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On the Cauchy problem for logarithmic fractional Schrödinger equation

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Abstract. We consider the fractional Schrödinger equation with a logarithmic nonlinearity, when the power of the Laplacian is between zero and one. We prove global existence results in three different functional spaces: the Sobolev space corresponding to the quadratic form domain of the fractional Laplacian, the energy space, and a space contained in the operator domain of the fractional Laplacian. For this last case, a finite momentum assumption is made, and the key step consists in estimating the Lie commutator between the fractional Laplacian and the multiplication by a monomial.

1. Introduction

We consider the logarithmic Schrödinger equation

$$i\partial_t u - (-\Delta)^s u = \lambda \log(|u|^2)u, \quad u_{|t=0} = u_0, \tag{1.1}$$

where $0 < s < 1$, $u = u(t,x)$ represents a complex-valued function defined on $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, with $d \geq 1$. The fractional Laplacian $(-\Delta)^s$ is defined through the Fourier transform as follows:

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s}\mathcal{F}u(\xi),$$

where the Fourier transform is given by

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} u(x)e^{-i\xi\cdot x}\,dx.$$

The fractional Laplacian $(-\Delta)^s$ is a self-adjoint operator acting on the space $L^2(\mathbb{R}^d)$, characterized by a quadratic form domain $H^s(\mathbb{R}^d)$ and an operator domain $H^{2s}(\mathbb{R}^d)$. The nonlocal operator $(-\Delta)^s$ serves as the infinitesimal generators in the context of Lévy stable diffusion processes, as outlined in [2]. Fractional derivatives of the Laplacian have applications in numerous equations in mathematical physics

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and related disciplines, as proposed in [28,29] in the case of linear Schrödinger equations; see also [2, 16, 22] and the associated references. Recently, there has been a strong focus on studying mathematical problems related to the fractional Laplacian purely from a mathematical perspective. Regarding specifically fractional nonlinear Schrödinger equations, important progress has been made in, e.g., [4,6,12,13,18–21].

The problem (1.1) does not seem to have physical motivations (so far), and was introduced in [15] as a generalization of the case $s = 1$, introduced in [5], and proposed in different physical contexts since (see, e.g., [25, 33]). Note also that the logarithmic nonlinearity may be obtained as the limit of a homogeneous nonlinearity $\lambda|u|^{2\sigma}u$ when σ goes to zero, at least when ground states are considered (case $\lambda < 0$; see [31] for $s = 1$, [1] in the fractional case).

In [3], the author addresses the nonlinear fractional logarithmic Schrödinger equation (1.1) with $\lambda = -1$ and $d \geq 2$, employing a compactness method to establish a unique global solution for the associated Cauchy problem within a suitable functional framework, inspired by [11] (for the logarithmic nonlinearity) and [13] (for the fractional Laplacian). In [32], the authors investigate the existence of a global weak solution to the problem (1.1) in the case of $\lambda = -1$, when the space variable x belongs to some smooth bounded domain, by using a combination of potential wells theory and the Galerkin method. In this paper, we complement the approach from [3,32] by adapting the strategy employed in [24] in the case of the standard Laplacian, $s = 1$.

Formally, (1.1) enjoys the conservations of mass, momentum, and energy,

$$\begin{aligned}
&M(u(t)) = \|u(t)\|^2_{L^2(\mathbb{R}^d)},\\
&J(u(t)) = \operatorname{Im}\int_{\mathbb{R}^d} \overline{u}(t,x)\nabla u(t,x)dx,\\
&E(u(t)) = \frac{1}{2}\|(-\Delta)^{s/2}u(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{\lambda}{2}\int_{\mathbb{R}^d} |u(t,x)|^2(\log|u(t,x)|^2 - 1)dx.
\end{aligned}\tag{1.2}$$

The energy is well defined in the subset of $H^s(\mathbb{R}^d)$,

$$W_1^s := \{u \in H^s(\mathbb{R}^d),\ x \mapsto |u(x)|^2 \log|u(x)|^2 \in L^1(\mathbb{R}^d)\}.$$

When $s = 1$, Hayashi and Ozawa [24] revisit the Cauchy problem for the logarithmic Schrödinger equation, constructing strong solutions in both H^1 and $W_1 = W_1^1$. This approach deliberately avoids relying on compactness arguments, demonstrating the convergence of a sequence of approximate solutions in a complete function space. The authors in [24] also address the existence in the H^2-energy space, as discussed below.

The main contributions of this paper can be summarized as follows:

(1) Construction of H^s strong solutions, without relying on the conservation of the energy.

(2) Construction of solutions in the energy space W_1^s.

(3) The higher H^{2s} regularity is established by assuming some further spatial decay of the initial data.

In all cases, no sign assumption is made on $\lambda \in \mathbb{R}$.

Theorem 1.1. *Let $\lambda \in \mathbb{R}$ and $0 < s < 1$. For any $\varphi \in H^s(\mathbb{R}^d)$, there exists a unique solution $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ to (1.1) in the sense of*

$$i\partial_t u - (-\Delta)^s u = \lambda \log(|u|^2)u \quad \text{in } H^{-s}(\Omega) \tag{1.3}$$

for all bounded open sets $\Omega \subset \mathbb{R}^d$ and all $t \in \mathbb{R}$, and with $u_{|t=0} = \varphi$. If in addition we assume $\varphi \in W_1^s$, this H^s-solution satisfies $u \in (C \cap L^\infty)(\mathbb{R}, W_1^s)$ if $\lambda < 0$ and $u \in C(\mathbb{R}, W_1^s)$ if $\lambda > 0$. Moreover, the W_1^s-solution u satisfies equation (1.3) in the sense of $(W_1^s)^*$, where $(W_1^s)^*$ is the dual space of W_1^s. Finally, if $\varphi \in H^1(\mathbb{R}^d)$, then the solution $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ to (1.1) satisfies in addition $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$.

The next result addresses on the construction of strong solutions in W_2^s, where

$$W_2^s := \left\{u \in H^{2s}(\mathbb{R}^d),\ x \mapsto u(x)\log|u(x)|^2 \in L^2(\mathbb{R}^d)\right\},$$

and this space is the natural counterpart of the space W_2 of the H^2-energy space introduced in [24] for the case $s = 1$. Note that considering this space is interesting especially when $s > 1/2$, since we have seen in Theorem 1.1 that the H^1 regularity is propagated, and $H^1(\mathbb{R}^d) \subset H^{2s}(\mathbb{R}^d)$ when $s \leq 1/2$.

In the fractional case, it seems delicate to adapt the strategy introduced in [24], as some algebraic structure is lost. More precisely, the strategy in [24] starts by showing that $\partial_t u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2)$, to eventually conclude that $\Delta u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2)$. At this level of generality, this is the standard approach, as presented in, e.g., [10], but the logarithmic nonlinearity actually requires some special care. The above line of reasoning needs, as an intermediary step, to know that $u \log|u|^2 \in L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2)$, which is by no means obvious, due to the region $\{|u| < 1\}$ where the nonlinearity is morally sublinear. This difficulty is overcome in [24] by a beautiful algebraic identity ([24, Lemma 3.3]), whose derivation involves an integration by parts in the term

$$\mathrm{Re}(\Delta u, u\log(|u| + \varepsilon))_{L^2} = -\mathrm{Re}\left(\overline{u}\nabla u, \frac{\nabla|u|}{|u| + \varepsilon}\right)_{L^2} + (|\nabla u|^2, \log(|u| + \varepsilon))_{L^2}.$$

In the present case, we would face

$$\mathrm{Re}((-\Delta)^s u, u\log(|u| + \varepsilon))_{L^2},$$

and the integration by parts would require to control a fractional derivative of $u\log(|u|+\varepsilon)$, at least in the case $s<1/2$ (for $s>1/2$, one could consider the gradient again).

