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Orlicz spaces with s-norms that are abstract M-spaces

Badik Hüseyin Uysal and Serap Öztop

Abstract. Let Φ be an Orlicz function and $L^{\Phi}(X, \Sigma, \mu)$ the corresponding Orlicz space on a non-atomic, σ-finite, complete measure space (X, Σ, μ). We describe those Orlicz spaces equipped with s-norms that are abstract M-spaces. Our study generalizes and unifies the results that have been obtained for the Orlicz norm, the Luxemburg norm, and the p-Amemiya norms separately.

1. Introduction

The notion of an abstract M-space plays a crucial role for the Banach lattice structure of a Banach function space and its description as an abstract L-space. Every abstract M-space is isomorphic as a Banach lattice with some closed vector sublattice of some suitable $C_{\mathbb{R}}(X)$ for some compact Hausdorff space X [17]. Also, the strong dual of an abstract M-space with unit is an abstract L-space. Consequently, an abstract M-space is an important geometric property of a Banach function space. Orlicz spaces comprise an important class of Banach spaces that are a kind of generalization of Lebesgue spaces. The theory of Orlicz spaces has been greatly developed because of their important theoretical properties and value in applications. Up to now, criteria that an Orlicz space equipped with the Orlicz norm or the Luxemburg norm to be an abstract M-space have been obtained [5]. In [18], using the concept of an outer function, M. Wisła presented a general and universal method of introducing norms in Orlicz spaces that covered the classical Orlicz and Luxemburg norms and p-Amemiya norms ($1 \leqslant p \leqslant \infty$). Recently, in [3], s-norms were classified with respect to the constant σ_s and real extreme points were described. It is known that the geometric properties of a Banach function space depends on the norm. Our aim in this work is to describe the abstract M-space of Orlicz spaces equipped with the s-norms.

The structure of this paper is as follows. In Section 2, we provide necessary definitions. In Section 3, we recall some technical results for Orlicz spaces equipped with s-norms that will be used and we make some observations from these known results. In Section 4, we present some properties of Orlicz functions which are related to s-norms and strictly increasing s-norms under some conditions. In Section 5, we describe abstract M-spaces of Orlicz spaces equipped with s-norms. In Theorems 5.3 and 5.7, we present criteria for abstract M-space of Orlicz spaces equipped with the s-norms.

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2. Preliminaries

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, Φ is not identically equal to zero, Φ is even and convex on the interval $(-b_\Phi, b_\Phi)$, and Φ is left continuous at b_Φ, where $b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}$. From these properties, it follows that an Orlicz function Φ is continuous on $(-b_\Phi, b_\Phi)$, is increasing on $[0, b_\Phi)$, and satisfies $\lim_{u\to\infty} \Phi(u) = \infty$. For any $u, v \in \mathbb{R}$,

$$\Phi(u) + \Phi(v) \leq \Phi(u + v).$$

This inequality is called the superadditivity of Φ [9].

If Φ is an Orlicz function, letting $a_\Phi = \sup\{u \geq 0 : \Phi(u) = 0\}$, then $a_\Phi = 0$ means that Φ vanishes only at 0 while $b_\Phi = \infty$ means that Φ takes only finite values. From the definition of an Orlicz function, we have $a_\Phi \leq b_\Phi$, $a_\Phi < \infty$, and $b_\Phi > 0$.

For an Orlicz function Φ, we define its complementary function Ψ by the formula

$$\Psi(v) = \sup_{u\geq 0}\{u|v| - \Phi(u)\}.$$

It is well known that the complementary function is an Orlicz function as well [16].

Throughout the paper, we will assume that (X, Σ, μ) is a measure space with a σ-finite, non-atomic, and complete measure μ and denote by $L^0(X, \Sigma, \mu)$ the space of all μ-equivalence classes of real-valued and Σ-measurable functions defined on X. For $1 \leq p < \infty$, we will denote by $L^p(X, \Sigma, \mu)$ or just by L^p the spaces of p-integrable functions with respect to (X, Σ, μ). For $p = \infty$, we also denote by $L^\infty(X, \Sigma, \mu)$ or just by L^∞ the spaces of essentially bounded function with respect to (X, Σ, μ). In addition, we use the conventions $\infty \cdot 0 = 0$, $\frac{1}{\infty} = 0$, and $\frac{1}{0} = \infty$.

For a given Orlicz function Φ, we define on $L^0(X, \Sigma, \mu)$ a convex functional I_Φ by

$$I_\Phi(f) = \int_X \Phi(f)\, d\mu.$$

The Orlicz space $L^\Phi(X, \Sigma, \mu)$ generated by an Orlicz function Φ is a linear space of measurable functions defined by [14]

$$L^\Phi(X, \Sigma, \mu) = \big\{f \in L^0(X, \Sigma, \mu) : I_\Phi\big(\lambda f\big) < \infty \text{ for some } \lambda > 0\big\}.$$

We denote the Orlicz space $L^\Phi(X, \Sigma, \mu)$ shortly by L^Φ.

