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Topological model for q-deformed rational numbers and categorification

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Abstract. Let $\mathbf{D}_3$ be a bigraded 3-decorated disk with an arc system $\mathbf{A}$. We associate a bigraded simple closed arc $\widehat{\eta}_{r/s}$ on $\mathbf{D}_3$ to any rational number $r/s \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. We show that the right (respectively, left) q-deformed rational numbers associated to r/s, in the sense of Morier-Genoud–Ovsienko (respectively, Bapat–Becker–Licata) can be naturally calculated by the $\mathfrak{q}$-intersection between $\widehat{\eta}_{r/s}$ and $\mathbf{A}$ (respectively, dual arc system $\mathbf{A}^*$). The Jones polynomials of rational knots can be also given by such intersections. Moreover, the categorification of $\widehat{\eta}_{r/s}$ is given by the spherical object $X_{r/s}$ in the Calabi–Yau-$\mathbb{X}$ category of Ginzburg dga of type A_2. Reducing to the CY-2 case, we recover result of Bapat–Becker–Licata with a slight improvement.

1. Introductions

1.1. Motivations

The notion of (right) q-deformed rational numbers $[r/s]^\sharp$ was originally introduced by Morier-Genoud and Ovsienko in [10] via continued fractions. Moreover, the q-deformations can also be described via the action of $\mathrm{PSL}_{2,\mathfrak{q}}(\mathbb{Z})$ by fractional linear transformations. They also extended the notion of q-deformation to irrational numbers in [11] by the convergency property. Such q-deformations own many good combinatorial properties and are related to a wide variety of areas, such as the Farey triangulation, F-polynomials of cluster algebras, and the Jones polynomial of rational (two-bridge) knots [10].

Motivated by the study of compactification of spaces of stability conditions, Bapat, Becker and Licata [1] developed a twin notion, the left q-deformation $[r/s]^\flat$, which has already been noticed in [11]. It also shares all the good properties of $[r/s]^\sharp$ and can also be described via the action of $\mathrm{PSL}_{2,\mathfrak{q}}(\mathbb{Z})$. Moreover, the Farey graph plays an important role in the definition of q-deformations, where the edges are assigned weights according to some iterative rules [10].

On the other hand, the homotopy classes of simple closed curves on the torus with at most one boundary component can be parameterized by $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. We aim to give a topological realization of q-deformations and their categorification.

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1.2. The topological model and categorification

The topological model. We study the decorated surface $\mathbf{S}_\triangle$ with bigrading introduced by Khovanov and Seidel in [7]. The bigrading of arcs provides bi-indices for their intersections, which we call $\mathfrak{q}$-intersections. We consider the A_2 case, where $\mathbf{S}_\triangle = \mathbf{D}_3$ is a disk with three decorations and the set of simple closed arcs on $\mathbf{D}_3$ can be parameterized by $\overline{\mathbb{Q}}$. We prove the following results, which imply that the right/left q-deformed rationals can be naturally calculated by the $\mathfrak{q}$-intersections between the corresponding arcs (see Theorems 3.18 and 3.21).

Theorem 1.1. *For any rational number $r/s \in \overline{\mathbb{Q}}$, we have*

$$\left[\frac{r}{s}\right]^{\flat}_{\mathfrak{q}} = \frac{\varepsilon \operatorname{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_0)}{\operatorname{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_\infty)}\bigg|_{\mathfrak{q}=q_1^{-1}q_2},$$

corresponding to the left $\mathfrak{q}$-deformation of r/s, where

$$\varepsilon = \begin{cases} q_1^{-1}, & \text{if } r/s \geq 0, \\ -1, & \text{if } r/s < 0. \end{cases}$$

Theorem 1.2. *For any rational number $r/s \in \overline{\mathbb{Q}}$, we have*

$$\left[\frac{r}{s}\right]^{\sharp}_{\mathfrak{q}} = \frac{\varepsilon \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{r/s})}{\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{r/s})}\bigg|_{\mathfrak{q}=q_1^{-1}q_2},$$

corresponding to the right $\mathfrak{q}$-deformation of r/s, where

$$\varepsilon = \begin{cases} 1, & \text{if } r/s \geq 0, \\ -q_1^{-1}, & \text{if } r/s < 0, \end{cases}$$

and the polynomials in the numerator and denominator are polynomials in $\mathbb{Z}[q_1^{-1}q_2]$.

We see that the left and right $\mathfrak{q}$-deformations are "dual" to each other in the sense that they are computed as the the $\mathfrak{q}$-intersections with dual arcs. Two examples of -2 and $3/2$ (see Figure 13) for the left/right $\mathfrak{q}$-deformations are illustrated in Examples 3.19 and 3.22. The topological realization directly implies many combinatorial properties of q-deformations, including positivity and specialization (Corollary 3.23). Surprisingly, the bi-index always collapses into one, which is not obvious from the construction/definition of $\mathfrak{q}$-intersection.

Categorification. For the categorification, we shall consider the Calabi–Yau-$\mathbb{X}$ category $\mathcal{D}_{\mathbb{X}}(\mathbf{S}_\triangle)$ associated to $\mathbf{S}_\triangle$ (cf. [5, 7]), which is the perfect valued derived category of the bigraded Ginzburg algebra constructed from $\mathbf{S}_\triangle$. The $\mathbb{X}$-spherical objects in $\mathcal{D}_{\mathbb{X}}(\mathbf{S}_\triangle)$ correspond to the bigraded simple closed arcs in $\mathbf{S}_\triangle$, and their $\mathfrak{q}$-dimensions of Hom-spaces equal to the $\mathfrak{q}$-intersections between the corresponding arcs [5, 7, 13]. As a result, we can restate the left/right $\mathfrak{q}$-deformations homologically, i.e., via $\mathfrak{q}$-dimension of bigraded Hom of corresponding objects (Corollary 4.6 and Corollary 4.7).

Moreover, one can specialize $\mathbb{X} = N$, and $\mathcal{D}_{\mathbb{X}}(\mathbf{S}_\triangle)$ becomes a Calabi–Yau-N category, for any integer $N \geq 2$. When $N = 3$, $\mathcal{D}_3(\mathbf{S}_\triangle)$ provides an additive categorification of cluster algebras of surface type (see, e.g., [12]). When $\mathbf{S}_\triangle = \mathbf{D}_3$ and $N = 2$, we recover the result of [1] (with a slight improvement).

1.3. Contents

The paper is organized as follows. In Section 2, we recall several equivalent definitions of left and right $\mathfrak{q}$-deformed rationals from [1, 10], via continued fractions, the braid twist action and the $\mathfrak{q}$-weighted Farey graph, respectively. In Section 3, we recall the graded decorated surface in the sense of [5, 7] and prove the main results. In Section 4, we give the categorification, and in Section 5, we discuss reduction and the relation with Jones polynomials.

2. $\mathfrak{q}$-deformed rationals and Farey graphs

In the paper, we fix the following conventions.

- Let $\mathfrak{q}$ be a formal parameter.
- A rational number always belongs to $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. We also denote $\overline{\mathbb{Q}^{\geq 0}} := \mathbb{Q}^{\geq 0} \cup \{\infty\}$. We usually state the results for $\overline{\mathbb{Q}}$ but prove the non-negative case since the negative case holds by applying the $\mathrm{PSL}_2(\mathbb{Z})$-actions.
- We denote a rational number by r/s, including the exceptional cases when $0 = 0/1$ and $\infty = 1/0$. We assume that r/s is irreducible.

2.1. Right and left $\mathfrak{q}$-deformed rationals

We first recall the definitions of right and left $\mathfrak{q}$-deformations of rational numbers via finite continued fractions and formulate their basic properties. For a positive rational r/s, it can be expressed as an expansion of continued fraction as

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}} := [a_1, \ldots, a_{2m}], \tag{2.1}$$

for $a_1 \in \mathbb{N}$ and $a_2, \ldots, a_{2m} \in \mathbb{N} \setminus \{0\}$, which is known as the (*regular*) continued fraction (expansion). For the exceptional cases, we denote $0 = [-1, 1]$ and $\infty = []$.

For a non-negative integer a, the *right* $\mathfrak{q}$*-deformation* is defined as

$$[a]^\sharp_{\mathfrak{q}} := \frac{1 - \mathfrak{q}^a}{1 - \mathfrak{q}} = 1 + \mathfrak{q} + \mathfrak{q}^2 + \cdots + \mathfrak{q}^{a-1},$$

and the corresponding *left* $\mathfrak{q}$*-deformation* is defined as

$$[a]^\flat_{\mathfrak{q}} := \frac{1 - \mathfrak{q}^{a-1} + \mathfrak{q}^a - \mathfrak{q}^{a+1}}{1 - \mathfrak{q}} = 1 + \mathfrak{q} + \cdots + \mathfrak{q}^{a-2} + \mathfrak{q}^a.$$

Definition 2.1 ([1, 10]). Let $r/s \in \mathbb{Q}^+$ be a rational number with continued fraction expansion $[a_1, \ldots, a_{2m}]$.

(1) We define its *right* $\mathfrak{q}$*-deformation* by the following formula:

$$\left[\frac{r}{s}\right]^{\sharp}_{\mathfrak{q}} := [a_1]^{\sharp}_{\mathfrak{q}} + \cfrac{\mathfrak{q}^{a_1}}{[a_2]^{\sharp}_{\mathfrak{q}^{-1}} + \cfrac{\mathfrak{q}^{-a_2}}{[a_3]^{\sharp}_{\mathfrak{q}} + \cfrac{\mathfrak{q}^{a_3}}{[a_4]^{\sharp}_{\mathfrak{q}^{-1}} + \cfrac{\mathfrak{q}^{-a_4}}{\cfrac{\ddots}{[a_{2m-1}]^{\sharp}_{\mathfrak{q}} + \cfrac{\mathfrak{q}^{a_{2m-1}}}{[a_{2m}]^{\sharp}_{\mathfrak{q}^{-1}}}}}}}}.$$

(2) We define its *left* $\mathfrak{q}$*-deformation* by the following formula:

$$\left[\frac{r}{s}\right]^{\flat}_{\mathfrak{q}} := [a_1]^{\sharp}_{\mathfrak{q}} + \cfrac{\mathfrak{q}^{a_1}}{[a_2]^{\sharp}_{\mathfrak{q}^{-1}} + \cfrac{\mathfrak{q}^{-a_2}}{[a_3]^{\sharp}_{\mathfrak{q}} + \cfrac{\mathfrak{q}^{a_3}}{[a_4]^{\sharp}_{\mathfrak{q}^{-1}} + \cfrac{\mathfrak{q}^{-a_4}}{\cfrac{\ddots}{[a_{2m-1}]^{\sharp}_{\mathfrak{q}} + \cfrac{\mathfrak{q}^{a_{2m-1}}}{[a_{2m}]^{\flat}_{\mathfrak{q}^{-1}}}}}}}}.$$

We normalize them as

$$\left[\frac{r}{s}\right]^{\sharp}_{\mathfrak{q}} = \frac{\mathbf{R}^{\sharp}_{\mathfrak{q}}(r/s)}{\mathbf{S}^{\sharp}_{\mathfrak{q}}(r/s)} \quad \text{and} \quad \left[\frac{r}{s}\right]^{\flat}_{\mathfrak{q}} = \frac{\mathbf{R}^{\flat}_{\mathfrak{q}}(r/s)}{\mathbf{S}^{\flat}_{\mathfrak{q}}(r/s)},$$

so that the denominators are polynomials of $\mathfrak{q}$ with lowest non-zero constant term. For 0 and ∞, we set

$$\mathbf{R}^{\sharp}_{\mathfrak{q}}(0) = 0, \quad \mathbf{S}^{\sharp}_{\mathfrak{q}}(0) = 1, \quad \mathbf{R}^{\flat}_{\mathfrak{q}}(0) = \mathfrak{q} - 1, \quad \mathbf{S}^{\flat}_{\mathfrak{q}}(0) = \mathfrak{q};$$
$$\mathbf{R}^{\sharp}_{\mathfrak{q}}(\infty) = 1, \quad \mathbf{S}^{\sharp}_{\mathfrak{q}}(\infty) = 0, \quad \mathbf{R}^{\flat}_{\mathfrak{q}}(\infty) = 1, \quad \mathbf{S}^{\flat}_{\mathfrak{q}}(\infty) = 1 - \mathfrak{q}.$$

Next, we consider the group

$$\mathrm{PSL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

which is generated by

$$t_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t_2 := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It acts on rational numbers $\overline{\mathbb{Q}}$ by linear fractional transformation as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \frac{r}{s} \end{pmatrix} = \frac{ar + bs}{cr + ds}, \tag{2.2}$$

where $r/s \in \overline{\mathbb{Q}}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$. For a rational number $r/s \in \overline{\mathbb{Q}^{\geq 0}}$ with continued fraction expansion as (2.1), it is well known (see Proposition 2.2 in [1]) that

$$\frac{r}{s} = t_1^{a_1} t_2^{-a_2} t_1^{a_3} t_2^{-a_4} \cdots t_1^{a_{2m-1}} t_2^{-a_{2m}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.3}$$

Proposition/Definition 2.2 ([1,10]). *Consider the $\mathfrak{q}$-deformation $\mathrm{PSL}_{2,\mathfrak{q}}(\mathbb{Z})$ of $\mathrm{PSL}_2(\mathbb{Z})$, which is generated by*

$$t_{1,\mathfrak{q}} = \begin{pmatrix} \mathfrak{q} & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t_{2,\mathfrak{q}} = \begin{pmatrix} 1 & 0 \\ -\mathfrak{q} & \mathfrak{q} \end{pmatrix}.$$

For a rational number $r/s \in \overline{\mathbb{Q}^{\geq 0}}$ with expression (2.1), we have

$$\begin{cases} \left[\dfrac{r}{s}\right]_{\mathfrak{q}}^{\sharp} = t_{1,\mathfrak{q}}^{a_1} t_{2,\mathfrak{q}}^{-a_2} t_{1,\mathfrak{q}}^{a_3} t_{2,\mathfrak{q}}^{-a_4} \cdots t_{1,\mathfrak{q}}^{a_{2m-1}} t_{2,\mathfrak{q}}^{-a_{2m}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \left[\dfrac{r}{s}\right]_{\mathfrak{q}}^{\flat} = t_{1,\mathfrak{q}}^{a_1} t_{2,\mathfrak{q}}^{-a_2} t_{1,\mathfrak{q}}^{a_3} t_{2,\mathfrak{q}}^{-a_4} \cdots t_{1,\mathfrak{q}}^{a_{2m-1}} t_{2,\mathfrak{q}}^{-a_{2m}} \begin{pmatrix} 1 \\ 1-\mathfrak{q} \end{pmatrix}. \end{cases}$$

2.2. $\mathfrak{q}$-deformations via Farey graphs

The classical *Farey graph* FG is an infinite graph with set of vertices

$$\mathrm{FG}_0 = \overline{\mathbb{Q}}.$$

There is an edge between p/q and u/v if and only if $pv - uq = \pm 1$ (see Figure 1). If p/q and u/v are connected by an edge, we define their *Farey sum* by

$$\frac{p}{q} \oplus \frac{u}{v} := \frac{p+u}{q+v}.$$

Moreover, FG_0 is parametrized by homotopy classes of simple closed arcs on the torus with at most one boundary component and the edges are those arcs with intersection number one. $\mathrm{PSL}_2(\mathbb{Z})$ acts on FG by (2.2) taking one edge to another. In particular, if $T \in \mathrm{PSL}_2(\mathbb{Z})$ takes the form

$$T_{r/s} = \begin{pmatrix} 1+rs & -r^2 \\ s^2 & 1-rs \end{pmatrix},$$

then it is a rotation which fixes r/s.