To overcome this issue, we adopt the approach considered in [7] for the case $s=1$, and rely on some finite momentum assumption. For $0<\alpha\le 1$, we have

$$\mathcal{F}(H^\alpha)=\left\{u\in L^2(\mathbb{R}^d),\ x\mapsto \langle x\rangle^\alpha u(x)\in L^2(\mathbb{R}^d)\right\},$$

where $\langle x\rangle := \sqrt{1+|x|^2}$, and this space is equipped with the norm

$$\|u\|_{\mathcal{F}(H^\alpha)} := \|\langle x\rangle^\alpha u(x)\|_{L^2(\mathbb{R}^d)}.$$

Denote, for $\alpha>0$, $X^{2s}_\alpha := H^{2s}\cap\mathcal{F}(H^\alpha)$: for any $\alpha>0$, $X^{2s}_\alpha\subset W^s_2$, as can be seen from the estimate, valid for any $\delta\in(0,1)$,

$$\left|u\log(|u|^2)\right|\lesssim |u|^{1-\delta}+|u|^{1+\delta}.$$

Theorem 1.2. *Let $\lambda\in\mathbb{R}$, $0<s<1$. Consider $0<\alpha<2s$ with $\alpha\le 1$. For any $\varphi\in X^{2s}_\alpha = H^{2s}\cap\mathcal{F}(H^\alpha)$, there exists a unique solution $u\in C_w\cap L^\infty_{\rm loc}(\mathbb{R},X^{2s}_\alpha)$ to (1.1) in the sense of*

$$i\partial_t u-(-\Delta)^s u=\lambda\log(|u|^2)u\quad \text{in } L^2(\Omega), \tag{1.4}$$

for all bounded open sets $\Omega\subset\mathbb{R}^d$ and a.e. $t\in\mathbb{R}$, with $u_{|t=0}=\varphi$. Moreover, when $\lambda<0$, $u\in C(\mathbb{R},X^{2s}_\alpha)$ and (1.4) holds in $L^2(\mathbb{R}^d)$ and for all $t\in\mathbb{R}$.

The new difficulty in proving the above result, compared to the case $s=1$, lies in the fact that the Lie bracket $[(-\Delta)^s,\langle x\rangle^\alpha]$ requires some extra care; see Lemma 2.3.

We underline the fact that we do not know whether the H^σ regularity is propagated by the flow of (1.1), when $\sigma=2s$ for $s>1/2$, like in the case of the regular Laplacian $s=1$.

Notations.

- $\int f$ is employed in place of $\int_{\mathbb{R}^d} f(x)dx$.
- The inner product in L^2 is denoted by

$$(f,g)_{L^2}=\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx=\int f\overline{g}.$$

- Let $C(I,X)$ (resp. $C_w(I,X)$) be the space strongly (resp. weakly) continuous functions from interval I ($\subseteq\mathbb{R}$) to X.
- Abbreviated notation: for $T>0$, we write

$$C_T(X)=C([-T,T],X),\quad L^\infty_T(X)=L^\infty((-T,T),X).$$

- $A\lesssim B$ represents the inequality $A\le CB$ with some constant $C>0$.

Content. The rest of the paper is organized as follows. In Section 2, we collect lemmas which are of constant use in this paper. Section 3 is dedicated to the study of the Cauchy problem for (1.1) in both H^s and W^1_s, proving Theorem 1.1. In Section 4, we consider higher regularity and prove Theorem 1.2.

2. Useful lemmas

The following lemma is a generalization of the inequality proven initially by Cazenave and Haraux [11] in the case $\varepsilon = \mu = 0$.

Lemma 2.1 ([24, Lemma A.1]). *For all $u, v \in \mathbb{C}$ and $\varepsilon, \mu \geq 0$, we have*

$$\left|\operatorname{Im}(u \log(|u| + \varepsilon) - v \log(|v| + \mu))(\overline{u} - \overline{v})\right| \leq |u - v|^2 + |\varepsilon - \mu||u - v|.$$

We will also use several times the fractional Leibniz rule. We state a simplified version of a result from [30], by using the fact that BMO contains L^∞, and considering only the L^2 setting.

Lemma 2.2 ([30, Corollary 1.4]). *For $\sigma > 0$, let A^σ be a differential operator such that its symbol $\widehat{A^\sigma}(\xi)$ is homogeneous of degree σ and $\widehat{A^\sigma}(\xi) \in C^\infty(\mathbb{S}^{d-1})$.*

- *If $0 < \sigma < 1$,*
$$\|A^\sigma(fg) - gA^\sigma f\|_{L^2} \lesssim \|f\|_{L^2}\|(-\Delta)^{\sigma/2} g\|_{L^\infty}.$$
- *If $1 \leq \sigma < 2$,*
$$\|A^\sigma(fg) - gA^\sigma f - \nabla g \cdot A^{\sigma,\nabla} f\|_{L^2} \lesssim \|f\|_{L^2}\|(-\Delta)^{\sigma/2} g\|_{L^\infty}$$
where $\widehat{A^{\sigma,\nabla} g}(\xi) = -i\nabla_\xi(\widehat{A^\sigma}(\xi))\widehat{g}(\xi)$.

We recall that the characterization of the H^s norm, for $0 < s < 1$, can be expressed as follows (see, e.g., [17]):

$$\begin{aligned}\|f\|^2_{H^s} &= \|f\|^2_{L^2} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} dy dx \\ &= \|f\|^2_{L^2} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x + y) - f(x)|^2}{|y|^{d+2s}} dy dx.\end{aligned}$$

We also have, for $0 < s < 1$ and $f \in \mathcal{S}(\mathbb{R}^d)$ (see, e.g., [17]),

$$(-\Delta)^s f(x) = c(d, s) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x + y) + f(x - y) - 2f(x)}{|y|^{d+2s}} dy dx, \tag{2.1}$$

for some constant $c(d, s)$ whose exact value is irrelevant here.

The following lemma will be crucial in the proof of Theorem 1.2.

Lemma 2.3. *Let* $0 < s < 1$*. If* $0 < \alpha < 2s$ *and* $\alpha \le 1$*, then the commutator* $[(-\Delta)^s, \langle x\rangle^\alpha]$ *is continuous from* $H^s(\mathbb{R}^d)$ *to* $L^2(\mathbb{R}^d)$*.*

Proof. The proof relies on the fractional Leibniz rule stated in Lemma 2.2, with $A^\sigma = (-\Delta)^s$, hence $\sigma = 2s$. Fix $f \in C_c^\infty(\mathbb{R}^d)$, and let $g(x) = \langle x\rangle^\alpha$.