The Orlicz space L^Φ is usually equipped with the Orlicz norm [14]

$$\|f\|_\Phi^o = \sup\left\{\int_X |f(t)g(t)|\, d\mu : g \in L^\Psi,\ I_\Psi(g) \leqslant 1\right\},$$

where Ψ is the complementary function to Φ, or with the equivalent Luxemburg norm [11]

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

At first sight, the Orlicz norm and the Luxemburg norm seem far from similar. In fact, in many cases, the geometric properties of Orlicz spaces under each of these norms differ from each other. I. Amemiya (see [13]) considered a norm defined by the formula

$$\|f\|_\Phi^A = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kf)).$$

Krasnosel'skii and Rutickii [9], Nakano [13], and Luxemburg and Zaanen [12] proved that the Orlicz norm is expressed exactly by the Amemiya formula, i.e., $\|f\|_\Phi^o = \|f\|_\Phi^A$. In the most general case of an Orlicz function Φ, a similar result was obtained by Hudzik and Maligranda [7]. Moreover, it is not difficult to verify that the Luxemburg norm can also be expressed by an Amemiya-like formula [4, 15], namely,

$$\|f\|_\Phi = \inf_{k>0} \frac{1}{k} \max\{1, I_\Phi(kf)\}.$$

The only difference between the two Amemiya formulas is the function under the infimum operation: for all $u \geq 0$ it is $1 + u$ (for the Orlicz norm) and $\max\{1, u\}$ (for the Luxemburg norm). In the paper [7], Hudzik and Maligranda proposed to investigate another class of norms given by Amemiya formula norms generated by the outer functions of the type

$$s_p(u) = \begin{cases} (1 + u^p)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \max\{1, u\} & \text{for } p = \infty. \end{cases}$$

The p-Amemiya norms for $f \in L^\Phi$ were defined by the formula

$$\|f\|_{\Phi,p} = \inf_{k>0} \frac{1}{k} s_p(I_\Phi(kf)),$$

where $1 \leq p \leq \infty$ [4].

In 2020, the notion of the s-norms was introduced by M. Wisła, and the following definitions can be found in [18].

Definition 2.1. A function $s : [0, \infty] \to [1, \infty]$ is called an outer function if it is convex and satisfies the inequality

$$\max\{u, 1\} \leq s(u) \leq u + 1$$

for all $u \geq 0$.

Let us note that an outer function s is continuous and increasing on $[0, \infty)$. Evidently, $s(0) = 1$, and we set $s(\infty) = \infty$.

Since it is convex, an outer function s has both right and left derivatives. Let s'_+ be the right derivative of s so that $s'_+ : [0, \infty) \to [0, 1]$ is an increasing function. Let ${s'_+}^{-1} : [0, 1] \to [0, \infty]$ be a general inverse of s'_+ as defined in [18]. Then, ${s'_+}^{-1}$ is an increasing function as well.

Let us give some examples of families of outer functions.

Example 2.2. (i) For $1 \le p \le \infty$,

$$s_p(u) = \begin{cases} (1+u^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1,u\} & \text{if } p = \infty. \end{cases} \tag{2.1}$$

(ii) For $0 \le c \le 1$,

$$s_c(u) = \max\{1, u+c\}. \tag{2.2}$$

Definition 2.3. Let s be an outer function and Φ an Orlicz function. Then, the s-norms of $f \in L^\Phi$ are defined by

$$\|f\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_\Phi(kf)).$$

The Orlicz space equipped with an s-norm will be denoted by L_s^Φ.

Remark 2.4. Observe that each of the families given in (2.1) and (2.2) generates both the Orlicz norm and the Luxemburg norm. In (2.1), if we take $s = s_1$, then $\|f\|_{\Phi,s_1} = \|f\|_\Phi^o$; if $s = s_\infty$, then $\|f\|_{\Phi,s_\infty} = \|f\|_\Phi$; if $s = s_p$ for $1 < p < \infty$, then $\|f\|_{\Phi,s_p} = \|f\|_{\Phi,p}$ (see [4, 7]).

Similarly, in (2.2), $c = 0$ gives the Luxemburg norm and $c = 1$ the Orlicz norm.

Definition 2.5. Let s be an outer function. For all $0 \le v \le 1$, define

$$w(v) = \int_0^v {s'_+}^{-1}(t)\,dt.$$

It is clear that w is a non-negative, increasing, and continuous function on $[0, 1]$.

Definition 2.6. Let s be an outer function. For all $0 \le u < \infty$ and $0 \le v \le \infty$, set

$$\beta_s(u,v) = 1 - w(s'_+(u)) - v s'_+(u).$$

Denote also $\beta_s(kf) = \beta_s(I_\Phi(kf), I_\Psi(p_+(k|f|)))$ for all $f \in L^\Phi$.

Note that the function $k \mapsto \beta_s(kf)$ is decreasing on $[0,\infty)$.

