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Schrödinger equation for Sturm–Liouville operator with singular propagation and potential

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Abstract. In this paper, we consider an initial/boundary value problem for the Schrödinger equation with the Hamiltonian involving the fractional Sturm–Liouville operator with singular propagation and potential. To construct a solution, first considering the coefficients in a regular sense, the method of separation of variables is used, which leads the solution of the equation to the eigenvalue and eigenfunction problem of the Sturm–Liouville operator. Next, using the Fourier series expansion in eigenfunctions, a solution to the Schrödinger equation is constructed. Important estimates related to the Sobolev space are also obtained. In addition, the equation is studied in the case where the initial data, propagation, and potential are strongly singular. For this case, the concept of "very weak solutions" is used. The existence, uniqueness, negligibility, and consistency of very weak solution of the Schrödinger equation are established.

1. Introduction

The main goal of this paper is to establish the existence of physical solutions for the Schrödinger equation, specifically when it involves the Sturm–Liouville operator with singular potentials. When tackling problems with strong singularities, a prior study by [8] introduced the concept of "very weak solutions". This approach is necessary because when the equation involves products of various terms, it can no longer be clearly defined in spaces of distributions. Consequently, we require an alternative way to determine the well posedness of the equation.

The development of very weak solutions for various types of problems continued in several works, such as [1–6, 13, 14, 16]. In the works [12, 15], the concept of very weak solutions of the wave equation for the Sturm–Liouville operator with singular potentials in bounded domains was expanded.

It is known that the Schrödinger equation can be simplified into ordinary linear equations using the "separation of variables" method; see, e.g., [9]. To present our main findings, we provide some initial information about the Sturm–Liouville operator with singular potentials. Savchuk and Shkalikov's study in [18] yielded eigenvalues and eigenfunctions for this operator. Additionally, studies in [11,17,19,20] explored the Sturm–Liouville

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operator with potential distributions. To establish the framework for very weak solutions, our focus is primarily on estimating solutions for more regular problems while also considering the impact of a regularization parameter on these solutions.

For further reasoning and obtaining our results, we need some preliminaries about the Sturm–Liouville operator with singular potentials. More specifically, we consider the Sturm–Liouville operator $\mathcal{L}$ generated on the interval (0,1) by the differential expression

$$\mathcal{L}y := -\frac{d^2}{dx^2}y + q(x)y, \tag{1.1}$$

with the boundary conditions

$$y(0) = y(1) = 0. \tag{1.2}$$

The potential q is defined as

$$q(x) = \nu'(x) \geq 0, \quad \nu \in L^2(0,1). \tag{1.3}$$

The eigenvalues of the Sturm–Liouville operator $\mathcal{L}$ generated on the interval (0,1) by the differential expression (1.1) with the boundary conditions (1.2) are real [10] and given by

$$\lambda_n = (\pi n)^2(1 + o(n^{-1})), \quad n = 1, 2, \ldots, \tag{1.4}$$

and the corresponding eigenfunctions are

$$\tilde{\phi}_n(x) = r_n(x)\sin\theta_n(x), \tag{1.5}$$

where

$$r_n(x) = \exp\left(-\int_0^x \nu(s)\cos 2\theta_n(s)ds + o(1)\right) = 1 + o(1),$$

$$\theta_n(x) = \sqrt{\lambda_n}x + o(1)$$

for $n \to \infty$. According to (1.3), (1.4), and (1.5), it is clear that the $\tilde{\phi}_n$ are real. Here and below, we will have the positive operator $\langle \mathcal{L}y, y\rangle \geq 0$, which implies that all eigenvalues λ_n are real and non-negative.

The first derivatives of $\tilde{\phi}_n$ are given by the formulas

$$\tilde{\phi}_n'(x) = \sqrt{\lambda_n}r_n(x)\cos(\theta_n(x)) + \nu(x)\tilde{\phi}_n(x). \tag{1.6}$$

According to [17, Theorem 2], we have

$$\tilde{\phi}_n(x) = \sin\sqrt{\lambda_n}x + \psi_n(x), \quad n = 1, 2, \ldots, \quad \sum_{n=1}^{\infty}\|\psi_n\|^2 \leq C\int_0^1 |\nu(x)|^2 dx.$$

On the other hand, we can estimate the $\|\tilde{\phi}_n\|_{L^2}$ using formula (1.5) as follows:

$$\|\tilde{\phi}_n\|_{L^2}^2 \lesssim \exp\left(\|\nu\|_{L^2} + \lambda^{-\frac{1}{2}}\|\nu\|_{L^2}^2\right) < \infty. \tag{1.7}$$

Also, according to [18, Theorem 4], we have

$$\widetilde{\phi}_n(x) = \sin(\pi n x) + o(1) \tag{1.8}$$

for sufficiently large n. Along with (1.5), we see that there exist some $C_0 > 0$ such that

$$0 < C_0 \le \|\widetilde{\phi}_n\|_{L^2} < \infty$$

for all n.

Since the eigenfunctions of the Sturm–Liouville operator form an orthogonal basis in $L^2(0,1)$, we normalise them for further use:

$$\phi_n(x) = \frac{\widetilde{\phi}_n(x)}{\sqrt{\langle \widetilde{\phi}_n, \widetilde{\phi}_n \rangle}} = \frac{\widetilde{\phi}_n(x)}{\|\widetilde{\phi}_n\|_{L^2}}. \tag{1.9}$$

2. Non-homogeneous Schrödinger equation

We consider the non-homogeneous Schrödinger equation with initial/boundary conditions

$$\begin{cases} i\partial_t u(t,x) + a(t)\mathcal{L}^s u(t,x) = f(t,x), & (t,x) \in [0,T] \times (0,1), \\ u(0,x) = u_0(x), & x \in (0,1), \\ u(t,0) = 0 = u(t,1), & t \in [0,T], \end{cases} \tag{2.1}$$

where $a(t) \ge a_0 > 0$ for $t \in [0,T]$ and $a \in L^\infty[0,T]$, $s \in \mathbb{R}$, with operator $\mathcal{L}$ defined by

$$\mathcal{L} = -\frac{\partial^2}{\partial x^2} + q(x), \quad x \in (0,1),$$

and $q = \nu' \ge 0$, $\nu \in L^2(0,1)$.

It is well known [12, 15] that the general solution to this equation is

$$u(t,x) = u_1(t,x) + u_2(t,x),$$

where $u_1(t,x)$ is the general solution to the homogeneous Schrödinger equation

$$i\partial_t u(t,x) + a(t)\mathcal{L}^s u(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \tag{2.2}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{2.3}$$

and with Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,1), \quad t \in [0,T], \tag{2.4}$$

and $u_2(t,x)$ is the particular solution to the non-homogeneous Schrödinger equation with initial/boundary conditions (2.1). In other words, to get a solution to (2.1), we need to consider problem (2.2)–(2.4).

In our results below, concerning the initial/boundary problem (2.2)–(2.4), as a preliminary step, we first carry out the analysis in the regular case for bounded $q \in L^\infty(0,1)$. In this case, we obtain the well posedness in the Sobolev spaces $W^k_{\mathcal{L}}$ associated to the operator $\mathcal{L}$: we define the Sobolev space $W^k_{\mathcal{L}}$ associated to $\mathcal{L}$, for any $k \in \mathbb{R}$, as the space

$$W^k_{\mathcal{L}} := \left\{ f \in \mathcal{D}'_{\mathcal{L}}(0,1) : \mathcal{L}^{k/2} f \in L^2(0,1) \right\},$$

with the norm $\|f\|_{W^k_{\mathcal{L}}} := \|\mathcal{L}^{k/2} f\|_{L^2}$. The global space of distributions $\mathcal{D}'_{\mathcal{L}}(0,1)$ is defined as follows.

The space $C^\infty_{\mathcal{L}}(0,1) := \mathrm{Dom}(\mathcal{L}^\infty)$ is called the space of test functions for $\mathcal{L}$, where we define

$$\mathrm{Dom}(\mathcal{L}^\infty) := \bigcap_{m=1}^{\infty} \mathrm{Dom}(\mathcal{L}^m),$$

where $\mathrm{Dom}(\mathcal{L}^m)$ is the domain of the operator $\mathcal{L}^m$, in turn defined as

$$\mathrm{Dom}(\mathcal{L}^m) := \left\{ f \in L^2(0,1) : \mathcal{L}^j f \in \mathrm{Dom}(\mathcal{L}),\ j = 0,1,2,\ldots,m-1 \right\}.$$

The Fréchet topology of $C^\infty_{\mathcal{L}}(0,1)$ is given by the family of norms

$$\|\phi\|_{C^m_{\mathcal{L}}} := \max_{j \le m} \|\mathcal{L}^j \phi\|_{L^2(0,1)}, \quad m \in \mathbb{N}_0,\ \phi \in C^\infty_{\mathcal{L}}(0,1).$$

The space of $\mathcal{L}$-distributions

$$\mathcal{D}'_{\mathcal{L}}(0,1) := \mathbf{L}(C^\infty_{\mathcal{L}}(0,1), \mathbb{C})$$

is the space of all linear continuous functionals on $C^\infty_{\mathcal{L}}(0,1)$. For $\omega \in \mathcal{D}'_{\mathcal{L}}(0,1)$ and $\phi \in C^\infty_{\mathcal{L}}(0,1)$, we will write

$$\omega(\phi) = \langle \omega, \phi \rangle.$$

For any $\psi \in C^\infty_{\mathcal{L}}(0,1)$, the functional

$$C^\infty_{\mathcal{L}}(0,1) \ni \phi \mapsto \int_0^1 \psi(x)\phi(x)dx$$

is an $\mathcal{L}$-distribution, which gives an embedding $\psi \in C^\infty_{\mathcal{L}}(0,1) \hookrightarrow \mathcal{D}'_{\mathcal{L}}(0,1)$.