We first show that under the assumptions of the lemma, $(-\Delta)^s g \in L^\infty(\mathbb{R}^d)$, by using the characterization (2.1). In the region $\{|y| \ge 1\}$, we write, since $0 < \alpha \le 1$,

$$|\langle x \pm y\rangle^\alpha - \langle x\rangle^\alpha| \le |\langle x \pm y\rangle - \langle x\rangle|^\alpha \lesssim |y|^\alpha,$$

hence

$$\left|\int_{|y|\ge 1} \frac{\langle x+y\rangle^\alpha + \langle x-y\rangle^\alpha - 2\langle x\rangle^\alpha}{|y|^{d+2s}} dy\right| \lesssim \int_{|y|\ge 1} \frac{|y|^\alpha}{|y|^{d+2s}} dy < \infty,$$

provided that $\alpha < 2s$. In the ball $\{|y| < 1\}$, Taylor's formula yields

$$\langle x+y\rangle^\alpha + \langle x-y\rangle^\alpha - 2\langle x\rangle^\alpha = \langle \nabla^2 g(x) y, y\rangle + \mathcal{O}(|y|^3),$$

where the remainder is uniform in $x \in \mathbb{R}^d$, as the third order derivatives of g are bounded. Also, the Hessian of g is bounded since $|\nabla^2 g(x)| \lesssim \langle x\rangle^{\alpha-2}$, and

$$\left|\int_{|y|\le 1} \frac{\langle x+y\rangle^\alpha + \langle x-y\rangle^\alpha - 2\langle x\rangle^\alpha}{|y|^{d+2s}} dy\right| \lesssim \int_{|y|\le 1} \frac{|y|^2}{|y|^{d+2s}} dy < \infty,$$

since $s < 1$.

First case: $0 < s < 1/2$. In view of the first case in Lemma 2.2,

$$\|[(-\Delta)^s, \langle x\rangle^\alpha] f\|_{L^2} \lesssim \|f\|_{L^2} \|(-\Delta)^s g\|_{L^\infty} \lesssim \|f\|_{L^2},$$

and $[(-\Delta)^s, \langle x\rangle^\alpha]$ is continuous from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

Second case: $1/2 \le s < 1$. In view of the second case in Lemma 2.2,

$$\|[(-\Delta)^s, \langle x\rangle^\alpha] f\|_{L^2} \lesssim \|\nabla g \cdot A^{2s,\nabla} f\|_{L^2} + \|f\|_{L^2} \|(-\Delta)^s g\|_{L^\infty}.$$

In view of the definition of $A^{2s,\nabla}$, with $A^{2s} = (-\Delta)^s$,

$$\|A^{2s,\nabla} f\|_{L^2} \lesssim \|f\|_{\dot{H}^{2s-1}} \lesssim \|f\|_{H^s},$$

since $0 < s < 1$. Recalling that since $\alpha \le 1$, $\nabla g \in L^\infty$, the lemma is proved. ∎

3. The Cauchy problem in H^s and the energy space

In this section, we prove Theorem 1.1, by resuming the strategy of [24], which requires very few adaptations to treat this fractional case (essentially, the fractional Leibniz rule).

3.1. Approximate problems

For $\varepsilon > 0$, we consider the approximate equation

$$i\partial_t u_\varepsilon - (-\Delta)^s u_\varepsilon = 2\lambda u_\varepsilon \log(|u_\varepsilon| + \varepsilon), \quad u_\varepsilon(0, x) = \varphi(x). \tag{3.1}$$

We set

$$g(u) = 2u \log |u|, \quad g_\varepsilon(u) = 2u \log(|u| + \varepsilon).$$

For $\sigma \geq 0$ we have

$$\int_0^\sigma g_\varepsilon(\tau) d\tau = \frac{1}{2}\sigma^2 \log((\sigma + \varepsilon)^2) - \frac{1}{2}\int_0^\sigma \frac{2\tau^2}{\tau + \varepsilon} d\tau.$$

We define $G_\varepsilon(u)$ by

$$G_\varepsilon(u) = \frac{1}{2}\int |u|^2 \log((|u| + \varepsilon)^2) - \frac{1}{2}\int \mu_\varepsilon(|u|), \quad \text{for } u \in H^s(\mathbb{R}^d),$$

where

$$\mu_\varepsilon(\sigma) := \int_0^\sigma \frac{2\tau^2}{\tau + \varepsilon} d\tau, \quad \text{for } \sigma \geq 0.$$

We define $E_\varepsilon(u)$ by

$$\begin{aligned} E_\varepsilon(u) &= \frac{1}{2}\int |(-\Delta)^{s/2} u|^2 + \lambda G_\varepsilon(u) \\ &= \frac{1}{2}\int |(-\Delta)^{s/2} u|^2 + \frac{\lambda}{2}\int |u|^2 \log((|u| + \varepsilon)^2) - \frac{\lambda}{2}\int \mu_\varepsilon(|u|). \end{aligned} \tag{3.2}$$

Lemma 3.1. *Let $\varphi \in H^s(\mathbb{R}^d)$ and $\varepsilon > 0$. Then (3.1) possesses a unique solution*

$$u_\varepsilon \in C(\mathbb{R}, H^s(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{-s}(\mathbb{R}^d)).$$

Moreover, the mass and energy are conserved: for all $t \in \mathbb{R}$,

$$\|u_\varepsilon(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2, \quad E_\varepsilon(u_\varepsilon(t)) = E_\varepsilon(\varphi).$$

Proof. Unlike in the case of the regular Laplacian, $s = 1$, it seems delicate to invoke Strichartz estimates independently of the space dimension d in order to solve (3.1) in H^s, since a loss of regularity is present when $0 < s < 1$, see [14], and [23, 26]. We rather adopt the approach of [13], which in turn resumes the arguments from [10]. A key step is to check that, for a given $T > 0$, (3.1) has at least one (weak) solution $u_\varepsilon \in L_T^\infty H^s \cap W_T^{1,\infty} H^{-s}$. By interpolation, such a solution belongs to $C_T L^2$, and if $u_\varepsilon, v_\varepsilon$ are two such solutions, $u_\varepsilon - v_\varepsilon$ solves

$$(i\partial_t - (-\Delta)^s)(u_\varepsilon - v_\varepsilon) = \lambda(u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - v_\varepsilon \log(|v_\varepsilon| + \varepsilon)).$$

We then proceed with the usual argument for L^2 estimates in Schrödinger equations: multiply by $\overline{u}_\varepsilon - \overline{v}_\varepsilon$, integrate over $\mathbb{R}^d$, and take the imaginary part. The term involving the fractional Laplacian vanishes by self-adjointness, and the nonlinear term is controlled thanks to Lemma 2.1 (with $\mu = \varepsilon$), so we get

$$\frac{d}{dt}\|u_\varepsilon - v_\varepsilon\|_{L^2}^2 \lesssim \|u_\varepsilon - v_\varepsilon\|_{L^2}^2,$$

hence $u_\varepsilon \equiv v_\varepsilon$ by Gronwall's lemma, since $\|u_\varepsilon(t) - v_\varepsilon(t)\|_{L^2}$ is continuous, and equal to 0 at $t = 0$. The existence of such a weak solution is given by [10, Theorem 3.3.5], which is readily adapted to the case of the fractional Laplacian, and since we note that for fixed $\varepsilon > 0$, there exists a function $C^\varepsilon(\cdot)$ such that if $\|u\|_{H^s}, \|v\|_{H^s} \leq M$, then

$$\|g_\varepsilon(u) - g_\varepsilon(v)\|_{H^{-s}} \leq \|g_\varepsilon(u) - g_\varepsilon(v)\|_{L^2} \leq C^\varepsilon(M)\|u - v\|_{H^s}.$$

With the above uniqueness property, we can resume the proof of [10, Theorem 3.3.9] and [10, Theorem 3.4.1] for the globalization, since, as we have, for every $\delta > 0$,

$$\big||u|^2 \log(|u| + \varepsilon) - \mu_\varepsilon(|u|)\big| \leq C_{\varepsilon,\delta}|u|^{2+\delta} + |u|,$$

Gagliardo–Nirenberg and Young inequalities yield

$$|\lambda G_\varepsilon(u)| \leq \frac{1}{4}\|u\|_{\dot{H}^s}^2 + C(\|u\|_{L^2}),$$

so we obtain the lemma. ■

3.2. Construction of weak H^s solutions

We initially establish a uniform estimate for approximate solutions within the H^s space.