Definition 2.7. Let s be an outer function and Φ an Orlicz function. For $f \in L^\Phi \setminus \{0\}$ and $0 < k < \infty$, we define the following functions:

$$\begin{aligned}
D &: L_s^\Phi \to \mathcal{P}([0,\infty)), & D(f) &= \{0 < k < \infty : I_\Phi(kf) < \infty\}, \\
\Theta &: L_s^\Phi \to [0,\infty), & \Theta(f) &= \inf D(f)^{-1}, \\
k^* &: L_s^\Phi \to (0,\infty], & k^*(f) &= \inf\{k \in D(f) : \beta_s(kf) \le 0\}, \\
k^{**} &: L_s^\Phi \to [0,\infty), & k^{**}(f) &= \sup\{k \in D(f) : \beta_s(kf) \ge 0\}.
\end{aligned}$$

It is easy to see that $0 < k^*(f) \le k^{**}(f) \le \infty$. Let us also define

$$K(f) = \left\{0 < k < \infty : k^*(f) \le k \le k^{**}(f)\right\}.$$

Obviously, $K(f) \neq \emptyset$ if and only if $k^*(f) < \infty$. If $k^*(f) < \infty$ for any $f \in L_s^\Phi$, then the s-norms are called k^*-finite; if $k^{**}(f) < \infty$ for any $f \in L_s^\Phi$, then the s-norms are called k^{**}-finite. Further, if $k^*(f) = k^{**}(f) < \infty$ for any $f \in L_s^\Phi$, then the s-norms are called k-unique.

Definition 2.8. Let s be an outer function. Define the constant σ_s by

$$\sigma_s = \sup\{u \geq 0 : s(u) = 1\}.$$

Note that $0 \leq \sigma_s \leq 1$, and it is obvious that s is strictly increasing on $[\sigma_s, \infty)$. We focus on the cases of $\sigma_s > 0$ and $\sigma_s = 0$ in the rest of this paper. The key point in defining this constant is that the equality $\sigma_s = 0$ provides an inverse function for the outer function s since this function is strictly increasing on the entire interval $[0, \infty)$ whenever $\sigma_s = 0$.

Let $\mathcal{S}$ denote the set of outer functions and define the sets

$$\mathcal{S}_0 = \{s \in \mathcal{S} : \sigma_s = 0\} \quad \text{and} \quad \mathcal{S}_+ = \{s \in \mathcal{S} : \sigma_s > 0\}.$$

The constants σ_s of the outer functions in (2.1) and (2.2) are obtained as follows.

(i) For s_p of (2.1),

$$\sigma_{s_p} = \begin{cases} 0, & 1 \leq p < \infty, \\ 1, & p = \infty. \end{cases}$$

(ii) For s_c of (2.2),

$$\sigma_{s_c} = \sup\{u \geq 0 : u + c \leq 1\} = 1 - c.$$

Note that $0 \leq c \leq 1$.

As a consequence, we can classify the given outer functions as follows. The outer functions $s_p, s_c \in \mathcal{S}_0$ for $1 \leq p < \infty$, $c = 1$ and $s_p, s_c \in \mathcal{S}_+$ for $p = \infty$, $0 \leq c < 1$.

For more information about Orlicz spaces, their geometry, and their applications, we refer to [2–4, 6–8, 10, 16, 18].

3. Auxiliary results

We recall some technical results that will be used in the rest of the paper.

Lemma 3.1 ([18, Lemma 3.2]). *For every outer function s and Orlicz function Φ,*

$$\|f\|_\Phi = \|f\|_{\Phi,s_\infty} \leq \|f\|_{\Phi,s} \leq \|f\|_{\Phi,s_1} = \|f\|_\Phi^o \leq 2\|f\|_{\Phi,s_\infty} = 2\|f\|_\Phi$$

for all $f \in L_s^\Phi$.

Note that the Orlicz space L_s^Φ is a Banach space with the s-norms.

Theorem 3.2 ([18, Theorem 7.3]). *Let s be an outer function and Φ an Orlicz function.*

(i) *The s-norm is k^*-finite if and only if one of the following conditions is satisfied.*

(a) Φ *takes infinite values, i.e.,* $b_\Phi < \infty$.

(b) $w\big(s'_+(u)\big) = 1$ *for some* $0 < u < \infty$.

(c) $w(1) = 1$ *and* Φ *is not linear on* $[0, \infty)$.

(d) Φ *does not admit an oblique asymptote.*

(ii) *The s-norm is k^{**}-finite if and only if one of the conditions (a), (c), or (d) is satisfied.*

(iii) *If Φ does not admit an oblique asymptote, then the s-norm is k^{**}-finite if and only if it is k^*-finite.*

Corollary 3.3 ([18, Corollary 6.2]). *Let s and Φ be an outer and an Orlicz function, respectively. The following hold for any $f \in L_s^\Phi \setminus \{0\}$.*

(i) *For every $k \in (0, \infty) \cap [k^*(f), k^{**}(f)]$, we have $\|f\|_{\Phi,s} = \frac{1}{k} s(I_\Phi(kf))$.*

(ii) *If $k^{**}(f) = \infty$, then $\|f\|_{\Phi,s} = \lim_{k\to\infty} \frac{1}{k} s(I_\Phi(kf))$.*

Lemma 3.4 ([3, Lemma 4.4]). *Let s be an outer function and Φ an Orlicz function.*

(i) *If $k^*(f) < \infty$ for $f \in L_s^\Phi \setminus \{0\}$, then $(k^*(f))^{-1} \leq \|f\|_{\Phi,s}$.*

(ii) *Let $f \in L_s^\Phi \setminus \{0\}$. If $k^{**}(f) = \infty$, then $k^{**}(f) = \frac{1}{\Theta(f)} = \infty$.*

(iii) *For all $f \in L_s^\Phi \setminus \{0\}$, we obtain $k^*(f) \leq k^{**}(f) \leq \frac{1}{\Theta(f)}$.*

4. Some properties of Orlicz spaces equipped with s-norms

In this section, we give some results for s-norms that generalize the results obtained for the Orlicz norm and the Luxemburg norm [5, 6].

