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Embeddings and pointwise multiplication in Kondratiev spaces on polyhedral type domains

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Abstract. In this paper, we investigate Kondratiev spaces on domains of polyhedral type. In particular, we will be concerned with necessary and sufficient conditions for continuous and compact embeddings and in addition, we will deal with pointwise multiplication in these spaces.

1. Introduction

Since the midsixties scales of weighted Sobolev spaces have become popular in the study of regularity of solutions to elliptic PDEs on polygonal and polyhedral domains in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. In this context, we refer to the pioneering work of Kondratiev [22, 23], see also the survey of Kondratiev and Oleinik [21]. Later on, these types of spaces, partly more general, have been considered by many authors. Let us mention just a few: Babuška, Guo [14], Bacuta, Mazzucato, Nistor, Zikatanov [3], Dauge [12], Kozlov, Maz'ya, Rossmann [24, 25], Kufner, Sändig [28], Maz'ya, Rossmann [30], Mazzucato, Nistor [32], and Nazarov, Plamenevskii [34]. Whereas in the mentioned references the weight was always chosen to be a power of the distance to the singular set of the boundary, there are also publications dealing with the weight being a power of the distance to the whole boundary. We refer, e.g., to Kufner, Sändig [28], Triebel [39, Section 3.2.3] and Lototsky [29].

Kondratiev spaces provide a very powerful tool in the context of the qualitative theory of elliptic and parabolic PDEs, especially on nonsmooth domains. In particular, on domains with edges and corners, these nonsmooth parts of the boundary induce singularities for the solution and its derivatives. By means of Kondratiev spaces, it is possible to describe very precisely the behaviour of these singularities. Moreover, these specific smoothness spaces allow for certain shift theorems in the following sense. Suppose that we are given a second order elliptic differential equation on a polygonal or polyhedral domain. Then, under certain conditions on the coefficients and on the domain, it turns out that if the right-hand side has smoothness $m-1$ in the scale of Kondratiev spaces, then the solution u of the PDE has smoothness $m+1$. We refer to [3] and particularly to [30] for further information. While for smooth domains similar statements also hold

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for classical smoothness spaces such as Sobolev spaces, the situation is completely different on the nonsmooth domains we are concerned with here. In this case, the singularities at the boundary diminish the Sobolev regularity. Let us in this context recall the famous $H^{3/2}$-theorem proved by Jerison and Kenig [20], which states that for the Poisson equation there exist Lipschitz domains and right-hand sides $f \in C^\infty$ such that the smoothness of the corresponding solution is limited by $3/2$.

The above remarks reflect that Kondratiev spaces have been shown to be an indispensable tool in the theory of elliptic equations, in particular, on non-convex polyhedral domains. Our intention in this paper is twofold: the first part has a survey character on the basic properties of Kondratiev spaces. We systematically present what is known about these classes with respect to continuous and compact embeddings. Moreover, we provide examples to illustrate the sharpness of the embedding results. In the second part we present new results concerning pointwise multiplication in Kondratiev spaces, where we would like to understand the mappings $u \mapsto u^n$, $n \in \mathbb{N}$, in the framework of these scales. To do this we will allow a greater generality. Moreover, we will give the final answer under which conditions Kondratiev spaces form algebras with respect to pointwise multiplication.

There is also an interesting relationship of Kondratiev spaces with important issues in numerical analysis. As is well known, the approximation order that can be achieved by adaptive and other nonlinear methods usually depends on the regularity of the exact solutions in scales of Besov spaces [7, 10, 11, 17]. Since there exist a lot of embeddings of Kondratiev spaces into Besov spaces, cf. [16, 17], Besov regularity estimates can very often be traced back to regularity questions in Kondratiev spaces. Therefore, the results presented in this paper will be used in a follow-up paper [9] in order to look at Besov regularity of solutions to nonlinear elliptic partial differential equations, e.g.,

$$-\Delta u(x) + u^n(x) = f(x), \quad x \in \Omega,\ n > 2,$$
$$u(x) = 0, \quad x \in \partial\Omega.$$

The paper is organized as follows. In Section 2, we will give the definition of the scales $\mathcal{K}^m_{a,p}(\Omega, M)$. Therein, we also discuss in detail which types of domains we are interested in, cf. Subsection 2.2. The next two sections are then devoted to the study of necessary and sufficient conditions for continuous and compact embeddings of Kondratiev spaces. In Section 5, we discuss pointwise multipliers for Kondratiev spaces in great detail. Whereas other parts of the paper have the character of a survey, the contents of this section are completely new. Firstly, we investigate under which conditions on the parameters m, p, and a a space $\mathcal{K}^m_{a,p}(\Omega, M)$ forms an algebra with respect to pointwise multiplication. Secondly, under certain conditions on the parameters, we also deal with the more general case of products of the form $\mathcal{K}^{m_1}_{a_1,p_1}(\Omega, M) \cdot \mathcal{K}^{m_2}_{a_2,p_2}(\Omega, M)$.

In almost all cases, the following strategy is used. In a first step, we deal with the corresponding problem for simplified Kondratiev spaces defined on $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$. Though our main interest is in domains in $\mathbb{R}^2$ and $\mathbb{R}^3$, those simplified (yet prototypical) spaces usually can be treated without additional technical effort in general dimensions. Afterwards, using linear and continuous extension operators, we extend the obtained results to Kondratiev spaces defined on smooth cones, nonsmooth cones and specific dihedral domains, see Cases I–III. In a third step, by making use of a simple decomposition, we are able to handle Kondratiev spaces defined on polyhedral cones, see Case IV. Furthermore, the decomposition from Lemma 10 allows us to extend everything to so-called domains of polyhedral type. Note, however, that in $\mathbb{R}^d$ with $d > 3$ our definition of such domains of polyhedral type is quite restrictive and allows only for very special situations like, e.g., domains with smooth boundary except for finitely many conical points.

Some final remarks concerning the choice of our weighted Sobolev spaces are in order. In our setting, see Definition 1 of Subsection 2.1, all derivatives that occur are weighted by some power of the distance to the singularity set, where the power depends on the order α of the corresponding derivative. This is of course not the only possible choice. Indeed, several authors worked with the scale J^m_γ, where the power does *not* depend on α. Let us just mention the work of Babuška and Guo [14] and Costabel, Dauge and Nicaise [6] (this list is clearly not complete). It is sometimes claimed that the scale J^m_γ is more versatile in order to describe the global regularity of solutions of PDEs. And indeed, these spaces have the advantage that for large enough m they may contain all polynomials, which is not true in our case. Consequently, based on (intersections of) these spaces, Babuška and Guo [14] and Guo [15] have been able to show exponential convergence of hp-versions of finite element methods. However, for our purposes, the Kondratiev scale as introduced in Section 2 is more suitable for several reasons. In particular, (complex) interpolation of these spaces is much simpler and the desired embeddings into scales of Besov spaces also arise very naturally.

2. Kondratiev spaces

Let us start by collecting some general notation used throughout the paper.

As usual, $\mathbb{N}$ stands for the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^d$, $d \in \mathbb{N}$, is the d-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^d$, denoting the Euclidean norm of x. Let $\mathbb{N}_0^d$, where $d \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}_0$ and

$$|\alpha| := \sum_{j=1}^{d} \alpha_j .$$

Furthermore, $B_\varepsilon(x)$ is the open ball of radius $\varepsilon > 0$ centered at x.

We denote by c a generic positive constant which is independent of the main parameters, but its value may change from line to line. The expression $A \lesssim B$ means that $A \le c\, B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded.

A domain Ω is an open bounded set in $\mathbb{R}^d$. Let $L_p(\Omega)$, $1 \le p \le \infty$, be the Lebesgue spaces on Ω as usual. Furthermore, for $m \in \mathbb{N}$ and $1 \le p \le \infty$, we denote by $W^m_p(\Omega)$ the standard Sobolev space on the domain Ω equipped with the norm

$$\|u \mid W_p^m(\Omega)\| := \Big(\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u(x)|^p \, dx \Big)^{1/p}$$

(with the usual modification if $p = \infty$). If $p = 2$ we will also write $H^m(\Omega)$ instead of $W_2^m(\Omega)$.

2.1. Definition and basic properties

Definition 1. Let Ω be a domain in $\mathbb{R}^d$ and let M be a non-trivial closed subset of its boundary $\partial\Omega$. Furthermore, let $m \in \mathbb{N}_0$ and $a \in \mathbb{R}$. We put

$$\rho(x) := \min\{1, \operatorname{dist}(x, M)\}, \quad x \in \Omega. \tag{2.1}$$

(i) Let $1 \le p < \infty$. We define the Kondratiev spaces $\mathcal{K}^m_{a,p}(\Omega, M)$ as the collection of all measurable functions which admit m weak derivatives in Ω satisfying

$$\|u|\mathcal{K}^m_{a,p}(\Omega, M)\| := \Big(\sum_{|\alpha|\le m} \int_\Omega |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \Big)^{1/p} < \infty.$$

(ii) The space $\mathcal{K}^m_{a,\infty}(\Omega, M)$ is the collection of all measurable functions which admit m weak derivatives in Ω satisfying

$$\|u \mid \mathcal{K}^m_{a,\infty}(\Omega, M)\| := \sum_{|\alpha|\le m} \big\|\rho^{|\alpha|-a} \partial^\alpha u \mid L_\infty(\Omega)\big\| < \infty.$$

Remark 2. (i) Many times the set M will be the *singularity set* S of the domain Ω, i.e., the set of all points $x \in \partial\Omega$ such that for any $\varepsilon > 0$ the set $\partial\Omega \cap B_\varepsilon(x)$ is not smooth.

(ii) We will not distinguish spaces which differ by an equivalent norm.

Basic properties

We collect basic properties of Kondratiev spaces that will be useful in what follows.

- $\mathcal{K}^m_{a,p}(\Omega, M)$ is a Banach space, see [26, 27].
- The scale of Kondratiev spaces is monotone in m and a, i.e.,

  $$\mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow \mathcal{K}^{m'}_{a,p}(\Omega, M) \quad \text{and} \quad \mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow \mathcal{K}^m_{a',p}(\Omega, M) \tag{2.2}$$

  if $m' < m$ and $a' < a$.
- Regularized distance function: there exist a function $\tilde{\varrho} : \overline{\Omega} \to [0, \infty)$, which is infinitely often differentiable in Ω, and positive constants A, B, C_α such that

  $$A\rho(x) \le \tilde{\varrho}(x) \le B\, \rho(x), \quad x \in \Omega$$

and for all $\alpha \in \mathbb{N}_0^d$,

$$\left|\partial^\alpha \widetilde{\varrho}(x)\right| \le C_\alpha \rho^{1-|\alpha|}(x), \quad x \in \Omega,$$

see Stein [38, Theorem VI.2.2] (the construction given there is valid for arbitrary closed subsets of $\mathbb{R}^d$).

- By using the previous item and replacing ρ by $\widetilde{\varrho}$ in the norm of $\mathcal{K}^m_{a,p}(\Omega, M)$ one can prove the following.
  Let $b \in \mathbb{R}$. Then, the mapping $T_b : u \mapsto \widetilde{\varrho}^b\, u$ yields an isomorphism of $\mathcal{K}^m_{a,p}(\Omega, M)$ onto $\mathcal{K}^m_{a+b,p}(\Omega, M)$.
- Let $a \ge 0$. Then, $\mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow L_p(\Omega)$.
- A function $\psi : \Omega \to \mathbb{R}$ such that the ordinary derivatives $\partial^\alpha \psi$ are continuous functions on Ω for all α, $|\alpha| \le m$,

$$\|\psi \mid C^m(\Omega)\| := \max_{|\alpha|\le m} \sup_{x\in\Omega} |\partial^\alpha \psi(x)| < \infty,$$

  is a pointwise multiplier for $\mathcal{K}^m_{a,p}(\Omega, M)$, i.e., $\psi \cdot u \in \mathcal{K}^m_{a,p}(\Omega, M)$ for all $u \in \mathcal{K}^m_{a,p}(\Omega, M)$.
- For $1 \le p < \infty$, the subspace

$$C^\infty_*(\Omega, M) = \left\{u|_\Omega : u \in C^\infty_0(\mathbb{R}^d \setminus M)\right\}$$

  is a dense subset of $\mathcal{K}^m_{a,p}(\Omega, M)$. A proof can be found in [40].

2.2. Domains of polyhedral type

In the sequel, we will mainly be interested in the case that d is either 2 or 3 and that Ω is a bounded domain of polyhedral type. The precise definition will be given below in Definitions 4 and 5. Essentially, we will consider domains for which the analysis of the associated Kondratiev spaces can be reduced to the following four basic cases:

- smooth cones;
- specific nonsmooth cones;
- specific dihedral domains;
- polyhedral cones.

Let $d \ge 2$. Below, an infinite smooth cone with vertex at the origin is the set

$$\left\{x \in \mathbb{R}^d : 0 < |x| < \infty, x/|x| \in \Omega\right\},$$

where Ω is a simply connected subdomain of the unit sphere S^{d-1} in $\mathbb{R}^d$ with C^∞ boundary.

Case I. Let K' be an infinite smooth cone in $\mathbb{R}^d$ as described above, and let $M := \{0\}$. Then, we define the truncated cone K by

$$K := K' \cap B_1(0) \tag{2.3}$$

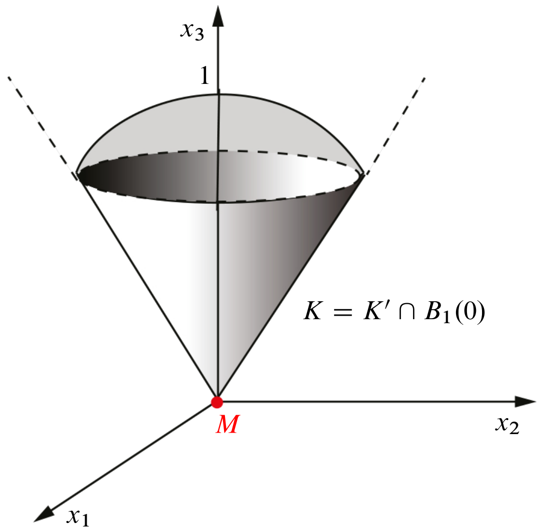


Figure 1. Kondratiev spaces on smooth cones.

and put

$$\|u|\mathcal{K}^m_{a,p}(K,M)\| := \left(\sum_{|\alpha|\leq m}\int_K \big||x|^{|\alpha|-a}\partial^\alpha u(x)\big|^p\,dx\right)^{1/p}. \tag{2.4}$$

Observe that M is just a part of the singular set of the boundary of the truncated cone K (see Figure 1). We further remark that the truncation in (2.3) not necessarily has to be done with the unit ball. Any smooth hypersurface in $\mathbb{R}^d$, particularly a hyperplane (see Case II below) would be sufficient (since it does not generate additional singularities).