Lemma 3.2. *Let* $0 < \alpha \leq 1$ *and* $\varphi \in H^s$. *For all* $t \in \mathbb{R}$ *we have*

$$\|(-\Delta)^{s/2}u_\varepsilon(t)\|_{L^2}^2 \leq e^{4|\lambda||t|}\|(-\Delta)^{s/2}\varphi\|_{L^2}^2. \tag{3.3}$$

Proof. We resume the energy estimate from [8]: in view of the conservation of the L^2-norm,

$$\begin{aligned}
&\frac{d}{dt}\|u_\varepsilon(t)\|_{H^s(\mathbb{R}^d)}^2 \\
&\quad = 2\,\mathrm{Re}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \overline{(u_\varepsilon(t,x+y) - u_\varepsilon(t,x))}\partial_t(u_\varepsilon(t,x+y) - u_\varepsilon(t,x))\frac{dxdy}{|y|^{d+2\alpha}} \\
&\quad = -2\,\mathrm{Im}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \overline{(u_\varepsilon(t,x+y) - u_\varepsilon(t,x))} \\
&\qquad\qquad \cdot (-\Delta)^s(u_\varepsilon(t,x+y) - u_\varepsilon(t,x))\frac{dxdy}{|y|^{d+2\alpha}}
\end{aligned}$$

$$- 4\lambda \operatorname{Im} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{(u_\varepsilon(t, x+y) - u_\varepsilon(t,x))} \cdot (g_\varepsilon(u_\varepsilon(t,x+y)) - g_\varepsilon(u_\varepsilon(t,x))) \frac{dxdy}{|y|^{d+2\alpha}}.$$

Here, the first term on the right-hand side of the equation vanishes, since $(-\Delta)^s$ is self-adjoint, and so the imaginary part of the integral in x is zero. By applying Lemma 2.1 with $\mu = \varepsilon$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|u_\varepsilon(t)\|^2_{H^s(\mathbb{R}^d)} \\ &\quad \le 4|\lambda| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \Big|\operatorname{Im}\big[\overline{(u_\varepsilon(t,x+y) - u_\varepsilon(t,x))} \\ &\qquad \cdot (g_\varepsilon(u_\varepsilon(t,x+y)) - g_\varepsilon(u_\varepsilon(t,x)))\big]\Big| \frac{dxdy}{|y|^{d+2\alpha}} \\ &\quad \le 4|\lambda| \iint_{\mathbb{R}^d \times \mathbb{R}^d} |u_\varepsilon(t,x+y) - u_\varepsilon(t,x)|^2 \frac{dxdy}{|y|^{d+2\alpha}} \le 4|\lambda| \|u_\varepsilon(t)\|^2_{H^s(\mathbb{R}^d)}. \end{aligned}$$

Gronwall's lemma then yields

$$\|u_\varepsilon(t)\|^2_{H^s(\mathbb{R}^d)} \le e^{4|\lambda t|}\|\varphi\|^2_{H^s(\mathbb{R}^d)}, \qquad \text{for all } t \in \mathbb{R},$$

hence the lemma. ∎

It follows from Lemma 3.1 and (3.3) that for any $T > 0$ we have

$$M_T := \sup_{0<\varepsilon<1} \|u_\varepsilon\|_{L^\infty_T(H^s)} \le C(T, \|\varphi\|_{H^s}). \tag{3.4}$$

Next we prove that $\{u_\varepsilon\}_{0<\varepsilon<1}$ forms a Cauchy sequence in $C_T(L^2_{\text{loc}}(\mathbb{R}^d))$ as $\varepsilon \downarrow 0$ for any $T > 0$. Take a function $\zeta \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2, \end{cases} \qquad 0 \le \zeta(x) \le 1 \quad \text{for all } x \in \mathbb{R}^d.$$

For $R > 0$ we set $\zeta_R := \zeta(x/R)$. For $\varepsilon, \mu \in (0,1)$, utilizing (3.1), (2.1), and (3.4), a direct computation indicates that

$$\begin{aligned} &\frac{d}{dt}\|\zeta_R(u_\varepsilon - u_\mu)\|^2_{L^2} \\ &\quad = 2\operatorname{Im}(i\zeta_R^2 \partial_t(u_\varepsilon - u_\mu), u_\varepsilon - u_\mu) \\ &\quad = 2\operatorname{Im}(\zeta_R^2(-\Delta)^s(u_\varepsilon - u_\mu), u_\varepsilon - u_\mu) \\ &\qquad + 4\lambda \operatorname{Im}(\zeta_R^2(u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - u_\mu \log(|u_\mu| + \mu)), u_\varepsilon - u_\mu). \end{aligned}$$

The first term on the right-hand side is estimated thanks to the fractional Leibniz rule recalled in (the first case of) Lemma 2.2, since

$$\operatorname{Im}((-\Delta)^{s/2}(u_\varepsilon - u_\mu), \zeta_R^2(-\Delta)^{s/2}(u_\varepsilon - u_\mu)) = 0,$$

by

$$\begin{aligned}&\left|\operatorname{Im}((-\Delta)^{s/2}(u_\varepsilon - u_\mu), (-\Delta)^{s/2}(\zeta_R^2(u_\varepsilon - u_\mu)))\right| \\ &\qquad \lesssim \|u_\varepsilon - u_\mu\|_{\dot H^s}\|u_\varepsilon - u_\mu\|_{L^2}\|(-\Delta)^{s/2}(\zeta_R^2)\|_{L^\infty}.\end{aligned}$$

The estimate $\|(-\Delta)^{s/2}\zeta_R^2\|_{L^\infty} \lesssim 1/R^s$ follows by homogeneity (using, e.g., Fourier transform), and thus

$$\begin{aligned}\frac{d}{dt}\|\zeta_R(u_\varepsilon - u_\mu)\|_{L^2}^2 \le{}& \frac{C}{R^s}\|u_\varepsilon - u_\mu\|_{\dot H^s}\|u_\varepsilon - u_\mu\|_{L^2} \\ &+ 4|\lambda|(\|\zeta_R(u_\varepsilon - u_\mu)\|_{L^2}^2 + |\varepsilon - \mu|\|\zeta_R^2(u_\varepsilon - u_\mu)\|_{L^1}).\end{aligned}$$

Gronwall's lemma implies

$$\|\zeta_R(u_\varepsilon - u_\mu)(t)\|_{L^2}^2 \le e^{4|\lambda|T}\left(\frac{C(M_T)}{R^s} + |\varepsilon - \mu||B_{2R}|^{1/2}\|\varphi\|_{L^2}\right), \tag{3.5}$$

for all $t \in [-T, T]$, where we have used

$$\|\zeta_R^2(u_\varepsilon - u_\mu)\|_{L^1} \le \|u_\varepsilon - u_\mu\|_{L^2(B_{2R})} \le 2|B_{2R}|^{1/2}\|\varphi\|_{L^2}.$$