We introduce the spaces $C^j([0,T], W^k_{\mathcal{L}}(0,1))$ given by the family of norms

$$\|f\|_{C^n([0,T],W^k_{\mathcal{L}}(0,1))} = \max_{0 \le t \le T} \sum_{j=0}^{n} \left\| \partial_t^j f(t,\cdot) \right\|_{W^k_{\mathcal{L}}},$$

where $k \in \mathbb{R},\ f \in C^j([0,T], W^k_{\mathcal{L}}(0,1))$.

Theorem 2.1. *Assume that $q \in L^\infty(0,1)$, $q \ge 0$, $a(t) \ge a_0 > 0$ for all $t \in [0,T]$, and $a \in L^\infty[0,T]$. For any $k \in \mathbb{R}$, if the initial condition satisfies $u_0 \in W^k_{\mathcal{L}}$, then the Schrödinger* *equation* (2.2) *with the initial/boundary conditions* (2.3)–(2.4) *has a unique solution* $u \in C([0,T], W^k_{\mathcal{L}})$. *We also have the following estimates:*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2}, \tag{2.5}$$

$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim \|a\|_{L^\infty[0,T]}\|u_0\|_{W^{2s}_{\mathcal{L}}}. \tag{2.6}$$

When $s = 1$, *we also have*

$$\|\partial_x u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{W^1_{\mathcal{L}}}(1 + \|\nu\|_{L^2}) + \|u_0\|_{L^2}\|\nu\|_{L^\infty}, \tag{2.7}$$

$$\|\partial_x^2 u(t,\cdot)\|_{L^2} \lesssim \|q\|_{L^\infty}\|u_0\|_{L^2} + \|u_0\|_{W^2_{\mathcal{L}}}, \tag{2.8}$$

$$\|u(t,\cdot)\|_{W^k_{\mathcal{L}}} \lesssim \|u_0\|_{W^k_{\mathcal{L}}}, \tag{2.9}$$

where the constants in these inequalities are independent of u_0, ν, q, *and* a.

We note that $q \in L^\infty(0,1)$ implies that $\nu \in L^\infty(0,1)$ and hence $\nu \in L^2(0,1)$ so that the formulas in the introduction hold true.

Proof. We apply the technique of the separation of variables (see, e.g., [9]). In particular, we are looking for a solution of the form

$$u(t,x) = T(t)X(x),$$

where $T(t)$, $X(x)$ are unknown functions that must be determined. Substituting $u(t,x) = T(t)X(x)$ into equation (2.2) and after simple transformations, we get for the function $T(t)$ the equation

$$T'(t) = i\mu a(t)T(t), \quad t \in [0,T], \tag{2.10}$$

and for the function $X(x)$, we get

$$\mathcal{L}^s X(x) = \mu X(x), \tag{2.11}$$

where μ is a spectral parameter. When $s = 1$, we obtain the Sturm–Liouville boundary value problem

$$\mathcal{L}X(x) := -X''(x) + q(x)X(x) = \lambda X(x), \tag{2.12}$$

$$X(0) = X(1) = 0. \tag{2.13}$$

Equation (2.12) with the boundary condition (2.13) has the eigenvalues of the form (1.4) with the corresponding eigenfunctions of the form (1.5) of the Sturm–Liouville operator $\mathcal{L}$ generated by the differential expression (1.1). Substituting

$$\mu_n = \lambda_n^s,$$

we get the eigenvalues of the form (1.4) and the corresponding eigenfunctions of the form (1.5) for equation (2.11), i.e.,

$$\mathcal{L}^s \phi_n(x) = \lambda_n^s \phi_n(x). \tag{2.14}$$

The solution to equation (2.10) with the initial conditions (2.3) is

$$T_n(t) = D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau},$$

where

$$D_n = \int_0^1 u_0(x)\phi_n(x)dx.$$

Taking into account the last expressions, we can write the solution of the homogeneous equation (2.2) with initial/boundary conditions (2.3)–(2.4) in the following form:

$$u(t,x) = \sum_{n=1}^{\infty} D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau}\phi_n(x).$$

Further, we will prove that $u \in C^2([0,T], L^2(0,1))$. By using the Cauchy–Schwarz inequality and fixed t, we can deduce that

$$\begin{aligned} \|u(t,\cdot)\|_{L^2}^2 &= \int_0^1 |u(t,x)|^2 dx = \int_0^1 \left| \sum_{n=1}^{\infty} D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau}\phi_n(x)\right|^2 dx \\ &\lesssim \int_0^1 \sum_{n=1}^{\infty} \left|D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau}\right|^2 |\phi_n(x)|^2 dx. \end{aligned} \tag{2.15}$$

According to (1.4), (1.9), using Euler's formula and Parseval's identity, we obtain

$$\begin{aligned} \|u(t,\cdot)\|_{L^2}^2 &\lesssim \int_0^1 \sum_{n=1}^{\infty} \left|D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau}\right|^2 |\phi_n(x)|^2 dx = \sum_{n=1}^{\infty} |D_n|^2 \int_0^1 |\phi_n(x)|^2 dx \\ &= \sum_{n=1}^{\infty} |D_n|^2 = \int_0^1 |u_0(x)|^2 dx = \|u_0\|_{L^2}^2. \end{aligned} \tag{2.16}$$

Since $a \in L^\infty[0,T]$ and using (2.16), we obtain

$$\begin{aligned} \|\partial_t u(t,\cdot)\|_{L^2}^2 &= \int_0^1 |\partial_t u(t,x)|^2 dt = \int_0^1 \left| \sum_{n=1}^{\infty} \left((i\lambda_n^s)a(t) D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau}\phi_n(x)\right)\right|^2 dx \\ &\lesssim \int_0^1 \sum_{n=1}^{\infty} |a(t)|^2 |\lambda_n^s D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau}|^2 |\phi_n(x)|^2 dx \\ &\leq \sum_{n=1}^{\infty} \|a\|_{L^\infty[0,T]}^2 |\lambda_n^s D_n|^2 \int_0^1 |\phi_n(x)|^2 dx \\ &= \|a\|_{L^\infty[0,T]}^2 \sum_{n=1}^{\infty} |\lambda_n^s D_n|^2. \end{aligned} \tag{2.17}$$

Since λ_n are eigenvalues and ϕ_n are eigenfunctions of the operator $\mathcal{L}$, using Parseval's identity, we obtain

$$\sum_{n=1}^{\infty}|\lambda_n^s D_n|^2 = \sum_{n=1}^{\infty}\left|\lambda_n^s \int_0^1 u_0(x)\phi_n(x)dx\right|^2 = \sum_{n=1}^{\infty}\left|\int_0^1 \lambda_n^s u_0(x)\phi_n(x)dx\right|^2$$
$$= \sum_{n=1}^{\infty}\left|\int_0^1 \mathcal{L}^s u_0(x)\phi_n(x)dx\right|^2 = \|\mathcal{L}^s u_0\|_{L^2}^2 = \|u_0\|_{W_{\mathcal{L}}^{2s}}^2. \quad (2.18)$$

Thus,

$$\|\partial_t u(t,\cdot)\|_{L^2}^2 \lesssim \|a\|_{L^\infty[0,T]}^2 \|u_0\|_{W_{\mathcal{L}}^{2s}}^2.$$

Let $s = 1$; then to estimate the norm of $\partial_x u(t,\cdot)$ in L^2 we use (1.6) and (1.9) for ϕ_n':

$$\|\partial_x u(t,\cdot)\|_{L^2}^2 = \int_0^1 |\partial_x u(t,x)|^2 dt = \int_0^1 \left|\sum_{n=1}^{\infty} D_n e^{i\lambda_n \int_0^t a(\tau)d\tau}\phi_n'(x)\right|^2 dx$$
$$= \int_0^1 \left|\sum_{n=1}^{\infty} D_n e^{i\lambda_n \int_0^t a(\tau)d\tau}\left(\frac{\sqrt{\lambda_n} r_n(x)\cos\theta_n(x)}{\|\widetilde{\phi}_n\|_{L^2}} + \nu(x)\phi_n(x)\right)\right|^2 dx.$$