Case II. Let K' denote a rotationally symmetric smooth cone with opening angle $\gamma \in (0, \pi/2)$ (the precise value of γ will be of no importance), and let K be its truncated version $K := K' \cap \{x \in \mathbb{R}^d : 0 < x_d < 1\}$ as in Case I. Moreover, we put

$$I := \big\{x \in \mathbb{R}^d : 0 < x_i < 1, i = 1,\dots,d\big\}, \tag{2.5}$$

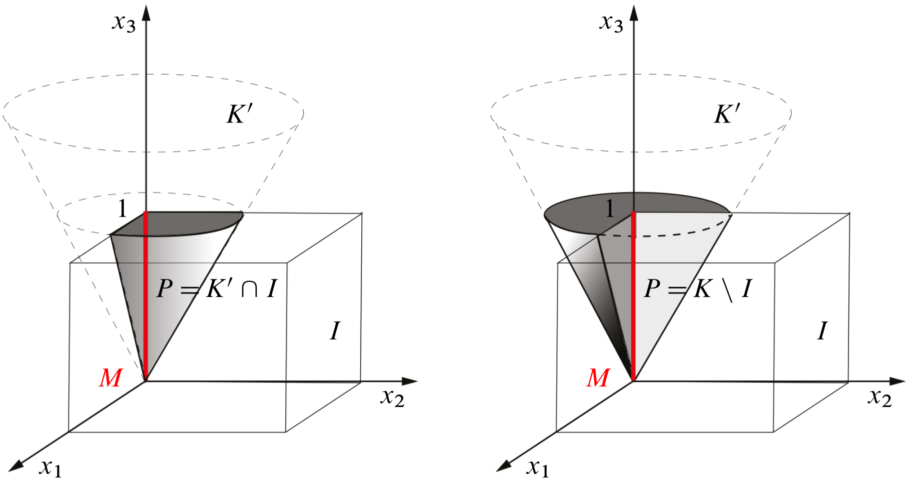
the unit cube in $\mathbb{R}^d$. Then, we define two (non-diffeomorphic) versions of specific non-smooth cones P: in both situations, we choose

$$M = \Gamma := \big\{x \in \mathbb{R}^d : x = (0,\dots,0,x_d), 0 \leq x_d \leq 1\big\}$$

and define

$$\|u \mid \mathcal{K}^m_{a,p}(P,\Gamma)\| := \left(\sum_{|\alpha|\leq m}\int_P \big|\rho(x)^{|\alpha|-a}\partial^\alpha u(x)\big|^p\,dx\right)^{1/p}, \tag{2.6}$$

where $\rho(x)$ denotes the distance of x to Γ, i.e., $\rho(x) = |(x_1,\dots,x_{d-1})|$. Also, in this case the set Γ is a proper subset of the singular set of P (see Figure 2).



(a) We consider $P = K' \cap I$, or equivalently $P = K \cap I$.

(b) Alternatively, also $P = K \setminus \bar{I}$ will be called a specific non-smooth cone.

Figure 2. Kondratiev spaces on specific nonsmooth cones.

Case III. Let $1 \leq \ell < d$ and let I be the unit cube defined in (2.5). For $x \in \mathbb{R}^d$, we write

$$x = (x', x'') \in \mathbb{R}^{d-\ell} \times \mathbb{R}^{\ell},$$

where $x' := (x_1, \dots, x_{d-\ell})$ as well as $x'' := (x_{d-\ell+1}, \dots, x_d)$. Hence, $I = I' \times I''$ with the obvious interpretation (see Figure 3).

Additionally, also sets of the form $I = I' \times K$, where $K \subset \mathbb{R}^{d-\ell}$ is a truncated cone as in Case I, are considered. Then, we choose

$$M_\ell := \{x \in \partial I : x_1 = \dots = x_{d-\ell} = 0\} \tag{2.7}$$

and define

$$\|u|\mathcal{K}^m_{a,p}(I, M_\ell)\| := \left(\sum_{|\alpha| \leq m} \int_I \big| |x'|^{|\alpha|-a} \partial^\alpha u(x) \big|^p \, dx \right)^{1/p}. \tag{2.8}$$

Case IV. Let K' be an infinite cone in $\mathbb{R}^3$ with vertex at the origin, such that $\bar{K}' \setminus \{0\}$ is contained in the half space $\{x \in \mathbb{R}^3 : x_3 > 0\}$. We assume that the boundary $\partial K'$ consists of the vertex $x = 0$, the edges (half lines) $M_1, \dots, M_n$, and smooth faces $\Gamma_1, \dots, \Gamma_n$ (see Figure 4). This means $\Omega := K' \cap S^2$ is a domain of polygonal type on the unit sphere with sides $\Gamma_k \cap S^2$. Therein, without loss of generality, we may assume that Ω is simply connected. We put

$$Q := K' \cap \{x \in \mathbb{R}^3 : 0 < x_3 < 1\}.$$

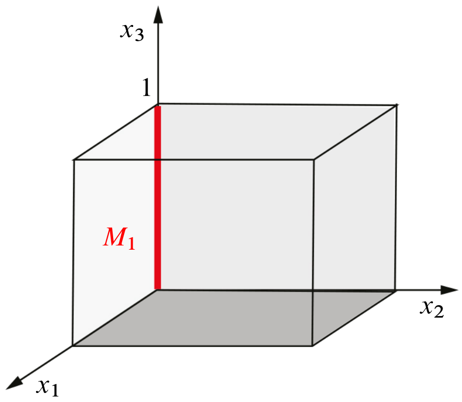


Figure 3. Kondratiev spaces on specific dihedral domains.

In this case, we choose

$$M := (M_1 \cup \dots \cup M_n) \cap \bar{Q}$$

and define

$$\|u|\mathcal{K}^m_{a,p}(Q,M)\| := \left(\sum_{|\alpha| \le m} \int_Q |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \right)^{1/p}, \tag{2.9}$$

where $\rho(x)$ denotes the distance of x to M.

Remark 3. The technical assumption $\bar{K}' \setminus \{0\} \subset \{x \in \mathbb{R}^3 : x_3 > 0\}$ immediately implies that the truncated cone Q is bounded (alternatively we can truncate the cone K' by intersecting with the unit ball $B_1(0)$). Moreover, if $\vartheta(x)$ denotes the angle between the line $\overrightarrow{0x}$ and the positive x_3-axis, then $\vartheta(x) \le \vartheta_0 < \pi/2$ for all $x \in Q$.

Based on these four cases, which we define the specific domains we will be concerned with in this paper.

Definition 4. Let D be a domain in $\mathbb{R}^2$ with singularity set S. Then, D is of *polyhedral type*, if there exist finite disjoint index sets Λ_1 and Λ_2 and a covering $(U_i)_i$ of bounded open sets such that

$$\bar{D} \subset \left(\bigcup_{i \in \Lambda_1} U_i \right) \cup \left(\bigcup_{j \in \Lambda_2} U_j \right),$$

where

(i) for $i \in \Lambda_1$ the set U_i is a ball such that $\overline{U_i} \cap S = \emptyset$;

(ii) for $j \in \Lambda_2$ there exists a C^∞-diffeomorphism $\eta_j : \overline{U_j} \to \eta_j(\overline{U_j}) \subset \mathbb{R}^d$ such that $\eta_j(U_j \cap D)$ is a smooth cone K as described in Case I. Moreover, we assume that for all $x \in U_j \cap D$ the distance to S is equivalent to the distance to the point $x^j := \eta_j^{-1}(0)$.

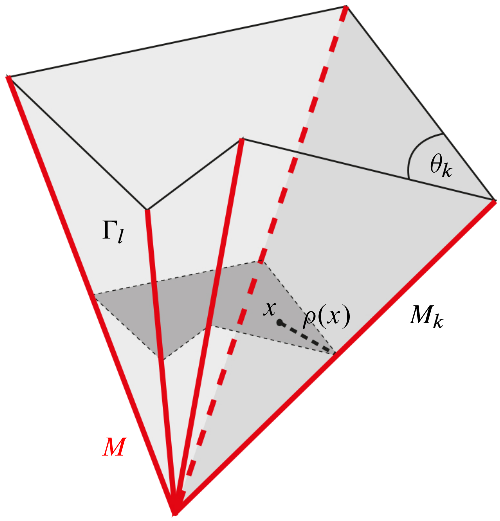


Figure 4. Kondratiev spaces on polyhedral cones.

Definition 5. Let D be a domain in $\mathbb{R}^3$ with singularity set S. Then, D is of *polyhedral type*, if there exist finite disjoint index sets $\Lambda_1, \ldots, \Lambda_5$ and a covering $(U_i)_i$ of bounded open sets such that

$$\bar{D} \subset \Big(\bigcup_{i\in\Lambda_1} U_i\Big) \cup \Big(\bigcup_{j\in\Lambda_2} U_j\Big) \cup \Big(\bigcup_{k\in\Lambda_3} U_k\Big) \cup \Big(\bigcup_{\ell\in\Lambda_4} U_\ell\Big) \cup \Big(\bigcup_{m\in\Lambda_5} U_m\Big),$$

where

(i) for $i \in \Lambda_1$ the set U_i is a ball such that $\overline{U_i} \cap S = \emptyset$;

(ii) for $j \in \Lambda_2$ there exists a C^∞-diffeomorphism $\eta_j : \overline{U_j} \to \eta_j(\overline{U_j}) \subset \mathbb{R}^3$ such that $\eta_j(U_j \cap D)$ is a smooth cone K as described in Case I. Moreover, we assume that for all $x \in U_j \cap D$ the distance to S is equivalent to the distance to the point $x^j := \eta_j^{-1}(0)$;

(iii) for $k \in \Lambda_3$ there exists a C^∞-diffeomorphism $\eta_k : \overline{U_k} \to \eta_k(\overline{U_k}) \subset \mathbb{R}^3$ such that $\eta_k(U_k \cap D)$ is the nonsmooth cone P as described in Case II. Moreover, we assume that for all $x \in U_k \cap D$ the distance to S is equivalent to the distance to the set $\Gamma^k := \eta_k^{-1}(\Gamma)$;

(iv) for $\ell \in \Lambda_4$ there exists a C^∞-diffeomorphism $\eta_\ell : \overline{U_\ell} \to \eta_\ell(\overline{U_\ell}) \subset \mathbb{R}^3$ such that $\eta_\ell(U_\ell \cap D)$ is a specific dihedral domain as described in Case III. Moreover, we assume that for all $x \in U_\ell \cap D$ the distance to S is equivalent to the distance to the set $M^\ell := \eta_\ell^{-1}(M_n)$ for some $n \in \{1, \ldots, d-1\}$;

(v) for $m \in \Lambda_5$ there exists a C^∞-diffeomorphism $\eta_m : \overline{U_m} \to \eta_m(\overline{U_m}) \subset \mathbb{R}^3$ such that $\eta_m(U_m \cap D)$ is a polyhedral cone as described in Case IV. Moreover, we assume that for all $x \in U_m \cap D$ the distance to S is equivalent to the distance to the set $M'_m := \eta_m^{-1}(M)$.

Remark 6. (i) Below we will not always distinguish between the cases $d = 2$ and $d = 3$, as clearly the case $d = 2$ is similar to the situation of a domain with conical points in $\mathbb{R}^3$, that is, a domain with $\Lambda_3 = \Lambda_4 = \Lambda_5 = \emptyset$.

(ii) Cases I–III are formulated for general d, and indeed, in the sequel, our arguments usually work in arbitrary dimensions. However, the counterpart of Definition 5 in $d > 3$ either leads to a very restricted class of domains (even the unit cube $[0, 1]^d$ would not be included) or it requires further standard situations beyond Cases I–IV. Therefore, a detailed discussion of higher-dimensional domains is beyond the scope of this paper. In most numerical applications the case $d = 3$ is the most interesting one anyway.

(iii) In our considerations, domains of polyhedral type are always bounded, as they can be covered by finitely many bounded sets U_i. While a number of results can be extended to unbounded domains, in the sequel we will not discuss this case.

(iv) In the literature, many different types of polyhedral domains are considered. As will be discussed below, in our context, only Cases I and III are essential. Therefore, our definition coincides with the one of Maz'ya and Rossmann [30, Definition 4.1.1]. Further variants can be found in Babuška, Guo [14], Bacuta, Mazzucato, Nistor, Zikatanov [3], and Mazzucato, Nistor [32].

(v) While the types of polyhedral domains coincide, in [30] more general weighted Sobolev spaces on those polyhedral domains are discussed. In particular, $\mathcal{K}^m_{a,p}(D, S)$ coincide with the classes $V^{\ell,p}_{\beta,\delta}(D)$ if $m = \ell$,

$$\beta = (\beta_1, \ldots, \beta_k) = (\ell - a, \ldots, \ell - a) \quad \text{and} \quad \delta = (\delta_1, \ldots, \delta_{k'}) = (\ell - a, \ldots, \ell - a).$$

For the meaning of k and k', we refer to [30, Lemma 4.1.1].

The definition contains certain redundancies. Cases II and III coincide when only functions with support within such subdomains are considered, e.g., when working with resolutions of unity as in Lemma 10 below; for restrictions of functions to subdomains, the cases are not equivalent. A little less obvious (though via still quite basic geometric arguments) it can be seen that also Case IV can be reduced to Cases I and II (see Lemma 11 below). However, this simple domain covering and the resulting norm decomposition are not applicable to every situation: the method does not allow the usage of a resolution of unity with compactly supported functions within the subdomains as discussed in Lemma 10—while a finite cover can be given, the compact supports of the functions from the resolution of unity prevent from getting arbitrarily close to the vertex of the polyhedral cone. Alternatively, one has to specifically include a neighbourhood of that vertex, on which the distance function is neither equivalent to the distance to an edge nor to the distance to the vertex. Fortunately, it turns out that the results presented in this article can be proven without the decomposition result from Lemma 10. One situation where the usage of a resolution of unity as in Lemma 10 is necessary arises when considering extension operators. This is discussed in detail in [18, 19]. Despite this redundancy, we still decided to include polyhedral cones in the above definition since they represent an important special case and, moreover, a number of results can be proved directly for such cones. This makes a reduction to other cases unnecessary and the presentation itself more accessible.

Remark 7. A discussion of a number of examples, as well as a slight generalization of Definition 5 to include also certain non-Lipschitz domains which the naïve geometric intuition would also label "polyhedral domain", can be found in [8].

Remark 8. In the literature, scales of weighted Sobolev spaces are also discussed in far more general domains. Exemplary, let us mention the work of Schrohe and Schulze [36, 37]. The authors discuss pseudodifferential operators on manifolds with conical singularities, that is, topological spaces $X \times [0, \infty)/X \times \{0\}$ with X being a smooth n-dimensional compact manifold. In this context, with the help of local coordinates (x, t), spaces $H^{s,\gamma}$ can be defined to contain functions for which

$$t^{\frac{n}{2}-\gamma}(t\partial_t)^k \partial_x^\alpha u(x,t) \in L_2$$

for all $k + |\alpha| \leq s \in \mathbb{N}$. In case of a smooth cone as in Case I (i.e., X being a smooth submanifold of $\mathbb{S}^n$) the coordinate t is equivalent to the weight function ρ, the (Euclidean) distance to the origin. Moreover, derivatives $\partial_x^\alpha u$ w.r.t. the spherical coordinates in X can be expressed in terms of $\rho^{|\beta|}\partial^\beta u$ w.r.t. the standard cartesian coordinates. In other words, on smooth cones and for $s \in \mathbb{N}$ those spaces $H^{s,\gamma}$ correspond to Kondratiev spaces $\mathcal{K}^s_{\gamma-\frac{n}{2},p}$. However, note that the general definition of the spaces $H^{s,\gamma}$ immediately allows for fractional smoothness parameters, i.e., $s \in \mathbb{R}$.

We continue with a few well-known properties of Kondratiev spaces.

Lemma 9. *Let D be a domain of polyhedral type with singularity set S. The space $\mathcal{K}^m_{a,p}(D, S)$ is invariant under C^∞ diffeomorphisms, i.e., if $\eta : D \to \eta(D) =: U$ denotes a C^∞ diffeomorphism, then the function $u : D \to \mathbb{R}$ belongs to $\mathcal{K}^m_{a,p}(D, S)$ if and only if the function $u \circ \eta^{-1} : U \to \mathbb{R}$ belongs to $\mathcal{K}^m_{a,p}(U, \eta(S))$. Furthermore, $\|u \mid \mathcal{K}^m_{a,p}(D, S)\|$ and $\|u \circ \eta^{-1} \mid \mathcal{K}^m_{a,p}(U, \eta(S))\|$ are equivalent.*

Proof. For convenience of the reader, we give a proof. For unweighted Sobolev spaces such a result is well known, we refer to Adams [1, Theorem 3.35].