Lemma/Definition 2.3 (Section 2.2 of [10]). *Let $r/s \in \mathbb{Q}^+$ be any rational number with continued fraction expansion as* (2.1)*. Then it can be uniquely written as the Farey sum of two rationals $p/q, u/v \in \overline{\mathbb{Q}^{\geq 0}}$:*

$$\frac{r}{s} = \frac{p}{q} \oplus \frac{u}{v},$$

with $uq - pv = 1$ and $p/q < r/s < u/v$. In fact,

$$\frac{p}{q} = \begin{cases} [a_1, a_2, \ldots, a_{2m-2} + 1], & \text{if } a_{2m-1} = 1 \text{ and } m > 1; \\ [a_1, a_2, \ldots, a_{2m-1} - 1, 1], & \text{otherwise}, \end{cases}$$

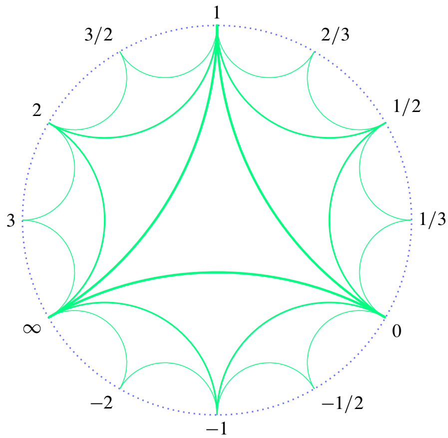

Figure 1. The Farey graph.

and

$$\frac{u}{v} = \begin{cases} [a_1, a_2, \ldots, a_{2m-1}, a_{2m} - 1], & \textit{if } a_{2m} \geq 2; \\ [a_1, a_2, \ldots, a_{2m-2}], & \textit{if } a_{2m} = 1. \end{cases}$$

Moreover, there is an associated integer defined as

$$l = l\left(\frac{r}{s}\right) = \begin{cases} 0, & \textit{if } a_{2m} \geq 2; \\ a_{2m-1}, & \textit{if } a_{2m} = 1. \end{cases}$$

In particular, we have $l(n+1 = (n/1) \oplus (1/0)) = n - 1.$

On the other hand, l can also be defined for an edge in FG connecting p/q and u/v, provided $p/q < u/v$. More precisely, $l(p/q, u/v) := l((p/q) \oplus (u/v))$.

As in [10], we assign a weight to each edge of the Farey graph, that goes along with the right or left $\mathfrak{q}$-deformations associated to vertices. Then the right and left $\mathfrak{q}$-deformations can also be defined via $\mathfrak{q}$-Farey sum.

Let us recall a standard fact. Let $r/s \in \mathbb{Q}^+$ be a rational with the decomposition $r/s = (p/q) \oplus (u/v)$ and $l = l(p/q, u/v)$ as above. For right $\mathfrak{q}$-deformation, we have

$$\mathbf{R}^\sharp_\mathfrak{q}\left(\frac{r}{s}\right) := \mathbf{R}^\sharp_\mathfrak{q}\left(\frac{p}{q}\right) + \mathfrak{q}^{l+1}\,\mathbf{R}^\sharp_\mathfrak{q}\left(\frac{u}{v}\right) \quad \text{and} \quad \mathbf{S}^\sharp_\mathfrak{q}\left(\frac{r}{s}\right) := \mathbf{S}^\sharp_\mathfrak{q}\left(\frac{p}{q}\right) + \mathfrak{q}^{l+1}\mathbf{S}^\sharp_\mathfrak{q}\left(\frac{u}{v}\right).$$

For left $\mathfrak{q}$-deformation, if we set $\bar{\boldsymbol{R}}^\flat_\mathfrak{q}(0) = \boldsymbol{S}^\flat_\mathfrak{q}(0)$, $\bar{\boldsymbol{R}}^\flat_\mathfrak{q}(\infty) = \boldsymbol{S}^\flat_\mathfrak{q}(\infty)$ and define

$$\bar{\mathbf{R}}^\flat_\mathfrak{q}\left(\frac{r}{s}\right) := \bar{\mathbf{R}}^\flat_\mathfrak{q}\left(\frac{p}{q}\right) + (\mathfrak{q}^{-1})^{l+1}\,\bar{\mathbf{R}}^\flat_\mathfrak{q}\left(\frac{u}{v}\right) \quad \text{and} \quad \bar{\mathbf{S}}^\flat_\mathfrak{q}\left(\frac{r}{s}\right) := \bar{\mathbf{S}}^\flat_\mathfrak{q}\left(\frac{p}{q}\right) + (\mathfrak{q}^{-1})^{l+1}\,\bar{\mathbf{S}}^\flat_\mathfrak{q}\left(\frac{u}{v}\right),$$

then we have

$$\left[\frac{r}{s}\right]^\flat_\mathfrak{q} = \frac{\bar{\mathbf{R}}^\flat_\mathfrak{q}(r/s)}{\bar{\mathbf{S}}^\flat_\mathfrak{q}(r/s)}.$$

We label a weight to each edge and a right or left $\mathfrak{q}$-deformation to each vertex in the Farey graph, which are drawn in Figure 2 and Figure 3, respectively. Here the integer $l = l(p/q, u/v)$ is as above.

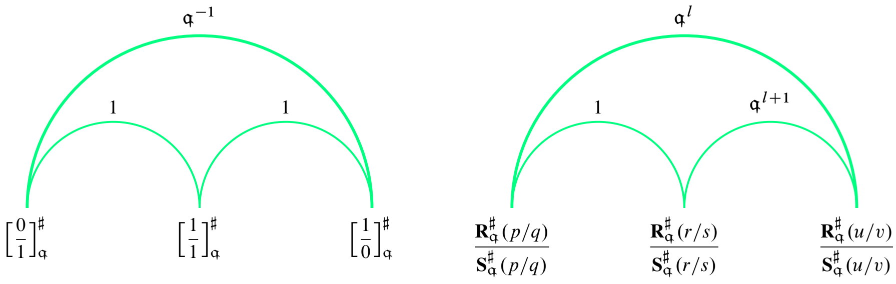

Figure 2. The right $\mathfrak{q}$-deformation via a $\mathfrak{q}$-Farey sum.

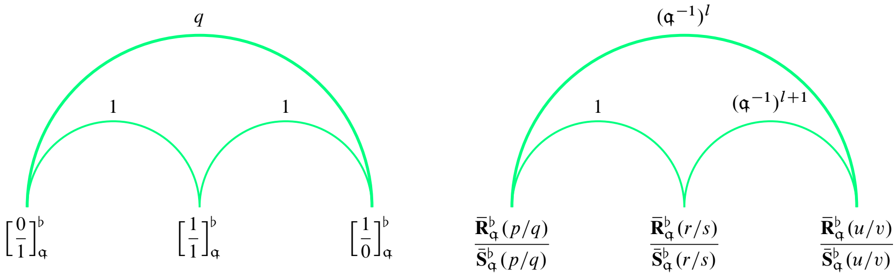

Figure 3. The left $\mathfrak{q}$-deformation via a $\mathfrak{q}$-Farey sum.

Remark 2.4. For $r/s \in \mathbb{Q}^+$ with continued fraction expression $[a_1, \ldots, a_{2m}]$, we have

$$-\frac{r}{s} = [-a_1, \ldots, -a_{2m}], \quad \text{and} \quad -\frac{r}{s} = t_1^{-a_1} t_2^{a_2} t_1^{-a_3} t_2^{a_4} \cdots t_1^{-a_{2m-1}} t_2^{a_{2m}} \binom{1}{0}.$$

The right and left $\mathfrak{q}$-deformations for negative rational numbers are defined as:

$$\begin{cases} \left[-\dfrac{r}{s}\right]^\sharp_\mathfrak{q} & = t_{1,\mathfrak{q}}^{-a_1} t_{2,\mathfrak{q}}^{a_2} t_{1,\mathfrak{q}}^{-a_3} t_{2,\mathfrak{q}}^{a_4} \cdots t_{1,\mathfrak{q}}^{-a_{2m-1}} t_{2,\mathfrak{q}}^{a_{2m}} \left(\dfrac{1}{0}\right), \\ \left[-\dfrac{r}{s}\right]^\flat_\mathfrak{q} & = t_{1,\mathfrak{q}}^{-a_1} t_{2,\mathfrak{q}}^{a_2} t_{1,\mathfrak{q}}^{-a_3} t_{2,\mathfrak{q}}^{a_4} \cdots t_{1,\mathfrak{q}}^{-a_{2m-1}} t_{2,\mathfrak{q}}^{a_{2m}} \left(\dfrac{1}{1-\mathfrak{q}}\right). \end{cases}$$

In fact, we can obtain $\mathfrak{q}$-deformations of negative rationals from positive ones by the following formula:

$$\left[-\frac{r}{s}\right]^*_\mathfrak{q} := -\mathfrak{q}^{-1}\left[\frac{r}{s}\right]^*_{\mathfrak{q}^{-1}},$$

where $* \in \{\sharp, \flat\}$.

3. The topological model

In this section, we introduce decorated (marked) surfaces as the topological model which we will use. We first summarize the setting and results in [5, 7] and then show that the $\mathfrak{q}$-intersections of certain arcs describe the left/right $\mathfrak{q}$-deformations for rational numbers.

3.1. Decorated surfaces

Let $\mathbf{S}$ be an oriented surface with non-empty boundary $\partial\mathbf{S}$ and denote its interior by $\mathbf{S}^\circ = \mathbf{S} \setminus (\partial\mathbf{S})$. We decorate $\mathbf{S}$ with a finite set $\triangle$ of points (*decorations*) in $\mathbf{S}^\circ$, denoted by $\mathbf{S}_\triangle$.

Let $\mathbf{S}^\circ_\triangle = \mathbf{S} \setminus (\partial\mathbf{S} \cup \triangle)$. An *arc* c in $\mathbf{S}_\triangle$ is a curve $c\colon [0,1] \to \mathbf{S}$ such that $c(t) \in \mathbf{S}^\circ_\triangle$ for any $t \in (0,1)$. The *inverse* $\bar{c}$ of an arc c is defined as $\bar{c}(t) = c(1-t)$ for any $t \in [0,1]$.

Definition 3.1. A *closed arc* c is an arc whose endpoints $c(0)$ and $c(1)$ are in $\triangle$. It is *simple* if moreover it satisfies $c(0) \neq c(1)$, without self-intersections in $\mathbf{S}^\circ_\triangle$. We denote by $\mathrm{CA}(\mathbf{S}_\triangle)$ the set of simple closed arcs.

In this paper, we always consider arcs up to taking inverse and homotopy relative to endpoints and exclude the arcs which are isotopic to a point in $\mathbf{S}_\triangle$. Two arcs are in *minimal position* if their intersection is minimal in the homotopy class. For three simple closed arcs, they form a *contractible triangle* if they bound a disk which is contractible.

For $\sigma, \tau \in \mathrm{CA}(\mathbf{S}_\triangle)$, their *intersection number*, which is in $\frac{1}{2}\mathbb{Z}$, is defined as follows:

$$\mathrm{Int}_{\mathbf{S}_\triangle}(\sigma,\tau) := \frac{1}{2}\cdot \mathrm{Int}_\triangle(\sigma,\tau) + \mathrm{Int}_{\mathbf{S}^\circ_\triangle}(\sigma,\tau),$$

where

$$\mathrm{Int}_\Pi(\sigma,\tau) = \min\{|\sigma' \cap \tau' \cap \Pi|, \sigma' \sim \sigma, \tau' \sim \tau\}$$

with $\Pi = \triangle$ or $\mathbf{S}^\circ_\triangle$. For example, as drawn in the middle of Figure 13, $\widehat{\eta}_{3/2}$ and $\widehat{\eta}_0$ intersect twice, where one is at z_* and the other is in the interior of $\mathbf{S}_\triangle$. Then their intersection number is counted as $\mathrm{Int}_{\mathbf{S}_\triangle}(\sigma,\tau) = 1/2 + 1 = 3/2$.

The *mapping class group* $\mathrm{MCG}(\mathbf{S}_\triangle)$ of a decorated surface $\mathbf{S}_\triangle$ consists of the isotopy classes of the homeomorphisms of $\mathbf{S}$ which fix $\partial\mathbf{S}$ pointwise and fix $\triangle$ setwise. For any $\alpha \in \mathrm{CA}(\mathbf{S}_\triangle)$, the associated *braid twist* $B_\alpha \in \mathrm{MCG}(\mathbf{S}_\triangle)$ is defined in Figure 4.

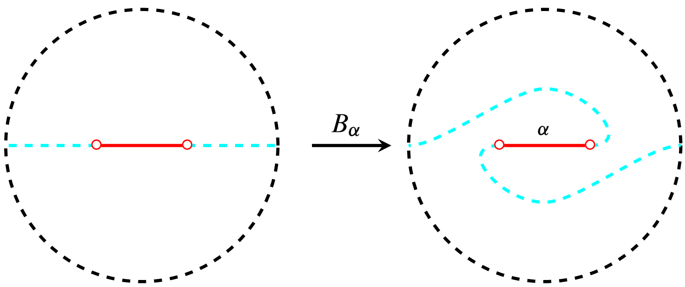

Figure 4. The braid twist.

