q^q p^{\frac{q}{p}}(q-p)}\right)^{\frac{p}{q}}.$$

Remark 3.3. Notice that since $X = W_0^{1,p}(\Omega)$ is a reflexive Banach space (for $1 < p < +\infty$), Ψ is also weakly sequentially continuous [36, Corollary 41.9]. Then, J_λ is a C^1-functional and it is weakly sequentially lower semi-continuous. For $1 < q \leq p$, we can obtain the existence of a solution for (P*) as global minimum of J_λ, given that, in this case, the energy functional is also coercive. Indeed, from estimates in (2.14), one has

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega H_{\lambda,\mu}(x,u)dx \geq \frac{1}{p}\|u\|^p - A_{\lambda,\mu}\left(2|\Omega| + S_q^q\|u\|^q\right),$$

and so, we have the following cases:

(1) if $1 < q < p$,

$$\lim_{\|u\|\to+\infty} J(u) = +\infty$$

for all $\lambda, \mu > 0$;

(2) if $q = p$,

$$\lim_{\|u\|\to+\infty} J(u) = +\infty$$

provided $\frac{1}{p} - A_{\lambda,\mu} S_p^p > 0$, i.e.,

$$\lambda\Big(\frac{c_1}{1-\gamma} + c_2\Big) + \mu(c_3 + c_4) < \frac{1}{S_p^p\, p}.$$

Then, by Tonelli–Weierstrass' theorem [14, Theorem 1.2], there exists a weak solution of (P*). Our approach (Theorem 2.12), allows us to guarantee the existence of a weak solution also for the super-linear case and, in all three cases, without the assumption of weakly lower semi-continuity on J_λ.

Now, we prove the multiplicity result.

Theorem 3.4. *Assume that (Q_1) and (Q_2) hold. Moreover, we suppose that there exist $\eta > p$ and $s_1 > 0$ such that*

$$\operatorname*{essinf}_{x\in\Omega} \int_0^s g(x,t)dt > 0, \quad 0 < \eta \int_{s_1}^s g(x,t)dt \le g(x,s)s. \tag{AR}$$

Then, if $p < q < p^*$, the problem $(P_{\lambda,\mu})$ admits at least two solutions: $u^*(\lambda,\mu)$ a local minimum and $\tilde{u}(\lambda,\mu)$ a mountain pass point of the energy functional J_λ, for all λ, μ such that

$$\lambda\Big(\frac{c_1}{1-\gamma} + c_2\Big) + \mu(c_3 + c_4) < Z_2,$$

with Z_2 being as in (3.2).

Proof. From Lemma 2.5, it is enough to obtain solution of problem (P*). Moreover, the energy functional

$$J_\lambda = \Phi - \lambda\Psi$$

satisfies the (PS)-condition (Lemma 2.9) and it is a C^1-functional (Lemma 2.6). So, by Theorem 3.1, there exists u^* local minimum of J_λ such that $J_\lambda(u^*) \le J_\lambda(u)$ for all $u \in \overline{B}(0,r)$ for a suitable $r > 0$ as in Theorem 3.1 (see also Remark 3.2). Since J_λ is unbounded from below (Lemma 2.10), [4, Theorem 2.1] ensures that it satisfies also the mountain pass geometry and the proof is completed, owing the Theorem 2.11. ∎

We desire to conclude our work by giving the following example.

Example 3.5. For every $\lambda, \mu > 0$ such that

$$2\lambda + \mu < \frac{5}{54}\sqrt[9]{\frac{3125}{64}\Big(\frac{77\pi}{1024}\Big)^4}.$$

Theorem 3.4 guarantees the existence of at least two solutions for the problem (1.1) proposed in the Introduction, where $\Omega \subset \mathbb{R}^5$ is a bounded domain with smooth boundary and

$|\Omega| = 1$, $p = 4$, $q = 9$, $4 < \eta \leq 9$ and $\gamma = \frac{1}{2}$. Moreover, taking into account Remark 3.2, the first solution u^* (corresponding to a local minimum of the energy functional) satisfies

$$\|u^*\| \leq \frac{10}{27} \sqrt[9]{\frac{8}{625}} \Big(\frac{77\pi}{1024}\Big)^4.$$

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Pasquale Candito
Dipartimento di Ingegneria Civile, dell’Energia, dell’Ambiente e dei Materiali, Università degli Studi Mediterranea di Reggio Calabria, via Zehender (Località Feo di Vito), 89122 Reggio Calabria, Italy; pasquale.candito@unirc.it

Giuseppe Failla
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra, Università degli Studi di Messina, Viale Ferdinando Stagno d’Alcontres, 98166 Messina, Italy; giuseppe.failla@studenti.unime.it

Roberto Livrea
Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi, 90123 Palermo, Italy; roberto.livrea@unipa.it