Step 1. For the time being we restrict ourselves to the standard situations described in Case I–Case IV. Recall that in these specialized situations we do not deal with the distance to the associated singularity set. We need a common notation. Let (R, N) refer to one of the above four cases. We will need a geometrical property of the underlying domain. Concentrating on Case I–Case IV, it is obvious that there exists some $\varepsilon > 0$ such that for all $x \in N$ and all $y \in B_\varepsilon(x) \cap R$ the lines connecting x and y are contained in R. Let $\eta = (\eta_1, \dots, \eta_d)$. For all such pairs x and y it follows that

$$|\eta(x) - \eta(y)| \leq \left(\sup_{\xi \in D} \max_{j,i=1,\dots,d} \left| \frac{\partial}{\partial \xi_j} \eta_i(\xi) \right| \right) |x - y|.$$

Of course, $\bar{R}$ and its image $\bar{U} = \eta(\bar{R})$ are compact. Hence, there must exist a constant $C_\eta > 0$ such that

$$\frac{1}{C_\eta} |x - y| \leq |\eta(x) - \eta(y)| \leq C_\eta |x - y|, \quad x, y \in \bar{R}. \tag{2.10}$$

Let $\tau := \eta^{-1}$. The Faà di Bruno formula for derivatives of the composition, cf. [4, Theorem 3.4], gives us

$$\partial^\alpha (u \circ \tau)(y) = \sum_{1 \leq |\gamma| \leq |\alpha|} (\partial^\gamma u)(\tau(y)) \sum_{\beta_1^1, \dots, \beta_d^{\gamma_d}} c_{\alpha, \beta_1^1, \dots, \beta_d^{\gamma_d}} \prod_{j=1}^{d} \prod_{k=1}^{\gamma_j} \partial^{\beta_j^k} \tau_j(y), \tag{2.11}$$

where the second sum runs over all multiindices

$$\beta_1^1, \dots, \beta_1^{\gamma_1}, \dots, \beta_d^1, \dots, \beta_d^{\gamma_d} \in \mathbb{N}_0^d \setminus \{0\} \text{ satisfying } \alpha = \sum_{j=1}^{d} \sum_{k=1}^{\gamma_j} \beta_j^k,$$

with appropriate positive constants $c_{\alpha, \beta_1^1, \dots, \beta_d^{\gamma_d}}$. We put

$$\rho_R(y) := \min(1, \operatorname{dist}(y, N)), \quad y \in R,$$
$$\rho_U(y) := \min(1, \operatorname{dist}(y, \eta(N))), \quad y \in U.$$

Let $x \in N$ be fixed. Hence, the boundedness of the derivatives $\partial^{\beta_j^k} \tau_j$ and a change of coordinates lead to

$$\begin{aligned} &\int_{\eta(B_\varepsilon(x) \cap R)} |\rho_U(y)^{|\alpha| - a} \partial^\alpha (u \circ \tau)(y)|^p \, dy \\ &\quad \lesssim \int_{\eta(B_\varepsilon(x) \cap R)} \left| \rho_U(y)^{|\alpha| - a} \sum_{1 \leq |\gamma| \leq |\alpha|} (\partial^\gamma u)(\tau(y)) \right|^p dy \\ &\quad \lesssim \sum_{1 \leq |\gamma| \leq |\alpha|} \int_{\eta(B_\varepsilon(x) \cap R)} \left| \rho_U(y)^{|\alpha| - a} (\partial^\gamma u)(\tau(y)) \right|^p dy \\ &\quad \lesssim \sum_{1 \leq |\gamma| \leq |\alpha|} \int_{B_\varepsilon(x) \cap R} \left| \rho_U(\eta(z))^{|\alpha| - a} (\partial^\gamma u)(z) \right|^p dz. \end{aligned} \tag{2.12}$$

Applying (2.10), we can replace $\rho_U(\eta(z))$ by $\rho_R(z)$ itself on the right-hand side. We define

$$N_{\varepsilon/2} := \bigcup_{x \in N} \overline{B_{\varepsilon/2}(x)}.$$

Obviously, $N_{\varepsilon/2}$ is a compact set which has an open covering given by $\bigcup_{x\in N} B_\varepsilon(x)$. The theorem of Heine–Borel yields the existence of finitely many points $x_1,\dots,x_J$ in N such that

$$N_{\varepsilon/2} \subset \bigcup_{j=1}^{J} B_\varepsilon(x_j).$$

Furthermore, let

$$R_0 := \big\{x \in R : \operatorname{dist}(x,N) \geq \varepsilon/2\big\}.$$

On the set R_0, the function ρ_R is equivalent to 1 and at the same time, the function ρ_U is equivalent to 1 on $U_0 := \eta(R_0)$. Hence, on this part of R, we may use the invariance with respect to the unweighted case, i.e.,

$$\|u\circ\tau \mid \mathcal{K}^m_{a,p}(U_0,\eta(N))\| \lesssim \|u \mid \mathcal{K}^m_{a,p}(R_0,N)\|, \tag{2.13}$$

see Adams [1, Theorem 3.35]. Clearly,

$$R \subset \Bigg(R_0 \cup \bigcup_{j=1}^{J} B_\varepsilon(x_j)\Bigg).$$

Finally, summing up the inequalities (2.12) with x replaced by x_j and taking into account (2.13), we get

$$\|u\circ\tau \mid \mathcal{K}^m_{a,p}(U,\eta(N))\| \lesssim \|u \mid \mathcal{K}^m_{a,p}(R,N)\|.$$

Interchanging the roles of τ and η, we obtain the reverse inequality.

Step 2. The necessary modifications for the general case are obvious. ■

Now, we are going to discuss the importance of the existence of an associated decomposition of unity.

Lemma 10. *Let D, $(U_i)_i$, $(\eta_i)_i$, and Λ_j with $j = 1,\dots,5$ be as in Definition 5. Moreover, denote by S the singularity set of D and let $(\varphi_i)_i$ be a decomposition of unity subordinate to our covering, i.e., $\varphi_i \in C^\infty$, $\operatorname{supp}\varphi_i \subset U_i$, $0 \leq \varphi_i \leq 1$, and*

$$\sum_i \varphi_i(x) = 1 \quad \textit{for all } x \in \bar{D}.$$

We put $u_i := u\cdot\varphi_i$ in D.

(i) *If $u \in \mathcal{K}^m_{a,p}(D,S)$, then*

$$\begin{aligned}
&\|u \mid \mathcal{K}^m_{a,p}(D,S)\|^* \\
&\quad := \max_{i\in\Lambda_1} \|u_i|W^m_p(D\cap U_i)\| + \max_{i\in\Lambda_2} \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{a,p}(K,\{0\})\| \\
&\qquad + \max_{i\in\Lambda_3} \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{a,p}(P,\Gamma)\| + \max_{i\in\Lambda_4} \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{a,p}(I,M_\ell)\| \\
&\qquad + \max_{i\in\Lambda_5} \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{a,p}(Q,M)\|
\end{aligned}$$

generates an equivalent norm on $\mathcal{K}^m_{a,p}(D,S)$.

(ii) *If $u : D \to \mathbb{C}$ is a function such that the pieces u_i satisfy*

(a) $u_i \in W_p^m(D \cap U_i)$, $i \in \Lambda_1$;

(b) $u_i(\eta_i^{-1}(\cdot)) \in \mathcal{K}^m_{a,p}(K, \{0\})$, $i \in \Lambda_2$;

(c) $u_i(\eta_i^{-1}(\cdot)) \in \mathcal{K}^m_{a,p}(P, \Gamma)$, $i \in \Lambda_3$;

(d) $u_i(\eta_i^{-1}(\cdot)) \in \mathcal{K}^m_{a,p}(I, M_\ell)$, $i \in \Lambda_4$;

(e) $u_i(\eta_i^{-1}(\cdot)) \in \mathcal{K}^m_{a,p}(Q, M)$, $i \in \Lambda_5$;

then $u \in \mathcal{K}^m_{a,p}(D, S)$ *and*

$$\|u \mid \mathcal{K}^m_{a,p}(D, S)\| \lesssim \|u \mid \mathcal{K}^m_{a,p}(D, S)\|^*.$$

Proof. Step 1. Proof of (i). Let $S_i := S \cap \overline{(U_i \cap D)}$. We claim

$$\min(1, \operatorname{dist}(x, S)) \asymp \min(1, \operatorname{dist}(x, S_i)), \quad x \in U_i \cap D. \tag{2.14}$$

To prove this, we argue by contradiction. Let us assume that there exists a real number $\varepsilon > 0$ and a sequence $(x_j)_{j=1}^\infty \subset U_1 \cap D$ such that

$$\operatorname{dist}(x_j, S_1) \geq \varepsilon \quad \text{and} \quad \lim_{j\to\infty} \operatorname{dist}(x_j, S) = 0.$$

Hence, there exists a subsequence $(x_{j_\ell})_\ell$ which is convergent with limit x_0. Necessarily, $x_0 \in S$ and $x_0 \in \overline{U_1 \cap D}$. But this implies $x_0 \in S_1$, which is a contradiction to our previously made assumption. The boundedness of D yields the claim (2.14). Observe that this implies

$$\begin{aligned}\|u \mid \mathcal{K}^m_{p,a}(D, S)\| &= \Bigg(\sum_{|\alpha|\leq m} \int_D |\rho(x)^{|\alpha|-a} \sum_i \partial^\alpha u_i(x)|^p \, dx\Bigg)^{1/p} \\ &\leq \sum_i \Bigg(\sum_{|\alpha|\leq m} \int_{U_i\cap D} |\rho(x)^{|\alpha|-a} \partial^\alpha u_i(x)|^p \, dx\Bigg)^{1/p} \\ &\lesssim \sum_i \|u_i \mid \mathcal{K}^m_{p,a}(U_i \cap D, S_i)\|. \end{aligned} \tag{2.15}$$

We split the summation on the right-hand side into the five sums $\sum_{i\in\Lambda_j}$, $j = 1, \ldots, 5$. In the first case, we will use

$$\operatorname{dist}(U_i \cap D, S) \geq c > 0, \quad i \in \Lambda_1.$$

This yields $c \leq \min_{i\in\Lambda_1} \operatorname{dist}(U_i \cap D, S)$ and consequently, by using $\rho \leq 1$, we have

$$\begin{aligned}\sum_{i\in\Lambda_1} \|u_i \mid \mathcal{K}^m_{p,a}(U_i \cap D, S_i)\| &\lesssim \sum_{i\in\Lambda_1} \Bigg(\sum_{|\alpha|\leq m} \int_{U_i\cap D} |\partial^\alpha u_i(x)|^p \, dx\Bigg)^{1/p} \\ &\lesssim \sum_{i\in\Lambda_1} \|u_i|W_p^m(D \cap U_i)\| \lesssim \max_{i\in\Lambda_1} \|u_i|W_p^m(D \cap U_i)\|. \end{aligned} \tag{2.16}$$

Concerning the remaining terms, we will apply Lemma 9 and find

$$\|u_i \mid \mathcal{K}^m_{p,a}(U_i \cap D, S_i)\| \lesssim \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{p,a}(K, \{0\})\|, \quad i \in \Lambda_2, \tag{2.17}$$

$$\|u_i \mid \mathcal{K}^m_{p,a}(U_i \cap D, S_i)\| \lesssim \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{p,a}(K, P)\|, \quad i \in \Lambda_3, \tag{2.18}$$

$$\|u_i \mid \mathcal{K}^m_{p,a}(U_i \cap D, S_i)\| \lesssim \|u_i(\eta_i^{-1}(\cdot)) \mid \mathcal{K}^m_{p,a}(I, M_l)\|, \quad i \in \Lambda_4, \tag{2.19}$$

$$\|u_i \mid \mathcal{K}^m_{p,a}(U_i \cap D, S_i)\| \lesssim \|u_i(\eta_i^{-1}(\cdot)) \,|\mathcal{K}^m_{p,a}(Q, M)\|, \quad i \in \Lambda_5. \tag{2.20}$$

Inserting (2.16)–(2.20) into (2.15), we have proved

$$\|u \mid \mathcal{K}^m_{p,a}(D, S)\| \lesssim \|u \mid \mathcal{K}^m_{p,a}(D, S)\|^*.$$

In view of Lemma 9, the reverse inequality is obvious.

Step 2. Proof of (ii). Lemma 9 yields $u_i \in \mathcal{K}^m_{p,a}(D, S)$ for all i; hence, $\sum_i u_i \in \mathcal{K}^m_{p,a}(D, S)$. On the other hand, we have

$$\sum_i u_i = u.$$

∎

Now, let us consider the case $d = 3$ in a little more detail. If we omit the usage of a resolution of unity, we can decompose a polyhedral domain without the explicit inclusion of polyhedral cones. Thus, let the polyhedral cone Q and the set M be as in Case IV, with edges $M_1, \dots, M_n$ and vertex in 0. The angles in the plane $x_3 = 1$ are denoted by $\theta_1, \dots, \theta_n$; see Figure 4.

Recall the definition of nonsmooth cones from Case II: for a rotationally symmetric smooth cone K', this notion refers to either the set $K' \cap I$ or $K \setminus \overline{I}$. Using diffeomorphisms, we can further replace the unit cube I by general parallelepipeds.

Then, we can construct a covering of a polyhedral cone as follows (see Figure 5). For every edge M_j with angle θ_j, we choose a nonsmooth cone P_j with axis M_j as described above with sufficiently small opening angle γ_j so that none of these cones intersect. Next, we choose a further nonsmooth cone $\widetilde{P}_j$ with axis M_j and opening angle $\gamma_j/2$. Then, there exists a smooth cone $\widetilde{K}$ (i.e., a cone whose intersection with S^2 is a smooth subset) such that

$$\Big(Q \Big\backslash \bigcup_{j=1}^n P_j\Big) \subset \widetilde{K} \subset \Big(Q \Big\backslash \bigcup_{j=1}^n \overline{\widetilde{P}_j}\Big).$$

Since the original cone Q had been simply connected, this cone $\widetilde{K}$ fits the requirements for Case I.

This construction ensures that $\widetilde{K}$ has a non-empty intersection with each cone P_j; on $P_j \subset Q$, the distance to M is equivalent to the distance to M_j; and on $\widetilde{K}$, the distance to M is equivalent to the distance to the vertex 0. Altogether, we have a decomposition

$$Q = \widetilde{K} \cup \bigcup_{j=1}^n P_j,$$

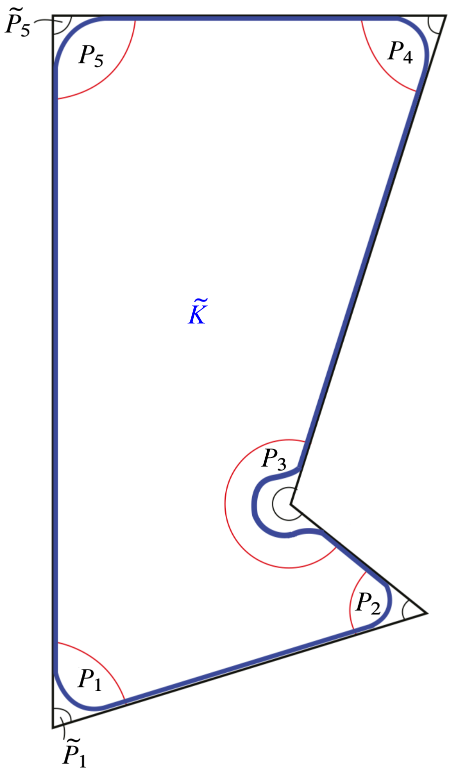


Figure 5. Covering of a polyhedral cone.

where each point $x \in Q$ belongs to at most 2 of the subsets. Let $\rho(x) := \min(1, \operatorname{dist}(x, M))$. The latter then immediately implies

$$\begin{aligned}
&\|u|\mathcal{K}^m_{a,p}(Q,M)\|^p \\
&\quad \sim \sum_{|\alpha|\le m} \left(\int_{\widetilde{K}} \left|\rho(x)^{|\alpha|-a} \partial^\alpha u(x)\right|^p dx + \sum_{j=1}^{n} \int_{P_j} \left|\rho(x)^{|\alpha|-a} \partial^\alpha u(x)\right|^p dx \right) \\
&\quad \sim \|u|\mathcal{K}^m_{a,p}(\widetilde{K},\{0\})\|^p + \max_{j=1,\dots,n} \|u|\mathcal{K}^m_{a,p}(P_j, M_j)\|^p .
\end{aligned}$$

We see that the above considerations lead to the following.