We have the formula

$$B_{\Psi(\alpha)} = \Psi \circ B_\alpha \circ \Psi^{-1}, \quad \text{for any } \alpha \in \mathrm{CA}(\mathbf{S}_\triangle) \text{ and } \Psi \in \mathrm{MCG}(\mathbf{S}_\triangle).$$

We define $\mathrm{BT}(\mathbf{S}_\triangle)$ to be the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by B_α for $\alpha \in \mathrm{CA}(\mathbf{S}_\triangle)$.

The braid twist can be illustrated by smoothing out.

Construction 3.2. For any $\sigma, \tau \in \mathrm{CA}(\mathbf{S}_\triangle)$ with $\sigma(0) = \tau(0) = z \in \triangle$, the extension $\sigma \wedge \tau$ of σ by τ (with respect to the common starting point), which is the operation of smoothing out the intersection moving from σ to τ clockwise, is defined in Figure 5.

Notice that if $\mathrm{Int}_{\mathbf{S}_\triangle}(\sigma, \tau) = 1/2$, i.e., if they only intersect at one decoration, then

$$\sigma \wedge \tau = B_\tau(\sigma) = B_\sigma^{-1}(\tau). \tag{3.1}$$

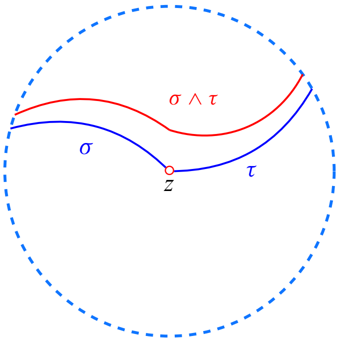

Figure 5. The extension as smoothing out.

Lemma 3.3. *Assume that* $|\triangle| \geq 3$. *For any* $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, *we have*

$$\mathrm{CA}(\mathbf{S}_\triangle) = \mathrm{BT}(\mathbf{S}_\triangle) \cdot \{\eta\}.$$

Proof. Let ξ be a simple closed arc in $\mathrm{CA}(\mathbf{S}_\triangle)$. We notice first that the intersection $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) \in \frac{1}{2} \cdot \mathbb{Z}^+$. When $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) > 0$, we use induction on it. For the starting case when $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) = 1/2$, we take $\alpha = \xi \wedge \eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, and then $\xi = B_\alpha(\eta)$ by (3.1).

Now suppose the assertion holds when $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) < k$ and consider the case when $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) = k \in \frac{1}{2} \cdot \mathbb{Z}^+$. There exists some decoration z which is not the endpoint of ξ (if the endpoints of η and ξ do not coincide, we take z to be an endpoint of η) and we connect z to ξ by l such that it intersects ξ at p (see Figure 6). We cut ξ at p and then smooth out the two resulting parts of ξ and l at p, respectively. Then we obtain $\xi_1, \xi_2 \in \mathrm{CA}(\mathbf{S}_\triangle)$ such that $\xi = \xi_1 \wedge \xi_2$ and

$$\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) = \mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi_1) + \mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi_2).$$

By assumption, there is $b \in \mathrm{BT}(\mathbf{S}_\triangle)$ such that $\xi_1 = b(\eta)$. Thus $\xi = B_{\xi_2}(\xi_1) = (B_{\xi_2} \cdot b)(\eta)$.

Finally, we consider the case when $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \xi) = 0$. We can choose a simple closed arc α such that $\mathrm{Int}_{\mathbf{S}_\triangle}(\eta, \alpha) = \mathrm{Int}_{\mathbf{S}_\triangle}(\xi, \alpha) = 1/2$. By the starting case, we have that $\alpha \in \mathrm{BT}(\mathbf{S}_\triangle) \cdot \eta$ and $\xi \in \mathrm{BT}(\mathbf{S}_\triangle) \cdot \alpha \subset \mathrm{BT}(\mathbf{S}_\triangle) \cdot \eta$. ∎

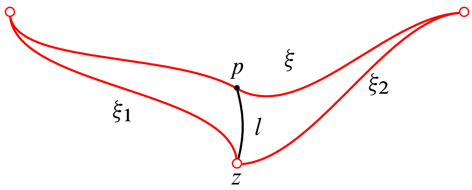

Figure 6. Decomposing ξ.

The branched double cover. Let $\Sigma_\triangle$ be a branched double cover of $\mathbf{S}_\triangle$ branching at decorations. We denote the covering map by $\pi\colon \Sigma_\triangle \to \mathbf{S}_\triangle$.

We consider the special case when $\mathbf{S}_\triangle = \mathbf{D}_3$ is a disk and $\triangle = \{z_\infty, z_*, z_0\}$. We fix two initial simple closed arcs η_0 and η_∞ such that $\eta_0 \cap \eta_\infty = \{z_*\}$, see Figure 7. Notice that there is an anti-clockwise angle from η_∞ to η_0. The branched double cover $\Sigma_\triangle$ is a torus with one boundary component ∂_Σ. For simplicity, we draw ∂_Σ as a puncture in the figures.

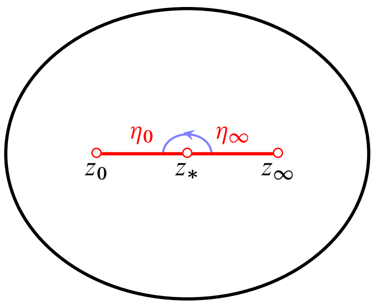

Figure 7. $\mathbf{D}_3$ is a disk with three decorations.

We take the $\mathbb{Z}^2$-covering $\widetilde{\Sigma}_\triangle$ of $\Sigma_\triangle$, where the white area is a fundamental domain (see Figure 8), and we denote the covering map by $\widetilde{\pi}\colon \widetilde{\Sigma}_\triangle \to \Sigma_\triangle$. When forgetting the punctures and decorations of $\widetilde{\Sigma}_\triangle$, this is the universal cover of the torus. Hence we embed $\widetilde{\Sigma}_\triangle$ into $\mathbb{R}^2$ such that all decorations and punctures are integer points, where its fundamental domain is a unit square. For each line (it is not allowed to pass through the punctures) in $\widetilde{\Sigma}_\triangle$ with rational slope $r/s \in \overline{\mathbb{Q}}$, it becomes a simple closed curve $C_{r/s}$ in $\Sigma_\triangle$ under the map $\widetilde{\pi}$.

Lemma 3.4 (Section 10 of [8]). *The set* $\mathrm{CA}(\mathbf{D}_3)$ *of simple closed arcs in* $\mathbf{D}_3$ *can be parameterized by rational numbers, i.e., there is a bijection*

$$\begin{tikzcd}
\mathrm{Br}_3 / Z(\mathrm{Br}_3) \arrow[r, "\iota"] \arrow[d, phantom] & \mathrm{PSL}_2(\mathbb{Z}) \\
\mathrm{CA}(\mathbf{D}_3) \arrow[r, "\cong"] \arrow[loop above] & \overline{\mathbb{Q}} \arrow[loop above]
\end{tikzcd}$$

sending $\eta_{r/s}$ *to* r/s.

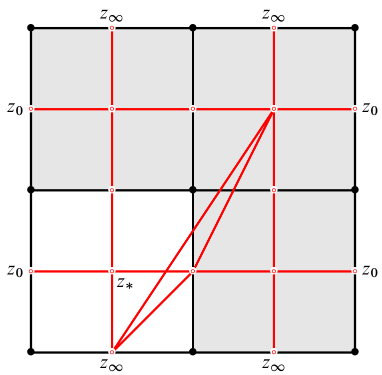

Figure 8. The $\mathbb{Z}^2$-covering of Σ_Δ and its fundamental domain.

Proof. We lift simple closed arcs in $\mathrm{CA}(\mathbf{S}_\triangle)$ to simple closed curves in Σ_Δ, which can be parametrized by rational numbers in $\overline{\mathbb{Q}}$. That is, for any $r/s \in \overline{\mathbb{Q}}$, there exists an $\eta_{r/s} \in \mathrm{CA}(\mathbf{S}_\triangle)$ which corresponds to a simple closed curve $C_{r/s}$ in Σ_Δ. Notice that the homology group $H_1(\Sigma_\Delta) = \mathbb{Z}[C_\infty] \oplus \mathbb{Z}[C_0]$. Thus the homology class $[C_{r/s}]$ corresponds to (r,s) in $H_1(\Sigma_\Delta) \cong \mathbb{Z}^2$. Notice that the braid twist $\mathrm{BT}(\mathbf{D}_3) \cong \mathrm{Br}_3$ lifts to a Dehn twist $\mathrm{DT}(\Sigma_\Delta) \subset \mathrm{MCG}(\Sigma_\Delta)$, which is generated by C_0 and C_∞. By the identification $\mathrm{DT}(\Sigma_\Delta)/\,Z(\mathrm{DT}(\Sigma_\Delta)) \cong \mathrm{PSL}_2(\mathbb{Z})$, the lemma holds. ■

Remark 3.5. Here is a consequence of the lemma above. For a rational number $r/s \in \mathbb{Q}^+$ with expression (2.3), the corresponding arc in $\mathrm{CA}(\mathbf{D}_3)$ is

$$\eta_{r/s} = B_{\eta_\infty}^{a_1} B_{\eta_0}^{-a_2} B_{\eta_\infty}^{a_3} B_{\eta_0}^{-a_4} \cdots B_{\eta_\infty}^{a_{2m-1}} B_{\eta_0}^{-a_{2m}} (\eta_\infty).$$

Moreover,

$$\eta_{-r/s} = B_{\eta_\infty}^{-a_1} B_{\eta_0}^{a_2} B_{\eta_\infty}^{-a_3} B_{\eta_0}^{a_4} \cdots B_{\eta_\infty}^{-a_{2m-1}} B_{\eta_0}^{a_{2m}} (\eta_\infty).$$

3.2. Bigraded arcs and $\mathfrak{q}$-intersections

Let $\mathbf{S}_\triangle$ be a decorated surface. In this section, we define the bigraded arcs and their $\mathfrak{q}$-intersections. Let $\mathbb{P}T\mathbf{S}_\triangle = \mathbb{P}T(\mathbf{S} \setminus \triangle)$ be the real projectivization of the tangent bundle of $\mathbf{S} \setminus \triangle$. We want to introduce a particular covering of $\mathbb{P}T\mathbf{S}_\triangle$ with covering group $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X} \cong \mathbb{Z}^2$, which is a rank two free module spanned by the basis 1 and $\mathbb{X}$.

A *grading* $\Lambda\colon \mathbf{S}_\triangle \to \mathbb{P}T\mathbf{S}_\triangle$ on $\mathbf{S}_\triangle$ is determined by a class in $\mathrm{H}_1(\mathbb{P}T\mathbf{S}_\triangle, \mathbb{Z} \oplus \mathbb{Z}\mathbb{X})$, with value 1 on each anti-clockwise loop $\{p\} \times \mathbb{RP}^1$ on $\mathbb{P}T_p\mathbf{S}_\triangle$ for $p \notin \triangle$ and value $-2+\mathbb{X}$ on each anti-clockwise loop $l_z \times \{x\}$ on $\mathbf{S}_\triangle$ around any $z \in \triangle, x \in \mathbb{RP}^1$. For any simple loop α on $\mathbf{S}_\triangle$, we denote $\Lambda_1(\alpha)$ the $\mathbb{Z}$ part of $\Lambda(\alpha)$ and denote $\Lambda_2(\alpha)$ the $\mathbb{Z}\mathbb{X}$ part of $\Lambda(\alpha)$. In fact, the first grading is a line field λ of $\mathbf{S}_\triangle$ which is determined by a class in $\mathrm{H}_1(\mathbb{P}T\mathbf{S}_\triangle, \mathbb{Z})$. The $\mathbb{X}$-grading is the Adams $\mathbb{Z}$-grading and we refer to the log surfaces in Figure 4 of [5] for more details. Define $\widehat{\mathbb{P}T\mathbf{S}_\triangle}$ to be the $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$ covering of $\mathbb{P}T\mathbf{S}_\triangle$ classified by the grading Λ, and denote the $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$ action on $\widehat{\mathbb{P}T\mathbf{S}_\triangle}$ by χ.

Definition 3.6. A *graded decorated surface* $\mathbf{S}_\triangle^\Lambda$ consists of a decorated surface $\mathbf{S}_\triangle$ and a grading Λ on $\mathbf{S}_\triangle$.

Let $\mathbf{S}_\triangle^\Lambda$ be a graded decorated surface and let $c\colon[0,1]\to\mathbf{S}$ be an arc in $\mathbf{S}$. There is a canonical section $s_c\colon c\setminus\triangle\to\mathbb{P}T\mathbf{S}_\triangle$ given by $s_c(z)=T_zc$. A *bigrading* on c is given by a lift $\widehat{c}$ of s_c to $\widehat{\mathbb{P}T\mathbf{S}_\triangle}$. The pair $(c,\widehat{c})$ is called a *bigraded arc*, and we usually denote it by $\widehat{c}$. Note that there are $\mathbb{Z}^2$ lifts of c, which are related by the $\mathbb{Z}^2$-action defined by χ. One is the shift grading such that $\widehat{c}[m](t)=\chi(m,0)\widehat{c}(t)$, and the other is the $\mathbb{X}$-grading such that $\widehat{c}\{m\}(t)=\chi(0,m)\widehat{c}(t)$ for any $m\in\mathbb{Z}$. For any $\eta\in\mathrm{CA}(\mathbf{S}_\triangle)$, we call any of its lifts $\widehat{\eta}$ in $\widehat{\mathbb{P}T\mathbf{S}_\triangle}$ a *bigraded simple closed arc*. Denote by $\widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$ the set of bigraded simple closed arcs.

