Z. Anal. Anwend. 44 (2025), 235–250
DOI 10.4171/ZAA/1789

Fractional Schrödinger–Poisson system involving concave and convex nonlinearities

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Abstract. In this paper, we investigate the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = P(x)|u|^{p-2}u - Q(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $1 < p < 2 < q < +\infty$, $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (\frac{3}{4}, 1)$, and $V(x)$, $P(x)$, and $Q(x)$ are given functions satisfying certain conditions. We aim to establish the existence of infinitely many solutions for this system, considering nonlinearities $P(x)|u|^{p-2}u$ and $Q(x)|u|^{q-2}u$ with varying growth rates, including subcritical, critical, and supercritical cases.

1. Introduction and main results

In this paper, we study the existence of solutions for the following fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = P(x)|u|^{p-2}u - Q(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $1 < p < 2 < q < +\infty$ and $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (\frac{3}{4}, 1)$. When $s = \frac{1}{2}$, this system becomes particularly intriguing from a physical perspective. It arises in the semi-relativistic theory within the context of repulsive Coulomb interactions in plasma physics (see, for instance, [1]). By substituting the second equation into the first, the system reduces to the semi-relativistic Hartree equation, which plays a significant role in the quantum theory of boson stars [19].

When $\phi(x) = 0$ in this system, equation (1.1) reduces to the fractional Schrödinger equation like:

$$(-\Delta)^s u + V(x)u = P(x)|u|^{p-2}u - Q(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

which is related to standing wave solutions of the fractional time-dependent Schrödinger

Mathematics Subject Classification 2020: 35J50 (primary); 35Q40 (secondary).
Keywords: fractional Schrödinger–Poisson equation, infinitely many solutions, variational methods.

equation of the form

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^s\psi + V(x)\psi - f(x,|\psi|), \quad x \in \mathbb{R}^N,$$

which is a fundamental equation in fractional quantum mechanics (see [18]). It is well known that, different from the classical Laplacian operator, the usual analysis tools for elliptic PDEs cannot be directly applied to (1.2) since $(-\Delta)^s$ is a nonlocal operator.

When $s = 1$, (1.1) is the following classical Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \mu\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

which was proposed by Benci and Fortunato [7] in 1998 on a bounded domain and is related to the Hartree equation [21]. In the past several years, the existence and multiplicity of solutions to the systems similar to (1.3) have been studied extensively by means of variational tools; we refer the interested readers to see [2, 4, 5, 9, 13, 15, 39] and the references therein. In particular, when $f(x,u) = u^{p-1}(2 < p < 6)$, $V \equiv 1$, and $\mu > 0$ is a positive parameter, Ruiz [26] obtained some general results about existence and nonexistence of positive solutions. In the case $p < 4$, the problem (1.3) becomes more delicate since the corresponding energy functional does not possess the mountain pass geometry in general. To overcome this difficulty, Ruiz considered a new constrained minimization problem on a new manifold which is obtained by combining the usual Nehari manifold and the Pohožaev's identity. After that, Wang, and Zhou [31] proved that (1.3) has a positive solution for μ small and has no nontrivial solution for μ large when the nonlinearity $f(x,u)$ is asymptotically linear with respect to u at infinity. The existence of solutions of (1.3) involving nonconstant positive potentials was considered independently in [6,40]. Ambrosetti and Ruiz [3] constructed multiple solutions to (1.3) with a potential vanishing at infinity. A system under the effect of a general nonlinear term was considered in [4,5]. The existence of sign-changing solutions for (1.3) was established in [10,12,14,16,27,32] under different conditions on $V(x)$ and $f(x,u)$. Recently, in [24], Liu, Wang and Zhang obtained the existence of infinitely many sign-changing solutions to (1.3) with a general nonlinearity $f(u) \sim |u|^{p-1}u$ $(3 < p < 5)$ and a coercive potential by using the method of invariant sets of descending flow.

Recently, there is an increasing interest in the existence of solutions to the fractional Schrödinger–Poisson system. A fractional Schrödinger–Poisson system with $V = 0$ and a general nonlinearity in the subcritical and critical case was considered in [38]. In [29, 30], Teng adapted the monotonicity trick to obtain the existence of ground state solutions to critical and subcritical cases, respectively. For the other results about existence and concentration of solutions, we refer to [20, 23, 25, 34–37] and the references therein.

In this paper, we are concerned with the existence of multiplicity of solutions for (1.1) with concave and convex nonlinearities which have subcritical, critical, or supercritical growth. To the best of our knowledge, there are no results in this direction for fractional problems. We assume that $V(x)$, $P(x)$, and $Q(x)$ are three measurable functions satisfying the following conditions.

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $0 < \inf_{x \in \mathbb{R}^3} V(x)$.

(P) $P \in L^{\frac{6}{6-(3-2s)p}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $P(x) \geq 0$ is not identically zero.

(Q) $Q \in L^\infty(\mathbb{R}^3)$ and $Q(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$.

Now, we state our main results as follows.