Lemma 11. *Let P be a polyhedral cone and let M be the particular set from Case IV.*

(i) *Then,*

$$\|u \mid \mathcal{K}^m_{a,p}(P,M)\|^{**} := \|u \mid \mathcal{K}^m_{a,p}(\widetilde{K},\{0\})\| + \max_{j=1,\dots,n} \|u \mid \mathcal{K}^m_{a,p}(P_j, M_j)\|$$

generates an equivalent norm on $\mathcal{K}^m_{a,p}(P,M)$.

(ii) *Let* $u_0 : \widetilde{K} \to \mathbb{C}$ *and* $u_j : P_j \to \mathbb{C}$, $j = 1, \dots, n$, *be given functions satisfying* $u_0(x) = u_j(x)$ *for* $x \in \widetilde{K} \cap P_j$ *and* $u_i(x) = u_j(x)$ *for* $x \in P_i \cap P_j$, *respectively, and*

- $u_0 \in \mathcal{K}^m_{a,p}(\widetilde{K}, \{0\})$,
- $u_j \in \mathcal{K}^m_{a,p}(P_j, M_j)$, $j = 1, \dots, n$.

Then, the function $u : P \to \mathbb{C}$, *piecewise-defined via* $u(x) = u_0(x)$, $x \in \widetilde{K}$, *and* $u(x) = u_j(x)$, $x \in P_j$, $j = 1, \dots, n$, *satisfies* $u \in \mathcal{K}^m_{a,p}(P, M)$ *and*

$$\|u \mid \mathcal{K}^m_{a,p}(P, M)\| \lesssim \|u \mid \mathcal{K}^m_{a,p}(P, M)\|^{**}.$$

Proof. It will be enough to comment on (ii). The compatibility condition implies that the function u is well defined. Moreover, since the sets $\widetilde{K}$, P_j are assumed to be open (hence their intersections are open as well), also the weak differentiability of the pieces u_j carries over. ∎

Remark 12. All domains of polyhedral type satisfy the cone condition, cf. [2, Definition 4.6]. But, in general, they do not have a Lipschitz boundary, see, e.g., the example [13, Example 6.5]. For our investigations within this article, we do not require any additional regularity assumptions on the domain (or its boundary) beyond being of polyhedral type.

2.3. Extensions

Stein's linear extension operator $\mathfrak{E}$, see [38, Section VI.3.2], has become a standard tool in the framework of Sobolev spaces on Lipschitz domains. It can be used in the framework of Kondratiev spaces as well. The following proposition contains particular cases of more general results which can be found in [17–19]. We need one more notation, compare with (2.7):

$$\mathbb{R}^\ell_* := \big\{x \in \mathbb{R}^d : x_1 = \cdots = x_{d-\ell} = 0\big\}. \tag{2.21}$$

Clearly, if $\ell = 1$, we will simply write $\mathbb{R}_*$. For brevity, we also put

$$\mathbb{R}^0_* := \{0\}, \quad \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*) := \mathcal{K}^m_{a,p}(\mathbb{R}^d \setminus \mathbb{R}^\ell_*, \mathbb{R}^\ell_*), \quad 0 \le \ell < d.$$

Proposition 13. *Let* $d \ge 2$, $1 \le p < \infty$, $a \in \mathbb{R}$, *and* $m \in \mathbb{N}$.

(i) *Let* K *be our smooth cone from Case I. Then, the Stein extension operator* $\mathfrak{E}$ *yields a linear and bounded mapping of* $\mathcal{K}^m_{a,p}(K, \{0\})$ *into* $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \{0\})$.

(ii) *Let* P *and* Γ *be as in Case II. Then, the Stein extension operator* $\mathfrak{E}$ *yields a linear and bounded mapping of* $\mathcal{K}^m_{a,p}(P, \Gamma)$ *into* $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}_*)$.

(iii) *Let* I *and* M_ℓ, $1 \le \ell < d$, *be as in Case III. Then, the Stein extension operator* $\mathfrak{E}$ *yields a linear and bounded mapping of* $\mathcal{K}^m_{a,p}(I, M_\ell)$ *into* $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

(vi) *Let* Q *and* M *be as in Case IV. Then, there exists a linear and bounded extension operator* $\mathfrak{E} : \mathcal{K}^m_{a,p}(Q, M) \to \mathcal{K}^m_{a,p}(\mathbb{R}^d, M)$.

With Proposition 13 and Lemma 10 at hand, we can reduce the basic properties of the spaces $\mathcal{K}^m_{a,p}(D,S)$ to the model cases $\mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)$, $0 \le \ell < d$. In this model setting, we find for the weight function that

$$\rho(x_1,\dots,x_{d-\ell},\dots,x_d) = \min\Big(1, \Big(\sum_{i=1}^{d-\ell} |x_i|^2\Big)^{1/2}\Big). \tag{2.22}$$

2.4. A localization principle

The following decomposition of the norm of weighted Sobolev spaces is in some sense standard. We will allow a slightly greater generality than before.

Let $\Omega \subset \mathbb{R}^d$ be an open, nontrivial connected set and let M be a closed nontrivial subset of the boundary. Then, we define

$$\Omega_j := \big\{x \in \mathbb{R}^d : 2^{-j-1} < \rho(x) < 2^{-j+1}\big\}, \quad j \in \mathbb{N}_0, \tag{2.23}$$

where $\rho(x) := \min(1, \operatorname{dist}(x, M))$, $x \in \mathbb{R}^d$. Next, we choose the largest number $j_0 \in \mathbb{N}_0$ such that

$$\big\{x \in \Omega : \rho(x) \ge 2^{-j_0+1}\big\} = \emptyset.$$

This implies

$$\Omega \cap \Omega_{j_0} = \big\{x \in \Omega : 2^{-j_0-1} < \rho(x) < 2^{-j_0+1}\big\} \ne \emptyset.$$

Because Ω is open and connected, the continuity of ρ yields

$$|\Omega \cap \Omega_j| > 0 \quad \text{for all } j \ge j_0. \tag{2.24}$$

Hence,

$$\Omega = \bigcup_{j=j_0}^{\infty} (\Omega \cap \Omega_j).$$

For technical reasons, we need to distinguish the following two cases:

(a) $\{x \in \Omega : 2^{-j_0} + 2^{-j_0-2} < \rho(x) < 2^{-j_0+2} - 2^{-j_0-2}\} = \emptyset$. Then, we define $j_1 := j_0$,

(b) $\{x \in \Omega : 2^{-j_0} + 2^{-j_0-2} < \rho(x) < 2^{-j_0+2} - 2^{-j_0-2}\} \ne \emptyset$. Then, we define $j_1 := j_0 - 1$.

Of course, we will need some regularity of Ω. We will use a condition guaranteeing that for $x \in \Omega$ an essential part of the ball centred at x and with radius proportional to the distance of x to M lies inside Ω. We put $\sigma := 2^{-j_0}$.

Proposition 14. *Let $1 \le p < \infty$, $a \in \mathbb{R}$, and $m \in \mathbb{N}$. Let Ω, M, ρ, and j_1 be as above. We assume that there exist two constants $c > 0$ and $t \in (0,1)$ such that*

- *for all $x \in \Omega$, $\operatorname{dist}(x, M) < \sigma$, the balls $B_\lambda(x)$ satisfy*

$$|B_\lambda(x) \cap \Omega| \ge c\,\lambda^d \quad \textit{for all } \lambda \in \Big[\frac{t}{32}\,\rho(x), t\,\rho(x)\Big]; \tag{2.25}$$

- *for all* $x \in \Omega$, $\operatorname{dist}(x, M) \geq \sigma$, *the balls* $B_\lambda(x)$ *satisfy*

$$|B_\lambda(x) \cap \Omega| \geq c\,\lambda^d \quad \textit{for all } \lambda \leq t\sigma. \tag{2.26}$$

Then, there exist positive constants A, B, *and a smooth decomposition of unity* $(\varphi_j)_{j \geq j_1}$ *such that*

- $\varphi_j \in C^\infty(\mathbb{R}^d)$, $j \geq j_1$;
- $\operatorname{supp} \varphi_j \subset \Omega_j$, $j \geq j_1$;
- $0 \leq \varphi_j(x) \leq 1$ *for all* $j \geq j_1$ *and all* $x \in \Omega$;

$$\sum_{j=j_1}^{\infty} \varphi_j(x) = 1 \quad \textit{for all } x \in \Omega,$$

and

$$A \|u \mid \mathcal{K}^m_{a,p}(\Omega, M)\|^p \leq \sum_{j=j_1}^{\infty} \|\varphi_j u \mid \mathcal{K}^m_{a,p}(\Omega, M)\|^p \leq B \|u \mid \mathcal{K}^m_{a,p}(\Omega, M)\|^p \tag{2.27}$$

for all $u \in \mathcal{K}^m_{a,p}(\Omega, M)$.

Proof. Step 1. The lengthy construction of the family $(\varphi_j)_{j=j_1}^{\infty}$ can be found in [8]. We only note that, in addition to the claimed properties, we also have

$$|\partial^\alpha \varphi_j(x)| \lesssim 2^{j|\alpha|}, \quad j \geq j_1. \tag{2.28}$$

Step 2. Because of $\operatorname{supp} \varphi_j \subset \Omega_j$ and on Ω_j only $\varphi_{j+\ell}$, $|\ell| \leq 1$, are not identically zero, we conclude that

$$\begin{aligned} \|u \mid \mathcal{K}^m_{a,p}(\Omega, M)\|^p &\lesssim \sum_{|\alpha| \leq m} \Bigg(\sum_{j=j_1}^{\infty} \int_{\Omega \cap \operatorname{supp} \varphi_j} |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \Bigg) \\ &\lesssim \sum_{|\alpha| \leq m} \Bigg(\sum_{j=j_1}^{\infty} 2^{-j(|\alpha|-a)p} \int_{\Omega_j \cap \Omega} |\partial^\alpha (u\varphi_j)(x)|^p dx \Bigg). \end{aligned}$$

Applying the estimate (2.28) to the subsequence $(\varphi_{2j})_{j \geq j_1/2}$, we find

$$\begin{aligned} &\sum_{|\alpha| \leq m} \sum_{j \geq j_1/2} 2^{-2j(|\alpha|-a)p} \int_{\Omega_{2j} \cap \Omega} \big|\partial^\alpha (u\varphi_{2j})(x)\big|^p \, dx \\ &\quad \lesssim \sum_{|\alpha| \leq m} \sum_{j \geq j_1/2} \int_{\operatorname{supp} \varphi_{2j} \cap \Omega} \big|\rho(x)^{|\alpha|-a}\, \partial^\alpha (u\varphi_{2j})(x)\big|^p \, dx \\ &\quad \lesssim \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} \sum_{j \geq j_1/2} \int_{\operatorname{supp} \varphi_{2j} \cap \Omega} \big|\rho(x)^{|\alpha|-a}\, \partial^\beta u(x)\, \partial^{\alpha-\beta} \varphi_{2j}(x)\big|^p \, dx \\ &\quad \lesssim \sum_{|\beta| \leq m} \sum_{j \geq j_1/2} \int_{\operatorname{supp} \varphi_{2j} \cap \Omega} \big|\rho(x)^{|\beta|-a}\, \partial^\beta u(x)\big|^p \, dx \\ &\quad \lesssim \sum_{|\beta| \leq m} \int_{\Omega} \big|\rho(x)^{|\beta|-a}\, \partial^\beta u(x)\big|^p \, dx = \big\|u \mid \mathcal{K}^m_{a,p}(\Omega, M)\big\|^p, \end{aligned}$$

where we used in the last line $\operatorname{supp}\varphi_{2j} \cap \operatorname{supp}\varphi_{2j+2} = \emptyset$. Taking a similar estimate with φ_{2j+1} instead of φ_{2j} into account, we obtain

$$\|u \mid \mathcal{K}^m_{a,p}(\Omega, M)\|^p \sim \sum_{|\alpha|\le m} \sum_{j=j_1}^{\infty} 2^{-j(|\alpha|-a)p} \int_{\Omega_j\cap\Omega} \left|\partial^\alpha (u\varphi_j)(x)\right|^p dx$$
$$\sim \sum_{j=j_1}^{\infty} \left\|\varphi_j u \mid \mathcal{K}^m_{a,p}(\Omega, M)\right\|^p, \tag{2.29}$$

as claimed. ∎

Examples and comments

Lemma 15. *Let $1 \le p < \infty$, $a \in \mathbb{R}$, and $m \in \mathbb{N}$. Let $0 \le \ell < d$. Then, $\mathbb{R}^d \setminus \mathbb{R}^\ell_*$ satisfies the restrictions (2.25) and (2.26) with respect to the set $M := \mathbb{R}^\ell_*$. The decomposition of unity, constructed in the proof of Proposition 14, has the following additional properties:*

- $j_0 = 0$, *i.e.,* $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ *for all* $x \in \mathbb{R}^d \setminus \mathbb{R}^\ell_*$;
- $\varphi_0(x) + \varphi_1(x) = 1$ *for all* $x \in \Omega_0$;
- $\varphi_{j-1}(x) + \varphi_j(x) + \varphi_{j+1}(x) = 1$ *for all* $x \in \Omega_j$, $j \in \mathbb{N}$;
- $\varphi_j(x) = \varphi_1(2^{j-1}x)$, $x \in \mathbb{R}^d$, $j \in \mathbb{N}$.

Proof. Almost all properties of the decomposition of unity are immediate except for probably the last one. The sets Ω_j with respect to the pair $(\mathbb{R}^d \setminus \mathbb{R}^\ell_*, \mathbb{R}^\ell_*)$ have a very simple geometric structure. The transformation $J : x \mapsto 2^{-j+1}x$ restricted to Ω_1, is a bijection onto Ω_j, $j \ge 1$. This is enough to show that

$$\varphi_j(x) = \varphi_1(2^{j-1}x), \quad x \in \mathbb{R}^d,\ j \in \mathbb{N}.$$
∎

Mutatis mutandis, one can prove also the following.

Lemma 16. *Proposition 14 is applicable with respect to the smooth cone, see Case I, with respect to the specific nonsmooth cone, see Case II, the specific dihedral domain, see Case III, and the polyhedral cone, see Case IV, always equipped with the appropriate sets M.*

Remark 17. (i) Those localized characterizations of Kondratiev spaces can be found also in Maz'ya, Rossmann [30, Lemmas 1.2.1, 2.1.4] for smooth cones and dihedral domains.

(ii) In [39, Section 3.2.3] Triebel discusses function spaces defined by localized norms as in (2.27). But he is working with $M := \partial\Omega$.

Remark 18. Let us stress the fact that a domain Ω needs not be a Lipschitz domain in order to satisfy the assumptions in Proposition 14, see, e.g., [13, Example 6.5]. Moreover, whereas a pair (Ω, M_1) may satisfy those restrictions, this is not necessarily true for an alternative choice (Ω, M_2). Explicit examples can be found in [8].

2.5. A general strategy

Beginning with the next subsection, we will employ Proposition 14 and its consequences, see Lemmas 15 and 16, always in the following way.

- First step: localization of the underlying domain D by means of Proposition 14.
- Second step: reduction to some standard situation (unweighted Sobolev spaces defined on Ω_0 or Ω_1) by using homogeneity arguments.
- Third step: application of some well-known properties of $W_p^m(\Omega_0)$, $W_p^m(\Omega_1)$.
- Fourth step: rescaling and a second application of Proposition 14.

2.6. A further equivalent norm

Given $x \in \Omega$, let $R(x)$ consist of all points y in Ω such that the line segment joining x to y lies entirely in Ω. Put

$$\Gamma(x) := \big\{y \in R(x) : |y - x| < 1\big\},$$

and let $|\Gamma(x)|$ denote the Lebesgue measure of $\Gamma(x)$. Then, Ω satisfies the *weak cone condition* if there exists a number $\delta > 0$ such that

$$|\Gamma(x)| \geq \delta \quad \text{for all } x \in \Omega.$$

For classical Sobolev spaces, it is known that

$$\|u|W_p^m(\Omega)\|_* := \|u|L_p(\Omega)\| + \sum_{|\alpha|=m} \|D^\alpha u|L_p(\Omega)\| \tag{2.30}$$

is an equivalent norm in $W_p^m(\Omega)$ as long as the underlying domain Ω satisfies the weak cone condition, cf. [2, Theorem 5.2].