For any bigraded arcs $\widehat{\sigma}$ and $\widehat{\tau}$ which are in minimal position with respect to each other, let $p=\sigma(t_1)=\tau(t_2)\in\mathbf{S}^\circ$ be the point where σ and τ intersect transversally. Fix a small circle $a\subset\mathbf{S}\setminus\triangle$ around p. Let $\alpha\colon[0,1]\to a$ be an embedded arc which moves anti-clockwise around p, such that α intersects σ and τ at $\alpha(0)$ and $\alpha(1)$, respectively (see Figure 9). If $p\in\triangle$, then α is unique up to a change of parametrization (see Figure 9, left picture); otherwise there are two possibilities, which are distinguished by their endpoints (right picture of Figure 9). Take a smooth path $\rho\colon[0,1]\to\mathbb{P}T\mathbf{S}_\triangle$ with $\rho(t)\in\mathbb{P}T_{\alpha(t)}\mathbf{S}_\triangle$ for all t, going from $\rho(0)=T_{\alpha(0)}\sigma$ to $\rho(1)=T_{\alpha(1)}\tau$, such that $\rho(t)\neq T_{\alpha(t)}l$ for all t. Lift ρ to a path $\widehat{\rho}\colon[0,1]\to\widehat{\mathbb{P}T\mathbf{S}_\triangle}$ with $\widehat{\rho}(0)=\widehat{\sigma}(\alpha(0))$. Then there exist some integers $\varrho,\varsigma\in\mathbb{Z}$ such that

$$\widehat{\tau}(\alpha(1))=\chi(\varrho+\varsigma\mathbb{X})\widehat{\rho}(1). \tag{3.2}$$

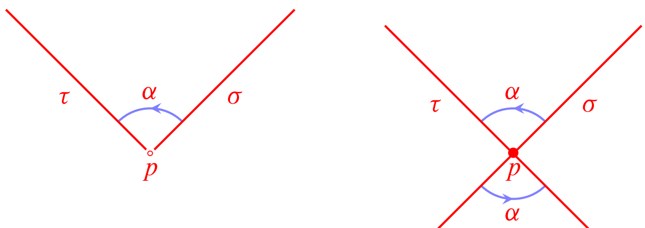

Figure 9. Intersection at p when p is a decoration or not.

Definition 3.7 ([7]). For any bigraded arcs $\widehat{\sigma}$ and $\widehat{\tau}$ in $\mathbf{S}_\triangle^\Lambda$, the *bi-index* of an intersection p of $\widehat{\sigma}$ and $\widehat{\tau}$ is given by

$$\mathrm{ind}_p^{\mathbb{Z}^2}(\widehat{\sigma},\widehat{\tau})=\mathrm{ind}_p(\widehat{\sigma},\widehat{\tau})+\mathrm{ind}_p^{\mathbb{X}}(\widehat{\sigma},\widehat{\tau})\mathbb{X},$$

where $\mathrm{ind}_p(\widehat{\sigma},\widehat{\tau}):=\varrho$ and $\mathrm{ind}_p^{\mathbb{X}}(\widehat{\sigma},\widehat{\tau}):=\varsigma$ are defined in (3.2).

We have the following equations among bi-indices, which will be used later.

Lemma 3.8 (Lemma 2.6 in [5]). *Let $\widehat{\sigma}$ and $\widehat{\tau}$ be bigraded arcs in $\mathbf{S}_\triangle^\Lambda$ with an intersection $p\in\mathbf{S}^\circ$. If $p\notin\triangle$, we have*

$$\mathrm{ind}_p^{\mathbb{Z}^2}(\widehat{\sigma},\widehat{\tau})+\mathrm{ind}_p^{\mathbb{Z}^2}(\widehat{\tau},\widehat{\sigma})=1.$$

If $p \in \triangle$, we have

$$\operatorname{ind}_p^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\tau}) + \operatorname{ind}_p^{\mathbb{Z}^2}(\widehat{\tau}, \widehat{\sigma}) = \mathbb{X}. \tag{3.3}$$

Lemma 3.9 (Lemma 2.7 in [5]). *Let $\widehat{\sigma}$, $\widehat{\tau}$ and $\widehat{\alpha}$ be bigraded arcs in $\mathbf{S}_\triangle^\Lambda$. If they are in the left case of Figure 10, we have*

$$\operatorname{ind}_p^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\alpha}) = \operatorname{ind}_p^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\gamma}) + \operatorname{ind}_p^{\mathbb{Z}^2}(\widehat{\gamma}, \widehat{\alpha}).$$

If they are in the left case of Figure 9, we have

$$\operatorname{ind}_p^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\tau}) = \operatorname{ind}_{\alpha(0)}^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\alpha}) - \operatorname{ind}_{\alpha(1)}^{\mathbb{Z}^2}(\widehat{\tau}, \widehat{\alpha}) = \operatorname{ind}_{\alpha(1)}^{\mathbb{Z}^2}(\widehat{\alpha}, \widehat{\tau}) - \operatorname{ind}_{\alpha(0)}^{\mathbb{Z}^2}(\widehat{\alpha}, \widehat{\sigma}).$$

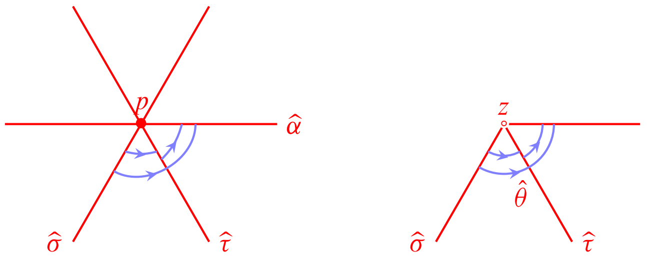

Figure 10. Bigraded arcs intersect at the same point (or decoration) in anti-clockwise

Lemma 3.10. *Let $\widehat{\sigma}$, $\widehat{\tau}$ and $\widehat{\alpha}$ be bigraded arcs on $\mathbf{S}_\triangle^\Lambda$ which share the same decoration z and sitting in anti-clockwise order in the right picture of Figure 10. We have*

$$\operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\alpha}) = \operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\tau}) + \operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\tau}, \widehat{\alpha}).$$

Proof. Fix a small circle $a \subset \mathbf{S} \setminus \triangle$ around z. Let $\hat{\theta}\colon [0,1] \to a$ be an embedded bigraded arc winding anti-clockwise at z such that the underlying arc θ intersect σ, τ and α at $\theta(0), \theta(1/2)$ and $\theta(1)$, respectively (see the right picture of Figure 10). The arc $\hat{\theta}$ is unique up to a change of parametrization. By Lemma 3.9, we have

$$\begin{aligned}
\operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\alpha}) &= \operatorname{ind}_{\theta(0)}^{\mathbb{Z}^2}(\widehat{\sigma}, \hat{\theta}) - \operatorname{ind}_{\theta(1)}^{\mathbb{Z}^2}(\widehat{\alpha}, \hat{\theta}) \\
&= [\operatorname{ind}_{\theta(0)}^{\mathbb{Z}^2}(\widehat{\sigma}, \hat{\theta}) - \operatorname{ind}_{\theta(1/2)}^{\mathbb{Z}^2}(\widehat{\tau}, \hat{\theta})] + [\operatorname{ind}_{\theta(1/2)}^{\mathbb{Z}^2}(\widehat{\tau}, \hat{\theta}) - \operatorname{ind}_{\theta(1)}^{\mathbb{Z}^2}(\widehat{\alpha}, \hat{\theta})] \\
&= \operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\tau}) + \operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\tau}, \widehat{\alpha}).
\end{aligned}$$ ∎

Definition 3.11. For $\widehat{\tau}, \widehat{\eta} \in \widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$ satisfying that $\operatorname{Int}_{\mathbf{S}_\triangle}(\tau, \eta) = 1/2$ and $\operatorname{ind}_z^{\mathbb{X}}(\widehat{\tau}, \widehat{\eta}) = a$, let $z \in \triangle$ be their common endpoint. Denote by $\widehat{\tau} \wedge \widehat{\eta}$ to be the bigraded arc in $\widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$ whose underlying arc is obtained by the smoothing out $\widehat{\tau} \cup \widehat{\eta}[(a-1)\mathbb{X}]$ at z and whose grading inherits from $\widehat{\tau}$. That is, we have $\operatorname{ind}_z^{\mathbb{Z}^2}(\widehat{\tau}, \widehat{\tau} \wedge \widehat{\eta}) = 0$, cf. Figure 11.

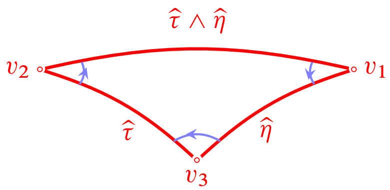

Figure 11. The sum of bi-indices of intersections between bigraded arcs via smoothing out

Proposition 3.12. *For $\widehat{\tau}, \widehat{\eta}$ and $\widehat{\tau} \wedge \widehat{\eta}$ in $\widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$ as in Definition 3.11, we have*

$$\mathrm{ind}^{\mathbb{Z}^2}_{v_2}(\widehat{\tau}, \widehat{\tau} \wedge \widehat{\eta}) + \mathrm{ind}^{\mathbb{Z}^2}_{v_1}(\widehat{\tau} \wedge \widehat{\eta}, \widehat{\eta}) + \mathrm{ind}^{\mathbb{Z}^2}_{v_3}(\widehat{\eta}, \widehat{\tau}) = 1.$$

Proof. We calculate the two gradings separately. For the first grading, it is a line field which is determined by $\mathrm{H}_1(\mathbb{P}T\mathbf{S}_\triangle, \mathbb{Z})$. We can identify all the projectization of the tangent space of any point in the contractible triangle formed by the three arcs (except the decorations) simultaneously. Hence, the sum of the first grading is 1 (rotating anti-clockwise). For the second grading, we use the log surface in the sense of Section 2.4 of [5]. By Definition 3.11, the segments of $\widehat{\tau} \wedge \widehat{\eta}$ and $\widehat{\eta}[(a-1)\mathbb{X}]$ near v_1 are in the same sheet of $\log(\mathbf{S}_\triangle)$ and the anti-clockwise angle does not cross the cut (cf. Figure 5 in [5]). Thus we have

$$\begin{aligned}\mathrm{ind}^{\mathbb{X}}_{v_2}(\widehat{\tau}, \widehat{\tau} \wedge \widehat{\eta}) + \mathrm{ind}^{\mathbb{X}}_{v_1}(\widehat{\tau} \wedge \widehat{\eta}, \widehat{\eta}) + \mathrm{ind}^{\mathbb{X}}_{v_3}(\widehat{\eta}, \widehat{\tau}) &= 0 + (a-1)\mathbb{X} + (1 - \mathrm{ind}^{\mathbb{X}}_{v_3}(\widehat{\tau}, \widehat{\eta})) \\ &= (a-1) + (1-a) = 0\end{aligned}$$

as required, where $\mathrm{ind}^{\mathbb{X}}_{?}$ denote the second grading. ∎

Notations 3.13. For $(\varrho, \varsigma) \in \mathbb{Z}^2$, we write $\bigcap^{\varrho+\varsigma\mathbb{X}}(\widehat{\sigma}, \widehat{\tau})$ for the set of intersections between $\widehat{\sigma}$ and $\widehat{\tau}$ with bi-index $\varrho + \varsigma\mathbb{X}$. We will use the notations

$$\begin{aligned}\mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{\Pi}(\widehat{\sigma}, \widehat{\tau}) &:= \left|\bigcap^{\varrho+\varsigma\mathbb{X}}(\widehat{\sigma}, \widehat{\tau}) \cap \Pi\right|, \\ \mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{\mathbf{S}_\triangle}(\widehat{\sigma}, \widehat{\tau}) &:= \frac{1}{2} \cdot \mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{\triangle}(\widehat{\sigma}, \widehat{\tau}) + \mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{\mathbf{S}^\circ_\triangle}(\widehat{\sigma}, \widehat{\tau}),\end{aligned}$$

for the bi-index $(\varrho + \varsigma\mathbb{X})$ intersection numbers at any proper subset Π of $\mathbf{S}_\triangle$ and at all of $\mathbf{S}^\circ$, respectively. The *total intersection*

$$\mathrm{Int}_{?}(\widehat{\sigma}, \widehat{\tau}) = \sum_{\varrho,\varsigma \in \mathbb{Z}} \mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{?}(\widehat{\sigma}, \widehat{\tau})$$

is the sum over all bi-indices, where $? = \triangle$ or $\mathbf{S}^\circ_\triangle$.

Definition 3.14 ([5,7]). Let q_1 and q_2 be two formal parameters. The $\mathbb{Z}^2$*-graded* $\mathfrak{q}$*-intersection* of $\widehat{\sigma}, \widehat{\tau} \in \widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$ is defined to be

$$\mathrm{Int}^{\mathfrak{q}}(\widehat{\sigma}, \widehat{\tau}) = \sum_{\varrho,\varsigma \in \mathbb{Z}} q_1^{\varrho} q_2^{\varsigma} \cdot \mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{\triangle}(\widehat{\sigma}, \widehat{\tau}) + (1 + q_1^{-1}q_2) \sum_{\varrho,\varsigma \in \mathbb{Z}} q_1^{\varrho} q_2^{\varsigma} \cdot \mathrm{Int}^{\varrho+\varsigma\mathbb{X}}_{\mathbf{S}^\circ_\triangle}(\widehat{\sigma}, \widehat{\tau}). \tag{3.4}$$

Note that we have $\mathrm{Int}^{\mathfrak{q}}(-,-) \mid_{q_1=q_2=1} = 2\,\mathrm{Int}_{\mathbf{S}_\triangle}(-,-) = \mathrm{Int}_{\Sigma_\triangle}(-,-)$, where $\Sigma_\triangle$ is the branched double cover of $\mathbf{S}_\triangle$.

3.3. Left $\mathfrak{q}$-deformations as $\mathfrak{q}$-intersections

Recall that $\mathbf{D}_3$ is a disk and $\triangle = \{z_\infty, z_*, z_0\}$. By Lemma 3.3, we can label simple closed arcs by $\overline{\mathbb{Q}}$ as follows. We fix two initial bigraded simple closed arcs and denote them $\widehat{\eta}_0$ and $\widehat{\eta}_\infty$ such that $\mathrm{ind}_{z_*}(\widehat{\eta}_\infty, \widehat{\eta}_0) = 1$ (see Figure 13).