According to (2.16), (1.7), and (1.8), there exist some $C_0 > 0$ such that $C_0 < \|\widetilde{\phi}_n\|_{L^2} < \infty$ so that

$$\|\partial_x u(t,\cdot)\|_{L^2}^2 \lesssim \sum_{n=1}^{\infty}\left(|\sqrt{\lambda_n}D_n|^2 \int_0^1 |r_n(x)|^2 dx\right) + \sum_{n=1}^{\infty}\left(|D_n|^2 \int_0^1 |\nu(x)\phi_n(x)|^2 dx\right).$$

Here, for $r_n(x)$ according to [17, Theorem 2], we have

$$r_n(x) = 1 + \rho_n(x), \quad \|\rho_n\|_{L^2} \lesssim \|\nu\|_{L^2},$$

where the constant is independent of ν and n. Therefore,

$$\int_0^1 |r_n(x)|^2 dx \lesssim 1 + \|\nu\|_{L^2}^2.$$

For the second term, we obtain

$$\int_0^1 |\nu(x)\phi_n(x)|^2 dx \le \|\nu\|_{L^\infty}^2 \|\phi_n\|_{L^2}^2 = \|\nu\|_{L^\infty}^2,$$

since $\{\phi_n\}$ is an orthonormal basis in L^2. Using the property of the operator $\mathcal{L}$ and the Parseval identity, we obtain

$$\sum_{n=1}^{\infty}|\sqrt{\lambda_n}D_n|^2 = \sum_{n=1}^{\infty}\left|\int_0^1 \sqrt{\lambda_n}u_0(x)\phi_n(x)dx\right|^2$$
$$= \sum_{n=1}^{\infty}\left|\int_0^1 \mathcal{L}^{\frac{1}{2}} u_0(x)\phi_n(x)dx\right|^2 = \left\|\mathcal{L}^{\frac{1}{2}} u_0\right\|_{L^2}^2 = \|u_0\|_{W_{\mathcal{L}}^1}^2.$$

Using the last relations, we obtain

$$\begin{aligned}\|\partial_x u(t,\cdot)\|^2_{L^2} &\lesssim \sum_{n=1}^{\infty} \left|\sqrt{\lambda_n} D_n\right|^2 \left(1+\|\nu\|^2_{L^2}\right) + \sum_{n=1}^{\infty} |D_n|^2 \|\nu\|^2_{L^\infty} \\ &\le \|u_0\|^2_{W^1_{\mathcal{L}}} \left(1+\|\nu\|^2_{L^2}\right) + \|u_0\|^2_{L^2}\|\nu\|^2_{L^\infty}, \end{aligned} \tag{2.19}$$

implying (2.7).

Let us get the next estimate by using the fact that $\phi_n''(x) = (q(x)-\lambda_n)\phi_n(x)$ in the case when $s = 1$,

$$\begin{aligned}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2} &= \int_0^1 \left|\partial_x^2 u(t,x)\right|^2 dx = \int_0^1 \left|\sum_{n=1}^{\infty} D_n e^{i\lambda_n \int_0^t a(\tau)d\tau} \phi_n''(x)\right|^2 dx \\ &\lesssim \int_0^1 \left(\sum_{n=1}^{\infty} \left|D_n e^{i\lambda_n \int_0^t a(\tau)d\tau}\right|^2 |(q(x)-\lambda_n)\phi_n(x)|^2\right) dx \\ &\lesssim \int_0^1 |q(x)|^2 \sum_{n=1}^{\infty} |D_n|^2 |\phi_n(x)|^2 dx + \int_0^1 \sum_{n=1}^{\infty} |\lambda_n D_n|^2 |\phi_n(x)|^2 dx \\ &\le \|q\|^2_{L^\infty} \sum_{n=1}^{\infty} |D_n|^2 + \sum_{n=1}^{\infty} |\lambda_n D_n|^2. \end{aligned} \tag{2.20}$$

Taking into account (2.18) for the last terms in (2.20), we obtain

$$\sum_{n=1}^{\infty} |\lambda_n D_n|^2 = \|u_0\|^2_{W^2_{\mathcal{L}}}.$$

Using the last expressions and (2.18), we finally get

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2} \lesssim \|q\|^2_{L^\infty} \|u_0\|^2_{L^2} + \|u_0\|^2_{W^2_{\mathcal{L}}},$$

implying (2.8).

Let us carry out the last estimate (2.9) using the fact that $\mathcal{L}^k u = \lambda_n^k u$ and Parseval's identity:

$$\begin{aligned}\|u(t,\cdot)\|^2_{W^k_{\mathcal{L}}} &= \left\|\mathcal{L}^{\frac{k}{2}} u(t,\cdot)\right\|^2_{L^2} = \int_0^1 \left|\mathcal{L}^{\frac{k}{2}} u(t,x)\right|^2 dx = \int_0^1 \left|\sum_{n=1}^{\infty} D_n e^{i\lambda_n t} \lambda_n^{\frac{k}{2}} \phi_n(x)\right|^2 dx \\ &\lesssim \sum_{n=1}^{\infty} \left|\lambda_n^{\frac{k}{2}} D_n\right|^2 = \left\|\mathcal{L}^{\frac{k}{2}} u_0\right\|^2_{L^2} = \|u_0\|^2_{W^k_{\mathcal{L}}}. \end{aligned}$$

The proof of Theorem 2.1 is complete. ■

The following statement removes the reliance on Sobolev spaces with respect to $\mathcal{L}$ while sacrificing the regularity of the data. This statement will be important for further analysis.

Corollary 2.2. *Let $s = 1$. Assume that q, $\nu \in L^\infty(0,1)$, $q \geq 0$, $a(t) \geq a_0 > 0$ for all $t \in [0,T]$, and $a \in L^\infty[0,T]$. If the initial condition satisfies $u_0 \in L^2(0,1)$ and $u_0'' \in L^2(0,1)$, then the Schrödinger equation (2.2) with the initial/boundary conditions (2.3)–(2.4) has a unique solution $u \in C([0,T], L^2(0,1))$ which satisfies the estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2}, \tag{2.21}$$

$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim \|a\|_{L^\infty[0,T]}(\|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2}), \tag{2.22}$$

$$\|\partial_x u(t,\cdot)\|_{L^2} \lesssim (\|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2})(1 + \|\nu\|_{L^2}) + \|u_0\|_{L^2}\|\nu\|_{L^\infty}, \tag{2.23}$$

$$\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2} \lesssim \|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2}, \tag{2.24}$$

where the constants in these inequalities are independent of u_0, q, and a.

Proof. The inequality (2.21) immediately follows from (2.5). Let us move on to estimating the inequality (2.22). In Theorem 2.1, we obtained estimates with respect to the operator $\mathcal{L}$, but here we want to obtain estimates with respect to the initial condition u_0 and potential $q(x)$.

By (2.17), we have

$$\|\partial_t u(t,\cdot)\|^2 \lesssim \|a\|^2_{L^\infty[0,T]} \sum_{n=1}^{\infty} |\lambda_n D_n|^2.$$

Since λ_n are the eigenvalues of the operator $\mathcal{L}$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |\lambda_n D_n|^2 &= \sum_{n=1}^{\infty} \left| \int_0^1 \lambda_n u_0(x)\phi_n(x)dx \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \int_0^1 (-u_0''(x) + q(x)u_0(x))\phi_n(x)dx \right|^2 \\
&\lesssim \sum_{n=1}^{\infty} \left| \int_0^1 u_0''(x)\phi_n(x)dx \right|^2 + \sum_{n=1}^{\infty} \left| \int_0^1 q(x)u_0(x)\phi_n(x)dx \right|^2.
\end{aligned} \tag{2.25}$$

Using Parseval's identity for the first and second terms in (2.25) and since $q \in L^\infty$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\lambda_n D_n|^2 &\lesssim \sum_{n=1}^{\infty} \left| \int_0^1 q(x)u_0(x)\phi_n(x)dx \right|^2 + \sum_{n=1}^{\infty} \left| \int_0^1 u_0''(x)\phi_n(x)dx \right|^2 \\
&= \sum_{n=1}^{\infty} |\langle (qu_0), \phi_n\rangle|^2 + \sum_{n=1}^{\infty} |u''_{0,n}|^2 = \|qu_0\|^2_{L^2} + \|u_0''\|^2_{L^2} \\
&\leq \|q\|^2_{L^\infty}\|u_0\|^2_{L^2} + \|u_0''\|^2_{L^2}.
\end{aligned} \tag{2.26}$$

Thus,

$$\|\partial_t u(t,\cdot)\|^2_{L^2} \lesssim \|a\|^2_{L^\infty[0,T]}\big(\|u_0''\|^2_{L^2} + \|q\|^2_{L^\infty}\|u_0\|^2_{L^2}\big),$$

proving (2.22).