We will show that also, in the case of Kondratiev spaces $\mathcal{K}^m_{a,p}(D, S)$ defined on domains of polyhedral type, it suffices to consider the extremal derivatives ($|\alpha| = m$ and $|\alpha| = 0$) to obtain an equivalent norm.

As a preparation, we will deal with the model case $\Omega := \mathbb{R}^d \setminus \mathbb{R}^l_*$ with $M := \mathbb{R}^l_*$.

Proposition 19. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$, $a \in \mathbb{R}$, and $0 \leq \ell < d$. Then,*

$$\|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|_* := \|\rho^{-a}\, u|L_p(\mathbb{R}^d)\| + \sum_{|\alpha|=m} \|\rho^{m-a}\, \partial^\alpha u|L_p(\mathbb{R}^d)\|$$

is an equivalent norm in $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

Proof. Recall that in this model case we have the explicit expression (2.22) for the weight function ρ. We will employ the partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ from Proposition 14 with

$$\operatorname{supp}\varphi_j \subset \Omega_j := \big\{x \in \mathbb{R}^d : 2^{-j-1} < \rho(x) < 2^{-j+1}\big\}, \quad j \in \mathbb{N}_0.$$

Moreover, $(\varphi_j)_{j\in\mathbb{N}_0}$ satisfies the properties from Lemma 15. In particular, the functions φ_j have finite overlap and $\varphi_j(x) = \varphi_1(2^{j-1}x)$ for all x and all $j \in \mathbb{N}$. With this, we estimate

$$\begin{aligned}
&\left\| u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^l_*)\right\| \\
&\quad \sim \left(\sum_{|\alpha|\le m} \sum_{j=0}^{\infty} \int_{2^{-j-1}<\rho(x)<2^{-j+1}} \left|\rho(x)^{(|\alpha|-a)}\, \partial^\alpha(\varphi_j u)(x)\right|^p dx \right)^{1/p} \\
&\quad \sim \left(\sum_{|\alpha|\le m} \sum_{j=0}^{\infty} \int_{\Omega_j} 2^{-(j-1)(|\alpha|-a)p} |\partial^\alpha(\varphi_j u)(x)|^p dx \right)^{1/p},
\end{aligned} \tag{2.31}$$

where for technical reasons we used $2^{-(j-1)(|\alpha|-a)p}$ instead of $2^{-j(|\alpha|-a)p}$ in the second step. A homogeneity argument, applied to the terms with $j \ge 1$, yields

$$\begin{aligned}
A &:= \left(\sum_{|\alpha|\le m} \int_{\Omega_j} 2^{-(j-1)(|\alpha|-a)p} |\partial^\alpha(\varphi_j u)(x)|^p dx \right)^{1/p} \\
&= \left(\sum_{|\alpha|\le m} 2^{(j-1)ap} 2^{-(j-1)d} \int_{\Omega_1} \left|\partial^\alpha((\varphi_j u)(2^{-j+1}\cdot))(y)\right|^p dy \right)^{1/p} \\
&= 2^{(j-1)a} 2^{-(j-1)d/p} \|(\varphi_j u)(2^{-j+1}\cdot)|W^m_p(\Omega_1)\|.
\end{aligned}$$

Since Ω_1 has the weak cone property, we conclude from (2.30) that

$$\begin{aligned}
A \sim \Bigg(&\sum_{|\alpha|=m} 2^{(j-1)ap} 2^{-(j-1)d} \int_{\Omega_1} |\partial^\alpha((\varphi_j u)(2^{-j+1}\cdot))(y)|^p dy \\
&+ 2^{(j-1)ap} 2^{-(j-1)d} \int_{\Omega_1} |(\varphi_j u)(2^{-j+1}y)|^p dy \Bigg)^{1/p}.
\end{aligned}$$

Now, it is easy to see that the right-hand side is equivalent to

$$\begin{aligned}
&\left(\sum_{|\alpha|=m} 2^{(j-1)ap} \int_{\Omega_j} |2^{-(j-1)|\alpha|}\, \partial^\alpha(\varphi_j u)(x)|^p dx + 2^{(j-1)ap} \int_{\Omega_j} |(\varphi_j u)(x)|^p dx \right)^{1/p} \\
&\quad \sim \left(\sum_{|\alpha|=m} \int_{\Omega_j} |\rho(x)^{|\alpha|-a}\, \partial^\alpha(\varphi_j u)(x)|^p dx + \int_{\Omega_j} |\rho(x)^{-a}(\varphi_j u)(x)|^p dx \right)^{1/p}.
\end{aligned}$$

On the other hand, for the term $j = 0$, we easily see that

$$\begin{aligned}
\left(\sum_{|\alpha|\le m} \int_{\Omega_0} |\rho(x)^{|\alpha|-a}\, \partial^\alpha(\varphi_0 u)(x)|^p dx \right)^{1/p} &\sim \left(\sum_{|\alpha|\le m} \int_{\Omega_0} |\partial^\alpha(\varphi_0 u)(x)|^p dx \right)^{1/p} \\
&= \|\varphi_0 u|W^m_p(\Omega_0)\|
\end{aligned}$$

and

$$\begin{aligned}
&\|\varphi_0 u|W_p^m(\Omega_0)\| \\
&\quad\sim \Big(\sum_{|\alpha|=m} \int_{\Omega_0} |\partial^\alpha(\varphi_0 u)(x)|^p dx + \int_{\Omega_0} |(\varphi_0 u)(x)|^p dx \Big)^{1/p} \\
&\quad\sim \Big(\sum_{|\alpha|=m} \int_{\Omega_0} |\rho(x)^{|\alpha|-a}\, \partial^\alpha(\varphi_0 u)(x)|^p dx + \int_{\Omega_0} |\rho(x)^{-a}(\varphi_0 u)(x)|^p dx \Big)^{1/p}.
\end{aligned}$$

Inserting these estimates into (2.31), we find

$$\begin{aligned}
&\|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^l_*)\| \\
&\quad\sim \Big(\sum_{|\alpha|=m} \int_{\mathbb{R}^d\setminus\mathbb{R}^l_*} |\rho(x)^{|\alpha|-a}\, \partial^\alpha u(x)|^p dx + \int_{\mathbb{R}^d\setminus\mathbb{R}^l_*} |\rho(x)^{-a} u(x)|^p dx \Big)^{1/p} \\
&\quad= \|u|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^l_*)\|_*.
\end{aligned}$$

The proof is complete. ∎

Theorem 20. *Let the pair (D, M) be a Lipschitz domain of polyhedral type. Furthermore, let $1 \le p < \infty$, $m \in \mathbb{N}$, and $a \in \mathbb{R}$. Then,*

$$\|u|\mathcal{K}^m_{a,p}(D, M)\|_* := \|\rho^{-a}\, u|L_p(D)\| + \sum_{|\alpha|=m} \|\rho^{m-a}\, \partial^\alpha u|L_p(D)\|$$

is an equivalent norm in $\mathcal{K}^m_{a,p}(D, M)$.

Proof. Step 1. For simplicity, we deal with Case I, the smooth cone K with $M = \{0\}$. As mentioned before, see the list of basic properties in Subsection 2.2, $C^\infty_*(K, \{0\})$ is a dense subset in $\mathcal{K}^m_{a,p}(K, \{0\})$. Let $u \in C^\infty_0(K, \{0\})$. Then, it is readily checked that $\mathfrak{E}u \in C^\infty_0(\mathbb{R}^d \setminus \{0\})$, where $\mathfrak{E}$ denotes Stein's extension operator. A closer inspection of the proof of Proposition 13 presented in [17] reveals that the estimates for the weighted L_p-norms of partial derivatives of $\mathfrak{E}u$ of order m involve only partial derivatives of u likewise of order m; thus, we find

$$\|\mathfrak{E}u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \{0\})\|_* \le c\|u \mid \mathcal{K}^m_{a,p}(K, \{0\})\|_*.$$

Hence, with the help of Proposition 19, we conclude that

$$\begin{aligned}
\|u \mid \mathcal{K}^m_{a,p}(K, \{0\})\| &\le \|\mathfrak{E}u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \{0\})\| \\
&\le c\|\mathfrak{E}u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \{0\})\|_* \le c\|u \mid \mathcal{K}^m_{a,p}(K, \{0\})\|_*.
\end{aligned}$$

Step 2. Clearly, Cases II and III can be handled in a similar fashion as in Step 1. The domain decomposition strategies from [18, 19] then can be used to transfer the result first to layers of polyhedral cones, i.e., intersections of polyhedral cones with sets $\{x \in \mathbb{R}^d : r < |x| < R\}$, and subsequently to polyhedral cones. Ultimately, general domains of polyhedral type D can be reduced to the standard situations with the help of Lemma 10. ∎

3. Continuous embeddings

With the help of the localization result from Subsection 2.4, Sobolev-type embeddings for Kondratiev spaces can now be traced back to corresponding results for unweighted Sobolev spaces. In a first step, we deal with the model case $\Omega := \mathbb{R}^d \setminus \mathbb{R}^\ell_*$ and $M := \mathbb{R}^\ell_*$.

3.1. Continuous embeddings in the model case

Again, we proceed as described in Subsection 2.5.

Proposition 21. *Let $1 \le p \le q < \infty$, $m \in \mathbb{N}$, $a \in \mathbb{R}$. Let $0 \le \ell < d$. Then, $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ is embedded into $\mathcal{K}^{m'}_{a',q}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ if and only if*

$$m - \frac{d}{p} \ge m' - \frac{d}{q} \quad \text{and} \quad a - \frac{d}{p} \ge a' - \frac{d}{q}. \tag{3.1}$$

Proof. Step 1. We will apply the decomposition of unity as constructed in Proposition 14 and with the additional properties as described in Lemma 15. Similar to (2.23) we put

$$D_j := \left\{x \in \mathbb{R}^d : 2^{-j-1} < \rho(x) < 2^{-j+1}\right\}, \qquad j \in \mathbb{N}_0 \tag{3.2}$$

and $\rho(x) = \min(1, |x'|)$, $x = (x', x'')$, $x' \in \mathbb{R}^{d-\ell}$, $x'' \in \mathbb{R}^\ell$. We choose $\varepsilon_j := 2^{-j-4}$, $j \in \mathbb{N}_0$. Recall, $\varphi_j(x) = \varphi_1(2^{j-1}x)$, $j \in \mathbb{N}$. In view of formula (2.29) we will consider the terms $\sum_{|\alpha| \le m'} 2^{-j(|\alpha|-a')q} \int_{D_j} |\, \partial^\alpha (u\varphi_j)(x)|^q \, dx$. For technical reasons, we replace $2^{-j(|\alpha|-a')q}$ by $2^{-(j-1)(|\alpha|-a')q}$. A transformation of coordinates $x := 2^{-j+1}y$ and the just mentioned homogeneity property of the system $(\varphi_j)_j$ yield

$$\begin{aligned}
&\sum_{|\alpha| \le m'} 2^{-(j-1)(|\alpha|-a')q} \int_{D_j} |\partial^\alpha (u\varphi_j)(x)|^q \, dx \\
&\quad = \sum_{|\alpha| \le m'} 2^{-(j-1)(|\alpha|-a')q} \int_{D_1} |\partial^\alpha (u\varphi_j)(2^{-j+1}y)|^q \, 2^{(-j+1)d} \, dy \\
&\quad = \sum_{|\alpha| \le m'} 2^{-(j-1)(|\alpha|-a')q} 2^{(-j+1)d} \, 2^{(j-1)|\alpha|q} \int_{D_1} \left|\partial^\alpha (u(2^{-j+1}\,\cdot)\varphi_1)(\cdot)\, \sqrt[n]{}(y)\right|^q dy \\
&\quad = 2^{(j-1)a'q} 2^{(-j+1)d} \| u(2^{-j+1}\,\cdot)\varphi_1 \mid W^{m'}_q(D_1)\|^q .
\end{aligned}$$

Here, $W^{m'}_q(D_1)$ denotes the standard Sobolev space with parameters m' and q on D_1. Clearly,

$$W^m_p(D_1) \hookrightarrow W^{m'}_q(D_1) \quad \text{if } m - d/p \ge m' - d/q, \ 1 \le p \le q < \infty, \tag{3.3}$$

see [2, Theorem 4.12]. Therefore, we obtain

$$\begin{aligned}
&\sum_{|\alpha| \le m'} 2^{-(j-1)(|\alpha|-a')q} \int_{D_j} |\partial^\alpha (u\varphi_j)(x)|^q \, dx \\
&\qquad \lesssim 2^{(j-1)a'q} 2^{(-j+1)d} \| u(2^{-j+1}\,\cdot)\varphi_1 | W^m_p(D_1)\|^q
\end{aligned} \tag{3.4}$$

with hidden constants independent of j and u. By applying the same homogeneity arguments as above, but in reversed order, we find

$$\begin{aligned}\|u(2^{-j+1}\cdot)\varphi_1 \mid W_p^m(D_1)\|^q &= \Big(\sum_{|\alpha|\le m}\int_{D_1}|2^{(-j+1)|\alpha|}\,\partial^\alpha(u\varphi_j)(2^{-j+1}y)|^p\,dy\Big)^{q/p}\\ &= \Big(\sum_{|\alpha|\le m}2^{(-j+1)|\alpha|p}\,2^{(j-1)d}\int_{D_j}|\partial^\alpha(u\varphi_j)(x)|^p\,dx\Big)^{q/p}.\end{aligned}$$

Inserting this into (3.4), we obtain

$$\begin{aligned}&\sum_{|\alpha|\le m'}2^{-(j-1)(|\alpha|-a')q}\int_{D_j}|\,\partial^\alpha(u\varphi_j)(x)|^q\,dx\\ &\quad\lesssim 2^{(j-1)a'q}2^{(-j+1)d}\Big(\sum_{|\alpha|\le m}2^{(-j+1)|\alpha|p}\,2^{(j-1)d}\int_{D_j}|\partial^\alpha(u\varphi_j)(x)|^p\,dx\Big)^{q/p}\\ &\quad\lesssim\Big(\sum_{|\alpha|\le m}2^{(j-1)(\frac{d}{p}-\frac{d}{q}-|\alpha|+a')p}\int_{D_j}|\partial^\alpha(u\varphi_j)(x)|^p\,dx\Big)^{q/p}\\ &\quad\lesssim\Big(\sum_{|\alpha|\le m}2^{(j-1)(\frac{d}{p}-\frac{d}{q}+a'-a)p}\int_{D_j}|2^{-j(|\alpha|-a)}\partial^\alpha(u\varphi_j)(x)|^p\,dx\Big)^{q/p}.\end{aligned}$$

Obviously, $2^{-j(|\alpha|-a)}\asymp\rho(x)^{|\alpha|-a}$ on D_j. By assumption $\frac{d}{p}-\frac{d}{q}+a'-a\le 0$. Hence,

$$\begin{aligned}&\sum_{|\alpha|\le m'}\sum_{j=1}^{\infty}2^{-j(|\alpha|-a')q}\int_{D_j}|\partial^\alpha(u\varphi_j)(x)|^q\,dx\\ &\qquad\lesssim\sum_{j=1}^{\infty}\Big(\sum_{|\alpha|\le m}\int_{D_j}|\rho(x)^{|\alpha|-a}\partial^\alpha(u\varphi_j)(x)|^p\,dx\Big)^{q/p}.\end{aligned}$$

Next, we will use that $\ell_1\hookrightarrow\ell_{q/p}$. This yields

$$\begin{aligned}&\sum_{|\alpha|\le m'}\sum_{j=1}^{\infty}2^{-j(|\alpha|-a')q}\int_{D_j}|\partial^\alpha(u\varphi_j)(x)|^q\,dx\\ &\qquad\lesssim\Big(\sum_{j=1}^{\infty}\sum_{|\alpha|\le m}\int_{D_j}|\rho(x)^{|\alpha|-a}\partial^\alpha(u\varphi_j)(x)|^p\,dx\Big)^{q/p}.\end{aligned}$$

For the term with $j=0$, it is enough to apply the Sobolev embedding (3.3) with D_1 replaced by D_0. In view of (2.29), this proves the sufficiency of our conditions.