Theorem 1.1. *Assume that $1 < p < 2 < q < +\infty$ and (V), (P), and (Q) hold; then, system (1.1) admits infinitely many solutions in $H^s(\mathbb{R}^3) \times \mathcal{D}^{s,2}(\mathbb{R}^3)$ with negative energy.*

As was remarked in a previous column, when $\phi(x) = 0$ in (1.1), the system (1.1) can reduce to equation (1.2); then, we have the following result.

Corollary 1.1. *Assume that $1 < p < 2 < q < +\infty$ and (V), (P), and (Q) hold; then, equation (1.2) admits infinitely many solutions in $H^s(\mathbb{R}^3)$ with negative energy.*

The main difficulty is the loss of compactness of the Sobolev embedding when we work on $\mathbb{R}^3$. Moreover, since $2 < p < +\infty$ is allowed to be supercritical, the usual space $H^s(\mathbb{R}^3)$ cannot be used as our framework for the study of problem (1.1). To overcome these difficulties, we introduce a new space which is similar to that in the paper [22, 28].

This paper is organized as follows. In Section 2, besides describing the functional setting to study problem (1.1), we prove some preliminary lemmas which will be used later. In Section 3, we give the proof of Theorem 1.1.

Notation. In this paper, we make use of the following notations.

- The letter C stands for any positive constants.
- "$\to$" and "$\rightharpoonup$" represent strong convergence and weak convergence, respectively.
- $o_n(1)$ is a quantity tending to 0 as $n \to \infty$.
- $\|\cdot\|_r$ is the usual norm of the space $L^r(\mathbb{R}^3)$.
- $B_r(x)$ denotes the open ball with center at x and radius r.

2. Variational settings and preliminary results

Firstly, fractional Sobolev spaces are the convenient setting for our problem, so we will give some sketches of the fractional-order Sobolev spaces and the complete introduction can be found in [11]. We recall that, for any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined as follows:

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} |\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi < \infty \right\},$$

whose norm is defined as

$$\|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \right)^{\frac{1}{2}} = [u]_{H^s(\mathbb{R}^3)}.$$

The embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_s}(\mathbb{R}^3)$ is continuous, and for any $s \in (0,1)$, there exists a best constant $S_s > 0$ such that

$$S_s := \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)}}{\|u\|^2_{2^*_s(\mathbb{R}^3)}}.$$

The fractional laplacian, $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3,$$

that is,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi\cdot x}\phi(x)dx,$$

for functions ϕ in the Schwartz class. Also, $(-\Delta)^s u$ can be equivalently represented [11] as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(s)\int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}dy, \quad \forall x \in \mathbb{R}^3,$$

where

$$C(s) = \left(\int_{\mathbb{R}^3} \frac{(1-cos\xi_1)}{|\xi|^{3+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Also, by the Plancherel formula in Fourier analysis, we have

$$[u]^2_{H^s(\mathbb{R}^3)} = \frac{2}{C(s)} \|(-\Delta)^{\frac{s}{2}} u\|^2_2.$$

As a consequence, the norms on $H^s(\mathbb{R}^3)$ defined above,

$$u \mapsto \left(\int_{\mathbb{R}^3} |u|^2 dx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \right)^{\frac{1}{2}},$$

$$u \mapsto \left(\int_{\mathbb{R}^3} (|\xi|^{2s}|\mathcal{F}(u)|^2 + |\mathcal{F}(u)|^2) d\xi \right)^{\frac{1}{2}},$$

$$u \mapsto \left(\int_{\mathbb{R}^3} |u|^2 dx + \|(-\Delta)^{\frac{s}{2}} u\|^2_2 \right)^{\frac{1}{2}}$$

are equivalent.

It is known that problem (1.1) can be reduced to a single equation; see [7]. In fact, for a fixed $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi_u^s \in \mathcal{D}^{s,2}(\mathbb{R}^3)$ which is the solution of

$$(-\Delta)^s \phi = u^2 \quad \text{in } \mathbb{R}^3.$$

We can write an integral expression for ϕ_u^s in the form

$$\phi_u^s(x) = C_s \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2s}} dy, \quad \forall x \in \mathbb{R}^3,$$

which is called s-Riesz potential (see [17]), where

$$C_s = \frac{1}{\pi^{\frac{3}{2}}} \frac{\Gamma(3-2s)}{2^{2s}\Gamma(s)}.$$

To be more precise about the solution ϕ of the fractional Poisson equation, we have the following lemma.

Lemma 2.1 ([29]). *For any $u \in H^s(\mathbb{R}^3)$ and $s \geq \frac{1}{2}$, we have the following:*

(i) $\phi_u^s \geq 0$;

(ii) $\phi_u^s : H^s(\mathbb{R}^3) \to \mathcal{D}^{s,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*

(iii) $\int_{\mathbb{R}^3} \phi_u^s u^2 dx \leq S_s^2 \|u\|^4_{\frac{12}{3+2s}} \leq C\|u\|^4$;

(iv) *If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^s \rightharpoonup \phi_u^s$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$;*

(v) *If $u_n \to u$ in $H^s(\mathbb{R}^3)$, then $\phi_{u_n}^s \to \phi_u^s$ in $\mathcal{D}^{s,2}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^s u^2 dx.$$

Define $N : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \int_{\mathbb{R}^3} \phi_u^s u^2 dx$$

it is clear that $N(u(\cdot + y)) = N(u)$ for any $y \in \mathbb{R}^3$, $u \in H^s(\mathbb{R}^3)$ and N is weakly lower semi-continuous in $H^s(\mathbb{R}^3)$. Moreover, similar to the well-known Brézis–Lieb lemma [8], we have the next lemma.