Taking into account (2.19), (1.4), using (2.26) and Parseval's identity, we get

$$\begin{aligned}\|\partial_x u(t,\cdot)\|_{L^2}^2 &\lesssim \sum_{n=1}^{\infty} \left|\sqrt{\lambda_n} D_n\right|^2 \left(1+\|\nu\|_{L^2}^2\right) + \sum_{n=1}^{\infty} |D_n|^2 \|\nu\|_{L^\infty}^2 \\ &\le \sum_{n=1}^{\infty} |\lambda_n D_n|^2 \left(1+\|\nu\|_{L^2}^2\right) + \sum_{n=1}^{\infty} |D_n|^2 \|\nu\|_{L^\infty}^2 \\ &\lesssim \left(\|u_0''\|_{L^2}^2 + \|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2\right)\left(1+\|\nu\|_{L^2}^2\right) + \|u_0\|_{L^2}^2 \|\nu\|_{L^\infty}^2,\end{aligned}$$

implying (2.23).

Using (2.20), (2.26), and Parseval's identity, we obtain

$$\begin{aligned}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2}^2 &\lesssim \|q\|_{L^\infty}^2 \sum_{n=1}^{\infty} |D_n|^2 + \sum_{n=1}^{\infty} |\lambda_n D_n|^2 \\ &\lesssim \|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2 + \|u_0''\|_{L^2}^2 + \|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2 \\ &= \|u_0''\|_{L^2}^2 + 2\|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2.\end{aligned}$$

The proof of Corollary 2.2 is complete. ∎

Using the statements for the homogeneous case, one can establish the following statements for the non-homogeneous Schrödinger initial/boundary problem (2.1).

Theorem 2.3. *Assume that $q \in L^\infty(0,1)$, $q \ge 0$, $a \in L^\infty[0,T]$, $a(t) \ge a_0 > 0$ for all $t \in [0,T]$, and $f \in C^1([0,T], W_{\mathcal{L}}^k(0,1))$ for some $k \in \mathbb{R}$. If the initial condition satisfies $u_0 \in W_{\mathcal{L}}^k(0,1)$, then the non-homogeneous Schrödinger equation with initial/boundary conditions (2.1) has the unique solution $u \in C([0,T], W_{\mathcal{L}}^k)$ which satisfies the estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + T\|f\|_{C([0,T],L^2(0,1))}, \tag{2.27}$$

$$\begin{aligned}\|\partial_t u(t,\cdot)\|_{L^2} &\lesssim \|a\|_{L^\infty[0,T]}\left(\|u_0\|_{W_{\mathcal{L}}^{2s}} + T\|f\|_{C^1([0,T],W_{\mathcal{L}}^{2s}(0,1))}\right) \\ &\quad + T\|f\|_{C^1([0,T],L^2(0,1))}.\end{aligned} \tag{2.28}$$

When $s = 1$, we also have

$$\|\partial_x u(t,\cdot)\|_{L^2} \lesssim (1+\|\nu\|_{L^\infty})(\|u_0\|_{W_{\mathcal{L}}^1} + T\|f\|_{C([0,T],W_{\mathcal{L}}^1(0,1))}), \tag{2.29}$$

$$\begin{aligned}\|\partial_x^2 u(t,\cdot)\|_{L^2} &\lesssim \|q\|_{L^\infty}(\|u_0\|_{L^2} + T\|f\|_{C([0,T],L^2(0,1))}) \\ &\quad + \|u_0\|_{W_{\mathcal{L}}^2} + T\|f\|_{C^1([0,T],W_{\mathcal{L}}^2(0,1))},\end{aligned} \tag{2.30}$$

where the constants in these inequalities are independent of u_0, q, a, and f.

Proof. We can use the eigenfunctions (1.5) of the corresponding (homogeneous) eigenvalue problem (2.11) and look for a solution in the series form

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x), \tag{2.31}$$

where

$$u_n(t) = \int_0^1 u(t,x)\phi_n(x)dx.$$

We can similarly expand the source function

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x), \quad f_n(t) = \int_0^1 f(t,x)\phi_n(x)dx. \tag{2.32}$$

Now, since we are looking for a twice differentiable function $u(t,x)$ that satisfies the homogeneous Dirichlet boundary conditions, we can use (2.14) to the Fourier series (2.31) term by term and using $\phi_n(x)$ satisfies equation (2.11) to obtain

$$\mathcal{L}^s u(t,x) = \sum_{n=1}^{\infty} \mathcal{L}^s(u_n(t)\phi_n(x)) = \sum_{n=1}^{\infty} u_n(t)\lambda_n^s\phi_n(x). \tag{2.33}$$

We can also differentiate the series (2.32) with respect to t to obtain

$$u_t(t,x) = \sum_{n=1}^{\infty} u_n'(t)\phi_n(x), \tag{2.34}$$

since the Fourier coefficients of $u_t(t,x)$ are

$$\int_0^1 u_t(t,x)\phi_n(x)dx = \frac{\partial}{\partial t}\left[\int_0^1 u(t,x)\phi_n(x)dx\right] = u_n'(t).$$

Differentiation under the above integral is allowed since the resulting integrand is continuous.

Substituting (2.34) and (2.33) into the equation and using (2.32), we have

$$i\sum_{n=1}^{\infty} u_n'(t)\phi_n(x) + a(t)\sum_{n=1}^{\infty} u_n(t)\lambda_n^s\phi_n(x) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x).$$

Due to the completeness,

$$iu_n'(t) + \lambda_n^s a(t)u_n(t) = f_n(t), \quad n = 1, 2, \ldots,$$

which are ODEs for the coefficients $u_n(t)$ of the series (2.31). By the method of variation of constants, we get

$$u_n(t) = D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} + e^{i\lambda_n^s \int_0^t a(\tau)d\tau}\int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds,$$

where

$$D_n = \int_0^1 u_0(x)\phi_n(x)dx.$$

Thus, we can write a solution to equation (2.1) in the form

$$u(t,x) = \sum_{n=0}^{\infty} D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \phi_n(x) + \sum_{n=0}^{\infty} e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x).$$

Let us estimate $\|u(t,\cdot)\|_{L^2}^2$. For this, we use the estimates

$$\begin{aligned}\int_0^1 |u(t,x)|^2 dx &= \int_0^1 \left| \sum_{n=0}^{\infty} D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \phi_n(x) + \sum_{n=0}^{\infty} e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \right|^2 dx \\ &\lesssim \int_0^1 \left| \sum_{n=0}^{\infty} D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \phi_n(x) \right|^2 dx \\ &\quad + \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \right|^2 dx \\ &= I_1 + I_2. \end{aligned} \tag{2.35}$$

For I_1, by using (2.15)–(2.16) for the homogeneous case, we have that

$$I_1 := \int_0^1 \left| \sum_{n=0}^{\infty} D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \phi_n(x) \right|^2 dx \lesssim \|u_0\|_{L^2}^2.$$

Now, we estimate I_2 in (2.35) taking into account that $s \in [0,t]$:

$$\begin{aligned} I_2 &:= \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \right|^2 dx \\ &\le \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n^s \int_0^t a(\tau)d\tau} e^{-i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t f_n(s)ds\phi_n(x) \right|^2 dx \\ &= \int_0^1 \left| \sum_{n=0}^{\infty} \int_0^t f_n(s)ds\phi_n(x) \right|^2 dx \lesssim \sum_{n=1}^{\infty} \left[\int_0^t |f_n(s)|ds \right]^2. \end{aligned}$$

Using Holder's inequality and taking into account that $t \in [0,T]$, we get

$$\left[\int_0^t |f_n(s)|ds \right]^2 \le \left[\int_0^T 1 \cdot |f_n(t)|dt \right]^2 \le T \int_0^T |f_n(t)|^2 dt,$$

since $f_n(t)$ is the Fourier coefficient of the function $f(t,x)$, and by Parseval's identity, we obtain

$$\sum_{n=1}^{\infty} T\int_0^T |f_n(t)|^2 dt = T\int_0^T \sum_{n=1}^{\infty} |f_n(t)|^2 dt = T\int_0^T \|f(t,\cdot)\|_{L^2}^2 dt.$$

Since

$$\|f\|_{C([0,T],L^2(0,1))} = \max_{0\le t\le T} \|f(t,\cdot)\|_{L^2},$$

we arrive at the inequality

$$T\int_0^T \|f(t,\cdot)\|_{L^2}^2 dt \le T^2 \|f\|_{C([0,T],L^2(0,1))}^2.$$

Thus,

$$\begin{aligned} I_2 &:= \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \right|^2 dx \\ &\lesssim T^2 \|f\|_{C([0,T],L^2(0,1))}^2. \end{aligned} \tag{2.36}$$

We finally get

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + T^2 \|f\|_{C([0,T],L^2(0,1))}^2,$$

implying (2.27).