Construction 3.15. We define a map

$$\begin{aligned} \widehat{\eta} : \overline{\mathbb{Q}} &\to \widehat{\mathrm{CA}}(\mathbf{D}_3) \\ \frac{r}{s} &\mapsto \widehat{\eta}_{r/s}, \end{aligned}$$

as follows. For any rational $r/s \in \mathbb{Q}^+$, by Lemma/Definition 2.3 we know that it can be uniquely written as

$$\frac{r}{s} = \frac{p}{q} \oplus \frac{u}{v}.$$

We iteratively define that

$$\widehat{\eta}_{r/s} := B_{\eta_{u/v}}(\widehat{\eta}_{p/q}) = \widehat{\eta}_{p/q} \wedge \widehat{\eta}_{u/v},$$

noticing that $\mathrm{Int}_{\mathbf{D}_3}(\eta_{p/q}, \eta_{u/v}) = 1/2$. For negative case, we set $\widehat{\eta}_{-\infty} := \widehat{\eta}_\infty$ and define

$$\widehat{\eta}_{-r/s} := B_{\eta_{-p/q}}(\widehat{\eta}_{-u/v}).$$

Thus, we get some bigraded closed arcs in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$ which are $\widehat{\eta}_{r/s}[\varrho + \varsigma\mathbb{X}]$, where $\varrho, \varsigma \in \mathbb{Z}$ and $r/s \in \overline{\mathbb{Q}}$.

Let $r/s \in \mathbb{Q}^+$ with $r/s = (p/q) \oplus (u/v)$. By Definition 3.11, the grading of the new arc $\widehat{\eta}_{(p/q)\oplus(u/v)}$ inherits the grading of $\widehat{\eta}_{p/q}$. That is, for any bigraded arc $\widehat{\sigma}$ intersecting $\widehat{\eta}_{r/s}$ and $\widehat{\eta}_{(p/q)\oplus(u/v)}$ at $p_1, p_2 \in \mathbf{S}^\circ$, respectively (see Figure 12), we have

$$\mathrm{ind}_z^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{(p/q)\oplus(u/v)}) = 0 \quad \text{and} \quad \mathrm{ind}_{p_1}^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\eta}_{p/q}) = \mathrm{ind}_{p_2}^{\mathbb{Z}^2}(\widehat{\sigma}, \widehat{\eta}_{(p/q)\oplus(u/v)}), \tag{3.5}$$

where $z \in \triangle$ is the common endpoint of $\widehat{\eta}_{p/q}$ and $\widehat{\eta}_{(p/q)\oplus(u/v)}$. The second equation in (3.5) follows from the fact that $\widehat{\eta}_{p/q}, \widehat{\eta}_{(p/q)\oplus(u/v)}$ and $\widehat{\eta}_{u/v}$ form a contractible triangle.

From Proposition 3.12, we know that these three simple closed arcs $\widehat{\eta}_{p/q}, \widehat{\eta}_{(p/q)\oplus(u/v)}$ and $\widehat{\eta}_{u/v}$ (take $\widehat{\tau} = \widehat{\eta}_{p/q}, \widehat{\eta} = \widehat{\eta}_{u/v}$ in Figure 11) satisfy

$$\mathrm{ind}_{v_2}^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{(p/q)\oplus(u/v)}) + \mathrm{ind}_{v_1}^{\mathbb{Z}^2}(\widehat{\eta}_{(p/q)\oplus(u/v)}, \widehat{\eta}_{u/v}) + \mathrm{ind}_{v_3}^{\mathbb{Z}^2}(\widehat{\eta}_{u/v}, \widehat{\eta}_{p/q}) = 1, \tag{3.6}$$

where v_1, v_2 and v_3 are the corresponding intersecting decorations.

We fix the following setting.

Setting 3.16. Recall that in Lemma/Definition 2.3, any fraction $r/s \in \mathbb{Q}^+$ can be uniquely written as

$$\frac{r}{s} = \frac{p}{q} \oplus \frac{u}{v},$$

with $p/q, u/v \in \overline{\mathbb{Q}^{\geq 0}}$, $uq - pv = 1$ and an associated integer $l(p/q, u/v)$. The corresponding arcs in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$ of these fractions are $\widehat{\eta}_{p/q}$, $\widehat{\eta}_{(p/q)\oplus(u/v)}$ and $\widehat{\eta}_{u/v}$, where $\widehat{\eta}_{p/q}$ and $\widehat{\eta}_{u/v}$ intersect at only one decoration z in $\triangle$. We do not distinguish between r/s and $(p/q) \oplus (u/v)$ in the following.

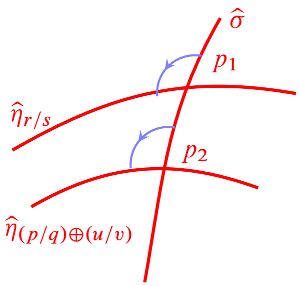

Figure 12. The bigrading inherited from braid twist.

Lemma 3.17. *For any two arcs $\widehat{\eta}_{p/q}$ and $\widehat{\eta}_{u/v}$ in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$ in Setting 3.16, the bi-index is of the form $l(1-\mathbb{X})$, where $l \in \mathbb{N}$, i.e.,*

$$\mathrm{ind}_z^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{u/v}) = l(1-\mathbb{X}), \tag{3.7}$$

except the special case

$$\mathrm{ind}_{z_*}^{\mathbb{Z}^2}(\widehat{\eta}_0, \widehat{\eta}_\infty) = \mathbb{X} - 1.$$

Here $l = l(p/q, u/v)$ in Lemma/Definition 2.3.

Proof. For the special case, we know that

$$\mathrm{ind}_{z_*}^{\mathbb{Z}^2}(\widehat{\eta}_0, \widehat{\eta}_\infty) = \mathbb{X} - \mathrm{ind}_{z_*}^{\mathbb{Z}^2}(\widehat{\eta}_\infty, \widehat{\eta}_0) = \mathbb{X} - 1,$$

from (3.3). In general, we prove the lemma by induction on l. For initial bigraded simple closed arcs $\widehat{\eta}_0, \widehat{\eta}_{\frac{1}{1}}$ and $\widehat{\eta}_\infty$ in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$, we have $\mathrm{ind}_{z_0}(\widehat{\eta}_0, \widehat{\eta}_{1/1}) = \mathrm{ind}_{z_\infty}(\widehat{\eta}_{1/1}, \widehat{\eta}_\infty) = 0$ and thus the lemma holds obviously. We assume that (3.7) holds for l and consider the $l+1$ case. For any $\widehat{\eta}_{p/q}$ and $\widehat{\eta}_{u/v}$ in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$ in Setting 3.16, we assume that they intersect at only one decoration v_3 with

$$\mathrm{ind}_{v_3}^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{u/v}) = l(1-\mathbb{X}),$$

where $l \in \mathbb{N}$. By (3.3), we have

$$\mathrm{ind}_{v_3}^{\mathbb{Z}^2}(\widehat{\eta}_{u/v}, \widehat{\eta}_{p/q}) = (l+1)\mathbb{X} - l.$$

Since the grading of $\widehat{\eta}_{(p/q)\oplus(u/v)}$ inherits the grading of $\widehat{\eta}_{p/q}$, we have

$$\mathrm{ind}_{v_1}^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{(p/q)\oplus(u/v)}) = 0. \tag{3.8}$$

By (3.6), we deduce that

$$\mathrm{ind}_{v_2}^{\mathbb{Z}^2}(\widehat{\eta}_{(p/q)\oplus(u/v)}, \widehat{\eta}_{u/v}) = 1 - 0 - [(l+1)\mathbb{X} - l] = (l+1) - (l+1)\mathbb{X}. \tag{3.9}$$

Finally, combining (3.8) and (3.9), the lemma is true. ∎

Theorem 3.18. *For any rational number $r/s \in \overline{\mathbb{Q}}$, we have*

$$\left[\frac{r}{s}\right]^{\flat}_{\mathfrak{q}} = \left.\frac{\varepsilon\,\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_0)}{\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_\infty)}\right|_{\mathfrak{q}=q_1^{-1}q_2}, \tag{3.10}$$

corresponding to the left $\mathfrak{q}$-deformation of r/s, where

$$\varepsilon = \begin{cases} q_1^{-1}, & \text{if } r/s \geq 0, \\ -1, & \text{if } r/s < 0. \end{cases}$$

In particular, for $r/s \in \overline{\mathbb{Q}^{\geq 0}}$, we have

$$\begin{cases} \bar{\boldsymbol{R}}^{\flat}_{\mathfrak{q}}(r/s) = q_1^{-1}\,\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_0)\big|_{\mathfrak{q}=q_1^{-1}q_2}, \\ \bar{\boldsymbol{S}}^{\flat}_{\mathfrak{q}}(r/s) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_\infty)\big|_{\mathfrak{q}=q_1^{-1}q_2}. \end{cases}$$

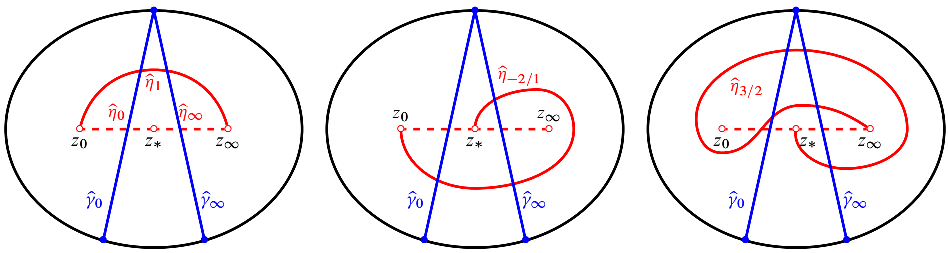

Figure 13. Red arcs: examples of bigraded closed arcs in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$.

Proof. For 0 and ∞, we have

$$\mathrm{Int}^{\mathfrak{q}}_{z_*}(\widehat{\eta}_0, \widehat{\eta}_\infty) = q_1^{-1}q_2 \quad \text{and} \quad \mathrm{Int}^{\mathfrak{q}}_{z_*}(\widehat{\eta}_\infty, \widehat{\eta}_0) = q_1$$

by definition. We set

$$\bar{\mathbf{R}}^{\flat}_{\mathfrak{q}}(0) = q_2 - q_1 \quad \text{and} \quad \bar{\mathbf{S}}^{\flat}_{\mathfrak{q}}(\infty) = 1 - q_1^{-1}q_2.$$

We prove the non-negative case by induction using Setting 3.16. At the starting step, the fractions in the theorem are $\frac{q_1^{-1}q_2-1}{q_1^{-1}q_2}$ and $\frac{1}{1-q_1^{-1}q_2}$, which coincide the left $\mathfrak{q}$-deformation of 0 and ∞, respectively, when $\mathfrak{q} = q_1^{-1}q_2$. We assume that the formula (3.10) holds for $p/q, u/v \in \overline{\mathbb{Q}^{\geq 0}}$ in Setting 3.16 which satisfy

$$\mathrm{ind}_z^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{u/v}) = l(1-\mathbb{X}),$$

where $l = l((p/q) \oplus (u/v)) \in \mathbb{N}$ and z is the common endpoint of $\widehat{\eta}_{p/q}$ and $\widehat{\eta}_{u/v}$. We use the $\mathbb{Z}^2$-covering $\widetilde{\Sigma}_\Delta$ to compute the $\mathfrak{q}$-intersection between $\widehat{\eta}_{(p/q)\oplus(u/v)}$ and $\widehat{\eta}_0$ (and $\widehat{\eta}_\infty$ is similar). We lift the arcs in $\widehat{\mathrm{CA}}(\mathbf{D}_3)$ to lines in $\widetilde{\Sigma}_\Delta$. Then $\widehat{\eta}_0$ becomes a series of horizontal lines which pass through z_* and z_0. The topological triangle T in $\mathbf{D}_3$ bounded by $\widehat{\eta}_{p/q}$, $\widehat{\eta}_{u/v}$ and $\widehat{\eta}_{(p/q)\oplus(u/v)}$ becomes a triangle $\widetilde{T}$ in $\mathbb{R}^2$ up to translation (or reflection). We may assume that the vertices of $\widetilde{T}$ are $\tilde{z}$, $\tilde{z}+(p,q)$ and $\tilde{z}+(r,s)$, which says that z', z and z'' are in T. The area of $\widetilde{T}$ equals to $1/2$ by the relation $uq-pv=1$. By Pick's theorem, there are no decorations or punctures in $\widetilde{T}$. The intersections between $\widehat{\eta}_{(p/q)\oplus(u/v)}$ and $\widehat{\eta}_0$ consist of two parts (see Figure 14):

- intersections below and on a, which inherit the bi-indices from $\widehat{\eta}_{p/q}$, and
- intersections above a, which are induced from $\widehat{\eta}_{u/v}$.

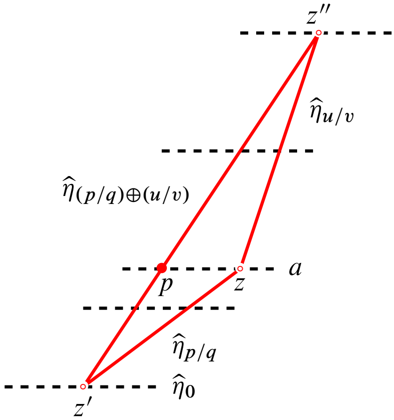

Figure 14. Intersections whose bi-indices inherit (respectively, are induced from) the one between $\widehat{\eta}_{p/q}$ (respectively, $\widehat{\eta}_{u/v}$) and $\widehat{\eta}_0$.