Lemma 4.1. *Let Φ be an Orlicz function and s an outer function. If $b_\Phi < \infty$, then $\|f\|_\infty \leq b_\Phi \|f\|_{\Phi,s}$ and $L_s^\Phi \subset L_\infty$.*

Proof. Assume that $b_\Phi < \infty$. For $f \in L_s^\Phi \setminus \{0\}$, let us define

$$A = \{t \in X : |f(t)| > b_\Phi \|f\|_{\Phi,s}\}.$$

Since $|f|\chi_A > b_\Phi \|f\|_{\Phi,s}$, we obtain $\Phi(\frac{f\chi_A}{\|f\|_{\Phi,s}}) = \infty$ by the definition of b_Φ. By using Lemma 3.1 and the definition of the Luxemburg norm, we have

$$\infty \cdot \mu(A) = I_\Phi\left(\frac{f\chi_A}{\|f\|_{\Phi,s}}\right) \leq I_\Phi\left(\frac{f}{\|f\|_{\Phi,s}}\right) \leq I_\Phi\left(\frac{f}{\|f\|_{\Phi,s_\infty}}\right) \leq 1.$$

Hence, $\mu(A)$ must be 0, i.e.,

$$|f(t)| \leq b_\Phi \, \|f\|_{\Phi,s} \quad \mu\text{ - a.e. } t \in X;$$

this gives us desired one. ∎

The next proposition concerns the situations in which the s-norms are strictly increasing.

Proposition 4.2. *If the Orlicz space L_s^Φ with an $s \in \mathcal{S}_0$ has $a_\Phi = 0$ and $b_\Phi < \infty$, then the s-norm is strictly increasing.*

Proof. Assume that $a_\Phi = 0$ and $b_\Phi < \infty$. By Theorem 3.2, we know that the s-norm is k^{**}-finite. Therefore, for all $f, g \in L_s^\Phi \setminus \{0\}$ with $f < g$, we know that $\|g\|_{\Phi,s} = \frac{1}{k_0} s(I_\Phi(k_0 g))$ for some positive $k_0 < k^{**}(g)$. By the superadditivity of Φ, we have

$$I_\Phi(k_0 g) = I_\Phi(k_0(g - f) + k_0 f) \geq I_\Phi(k_0(g - f)) + I_\Phi(k_0 f).$$

By the definition of the set of outer functions $\mathcal{S}_0$, we have that σ_s is 0. Hence, s is strictly increasing on the half-line $[0, \infty)$. By using that fact that $a_\Phi = 0$, then we obtain

$$\begin{aligned}\|g\|_{\Phi,s} &= \frac{1}{k_0} s(I_\Phi(k_0 g)) > \frac{1}{k_0} s(I_\Phi(k_0(g - f)) + I_\Phi(k_0 f)) > \frac{1}{k_0} s(I_\Phi(k_0 f)) \\ &\geq \inf_{k>0} \frac{1}{k} s(I_\Phi(k f)) = \|f\|_{\Phi,s}.\end{aligned}$$

Therefore, s-norm is strictly increasing. ∎

5. Abstract M-spaces

In this section, we give a description of abstract M-spaces of Orlicz spaces equipped with s-norms. Criteria for abstract M-space of Orlicz spaces equipped with the Orlicz norm and the Luxemburg norm are investigated in [5]. Our results will unify and extend these two classical cases. Our study is based on $s \in \mathcal{S}_0$ and $s \in \mathcal{S}_+$ given in Theorems 5.3 and 5.7.

Definition 5.1. An Orlicz space called L_s^Φ is an abstract M-space if

$$\|\max\{f, g\}\|_{\Phi,s} = \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\} \quad \text{for all } 0 \leq f, g \in L_s^\Phi.$$

We have an equivalent and very useful condition for an abstract M-space in [1]. It says that an Orlicz space is an abstract M-space if and only if

$$\|f + g\|_{\Phi,s} = \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\} \quad \text{for all } f, g \in L_s^\Phi \text{ with } f \perp g,$$

where $f \perp g$ means that $\mu(\operatorname{supp} f \cap \operatorname{supp} g) = 0$ and the support of a function $f \in L_s^\Phi$ is defined by the formula

$$\operatorname{supp} f = \{t \in X : f(t) \neq 0\}.$$

Throughout this paper, abstract M-spaces will be denoted by AM-spaces.

Lemmas 5.2 and 5.6 allow us to extend some results given for the Orlicz norm and the Luxemburg norm in [5]. They will be very useful for proving the main results of this section.

