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Measure-valued solutions to a generalized Landau–Lifshitz–Bloch equation

Chahid Ayouch, Driss Meskine, and Mouhcine Tilioua

Abstract. This paper considers a generalized version of the Landau–Lifshitz–Bloch equation. We prove existence of measure-valued solutions for the model in a bounded domain of $\mathbb{R}^d$ $(d \geq 1)$. The main difficulties in this study are due to the loss of compactness in the equation and the presence of a nonlinear term of type $\boldsymbol{u} \wedge \operatorname{div}(a(\nabla \boldsymbol{u}))$ which does not satisfy the monotonicity assumption in the sense of Leray–Lions. We use a compactness result proved in Landes et al. [Ark. Mat. 25 (1987), 29–40] and the concept of measure-valued solutions which are appropriate to solve this problem.

1. Introduction

The influence of thermal excitations on magnetic materials is an increasingly relevant topic in the theory of magnetism. The Landau–Lifshitz–Bloch (LLB) dynamical equation of motion for the macroscopic magnetization vector [9] has been shown to be a valid micromagnetic approach at high temperatures [8], particularly useful for temperatures θ close to the Curie temperature θ_c $(\theta > 3\theta_c/4)$ and ultrafast timescales. This approach has proven to be necessary for several new and exciting magnetic phenomena. These include laser-induced ultrafast demagnetization, heat-induced domain wall motion, spin torque effect at high temperatures, or heat-assisted magnetic recording. We refer to [2, 7] for physical issues and derivation of the LLB model. The LLB equation essentially interpolates between the Landau–Lifshitz–Gilbert (LLG) equation at low temperature and the Ginzburg–Landau theory of phase transitions. It is valid not only below but also above the Curie temperature θ_c. An important property of the LLB equation is that the magnetization amplitude is no longer conserved but is a dynamic variable. The known form of the Landau–Lifshitz–Bloch equation describing the dynamics of three-dimensional spin polarization vector $\boldsymbol{u} = (u_1, u_2, u_3)$ is written as

$$\partial_t \boldsymbol{u} = \gamma \boldsymbol{u} \wedge \mathcal{H}_{\text{eff}}(\boldsymbol{u}) + \frac{L_1}{|\boldsymbol{u}|^2}(\boldsymbol{u} \cdot \mathcal{H}_{\text{eff}}(\boldsymbol{u}))\boldsymbol{u} - \frac{L_2}{|\boldsymbol{u}|^2}\boldsymbol{u} \wedge (\boldsymbol{u} \wedge \mathcal{H}_{\text{eff}}(\boldsymbol{u})), \tag{1.1}$$

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where $\gamma > 0$ is the gyromagnetic ratio, the symbol $\wedge$ denotes the vector cross product in $\mathbb{R}^3$, L_1 and L_2 are the longitudinal and transverse damping parameters, respectively. The so-called effective field $\mathcal{H}_{\text{eff}}(\boldsymbol{u})$ is given by

$$\mathcal{H}_{\text{eff}}(\boldsymbol{u}) = \Delta \boldsymbol{u} - \frac{1}{\chi_{\|}}\left(1 + \frac{3}{5}\frac{\theta}{\theta - \theta_c}|\boldsymbol{u}|^2\right)\boldsymbol{u},$$

where $\chi_{\|}$ is the longitudinal susceptibility. In the case where the temperature θ is higher than the Curie temperature θ_c, the longitudinal L_1 and transverse L_2 damping parameters are equal; in this case, we can rewrite (1.1) in the following form:

$$\partial_t \boldsymbol{u} = \kappa_1 \Delta \boldsymbol{u} + \gamma \boldsymbol{u} \wedge \Delta \boldsymbol{u} - \kappa_2(1 + \mu|\boldsymbol{u}|^2)\boldsymbol{u}, \tag{1.2}$$

where $\kappa_1 := L_1 = L_2$, $\kappa_2 := \frac{\kappa_1}{\chi_{\|}}$, and $\mu := \frac{3\theta}{5(\theta-\theta_c)}$. In this paper, we are interested in a general version of the effective field $\mathcal{H}_{\text{eff}}(\boldsymbol{u})$ which is given by

$$\mathcal{H}_{\text{eff}}(\boldsymbol{u}) = \operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u})) - \frac{\kappa_2}{\kappa_1}(1 + \mu|\boldsymbol{u}|^2)\boldsymbol{u}, \tag{1.3}$$

where we assume that $\boldsymbol{a}$ satisfies the following assumptions:

(A_1) $\boldsymbol{a} : \mathbb{R}^{d\times 3} \to \mathbb{R}^{d\times 3}$ is continuous;

(A_2) $\forall \boldsymbol{\xi} \in \mathbb{R}^{d\times 3}$, $\boldsymbol{a}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq C_1(|\boldsymbol{\xi}|^p - 1)$;

(A_3) $\forall \boldsymbol{\xi} \in \mathbb{R}^{d\times 3}$, $|a_{ij}(\boldsymbol{\xi})| \leq C_2(1 + |\boldsymbol{\xi}|)^{p-1}$;

(A_4) $\forall \boldsymbol{\xi} \in \mathbb{R}^{d\times 3}$, $\boldsymbol{a}(\boldsymbol{\xi}) \wedge \boldsymbol{\xi} = 0$.

Examples

If $\boldsymbol{a} = \mathrm{Id}_{\mathbb{R}^{d\times 3}}$, we obtain $\operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u})) = \Delta \boldsymbol{u}$; in this case, the problem (1.4) corresponds to the usual Landau–Lifshitz–Bloch equation studied in [14].

If $\boldsymbol{a}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$ with $1 < p < +\infty$, then $\operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u})) = \Delta_p \boldsymbol{u}$.