For the first case, if the two intersections are both either in the interior or at decorations, the bi-indices are the same. For the second case, the two bi-indices differ from $\mathrm{ind}^{\mathbb{Z}^2}_{z''}(\widehat{\eta}_{(p/q)\oplus(u/v)},\widehat{\eta}_{u/v})=(l+1)\cdot(1-\mathbb{X})$. Thus we have

$$\mathrm{Int}^{\mathfrak{q}}_{\mathbf{D}_3\setminus\{p\}}(\widehat{\eta}_{(p/q)\oplus(u/v)},\widehat{\eta}_0)=\mathrm{Int}^{\mathfrak{q}}_{\mathbf{D}_3\setminus\{z\}}(\widehat{\eta}_{p/q},\widehat{\eta}_0)+q_1^{l+1}q_2^{-l-1}\cdot\mathrm{Int}^{\mathfrak{q}}_{\mathbf{D}_3\setminus\{z\}}(\widehat{\eta}_{u/v},\widehat{\eta}_0).$$

For the intersection on a, by inheritance, we have bi-index given directly by

$$\mathrm{ind}^{\mathbb{Z}^2}_{p}(\widehat{\eta}_0,\widehat{\eta}_{(p/q)\oplus(u/v)})=\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_0,\widehat{\eta}_{p/q}).$$

Hence we have

$$\begin{aligned}\mathrm{ind}^{\mathbb{Z}^2}_{p}(\widehat{\eta}_{(p/q)\oplus(u/v)},\widehat{\eta}_0)&=1-\mathrm{ind}^{\mathbb{Z}^2}_{p}(\widehat{\eta}_0,\widehat{\eta}_{(p/q)\oplus(u/v)})=1-\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_0,\widehat{\eta}_{p/q})\\&=1-\mathbb{X}+\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_0).\end{aligned}$$

Thus, the bi-index in $\mathbf{D}_3^\circ$ contributes $(1-\mathbb{X})+\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_0)$ and $\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_0)$ to the $\mathfrak{q}$-intersection in (3.4).

On the other hand, we have

$$\begin{aligned}\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{u/v},\widehat{\eta}_0)&=\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_0)-\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_{u/v})\\&=[\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_0)+(1-\mathbb{X})]-[\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_{u/v})+(1-\mathbb{X})]\\&=[\mathrm{ind}^{\mathbb{Z}^2}_{z}(\widehat{\eta}_{p/q},\widehat{\eta}_0)+(1-\mathbb{X})]-(l+1)\cdot(1-\mathbb{X}).\end{aligned}$$

Thus, we deduce that

$$\mathrm{Int}_z^{\mathfrak{q}}(\widehat{\eta}_{(p/q)\oplus(u/v)}, \widehat{\eta}_0) = \mathrm{Int}_z^{\mathfrak{q}}(\widehat{\eta}_{p/q}, \widehat{\eta}_0) + q_1^{l+1} q_2^{-l-1} \cdot \mathrm{Int}_z^{\mathfrak{q}}(\widehat{\eta}_{u/v}, \widehat{\eta}_0).$$

Therefore, we have

$$\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{(p/q)\oplus(u/v)}, \widehat{\eta}_0) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{p/q}, \widehat{\eta}_0) + q_1^{l+1} q_2^{-l-1} \cdot \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{u/v}, \widehat{\eta}_0),$$

which coincides with $\overline{\mathbf{R}}_{\mathfrak{q}}^{\flat}(r/s)$ after multiplication by q_1^{-1} if we take $\mathfrak{q} = q_1^{-1} q_2$. Similarly, we deduce that

$$\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{(p/q)\oplus(u/v)}, \widehat{\eta}_\infty) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{p/q}, \widehat{\eta}_\infty) + q_1^{l+1} q_2^{-l-1} \cdot \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{u/v}, \widehat{\eta}_\infty).$$

If we take $q_1^{-1} q_2 = \mathfrak{q}$, the fraction we get in the theorem coincides the left $\mathfrak{q}$-deformation of $r/s = (p/q) \oplus (u/v)$, which completes the proof. ∎

Example 3.19. We give examples of -2 and $3/2$ (see Figure 13) for the left $\mathfrak{q}$-deformations. We know that

$$-\frac{2}{1} = -\frac{1}{1} \oplus -\frac{1}{0} \quad \text{and} \quad \frac{3}{2} = \frac{1}{1} \oplus \frac{2}{1}.$$

By the strategy of our proof, we compute that

$$\begin{cases} \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{-2/1}, \widehat{\eta}_0) = q_1^3 q_2^{-2} + q_1, \\ \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{-2/1}, \widehat{\eta}_\infty) = q_2; \end{cases}$$

and

$$\begin{cases} \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{3/2}, \widehat{\eta}_0) = q_1 + q_2 + q_1^3 q_2^{-2}, \\ \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{3/2}, \widehat{\eta}_\infty) = 1 + q_1^2 q_2^{-2}. \end{cases}$$

By Theorem 3.18, we have

$$\left[-\frac{2}{1}\right]_{\mathfrak{q}}^{\flat} = -\frac{\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{-2/1}, \widehat{\eta}_0)}{\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{-2/1}, \widehat{\eta}_\infty)}\Big|_{\mathfrak{q}=q_1^{-1}q_2} = -\frac{\mathfrak{q}^2 + 1}{\mathfrak{q}^3}$$

and

$$\left[\frac{3}{2}\right]_{\mathfrak{q}}^{\flat} = \frac{q_1^{-1}\,\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{3/2}, \widehat{\eta}_0)}{\mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}_{3/2}, \widehat{\eta}_\infty)}\Big|_{\mathfrak{q}=q_1^{-1}q_2} = \frac{\mathfrak{q}^3 + \mathfrak{q}^2 + 1}{\mathfrak{q}^2 + 1}.$$

3.4. Right $\mathfrak{q}$-deformations as $\mathfrak{q}$-intersections

In this section, we add a finite set $\mathbf{M}$ of (*open*) *marked points* to $\partial\mathbf{S}$ satisfying $|\mathbf{M}| = |\triangle|$ and get a decorated marked surface (or *DMS* for short). We still denote the DMS by $\mathbf{S}_\triangle$. An arc c is called *open* if $c(0)$ and $c(1)$ are in $\mathbf{M}$, without self-intersections in $\mathbf{S}_\triangle^\circ$. We call two open arcs do not cross each other if they do not have intersections in $\mathbf{S}_\triangle^\circ$.

We also have bigraded open arcs as before. For an open arc γ, we define the $\mathbb{Z}^2$-*graded* $\mathfrak{q}$-*intersection* between a lift $\widehat{\gamma}$ of γ and $\widehat{\tau} \in \widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$ to be

$$\mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}, \widehat{\tau}) = \sum_{\varrho,\varsigma \in \mathbb{Z}} q_1^{\varrho} q_2^{\varsigma} \cdot \mathrm{Int}_{\mathbf{S}_\triangle^\circ}^{\varrho+\varsigma\mathbb{X}}(\widehat{\gamma}, \widehat{\tau}).$$

Note that we have $\mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}, \widehat{\tau})\,|_{q_1=q_2=1} = \mathrm{Int}_{\mathbf{S}_\triangle}(\gamma, \tau)$. We define a special class of open (bigraded) arcs.

Definition 3.20. An *open full formal arc system* $\mathbf{A} = \{\gamma_1, \ldots, \gamma_n\}$ of a DMS $\mathbf{S}_\triangle$ is a collection of pairwise non-crossing open arcs that divides the surface $\mathbf{S}_\triangle$ into polygons, called $\mathbf{A}$-*polygons*, satisfying that each $\mathbf{A}$-polygon contains exactly one decoration. We call $\sigma \in \mathrm{CA}(\mathbf{S}_\triangle)$ the *dual* to γ_i if γ_i intersects it once and γ_j does not intersect it for any $j \neq i$. Denote s_i the dual to γ_i and let $\mathbf{A}^* = \{s_1, \ldots, s_n\}$.

Let $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_n, \widehat{s}_1, \ldots, \widehat{s}_n$ be their bigraded lifts with

$$\mathrm{ind}^{\mathbb{Z}^2}(\widehat{\gamma}_i, \widehat{s}_i) = 0.$$

We add three open marked points to the boundary of $\mathbf{D}_3$ in Section 3.3. Let γ_0, γ_∞ be two open arcs which form an open arc system in $\mathbf{D}_3$ and intersect with η_0 and η_∞ transitively only once respectively. Let $\widehat{\gamma}_0, \widehat{\gamma}_\infty$ be their bigraded lifts respectively which satisfy

$$\mathrm{ind}^{\mathbb{Z}^2}(\widehat{\gamma}_\infty, \widehat{\eta}_\infty) = \mathrm{ind}^{\mathbb{Z}^2}(\widehat{\gamma}_0, \widehat{\eta}_0) = 0.$$

We draw them as blue arcs in Figure 13.

Theorem 3.21. *For any rational number $r/s \in \overline{\mathbb{Q}}$, we have*

$$\left[\frac{r}{s}\right]^\sharp_{\mathfrak{q}} = \frac{\varepsilon\,\mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{r/s})}{\mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{r/s})}\Big|_{\mathfrak{q}=q_1^{-1}q_2}, \tag{3.11}$$

corresponding to the right $\mathfrak{q}$-deformation of r/s, where

$$\varepsilon = \begin{cases} 1, & \text{if } r/s \geq 0, \\ -q_1^{-1}, & \text{if } r/s < 0, \end{cases}$$

and the polynomials in the numerator and denominator are polynomials in $\mathbb{Z}[q_1^{-1}q_2]$. In particular, for $r/s \in \overline{\mathbb{Q}^{\geq 0}}$, we have

$$\begin{cases} \boldsymbol{R}^\sharp_{\mathfrak{q}}(r/s) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{r/s})\big|_{\mathfrak{q}=q_1^{-1}q_2}, \\ \boldsymbol{S}^\sharp_{\mathfrak{q}}(r/s) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{r/s})\big|_{\mathfrak{q}=q_1^{-1}q_2}. \end{cases}$$

Proof. The theorem follows the same way of the left version, and we prove the non-negative case by induction using Setting 3.16. For the starting case, we have

$$\mathrm{Int}(\widehat{\gamma}_\infty, \widehat{\eta}_0) = \mathrm{Int}(\widehat{\gamma}_0, \widehat{\eta}_\infty) = 0,$$
$$\mathrm{Int}(\widehat{\gamma}_\infty, \widehat{\eta}_\infty) = \mathrm{Int}(\widehat{\gamma}_0, \widehat{\eta}_0) = q_1^0 q_2^0 = 1.$$

Thus, they coincide with the right $\mathfrak{q}$-deformation of 0 and ∞, respectively. We assume that the formula (3.11) holds for $p/q, u/v \in \overline{\mathbb{Q}^{\geq 0}}$ in Setting 3.16 which satisfy

$$\mathrm{ind}_z^{\mathbb{Z}^2}(\widehat{\eta}_{p/q}, \widehat{\eta}_{u/v}) = l(1 - \mathbb{X}),$$

where $l = l((p/q) \oplus (u/v)) \in \mathbb{N}$.

As in the left version, the intersection between $\widehat{\gamma}_0$ and $\widehat{\eta}_{(p/q)\oplus(u/v)}$ consist of two parts. One inherits the bi-indices from $\widehat{\eta}_{p/q}$, whose bi-indices are the same; and the other one is induced from $\widehat{\eta}_{u/v}$, whose bi-indices differ from $-\operatorname{ind}_{z''}^{\mathbb{Z}^2}(\widehat{\eta}_{(p/q)\oplus(u/v)}, \widehat{\eta}_{u/v}) = (l+1)\cdot(\mathbb{X}-1)$. Notice that the intersections are all in the interior, which simplifies things a lot. Thus, we have

$$\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{(p/q)\oplus(u/v)}) = \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{p/q}) + q_1^{-l-1}q_2^{l+1}\cdot \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{u/v}),$$

which coincides with the denominator of the right $\mathfrak{q}$-deformation of $(p/q)\oplus(u/v)$ if we take $q_1^{-1}q_2 = \mathfrak{q}$. Similarly, we deduce that

$$\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{(p/q)\oplus(u/v)}) = \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{p/q}) + q_1^{-l-1}q_2^{l+1}\cdot \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{u/v}).$$

Thus, we finish the proof. ∎

Example 3.22. We continue the examples of -2 and $3/2$ (see Figure 13) for the right $\mathfrak{q}$-deformations. We compute that

$$\begin{cases} \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{-2/1}) = 1 + q_1^{-1}q_2, \\ \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{-2/1}) = q_1^{-3}q_2^2; \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{3/2}) = 1 + q_1^{-1}q_2 + q_1^{-2}q_2^2, \\ \operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{3/2}) = 1 + q_1^{-1}q_2. \end{cases}$$

By Theorem 3.21, we have

$$\left[-\frac{2}{1}\right]_{\mathfrak{q}}^{\sharp} = -\frac{q_1^{-1}\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{-2/1})}{\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{-2/1})}\bigg|_{\mathfrak{q}=q_1^{-1}q_2} = -\frac{\mathfrak{q}+1}{\mathfrak{q}^2}$$

and

$$\left[\frac{3}{2}\right]_{\mathfrak{q}}^{\sharp} = \frac{\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_\infty, \widehat{\eta}_{3/2})}{\operatorname{Int}^{\mathfrak{q}}(\widehat{\gamma}_0, \widehat{\eta}_{3/2})}\bigg|_{\mathfrak{q}=q_1^{-1}q_2} = \frac{\mathfrak{q}^2+\mathfrak{q}+1}{\mathfrak{q}+1}.$$

3.5. Combinatorial properties via $\mathfrak{q}$-intersections

Next, we give some topological explanation of some properties in [10] via $\mathfrak{q}$-intersections. We draw γ_0 and γ_∞ as foliations in the branched double cover Σ_Δ, which intersect C_0 and C_∞ only once, respectively (see Figure 15). We obtain the following corollary, where there is a statement much stronger than that of [10].

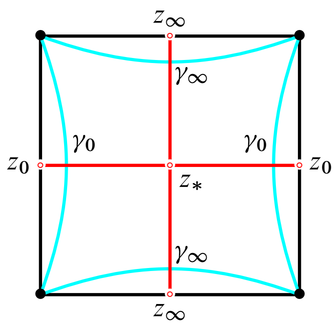

Figure 15. The branched double cover Σ_Δ of $\mathbf{D}_3$ and its foliations.

Corollary 3.23 ([10]). *For any rational number $r/s \in \overline{\mathbb{Q}^{\geq 0}}$, the right and left $\mathfrak{q}$-deformations satisfy the following properties.*

(Positivity) *The polynomials $\boldsymbol{R}_{\mathfrak{q}}^{\sharp}(r/s)$, $\boldsymbol{S}_{\mathfrak{q}}^{\sharp}(r/s)$, $\boldsymbol{R}_{\mathfrak{q}}^{\flat}(r/s)$ and $\boldsymbol{S}_{\mathfrak{q}}^{\flat}(r/s)$ have positive integer coefficients.*

(Specialization) *If we take $\mathfrak{q} = 1$, we have*

$$\begin{cases} \boldsymbol{R}_{\mathfrak{q}}^{\sharp}(r/s)|_{\mathfrak{q}=1} = \boldsymbol{R}_{\mathfrak{q}}^{\flat}(r/s)|_{\mathfrak{q}=1} = r, \\ \boldsymbol{S}_{\mathfrak{q}}^{\sharp}(r/s)|_{\mathfrak{q}=1} = \boldsymbol{S}_{\mathfrak{q}}^{\flat}(r/s)|_{\mathfrak{q}=1} = s. \end{cases}$$

Proof. The positivity follows from Theorem 3.21, Theorem 3.18 and the fact that the intersection numbers are all positive.