Lemma 5.2. *If the Orlicz space L_s^Φ with $s \in \mathcal{S}_0$ is an AM-space, then $b_\Phi < \infty$.*

Proof. Suppose that L_s^Φ with $s \in \mathcal{S}_0$ is an AM-space. Let us prove $b_\Phi < \infty$. Assume that $b_\Phi = \infty$. For any $c > a_\Phi$, there exists $\varepsilon > 0$ such that $(1+\varepsilon)(a_\Phi + \varepsilon) < c$. Since our measure is non-atomic, choose a measurable subset A of X such that $0 < \mu(A) < \infty$ and

$$s(I_\Phi(c\chi_A)) = s(\Phi(c)\mu(A)) < 1 + \varepsilon.$$

For all $u \geq 0$, we obtain

$$\begin{aligned} \|\chi_A\|_{\Phi,s} &= \inf_{k>0} \frac{1}{k} s(I_\Phi(k\chi_A)) \leq \frac{1}{c} s(I_\Phi(c\chi_A)) \\ &= \frac{1}{c} s(\Phi(c)\mu(A)) < \frac{1+\varepsilon}{c} < \frac{1}{a_\Phi + \varepsilon}. \end{aligned} \tag{5.1}$$

Now, we divide the proof into two cases.

Case 1. Suppose that $k^{**}(\chi_A) < \infty$. Since $k^{**}(\chi_A) < \infty$, there exists $0 < k_0 \leq k^{**}(\chi_A)$ such that $\|\chi_A\|_{\Phi,s} = \frac{1}{k_0} s(I_\Phi(k_0\chi_A))$ by Corollary 3.3.

First, we will show that $I_\Phi(k_0\chi_A) > 0$. If $a_\Phi = 0$, then it is obviously true by the definition of a_Φ. Let $a_\Phi > 0$. Assume that $I_\Phi(k_0\chi_A) = 0$. Then, we have $\Phi(k_0)\mu(A) = 0$, and so, $k_0 \leq a_\Phi$. Therefore, by using the fact that $s(0) = 1$, we obtain

$$\|\chi_A\|_{\Phi,s} = \frac{1}{k_0} s(I_\Phi(k_0\chi_A)) = \frac{1}{k_0} s(0) = \frac{1}{k_0} \geq \frac{1}{a_\Phi},$$

a contradiction to (5.1). Hence, we have $I_\Phi(k_0\chi_A) > 0$. This gives that $0 < \Phi(k_0) < \infty$. Let us take a measurable set $B \subset A$ such that

$$\mu(B) = \mu(A \setminus B) = \frac{1}{2}\mu(A).$$

By assumption, we have $\sigma_s = 0$; then we know that s is a strictly increasing function on the half-line $[0, \infty)$. Therefore,

$$\|\chi_A\|_{\Phi,s} = \frac{1}{k_0} s(I_\Phi(k_0\chi_A)) > \frac{1}{k_0} s(I_\Phi(k_0\chi_B)) \geq \inf_{k>0} \frac{1}{k} s(I_\Phi(k\chi_B)) = \|\chi_B\|_{\Phi,s}.$$

We can prove in the same way that

$$\|\chi_A\|_{\Phi,s} > \|\chi_{A\setminus B}\|_{\Phi,s},$$

and so,

$$\|\chi_B + \chi_{A\setminus B}\|_{\Phi,s} = \|\chi_A\|_{\Phi,s} > \max\{\|\chi_B\|_{\Phi,s}, \|\chi_{A\setminus B}\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ is not an AM-space.

Case 2. Assume that $k^{**}(\chi_A) = \infty$. By Corollary 3.3, we obtain

$$\|\chi_A\|_{\Phi,s} = \lim_{k\to\infty} \frac{1}{k} s(I_\Phi(k\chi_A)) = \lim_{k\to\infty} \frac{1}{k} s(\Phi(k)\mu(A)).$$

Then, for a measurable set $B \subset A$ with $\mu(B) = \mu(A \setminus B) = \frac{1}{2}\mu(A)$, by using the fact that $u \le s(u) \le 1 + u$ for all $u \ge 0$, by Corollary 3.3, we have

$$\begin{aligned}\|\chi_A\|_{\Phi,s} &= \lim_{k\to\infty} \frac{1}{k} s(\Phi(k)\mu(A)) \ge \lim_{k\to\infty} \frac{1}{k}\Phi(k)\mu(A) = 2\lim_{k\to\infty} \frac{1}{k}\Phi(k)\mu(B)\\ &= 2\lim_{k\to\infty} \frac{1}{k}(I_\Phi(k\chi_B)) = 2\lim_{k\to\infty} \frac{1}{k}(1 + I_\Phi(k\chi_B)) \ge 2\lim_{k\to\infty} \frac{1}{k} s(I_\Phi(k\chi_B))\\ &\ge 2\inf_{k>0} \frac{1}{k} s(I_\Phi(k\chi_B)) > \inf_{k>0} \frac{1}{k} s(I_\Phi(k\chi_B)) = \|\chi_B\|_{\Phi,s}.\end{aligned}$$

Similarly, we can prove that $\|\chi_A\|_{\Phi,s} > \|\chi_{A\setminus B}\|_{\Phi,s}$. Therefore,

$$\|\chi_B + \chi_{A\setminus B}\|_{\Phi,s} = \|\chi_A\|_{\Phi,s} > \max\{\|\chi_B\|_{\Phi,s}, \|\chi_{A\setminus B}\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ with an $s \in \mathcal{S}_0$ is not an AM-space, and the proof is complete. ■

We have the following criteria for the Orlicz space equipped with s-norms to be an AM-space in the case $s \in \mathcal{S}_0$.