We now fix $R_0 > 0$ and take $R \in (R_0, \infty)$ as a parameter. It follows from (3.5) that

$$\begin{aligned}\|u_\varepsilon - u_\mu\|_{C_T(L^2(B_{R_0}))}^2 &\le \|\zeta_R(u_\varepsilon - u_\mu)\|_{C_T(L^2)}^2 \\ &\le C(T, \|\varphi\|_{H^s})\left(\frac{1}{R^s} + |\varepsilon - \mu||B_{2R}|^{1/2}\right),\end{aligned}$$

which yields

$$\limsup_{\varepsilon,\mu\downarrow 0}\|u_\varepsilon - u_\mu\|_{C_T(L^2(B_{R_0}))}^2 \le \frac{C(T, \|\varphi\|_{H^s})}{R^s} \xrightarrow[R\to\infty]{} 0.$$

As $R_0 > 0$ is arbitrary, we conclude that the sequence $\{u_\varepsilon\}_{0<\varepsilon<1}$ constitutes a Cauchy sequence in $C_T(L^2_{\mathrm{loc}}(\mathbb{R}^d))$. When combining this with Lemma 3.1, this entails that there exists a function $u \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$ such that

$$u_\varepsilon \to u \quad \text{in } C_T(L^2_{\mathrm{loc}}(\mathbb{R}^d)) \quad \text{as } \varepsilon \downarrow 0, \tag{3.6}$$

for all $T > 0$.

Lemma 3.3. *We have* $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, H^s(\mathbb{R}^d))$ *and*

$$u_\varepsilon(t) \rightharpoonup u(t) \quad \text{in } H^s(\mathbb{R}^d), \text{ for all } t \in \mathbb{R}. \tag{3.7}$$

Proof. First it follows from (3.6) that

$$u_\varepsilon(t) \rightharpoonup u(t) \quad \text{in } L^2(\mathbb{R}^d), \text{ for all } t \in \mathbb{R}. \tag{3.8}$$

To prove $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, H^s(\mathbb{R}^d))$, we use the characterization of H^s functions by duality. For any $\psi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [-T, T]$ we obtain from (3.4) that

$$\left| \int u_\varepsilon(t)(-\Delta)^{s/2}\psi \right| = \left| \int (-\Delta)^{s/2} u_\varepsilon(t)\psi \right| \le \|u_\varepsilon(t)\|_{\dot{H}^s} \|\psi\|_{L^2} \le M_T \|\psi\|_{L^2}.$$

Then it follows from (3.8) that

$$\left| \int u(t)(-\Delta)^{s/2}\psi \right| \le M_T \|\psi\|_{L^2} \quad \text{for all } t \in [-T, T].$$

We infer that for all $t \in [-T, T]$,

$$u(t) \in H^s(\mathbb{R}^d) \quad \text{and} \quad \|(-\Delta)^{s/2}u(t)\|_{L^2} \le M_T,$$

hence $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, H^s(\mathbb{R}^d))$. Also, in view of (3.8),

$$\int (-\Delta)^{s/2} u_\varepsilon(t)\psi \to \int (-\Delta)^{s/2} u(t)\psi,$$

for any $\psi \in C_c^\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$. Using (3.4) and a density argument, we deduce that

$$(-\Delta)^{s/2}u_\varepsilon(t) \rightharpoonup (-\Delta)^{s/2}u(t) \quad \text{in } L^2(\mathbb{R}^d), \text{ for all } t \in \mathbb{R},$$

hence the lemma. ■

Next, we prove the convergence of the nonlinear term.

Lemma 3.4. *For all* $t \in \mathbb{R}$ *we have*

$$g_\varepsilon(u_\varepsilon(t)) \to g(u(t)) \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{as } \varepsilon \downarrow 0.$$

Proof. We show that for any $\Omega \subset\subset \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$u_\varepsilon(t) \log(|u_\varepsilon(t)| + \varepsilon) \to u(t) \log |u(t)| \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \downarrow 0.$$

In view of [24, Lemma A.2], we know that for $\alpha \in (0, 1)$, there exists $C(\alpha) > 0$ such that for all $u, v \in \mathbb{C}$, $\varepsilon \in (0, 1)$

$$\begin{aligned} \big|v \log(|v| + \varepsilon) - u \log |u|\big| &\le \varepsilon + |u - v| + C(\alpha) \\ &\quad \times (1 + |u|^{1-\alpha} \log^+ |u| + |v|^{1-\alpha} \log^+ |v|)|u - v|^\alpha, \end{aligned}$$

where $\log^+ x := \max(\log x, 0)$. Hence, for any $\delta > 0$ small, there exists $C(\delta) > 0$ such that

$$\begin{aligned}\big|u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - u \log|u|\big| &\leq \varepsilon + |u_\varepsilon - u| + C(\alpha)\\ &\quad \times (1 + |u_\varepsilon|^{1/2+\delta} + |u|^{1/2+\delta})|u_\varepsilon - u|^{1/2}.\end{aligned}$$

Fixing $\delta > 0$ sufficiently small so that $H^s(\mathbb{R}^d) \subset L^{2+4\delta}(\mathbb{R}^d)$, we have

$$\begin{aligned}\||u_\varepsilon|^{1/2+\delta}|u_\varepsilon - u|^{1/2}\|^2_{L^2(\Omega)} = \int_\Omega |u_\varepsilon|^{1+2\delta}|u_\varepsilon - u| &\leq \|u_\varepsilon\|^{1+2\delta}_{L^{2+4\delta}}\|u_\varepsilon - u\|_{L^2(\Omega)}\\ &\lesssim \|u_\varepsilon\|^{1+2\delta}_{H^s}\|u_\varepsilon - u\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, the results follow from (3.4) and (3.6). ∎

From (3.1) it follows that for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and every $\phi \in C_c^1(\mathbb{R})$,

$$\begin{aligned}&\int_\mathbb{R} (iu_\varepsilon, \psi)_{L^2}\phi'(t)dt\\ &\quad = -\int_\mathbb{R} \langle i\partial_t u_\varepsilon, \psi\rangle_{H^{-s},H^s}\phi(t)dt\\ &\quad = -\int_\mathbb{R} \langle(-\Delta)^s u_\varepsilon + 2\lambda u_\varepsilon \log(|u_\varepsilon| + \varepsilon), \psi\rangle_{H^{-s},H^s}\phi(t)dt\\ &\quad = -\int_\mathbb{R} \{((-\Delta)^{s/2}u_\varepsilon, (-\Delta)^{s/2}\psi)_{L^2} + (\lambda g_\varepsilon(u_\varepsilon), \psi)_{L^2}\}\phi(t)dt.\end{aligned}$$

From (3.7), $u_\varepsilon(t) \rightharpoonup u(t)$ in $H^s(\mathbb{R}^d)$. In view of Lemma 3.4, taking the limit $\varepsilon \downarrow 0$ yields

$$\int_\mathbb{R} (iu, \psi)_{L^2}\phi'(t)dt = -\int_\mathbb{R} \{((-\Delta)^{s/2}u, (-\Delta)^{s/2}\psi)_{L^2} + (\lambda g(u), \psi)_{L^2}\}\phi(t)dt.$$

It can be easily verified for any $\Omega \subset\subset \mathbb{R}^d$,

$$u \in L^\infty_{\text{loc}}(\mathbb{R}, H^s(\mathbb{R}^d)) \cap W^{s,\infty}_{\text{loc}}(\mathbb{R}, H^{-s}(\Omega))$$

and

$$i\partial_t u - (-\Delta)^s u = \lambda g(u) \quad \text{in } H^{-s}(\Omega), \tag{3.9}$$

for almost all $t \in \mathbb{R}$.