For specialization, we consider the branched double covering Σ_Δ of $\mathbf{D}_3$. Then the closed arc $\eta_{r/s}$ becomes the simple closed curve $C_{r/s}$, whose preimage under $\widetilde{\pi}$ is a line with slope r/s, on Σ_Δ through these corresponding decorations which are endpoints of $\eta_{r/s}$. When we take $q_1 = q_2 = 1$, the $\mathfrak{q}$-intersection degenerates to usual intersection. By the construction above, we have

$$\begin{aligned} &\mathrm{Int}_{\mathbf{D}_3}(\eta_{r/s}, \eta_0) = \tfrac{1}{2}\cdot \mathrm{Int}_{\Sigma_\Delta}(C_{r/s}, C_0) = r, \quad \mathrm{Int}_{\mathbf{D}_3}(\eta_{r/s}, \eta_\infty) = \tfrac{1}{2}\cdot \mathrm{Int}_{\Sigma_\Delta}(C_{r/s}, C_\infty) = s; \\ &\mathrm{Int}_{\mathbf{D}_3}(\gamma_\infty, \eta_{r/s}) = \tfrac{1}{2}\cdot \mathrm{Int}_{\Sigma_\Delta}(\gamma_\infty, C_{r/s}) = r, \quad \mathrm{Int}_{\mathbf{D}_3}(\gamma_0, \eta_{r/s}) = \tfrac{1}{2}\cdot \mathrm{Int}_{\Sigma_\Delta}(\gamma_0, C_{r/s}) = s. \end{aligned}$$

Therefore, the results follows from Theorem 3.18 and Theorem 3.21. ∎

Example 3.24. We notice that in the example of $3/s$, $\eta_{3/2}$ hits γ_∞ three times and hits γ_0 twice in $\mathbf{S}_\triangle$, which implies that $\mathbf{R}_{\mathfrak{q}}^{\sharp}(3/2)\ |_{\mathfrak{q}=1} = 3$ and $\mathbf{S}_{\mathfrak{q}}^{\sharp}(3/2)\ |_{\mathfrak{q}=1} = 2$.

4. Categorification

4.1. Ginzburg algebra and derived categories

Definition 4.1 ([4, 6]). Let $Q = (Q_0, Q_1)$ be a finite quiver with set of vertices $Q_0 = \{1, 2, \ldots, n\}$ and arrows set Q_1. The *Ginzburg Calabi–Yau-$\mathbb{X}$ differential bigraded algebra* (dbg algebra for short) $\Gamma_{\mathbb{X}} Q := (\mathbf{k}\overline{Q}, d)$ is defined as follows. We define a $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$-graded quiver $\overline{Q}$ with the same vertices set as Q_0 and the following arrows:

- original arrows $a\colon i \to j \in Q_1$ with degree 0;
- opposite arrows $a^*\colon j \to i \in Q_1$ associated to $a \in Q_1$ with degree $2 - \mathbb{X}$;
- a loop e_i^* for each $i \in Q_0$ with degree $1 - \mathbb{X}$, where e_i is the idempotent at i.

Let $\mathbf{k}\overline{Q}$ be a $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$-graded path algebra of $\overline{Q}$, and define a differential $d\colon \mathbf{k}\overline{Q} \to \mathbf{k}\overline{Q}$ of degree 1 by

- $da = da^* = 0$ for $a \in Q_1$;
- $de_i^* = e_i\big(\sum_{a\in Q_1}(aa^* - a^*a)\big)e_i$.

We denote by $\mathcal{D}_{\mathbb{X}}(Q) := \mathrm{pvd}\,\Gamma_{\mathbb{X}} Q$ the perfect value derived category of $\Gamma_{\mathbb{X}} Q$, which is the same as the finite-dimensional derived category of $\Gamma_{\mathbb{X}} Q$. We consider the A_2 case, where A_2 is a quiver with vertices set $\{1, 2\}$ and an arrow $1 \to 2$, and the corresponding category is denoted by $\mathcal{D}_{\mathbb{X}}(A_2)$.

4.2. Rational case via spherical objects

In this section, we aim to find spherical objects in some category which correspond to rational numbers and represent their hom space via right and left $\mathfrak{q}$-deformations. We particularly consider the case of the Calabi–Yau-$\mathbb{X}$ category of the A_2 quiver. Recall that a triangulated category $\mathcal{D}$ is called *Calabi–Yau-*$\mathbb{X}$ if for any objects L, M in $\mathcal{D}$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(L, M) \cong D\,\mathrm{Hom}_{\mathcal{D}}(M, L[\mathbb{X}]),$$

where $D = \mathrm{Hom}(-, \mathbf{k})$ is the dual functor and $\mathbf{k}$ is an algebraically closed field. In particular, $\mathcal{D}_{\mathbb{X}}(A_2)$ is a Calabi–Yau-$\mathbb{X}$ category with a distinguish auto-equivalence

$$\mathbb{X} : \mathcal{D}_{\mathbb{X}}(A_2) \to \mathcal{D}_{\mathbb{X}}(A_2).$$

Definition 4.2 ([5]). For any $M, N \in \mathcal{D}$, we define the *bigraded Hom* as

$$\mathrm{Hom}^{\mathbb{Z}^2}(M, N) := \bigoplus_{\varrho,\varsigma\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, N[\varrho + \varsigma\mathbb{X}]),$$

and its $\mathfrak{q}$-dimension as

$$\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(M, N) := \sum_{\varrho,\varsigma\in\mathbb{Z}} q_1^{\varrho} q_2^{\varsigma} \cdot \dim \mathrm{Hom}_{\mathcal{D}}(M, N[\varrho + \varsigma\mathbb{X}]). \tag{4.1}$$

When $M = N$, $\mathrm{Hom}^{\mathbb{Z}^2}(M, M)$ becomes a $\mathbb{Z}^2$-graded algebra, called the Ext-algebra of M and denoted by $\mathrm{Ext}^{\mathbb{Z}^2}(M, M)$.

By definition, we directly have

$$\dim \mathrm{Hom}^{\mathbb{Z}^2}(M, N) = \dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(M, N) \mid_{q_1=q_2=1} .$$

Definition 4.3 ([5]). An object S is called $\mathbb{X}$-*spherical* if

$$\mathrm{Hom}^{\bullet}(S, S) = \mathbf{k} \oplus \mathbf{k}[-\mathbb{X}].$$

For any spherical object S in a Calabi–Yau-$\mathbb{X}$ category $\mathcal{D}$, there is an associated auto-equivalence, namely the *twist functor* $\phi_S \colon \mathcal{D} \to \mathcal{D}$, defined by

$$\phi_S(X) = \mathrm{Cone}(S \otimes \mathrm{Hom}^{\bullet}(S, X) \to X)$$

with inverse

$$\phi_S^{-1}(X) = \mathrm{Cone}(X \to S \otimes \mathrm{Hom}^{-\bullet}(X, S))[-1].$$

By Lemma 2.11 in [14], we have the formula

$$\phi_{\psi(M)} = \psi \circ \phi_M \circ \psi^{-1}$$

for any spherical object M and any automorphism ψ in $\mathrm{Aut}\,\mathcal{D}$.

We define $\widehat{\mathrm{Sph}}(\Gamma_{\mathbb{X}} A_2)$ to be the set of all spherical objects in $\mathcal{D}_{\mathbb{X}}(A_2)$ which are simples in some hearts (see Section 10 in [8]). Let

$$\mathrm{Sph}(\Gamma_{\mathbb{X}} A_2) := \widehat{\mathrm{Sph}}(\Gamma_{\mathbb{X}} A_2)/\langle [1], [\mathbb{X}] \rangle.$$

We let $\mathrm{ST}(A_2)$ be the subgroup of $\mathrm{Aut}\,\mathcal{D}_{\mathbb{X}}(A_2)$ generated by ϕ_S for any $S \in \widehat{\mathrm{Sph}}(\Gamma_{\mathbb{X}} A_2)$.

Let $\mathbf{D}_3$ be the three decorated disk as before. There are reachable spherical objects up to shifts [1] and $\mathbb{X}$ in $\mathcal{D}_{\mathbb{X}}(A_2)$ corresponding to rational numbers. We have a categorification of Lemma 3.4 as follows.

Proposition 4.4 (Section 4 of [5]). *There are a bijection X and an isomorphism ι which fit into the following:*

$$\begin{tikzcd} \mathrm{BT}(\mathbf{D}_3) \arrow[r, "\iota"] & \mathrm{ST}(A_2) \\ \widehat{\mathrm{CA}}(\mathbf{D}_3) \arrow[loop above] \arrow[r, "X"] & \widehat{\mathrm{Sph}}(\Gamma_{\mathbb{X}} A_2) \arrow[loop above] \end{tikzcd}$$

sending $\widehat{\eta}_{\pm r/s}$ to $X_{\pm r/s}$ and $B_{\eta_{\pm r/s}}$ to $\phi_{X_{\pm r/s}}$, and satisfying

$$X_{B_{\eta_{u/v}}(\widehat{\eta}_{p/q})} = \phi_{X_{u/v}}(X_{p/q}) \quad \text{and} \quad X_{B_{\eta_{-p/q}}(\widehat{\eta}_{-u/v})} = \phi_{X_{-p/q}}(X_{-u/v})$$

for $r/s \in \overline{\mathbb{Q}^{\geq 0}}$. Hence, (3.7) *translates to the triangle*

$$X_{p/q} \longrightarrow X_{(p/q)\oplus(u/v)} \longrightarrow X_{u/v}[(l+1)(1-\mathbb{X})] \longrightarrow X_{p/q}[1], \tag{4.2}$$

where $l = l((p/q) \oplus (u/v))$ is the integer in Setting 3.16.

In fact, when we draw $\{X_{r/s}\}_{r/s \in \overline{\mathbb{Q}^{\geq 0}}}$ on the weighted Farey graph in the right picture of Figure 16, $\mathfrak{q}^l$, $l \geq 0$, represents that there is a morphism of degree $l(1-\mathbb{X})$ between the connected spherical objects.

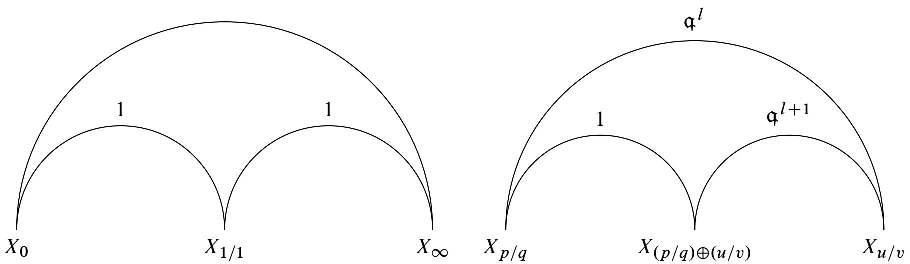

Figure 16. The categorification.

Here is the iterative construction of Proposition 4.4. Let X_0 and X_∞ be two spherical objects which are simple in some canonical heart with $\mathrm{Ext}^1(X_\infty, X_0) \neq 0$. Let $X_{1/1} = \phi_{X_\infty}(X_0)$; we deduce that $X_{1/1}$ is also a spherical object. We have a triangle

$$X_0 \longrightarrow X_{1/1} \longrightarrow X_\infty \longrightarrow X_0[1]$$

by construction. We assume that the triangle in (4.2) holds for l and consider the $l+1$ case. By the Calabi–Yau-$\mathbb{X}$ duality, we have non-zero morphisms $X_{(p/q)\oplus(u/v)} \to X_{p/q}[\mathbb{X}]$ and $X_{u/v} \to X_{(p/q)\oplus(u/v)}[(l+1)\mathbb{X} - l]$. Hence we can extend them to triangles:

$$X_{p/q} \longrightarrow Y \longrightarrow X_{(p/q)\oplus(u/v)}[1-\mathbb{X}] \longrightarrow X_{p/q}[1]$$

and

$$X_{(p/q)\oplus(u/v)} \longrightarrow Y' \longrightarrow X_{u/v}[(l+1)(1-\mathbb{X})] \longrightarrow X_{(p/q)\oplus(u/v)}[1],$$

where these new spherical objects are

$$Y = \phi_{X_{(p/q)\oplus(u/v)}[1-\mathbb{X}]}(X_{p/q}) = \phi_{X_{(p/q)\oplus(u/v)}}(X_{p/q})$$

and

$$Y' = \phi_{X_{u/v}[(l+1)(1-\mathbb{X})]}(X_{(p/q)\oplus(u/v)}) = \phi_{X_{u/v}}(X_{(p/q)\oplus(u/v)}).$$

Thus we construct the spherical objects associated to non negative rational numbers and the negative case is similar.

Theorem 4.5 (Lemma 3.4 and Proposition 4.6 in [5]). *For any $\widehat{\eta}, \widehat{\eta}' \in \widehat{\mathrm{CA}}(\mathbf{D}_3)$ satisfying $\mathrm{Int}_{\mathbf{D}_3^\circ}(\widehat{\eta}, \widehat{\eta}') = 0$, and $\widehat{\gamma}_i \in \mathbf{A}$, we have*

$$\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(P_i, X_{\widehat{\eta}}) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\gamma}_i, \widehat{\eta})$$

and

$$\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(X_{\widehat{\eta}}, X_{\widehat{\eta}'}) = \mathrm{Int}^{\mathfrak{q}}(\widehat{\eta}, \widehat{\eta}'),$$

where P_i is the indecomposable projective module corresponding to $\widehat{\gamma}_i$.

By Theorems 3.18, 3.21 and 4.5, we have the following direct corollaries.