Theorem 5.3. *The Orlicz space L_s^Φ with $s \in \mathcal{S}_0$ is an AM-space if and only if*

$$0 < a_\Phi, \quad b_\Phi < \infty, \quad \text{and} \quad a_\Phi = b_\Phi. \tag{5.2}$$

Proof. Sufficiency. Suppose that Φ satisfies condition (5.2). By the definition of a_Φ and b_Φ, we obtain

$$\Phi(u) = \begin{cases} 0 & \text{for} \quad 0 \le u \le u_0,\\ \infty & \text{for} \quad u > u_0,\end{cases}$$

where $u_0 = a_\Phi = b_\Phi$. Moreover, for all $f \in L_s^\Phi \setminus \{0\}$ and $k > u_0\|f\|_\infty^{-1}$, we have

$$I_\Phi(kf) = \infty.$$

Therefore, by using the fact that $s(0) = 1$, we obtain

$$\|f\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_\Phi(kf)) = \inf_{0<k\le u_0\|f\|_\infty^{-1}} \frac{1}{k} s(0) = u_0^{-1}\|f\|_\infty$$

from [18]. Hence, we have $\|f\|_{\Phi,s} = u_0^{-1}\|f\|_\infty$. Then, L_s^Φ is linearly isometric to L^∞. By [1], we know that L^∞ is an AM-space. Then, we conclude that L_s^Φ is an AM-space.

Necessity. Suppose that L_s^Φ is an AM-space for any $s \in \mathcal{S}_0$; then we know that $b_\Phi < \infty$ by Lemma 5.2. Assume that $a_\Phi = 0$. Then, we know that s-norm is strictly increasing by Proposition 4.2. Therefore, we obtain

$$\|f + g\|_{\Phi,s} > \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\}$$

for any $f, g \in L_s^\Phi \setminus \{0\}$. This gives the fact that L_s^Φ is not an AM-space, which is a contradiction. Hence, $a_\Phi > 0$.

Now, assume that $0 < a_\Phi < b_\Phi < \infty$. Take $\varepsilon > 0$ such that $(1+\varepsilon)a_\Phi < b_\Phi - \varepsilon$ and choose a measurable subset A of X with $0 < \mu(A) < \infty$ such that

$$I_\Phi((b_\Phi - \varepsilon)\chi_A) = \Phi(b_\Phi - \varepsilon)\mu(A) \leq \varepsilon.$$

Then, we obtain

$$\begin{aligned}\|\chi_A\|_{\Phi,s} &= \inf_{k>0} \frac{1}{k} s(I_\Phi(k\chi_A)) \leq \frac{1}{b_\Phi - \varepsilon} s(I_\Phi((b_\Phi - \varepsilon)\chi_A)) \\ &\leq \frac{1}{b_\Phi - \varepsilon}(1 + I_\Phi((b_\Phi - \varepsilon)\chi_A)) \leq \frac{1+\varepsilon}{b_\Phi - \varepsilon} < \frac{1}{a_\Phi}.\end{aligned}$$

Similarly, as in the proof of Lemma 5.2,

$$\|\chi_B + \chi_{A\setminus B}\|_{\Phi,s} = \|\chi_A\|_{\Phi,s} > \max\{\|\chi_B\|_{\Phi,s}, \|\chi_{A\setminus B}\|_{\Phi,s}\},$$

where for a measurable set $B \subset A$ with $\mu(B) = \mu(A \setminus B) = \frac{1}{2}\mu(A)$. Hence, we conclude that L_s^Φ with any $s \in \mathcal{S}_0$ is not an AM-space, and the proof is complete. ∎

By [1], we know that L^∞ is an AM-space. Then, we conclude that the following corollary from Theorem 5.3.

Corollary 5.4. *The Orlicz space L_s^Φ with $s \in \mathcal{S}_0$ is an AM-space if and only if L_s^Φ is linearly isometric to L^∞.*

We can give the following example.

Example 5.5. For $u \in \mathbb{R}$, let $s_p(u) = (1+u^p)^{\frac{1}{p}}$, where $1 \leq p < \infty$ and

$$\Phi(u) = \begin{cases} 0 & \text{for } |u| \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

We have $s_p \in \mathcal{S}_0$ and $a_\Phi = b_\Phi = 1$. By Theorem 5.3, $L_{s_p}^\Phi$ is an AM-space. Hence, $L_{s_p}^\Phi$ is just equal to L^∞.