3.3. Uniqueness and regularity

Following [11, Lemme 2.2.1], we have the next lemma.

Lemma 3.5. *Assume that, for some $T > 0$, $u, v \in L^\infty_T(H^s(\mathbb{R}^d))$ solve (1.1) in the distribution sense. Then $u = v$.*

Proof. We set

$$M := \max\{\|u\|_{L^\infty_T(H^s)}, \|v\|_{L^\infty_T(H^s)}\}.$$

As mentioned above, u, v satisfy the equation in the sense of (3.9). Resuming the cut-off function ζ_R, and the computations from Section 3.2 (with u_ε replaced by u and u_μ replaced by v), Gronwall's lemma yields, like for (3.5) (with now $\varepsilon = \mu = 0$),

$$\|\zeta_R(u-v)(t)\|_{L^2}^2 \le e^{4|\lambda|T}\left(\|\zeta_R(u(0)-v(0))\|_{L^2}^2 + \frac{C(M)}{R^s}T\right) \quad \text{for all } t \in [-T,T].$$

By Fatou's Lemma,

$$\|(u-v)(t)\|_{L^2}^2 \le \liminf_{R\to\infty} \|\zeta_R(u-v)(t)\|_{L^2}^2 \le 0,$$

for all $t \in [-T,T]$. Therefore, $u = v$ on $[-T,T]$. ■

Continuity in time and strong L^2 convergence are established like in the proof of [24, Lemma 2.10].

Lemma 3.6. *We have* $u \in C_w(\mathbb{R}, H^s(\mathbb{R}^d)) \cap C(\mathbb{R}, L^2(\mathbb{R}^d))$ *and*

$$u_\varepsilon(t) \to u(t) \quad \text{in } L^2(\mathbb{R}^d).$$

Proof. First we note that $u \in C_w(\mathbb{R}, H^s(\mathbb{R}^d))$. Indeed, this easily follows from Lemma 3.3 and $u \in C(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^d))$. Next, we obtain from Lemma 3.1 and (3.8) that

$$\|u(t)\|_{L^2}^2 \le \liminf_{\varepsilon\to 0} \|u_\varepsilon(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}.$$

Uniqueness of solutions yields that

$$\|u(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}. \tag{3.10}$$

As $u \in C_w(\mathbb{R}, L^2(\mathbb{R}^d))$, we deduce that $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$. Since no mass is lost in the weak convergence (3.8), the convergence is strong in L^2. ■

Lemma 3.7. *We have* $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$.

Proof. We just need to show the continuity $t \mapsto u(t) \in H^s(\mathbb{R}^d)$ at $t = 0$. It follows from (3.3), (3.7), and the weak lower semicontinuity of the norm that

$$\|u(t)\|_{\dot H^s}^2 \le e^{4|\lambda||t|}\|\varphi\|_{\dot H^s}^2.$$

Passing to the limit as $t \to 0$ we have

$$\limsup_{t\to 0} \|u(t)\|_{\dot H^s}^2 \le \|\varphi\|_{\dot H^s}^2.$$

On the other hand, it follows from the weak continuity $t \mapsto u(t) \in H^s(\mathbb{R}^d)$ at $t = 0$ that

$$\|\varphi\|_{\dot H^s}^2 \le \liminf_{t\to 0} \|u(t)\|_{\dot H^s}^2.$$

So we obtain

$$\lim_{t\to 0} \|u(t)\|_{\dot H^s}^2 = \|\varphi\|_{\dot H^s}^2.$$

Therefore, the weak convergence in (3.7) is actually strong. ■

3.4. Construction of solutions in W_1^s

We now assume that $\varphi \in W_1^s \subset H^s(\mathbb{R}^d)$. From the dominated convergence theorem we have

$$E_\varepsilon(\varphi) \to E(\varphi) \quad \text{as } \varepsilon \downarrow 0,$$

recalling that $E_\varepsilon(\varphi)$ and $E(\varphi)$ are defined in (3.2) and (1.2), respectively. Let $\theta \in C_c^1(\mathbb{C}, \mathbb{R})$ satisfying

$$\theta(z) = \begin{cases} 1 & \text{if } |z| \le 1/4, \\ 0 & \text{if } |z| \ge 1/2, \end{cases} \quad 0 \le \theta(z) \le 1 \quad \text{for } z \in \mathbb{C},$$

and set, for $\varepsilon > 0$,

$$\begin{aligned} F_{1\varepsilon}(u) &= \theta(u)|u|^2 \log((|u|+\varepsilon)^2), & F_{2\varepsilon}(u) &= (1-\theta(u))|u|^2 \log((|u|+\varepsilon)^2), \\ F_1(u) &= \theta(u)|u|^2 \log(|u|^2), & F_2(u) &= (1-\theta(u))|u|^2 \log(|u|^2). \end{aligned}$$

In the subsequent discussion, we confine the range of ε to $(0, 1/2)$. The energy expressed in equation (3.1) is denoted as

$$E_\varepsilon(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{\lambda}{2}\int F_{1\varepsilon}(u) + \frac{\lambda}{2}\int F_{2\varepsilon}(u) - \frac{\lambda}{2}\int \mu_\varepsilon(|u|).$$

Taking $\delta > 0$ sufficiently small,

$$\int |F_2(u)| \lesssim \int |u|^{2+\delta} \lesssim (\|u\|_{L^2}^{1-\eta}\|u\|_{\dot H^s}^{\eta})^{2+\delta}, \quad \eta = \frac{d}{s}\left(\frac{1}{2} - \frac{1}{2+\delta}\right) \in (0,1). \tag{3.11}$$

In particular,

$$\text{for } u \in H^s(\mathbb{R}^d), \quad u \in W_1^s \Longleftrightarrow \int |F_1(u)| < \infty.$$

Lemma 3.8. *For all $t \in \mathbb{R}$ we have, as $\varepsilon \to 0$,*

$$\int \mu_\varepsilon(|u_\varepsilon(t)|) \to \int |u(t)|^2, \quad \int F_{2\varepsilon}(u_\varepsilon(t)) \to \int F_2(u(t)).$$

The proof of this lemma is found in [24, Lemma 2.13], and relies on the observation that for any $\delta \in (0,1)$ there exists $C(\delta) > 0$ such that

$$|F_{2\varepsilon}(z) - F_2(w)| \leq C(\delta)(|z|^{1+\delta} + |w|^{1+\delta})|z-w| \quad \text{for all } z, w \in \mathbb{C}.$$

Proposition 3.9. *Let $\lambda < 0$. Then, $u \in (C \cap L^\infty)(\mathbb{R}, W_1^s)$ and $E(u(t)) = E(\varphi)$ for all $t \in \mathbb{R}$.*

Proof. For $\varepsilon \in (0, 1/2)$, we have $F_{1\varepsilon}(u) \leq 0$, and we can rewrite the first two terms of $E_\varepsilon(u)$ as

$$\frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{\lambda}{2}\int F_{1\varepsilon}(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{|\lambda|}{2}\int |F_{1\varepsilon}(u)|.$$