Note that the hypothesis (A_4) covers the more important physical case given in the first example. The main goal of this paper is to prove existence of measure-valued solutions to the following generalized Landau–Lifshitz–Bloch (GLLB) equation:

$$\begin{cases} \partial_t \boldsymbol{u}(t,x) = \kappa_1 \operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u})) + \gamma \boldsymbol{u} \wedge \operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u})) - \kappa_2(1 + \beta|\boldsymbol{u}|^2)\boldsymbol{u} & \text{in } Q = I \times \Omega, \\ \boldsymbol{u} = 0 & \text{on } I \times \partial\Omega, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $I = (0,T)$ and $\partial_t \boldsymbol{u}(t,x)$ is the time derivative of $\boldsymbol{u}$. The main difficulty in this equation is the presence of the nonlinear term $\boldsymbol{u} \wedge \operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u}))$ which leads to non-sufficient estimates on the approximate solutions to obtain the existence of weak solutions; more precisely, we do not have the convergence almost everywhere of the gradient for a subsequence of approximate solutions in order to pass to the limit in the nonlinear term.

In addition, the classical Leray–Lions monotonicity condition is not satisfied for operators of this kind. Another difficulty is that it is not easy to get an estimate for the time weak derivative of $\boldsymbol{u}_n$. We use a compactness result proved by Landes in [13] (see Proposition 1) instead of the classical Aubin–Lions lemma to obtain the almost everywhere convergence for a subsequence of approximate solutions. For all these reasons, the notion of measure-valued solutions is better suitable for our problem. Before dealing with existence of measure-valued solutions to (1.4), let us first review some results on the classical LLB equation. Despite its importance, very little is known about solutions to the deterministic LLB equation. A pioneering work on the existence of weak solutions to the deterministic LLB equation in a bounded domain is carried out in [14]. In this paper, a Faedo–Galerkin approximation was introduced and the method of compactness was used to prove the existence of a weak solution for the LLB equation and its regularity properties. In the framework of fractional differential operators, a global existence of weak solutions of a time-space fractional LLB equation involving the weak Caputo derivative and a fractional Laplacian is proved in [4] by using Faedo–Galerkin method with some commutator estimates. The uniqueness is also discussed in a special one dimensional case. In [3], a finite difference scheme for temporal discretization of the time-fractional LLB equation, such as the fractional time derivative of order δ is taken in the sense of Caputo. An existence result is established for the semi-discrete problem by Schaefer's fixed point theorem. Stability and error analyses are then provided, showing that the temporal accuracy is of order $2-\delta$. More recently, the paper [11] considered strong solutions for the LLB equation both for dynamic and static models. In this work, the authors discuss existence of solutions of the LLB equation describing the dynamics of the magnetization for the whole range of the temperature. By using energy methods, they prove global existence of strong solutions for given initial data, existence of time-periodic solutions as well as existence of steady-state solutions of the equation. In [12], the authors considered the LLB equation on an m-dimensional closed Riemannian manifold and prove that it admits a unique local solution. In addition, if $m \geq 3$ and the L^∞-norm of initial datum is sufficiently small, the solution can be extended globally. Moreover, if $m = 2$, it is proved that the unique solution is global without assuming small initial data. The paper [6] addresses the local existence and uniqueness of regular solution to the LLB equation with applied current in a bounded domain of $\mathbb{R}^3$, and global existence of regular solution is also obtained in dimension two without any restriction on the initial data. The work [15] provides a comprehensive analysis of weak and strong solutions to the LLB equation coupled to the Maxwell equations with polarization. Some asymptotic behavior results for the LLB equation are presented in [10]. Recently, in [5], the Landau–Lifshitz–Baryakhtar (LLBar) equation, which is a generalization of the LLB equation for magnetization dynamics in ferrimagnets, is considered. Global existence of periodic solutions as well as local existence and uniqueness of regular solutions are proved. The relationships between the (LLBar) equation and both LLB and harmonic map equations are revealed.

This paper is organized as follows. In the next section, we introduce some notation and we recall some definitions and results concerning the notion of Young measures which are necessary to state the main result of this paper. Section 3 is devoted to the existence of measure-valued solutions for the problem (1.4).

2. Some notation and mathematical tools

In this section, we introduce some notation and we recall some definitions and results concerning the Radon and Young measures which are necessary to define the concept of measure-valued solutions for the problem (1.4); for more details, we refer, for example, to [16, Chapter 4, Theorem 2.1 and Corollary 2.10].

2.1. Notation

Throughout this paper, we adopt the following notation. For a matrix $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{d\times 3}$, we define $\boldsymbol{a}(\boldsymbol{\xi}) \wedge \boldsymbol{\eta} \in \mathbb{R}^{d\times 3}$ to be a matrix such that for the ith row there holds

$$(\boldsymbol{a}(\boldsymbol{\xi}) \wedge \boldsymbol{\eta})^{(i)} = \boldsymbol{a}^{(i)}(\boldsymbol{\xi}) \wedge \boldsymbol{\eta}^{(i)},$$

where

$$\boldsymbol{\eta}^{(i)} := \sum_{j=1}^{3} \eta_{ij} e_j \quad \text{and} \quad \boldsymbol{a}^{(i)}(\boldsymbol{\xi}) := \sum_{j=1}^{3} a_{ij}(\boldsymbol{\xi}) e_j.$$

We define

$$\operatorname{div}(\boldsymbol{a}(\nabla \boldsymbol{u})) := \sum_{j=1}^{3} \sum_{i=1}^{d} \partial_i [a_{ij}(\nabla \boldsymbol{u})] e_j.$$

Moreover,

$$\left[\boldsymbol{u} \wedge \boldsymbol{a}(\nabla \boldsymbol{u})\right]^{(i)} := \boldsymbol{u} \wedge \boldsymbol{a}^{(i)}(\nabla \boldsymbol{u}) \quad \forall i = 1, \dots, d.$$

For $1 \le k \le +\infty$ and $1 \le p < +\infty$, $\mathbb{L}^k(\Omega) := L^k(\Omega; \mathbb{R}^3)$, $\mathbb{W}^{m,p}(\Omega) := W^{m,p}(\Omega; \mathbb{R}^3)$ are the usual Lebesgue and Sobolev spaces. The norm in $\mathbb{W}^{m,p}$ is denoted by $\|\cdot\|_{m,p}$. Sometimes, for simplicity, we use the notation $\|\cdot\|_k$ for the norm $\|\cdot\|_{\mathbb{L}^k(\Omega)}$. Finally, the inner product in $\mathbb{L}^2(\Omega)$ is denoted by $(\cdot,\cdot)$.