Corollary 4.6. *For any rational number $r/s \in \mathbb{Q} \setminus \{0\}$, the fraction*

$$\left.\frac{\varepsilon \dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(X_{r/s}, X_0)}{\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(X_{r/s}, X_\infty)}\right|_{\mathfrak{q}=q_1^{-1}q_2} \tag{4.3}$$

corresponds to the left $\mathfrak{q}$-deformation of r/s, where

$$\varepsilon = \begin{cases} q_1^{-1}, & \text{if } r/s \geq 0, \\ -1, & \text{if } r/s < 0. \end{cases}$$

Corollary 4.7. *For any rational number $r/s \in \overline{\mathbb{Q}}$, the fraction*

$$\left.\frac{\varepsilon \dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(P_\infty, X_{r/s})}{\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(P_0, X_{r/s})}\right|_{\mathfrak{q}=q_1^{-1}q_2} \tag{4.4}$$

corresponds to the right $\mathfrak{q}$-deformation of r/s, where P_0 and P_∞ are the corresponding indecomposable projectives satisfying that $\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(P_i, X_j) = \delta_{i,j}, i, j \in \{0, \infty\}$. Here

$$\varepsilon = \begin{cases} 1, & \text{if } r/s \geq 0, \\ -q_1^{-1}, & \text{if } r/s < 0. \end{cases}$$

5. Applications

5.1. Reduction to single grading as foliations

Let $N \geq 2$ be an integer. We collapse the double grading Λ on $\mathbf{S}_\triangle$ to a single grading λ, which is a line field (or foliation) in $\mathbb{P}T\mathbf{S}_\triangle$ by setting $\mathbb{X} = N$. More precisely, a double grading $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$ collapses into $a + bN \in \mathbb{Z}$. The foliations in such cases are given by quadratic differentials

$$(z^3 + az + b)^{N-2}\, \mathrm{d}z^{\otimes 2}$$

on $\mathbb{CP}^1$ with real blow-up at ∞, cf. Figure 17 for $N = 2, 3, 4$ and $\mathbf{S}_\triangle = \mathbf{D}_3$. Notice that the foliations come from quadratic differential (cf. [2, 3]). Then $q_2 = q_1^N$ and the $\mathfrak{q}$-intersection formula (3.4) reduces to

$$\mathrm{Int}^{\mathfrak{q}}(\widehat{\sigma}, \widehat{\tau}) = \sum_{k \in \mathbb{Z}} q_1^k \cdot \mathrm{Int}_\triangle^k(\widehat{\sigma}, \widehat{\tau}) + (1 + q_1^{N-1}) \sum_{k \in \mathbb{Z}} q_1^k \cdot \mathrm{Int}_{\mathbf{S}_\triangle^\circ}^k(\widehat{\sigma}, \widehat{\tau}).$$

for $\widehat{\sigma}, \widehat{\tau} \in \widehat{\mathrm{CA}}(\mathbf{S}_\triangle)$. Moreover, $\mathfrak{q} = q_1^{N-1}$ in Theorem 3.18 and Theorem 3.21. When $N = 2$, $\mathfrak{q} = q_1$ and no specialization is required.

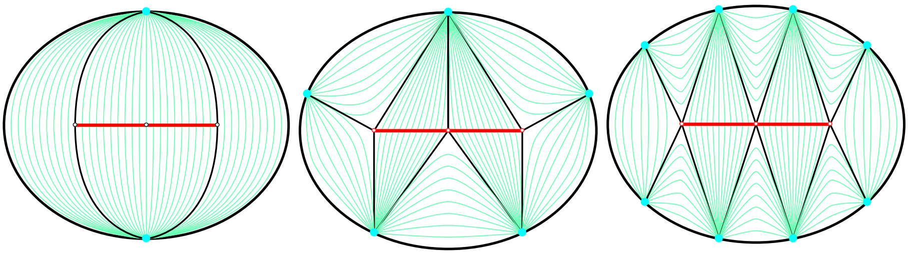

Figure 17. The foliations of the CY-2,3,4 case.

5.2. Relation with BBL's results

In Definition 4.1, if we replace $\mathbb{X}$ by an integer $N \geq 2$, we obtain the N-Ginzburg dg algebra $\Gamma_N Q$ and the corresponding Calabi–Yau-N category $\mathcal{D}_N(Q)$. That is, there is a projection

$$\pi_N : \Gamma_{\mathbb{X}} Q \to \Gamma_N Q$$

collapsing the double grading $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$ into $\mathbb{Z}$ by setting $\mathbb{X} = N$ similar as above. It induces a functor $\pi_N \colon \mathcal{D}_{\mathbb{X}}(Q) \to \mathcal{D}_N(Q)$. We consider the case when $Q = A_2$. For $r/s \in \overline{\mathbb{Q}^{\geq 0}}$, we claim that

$$\dim^{\mathfrak{q}} \mathrm{Hom}^{\mathbb{Z}^2}(P_?, X_{r/s}) = \sum_{k \in \mathbb{Z}} m\left(\frac{r}{s}, k\right) q_1^{-k} q_2^k,$$

where $m(r/s,k)$ is the occurrence times of $X_?[k-k\mathbb{X}]$ in the HN-filtration of $X_{r/s}$, $? \in \{0,\infty\}$. This follows from the fact that

$$\dim^{\mathfrak{q}} \operatorname{Hom}^{\mathbb{Z}^2}(P_i, X_j[\varrho + \varsigma\mathbb{X}]) = \delta_{i,j}\,\delta_{\varrho,0}\,\delta_{\varsigma,0} \tag{5.1}$$

where $i,j \in \{0,\infty\}$ and induction on $l(r/s)$.

Consider the case when $N=2$ again with $\mathfrak{q}=q_1$, and write $q_1=q$. In [1], they define two functionals

$$\operatorname{occ}_q, \overline{\operatorname{hom}}_q : \widehat{\operatorname{Sph}}^{\mathbb{Z}}(\Gamma_2 A_2) \times \widehat{\operatorname{Sph}}^{\mathbb{Z}}(\Gamma_2 A_2) \to \mathbb{Z}[q^{-1},q],$$

where $\widehat{\operatorname{Sph}}^{\mathbb{Z}}(\Gamma_2 A_2)$ is the set of spherical objects in $\mathcal{D}_2(A_2)$. The first one, $\operatorname{occ}_q(X_?, X_{r/s})$, counts the occurrences of $X_?$ in the HN-filtration of $X_{r/s}$ for $? \in \{0,\infty\}$. By (5.1), we deduce that

$$\operatorname{occ}_q(X_?, X) = \dim^{\mathfrak{q}} \operatorname{Hom}^{\mathbb{Z}^2}(P_?, X)\ |_{q_1^{-1}q_2=q^{-1}}\ .$$

The second one is

$$\overline{\operatorname{hom}}_q(L,M) := \begin{cases} q^k(q^{-2}-q^{-1}), & \text{if } M \cong L[k], \\ \sum_{k\in\mathbb{Z}} \dim \operatorname{Hom}(L, M[k])q^{-k}, & \text{otherwise.} \end{cases}$$

Recall, from Remark 2.4, that the q-deformations for negative rational numbers are defined as

$$\left[-\frac{r}{s}\right]_q^* := -q^{-1}\left[\frac{r}{s}\right]_{q^{-1}}^*,$$

where $r/s \in \mathbb{Q}^+ \cup \{\infty\}$ and $* \in \{\sharp, \flat\}$.

Corollary 5.1. *When specializing $\mathbb{X}=2$, the formulae (4.4) and (4.3) in Corollary 4.7 and Corollary 4.6 become the formulae in Theorem 3.7 and Theorem 3.8 of [1], respectively. Notice that the condition $X \geq 0$ corresponds to our $X_{-r/s}$ with $r/s \geq 0$.*

5.3. Grothendieck group interpretation

Recall that $\mathcal{D}_{\mathbb{X}}(A_2)$ is a Calabi–Yau-$\mathbb{X}$ category. The Grothendieck group $\mathrm{K}(\mathcal{D}_{\mathbb{X}}(A_2))$ admits a basis $\{[X_0],[X_\infty]\}$ and is a $\mathbb{Z}[\mathfrak{q}^{\pm 1}]$-module defined by the action

$$q_1^l q_2^k \cdot [E] := [E[-l-k\mathbb{X}]].$$

We have the following result.

Proposition 5.2. *For any $r/s \in \overline{\mathbb{Q}}$, we have*

$$[X_{r/s}] = \boldsymbol{R}^{\sharp}_{q_1^{-1}q_2}\left(\frac{r}{s}\right)[X_0] + \boldsymbol{S}^{\sharp}_{q_1^{-1}q_2}\left(\frac{r}{s}\right)[X_\infty],$$

where $\boldsymbol{R}^{\sharp}_{q_1^{-1}q_2}$ (respectively, $\boldsymbol{S}^{\sharp}_{q_1^{-1}q_2}$) is a polynomial of $q_1^{-1}q_1$ if we take $\mathfrak{q}=q_1^{-1}q_1$ in $\boldsymbol{R}^{\sharp}_{\mathfrak{q}}$ (respectively, $\boldsymbol{S}^{\sharp}_{\mathfrak{q}}$).

Proof. We only prove the non-negative case by induction on $l(r/s)$. For the initial case, it holds obviously for X_0 and X_∞. We assume that it holds for $X_{p/q}$ and $X_{u/v}$, where p/q and u/v are in Setting 3.16. For $X_{(p/q)\oplus(u/v)}$, we have

$$\begin{aligned}
[X_{(p/q)\oplus(u/v)}] &= [X_{p/q}] + [X_{u/v}[(l+1)(1-\mathbb{X})]] = [X_{p/q}] + q_1^{-l-1}q_2^{l+1}[X_{u/v}] \\
&= \mathbf{R}^\sharp_{q_1^{-1}q_2}\Big(\frac{p}{q}\Big)[X_0] + \mathbf{S}^\sharp_{q_1^{-1}q_2}\Big(\frac{p}{q}\Big)[X_\infty] + q_1^{-l-1}q_2^{l+1}\Big(\mathbf{R}^\sharp_{q_1^{-1}q_2}\Big(\frac{u}{v}\Big)[X_0] + \mathbf{S}^\sharp_{q_1^{-1}q_2}\Big(\frac{u}{v}\Big)[X_\infty]\Big) \\
&= \mathbf{R}^\sharp_{q_1^{-1}q_2}\Big(\frac{p}{q}\oplus\frac{u}{v}\Big)[X_0] + \mathbf{S}^\sharp_{q_1^{-1}q_2}\Big(\frac{p}{q}\oplus\frac{u}{v}\Big)[X_\infty],
\end{aligned}$$

which implies the result. ■

5.4. Relation to Jones polynomials for rational case

For every rational number $r/s > 1$ with continued fraction expansion $[a_1, a_2, \ldots, a_{2m}]$, there is an associated rational (two-bridge) knot/link $C(r/s)$ defined as follows (see Section 4 of [9] for more details). The knot or link is composed of $2m$ segments, each representing a 2-strand braid with a_i crossings, where $i = 1, 2, \ldots, n$. These segments are connected in a manner that ensures the link is alternating. We refer to Figure 18 for an illustrative example.

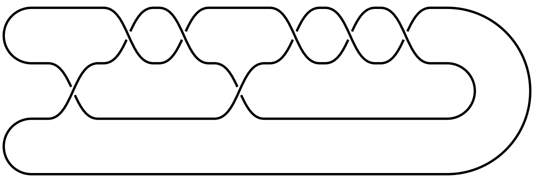

Figure 18. The knot associated to $15/11 = [1, 2, 1, 3]$.

For an oriented link, we define the associated *Jones polynomial* recursively. Specifically, the Jones polynomial of the unknot is 1. For any three oriented links L_-, L_+ and L_0 that are identical except in the neighborhood of a point where they appear as shown in Figure 19, they satisfy the following skein relation:

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

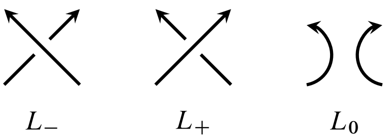

Figure 19. Definition of the Jones polynomial.

There are two ways to orient a knot or each component of a link. If we fix the orientation, then there is a natural orientation for $C(r/s)$ as described in Section 4.1 of [9]. The associated Jones polynomial is denoted by $V_{r/s}(t)$. By multiplying $V_{r/s}(t)$ by an appropriate power of t so that the highest degree term becomes constant, and setting $\mathfrak{q} = -t^{-1}$, we obtain a polynomial $J_{r/s}(\mathfrak{q}) \in \mathbb{Z}[\mathfrak{q}]$, called the *normalized Jones polynomial*. The following corollary holds.

Corollary 5.3. *For every rational $r/s > 1$, we have*

$$J_{r/s}(\mathfrak{q}) = \left| q_1^{-1} \operatorname{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_0)|_{\mathfrak{q}=q_1^{-1}q_2} \right|,$$

where $|\cdot|$ is the normalized polynomial of $\mathfrak{q}$ with lowest non-zero constant term.

Proof. By Theorem A.3 in [1] and Theorem 3.18, we have that

$$J_{r/s}(\mathfrak{q}) = \mathbf{R}^{\flat}_{\mathfrak{q}}(r/s) = \left| q_1^{-1} \operatorname{Int}^{\mathfrak{q}}(\widehat{\eta}_{r/s}, \widehat{\eta}_0)|_{\mathfrak{q}=q_1^{-1}q_2} \right|. \qquad \blacksquare$$

Example 5.4. We have the following examples.

- For the Hopf link $C(2)$ in Figure 20, we have $V_2(t) = -t^{5/2}(1 + t^{-2})$ and $\mathbf{R}^{\flat}_{\mathfrak{q}}(2) = 1 + q^2$.
- For th eleft-handed Trefoil $C(3)$ in Figure 21, we have $V_3(t) = t^{-1}(1 + t^{-2} - t^{-3})$ and $\mathbf{R}^{\flat}_{\mathfrak{q}}(3) = 1 + q^2 + q^3$.

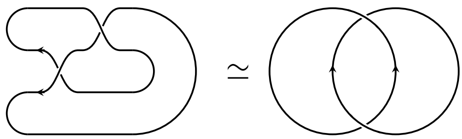

Figure 20. The Hopf link, which is associated to $2 = [1, 1]$.

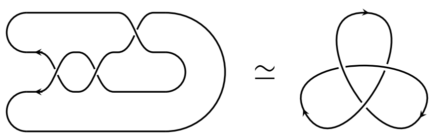

Figure 21. The left-handed Trefoil, which is associated to $3 = [2, 1]$.

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