The weak lower semicontinuity of the norm, Fatou's lemma (for the second term), and Lemma 3.8 imply

$$\begin{aligned} \frac{1}{2}\int |(-\Delta)^{s/2}u(t)|^2 &+ \frac{|\lambda|}{2}\int |F_1(u(t))| \\ &\leq \liminf_{\varepsilon\to 0}\Big(E_\varepsilon(u_\varepsilon(t)) - \frac{\lambda}{2}\int F_{2\varepsilon}(u_\varepsilon(t)) + \frac{\lambda}{2}\int \mu_\varepsilon(u_\varepsilon(t))\Big) \\ &\leq E(\varphi) - \frac{\lambda}{2}\int F_2(u(t)) + \frac{\lambda}{2}\int |u(t)|^2, \end{aligned}$$

for all $t \in \mathbb{R}$. It implies that

$$u(t) \in W_1^s, \quad E(u(t)) \leq E(\varphi), \quad \text{for all } t \in \mathbb{R}.$$

Invoking Lemma 3.6, we obtain that the conservation of the energy

$$E(u(t)) = E(\varphi), \quad \text{for all } t \in \mathbb{R}. \tag{3.12}$$

From inequality (3.11) with $(2+\delta)\eta < 2$, and the identity (3.10) we get

$$\int |(-\Delta)^{s/2}u(t)|^2 + \int |F_1(u(t))| \leq C(E(\varphi), \|\varphi\|_{L^2}),$$

for all $t \in \mathbb{R}$. Therefore we deduce that

$$u \in L^\infty(\mathbb{R}, H^s(\mathbb{R}^d)) \quad \text{and} \quad t \mapsto \int |u(t)|^2 \log(|u(t)|^2) \in L^\infty(\mathbb{R}),$$

and thus $u \in L^\infty(\mathbb{R}, W_1^s)$. Moreover, from (3.12) and Lemma (3.7), we know that

$$t \mapsto \int |u(t)|^2 \log(|u(t)|^2) \in C(\mathbb{R}) \iff u \in C(\mathbb{R}, W_1^s),$$

which completes the proof. ∎

Proposition 3.10. *Let $\lambda > 0$. Then, $u \in C(\mathbb{R}, W_1^s)$.*

Proof. Step 1. We show that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, W_1^s)$. It follows from (3.2) and (3.12) that for any $T > 0$ and $t \in [-T, T]$,

$$\begin{aligned}\frac{|\lambda|}{2}\int |F_{1\varepsilon}(u_\varepsilon(t))| &= -\frac{\lambda}{2}\int F_{1\varepsilon}(u_\varepsilon(t)) \\ &= -E_\varepsilon(u_\varepsilon(t)) + \frac{1}{2}\int |(-\Delta)^{s/2}u_\varepsilon(t)|^2 \\ &\quad + \frac{\lambda}{2}\Big(\int F_{2\varepsilon}(u_\varepsilon(t)) - \int \mu_\varepsilon(|u_\varepsilon(t)|)\Big).\end{aligned}$$

Fatou's Lemma and (3.4) imply

$$\frac{|\lambda|}{2}\int |F_1(u(t))| \le \liminf_{\varepsilon\to 0}\frac{|\lambda|}{2}\int |F_{1\varepsilon}(u_\varepsilon(t))| \le -E(\varphi) + C(M_T),$$

for all $t \in [-T, T]$. This entails

$$t \mapsto \int |u(t)|^2 \log(|u(t)|^2) \in L^\infty_{\mathrm{loc}}(\mathbb{R}),$$

hence the claim.

Step 2. We show that $u \in C(\mathbb{R}, W_1^s)$. We check that the map $t \mapsto \int F_2(u(t))$ is continuous, and then we need to show that so is $t \mapsto \int F_1(u(t))$. As in the proof of Lemma 3.7, we consider continuity at $t = 0$ only. Resuming the computation for the preceding paragraph, we derive

$$\begin{aligned}\frac{|\lambda|}{2}\int |F_{1\varepsilon}(u_\varepsilon(t))| &= -E_\varepsilon(u_\varepsilon(t)) + \frac{1}{2}\int |(-\Delta)^{s/2}u_\varepsilon(t)|^2 \\ &\quad + \frac{\lambda}{2}\Big(\int F_{2\varepsilon}(u_\varepsilon(t)) - \int \mu_\varepsilon(|u_\varepsilon(t)|)\Big) \\ &\le -E_\varepsilon(\varphi) + \frac{1}{2}e^{4|\lambda||t|}\|(-\Delta)^{s/2}\varphi\|_{L^2}^2 \\ &\quad + \frac{\lambda}{2}\Big(\int F_{2\varepsilon}(u_\varepsilon(t)) - \int \mu_\varepsilon(|u_\varepsilon(t)|)\Big).\end{aligned}$$

In view of Fatou's Lemma and Lemma 3.8, we infer

$$\begin{aligned}\frac{|\lambda|}{2}\int |F_1(u(t))| &\le -E(\varphi) + \frac{1}{2}e^{4|\lambda||t|}\|(-\Delta)^{s/2}\varphi\|_{L^2}^2 \\ &\quad + \frac{\lambda}{2}\int F_2(u(t)) - \frac{\lambda}{2}\int |u(t)|^2.\end{aligned}$$

Passing to the limit $t \to 0$ yields

$$\limsup_{t\to 0} \frac{|\lambda|}{2}\int |F_1(u(t))| \le -E(\varphi) + \frac{1}{2}\|(-\Delta)^{s/2}\varphi\|_{L^2}^2 + \frac{\lambda}{2}\int F_2(\varphi) - \frac{\lambda}{2}\int |\varphi|^2$$
$$= -\frac{\lambda}{2}\int F_1(\varphi) = \frac{|\lambda|}{2}\int |F_1(\varphi)|.$$

Thanks to Fatou's Lemma,

$$\int |F_1(\varphi)| \le \liminf_{t\to 0} |F_1(u(t))|,$$

hence the proposition. ∎

Since regardless of the sign of λ, $u \in C(\mathbb{R}, W_1^s)$, arguing like in the proof of [9, Lemma 2.6], we infer

$$i\partial_t u - (-\Delta)^s u = \lambda u \log(|u|^2) \quad \text{in } (W_1^s)^*.$$

3.5. The H^1 case

To conclude the proof of Theorem 1.1, we now assume $\varphi \in H^1(\mathbb{R}^d)$. Since $0 < s < 1$, we already know that (1.1) has a unique solution $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$. We note that the solution u_ε to (3.1) is bounded in $H^1(\mathbb{R}^d)$, uniformly on any time interval $[-T, T]$ and in $\varepsilon \in (0, 1]$. Indeed, applying the gradient to (3.1) yields

$$i\partial_t \nabla u_\varepsilon - (-\Delta)^s \nabla u_\varepsilon = 2\lambda \nabla u_\varepsilon \log(|u_\varepsilon| + \varepsilon) + 2\lambda \frac{u_\varepsilon}{|u_\varepsilon| + \varepsilon}\nabla |u_\varepsilon|,$$

and the standard L^2 estimate readily provides

$$\frac{d}{dt}\|\nabla u_\varepsilon\|_{L^2}^2 \le 4|\lambda| \|\nabla u_\varepsilon\|_{L^2} \|\nabla |u_\varepsilon|\|_{L^2} \le 4|\lambda| \|\nabla u_\varepsilon\|_{L^2}^2.$$

The conclusion of Theorem 1.1 then follows from the same arguments as above, when we proved that $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$.