Let us estimate $\|\partial_t u(t,\cdot)\|_{L^2}$; for this, we calculate $\partial_t u(t,x)$ as follows:

$$\begin{aligned} \partial_t u(t,x) = &\sum_{n=0}^{\infty} i\lambda_n^s a(t) D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \phi_n(x) \\ &+ \sum_{n=0}^{\infty} i\lambda_n^s a(t) e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \\ &+ \sum_{n=0}^{\infty} f_n(t)\phi_n(x). \end{aligned}$$

Then,

$$\begin{aligned} \|\partial_t u(t,\cdot)\|_{L^2}^2 = \int_0^1 |\partial_t u(t,x)|^2 dx \lesssim &\int_0^1 \left| \sum_{n=0}^{\infty} i\lambda_n^s a(t) D_n e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \phi_n(x) \right|^2 dx \\ &+ \int_0^1 \left| \sum_{n=0}^{\infty} i\lambda_n^s a(t) e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \right|^2 dx \\ &+ \int_0^1 \left| \sum_{n=0}^{\infty} f_n(t)\phi_n(x) \right|^2 dx = J_1 + J_2 + J_3. \end{aligned} \tag{2.37}$$

Here, for J_1 by using (2.17) and (2.18) and taking into account (2.32) for the function $f(t,x)$ in J_3, we obtain

$$\|\partial_t u(t,\cdot)\|_{L^2}^2 \lesssim \|a\|_{L^\infty[0,T]}^2 \|u_0\|_{W_{\mathcal{L}}^{2s}}^2 + J_2 + \|f(t,\cdot)\|_{L^2}^2.$$

To estimate J_2, conducting evaluations as in (2.36) and taking into account (2.32), we obtain

$$\begin{aligned}
J_2 &:= \int_0^1 \left| \sum_{n=0}^{\infty} i\lambda_n^s a(t) e^{i\lambda_n^s \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n^s \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x) \right|^2 dx \\
&\lesssim \int_0^1 \sum_{n=0}^{\infty} \left| \lambda_n^s a(t) \int_0^t f_n(s)ds\phi_n(x) \right|^2 dx \le \|a\|_{L^\infty[0,T]}^2 \sum_{n=0}^{\infty} \left| \int_0^t \lambda_n^s f_n(s)ds \right|^2 \\
&\le \|a\|_{L^\infty[0,T]}^2 \sum_{n=0}^{\infty} \left| \int_0^T \lambda_n^s f_n(t)dt \right|^2 \le T\|a\|_{L^\infty[0,T]}^2 \int_0^T \sum_{n=0}^{\infty} |\lambda_n^s f_n(t)|^2 dt \\
&= T\|a\|_{L^\infty[0,T]}^2 \int_0^T \sum_{n=0}^{\infty} \left| \int_0^1 \lambda_n^s f(t,x)\phi_n(x)dx \right|^2 dt \\
&= T\|a\|_{L^\infty[0,T]}^2 \int_0^T \|\mathcal{L}^{2s} f(t,\cdot)\|_{L^2}^2 dt \\
&\le T^2 \|a\|_{L^\infty[0,T]}^2 \|f\|_{C([0,T],W_{\mathcal{L}}^{2s}(0,1))}^2.
\end{aligned}$$

Therefore,

$$\|\partial_t u(t,\cdot)\|_{L^2}^2 \lesssim \|a\|_{L^\infty[0,T]}^2 \big(\|u_0\|_{W_{\mathcal{L}}^{2s}}^2 + T^2\|f\|_{C([0,T],W_{\mathcal{L}}^{2s}(0,1))}^2\big) + \|f\|_{C([0,T],L^2(0,1))}^2,$$

implying (2.28).

Let $s=1$. Then, we carry out the next estimate as follows:

$$\begin{aligned}
\|\partial_x u(t,\cdot)\|_{L^2}^2 = \int_0^1 |\partial_x u(t,x)|^2 dx &\lesssim \int_0^1 \left| \sum_{n=0}^{\infty} D_n e^{i\lambda_n \int_0^t a(\tau)d\tau} \phi_n'(x) \right|^2 dx \\
&+ \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n'(x) \right|^2 dx \\
&= K_1 + K_2.
\end{aligned}$$

Using (2.7), we get

$$\begin{aligned}
K_1 &:= \int_0^1 \left| \sum_{n=0}^{\infty} D_n e^{i\lambda_n \int_0^t a(\tau)d\tau} \phi_n'(x) \right|^2 dx \\
&\lesssim \|u_0\|_{W_{\mathcal{L}}^1}^2 (1 + \|\nu\|_{L^2}^2) + \|u_0\|_{L^2}^2 \|\nu\|_{L^\infty}^2.
\end{aligned}$$

For K_2, using (2.36), (2.7), and (1.6), (1.9) for ϕ_n', we obtain

$$\begin{aligned} K_2 &:= \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n'(x) \right|^2 dx \\ &\le \int_0^1 \left| \sum_{n=1}^{\infty} \int_0^t f_n(s)ds \left(\frac{\sqrt{\lambda_n} r_n(x)\cos\theta_n(x)}{\|\tilde{\phi}_n\|_{L^2}} + \nu(x)\phi_n(x) \right) \right|^2 dx \\ &\lesssim \sum_{n=0}^{\infty} \left| \int_0^t |\sqrt{\lambda_n} f_n(s)|ds \right|^2 (1 + \|\nu\|_{L^2}^2) + \|\nu\|_{L^\infty}^2 T^2 \|f\|^2_{C([0,T],L^2(0,1))}. \end{aligned}$$

Taking into account (2.32) and (2.36), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \int_0^t |\sqrt{\lambda_n} f_n(s)|ds \right|^2 &\le \sum_{n=0}^{\infty} \left| \int_0^T |\sqrt{\lambda_n} f_n(t)|dt \right|^2 \le T \int_0^T \sum_{n=0}^{\infty} \left|\sqrt{\lambda_n} f_n(t)\right|^2 dt \\ &= T \int_0^T \sum_{n=0}^{\infty} \left| \int_0^1 \sqrt{\lambda_n} f(t,x)\phi_n(x)dx \right|^2 dt \\ &= T \int_0^T \|\mathcal{L}^{\frac{1}{2}} f(t,\cdot)\|_{L^2}^2 dt \le T^2 \|f\|^2_{C([0,T],W^1_{\mathcal{L}}(0,1))}, \end{aligned}$$

and we finally obtain

$$\begin{aligned} \|\partial_x u(t,\cdot)\|_{L^2}^2 &\lesssim \left(\|u_0\|^2_{W^1_{\mathcal{L}}} + T^2\|f\|^2_{C([0,T],W^1_{\mathcal{L}}(0,1))}\right)\left(1 + \|\nu\|_{L^2}^2\right) \\ &\quad + \left(\|u_0\|_{L^2}^2 + T^2\|f\|^2_{C([0,T],L^2(0,1))}\right)\|\nu\|^2_{L^\infty} \\ &\lesssim \left(1 + \|\nu\|^2_{L^\infty}\right)\left(\|u_0\|^2_{W^1_{\mathcal{L}}} + T^2\|f\|^2_{C([0,T],W^1_{\mathcal{L}}(0,1))}\right), \end{aligned}$$

which gives (2.29).

For the estimate $\|\partial_x^2 u(t,\cdot)\|_{L^2}$, we use the fact that $\phi_n''(x) = (q(x) - \lambda_n)\phi_n(x)$ to deduce

$$\begin{aligned} &\|\partial_x^2 u(t,\cdot)\|_{L^2}^2 \\ &\quad = \int_0^1 |\partial_x^2 u(t,x)|^2 dx \\ &\quad \lesssim \int_0^1 \left| \sum_{n=0}^{\infty} D_n e^{i\lambda_n \int_0^t a(\tau)d\tau} (q(x) - \lambda_n)\phi_n(x) \right|^2 dx \\ &\qquad + \int_0^1 \left| \sum_{n=0}^{\infty} e^{i\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} f_n(s)ds(q(x) - \lambda_n)\phi_n(x) \right|^2 dx, \end{aligned}$$

and using (2.8), (2.36), we arrive at the estimates

$$\begin{aligned} \|\partial_x^2 u(t,\cdot)\|_{L^2}^2 &\lesssim \|q\|^2_{L^\infty}\left(\|u_0\|^2_{L^2} + T^2\|f\|^2_{C([0,T],L^2(0,1))}\right) \\ &\quad + \|u_0\|^2_{W^2_{\mathcal{L}}} + T^2\|f\|^2_{C^1([0,T],W^2_{\mathcal{L}}(0,1))}. \end{aligned}$$

The proof of Theorem 2.3 is complete. ∎

Corollary 2.4. *Let $s = 1$. Assume that $q \in L^2(0,1)$, $q \geq 0$, $a \in L^\infty[0,T]$, $a(t) \geq a_0 > 0$ for all $t \in [0,T]$, and $f \in C^1([0,T], L^2(0,1))$. If the initial condition satisfies $u_0 \in L^2(0,1)$ and $u_0'' \in L^2(0,1)$, then the non-homogeneous Schrödinger equation with initial/boundary conditions (2.1) has a unique solution $u \in C([0,T], L^2(0,1))$ such that*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + T\|f\|_{C([0,1],L^2(0,1))}, \tag{2.38}$$

$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim \|a\|_{L^\infty[0,T]}\left(\|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2} + \frac{T}{a_0}\|f\|_{C^1([0,T],L^2(0,1))}\right) + \|f\|_{C([0,T],L^2(0,1))}, \tag{2.39}$$

$$\|\partial_x u(t,\cdot)\|_{L^2} \lesssim \left(\|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2} + \frac{T}{a_0}\|f\|_{C^1([0,T],L^2(0,1))}\right)(1 + \|\nu\|_{L^2}) + \|\nu\|_{L^\infty}(\|u_0\|_{L^2} + T\|f\|_{C([0,T],L^2(0,1))}), \tag{2.40}$$

$$\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2} \lesssim \|u_0''\|_{L^2} + \frac{T}{a_0}\|f\|_{C^1([0,T],L^2(0,1))} + \|q\|_{L^\infty}\big(\|u_0\|_{L^2} + T|f\|_{C([0,T],L^2(0,1))}\big), \tag{2.41}$$

where the constants in these inequalities are independent of u_0, q, a, and f.