2.2. Signed Radon measures

Let Ω be a bounded domain of $\mathbb{R}^d$. We denote by $M(\Omega)$ the space of the so-called signed Radon measures defined as the dual space of $C(\overline{\Omega})$. Obviously, $L^1(\Omega) \hookrightarrow M(\Omega)$, since for $f \in L^1(\Omega)$, $g : \phi \mapsto \int_\Omega f(x)\phi(x)\mathrm{d}x$ for all $\phi \in C(\overline{\Omega})$ defines a continuous linear functional on $C(\overline{\Omega})$ and consequently $g \in M(\Omega)$. Moreover, $\|g\|_{M(\Omega)} = \|f\|_1$, where $\|\cdot\|_{M(\Omega)}$ is defined as a dual norm. If $\Omega = \mathbb{R}^d$, we define

$$C_0(\mathbb{R}^d) = \Big\{u \in C(\mathbb{R}^d),\ \lim_{|x|\to+\infty} u(x) = 0\Big\}.$$

Note that

$$C_0(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{\|\cdot\|_\infty}.$$

The space of finite (signed) Radon measures is defined as

$$M(\mathbb{R}^d) = \big\{\mu : C_0(\mathbb{R}^d) \to \mathbb{R};\ \text{linear such that } \exists c > 0, |\mu(f)| \le c\|f\|_\infty\, \forall f \in \mathcal{D}(\mathbb{R}^d)\big\}.$$

Let us define

$$\|\mu\|_{M(\mathbb{R}^d)} := \sup_{f\in\mathcal{D}(\mathbb{R}^d), \|f\|_\infty \le 1} |\mu(f)|.$$

If $\mu \in M(\mathbb{R}^d)$, $\mu(f) \ge 0$ for all $f \in \mathcal{D}(\mathbb{R}^d)$ with $f \ge 0$, we say that μ is a non-negative bounded Radon measure. The space of probability measures is then defined by

$$\mathrm{Prob}(\mathbb{R}^d) = \big\{\mu \in M(\mathbb{R}^d), \mu \text{ is non-negative}, \|\mu\|_{M(\mathbb{R}^d)} = 1\big\}.$$

2.3. Young measures

The following theorem introduces the concept of Young measures which turns out to be an appropriate tool for describing composite limits of smooth nonlinearities with weakly convergent sequences.

Theorem 1 ([16]). *Let $\boldsymbol{u}_n : \mathbb{R}^m \to \mathbb{R}^3$ $(m \ge 1)$ be an arbitrary sequence of measurable functions for which*

$$\|\boldsymbol{u}_n\|_{\mathbb{L}^\infty(\mathbb{R}^m)} \le C.$$

Then, there exist a weakly-$*$ convergent subsequence $(\boldsymbol{u}_{n_k})_k$ of $(\boldsymbol{u}_n)_n$ and a family of probability measures $\{\nu_y\}_{y\in\mathbb{R}^m}$ called Young measures, supported uniformly in a compact set $K \subset \mathbb{R}^3$:

$$\{\nu_y\}_{y\in\mathbb{R}^m} \subset \mathrm{Prob}(\mathbb{R}^3), \quad \mathrm{supp}(\nu_y) \subset K \quad \textit{for a.e. } y \in \mathbb{R}^m,$$

which represents the subsequence $(\boldsymbol{u}_{n_k})_k$ in the following sense.

For any $\boldsymbol{g} \in C(\mathbb{R}^3, \mathbb{R}^s)$, we have

$$\bar{\boldsymbol{g}}(\boldsymbol{u}_{n_k}) \overset{*}{\rightharpoonup} \bar{\boldsymbol{g}} \quad \textit{in } L^\infty(\mathbb{R}^m, \mathbb{R}^s)$$

and

$$\bar{\boldsymbol{g}}(y) = \int_{\mathbb{R}^3} \boldsymbol{g}(\boldsymbol{\lambda})\mathrm{d}\nu_y(\boldsymbol{\lambda}) = \langle \nu_y, \boldsymbol{g}\rangle \quad \textit{for a.e. } y \in \mathbb{R}^m.$$

Definition 1 ([16]). Let $Q \subset \mathbb{R}^m$ $(m \ge 1)$ be an open set. The mapping $\nu : Q \mapsto M(\mathbb{R}^3)$ is said to be weak-$*$ measurable if for all $F \in L^1(Q, C_0(\mathbb{R}^3))$ the function

$$x \mapsto \langle \nu_x, F(x,\cdot)\rangle = \int_{\mathbb{R}^3} F(x,\boldsymbol{\lambda})\mathrm{d}\nu_x(\boldsymbol{\lambda})$$

is measurable. The set of bounded weak-$*$ measurable maps $\mu : Q \mapsto M(\mathbb{R}^3)$ is called $L^\infty(Q, M(\mathbb{R}^3))$.

Now, we give a result about Young measures which will play a key role in the construction of measure-valued solutions for problem (1.4).

Corollary 1 ([16]). *Let $Q \subset \mathbb{R}^m$ ($m \geq 1$) be a bounded open set. Let $(\boldsymbol{z}_j)_j$ be uniformly bounded in $\mathbb{L}^p(Q)$. Then, there exists a subsequence still denoted by $\boldsymbol{z}_j$ and a measure-valued function $\nu \in L^\infty_\omega(Q, \mathrm{Prob}(\mathbb{R}^3))$ such that for all $\tau : \mathbb{R}^m \to \mathbb{R}$ continuous and satisfying for some $q > 0$ the growth condition:*

$$|\tau(\boldsymbol{\xi})| \leq C(1 + |\boldsymbol{\xi}|)^q \quad \forall \boldsymbol{\xi} \in \mathbb{R}^m,$$

we have

$$\tau(\boldsymbol{z}_j) \rightharpoonup \bar{\tau}, \quad \text{weakly in } \mathbb{L}^r(Q), \quad \bar{\tau}(y) = \langle \nu_y, \tau \rangle, \text{ a.e.}$$

provided that

$$1 < r \leq \frac{p}{q}.$$

We finish this subsection with this useful theorem for differential systems; see, for example, [1, Chapter 2, Section 7, Theorem 7.6]

Theorem 2. *Let $\boldsymbol{F} : (a, b) \times D \mapsto \mathbb{R}^s$ ($s \geq 1$) where $D \subset \mathbb{R}^s$ a domain, satisfying the following Carathéodory conditions:*