Lemma 2.2 ([29]). *Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$ with $s > \frac{3}{4}$. Then,*

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;

(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $(H^s(\mathbb{R}^3))^{-1}$.

Putting $\phi = \phi_u^s$ into the first equation of (1.1), we obtain a semilinear elliptic equation

$$(-\Delta)^s u + V(x)u + \phi_u^s u = P(x)|u|^{p-2}u - Q(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3,$$

with a nonlocal term. Note that if $s \geq \frac{1}{2}$, there holds $2 \leq \frac{12}{3+2s} \leq 2_s^*$ and thus $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2s}}(\mathbb{R}^3)$; then, by Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \phi_u^s u^2 dx &\leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2s}} dx\right)^{\frac{3+2s}{6}} \left(\int_{\mathbb{R}^3} |\phi_u^s|^{2_s^*} dx\right)^{\frac{1}{2_s^*}} \\ &\leq S_s^{-\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2s}} dx\right)^{\frac{3+2s}{6}} \|\phi_u^s\|_{\mathcal{D}^{s,2}} \\ &\leq C\|u\|^2 \|\phi_u^s\|_{\mathcal{D}^{s,2}} < \infty.\end{aligned}$$

In view of the presence of potential $V(x)$, we introduce the subspace

$$H = \left\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\right\},$$

which is a Hilbert space equipped with the inner product

$$(u, v)_H = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^3} V(x)uv dx,$$

and the norm

$$\|u\|_H^2 = (u, u) = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^3} V(x)u^2 dx.$$

Unfortunately, since $1 < p < 2 < q < +\infty$, which means the nonlinearities have subcritical, critical, or supercritical growth, we cannot use H as our framework for the study of system (1.1); then, we have to introduce a new space. For the nonnegative measurable function $P(x)$ and $1 < p < +\infty$, we define the weighted Lebesgue space

$$L_P^p(\mathbb{R}^3) := \left\{u \text{ is measurable} : \int_{\mathbb{R}^3} P(x)|u|^p dx < \infty\right\}$$

and associate with the seminorm

$$|u|_{p,P}^p = \int_{\mathbb{R}^3} P(x)|u|^p dx. \tag{2.1}$$

Motivated by [22], let $L_P^p(\mathbb{R}^3)$ be defined as (2.1) and E the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\| = \|u\|_H + |u|_{p,P}.$$

Then, E is a Banach space; however, it is difficult to judge whether E is reflexive as pointed out in [22]. The following inclusions also hold: $C_0^\infty \subset E \subset \mathcal{D}^{s,2} \cap L_P^p$. Moreover, from the following lemma, we have some compactness of E.

Lemma 2.3. *Suppose $P(x) \in L^{\frac{2N}{2N-(N-2)p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $1 < p < 2_s^*$. Then, for every $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, the equation*

$$(-\Delta)^s w = P(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

possesses a unique solution $w \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Further, the operator

$$K_P^p : \mathcal{D}^{s,2}(\mathbb{R}^N) \mapsto \mathcal{D}^{s,2}(\mathbb{R}^N)$$

defined by $K_P^p(u) = w$ is compact.

Proof. We define the linear form

$$\Psi v = \int_{\mathbb{R}^N} P(x)|u|^{p-2}uv\,dx, \quad \forall u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Obviously, Ψ is continuous, and by Riesz's representation lemma, there exists a unique $w \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} w (-\Delta)^{\frac{s}{2}} v\,dx = \int_{\mathbb{R}^N} P(x)|u|^{p-2}uv\,dx, \quad \forall v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

which means that w is a weak solution of

$$(-\Delta)^s w = P(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Next, we prove the compactness of the operator K_P^p; taking $\{u_n\} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ as a bounded sequence, up to a subsequence, we may assume that $u_n \rightharpoonup u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Let $w_n = K_P^p(u_n)$ and $w = K_P^p(u)$; we have

$$\begin{aligned}\|w_n - w\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(w_n - w)\,dx \\ &\le \big\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|_{\frac{2N}{N+2s}} \|w_n - w\|_{\frac{2N}{N-2s}} \\ &\le C_1 \big\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|_{\frac{2N}{N+2s}} \|w_n - w\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)};\end{aligned}$$

then,

$$\|w_n - w\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le C_1 \big\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|_{\frac{2N}{N+2s}}.$$

After that, we only need to prove that the last norm tends to zero. In fact, for an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned}\big\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|^{\frac{2N}{N+2s}}_{\frac{2N}{N+2s}} &= \big\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|^{\frac{2N}{N+2s}}_{L^{\frac{2N}{N+2s}}(B_R)} \\ &\quad + \big\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|^{\frac{2N}{N+2s}}_{L^{\frac{2N}{N+2s}}(\mathbb{R}^N \setminus B_R)}.\end{aligned}$$