Proof. The inequality (2.38) follows from Theorem 2.3. For $\|\partial_t u(t,\cdot)\|_{L^2}$, using (2.37), we have

$$\begin{aligned}\|\partial_t u(t,\cdot)\|_{L^2} \lesssim & \int_0^1 \left|\sum_{n=0}^{\infty} i\lambda_n a(t) D_n e^{i\lambda_n \int_0^t a(\tau)d\tau}\phi_n(x)\right|^2 dx \\ & + \int_0^1 \left|\sum_{n=0}^{\infty} i\lambda_n a(t) e^{i\lambda_n \int_0^t a(\tau)d\tau}\int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} f_n(s)ds\phi_n(x)\right|^2 dx \\ & + \int_0^1 \left|\sum_{n=0}^{\infty} f_n(t)\phi_n(x)\right|^2 dx = J_1 + J_2 + J_3.\end{aligned}$$

According to (2.17), (2.18), and (2.26), we get

$$\begin{aligned}J_1 := & \int_0^1 \left|\sum_{n=0}^{\infty} i\lambda_n a(t) D_n e^{i\lambda_n \int_0^t a(\tau)d\tau}\phi_n(x)\right|^2 dx \\ & \lesssim \|a\|^2_{L^\infty[0,T]}\big(\|q\|^2_{L^\infty}\|u_0\|^2_{L^2} + \|u_0''\|^2_{L^2}\big).\end{aligned}$$

For J_3, taking into account (2.32) and Parseval's identity, we obtain

$$\begin{aligned}J_3 := & \int_0^1 \left|\sum_{n=0}^{\infty} f_n(t)\phi_n(x)\right|^2 dx = \left|\sum_{n=0}^{\infty} f_n(t)\right|^2 \lesssim \sum_{n=0}^{\infty} |f_n(t)|^2 = \|f(t,\cdot)\|^2_{L^2} \\ & \leq \|f\|_{C([0,T],L^2(0,1))}.\end{aligned}$$

To estimate J_2, integrating by parts, we obtain

$$
\begin{aligned}
J_2 &= \int_0^1 \left| \sum_{n=0}^{\infty} i\lambda_n a(t) e^{i\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} f_n(s) ds \phi_n(x) \right|^2 dx \\
&= \int_0^1 \left| \sum_{n=0}^{\infty} \left(i\lambda_n a(t) e^{i\lambda_n \int_0^t a(\tau)d\tau} \frac{i}{\lambda_n} e^{-i\lambda_n \int_0^s a(\tau)d\tau} \frac{f_n(s)}{a(s)} \right|_0^t \right. \\
&\qquad \left. + a(t) e^{i\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{-i\lambda_n \int_0^s a(\tau)d\tau} \left(\frac{f_n(s)}{a(s)} \right)' ds \right) \phi_n(x) \Bigg|^2 dx \\
&\lesssim \sum_{n=0}^{\infty} \left| a(t) \frac{f_n(s)}{a(s)} \Big|_0^t \right|^2 + \sum_{n=1}^{\infty} \left| a(t) \int_0^t \left(\frac{f_n'(s)}{a(s)} - \frac{f_n(s)a'(s)}{a^2(s)} \right) ds \right|^2 \\
&= J_{21} + J_{22}.
\end{aligned}
$$

Since $a(x) \geq a_0 > 0$ in $t \in [0, T]$, we can use the estimate $|\frac{1}{a(t)}|^2 \leq \frac{1}{a_0^2}$ for all $t \in [0, T]$, and using Parseval's identity, we obtain

$$
\begin{aligned}
J_{21} &:= \sum_{n=0}^{\infty} \left| a(t) \frac{f_n(s)}{a(s)} \Big|_0^t \right|^2 \\
&\lesssim \|a\|^2_{L^\infty[0,T]} \left(\sum_{n=1}^{\infty} \left| \frac{f_n(t)}{a(t)} \right|^2 + \sum_{n=1}^{\infty} \left| \frac{f_n(0)}{a(0)} \right|^2 \right) \\
&\leq \frac{1}{a_0^2} \|a\|^2_{L^\infty[0,T]} \left(\sum_{n=1}^{\infty} |f_n(t)|^2 + \sum_{n=1}^{\infty} |f_n(0)|^2 \right) \\
&= \frac{1}{a_0^2} \|a\|^2_{L^\infty[0,T]} \left(\|f(t,\cdot)\|^2_{L^2} + \|f(0,\cdot)\|^2_{L^2} \right).
\end{aligned}
$$

Carrying out similar reasoning, integrating by parts and using (2.36), we get

$$
\begin{aligned}
J_{22} &:= \sum_{n=1}^{\infty} \left| a(t) \int_0^t \left(\frac{f_n'(s)}{a(s)} - \frac{f_n(s)a'(s)}{a^2(s)} \right) ds \right|^2 \\
&= \sum_{n=0}^{\infty} \left| a(t) \left(\int_0^t \frac{f_n'(s)}{a(s)} ds + \int_0^t f_n(s) d\left(\frac{1}{a(s)} \right) \right) \right|^2 \\
&\lesssim \frac{1}{a_0^2} \|a\|^2_{L^\infty[0,T]} \left| \sum_{n=1}^{\infty} \int_0^t |f_n'(s)| ds \right|^2 + \|a\|^2_{L^\infty[0,T]} \left| \sum_{n=1}^{\infty} \left(\frac{f_n(s)}{a(s)} \Big|_0^t - \int_0^t \frac{f'(s)}{a(s)} ds \right) \right|^2 \\
&\lesssim \frac{T^2}{a_0^2} \|a\|^2_{L^\infty[0,T]} \|f'\|^2_{C([0,T],L^2(0,1))} + \frac{1}{a_0^2} \|a\|^2_{L^\infty[0,T]} \left(\|f(t,\cdot)\|^2_{L^2} + \|f(0,\cdot)\|^2_{L^2} \right) \\
&\quad + \frac{T^2}{a_0^2} \|a\|^2_{L^\infty[0,T]} \|f'\|^2_{C([0,T],L^2(0,1))}.
\end{aligned}
$$

And finally, for J_2, we have

$$J_2 := \int_0^1 \left| \sum_{n=0}^{\infty} \lambda_n e^{-\lambda_n \int_0^t a(\tau)d\tau} \int_0^t e^{\lambda_n \int_0^s a(\tau)d\tau} f_n(s) ds \phi_n(x) \right|^2 dx$$
$$\lesssim \frac{2}{a_0^2} \|a\|^2_{L^\infty[0,T]} \big(\|f(t,\cdot)\|^2_{L^2} + \|f(0,\cdot)\|^2_{L^2} \big) + 2\frac{T^2}{a_0^2} \|a\|^2_{L^\infty[0,T]} \|f'\|^2_{C([0,T],L^2(0,1))}$$
$$\le \frac{2T^2}{a_0^2} \|a\|^2_{L^\infty[0,T]} \|f\|^2_{C^1([0,T],L^2(0,1))}.$$

Therefore,

$$\|\partial_t u(t,\cdot)\|^2_{L^2} \lesssim \|a\|^2_{L^\infty[0,T]} \left(\|u_0''\|^2_{L^2} + \|q\|^2_{L^\infty} \|u_0\|^2_{L^2} + \frac{2T^2}{a_0^2} \|f\|^2_{C^1([0,T],L^2(0,1))} \right) + \|f\|^2_{C([0,T],L^2(0,1))},$$

implying (2.39). Taking into account Corollary 2.2 and similar to previous estimates, we obtain the inequalities (2.40) and (2.41).

The proof of Corollary 2.4 is complete. ■

3. Very weak solutions

In this section, we consider the differential case $s = 1$. We will analyse the solutions for less regular coefficients q, a and the initial condition u_0. For this, we will be using the notion of very weak solutions.

Assume that the coefficient q and initial condition u_0 are the distributions on $(0, 1)$; the coefficient a is the distribution on $[0, T]$. To regularise distributions, we introduce the following definition.