- *$t \mapsto F_i(t, \boldsymbol{u})$ is measurable for all $i = 1, \ldots, n$ and for all $\boldsymbol{u} \in D$,*
- *$\boldsymbol{u} \mapsto F_i(t, \boldsymbol{u})$ is continuous for almost all $t \in (t_-, t_+)$,*
- *there exists an integrable function $G : (t_-, t_+) \mapsto \mathbb{R}$ such that*

$$|F_i(t, \boldsymbol{u})| \leq G(t) \quad \forall (t, \boldsymbol{u}) \in (t_-, t_+) \times D.$$

Then, there exists $a < t_- < t_0 < t_+ < b$ and a continuous, a.e. differentiable, and nonextendable solution $\boldsymbol{u} : (t_-, t_+) \to \mathbb{R}^s$ of

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{F}(t, \boldsymbol{u}), & t \in (t_-, t_+), \\ \boldsymbol{u}(t_0) = \boldsymbol{u}_0 \in \mathbb{R}^s. \end{cases}$$

Further, if $t_+ < b$, then one has

$$\lim_{t \to t_+} \min \left\{ \mathrm{dist}(u(t), D^c), \frac{1}{1 + |u(t)|} \right\} = 0. \tag{2.1}$$

2.4. Compactness lemma

Usually the compactness, for example, in the space $L^p(0, T, \mathbb{W}_0^{m,p}(\Omega))$ is obtained by a priori bounds of $(\partial_t \boldsymbol{u}_n)_n$ in some distribution spaces using the classical Aubin–Lions lemma. These bounds are replaced here by the hypothesis $\boldsymbol{u}_n(t, \cdot) \rightharpoonup \boldsymbol{u}(t, \cdot)$ weakly in $\mathbb{L}^1(\Omega)$ and $\|\boldsymbol{u}_n(t, \cdot)\|_1 \leq C$, which can be verified for the sequence of Galerkin solutions. The following result is very suitable for obtaining the compactness of a Galerkin subsequence. For the proof, we refer to [13].

Proposition 1. *Suppose that $(\boldsymbol{u}_n)_n$ be a bounded sequence in $L^p(0, T, \mathbb{W}_0^{m,p}(\Omega)) \cap L^\infty(0, T, \mathbb{L}^1(\Omega))$ with $1 < p < +\infty$ and $m \geq 1$. If $\boldsymbol{u}_n(t) \rightharpoonup \boldsymbol{u}(t)$ in $\mathbb{L}^1(\Omega)$ for a.e. $t \in [0, T]$, then $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^p(0, T, \mathbb{W}_0^{m-1,p}(\Omega))$ and for a.e. $(t, x) \in Q$ for some subsequence of $(\boldsymbol{u}_n)_n$.*

3. Existence of measure-valued solutions to (1.4)

In this section, we will show the existence of measure-valued solutions for the problem (1.4).

3.1. Definition of measure-valued solutions and main result

Before stating our main result, we first give the definition of measure-valued solutions to problem (1.4).

Definition 2. Let $1 < p < +\infty$ and $\boldsymbol{u}_0 \in \mathbb{L}^2(\Omega)$. The pair $(\boldsymbol{u}, \nu)$ is called a measure-valued solution of (1.4) if

$$\boldsymbol{u} \in L^\infty(I, \mathbb{L}^2(\Omega)) \cap L^p(I, \mathbb{W}_0^{1,p}(\Omega)) \cap \mathbb{L}^4(Q),$$
$$\nu \in L^\infty_\omega(Q, \mathrm{Prob}(\mathbb{R}^{d\times 3})),$$

and $(\boldsymbol{u}, \nu)$ satisfies

$$\int_Q \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = \kappa_1 \int_Q \nabla\boldsymbol{\varphi} \cdot \langle \nu_{t,x}, \boldsymbol{a}\rangle \mathrm{d}x\,\mathrm{d}t + \gamma \int_Q \boldsymbol{u} \wedge \nabla\boldsymbol{\varphi} \cdot \langle \nu_{t,x}, \boldsymbol{a}\rangle \mathrm{d}x\,\mathrm{d}t$$
$$+ \kappa_2 \int_Q (1 + \mu|\boldsymbol{u}|^2)\boldsymbol{u} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(Q)$. Additionally, we require that

$$\nabla \boldsymbol{u}(t,x) = \int_{\mathbb{R}^{d\times 3}} \boldsymbol{\lambda} \,\mathrm{d}\nu_{t,x}(\boldsymbol{\lambda}) \quad \text{a.e. } (t,x) \in Q.$$

Remark 1. A more regular solution for (1.4) than in the above Definition cannot be expected in general due to the fact that $\boldsymbol{u} \wedge \mathrm{div}(\boldsymbol{a}(\nabla\boldsymbol{u})) = \mathrm{div}(\boldsymbol{u} \wedge \boldsymbol{a}(\nabla\boldsymbol{u}))$ and $(\boldsymbol{\eta}, \boldsymbol{\xi}) \mapsto \boldsymbol{\eta} \wedge \boldsymbol{a}(\boldsymbol{\xi})$ does not satisfy the monotonicity assumption in the sense of Leray–Lions. However, in the particular case where $\boldsymbol{a} = \mathrm{Id}_{\mathbb{R}^{d\times 3}}$, we obtain using the principle of weak-strong convergences for the Galerkin sequence $(\boldsymbol{u}_n)_n$ that $\boldsymbol{u}_n \wedge \nabla\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \wedge \nabla\boldsymbol{u}$ in $\mathbb{L}^1(Q)$, then we can prove the existence of weak solutions to the problem (1.4) without using the concept of Young measure, but less regular than those found in [14] since the initial data $\boldsymbol{u}_0$ belongs only to $\mathbb{L}^2(\Omega)$.

The main result of this paper is stated as follows.