Step 2. Necessity. The necessity of $m-d/p\ge m'-d/q$ is part of the classical Sobolev theory, we refer to [1, Theorem 5.4]. It remains to prove necessity of $a-d/p\ge a'-d/q$.

Therefore, we choose a non-trivial function $u \in C_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} u \subset \{x \in \mathbb{R}^d : 0 < |x'| < 1\}$. Such a function and all dilated versions $u(\lambda\, \cdot)$, $\lambda > 0$, belong to all spaces $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$. Observe the following homogeneity property for values $\lambda > 1$:

$$\begin{aligned} \|u(\lambda\, \cdot)|\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p &= \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} \lambda^{|\alpha|p} |\rho(x)^{|\alpha|-a}\, \partial^\alpha u(\lambda x)|^p \, dx \\ &= \sum_{|\alpha|\le m} \lambda^{|\alpha|p-d} \int_{\operatorname{supp} u} |\rho(y/\lambda)^{|\alpha|-a}\, \partial^\alpha u(y)|^p \, dy \\ &\sim \sum_{|\alpha|\le m} \lambda^{ap-d} \int_{\operatorname{supp} u} |\rho(y)^{|\alpha|-a}\, \partial^\alpha u(y)|^p \, dy \\ &= \lambda^{ap-d} \|u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p. \end{aligned} \tag{3.5}$$

We assume $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*) \hookrightarrow \mathcal{K}^{m'}_{a',q}(\mathbb{R}^d, \mathbb{R}^\ell_*)$. This implies the existence of a positive constant c such that

$$\|u(\lambda\, \cdot) \mid \mathcal{K}^{m'}_{a',q}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \le c \|u(\lambda\, \cdot) \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|$$

holds for all $\lambda \ge 1$ and in view of (3.5) $a' - d/q \le a - d/p$, as claimed. ∎

The counterpart for $q = \infty$ can be formulated as follows.

Proposition 22. *Let $a \in \mathbb{R}$, $m \in \mathbb{N}$, and $0 \le \ell < d$.*

(i) *Let $1 < p < \infty$. Then, $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ is embedded into $\mathcal{K}^{m'}_{a',\infty}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ if and only if*

$$m - \frac{d}{p} > m' \quad \text{and} \quad a - \frac{d}{p} \ge a'.$$

(ii) *Let $p = 1$. Then, $\mathcal{K}^m_{a,1}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ is embedded into $\mathcal{K}^{m'}_{a',\infty}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ if and only if*

$$m - d \ge m' \quad \text{and} \quad a - d \ge a'.$$

Remark 23. Note that the different condition in Proposition 22 (i) is due to the fact that the corresponding Sobolev embedding (3.3) is not valid in the limiting case, cf. [2, Theorem 4.12].

3.2. Continuous embeddings for Kondratiev spaces on domains of polyhedral type

Theorem 24. *Let $1 \le p \le q < \infty$, $m \in \mathbb{N}$, and $a \in \mathbb{R}$. Let the pair (D, M) be of polyhedral type. Then, $\mathcal{K}^m_{a,p}(D, M)$ is embedded into $\mathcal{K}^{m'}_{a',q}(D, M)$ if and only if*

$$\begin{aligned} m - \frac{d}{p} &\ge m' - \frac{d}{q}, \\ a - \frac{d}{p} &\ge a' - \frac{d}{q}. \end{aligned}$$

Proof. Step 1. Let (D, M) be as in Cases I–III. The linear and continuous extension operator $\mathfrak{E}$ in Proposition 13 allows to reduce sufficiency to Proposition 21. Concerning necessity, we mention that our argument in Proposition 21 relied on a function $u \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u \subset \{x \in \mathbb{R}^d : 0 < |x'| < 1\}$. For notational convenience, we consider Case I with $D = K$ and $M = \{0\}$. Let $u \in C_0^\infty(K)$. Again, all functions $u(\lambda\,\cdot)$, $\lambda > 1$, belong to $C_0^\infty(K)$. Now, similar to the proof of Theorem 20, we can reduce all to Proposition 21.

Step 2. Let D be a polyhedral cone as in Case IV. It is enough to combine Lemma 11 with Step 1.

Step 3. Let D be a domain of polyhedral type with singularity set S. Then, we make use of Lemma 10 to reduce the problem to an application of Steps 1 and 2. ■

Arguing as in case $q < \infty$, we can derive also the following for $q = \infty$.

Theorem 25. *Let $a \in \mathbb{R}$ and $m \in \mathbb{N}$. Let the pair (D, M) be of polyhedral type.*

(i) *Let $1 < p < \infty$. Then, $\mathcal{K}^m_{a,p}(D, M)$ is embedded into $\mathcal{K}^{m'}_{a',\infty}(D, M)$ if and only if*

$$m - \frac{d}{p} > m' \quad \textit{and} \quad a - \frac{d}{p} \geq a'.$$

(ii) *Let $p = 1$. Then, $\mathcal{K}^m_{a,1}(D, M)$ is embedded into $\mathcal{K}^{m'}_{a',\infty}(D, M)$ if and only if*

$$m - d \geq m' \quad \textit{and} \quad a - d \geq a'.$$

Remark 26. Embeddings of Kondratiev spaces have been proved also in Maz'ya and Rossmann [30]. We refer to Lemma 1.2.2 and Lemma 1.2.3 (smooth cones), Lemma 2.1.1 (dihedron), Lemma 3.1.3 and Lemma 3.1.4 (cones with edges) and Lemma 4.1.2 (domains of polyhedral type). Only sufficiency is discussed there. Except for smooth cones the case of equality of $m - d/p$ and $m' - d/q$ is always excluded.

4. Compact embeddings

Having dealt with continuous embeddings within the scale of Kondratiev spaces so far, we now investigate when these embeddings are compact. Roughly speaking, it turns out that whenever we deal with strict inequalities in Theorems 24 and 25 we obtain compact embeddings. Recall that in a pair (D, M) of polyhedral type D is a bounded domain.

Theorem 27. *Let $1 \leq p \leq q \leq \infty$, $m \in \mathbb{N}$, and $a \in \mathbb{R}$. Let (D, M) be either (K, M) (Case I) or (P, Γ) (Case II) or (I, M_ℓ) (Case III) or (Q, M) (Case IV) or a domain of polyhedral type with S being the singularity set of D. Then, $\mathcal{K}^m_{a,p}(D, M)$ is compactly embedded into $\mathcal{K}^{m'}_{a',q}(D, M)$ if and only if*

$$m - \frac{d}{p} > m' - \frac{d}{q} \quad \textit{and} \quad a - \frac{d}{p} > a' - \frac{d}{q}.$$

Proof. Step 1. Sufficiency. Here, we will follow Maz'ya, Rossmann [30, Lemma 4.1.4].

We fix $\varepsilon > 0$. Moreover, for $\delta > 0$, we put

$$D_\delta := \{x \in D : \rho(x) < \delta\}.$$

If δ is small enough, then, as D itself, $D \setminus \bar{D}_\delta$ has the cone property. This implies the compactness of the embedding $W_p^m(D \setminus \bar{D}_\delta) \hookrightarrow\hookrightarrow W_q^{m'}(D \setminus \bar{D}_\delta)$ and therefore (with a slight abuse of notation) $\mathcal{K}^m_{a,p}(D \setminus \bar{D}_\delta, M) \hookrightarrow\hookrightarrow \mathcal{K}^{m'}_{a',q}(D \setminus \bar{D}_\delta, M)$.

Let U denote the unit ball in $\mathcal{K}^m_{a,p}(D, M)$. Then, this compact embedding, together with

$$\sup_{u\in U} \|u \mid \mathcal{K}^{m'}_{a',q}(D \setminus \bar{D}_\delta, M)\| \le \sup_{u\in U} \|u \mid \mathcal{K}^{m'}_{a',q}(D, M)\| = C_1 < \infty,$$

which in turn is a consequence of the continuity of the embedding $\mathcal{K}^m_{a,p}(D, M) \hookrightarrow \mathcal{K}^{m'}_{a',q}(D, M)$ (see Theorems 24, 25), implies the existence of a finite ε-net $u_1, \ldots, u_N \in U$ such that for all $u \in U$, we have

$$\min_{i=1,\ldots,N} \|u - u_i \mid \mathcal{K}^{m'}_{a',q}(D \setminus \bar{D}_\delta, M)\| < \varepsilon.$$

Next, we define $\sigma := a - \frac{d}{p} + \frac{d}{q} - a'$. By assumption $\sigma > 0$. If $u \in U$, applying Theorems 24 and 25, we conclude that

$$\begin{aligned}
\|u|\mathcal{K}^{m'}_{a',q}(D_\delta, M)\| &= \Big(\sum_{|\alpha|\le m'} \int_{D_\delta} |\rho(x)^{|\alpha|-(a-\frac{d}{p}+\frac{d}{q}-\sigma)} \partial^\alpha u(x)|^q\, dx\Big)^{1/q} \\
&\le \delta^\sigma \Big(\sum_{|\alpha|\le m'} \int_{D_\delta} |\rho(x)^{|\alpha|-(a-\frac{d}{p}+\frac{d}{q})} \partial^\alpha u(x)|^q\, dx\Big)^{1/q} \\
&\le \delta^\sigma \|u \mid \mathcal{K}^{m'}_{a-\frac{d}{p}+\frac{d}{q},q}(D, M)\| \\
&\le \delta^\sigma C_1 \|u \mid \mathcal{K}^m_{a,p}(D, M)\| \\
&\le \delta^\sigma C_1.
\end{aligned}$$

Choosing δ so small such that $\delta^\sigma C_1 < \varepsilon$, we get

$$\min_{i=1,\ldots,N} \|u - u_i \mid \mathcal{K}^{m'}_{a',q}(D, M)\| < 3\,\varepsilon.$$

Hence, the embedding is compact.

Step 2. We deal with necessity.

Substep 2.1. The necessity of $m - \frac{d}{p} > m' - \frac{d}{q}$ follows from the necessity of this condition for the compactness of the embedding $W_p^m(\Omega) \hookrightarrow\hookrightarrow W_q^{m'}(\Omega)$, where Ω is a domain in $\mathbb{R}^d$ satisfying a cone condition. It is enough to choose Ω as a ball contained in D such that $\operatorname{dist}(\Omega, M) \in [A, B]$ for $0 < A < B < \infty$.

Substep 2.2. Necessity of $a - \frac{d}{p} > a' - \frac{d}{q}$. Let $u \in C_0^\infty(\mathbb{R}^d)$ be a nontrivial function such that $\operatorname{supp} u \subset B_1(0)$. Next, we select a sequence $(x^j)_j \subset D$ such that

$$B_{2^{-(j+4)}}(x^j) \subset \{x \in D : 2^{-j} < \rho(x) < 2^{-j+1}\}, \quad j \geq j_0(D).$$

For D being a domain of polyhedral type it is clear that such a sequence exists if $j_0(D)$ is chosen sufficiently large. We put

$$u_j(x) := u(2^j(x - x^j)), \quad x \in \mathbb{R}^d, \ j \geq j_0(D).$$

It follows that

$$\begin{aligned} \|u_j \mid \mathcal{K}^m_{a,p}(D, M)\|^p &\asymp \sum_{|\alpha| \leq m} 2^{-j(|\alpha|-a)p} \int_D |\partial^\alpha u_j(x)|^p \, dx \\ &\asymp \sum_{|\alpha| \leq m} 2^{-j(|\alpha|-a)p} 2^{j|\alpha|p} \int_{B_1(0)} |\partial^\alpha u(y)|^p \, 2^{-jd} \, dy \\ &\asymp 2^{j(ap-d)} \|u \mid W^m_p(B_1(0))\|. \end{aligned}$$

Hence, we obtain

$$\|2^{-j(a-\frac{d}{p})} u_j \mid \mathcal{K}^m_{a,p}(D, M)\| \asymp 1, \quad j \geq j_0(D).$$

Let $a'' := a - \frac{d}{p} + \frac{d}{q}$. Then,

$$\|2^{-j(a''-\frac{d}{q})} u_j \mid \mathcal{K}^m_{a'',q}(D, M)\| = \|2^{-j(a-\frac{d}{p})} u_j \mid \mathcal{K}^m_{a'',q}(D, M)\| \geq c > 0, \quad j \geq j_0(D).$$

Observe that

$$\operatorname{supp} u_j \cap \operatorname{supp} u_{j+2} = \emptyset, \quad j \geq j_0(D).$$

Consequently, $(2^{-2j(a-\frac{d}{p})} u_{2j})_j$ is a bounded sequence in $\mathcal{K}^m_{a,p}(D, M)$ which does not have a convergent subsequence in $\mathcal{K}^m_{a'',q}(D, M)$. ∎

Remark 28. For Sobolev spaces this result, usually called Rellich–Kondrachov theorem, has been known for a long time, we refer to Adams [1, Theorem 6.2]. The sufficiency part of Theorem 27 is essentially contained in Maz'ya, Rossmann [30, Lemma 4.1.4].

5. Pointwise multiplication in Kondratiev spaces

5.1. Pointwise multiplication in the model case

First, we deal with our model case $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$. As before, the main idea consists in tracing everything back to the standard Sobolev case.

5.1.1. Algebras with respect to pointwise multiplication. Recall that $W_p^m(\mathbb{R}^d)$, $m \in \mathbb{N}$, $1 \leq p < \infty$, is an algebra with respect to pointwise multiplication if and only if either $1 < p < \infty$ and $m > d/p$ or $p = 1$ and $m \geq d$, cf. [1, Theorem 5.23] and also [31, Section 6.1] for the limiting case. In particular,

$$\|u \cdot v \mid W_p^m(\mathbb{R}^d)\| \leq c\|u \mid W_p^m(\mathbb{R}^d)\|\|v \mid W_p^m(\mathbb{R}^d)\| \tag{5.1}$$

for all $u, v \in W_p^m(\mathbb{R}^d)$ and some $c > 0$. Observe, that these conditions are equivalent with the L_∞-embedding of the Sobolev space.

As a first step, we now consider the following more general estimate of a product. The strategy of the proof will be essential for the following results.