Definition 3.1. (i) A net of functions $(u_\varepsilon = u_\varepsilon(t, x))$ is said to be uniformly L^2-moderate if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\|u_\varepsilon(t,\cdot)\|_{L^2} \le C\varepsilon^{-N} \quad \text{for all } t \in [0,T].$$

(ii) A net of functions $(u_{0,\varepsilon} = u_{0,\varepsilon}(x))$ is said to be H^2-moderate if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\|u_{0,\varepsilon}\|_{L^2} \le C\varepsilon^{-N}, \quad \|u''_{0,\varepsilon}\|_{L^2} \le C\varepsilon^{-N}.$$

Definition 3.2. (i) A net of functions $(q_\varepsilon = q_\varepsilon(x))$ is said to be L^∞-moderate if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\|q_\varepsilon\|_{L^\infty(0,1)} \le C\varepsilon^{-N}.$$

(ii) A net of functions $(a_\varepsilon = a_\varepsilon(t))$ is said to be L^∞-moderate if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\|a_\varepsilon\|_{L^\infty[0,T]} \le C\varepsilon^{-N}.$$

Remark 3.3. We note that such assumptions are natural for distributional coefficients in the sense that regularisations of distributions are moderate. Precisely, by the structure theorems for distributions (see, e.g., [7]), we know that distributions

$$\mathcal{D}'(0,1) \subset \{L^\infty(0,1) - \text{moderate families}\}, \tag{3.1}$$

and we see from (3.1) that a solution to an initial/boundary problem may not exist in the sense of distributions, while it may exist in the set of L^∞-moderate functions.

To give an example, let us take $f \in L^2(0,1)$, $f : (0,1) \to \mathbb{C}$. We introduce the function

$$\tilde{f} = \begin{cases} f, & \text{on } (0,1), \\ 0, & \text{on } \mathbb{R} \setminus (0,1); \end{cases}$$

then $\tilde{f} : \mathbb{R} \to \mathbb{C}$, and $\tilde{f} \in \mathcal{E}'(\mathbb{R})$.

Let $\tilde{f}_\varepsilon = \tilde{f} * \psi_\varepsilon$ be obtained as the convolution of $\tilde{f}$ with a Friedrich mollifier ψ_ε, where

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right), \quad \text{for } \psi \in C_0^\infty(\mathbb{R}), \ \int \psi = 1.$$

Then, the regularising net $(\tilde{f}_\varepsilon)$ is L^p-moderate for any $p \in [1,\infty)$, and it approximates f on $(0,1)$:

$$0 \leftarrow \|\tilde{f}_\varepsilon - \tilde{f}\|^p_{L^p(\mathbb{R})} \approx \|\tilde{f}_\varepsilon - f\|^p_{L^p(0,1)} + \|\tilde{f}_\varepsilon\|^p_{L^p(\mathbb{R}\setminus(0,1))}.$$

Now, let us introduce the notion of a very weak solution to the initial/boundary problem (2.2)–(2.4).

Definition 3.4. Let $q \in \mathcal{D}'(0,1)$, $a \in \mathcal{D}'[0,T]$. The net $(u_\varepsilon)_{\varepsilon>0}$ is said to be a very weak solution to the initial/boundary problem (2.2)–(2.4) if there exists an L^∞-moderate regularisation q_ε of q, L^∞-moderate regularisation a_ε of a, and an H^2-moderate regularisation $u_{0,\varepsilon}$ of u_0 such that

$$\begin{cases} i\partial_t u_\varepsilon(t,x) + a_\varepsilon(t)(-\partial_x^2 u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x)) = 0, & (t,x) \in [0,T] \times (0,1), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in (0,1), \\ u_\varepsilon(t,0) = 0 = u_\varepsilon(t,1), & t \in [0,T], \end{cases} \tag{3.2}$$

and (u_ε) and $(\partial_t u_\varepsilon)$ are uniformly L^2-moderate.

Then, we have the following properties of very weak solutions.

Theorem 3.5 (Existence). *Let the coefficients q and initial condition u_0 be distributions in $(0,1)$, $q \geq 0$, and let the coefficient a be distribution in $[0,T]$ and there exists $a_0 > 0$ such that $a \geq a_0 > 0$ in the sense that $\langle a - a_0, \phi \rangle \geq 0$ for any $\phi \geq 0$. Then, the initial/boundary problem (2.2)–(2.4) has a very weak solution.*

Proof. Since the formulation of (2.2)–(2.4) in this case might be impossible in the distributional sense due to issues related to the product of distributions, we replace (2.2)–(2.4) with a regularised equation. In other words, we regularise q and u_0 by some corresponding sets $q_\varepsilon \geq 0$ and $u_{0,\varepsilon}$ of smooth functions from $C^\infty(0,1)$, and a by the set a_ε of smooth functions from $C^\infty[0,T]$.

Hence, q_ε, a_ε are L^∞-moderate regularisations and $u_{0,\varepsilon}$ is an H^2-moderate regularisation of the coefficients q, a and the Cauchy condition u_0, respectively. So, by Definition 3.1, there exist $N \in \mathbb{N}_0$ and $C_1 > 0$, $C_2 > 0$, C_3, C_4 such that

$$\|q_\varepsilon\|_{L^\infty} \leq C_1\varepsilon^{-N}, \quad \|u_{0,\varepsilon}\|_{L^2} \leq C_2\varepsilon^{-N}, \quad \|u''_{0,\varepsilon}\|_{L^2} \leq C_3\varepsilon^{-N}, \quad \|a\|_{L^\infty} \leq C_4\varepsilon^{-N}.$$

Now, we fix $\varepsilon \in (0,1]$, and consider the regularised problem (3.2). Then, all discussions and calculations of Theorem 2.1 are valid. Thus, by Corollary 2.2, the equation (3.2) has a unique solution $u_\varepsilon(t,x)$ in the space $C([0,T];L^2(0,1))$.

By Corollary 2.2, there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\|u_\varepsilon(t,\cdot)\|_{L^2} \lesssim \|u_{0,\varepsilon}\|_{L^2} \leq C\varepsilon^{-N},$$
$$\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2} \lesssim \|a\|^2_{L^\infty[0,T]}(\|u''_{0,\varepsilon}\|_{L^2} + \|q_\varepsilon\|_{L^\infty}\|u_{0,\varepsilon}\|_{L^2}) \leq C\varepsilon^{-N},$$

where the constants in these inequalities are independent of u_0, q, and a. Therefore, (u_ε) is uniformly L^2-moderate, and the proof of Theorem 3.5 is complete. ■

Describing the uniqueness of the very weak solutions amounts to "measuring" the changes of involved associated nets: negligibility conditions for nets of functions/distributions read as follows.

Definition 3.6 (Negligibility). Let (u_ε), $(\tilde{u}_\varepsilon)$ be two nets in $L^2(0,1)$. Then, the net $(u_\varepsilon - \tilde{u}_\varepsilon)$ is called L^2-negligible if for every $N \in \mathbb{N}$ there exist $C > 0$ such that the following condition is satisfied:

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2} \leq C\varepsilon^N$$

for all $\varepsilon \in (0,1]$. In the case where $u_\varepsilon = u_\varepsilon(t,x)$ is a net depending on $t \in [0,T]$, the uniformly L^2-negligibility condition can be described as follows:

$$\|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \leq C\varepsilon^N,$$

uniformly in $t \in [0,T]$. The constant C can depend on N but not on ε.

Let us state the "ε-parameterised problems" to be considered:

$$\begin{cases} i\partial_t u_\varepsilon(t,x) + a_\varepsilon(t)(-\partial_x^2 u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x)) = 0, & (t,x) \in [0,T]\times(0,1), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in (0,1), \\ u_\varepsilon(t,0) = 0 = u_\varepsilon(t,1), & t \in [0,T], \end{cases} \tag{3.3}$$

and

$$\begin{cases} i\partial_t \tilde{u}_\varepsilon(t,x) + \tilde{a}_\varepsilon(t)(-\partial_x^2 \tilde{u}_\varepsilon(t,x) + \tilde{q}_\varepsilon(x)\tilde{u}_\varepsilon(t,x)) = 0, & (t,x) \in [0,T]\times(0,1), \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x), & x \in (0,1), \\ \tilde{u}_\varepsilon(t,0) = 0 = \tilde{u}_\varepsilon(t,1), & t \in [0,T]. \end{cases} \tag{3.4}$$

Definition 3.7 (Uniqueness of the very weak solution). Let $q \in \mathcal{D}'(0,1)$, $a \in \mathcal{D}'[0,T]$. We say that initial/boundary problem (2.2)–(2.4) has a unique very weak solution, if for all L^∞-moderate nets q_ε, $\tilde{q}_\varepsilon$, such that $(q_\varepsilon - \tilde{q}_\varepsilon)$ is L^∞-negligible; for all L^∞-moderate nets a_ε, $\tilde{a}_\varepsilon$, such that $(a_\varepsilon - \tilde{a}_\varepsilon)$ is L^∞-negligible; and for all H^2-moderate regularisations $u_{0,\varepsilon}$, $\tilde{u}_{0,\varepsilon}$, such that $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})$, is H^2-negligible, we have that $u_\varepsilon - \tilde{u}_\varepsilon$ is uniformly L^2-negligible.