Theorem 3. *Let $T > 0$ be arbitrary and Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 1)$ and $\boldsymbol{a}$ satisfying the assumptions:* (A$_1$), (A$_2$), (A$_3$), *and* (A$_4$). *Then, there exists a measure-valued solution $\boldsymbol{u}$ on Q to problem* (1.4) *for any $\boldsymbol{u}_0 \in \mathbb{L}^2(\Omega)$.*

3.2. Approximate solutions

We will show the existence of measure-valued solution to the problem (1.4) via Galerkin approximation. For this purpose, we choose the sequence $\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots\}$ in $C_0^\infty(\Omega; \mathbb{R}^3)$

orthonormal with respect to the Hilbert space $\mathbb{L}^2(\Omega)$ such that $\bigcup_{n\geq 1} V_n$ with $V_n = \text{span}\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_n\}$ is dense in $\mathbb{L}^2(\Omega)$. Then, for any $\boldsymbol{v} \in \mathbb{L}^2(\Omega)$ there exists a sequence $(\boldsymbol{v}_k)_k \subset \bigcup_{n\geq 1} V_n$ such that $\boldsymbol{v}_k \to \boldsymbol{v}$ in $\mathbb{L}^2(\Omega)$.

Definition 3. A function $\boldsymbol{u}_n \in \mathcal{C}([0,T], V_n)$ is called a Galerkin solution of (1.4), if $\partial_t \boldsymbol{u}_n \in L^1(0,T,V_n)$, $\boldsymbol{u}_n(0,\cdot) = \boldsymbol{u}_{0n}$ and for all $t \in (0,T]$, there holds

$$\int_{Q_t} \partial_t \boldsymbol{u}_n \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s = -\kappa_1 \int_{Q_t} \boldsymbol{a}(\nabla \boldsymbol{u}_n)\cdot \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s - \gamma \int_{Q_t} \boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n)\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s$$
$$- \kappa_2 \int_{Q_t} (1+\beta|\boldsymbol{u}_n|^2)\boldsymbol{u}_n\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s$$

for all $\boldsymbol{\varphi} \in \mathcal{C}([0,T], V_n)$, where $\boldsymbol{u}_{0n} \in V_n$ and $\boldsymbol{u}_{0n} \to \boldsymbol{u}_0$ in $\mathbb{L}^2(\Omega)$.

The existence of Galerkin solutions and some of their properties is the purpose of Lemmas 1 and 2.

Lemma 1. *For any $n \geq 1$, there exists a Galerkin solution $\boldsymbol{u}_n \in \mathcal{C}([0,T], V_n)$ such that*

$$\|\boldsymbol{u}_n\|_{L^\infty(I,\mathbb{L}^2(\Omega))} \leq C, \tag{3.1}$$
$$\|\boldsymbol{u}_n\|_{L^4(I,\mathbb{L}^4(\Omega))} \leq C, \tag{3.2}$$
$$\|\boldsymbol{u}_n\|_{L^p(I,\mathbb{W}_0^{1,p}(\Omega))} \leq C, \tag{3.3}$$

where C is a positive constant not depending on n.

Proof. We define

$$\boldsymbol{u}_n(t,x) = \sum_{i=1}^{n} c_i^n(t)\boldsymbol{\omega}_i(x),$$

where the coefficients $c_i^n(t)$ are such that

$$(\partial_t \boldsymbol{u}_n, \boldsymbol{\omega}_i) = -\kappa_1 \int_\Omega \boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{\omega}_i\,\mathrm{d}x - \gamma\int_\Omega \boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{\omega}_i\,\mathrm{d}x$$
$$- \kappa_2((1+\beta|\boldsymbol{u}_n|^2)\boldsymbol{u}_n, \boldsymbol{\omega}_i),$$
$$\int_\Omega \boldsymbol{u}_{0n}\cdot\boldsymbol{\omega}_i\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\omega}_i\,\mathrm{d}x, \quad i = 1,\ldots,n. \tag{3.4}$$

Due to the orthonormality of $\{\boldsymbol{\omega}_i\}_{i\geq 1}$ in $\mathbb{L}^2(\Omega)$, the system (3.4) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} c_i^n = \mathcal{F}_i(c_1^n,\ldots,c_n^n,t),$$
$$c_i^n(0) = (\boldsymbol{u}_0, \boldsymbol{\omega}_i), \tag{3.5}$$

where

$$\mathcal{F}_i(c_1^n,\ldots,c_n^n,t) := -\kappa_1\int_\Omega \boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{\omega}_i\,\mathrm{d}x - \gamma\int_\Omega \boldsymbol{u}_n\wedge\boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{\omega}_i\,\mathrm{d}x$$
$$- \kappa_2((1+\beta|\boldsymbol{u}_n|^2)\boldsymbol{u}_n, \boldsymbol{\omega}_i), \quad i = 1,\ldots,n.$$

Before discussing the solvability of (3.5), we derive an a priori estimate. Multiplying the ith equation of the Galerkin system (3.4) by $c_i^n(t)$ and add the equations, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_2^2 + \kappa_1 \int_\Omega \boldsymbol{a}(\nabla \boldsymbol{u}_n)\cdot\nabla\boldsymbol{u}_n\,\mathrm{d}x + \kappa_2\|\boldsymbol{u}_n\|_2^2 + \kappa_2\beta\|\boldsymbol{u}_n\|_4^4 = -\gamma\int_\Omega \boldsymbol{u}_n\wedge\boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{u}_n\,\mathrm{d}x.$$

On the other hand, the following computation hold:

$$\begin{aligned}\int_\Omega \boldsymbol{u}_n\wedge\boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{u}_n\,\mathrm{d}x &= \int_\Omega\sum_{i=1}^d\big(\boldsymbol{u}_n\wedge\boldsymbol{a}(\nabla\boldsymbol{u}_n)\big)^{(i)}\cdot(\nabla\boldsymbol{u}_n)^{(i)}\,\mathrm{d}x\\ &= \int_\Omega\sum_{i=1}^d(\boldsymbol{u}_n\wedge\boldsymbol{a}^{(i)}(\nabla\boldsymbol{u}_n))\cdot(\nabla\boldsymbol{u}_n)^{(i)}\,\mathrm{d}x\\ &= -\int_\Omega\sum_{i=1}^d((\nabla\boldsymbol{u}_n)^{(i)}\wedge\boldsymbol{a}^{(i)}(\nabla\boldsymbol{u}_n))\cdot\boldsymbol{u}_n\,\mathrm{d}x\\ &= 0,\end{aligned}$$

where (A_4) is used in the last equality.