Note that

$$\big\|\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\big\|_{L^{\frac{2N}{(N-2s)(r-1)}}(\mathbb{R}^N \setminus B_R)} \le C_2;$$

we can choose a ball B_R with enough large radius to obtain

$$\|P(x)\|^{\frac{2N}{N+2s}}_{L^{\frac{2N}{2N-(N-2s)r}}(\mathbb{R}^N \setminus B_R)} \le \frac{\varepsilon}{2C_2^{\frac{2N}{N+2s}}}.$$

Therefore,

$$\begin{aligned}&\left\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\right\|_{\frac{2N}{N+2s}}^{\frac{2N}{N+2s}} \\ &\quad \le \left\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\right\|_{L^{\frac{2N}{N+2s}}(B_R)}^{\frac{2N}{N+2s}} + \frac{\varepsilon}{2}.\end{aligned}$$

Moreover, due to the compact embedding of $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^r(B_R)$ with $r \in [2, 2_s^*)$ (see, [11]), we have $u_n \to u$ in $L^{\frac{2N}{N+2s}}(B_R)$, and then,

$$\left\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\right\|_{L^{\frac{2N}{N+2s}}(B_R)}^{\frac{2N}{N+2s}} < \frac{\varepsilon}{2}$$

for n sufficiently large. Thus, we deduce that

$$\left\| P(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)\right\|_{\frac{2N}{N+2s}}^{\frac{2N}{N+2s}} < \varepsilon$$

for n sufficiently large. This means $\|P(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{\frac{2N}{N+2s}} \to 0$, and we conclude that $w_n \to w$, which completes the proof. ■

Lemma 2.4. *If (V) and (P) hold, then the embedding $E \hookrightarrow L_P^p(\mathbb{R}^3)$ is compact.*

Proof. It follows from Lemma 2.3 that the embedding $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L_P^p(\mathbb{R}^3)$ $(p \in (1,2))$ is compact. Then, the compactness of $E \hookrightarrow L_P^p(\mathbb{R}^3)$ follows from the compactness of $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L_P^p(\mathbb{R}^3)$ $(p \in (1,2))$ and the continuity of $E \hookrightarrow \mathcal{D}^{s,2}(\mathbb{R}^3)$ (see [11]). ■

The corresponding functional $I : E \to \mathbb{R}$ is defined by

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3} P(x)|u|^p dx + \frac{1}{q}\int_{\mathbb{R}^3} Q(x)|u|^q dx \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} P(x)|u|^p dx + \frac{1}{q}\int_{\mathbb{R}^3} Q(x)|u|^q dx. \end{aligned}$$

Therefore, by Lemma 2.4, the functional I is well defined for every $u \in E$ and belongs to $C^1(E, \mathbb{R})$. Moreover, for any $u, v \in E$, we have

$$\begin{aligned} \langle I'(u), v\rangle = &\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v dx + \int_{\mathbb{R}^3} V(x)uv dx + \int_{\mathbb{R}^3} \phi_u^s uv dx \\ &- \int_{\mathbb{R}^3} |u|^{p-2}uv dx - \int_{\mathbb{R}^3} |u|^{q-2}uv dx. \end{aligned}$$

It is standard to verify that a critical point u of the functional I corresponds to a weak solution (u, ϕ) of (1.1) with $\phi = \phi_u^s$. Hence, in the following, we consider critical points of I using the variational method.

3. Proof of Theorem 1.1

As has been mentioned in above, we only need to prove that I has infinitely many critical points. Let E be a real Banach space and the functional $I \in C^1(E, \mathbb{R})$. Recall that the functional I is said to satisfy Palais–Smale (for short (PS)) condition if every sequence $\{u_n\} \subset E$ with

$$I(u_n) \text{ being bounded}, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty, \tag{3.1}$$

possesses a convergent subsequence. Moreover, we need the following critical point theorem to complete our proof.

Theorem 3.1 ([33]). *Suppose $I(0) = 0$ and I satisfies the (PS) condition and is even and bounded from below. If, for any $n \in \mathbb{N}$, there exists a n-dimensional subspace E^n and $\rho_n > 0$ such that*

$$\sup_{E^n \cap S_{\rho_n}} I < 0,$$

where $S_{\rho_n} = \{u \in E : \|u\| = \rho_n\}$, then I has a sequence of critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

To apply the above critical point theorem, the (PS) condition is important. First, we will verify this condition for $I : E \to \mathbb{R}$. For this, we have the following useful inequalities.