Theorem 3.8 (Uniqueness of the very weak solution). *Let the coefficient q and initial condition u_0 be distributions in $(0,1)$, $q \geq 0$, the coefficient a be a distribution in $[0,T]$ and there exists $a_0 > 0$ such that $a \geq a_0 > 0$ in the sense that $\langle a - a_0, \phi\rangle \geq 0$ for any $\phi \geq 0$. Then, the very weak solution to the initial/boundary problem (2.2)–(2.4) is unique.*

Proof. We denote by u_ε and $\tilde{u}_\varepsilon$ the families of solutions to the initial/boundary problems (3.3) and (3.4), respectively. Setting U_ε to be the difference of these nets $U_\varepsilon := u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)$, then U_ε solves

$$\begin{cases} i\partial_t U_\varepsilon(t,x) + a_\varepsilon(t)(-\partial_x^2 U_\varepsilon(t,x) + q_\varepsilon(x)U_\varepsilon(t,x)) = f_\varepsilon(t,x), & (t,x)\in[0,T]\times(0,1), \\ U_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), & x \in (0,1), \\ U_\varepsilon(t,0) = 0 = U_\varepsilon(t,1), \end{cases} \tag{3.5}$$

where we set

$$\begin{aligned} f_\varepsilon(t,x) &:= (a_\varepsilon(t) - \tilde{a}_\varepsilon(t))\partial_x^2 \tilde{u}_\varepsilon(t,x) \\ &\quad + \{\tilde{a}_\varepsilon(t)(\tilde{q}_\varepsilon(x) - q_\varepsilon(x)) + q_\varepsilon(x)(\tilde{a}_\varepsilon(x) - a_\varepsilon(x))\}\tilde{u}_\varepsilon(t,x) \end{aligned}$$

for the forcing term to the non-homogeneous initial/boundary problem (3.5).

Passing to the L^2-norm of the U_ε, by using (2.38), we obtain

$$\|U_\varepsilon(t,\cdot)\|_{L^2}^2 \lesssim \|U_\varepsilon(0,\cdot)\|_{L^2}^2 + T^2\|f_\varepsilon\|_{C([0,T],L^2(0,1))}^2.$$

For the $\|f_\varepsilon\|_{C([0,T],L^2(0,1))}^2$ by using (2.16) and (2.24), we get

$$\begin{aligned} \|f_\varepsilon\|_{C([0,T],L^2(0,1))}^2 \lesssim \|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty[0,T]}^2 &\big(\|\tilde{u}_{0,\varepsilon}''\|_{L^2}^2 + 2\|\tilde{q}_\varepsilon\|_{L^\infty}^2\|\tilde{u}_{0,\varepsilon}\|_{L^2}^2\big) \\ &+ \|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty}^2\|\tilde{a}_\varepsilon\|_{L^\infty[0,T]}^2\|\tilde{u}_\varepsilon\|_{C([0,T],L^2(0,1))}^2 \\ &+ \|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty[0,T]}^2\|q_\varepsilon\|_{L^\infty}^2\|\tilde{u}_\varepsilon\|_{C([0,T],L^2(0,1))}^2. \end{aligned}$$

Next, using the initial condition of (3.5), we obtain

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2}^2 \lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2}^2 + T^2\|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty[0,T]}^2 &\big(\|\tilde{u}_{0,\varepsilon}''\|_{L^2}^2 + 2\|\tilde{q}_\varepsilon\|_{L^\infty}^2\|\tilde{u}_{0,\varepsilon}\|_{L^2}^2\big) \\ + T^2\|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty}^2\|\tilde{a}_\varepsilon\|_{L^\infty[0,T]}^2\|\tilde{u}_\varepsilon\|_{C([0,T],L^2(0,1))}^2& \\ + T^2\|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty[0,T]}^2\|q_\varepsilon\|_{L^\infty}^2\|\tilde{u}_\varepsilon\|_{C([0,T],L^2(0,1))}^2.& \end{aligned}$$

Taking into account the negligibility of the nets $u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}$, $q_\varepsilon - \tilde{q}_\varepsilon$, and $a_\varepsilon - \tilde{a}_\varepsilon$, we get

$$\|U_\varepsilon(t,\cdot)\|^2_{L^2} \le C_1\varepsilon^{N_1} + \varepsilon^{N_2}\big(C_2\varepsilon^{-N_3} + C_3\varepsilon^{-N_4}\big) + \varepsilon^{N_5}\big(C_4\varepsilon^{-N_6} + C_5\varepsilon^{-N_7}\big)$$

for some $C_1 > 0, C_2 > 0, C_3 > 0, C_4 > 0, C_5 > 0, N_3, N_4, N_6, N_7 \in \mathbb{N}$ and all $N_1, N_2, N_5 \in \mathbb{N}$, since $\tilde{u}_\varepsilon$ is moderate. Then, for some $C_M > 0$ and all $M \in \mathbb{N}$,

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \le C_M\varepsilon^M.$$

The last estimate holds true uniformly in t, and this completes the proof of Theorem 3.8. ∎

Theorem 3.9 (Consistency). *Assume that $q \in L^\infty(0,1)$, $q \ge 0$, $a(t) \ge a_0 > 0$ for all $t \in [0,T]$, and let (q_ε) be any L^∞-regularisation of q and (a_ε) any L^∞-regularisation of a, that is, $\|q_\varepsilon - q\|_{L^\infty} \to 0$, $\|a_\varepsilon - a\|_{L^\infty[0,T]} \to 0$ as $\varepsilon \to 0$. Let the initial condition satisfy $u_0 \in L^2(0,1)$. Let u be a very weak solution to the initial/boundary problem (2.2)–(2.4). Then, for any families q_ε, a_ε, $u_{0,\varepsilon}$ such that $\|q - q_\varepsilon\|_{L^\infty} \to 0$, $\|a - a_\varepsilon\|_{L^\infty[0,T]} \to 0$, $\|u_0 - u_{0,\varepsilon}\|_{L^2} \to 0$ as $\varepsilon \to 0$, any representative (u_ε) of the very weak solution converges as*

$$\sup_{0\le t\le T}\|u(t,\cdot) - u_\varepsilon(t,\cdot)\|_{L^2} \to 0$$

for $\varepsilon \to 0$ to the unique classical solution $u \in C([0,T]; L^2(0,1))$ to the initial/boundary problem (2.2)–(2.4) given by Theorem 2.1.

Proof. For u and for u_ε, as in our assumption, we introduce an auxiliary notation

$$V_\varepsilon(t,x) := u(t,x) - u_\varepsilon(t,x).$$

Then, the net V_ε is a solution to the initial/boundary problem

$$\begin{cases} i\partial_t V_\varepsilon(t,x) + a_\varepsilon(t)(-\partial_x^2 V_\varepsilon(t,x) + q_\varepsilon(x)V_\varepsilon(t,x)) = f_\varepsilon(t,x), & \\ V_\varepsilon(0,x) = (u_0 - u_{0,\varepsilon})(x), & x \in (0,1), \\ V_\varepsilon(t,0) = 0 = V_\varepsilon(t,1), & t \in [0,T], \end{cases}$$

where

$$f_\varepsilon(t,x) := (a(t) - a_\varepsilon(t))\partial_x^2 u(t,x) + \{a_\varepsilon(t)(q_\varepsilon(x) - q(x)) + q(x)(a_\varepsilon(t) - a(t))\}u(t,x).$$

Analogously to Theorem 3.8, we have that

$$\begin{aligned} \|V_\varepsilon(t,\cdot)\|^2_{L^2} \lesssim \|u_0 - u_{0,\varepsilon}\|^2_{L^2} &+ T^2\|a - a_\varepsilon\|^2_{L^\infty[0,T]}\big(\|u''\|^2_{L^2} + 2\|q\|^2_{L^\infty}\|u\|^2_{L^2}\big) \\ &+ T^2\|a - a_\varepsilon\|^2_{L^\infty[0,T]}\|q\|^2_{L^\infty}\|u\|^2_{C([0,T],L^2(0,1))} \\ &+ T^2\|q - q_\varepsilon\|^2_{L^\infty}\|a_\varepsilon\|^2_{L^\infty[0,T]}\|u\|^2_{C([0,T],L^2(0,1))}. \end{aligned}$$

Since

$$\|u_0 - u_{0,\varepsilon}\|_{L^2} \to 0, \quad \|q_\varepsilon - q\|_{L^\infty} \to 0, \quad \|a - a_\varepsilon\|_{L^\infty[0,T]} \to 0$$

for $\varepsilon \to 0$ and u is a very weak solution to the initial/boundary problem (2.2)–(2.4), we get

$$\|V_\varepsilon(t,\cdot)\|_{L^2} \to 0$$

for $\varepsilon \to 0$. This proves Theorem 3.9. ■

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