Theorem 29. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$, $0 \leq \ell < d$, and $a \in \mathbb{R}$. In case that $W_p^m(\mathbb{R}^d)$ is an algebra with respect to pointwise multiplication there exists a constant c such that*

$$\|u \cdot v \mid \mathcal{K}^m_{2a-\frac{d}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \leq c\|u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|\|v \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|$$

holds for all $u, v \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

Proof. Step 1. Some preliminary estimates. Recall the definition of the sets D_j in (3.2) and the weight function ρ from (2.22). Fix $j \geq 1$. We start with

$$\begin{aligned}\int_{D_j} |\partial^\beta w(x)|^p\, dx &\lesssim 2^{-dj} \int_{D_1} \left|\partial^\beta w(2^{-j+1}y)\right|^p dy \\ &\lesssim 2^{j|\beta|p}\, 2^{-dj} \int_{D_1} \left|\partial^\beta \big(w(2^{-j+1}\,\cdot)\big)(y)\right|^p dy,\end{aligned}$$

where we used the transformation of coordinates $y := 2^{j-1}x$. Summation over β then implies

$$\sum_{|\beta|\leq m} 2^{-j|\beta|p} \int_{D_j} |\partial^\beta w(x)|^p\, dx \lesssim 2^{-dj}\|w(2^{-j+1}\,\cdot) \mid W_p^m(D_1)\|^p. \tag{5.2}$$

We further note that

$$\begin{aligned}\|w(2^{-j+1}\,\cdot) \mid W_p^m(D_1)\|^p &= \sum_{|\beta|\leq m} 2^{-(j-1)|\beta|p} \int_{D_1} |\partial^\beta w(2^{-j+1}y)|^p\, dy \\ &= \sum_{|\beta|\leq m} 2^{-(j-1)|\beta|p} \int_{D_j} |\partial^\beta w(x)|^p\, 2^{d(j-1)}\, dx.\end{aligned} \tag{5.3}$$

Step 2. Let $u, v \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ such that

$$\operatorname{supp} u, \operatorname{supp} v \subset \big\{x = (x', x'') \in \mathbb{R}^d : |x'| \leq 3/4\big\}.$$

Since $W_p^m(\mathbb{R}^d)$ is an algebra, (5.2) applied to $w = u \cdot v$ leads to

$$\begin{aligned}
&\|u \cdot v \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p \\
&\quad\lesssim \sum_{j=1}^{\infty} \sum_{|\beta|\le m} 2^{-j(|\beta|-a)p} \int_{D_j} |\partial^\beta (u \cdot v)(x)|^p \, dx \\
&\quad\lesssim \sum_{j=1}^{\infty} 2^{jap}\, 2^{-dj} \, \|u(2^{-j+1}\cdot) v(2^{-j+1}\cdot) \mid W_p^m(D_1)\|^p \\
&\quad\lesssim \sum_{j=1}^{\infty} 2^{jap}\, 2^{-dj} \, \|u(2^{-j+1}\cdot) \mid W_p^m(D_1)\|^p \|v(2^{-j+1}\cdot) \mid W_p^m(D_1)\|^p . \qquad (5.4)
\end{aligned}$$

In view of (5.3), we finally find

$$\begin{aligned}
\|u \cdot v \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p &\lesssim \sum_{j=1}^{\infty} 2^{jap}\, 2^{-dj} \Big(\sum_{|\beta|\le m} 2^{-j|\beta|p}\, 2^{dj} \int_{D_j} |\partial^\beta u(x)|^p dx \Big) \\
&\qquad \times \Big(\sum_{|\beta|\le m} 2^{-j|\beta|p}\, 2^{dj} \int_{D_j} |\partial^\beta v(x)|^p \, dx \Big) \\
&\lesssim \sum_{j=1}^{\infty} \Big(\sum_{|\beta|\le m} 2^{-j|\beta|p}\, 2^{j(\frac{a}{2}+\frac{d}{2p})p} \int_{D_j} |\, \partial^\beta u(x)|^p \, dx \Big) \\
&\qquad \times \sup_{j\ge 1} \Big(\sum_{|\beta|\le m} 2^{-j|\beta|p}\, 2^{j(\frac{a}{2}+\frac{d}{2p})p} \int_{D_j} |\, \partial^\beta v(x)|^p \, dx \Big) \\
&\lesssim \|u \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p \|v \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p .
\end{aligned}$$

Step 3. Let $u, v \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ such that

$$\operatorname{supp} u, \operatorname{supp} v \subset \big\{ x = (x', x'') \in \mathbb{R}^d : |x'| \ge 1/4 \big\}.$$

In this situation, the weight does not play any role and we may apply the result for Sobolev spaces directly.

Step 4. Let $u, v \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$. There exists a smooth function $\eta \in C^m(\mathbb{R}^d)$ with the following properties:

$$\eta(x) = 1 \quad \text{if } |x'| \le 1/2$$

and

$$\operatorname{supp} \eta \subset \big\{ x \in \mathbb{R}^d : |x'| \le 3/4 \big\}.$$

Let $\tau \in C^m(\mathbb{R}^d)$ be a function such that $\tau = 1$ on $\operatorname{supp}(1-\eta^2)$ and $\operatorname{supp} \tau \subset \{x \in \mathbb{R}^d : |x'| \ge 1/4\}$. Making use of the basic properties of the Kondratiev spaces in Subsection 2.1

and the results of Steps 2 and 3, we obtain

$$\begin{aligned}
&\big\|u\cdot v \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \\
&= \big\|u\cdot v\cdot \eta^2 + u\cdot v\cdot(1-\eta^2) \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \\
&\lesssim \big\|(u\cdot\eta)\cdot(v\cdot\eta) \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p + \big\|(u\cdot(1-\eta^2))\cdot(v\cdot\tau) \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \\
&\lesssim \big\|u\cdot\eta \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \big\|v\cdot\eta \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \\
&\quad + \big\|u\cdot(1-\eta^2) \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \big\|v\cdot\tau \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \\
&\lesssim \big\|u \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p \big\|v \mid \mathcal{K}^m_{\frac{a}{2}+\frac{d}{2p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\big\|^p.
\end{aligned}$$

The proof is complete. ∎

Corollary 30. *Let* $1 \le p < \infty$, $m \in \mathbb{N}$, $a \in \mathbb{R}$, *and* $0 \le \ell < d$.

(i) *Let* $a \ge \frac{d}{p}$ *and either* $1 < p < \infty$ *and* $m > d/p$ *or* $p = 1$ *and* $m \ge d$. *Then, the Kondratiev space* $\mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)$ *is an algebra with respect to pointwise multiplication, i.e., there exists a constant* c *such that*

$$\|u\cdot v \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\| \le c\|u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\|\|v \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)\|$$

holds for all $u, v \in \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*)$.

(ii) *Let* $\ell = 0$. *Then, the Kondratiev space* $\mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^0_*)$ *is an algebra with respect to pointwise multiplication if and only if* $a \ge \frac{d}{p}$ *and either* $1 < p < \infty$ *and* $m > d/p$ *or* $p = 1$ *and* $m \ge d$.

Proof. Step 1. Sufficiency. As mentioned in Subsection 2.1, the spaces $\mathcal{K}^m_{a,p}(\Omega, M)$ are monotone in a. Since

$$\mathcal{K}^m_{2a-\frac{d}{p},p}(\mathbb{R}^d,\mathbb{R}^\ell_*) \hookrightarrow \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^\ell_*) \quad \text{if } a \ge \frac{d}{p},$$

the claim follows from Theorem 29.

Step 2. Necessity in case $\ell = 0$. The necessity of the conditions $1 < p < \infty$ and $m > d/p$ or $p = 1$ and $m \ge d$ can be reduced to the necessity in case of Sobolev spaces by using an obvious cut-off argument. It remains to prove the necessity of $a \ge d/p$. Therefore, we construct a counterexample in case $a < d/p$. Employing Lemma 48 in the appendix, we conclude that

$$\tilde{\varrho}^b \cdot \psi \in \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^0_*) \quad \text{if and only if } a - b < \frac{d}{p}$$

as well as

$$(\tilde{\varrho}^b \cdot \psi)^2 \in \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^0_*) \quad \text{if and only if } a - 2b < \frac{d}{p}.$$

We choose $b < 0$ such that

$$a - \frac{d}{p} < b < \frac{a - d/p}{2}.$$

Then, $\tilde{\varrho}^b \cdot \psi \in \mathcal{K}^m_{a,p}(\mathbb{R}^d,\mathbb{R}^0_*)$ but $(\tilde{\varrho}^b \cdot \psi)^2$ does not belong to it. ∎

As the proof of Theorem 29 shows, we also have the following slightly more general version.

Corollary 31. *Let* $1 \le p < \infty$, $m, m_1, m_2 \in \mathbb{N}_0$, $a_1, a_2 \in \mathbb{R}$, *and* $0 \le \ell < d$. *If either* $1 < p < \infty$ *and* $\min(m_1, m_2) \ge m > d/p$ *or* $p = 1$ *and* $\min(m_1, m_2) \ge m \ge d$, *then there exists a constant* c *s.t.*

$$\|u \cdot v \mid \mathcal{K}^{m}_{a_1+a_2-\frac{d}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \le c\|u \mid \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \|v \mid \mathcal{K}^{m_2}_{a_2,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \quad (5.5)$$

holds for all $u \in \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ *and* $v \in \mathcal{K}^{m_2}_{a_2,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

5.1.2. Multiplication with unbounded functions. There are two known possibilities to extend (5.1) to unbounded functions. The first one is as follows. Let $1 < p < \infty$, $m_0 \in \mathbb{N}$, $m_1 \in \mathbb{N}_0$, and

$$\frac{d}{2p} \le m_0 < \frac{d}{p}. \quad (5.6)$$

Then, there exists a constant $c > 0$ such that

$$\|u \cdot v \mid W^{m_1}_p(\mathbb{R}^d)\| \le c\|u \mid W^{m_0}_p(\mathbb{R}^d)\| \|v \mid W^{m_0}_p(\mathbb{R}^d)\| \quad (5.7)$$

holds for all $u, v \in W^{m_0}_p(\mathbb{R}^d)$, where

$$m_1 \le 2m_0 - \frac{d}{p}. \quad (5.8)$$

Observe that these restrictions are natural in such a context. To see this, one may use the following family of test functions:

$$f_\alpha(x) := |x - x^0|^{-\alpha}\, \psi(x - x^0), \quad x \in \mathbb{R}^d,\ \alpha > 0.$$

Here, x^0 is an arbitrary point in $\mathbb{R}^d$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi(0) > 0$. Elementary calculations yield $f_\alpha \in W^m_p(\mathbb{R}^d)$ if and only if $\alpha < \frac{d}{p} - m$, cf. [35, Lemma 2.3.1/1]. First, we comment on the lower bound in (5.6). If $m_0 = d/(2p) - \varepsilon$ for some $\varepsilon > 0$, then we may choose $\alpha = d/(2p)$ getting $f_\alpha \in W^{m_0}_p(\mathbb{R}^d)$. But the product $f_\alpha \cdot f_\alpha$ does not belong to $L_p(\mathbb{R}^d)$ since the order of the singularity is d/p.

Secondly, we deal with (5.8). Let m_0 satisfy (5.6) and choose $\alpha = \frac{d}{p} - m_0 - \varepsilon$ with $\varepsilon > 0$ small. For the product $f_\alpha \cdot f_\alpha$, we conclude that it belongs to $W^{m_1}_p(\mathbb{R}^d)$ if

$$m_1 < \frac{d}{p} - 2\alpha = -\frac{d}{p} + 2m_0 + 2\varepsilon.$$

For $\varepsilon \downarrow 0$, the restriction (5.8) follows.

For all this, we refer to [35], in particular, Corollary 4.5.2.

In terms of Kondratiev spaces, the following can be shown.

Theorem 32. *Let* $1 < p < \infty$, $m_0 \in \mathbb{N}$, $m_1 \in \mathbb{N}_0$, $0 \le \ell < d$, *and* $a_0 \in \mathbb{R}$. *Suppose*

$$\frac{d}{2p} \le m_0 < \frac{d}{p}, \quad m_1 \le 2m_0 - \frac{d}{p}, \quad a_1 := 2\Big(a_0 - \frac{d}{2p}\Big).$$

Then, there exists a constant c such that

$$\big\|u \cdot v \mid \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\big\| \le c \big\|u \mid \mathcal{K}^{m_0}_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\big\| \|v \mid \mathcal{K}^{m_0}_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \tag{5.9}$$

holds for all $u, v \in \mathcal{K}^{m_0}_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

Proof. We follow the same strategy as used in the proof of Theorem 29. Therefore, only some comments to the modification are made. We have to modify (5.4). Using (5.7) instead of (5.1), we find

$$\begin{aligned} &\|u \cdot v \mid \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p \\ &\quad \lesssim \sum_{j=1}^{\infty} 2^{j a_1 p} 2^{-dj} \, \|u(2^{-j+1}\,\cdot) \mid W^{m_0}_p(D_1)\|^p \|v(2^{-j+1}\,\cdot) \mid W^{m_0}_p(D_1)\|^p. \end{aligned}$$

Now, we continue as before and obtain

$$\begin{aligned} \|u \cdot v \mid \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^p &\lesssim \sum_{j=1}^{\infty} 2^{j a_1 p}\, 2^{-dj} \Big(\sum_{|\beta| \le m_0} 2^{-j|\beta|p}\, 2^{dj} \int_{D_j} |\, \partial^\beta u(x)|^p dx \Big) \\ &\qquad \times \Big(\sum_{|\beta| \le m_0} 2^{-j|\beta|p}\, 2^{dj} \int_{D_j} |\, \partial^\beta v(x)|^p \, dx \Big) \\ &\lesssim \sum_{j=1}^{\infty} \Big(\sum_{|\beta| \le m_0} 2^{-j|\beta|p}\, 2^{j(\frac{a_1}{2} + \frac{d}{2p})p} \int_{D_j} |\partial^\beta u(x)|^p \, dx \Big) \\ &\qquad \times \sup_{j \ge 0} \Big(\sum_{|\beta| \le m_0} 2^{-j|\beta|p}\, 2^{j(\frac{a_1}{2} + \frac{d}{2p})p} \int_{D_j} |\partial^\beta v(x)|^p \, dx \Big) \\ &\lesssim \big\|u \mid \mathcal{K}^{m_0}_{\frac{a_1}{2} + \frac{d}{2p}, p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\big\|^p \big\|v \mid \mathcal{K}^{m_0}_{\frac{a_1}{2} + \frac{d}{2p}, p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\big\|^p. \end{aligned}$$

This proves the claim. ∎

Remark 33. Observe that we have to accept a loss in regularity in (5.9) since, by (5.6) we always have $m_1 < m_0$.

As before, there exists a generalization in the dependence with respect to the parameter a. It will be based on the following product estimate in case of Sobolev spaces, see [35, Corollary 4.5.2]. Let $1 < p < \infty$, $m_0, m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}_0$,

$$\frac{d}{p} \le m_0 + m_1, \quad m_0, m_1 < \frac{d}{p}. \tag{5.10}$$

Then, there exists a constant $c > 0$ such that

$$\|u \cdot v \mid W_p^{m_2}(\mathbb{R}^d)\| \le c \|u \mid W_p^{m_0}(\mathbb{R}^d)\| \|v \mid W_p^{m_1}(\mathbb{R}^d)\| \tag{5.11}$$

holds for all $u \in W_p^{m_0}(\mathbb{R}^d)$ and $v \in W_p^{m_1}(\mathbb{R}^d)$, where

$$m_2 \le m_0 + m_1 - \frac{d}{p}. \tag{5.12}$$

Proposition 34. *Let $1 < p < \infty$, $m_0, m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}_0$, $0 \le \ell < d$, and $a_0, a_1 \in \mathbb{R}$. Suppose (5.10) and (5.12). We put*

$$a_2 := a_0 + a_1 - \frac{d}{p}.$$

Then, there exists a constant c such that

$$\|u \cdot v \mid \mathcal{K}^{m_2}_{a_2,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \le c \|u \mid \mathcal{K}^{m_0}_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \|v \mid \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|$$

holds for all $u \in \mathcal{K}^{m_0}_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ and all $v \in \mathcal{K}^{m_1}_{a_1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

Proof. The proof follows along the same lines as the proof of Theorem 32. ∎

There is a second possibility. Instead of a shift in the smoothness parameters as above, we will now allow a shift in the integrability. Again, our approach will be based on Theorem 4.5.2 and Corollary 4.5.2 in [35]. Let $1 < p < \infty$, $m \in \mathbb{N}$, and

$$2d\Big(\frac{1}{p} - \frac{1}{2}\Big) < m < \frac{d}{p}. \tag{5.13}$$

We put

$$t := \frac{d}{2\frac{d}{p} - m}. \tag{5.14}$$

Then, there exists a constant $c > 0$ such that

$$\|u \cdot v \mid W_t^m(\mathbb{R}^d)\| \le c \|u | W_p^m(\mathbb{R}^d)\| \, \|v \mid W_p^m(\mathbb{R}^d)\| \tag{5.15}$$

holds for all $u, v \in W_p^m(\mathbb{R}^d)$. It is easy to see that the left-hand side of (5.13) is equivalent to $t > 1$. The optimality of t follows by employing the family f_α. Let $t < t'$ and put

$$\varepsilon := \frac{d}{2}\Big(\frac{1}{t} - \frac{1}{t'}\Big).$$

Then, the function f_α with $\alpha = \frac{d}{p} - m - \varepsilon$ belongs to $W_p^m(\mathbb{R}^d)$ and $f_\alpha \cdot f_\alpha \notin W_{t'}^m(\mathbb{R}^d)$ because of $2\alpha = \frac{d}{t'} - m$.