Due to assumption (A_2), it yields

$$\int_\Omega \boldsymbol{a}(\nabla\boldsymbol{u}_n)\cdot\nabla\boldsymbol{u}_n\,\mathrm{d}x \geq C_1(\|\nabla\boldsymbol{u}_n\|_p^p - |\Omega|).$$

Then,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_2^2 + \kappa_1 C_1\|\nabla\boldsymbol{u}_n\|_p^p + \kappa_2\|\boldsymbol{u}_n\|_2^2 + \kappa_2\beta\|\boldsymbol{u}_n\|_4^4 \leq \kappa_1 C_1|\Omega|. \tag{3.6}$$

Integrating (3.6) between 0 and t, one has

$$\begin{aligned}&\|\boldsymbol{u}_n(t)\|_2^2 + 2\kappa_1 C_1\int_0^t\|\nabla\boldsymbol{u}_n\|_p^p\,\mathrm{d}s + 2\kappa_2\int_0^t\|\boldsymbol{u}_n\|_2^2\,\mathrm{d}s + 2\kappa_2\beta\int_0^t\|\boldsymbol{u}_n\|_4^4\,\mathrm{d}s\\ &\quad\leq \|\boldsymbol{u}_{0n}\|_2^2 + 2\kappa_1 C_1 T|\Omega|.\end{aligned} \tag{3.7}$$

Since $\boldsymbol{u}_{0n}\to\boldsymbol{u}_0$ in $\mathbb{L}^2(\Omega)$, we conclude that the right-hand side of (3.7) is uniformly bounded with respect to n. Therefore, by Poincaré's inequality, there exists a positive constant C which does not depend to n such that

$$\begin{aligned}\|\boldsymbol{u}_n\|_{L^\infty(I,\mathbb{L}^2(\Omega))} &\leq C,\\ \|\boldsymbol{u}_n\|_{L^4(I,\mathbb{L}^4(\Omega))} &\leq C,\\ \|\boldsymbol{u}_n\|_{L^p(I,\mathbb{W}_0^{1,p}(\Omega))} &\leq C.\end{aligned} \tag{3.8}$$

The a priori estimate (3.8) implies that

$$|\boldsymbol{c}_n(t)|^2 \leq C \quad \text{for all } t\in I. \tag{3.9}$$

Since $\mathcal{F}_i$, $i = 1, \ldots, n$ satisfy the Carathéodory conditions in Theorem 2, we obtain the local existence of a continuous functions $\boldsymbol{c}_n : (0, T^*) \to \mathbb{R}^n$ with $T^* < T$ solving the system (3.5), but due to the uniform boundedness (3.9) and thanks to (2.1), we can extend the solution to the interval $(0, T)$. ∎

Lemma 2. *If the sequence $(\boldsymbol{u}_n)_n$ satisfies (3.1), (3.2), and (3.3) in Lemma 1, then there is a positive constant C independent of n such that the following statements hold.*

- *If $1 < p < d$, then for all q such that $1 < q \leq \frac{p+\frac{2p}{d}}{p+\frac{2(p-1)}{d}}$, one has*

$$\int_Q |\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n)|^q \,\mathrm{d}x \,\mathrm{d}t \leq C.$$

- *If $p \geq d$, then for all r such that $1 < r \leq \frac{p^2+p}{p^2+p-1}$, we have*

$$\int_Q |\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n)|^r \,\mathrm{d}x \,\mathrm{d}t \leq C.$$

Proof. By assumption (A_3), it yields

$$\begin{aligned} |\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n)| &\leq |\boldsymbol{u}_n||\boldsymbol{a}(\nabla \boldsymbol{u}_n)| \\ &\leq C_1|\boldsymbol{u}_n|(1 + |\nabla \boldsymbol{u}_n|)^{p-1}; \end{aligned}$$

then, it suffices to establish the lemma for the term $|\boldsymbol{u}_n||\nabla \boldsymbol{u}_n|^{(p-1)}$.

Let $1 < p < d$; we obtain

$$\int_Q |\boldsymbol{u}_n|^{\frac{p(d+2)}{d}} \,\mathrm{d}x \,\mathrm{d}t = \int_Q |\boldsymbol{u}_n|^{\frac{2p}{d}} |\boldsymbol{u}_n|^p \,\mathrm{d}x \,\mathrm{d}t.$$

Thus, by using Hölder's inequality with $\frac{d}{p}$ and its conjugate, one has

$$\int_Q |\boldsymbol{u}_n|^{\frac{p(d+2)}{d}} \,\mathrm{d}x \,\mathrm{d}t \leq \operatorname{ess\,sup}_{t \in I} \|\boldsymbol{u}_n\|_2^{\frac{2p}{d}} \int_Q |\nabla \boldsymbol{u}_n|^p \,\mathrm{d}x \,\mathrm{d}t \leq C; \tag{3.10}$$

consequently, for $1 < q \leq \frac{p+\frac{2p}{d}}{p+\frac{2(p-1)}{d}}$ and by Hölder's inequality with exponent $\frac{p}{q(p-1)}$ and its conjugate, we deduce thanks to (3.10) that

$$\int_Q |\boldsymbol{u}_n|^q |\nabla \boldsymbol{u}_n|^{q(p-1)} \,\mathrm{d}x \,\mathrm{d}t \leq C.$$

Let $p \geq d$. By using the continuous embedding $\mathbb{W}_0^{1,p}(\Omega) \hookrightarrow \mathbb{L}^{2p}(\Omega)$, we get

$$\int_Q |\boldsymbol{u}_n|^{p+1} \,\mathrm{d}x \,\mathrm{d}t \leq \operatorname{ess\,sup}_{t \in I} \|\boldsymbol{u}_n\|_2 \int_Q |\nabla \boldsymbol{u}_n|^p \,\mathrm{d}x \,\mathrm{d}t \leq C.$$

Hence, for r such that $1 < r \leq \frac{p^2+p}{p^2+p-1}$ and Hölder's inequality with the exponent $\frac{p+1}{r}$ and its conjugate implies that

$$\int_Q |\boldsymbol{u}_n|^r |\nabla \boldsymbol{u}_n|^{r(p-1)} \,\mathrm{d}x\,\mathrm{d}t \leq C,$$

which finishes the proof of Lemma 2. ∎

The following lemma is useful for applying Proposition 1.