Lemma 3.1. *Given $\alpha, \beta > 0$, there is a $C > 0$ such that*

$$\alpha\|u\|_H^2 + \beta \int_{\mathbb{R}^3} Q(x)|u|^q dx \le C(\|u\|^2 + \|u\|^q)$$

and

$$\alpha\|u\|_H^2 + \beta \int_{\mathbb{R}^3} Q(x)|u|^q dx \ge \begin{cases} C\|u\|^2 & \text{if } \|u\| \ge 1, \\ C\|u\|^q & \text{if } \|u\| \le 1. \end{cases}$$

Proof. This conclusion follows easily from the definition of $\|\cdot\|_H$ and $\|\cdot\|$. ∎

Lemma 3.2. *Under the assumptions of Theorem 1.1, the function I is coercive on E.*

Proof. By the best Sobolev constant and (P), we have

$$\int_{\mathbb{R}^3} P(x)|u|^p dx \le \|P\|_{\frac{6}{6-(3-2s)p}} S_s^{-\frac{p}{2}} \|u\|^p_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le C\|u\|^p. \tag{3.2}$$

For $\|u\|$ large enough, Lemma 3.1 together with (3.2) implies

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} P(x)|u|^p dx + \frac{1}{q}\int_{\mathbb{R}^3} Q(x)|u|^q dx \\ &\ge C\|u\|^2 - C\|u\|^p, \end{aligned}$$

which implies $I(u) \to +\infty$ as $\|u\| \to +\infty$. ∎

In general, to prove the (PS) condition, the reflexivity of the space is needed. However, we do not know whether E is reflexive. We borrow some ideas from [22] to avoid this difficulty.

Lemma 3.3. *Under the assumptions of Theorem 1.1, the function I satisfies the (PS) condition.*

Proof. It follows from Lemma 3.2 that every sequence $\{u_n\}$ satisfying (3.1) is bounded in E. Thus, $\{u_n\}$ is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^3)$, and we can assume that for some $u \in E$, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H, \\ u_n &\to u \text{ in } L^r_{\mathrm{loc}}(\mathbb{R}^3), \quad 2 \le r < 2^*_s, \\ u_n(x) &\to u(x) \; a.e. \text{ in } \mathbb{R}^3. \end{aligned} \tag{3.3}$$

Claim 1. $I'(u) = 0$.

For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, since

$$\langle I'(u_n), \varphi \rangle = o_n(1)\|\varphi\|,$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi + V(x) u_n \varphi) dx + \int_{\mathbb{R}^3} \phi^s_{u_n} u_n \varphi dx \\ &- \int_{\mathbb{R}^3} P(x)|u_n|^{p-2} u_n \varphi dx + \int_{\mathbb{R}^3} Q(x)|u_n|^{q-2} u_n \varphi dx \\ &= o_n(1)\|\varphi\|. \end{aligned} \tag{3.4}$$

We only need to prove the following convergence:

$$\int_{\mathbb{R}^3} \phi^s_{u_n} u_n \varphi dx \to \int_{\mathbb{R}^3} \phi^s_u u \varphi dx, \tag{3.5}$$

$$\int_{\mathbb{R}^3} P(x)|u_n|^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^3} P(x)|u|^{p-2} u \varphi dx, \tag{3.6}$$

and

$$\int_{\mathbb{R}^3} Q(x)|u_n|^{q-2} u_n \varphi dx \to \int_{\mathbb{R}^3} Q(x)|u|^{q-2} u \varphi dx. \tag{3.7}$$

In fact, it follows from Lemma 2.1 that (3.5) holds. Then, we prove (3.6) and (3.7). Since $\{u_n\}$ is bounded in E, then $\{P^{\frac{p-1}{p}} |u_n|^{p-2} u_n\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$. From (3.3), up to a subsequence, we can assume that

$$P^{\frac{p-1}{p}} |u_n|^{p-2} u_n \rightharpoonup P^{\frac{p-1}{p}} |u|^{p-2} u \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^3).$$

This together with $P^{\frac{1}{p}} \varphi \in L^p(\mathbb{R}^3)$ implies that (3.6). Similarly, (3.7) holds.

Therefore, letting $n \to \infty$ in (3.4), we have

$$\int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + V(x) u \varphi) dx + \int_{\mathbb{R}^3} \phi_u^s u \varphi dx$$
$$- \int_{\mathbb{R}^3} P(x)|u|^{p-2} u \varphi dx + \int_{\mathbb{R}^3} Q(x)|u|^{q-2} u \varphi dx = 0,$$

which implies that $I'(u) = 0$.

Claim 2. $u_n \to u$ in H.

From $\langle I'(u_n), u_n \rangle = o_n(1)\|u_n\|$, $\langle I'(u), u \rangle = 0$ and Lemma 2.4, we have

$$\begin{aligned} \lim_{n\to\infty} &\left(\|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx + \int_{\mathbb{R}^3} Q(x)|u_n|^q dx \right) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^3} P(x)|u_n|^p dx \\ &= \int_{\mathbb{R}^3} P(x)|u|^p dx \\ &= \|u\|_H^2 + \int_{\mathbb{R}^3} \phi_u^s u^2 dx + \int_{\mathbb{R}^3} Q(x)|u|^q dx. \end{aligned} \tag{3.8}$$

Note that $\|\phi_{u_n}^s\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le C\|u_n\|^2_{\frac{12}{3+2s}}$; we know that $\phi_{u_n}^s$ is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^3)$ and then

$$\phi_{u_n}^s \rightharpoonup \phi_u^s \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^3).$$

Thus, by the weak semi-continuity of norm, we have

$$\int_{\mathbb{R}^3} \phi_u^s u^2 dx = \|\phi_u^s\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} \le \liminf_{n\to\infty} \|\phi_{u_n}^s\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \liminf_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx. \tag{3.9}$$