Theorem 35. *Let $1 < p < \infty$, $m \in \mathbb{N}$, $0 \le \ell < d$, and $a_0 \in \mathbb{R}$. Let t be defined as in (5.14) and put $\tilde{a}_1 := 2a_0 - m$. Suppose (5.13). Then, for any $a_1 < \tilde{a}_1$, there exists a constant c such that*

$$\|u \cdot v \mid \mathcal{K}^m_{a_1,t}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \le c \|u \mid \mathcal{K}^m_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \, \|v \mid \mathcal{K}^m_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|$$

holds for all $u, v \in \mathcal{K}^m_{a_0,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$.

Proof. We follow the same strategy as used in the proof of Theorem 29. Again, it suffices to comment on the modifications needed in comparison with the proof of Theorem 29. We need to modify (5.4) by using (5.15), which gives

$$\begin{aligned}
&\|u \cdot v \mid \mathcal{K}^m_{a_1,t}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^t \\
&\qquad \lesssim \sum_{j=1}^{\infty} 2^{ja_1 t}\, 2^{-dj} \|u(2^{-j+1}\cdot) \mid W^m_p(D_1)\|^t \, \|v(2^{-j+1}\cdot) \mid W^m_p(D_1)\|^t .
\end{aligned}$$

Let $\varepsilon := \frac{t}{2}(\tilde{a}_1 - a_1) > 0$ and $a_2 := \frac{a_1}{2} + d(\frac{1}{p} - \frac{1}{2t})$. Observe $d(\frac{1}{p} - \frac{1}{2t}) = \frac{m}{2}$. It follows

$$\begin{aligned}
&\|u \cdot v \mid \mathcal{K}^m_{a_1,t}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^t \\
&\lesssim \sum_{j=1}^{\infty} 2^{ja_1 t}\, 2^{-dj} \Big(\sum_{|\beta| \le m} 2^{-j|\beta|p}\, 2^{dj} \int_{D_j} |\partial^\beta u(x)|^p dx \Big)^{t/p} \\
&\quad \times \Big(\sum_{|\beta| \le m} 2^{-j|\beta|p}\, 2^{dj} \int_{D_j} |\partial^\beta v(x)|^p \, dx \Big)^{t/p} \\
&\lesssim \sum_{j=1}^{\infty} 2^{-j\varepsilon} \sup_{j \ge 0} 2^{j\varepsilon} \Big(\sum_{|\beta| \le m} 2^{-j|\beta|p}\, 2^{j(\frac{a_1}{2} + d(\frac{1}{p} - \frac{1}{2t}))p} \int_{D_j} |\, \partial^\beta u(x)|^p \, dx \Big)^{t/p} \\
&\quad \times \sup_{j \ge 0} \Big(\sum_{|\beta| \le m} 2^{-j|\beta|p}\, 2^{j(\frac{a_1}{2} + d(\frac{1}{p} - \frac{1}{2t}))p} \int_{D_j} |\partial^\beta v(x)|^p \, dx \Big)^{t/p} \\
&\lesssim \|u \mid \mathcal{K}^m_{a_2 + \varepsilon/t, p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^t \, \|v \mid \mathcal{K}^m_{a_2,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^t .
\end{aligned}$$

Because of $a_0 = a_2 + \varepsilon/t$ and the monotonicity of Kondratiev spaces with respect to a, i.e., $\mathcal{K}^m_{a_2 + \frac{\varepsilon}{t},p}(\Omega, M) \hookrightarrow \mathcal{K}^m_{a_2,p}(\Omega, M)$, the claim follows. ∎

Remark 36. We compare Theorems 32 and 35. Observe that

$$\mathcal{K}^m_{2a_0 - m, t}(\mathbb{R}^d, \mathbb{R}^\ell_*) \hookrightarrow \mathcal{K}^{m_1}_{2(a_0 - d/(2p)),p}(\mathbb{R}^d, \mathbb{R}^\ell_*),$$

see Proposition 21. However, again by Proposition 21, it follows

$$\mathcal{K}^m_{2a_0 - m - \varepsilon, t}(\mathbb{R}^d, \mathbb{R}^\ell_*) \not\subset \mathcal{K}^{m_1}_{2(a_0 - d/(2p)),p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$$

for any $\varepsilon > 0$. In addition, the assumptions in Theorem 32 and Theorem 35 are different.

As already mentioned in the introduction, we are interested in some semilinear elliptic PDEs with nonlinearity given by u^n for some $n > 1$. For later use, we now collect the consequences of our previous results for the mapping $u \mapsto u^n$.

Corollary 37. *Let $m_0, m \in \mathbb{N}$, $m_1 \in \mathbb{N}_0$, $a \in \mathbb{R}$, and $0 \leq \ell < d$.*

(i) *Let either $1 < p < \infty$ and $m > d/p$ or $p = 1$ and $m \geq d$. Then, we have for all natural numbers $n > 1$,*

$$u^n \in \mathcal{K}^m_{na-\frac{d(n-1)}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*) \quad \textit{for all } u \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*),$$

together with the estimate

$$\|u^n \mid \mathcal{K}^m_{na-\frac{d(n-1)}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \leq c^{n-1} \|u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^n,$$

where c is the constant in (5.5).

(ii) *Let $1 < p < \infty$ and $d/(2p) \leq m_0 < d/p$. Let $m_1 \leq 2m_0 - d/p$. Then, we have*

$$u^2 \in \mathcal{K}^{m_1}_{2a-\frac{d}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*) \quad \textit{for all } u \in \mathcal{K}^{m_0}_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*),$$

together with the estimate

$$\|u^2 \mid \mathcal{K}^{m_1}_{2a-\frac{d}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \leq c \|u \mid \mathcal{K}^{m_0}_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^2,$$

where c is the constant in (5.9).

(iii) *Let $1 < p < \infty$, t as in (5.14) and*

$$2d\left(\frac{1}{p} - \frac{1}{2}\right) < m < \frac{d}{p}.$$

Then, for any $a_1 < 2a - m$, there exists a constant c such that

$$u^2 \in \mathcal{K}^m_{a_1,t}(\mathbb{R}^d, \mathbb{R}^\ell_*) \quad \textit{for all } u \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*),$$

together with the estimate

$$\|u^2 \mid \mathcal{K}^m_{a_1,t}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \leq c \|u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^2.$$

Proof. The result in (i) follows by induction upon applying Corollary 31 to u and u^{n-1}. Parts (ii) and (iii) are just special cases of Theorems 32 and 35, respectively. ∎

Remark 38. Applying Proposition 34, one can use the same induction argument as for the proof of part (i). However, the results obtained in this way seem to make sense only in the very special situation that d/p is a natural number. The reason for that can be found in the restriction (5.12). If d/p is not a natural number, one is losing too much regularity through the induction process. For $d/p \in \{1, \ldots, d-1\}$ and $(n-1)\frac{d}{p} \leq nm$ one obtains

$$u^n \in \mathcal{K}^{nm-(n-1)\frac{d}{p}}_{na-(n-1)\frac{d}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*) \quad \text{for all } u \in \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*),$$

together with the estimate

$$\left\| u^n \middle| \mathcal{K}^{nm-(n-1)\frac{d}{p}}_{na-(n-1)\frac{d}{p},p}(\mathbb{R}^d, \mathbb{R}^\ell_*) \right\| \leq c^{n-1} \|u \mid \mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\|^n.$$

5.1.3. Moser-type inequalities. Moser [33] was the first to see that one can improve the estimates of products in case one knows that both factors are bounded. In fact, the following is true: let $1 < p < \infty$ and $m \in \mathbb{N}$. Then, $W_p^m(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ is an algebra with respect to pointwise multiplication and there exists a constant c such that

$$\|u \cdot v \mid W_p^m(\mathbb{R}^d)\| \le c\big(\|u \mid W_p^m(\mathbb{R}^d)\| \, \|v \mid L_\infty(\mathbb{R}^d)\| \\ + \|v \mid W_p^m(\mathbb{R}^d)\| \, \|u \mid L_\infty(\mathbb{R}^d)\|\big)$$

holds for all $u, v \in W_p^m(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, see [35, Section 4.6.4].

Inserting this modification into Step 2 of the proof of Theorem 29, we obtain the following.

Theorem 39. *Let $1 < p < \infty$, $m \in \mathbb{N}$, $0 \le \ell < d$, and $a \in \mathbb{R}$. Then, there exists a constant c such that*

$$\|u \cdot v \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \le c\big(\|u \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \, \|v \mid L_\infty(\mathbb{R}^d)\| \\ + \|v \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \, \|u \mid L_\infty(\mathbb{R}^d)\|\big)$$

holds for all $u, v \in \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell) \cap L_\infty(\mathbb{R}^d)$.

5.1.4. Products of Kondratiev spaces with different p. In the sequel, we are not interested in the most general situation, only in a few special cases. It is quite easy to prove the following.

Proposition 40. *Let $1 \le p < \infty$, $m \in \mathbb{N}$, $0 \le \ell < d$, and $a \in \mathbb{R}$. Then, there exists a constant c such that*

$$\|u \cdot v \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \le c\|v \mid \mathcal{K}_{0,\infty}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \, \|u \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\|$$

holds for all $u \in \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)$ and $v \in \mathcal{K}_{0,\infty}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)$.

Proof. The result follows by a modification of the arguments in the proof of Theorem 29; details can be found in [8]. ∎

In our intended application in [9], we can also allow for a shift in the regularity parameter m in the pointwise multiplier assertion, i.e., for the product $u \cdot v$, we require less weak derivatives than for u and/or v. This aspect leads to the following modification.

Corollary 41. *Let $m, n \in \mathbb{N}$ and $0 \le \ell < d$.*

(i) *Let $\max(1, d/n) < p < \infty$ and $a \ge \frac{d}{p} - n$. Then, there exists a constant c such that*

$$\|u \cdot v \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \le c\|v \mid \mathcal{K}_{a+n,p}^{m+n}(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \, \|u \mid \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)\| \qquad (5.16)$$

holds for all $u \in \mathcal{K}_{a,p}^m(\mathbb{R}^d, \mathbb{R}_*^\ell)$ and $v \in \mathcal{K}_{a+n,p}^{m+n}(\mathbb{R}^d, \mathbb{R}_*^\ell)$.

(ii) *Let $p = 1$, $d \le n$, and $a \ge d - n$. Then, (5.16) is true as well.*

Proof. In view of Proposition 40 it is enough to apply $\mathcal{K}^{m+n}_{a+n,p}(\mathbb{R}^d, \mathbb{R}^\ell_*) \hookrightarrow \mathcal{K}^m_{0,\infty}(\mathbb{R}^d, \mathbb{R}^\ell_*)$, which by Corollary 25 is the case if

$$m + n - \frac{d}{p} > m;$$

i.e.,

$$n > \frac{d}{p}, \quad a + n - \frac{d}{p} \geq 0.$$

By our assumptions on p and a the proof is complete. ■

Remark 42. (i) Observe that in Corollary 41, we do not need that m is large. We only use that a is sufficiently large.

(ii) A simple reformulation yields in case $n = 2$:

$$\|u \cdot v \mid \mathcal{K}^{m-1}_{a-1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \leq c \|v \mid \mathcal{K}^{m+1}_{a+1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \, \|u \mid \mathcal{K}^{m-1}_{a-1,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)\| \quad (5.17)$$

if $m \in \mathbb{N}$, $\max(1, d/2) < p < \infty$, and $a \geq \frac{d}{p} - 1$.

5.2. Pointwise multiplication in Kondratiev spaces on domains of polyhedral type

In order to shift the results on multiplication obtained for $\mathcal{K}^m_{a,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ to $\mathcal{K}^m_{a,p}(D, M)$ with (D, M) being a pair of polyhedral type, we proceed as in case of the continuous embeddings. As a first step, we employ Proposition 13 and conclude that all sufficient conditions in Subsection 5.1 remain sufficient conditions for Kondratiev spaces on pairs (D, M) as in Cases I–III. Next, we use Lemma 11 to show the same for Kondratiev spaces on polyhedral cones. Then, we are finally ready to prove sufficiency also for Kondratiev spaces on domains of polyhedral type by making use of Lemma 10.

Theorem 43. *Theorem 29, Corollary 30* (i)*, Corollary 31, Theorem 32, Proposition 34, Theorem 35, Corollary 37, Theorem 39, Proposition 40, and Corollary 41 carry over to Kondratiev spaces with respect to pairs (D, M) of polyhedral type.*

Also, Corollary 30 (ii) has a counterpart, but restricted to spaces on smooth cones.

Corollary 44. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$, and $a \in \mathbb{R}$. Furthermore, let K be the smooth cone from Case I.*

The Kondratiev space $\mathcal{K}^m_{a,p}(K, \{0\})$ is an algebra with respect to pointwise multiplication if and only if $a \geq \frac{d}{p}$ with either $1 < p < \infty$ and $m > d/p$ or $p = 1$ and $m \geq d$.

6. Appendix: concrete examples

The following three lemmas are based on straightforward elementary calculations. Therefore, all details are omitted.

Lemma 45. *Let $1 \le p \le \infty$, $a \in \mathbb{R}$, and $m \in \mathbb{N}_0$.*

(i) *Let K be a smooth cone as in Case I. Then, $1 \in \mathcal{K}^m_{a,p}(K, \{0\})$ if and only if either $a < d/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

(ii) *Let P be a specific nonsmooth cone as in Case II. Then, $1 \in \mathcal{K}^m_{a,p}(P, \Gamma)$ if and only if either $a < (d-1)/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

(iii) *Let I be a specific dihedral domain as in Case III. Then, $1 \in \mathcal{K}^m_{a,p}(I, M_\ell)$ if and only if either $a < (d-\ell)/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

(iv) *Let Q be a polyhedral cone as in Case IV. Then, $1 \in \mathcal{K}^m_{a,p}(Q, M)$ if and only if either $a < 2/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

Remark 46. The case of a smooth cone is considered in [5, Lemma 2.5].

Let $\widetilde{\varrho}$ denote the regularized distance function. Then, for any $b \in \mathbb{R}$, the mapping $T_b : u \mapsto \widetilde{\varrho}^b u$ yields an isomorphism of $\mathcal{K}^m_{a,p}(\Omega, M)$ onto $\mathcal{K}^m_{a+b,p}(\Omega, M)$. As an immediate conclusion of Lemma 45 we obtain the following.

Lemma 47. *Let $1 \le p \le \infty$, $a, b \in \mathbb{R}$, $0 \le l < d$, and $m \in \mathbb{N}_0$.*

(i) *Let K be a smooth cone as in Case I. Then, $\widetilde{\varrho}^b \in \mathcal{K}^m_{a+b,p}(K, \{0\})$ if and only if either $a < d/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

(ii) *Let P be a specific nonsmooth cone as in Case II. Then, $\widetilde{\varrho}^b \in \mathcal{K}^m_{a+b,p}(P, \Gamma)$ if and only if either $a < (d-1)/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

(iii) *Let I be a specific dihedral domain as in Case III. Then, $\widetilde{\varrho}^b \in \mathcal{K}^m_{a+b,p}(I, M_\ell)$ if and only if either $a < (d-\ell)/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

(iv) *Let Q be a polyhedral cone as in Case IV. Then, $\widetilde{\varrho}^b \in \mathcal{K}^m_{a+b,p}(Q, M)$ if and only if either $a < 2/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

In our model case $(\mathbb{R}^d, \mathbb{R}^\ell_*)$, we need a modification. Let ψ denote a smooth cut-off function, i.e.,

$$\psi(x) = 1, \quad |x| \le 1$$

and

$$\psi(x) = 0 \quad \text{if } |x| \ge 3/2.$$

Lemma 48. *Let $1 \le p \le \infty$, $a, b \in \mathbb{R}$, $0 \le \ell < d$, $m \in \mathbb{N}_0$. Then, $\widetilde{\varrho}^b \cdot \psi \in \mathcal{K}^m_{a+b,p}(\mathbb{R}^d, \mathbb{R}^\ell_*)$ if and only if either $a < (d-\ell)/p$ for $p < \infty$ or if $a \le 0$ for $p = \infty$.*

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