Lemma 3. *There exists some $\boldsymbol{u} \in \mathbb{L}^p(Q)$ such that (up to a subsequence) $(\boldsymbol{u}_n)_n$ converges to $\boldsymbol{u}$ in $\mathbb{L}^p(Q)$ and a.e.*

Proof. One can choose a subsequence of $(\boldsymbol{u}_n)_n$ such that (without relabeling)

- $(\boldsymbol{a}(\nabla \boldsymbol{u}_n))_n$ converges weakly in $\mathbb{L}^{p'}(Q)$ to some $\boldsymbol{\rho}$. This is due to Assumption (A$_2$) and the fact that $(\boldsymbol{u}_n)_n$ is bounded in $L^p(I, \mathbb{W}_0^{1,p}(\Omega))$,
- $(\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n))_n$ converges weakly in $\mathbb{L}^s(Q)$ for some $s > 1$, cf. Lemma 2. So, there exists $\boldsymbol{\delta} \in \mathbb{L}^s(Q)$ such that $\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n) \rightharpoonup \boldsymbol{\delta}$ weakly in $\mathbb{L}^s(Q)$,
- $(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n)_n$ converges weakly in $\mathbb{L}^{\frac{4}{3}}(Q)$ since $(\boldsymbol{u}_n)_n$ is bounded in $\mathbb{L}^4(Q)$,
- $(\boldsymbol{u}_n)_n$ converges weakly in $\mathbb{L}^2(Q)$ which is a consequence of the bound (3.8).

Now, we invoke Proposition 1 to get that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathbb{L}^p(Q)$ and a.e. for some further subsequence. Indeed, since $(\boldsymbol{u}_n)_n$ is bounded in $L^\infty(I, \mathbb{L}^2(\Omega))$ then it is bounded also in $L^\infty(I, \mathbb{L}^1(\Omega))$. Let $\boldsymbol{\varphi} \in \bigcup_{n\geq 1} V_n$ be arbitrary and $t \in [0, T]$. Then, from (3.4), one has

$$\begin{aligned}
&\left| \int_\Omega \big(\boldsymbol{u}_n(t,x) - \boldsymbol{u}_k(t,x)\big) \cdot \boldsymbol{\varphi}(x) \,\mathrm{d}x \right| \\
&= \left| \int_\Omega \int_0^t \partial_t \big(\boldsymbol{u}_n(s,x) - \boldsymbol{u}_k(s,x)\big) \cdot \boldsymbol{\varphi}(x) \,\mathrm{d}s\,\mathrm{d}x \right| \\
&\leq \kappa_1 \left| \int_0^t \int_\Omega (\boldsymbol{a}(\nabla \boldsymbol{u}_n) - \boldsymbol{a}(\nabla \boldsymbol{u}_k)) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad + \gamma \left| \int_0^t \int_\Omega (\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n) - \boldsymbol{u}_k \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_k)) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad + \kappa_2 \left| \int_0^t \int_\Omega \big(\boldsymbol{u}_n - \boldsymbol{u}_k\big) \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right| + \kappa_2 \beta \left| \int_0^t \int_\Omega (|\boldsymbol{u}_n|^2 \boldsymbol{u}_n - |\boldsymbol{u}_k|^2 \boldsymbol{u}_k) \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right|.
\end{aligned} \tag{3.11}$$

By employing the above convergences, one gets successively

$$\lim_{n,k\to+\infty} \left| \int_0^t \int_\Omega (\boldsymbol{a}(\nabla \boldsymbol{u}_n) - \boldsymbol{a}(\nabla \boldsymbol{u}_k)) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right| = 0,$$

$$\lim_{n,k\to+\infty} \left| \int_0^t \int_\Omega (\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n) - \boldsymbol{u}_k \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_k)) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right| = 0,$$

$$\lim_{n,k\to+\infty} \left| \int_0^t \int_\Omega (|\boldsymbol{u}_n|^2 \boldsymbol{u}_n - |\boldsymbol{u}_k|^2 \boldsymbol{u}_k) \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \right| = 0$$

and

$$\lim_{n,k\to+\infty}\left|\int_0^t\int_\Omega(\boldsymbol{u}_n-\boldsymbol{u}_k)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\right|=0.$$

Moreover, for any $\boldsymbol{v}\in\mathbb{L}^2(\Omega)$, there exists an approximating sequence $(\boldsymbol{\varphi}_i)_i\subset\bigcup_{n\geq 1}V_n$ such that $\boldsymbol{\varphi}_i\to\boldsymbol{v}$ in $\mathbb{L}^2(\Omega)$. Therefore, from (3.11), we conclude that $(\boldsymbol{u}_n(t,\cdot))_n$ is a Cauchy sequence in the weak topology of $\mathbb{L}^2(\Omega)$, for all $t\in[0,T]$. Hence, for each $t\in[0,T]$ there exists some $\bar{\boldsymbol{u}}(t)\in\mathbb{L}^2(\Omega)$ such that $\boldsymbol{u}_n(t)\rightharpoonup\bar{\boldsymbol{u}}(t)$ in $\mathbb{L}^2(\Omega)$. But since $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in $\mathbb{L}^2(Q)$ it is easy to see that

$$\boldsymbol{u}(t,\cdot)=\bar{\boldsymbol{u}}(t)\quad\text{a.e. in }[0,T].$$

Thus, Proposition 1 implies that $\boldsymbol{u}_n\to\boldsymbol{u}$ in $L^p(Q)$ and therefore $\boldsymbol{u}_n\to\boldsymbol{u}$ a.e. in Q for a further subsequence. ∎

3.3. Convergence of the approximate solutions

This subsection is dedicated to compactness arguments. We will choose from the above solutions a subsequence and prove that when we let $n\to\infty$, they converge to a measure-valued solution for the problem (1.4). Let $\boldsymbol{\varphi}$ be a test function lying in $\mathcal{D}(Q)$. By using integration by parts and the fact that $\boldsymbol{\varphi}(0,\cdot)=\boldsymbol{\varphi}(T,\cdot)=0$, we have

$$\int_0^T\int_\Omega\partial_t\boldsymbol{u}_n\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=-\int_0^T\int_\Omega\boldsymbol{u}_n\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t.$$