Further, by Fatou's lemma, one has

$$\int_{\mathbb{R}^3} Q(x)|u|^q dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} Q(x)|u_n|^q dx. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} \lim_{n\to\infty} &\left(\|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx + \int_{\mathbb{R}^3} Q(x)|u_n|^q dx \right) \\ &\ge \liminf_{n\to\infty} \|u_n\|_H^2 + \liminf_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx + \liminf_{n\to\infty} \int_{\mathbb{R}^3} Q(x)|u_n|^q dx \\ &\ge \liminf_{n\to\infty} \|u_n\|_H^2 + \int_{\mathbb{R}^3} \phi_u^s u^2 dx + \int_{\mathbb{R}^3} Q(x)|u|^q dx. \end{aligned}$$

This together with (3.8) gives

$$\|u\|_H^2 \ge \liminf_{n\to\infty} \|u_n\|_H^2.$$

From this and the weak low semi-continuity, we have that

$$\|u_n\|_H^2 \to \|u\|_H^2, \tag{3.11}$$

which means that $u_n \to u$ in H.

Claim 3. $u_n \to u$ in E.

In fact, it suffices to prove that

$$\int_{\mathbb{R}^3} Q(x)|u_n|^q dx \to \int_{\mathbb{R}^3} Q(x)|u|^q dx. \tag{3.12}$$

Note that $u_n \to u$ in H implies that $u_n \to u$ in $L^{\frac{12}{3+2s}}(\mathbb{R}^3)$ and $\phi_{u_n}^s \rightharpoonup \phi_u^s$ in $L^{2_s^*}(\mathbb{R}^3)$. Thus, by Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \int_{\mathbb{R}^3} \phi_u^s u^2 dx \right| \\
&\quad \le \int_{\mathbb{R}^3} \phi_{u_n}^s (u_n^2 - u^2) dx + \int_{\mathbb{R}^3} (\phi_{u_n}^s - \phi_u^s) u^2 dx \\
&\quad \le \|\phi_{u_n}^s\|_{2_s^*} \|u_n - u\|_{\frac{12}{3+2s}} \|u_n + u\|_{\frac{12}{3+2s}} + \int_{\mathbb{R}^3} (\phi_{u_n}^s - \phi_u^s) u^2 dx \\
&\quad \le C \|u_n - u\|_{\frac{12}{3+2s}} + \int_{\mathbb{R}^3} (\phi_{u_n}^s - \phi_u^s) u^2 dx \\
&\quad \to 0;
\end{aligned}$$

this together with (3.8) and (3.11) implies that (3.12). The proof is completed. ■

Let

$$\Omega := \{x \in \mathbb{R}^3 : P(x) = 0\},$$

and

$$\Gamma := \{u \in E : u(x) = 0 \text{ a.e. } x \in \Omega\},$$

then, we can see that Γ is an infinite-dimensional linear subspace of E. After having verified the (PS) condition, we will investigate the geometry of I and complete our proof.

Lemma 3.4. *The seminorm* $|u|_{p,P} = (\int_{\mathbb{R}^3} P(x)|u|^p dx)^{\frac{1}{p}}$ *is a norm on* Γ.

Proof. We only need to show that

$$u \in \Gamma, \quad |u|_{p,P} = 0 \Rightarrow u = 0.$$

In fact, since $P(x) \geqslant 0$, we have

$$0 = |u|_{p,P} = \int_{\mathbb{R}^3} P|u|^p dx = \int_{P>0} P|u|^p.$$

We see that $u = 0$ a.e. on $\{x \in \mathbb{R}^3 \mid P(x) > 0\}$. But $u \in \Gamma$, that is, $u = 0$ a.e. on Ω. So, $u = 0$ a.e. on $\mathbb{R}^3$. ■

Proof of Theorem 1.1. From Lemmas 3.2 and 3.3, we know that I satisfies the (PS) condition and is bounded from below. In order to apply Theorem 3.1, we only need to prove that, for any $n \in \mathbb{N}$, there exists an n-dimensional subspace E^n and $\rho_n > 0$ such that

$$\sup_{E^n \cap S_{\rho_n}} I < 0,$$

where $S_{\rho_n} = \{u \in E : \|u\| = \rho_n\}$. Let $|\Omega|$ denote the Lebesgue measure of Ω; we have the following two cases.

Case 1. $|\Omega| = 0$.

In this case, we have $P > 0$ a.e. on $\mathbb{R}^3$. By Lemmas 2.1 and 3.1, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_H^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} P(x)|u|^p dx + \frac{1}{q}\int_{\mathbb{R}^3} Q(x)|u|^q dx \\ &\le C\|u\|^2 + C\|u\|^q + c\|u\|^4 - \frac{1}{p}|u|_{p,P}^p. \end{aligned}$$

For any $n \in \mathbb{N}$, we can choose an n-dimensional subspace E^n in E, and $|\cdot|_{p,P}$ is a norm of E^n. Then, for $u \in E^n$, using the fact that all norms on finite-dimension space are equivalent and $1 < p < 2 < q$, there exists $\rho_n > 0$ small such that, for $\|u\| = \rho_n$,

$$I(u) \le C\|u\|^2 + C\|u\|^q + c\|u\|^4 - \frac{1}{p}|u|_{p,P}^p < 0.$$

Therefore, we completed the proof for this case.