4. The Cauchy problem in the H^{2s} regularity

In this section, we show that if $\varphi \in X_\alpha^{2s} = H^{2s} \cap \mathcal{F}(H^\alpha)$, then the solution $u \in C(\mathbb{R}, H^s)$ provided by Theorem 1.1 actually belongs to $C_w \cap L^\infty_{\text{loc}}(\mathbb{R}, X_\alpha^{2s})$ (note the obvious relation $X_\alpha^{2s} \subset H^s$).

The strategy is inspired by the classical one in the case of the nonlinear Schrödinger equation, when H^2 regularity is addressed, see [27] (see also [10]): we first prove that $\partial_t u \in L^\infty_{\text{loc}}(\mathbb{R}, L^2)$, and eventually use equation (1.1) to conclude that $(-\Delta)^s u \in L^\infty_{\text{loc}}(\mathbb{R}, L^2)$. The intermediate step consists in considering the nonlinearity, to show that $u \log |u|^2 \in L^\infty_{\text{loc}}(\mathbb{R}, L^2)$: due to the singularity of the logarithm at the origin, this is by no means obvious (in particular, the information $u \in C(\mathbb{R}, H^s)$ and the Sobolev embedding do not suffice to conclude). The first step is indeed the following.

Lemma 4.1. *Let $\alpha > 0$, $\varphi \in X_\alpha^{2s}$, and, for $\varepsilon > 0$, let u_ε solve (3.1). For all $t \in \mathbb{R}$ we have*

$$\|\partial_t u_\varepsilon(t)\|_{L^2}^2 \leq e^{4|\lambda t|}\|\partial_t u_\varepsilon(0)\|_{L^2}^2,$$

and there exists a map C independent of $\varepsilon \in (0,1)$ such that

$$\|\partial_t u_\varepsilon(0)\|_{L^2} \leq C(\|\varphi\|_{H^{2s}}, \|\langle x\rangle^\alpha \varphi\|_{L^2}).$$

Proof. For the first part of the lemma, we compute

$$\begin{aligned}\frac{d}{dt}\|\partial_t u_\varepsilon\|_{L^2}^2 &= 2\operatorname{Re}(\partial_t^2 u_\varepsilon, \partial_t u_\varepsilon)\\ &= -2\operatorname{Im}(\partial_t\{(-\Delta)^s u_\varepsilon + 2\lambda u_\varepsilon \log(|u_\varepsilon| + \varepsilon)\}, \partial_t u_\varepsilon)\\ &= -4\lambda \operatorname{Im}\left(\frac{u_\varepsilon}{|u_\varepsilon| + \varepsilon}\partial_t|u_\varepsilon|, \partial_t u_\varepsilon\right) \leq 4|\lambda|\|\partial_t u_\varepsilon(t)\|_{L^2}^2,\end{aligned}$$

hence the announced inequality by Gronwall's lemma. Now in view of (3.1),

$$\begin{aligned}\|\partial_t u_\varepsilon(0)\|_{L^2} &\leq \|(-\Delta)^s u_\varepsilon(0)\|_{L^2} + 2|\lambda|\|u_\varepsilon(0)\log(|u_\varepsilon(0)| + \varepsilon)\|_{L^2}\\ &\leq \|\varphi\|_{H^{2s}} + 2|\lambda|\|\varphi\log(|\varphi| + \varepsilon)\|_{L^2}.\end{aligned}$$

For $\delta > 0$,

$$|\varphi\log(|\varphi| + \varepsilon)| \lesssim |\varphi|((|\varphi| + \varepsilon)^{-\delta} + (|\varphi| + \varepsilon)^\delta) \lesssim |\varphi|^{1-\delta} + |\varphi|(|\varphi|^\delta + 1),$$

and, provided that $\delta > 0$ is sufficiently small (in terms of s and α),

$$\||\varphi|^{1-\delta}\|_{L^2} \lesssim \|\langle x\rangle^\alpha\varphi\|_{L^2}^{1-\delta}, \quad \|\varphi(|\varphi|^\delta + 1)\|_{L^2} \lesssim \|\varphi\|_{H^{2s}}^{1+\delta} + \|\varphi\|_{L^2},$$

hence the lemma. ∎

Combined with (3.4),

$$N_T := \sup_{\varepsilon\in(0,1)}(\|u_\varepsilon\|_{C_T(H^s)} + \|\partial_t u_\varepsilon\|_{C_T(L^2)}) \leq C(T, \|\varphi\|_{X_\alpha^{2s}}). \tag{4.1}$$

The unique solution $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ to (1.1) was constructed in Section 3, obtained as the limit of u_ε as $\varepsilon \to 0$, and we deduce from (4.1) that

$$u \in W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}, L^2(\mathbb{R}^d)), \quad \partial_t u_\varepsilon(t) \rightharpoonup \partial_t u(t) \quad \text{in } L^2(\mathbb{R}^d).$$

As announced above, the next step consists in showing that $u\log|u|^2$ belongs to $L_{\mathrm{loc}}^\infty(\mathbb{R}, L^2)$. Using the same estimates as in the proof of Lemma 4.1, it suffices to prove the following result.

Lemma 4.2. *Let* $0 < s < 1$, $0 < \alpha < 2s$ *with* $\alpha \leq 1$, *and* $\varphi \in X_\alpha^{2s}$. *Then the solution* $u \in C(\mathbb{R}, H^s)$ *provided by Theorem* 1.1 *also belongs to* $C_w \cap L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{F}(H^\alpha))$.

Proof. Let $\varepsilon > 0$: multiplying (3.1) by $\langle x\rangle^\alpha$, we find

$$i\partial_t(\langle x\rangle^\alpha u_\varepsilon) - \langle x\rangle^\alpha(-\Delta)^s u_\varepsilon = 2\lambda\langle x\rangle^\alpha u_\varepsilon \log(|u_\varepsilon| + \varepsilon),$$

which can be rewritten as

$$i\partial_t(\langle x\rangle^\alpha u_\varepsilon) - (-\Delta)^s(\langle x\rangle^\alpha u_\varepsilon) = 2\lambda\langle x\rangle^\alpha u_\varepsilon \log(|u_\varepsilon| + \varepsilon) - [(-\Delta)^s, \langle x\rangle^\alpha]u_\varepsilon.$$

Multiplying the above equation by $\langle x\rangle^\alpha \overline{u_\varepsilon}$, integrating over $\mathbb{R}^d$ and taking the imaginary part, we obtain, since $(-\Delta)^s$ is self-adjoint,

$$\frac{d}{dt}\|\langle x\rangle^\alpha u_\varepsilon\|_{L^2}^2 \leq 2\|\langle x\rangle^\alpha u_\varepsilon\|_{L^2}\|[(-\Delta)^s, \langle x\rangle^\alpha]u_\varepsilon\|_{L^2}.$$

The last factor is estimated thanks to Lemma 2.3: for $T > 0$ and $t \in [-T, T]$,

$$\|[(-\Delta)^s, \langle x\rangle^\alpha]u_\varepsilon(t)\|_{L^2} \lesssim \|u_\varepsilon(t)\|_{H^s} \lesssim M_T \lesssim N_T.$$

Gronwall's lemma implies that u_ε is uniformly bounded in $L^\infty_T\mathcal{F}(H^\alpha)$, and the lemma follows by the same arguments as in Section 3. ∎

As explained above, we conclude that $(-\Delta)^s u \in C_w \cap L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2)$, and Theorem 1.2 follows, keeping Lemma 4.2 in mind.

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