Due to the following convergence:

$$\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}\quad\text{weakly in }L^r(I,\mathbb{L}^2(\Omega))\cap L^p(I,\mathbb{W}_0^{1,p}(\Omega))$$

for all $r>1$, we obtain

$$\lim_{n\to+\infty}\int_0^T\int_\Omega\partial_t\boldsymbol{u}_n\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=-\int_0^T\int_\Omega\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t.$$

For the last term in (3.4), we have that $(|\boldsymbol{u}_n|^2\boldsymbol{u}_n)_n$ converges to $|\boldsymbol{u}|^2\boldsymbol{u}$ pointwise a.e. in Q. Furthermore, for each measurable subset $E\subset Q$, one has

$$\int_E\big||\boldsymbol{u}_n|^2\boldsymbol{u}_n\big|\,\mathrm{d}x\,\mathrm{d}t\leq|E|^{\frac14}\left(\int_E|\boldsymbol{u}_n|^4\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac34}\leq|E|^{\frac14}\sup_{m\in\mathbb{N}}\|\boldsymbol{u}_m\|_{\mathbb{L}^4(Q)}.$$

Thus, by applying Vitali's theorem, we infer that $|\boldsymbol{u}_n|^2\boldsymbol{u}_n\to|\boldsymbol{u}|^2\boldsymbol{u}$ strongly in $\mathbb{L}^1(Q)$. Moreover,

$$\begin{aligned}&\left|\int_Q\left[(1+\beta|\boldsymbol{u}_n|^2)\boldsymbol{u}_n-(1+\beta|\boldsymbol{u}|^2)\boldsymbol{u}\right]\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\right|\\&\quad\leq\left|\int_Q(\boldsymbol{u}_n-\boldsymbol{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\right|+\beta\left|\int_Q(|\boldsymbol{u}_n|^2\boldsymbol{u}_n-|\boldsymbol{u}|^2\boldsymbol{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\right|\to 0,\quad\text{as }n\to+\infty.\end{aligned}$$

It remains to find the limit of the nonlinear term given by $\boldsymbol{a}$. According to (3.3), the sequence $(\nabla \boldsymbol{u}_n)_n$ is bounded in $\mathbb{L}^p(Q)$. Because the components of the nonlinear continuous function $\boldsymbol{a}$ have $(p-1)$ growth, we can use Corollary 1 with

$$\boldsymbol{z}_j = \nabla \boldsymbol{u}_j, \quad q = p-1.$$

We obtain the existence of a measure valued function $\nu : Q \mapsto \mathrm{Prob}(\mathbb{R}^{d\times 3})$ such that

$$\boldsymbol{a}(\nabla \boldsymbol{u}_n) \rightharpoonup \bar{\boldsymbol{a}} \quad \text{in } \mathbb{L}^{\frac{p}{p-1}}(Q), \tag{3.12}$$

where

$$\bar{\boldsymbol{a}} = \int_{\mathbb{R}^{d\times 3}} \boldsymbol{a}(\boldsymbol{\lambda})\, \mathrm{d}\nu_{t,x}(\boldsymbol{\lambda}).$$

This means that

$$\int_Q \boldsymbol{a}(\nabla \boldsymbol{u}_n)\cdot \nabla \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t \to \int_Q \nabla \boldsymbol{\varphi} \cdot \int_{\mathbb{R}^{d\times 3}} \boldsymbol{a}(\boldsymbol{\lambda})\, \mathrm{d}\nu_{t,x}(\boldsymbol{\lambda})\, \mathrm{d}x\, \mathrm{d}t \quad \forall \boldsymbol{\varphi} \in \mathcal{D}(Q).$$

Now, for the term $\int_Q \boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n)\cdot \nabla \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t$, due to the strong convergence

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } \mathbb{L}^p(Q),$$

and thanks to (3.12), we deduce that

$$\boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n) \rightharpoonup \boldsymbol{u} \wedge \bar{\boldsymbol{a}} \quad \text{weakly in } \mathbb{L}^1(Q).$$

Therefore,

$$\int_Q \boldsymbol{u}_n \wedge \boldsymbol{a}(\nabla \boldsymbol{u}_n)\cdot \nabla \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t \to \int_Q \boldsymbol{u} \wedge \bar{\boldsymbol{a}} \cdot \nabla \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t \quad \forall \boldsymbol{\varphi} \in \mathcal{D}(Q).$$

Using again Corollary 1 with $\boldsymbol{\tau} = \mathrm{Id}$, $r = p$ and $q = 1$, we obtain for all $\boldsymbol{\varphi} \in \mathbb{L}^{p'}(Q)$

$$\int_Q \nabla \boldsymbol{u}_n \cdot \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t \to \int_Q \boldsymbol{\varphi} \cdot \int_{\mathbb{R}^{d\times 3}} \boldsymbol{\lambda}\, \mathrm{d}\nu_{t,x}(\boldsymbol{\lambda})\, \mathrm{d}x\, \mathrm{d}t.$$

Since $\nabla \boldsymbol{u}_n \rightharpoonup \nabla \boldsymbol{u}$ weakly in $\mathbb{L}^p(Q)$, it yields

$$\nabla \boldsymbol{u}(t,x) = \int_{\mathbb{R}^{d\times 3}} \boldsymbol{\lambda}\, \mathrm{d}\nu_{t,x}(\boldsymbol{\lambda}) \quad \text{a.e. in } Q.$$

This finishes the proof of the main result (Theorem 3).

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Chahid Ayouch
FST Marrakech, LAMAI Laboratory, Cadi Ayyad University, Guéliz, 40000 Marrakesh, Morocco; c.ayouch@uca.ac.ma

Driss Meskine
Ecole Supérieure de Technologie d'Essaouira, Cadi Ayyad University, Essaouira El Jadida, Essaouira, Morocco; dr.meskine@uca.ac.ma

Mouhcine Tilioua
FST Errachidia, MAIS Laboratory, MAMCS Group, Moulay Ismail University of Meknes, Boutalamine, 52000 Errachidia, Morocco; m.tilioua@umi.ac.ma