Lemma 5.6. *If the Orlicz space L_s^Φ with $s \in \mathcal{S}_+$ is an AM-space, then $b_\Phi < \infty$.*

Proof. Suppose that L_s^Φ with an $s \in \mathcal{S}_+$ is an AM-space. Let us prove $b_\Phi < \infty$. First, assume that $b_\Phi = \infty$. Take disjoint measurable subsets A, B of X and a number $c > a_\Phi$ such that $\Phi(c)\mu(A) = \sigma_s$ and $\Phi(c)\mu(B) = \sigma_s$. Define

$$f = c\chi_A, \quad g = c\chi_B.$$

Then, we have $I_\Phi(f) = I_\Phi(g) = \sigma_s$ and $I_\Phi(f+g) > \sigma_s$. Now, we divide the proof into two cases.

Case 1. Assume that $k^{**}(f+g) < \infty$. We investigate the following three situations. Assume that $k^*(f+g) \leq 1 \leq k^{**}(f+g)$. Note that s is strictly increasing on $[\sigma_s, \infty)$, so by Corollary 3.3, we have

$$\|f+g\|_{\Phi,s} = s(I_\Phi(f+g)) > s(I_\Phi(f)) \geq \inf_{k>0} \frac{1}{k} s(I_\Phi(kf)) = \|f\|_{\Phi,s}.$$

We can prove in the same way that

$$\|f+g\|_{\Phi,s} > \|g\|_{\Phi,s},$$

and so,

$$\|f+g\|_{\Phi,s} > \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ with an $s \in \mathcal{S}_+$ is not an AM-space.

Second, assume that $k^{**}(f+g) < 1$. We obtain

$$1 < (k^{**}(f+g))^{-1} \leq \|f+g\|_{\Phi,s}$$

from Lemma 3.4. By using the fact that

$$I_\Phi(f) = I_\Phi(g) = \sigma_s,$$

then we have $\|f\|_{\Phi,s} \leq s(I_\Phi(f)) = 1$, and similarly, $\|g\|_{\Phi,s} \leq 1$. Therefore, we have

$$\|f+g\|_{\Phi,s} > \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ with any $s \in \mathcal{S}_+$ is not an AM-space.

Finally, assume that $k^*(f+g) > 1$. We have $I_\Phi(k^*(f+g)(f+g)) > \sigma_s$ and $I_\Phi(k^*(f+g)f) \geq \sigma_s$. Note that s is a strictly increasing function on $[\sigma_s, \infty)$, so by Corollary 3.3, we have

$$\begin{aligned}\|f\|_{\Phi,s} &\leq \frac{1}{k^*(f+g)} s(I_\Phi(k^*(f+g)f)) \\ &< \frac{1}{k^*(f+g)} s(I_\Phi(k^*(f+g)(f+g))) = \|f+g\|_{\Phi,s}.\end{aligned}$$

We can prove in the same way that

$$\|f+g\|_{\Phi,s} > \|g\|_{\Phi,s},$$

and so,

$$\|f+g\|_{\Phi,s} > \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ with any $s \in \mathcal{S}_+$ is not an AM-space.

Case 2. Assume that $k^{**}(f+g)=\infty$. Therefore, by using the fact that $u \le s(u) \le 1+u$ for all $u \ge 0$, by Corollary 3.3, we have

$$\begin{aligned}\|f+g\|_{\Phi,s} &= \lim_{k\to\infty}\frac{1}{k}s(I_\Phi(k(f+g))) \ge \lim_{k\to\infty}\frac{1}{k}(\Phi(kc)\mu(A\cup B))\\ &= 2\lim_{k\to\infty}\frac{1}{k}(\Phi(kc)\mu(A)) = 2\lim_{k\to\infty}\frac{1}{k}(I_\Phi(kf))\\ &= 2\lim_{k\to\infty}\frac{1}{k}(1+I_\Phi(kf)) \ge 2\lim_{k\to\infty}\frac{1}{k}s(I_\Phi(kf))\\ &\ge 2\inf_{k>0}\frac{1}{k}s(I_\Phi(kf)) > \inf_{k>0}\frac{1}{k}s(I_\Phi(kf)) = \|f\|_{\Phi,s}.\end{aligned}$$

Similarly, $\|f+g\|_{\Phi,s} > \|g\|_{\Phi,s}$. Therefore,

$$\|f+g\|_{\Phi,s} > \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ with any $s \in \mathcal{S}_+$ is not an AM-space, and the proof is complete. ∎

Note that if Orlicz space L_s^Φ is an AM-space, then $k^*(f)$ is finite by Theorem 3.2.

We have the following criteria for the Orlicz space equipped with s-norms to be AM-space in the case $s \in \mathcal{S}_+$.