Case 2. $|\Omega| > 0$.

In this case, Lemma 3.4 implies that the seminorm $|\cdot|_{p,P}$ is a norm on the space Γ. Since $\dim \Gamma = \infty$, given $n \in \mathbb{N}$, let E^n be an n-dimensional subspace in Γ; then, for $u \in E^n$, Lemma 3.1 means that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_H^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} P(x)|u|^p dx + \frac{1}{q}\int_{\mathbb{R}^3} Q(x)|u|^q dx \\ &\le C\|u\|^2 + C\|u\|^q + c\|u\|^4 - \frac{1}{p}|u|_{p,P}^p \\ &\le C\|u\|^2 + C\|u\|^q + c\|u\|^4 - C\|u\|^p; \end{aligned}$$

then, there exists a $\rho_n > 0$ such that

$$I(u) < 0, \quad \text{for } \|u\| = \rho_n, \text{ with } u \in E^n.$$

The proof is completed. ∎

Acknowledgments. The authors would like to thank the anonymous referee for their careful reading of their manuscript and the useful comments made for its improvement.

Funding. This work was partially supported by the National Natural Science Foundation of China (12301145, 12261107), Yunnan Province Applied Basic Research for Youth Projects (202301AU070144), Yunnan Province Applied Basic Research for General Projects (202401AU070123), Scientific Research Fund of Yunnan Educational Commission (2023J0199), and Yunnan Key Laboratory of Modern Analytical Mathematics and Applications.

References

[1] W. Abou Salem, T. Chen, and V. Vougalter, Existence and nonlinear stability of stationary states for the semi-relativistic Schrödinger–Poisson system. *Ann. Henri Poincaré* **15** (2014), no. 6, 1171–1196 Zbl 1292.81046 MR 3205749

[2] A. Ambrosetti, On Schrödinger–Poisson systems. *Milan J. Math.* **76** (2008), 257–274 Zbl 1181.35257 MR 2465993

[3] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger–Poisson problem. *Commun. Contemp. Math.* **10** (2008), no. 3, 391–404 Zbl 1188.35171 MR 2417922

[4] A. Azzollini, Concentration and compactness in nonlinear Schrödinger–Poisson system with a general nonlinearity. *J. Differential Equations* **249** (2010), no. 7, 1746–1763 Zbl 1197.35096 MR 2677814

[5] A. Azzollini, P. d'Avenia, and A. Pomponio, On the Schrödinger–Maxwell equations under the effect of a general nonlinear term. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **27** (2010), no. 2, 779–791 Zbl 1187.35231 MR 2595202

[6] A. Azzollini and A. Pomponio, Ground state solutions for the nonlinear Schrödinger–Maxwell equations. *J. Math. Anal. Appl.* **345** (2008), no. 1, 90–108 Zbl 1147.35091 MR 2422637

[7] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equations. *Topol. Methods Nonlinear Anal.* **11** (1998), no. 2, 283–293 Zbl 0926.35125 MR 1659454

[8] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.* **88** (1983), no. 3, 486–490 Zbl 0526.46037 MR 0699419

[9] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger–Poisson systems. *J. Differential Equations* **248** (2010), no. 3, 521–543 Zbl 1183.35109 MR 2557904

[10] S. Chen and X. Tang, Ground state sign-changing solutions for a class of Schrödinger–Poisson type problems in $\mathbb{R}^3$. *Z. Angew. Math. Phys.* **67** (2016), no. 4, article no. 102 Zbl 1362.35142 MR 3531753

[11] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), no. 5, 521–573 Zbl 1252.46023 MR 2944369

[12] I. Ianni, Sign-changing radial solutions for the Schrödinger–Poisson–Slater problem. *Topol. Methods Nonlinear Anal.* **41** (2013), no. 2, 365–385 Zbl 1330.35128 MR 3114313

[13] I. Ianni and D. Ruiz, Ground and bound states for a static Schrödinger–Poisson–Slater problem. *Commun. Contemp. Math.* **14** (2012), no. 1, article no. 1250003 Zbl 1237.35146 MR 2902293

[14] I. Ianni and G. Vaira, Non-radial sign-changing solutions for the Schrödinger–Poisson problem in the semiclassical limit. *NoDEA Nonlinear Differential Equations Appl.* **22** (2015), no. 4, 741–776 Zbl 1327.35009 MR 3385620

[15] Y. Jiang and H.-S. Zhou, Schrödinger–Poisson system with steep potential well. *J. Differential Equations* **251** (2011), no. 3, 582–608 Zbl 1233.35086 MR 2802025

[16] S. Kim and J. Seok, On nodal solutions of the nonlinear Schrödinger–Poisson equations. *Commun. Contemp. Math.* **14** (2012), no. 6, article no. 1250041 Zbl 1263.35197 MR 2989645

[17] N. S. Landkof, *Foundations of modern potential theory*. Grundlehren Math. Wiss. 180, Springer, New York, 1972 Zbl 0253.31001 MR 0350027

[18] N. Laskin, Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A* **268** (2000), no. 4-6, 298–305 Zbl 0948.81595 MR 1755089