Theorem 5.7. *For an Orlicz function Φ and an $s \in \mathcal{S}_+$, the following are equivalent.*

(i) *The Orlicz space L_s^Φ is an AM-space.*

(ii) *$b_\Phi < \infty$ and $\Phi(b_\Phi)=0$ if $\mu(X)=\infty$ or $\Phi(b_\Phi)\mu(X) \le \sigma_s$ if $\mu(X)<\infty$.*

(iii) *L_s^Φ is linearly isometric to L^∞.*

Proof. Let us prove (i)⇒(ii). Suppose that L_s^Φ with an $s \in \mathcal{S}_+$ is an AM-space. By using Lemma 5.6, we have $b_\Phi < \infty$. Assume that $\Phi(b_\Phi)\mu(X) > \sigma_s$. Then, we obtain $\Phi(b_\Phi) > 0$. Take disjoint measurable subsets A, B of X and a number $c > a_\Phi$ such that $\Phi(c)\mu(A) = \sigma_s$ and $\Phi(c)\mu(B) = \sigma_s$. Define

$$f = c\chi_A, \quad g = c\chi_B.$$

Then, we have $I_\Phi(f) = I_\Phi(g) = \sigma_s$ and $I_\Phi(f+g) > \sigma_s$. By using the fact that $b_\Phi < \infty$, then we know that s-norm is k^{**}-finite from Corollary 3.3. Since $k^{**}(f+g) < \infty$, similarly, as in the proof of Lemma 5.6,

$$\|f+g\|_{\Phi,s} > \max\{\|f\|_{\Phi,s}, \|g\|_{\Phi,s}\}.$$

Hence, we conclude that L_s^Φ with an $s \in \mathcal{S}_+$ is not an AM-space.

Now, let us prove (ii)⇒(iii). Assume that $f \in L^\infty$. Then, by (ii),

$$I_\Phi\left(\frac{b_\Phi|f|}{\|f\|_\infty}\right) = \int_X \Phi\left(\frac{b_\Phi|f|}{\|f\|_\infty}\right) \le \int_X \Phi(b_\Phi) = \Phi(b_\Phi)\mu(X) \le \sigma_s.$$

Therefore,

$$\|f\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_\Phi(kf)) \le \frac{\|f\|_\infty}{b_\Phi} s\left(I_\Phi\left(\frac{b_\Phi |f|}{\|f\|_\infty}\right)\right) \le \frac{\|f\|_\infty}{b_\Phi}.$$

Then, $\|f\|_{\Phi,s} \le b_\Phi^{-1}\|f\|_\infty$, and so, $f \in L_s^\Phi$. On the other hand, take $f \in L_s^\Phi$. Since $b_\Phi < \infty$, by using Lemma 4.1, we obtain

$$b_\Phi^{-1}\|f\|_\infty \le \|f\|_{\Phi,s}.$$

Since the opposite inequality was also proved, we obtain the equality

$$\|f\|_{\Phi,s} = b_\Phi^{-1}\|f\|_\infty.$$

Therefore, L_s^Φ is linearly isometric to L^∞.

Finally, let us prove (iii)⇒(i). By using the fact that L^∞ is an AM-space [1], we obtain that L_s^Φ is an AM-space. ∎

In the following corollary, we specialize Theorems 5.3 and 5.7 to some special outer functions.

Corollary 5.8. (i) *For $s = s_p$ with $p = 1$, the corresponding Orlicz space is an AM-space if and only if $0 < a_\Phi, b_\Phi < \infty$, and $a_\Phi = b_\Phi$.*

(ii) *For $s = s_p$ with $1 < p < \infty$, the corresponding Orlicz space is an AM-space if and only if $0 < a_\Phi, b_\Phi < \infty$, and $a_\Phi = b_\Phi$.*

(iii) *For $s = s_p$ with $p = \infty$, the corresponding Orlicz space is an AM-space if and only if it provides one of the following.*

(a) *$b_\Phi < \infty$ and $\Phi(b_\Phi) = 0$ if $\mu(X) = \infty$ or $\Phi(b_\Phi)\mu(X) \le 1$ if $\mu(X) < \infty$.*

(b) *L_s^Φ is linearly isometric to L^∞.*

(iv) *For $s = s_c$ with $0 \le c < 1$, the corresponding Orlicz space is an AM-space if and only if it provides one of the following.*

(a) *$b_\Phi < \infty$ and $\Phi(b_\Phi) = 0$ if $\mu(X) = \infty$ or $\Phi(b_\Phi)\mu(X) \le 1 - c$ if $\mu(X) < \infty$.*

(b) *L_s^Φ is linearly isometric to L^∞.*

The statements (i) and (iii) of this corollary are obtained in [5]. Theorems 5.3 and 5.7 generalize and unify all the cases of Orlicz, Luxemburg, and p-Amemiya norms for $1 < p < \infty$.

We will indicate the connection between the relations of Theorems 5.3 and 5.7 in the following corollary.

Corollary 5.9. *For any Orlicz function Φ, measure space X, and outer function s, the Orlicz space L_s^Φ is an AM-space if and only if it is linearly isometric to the space L^∞.*

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Badik Hüseyin Uysal
Department of Mathematics, Faculty of Science, İstanbul University, Vezneciler, 34134 İstanbul; Institute of Graduate Studies in Sciences, İstanbul University, Süleymaniye, 34116 İstanbul, Turkey; huseyinuysal@istanbul.edu.tr

Serap Öztop
Department of Mathematics, Faculty of Science, İstanbul University, Vezneciler, 34134 İstanbul, Turkey; oztops@istanbul.edu.tr