[19] E. Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type. *Math. Phys. Anal. Geom.* **10** (2007), no. 1, 43–64 Zbl 1171.35474 MR 2340532

[20] K. Li, Existence of non-trivial solutions for nonlinear fractional Schrödinger–Poisson equations. *Appl. Math. Lett.* **72** (2017), 1–9 Zbl 1398.35272 MR 3652422

[21] P.-L. Lions, Solutions of Hartree–Fock equations for Coulomb systems. *Comm. Math. Phys.* **109** (1987), no. 1, 33–97 Zbl 0618.35111 MR 0879032

[22] S. Liu and S. Li, An elliptic equation with concave and convex nonlinearities. *Nonlinear Anal.* **53** (2003), no. 6, 723–731 Zbl 1217.35067 MR 1971024

[23] W. Liu and M. Niu, Multi-peak positive solutions for the fractional Schrödinger–Poisson system. *Commun. Contemp. Math.* **20** (2018), no. 3, article no. 1750017 Zbl 1390.35065 MR 3903223

[24] Z. Liu, Z.-Q. Wang, and J. Zhang, Infinitely many sign-changing solutions for the nonlinear Schrödinger–Poisson system. *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 3, 775–794 Zbl 1341.35041 MR 3500305

[25] Z. Liu and J. Zhang, Multiplicity and concentration of positive solutions for the fractional Schrödinger–Poisson systems with critical growth. *ESAIM Control Optim. Calc. Var.* **23** (2017), no. 4, 1515–1542 Zbl 1516.35467 MR 3716931

[26] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.* **237** (2006), no. 2, 655–674 Zbl 1136.35037 MR 2230354

[27] W. Shuai and Q. Wang, Existence and asymptotic behavior of sign-changing solutions for the nonlinear Schrödinger–Poisson system in $\mathbb{R}^3$. *Z. Angew. Math. Phys.* **66** (2015), no. 6, 3267–3282 Zbl 1332.35114 MR 3428464

[28] M. Sun, J. Su, and L. Zhao, Infinitely many solutions for a Schrödinger–Poisson system with concave and convex nonlinearities. *Discrete Contin. Dyn. Syst.* **35** (2015), no. 1, 427–440 Zbl 1304.35264 MR 3286966

[29] K. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent. *J. Differential Equations* **261** (2016), no. 6, 3061–3106 Zbl 1386.35458 MR 3527623

[30] K. Teng, Ground state solutions for the non-linear fractional Schrödinger–Poisson system. *Appl. Anal.* **98** (2019), no. 11, 1959–1996 Zbl 1447.35363 MR 3982764

[31] Z. Wang and H.-S. Zhou, Positive solution for a nonlinear stationary Schrödinger–Poisson system in $\mathbb{R}^3$. *Discrete Contin. Dyn. Syst.* **18** (2007), no. 4, 809–816 Zbl 1133.35427 MR 2318269

[32] Z. Wang and H.-S. Zhou, Sign-changing solutions for the nonlinear Schrödinger–Poisson system in $\mathbb{R}^3$. *Calc. Var. Partial Differential Equations* **52** (2015), no. 3-4, 927–943 Zbl 1311.35300 MR 3311919

[33] Z.-Q. Wang, Nonlinear boundary value problems with concave nonlinearities near the origin. *NoDEA Nonlinear Differential Equations Appl.* **8** (2001), no. 1, 15–33 Zbl 0983.35052 MR 1828946

[34] Z. Yang, Y. Yu, and F. Zhao, The concentration behavior of ground state solutions for a critical fractional Schrödinger–Poisson system. *Math. Nachr.* **292** (2019), no. 8, 1837–1868 Zbl 1426.35126 MR 3994307

[35] Z. Yang, Y. Yu, and F. Zhao, Concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system involving critical exponent. *Commun. Contemp. Math.* **21** (2019), no. 6, article no. 1850027 Zbl 1428.35427 MR 3996970

[36] Y. Yu, F. Zhao, and L. Zhao, The concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system. *Calc. Var. Partial Differential Equations* **56** (2017), no. 4, article no. 116 Zbl 1437.35699 MR 3671610

[37] Y. Yu, F. Zhao, and L. Zhao, The existence and multiplicity of solutions of a fractional Schrödinger–Poisson system with critical growth. *Sci. China Math.* **61** (2018), no. 6, 1039–1062 Zbl 1391.35403 MR 3802758

[38] J. Zhang, J. M. do Ó, and M. Squassina, Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity. *Adv. Nonlinear Stud.* **16** (2016), no. 1, 15–30 Zbl 1334.35407 MR 3456743

[39] L. Zhao, H. Liu, and F. Zhao, Existence and concentration of solutions for the Schrödinger–Poisson equations with steep well potential. *J. Differential Equations* **255** (2013), no. 1, 1–23 Zbl 1286.35103 MR 3045632

[40] L. Zhao and F. Zhao, On the existence of solutions for the Schrödinger–Poisson equations. *J. Math. Anal. Appl.* **346** (2008), no. 1, 155–169 Zbl 1159.35017 MR 2428280

Received 1 January 2024; revised 11 September 2024.

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