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On the topologies of the exponential

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Abstract. Factorization algebras have been defined using three different topologies on the Ran space. We study these three different topologies on the exponential, which is the union of the Ran space and the empty configuration, and show that an exponential property is satisfied in each case. As a consequence, we describe the weak homotopy type of the exponential $\mathrm{Exp}(X)$ for each topology, in the case where X is not (necessarily) connected.

We also study these exponentials as stratified spaces and show that the metric exponential is conically stratified for a general class of spaces. As a corollary, we obtain that locally constant factorization algebras defined by Beilinson–Drinfeld are equivalent to locally constant factorization algebras defined by Lurie.

Introduction

Roughly speaking, a factorization algebra $\mathcal{A}$ on a space X with values in a symmetric monoidal category $\mathcal{C}^{\otimes}$ is a gadget associating to each finite subset of points $S \subset X$ an object $\mathcal{A}_S \in \mathcal{C}$, such that

(*factorization*) $\mathcal{A}_{\bigsqcup_{i \in I} S_i} \cong \bigotimes_{i \in I} \mathcal{A}_{S_i}$ for every finite family I of disjoint finite subsets $S_i \subset X$;

(*continuity*) the assignment $S \mapsto \mathcal{A}_S$ be continuous.

Yet, to be able to say that $\mathcal{A}_S$ varies continuously with S, one first needs to answer the question:

What is the topology on the set of all finite subsets $S \subset X$?

The set of all finite subsets of X is called the exponential $\mathrm{Exp}(X)$ of X. The literature on factorization algebras provides three different candidates to topologize $\mathrm{Exp}(X)$:

(2004) In *Chiral algebras* [4, Sect. 3.4.1], Beilinson and Drinfeld endow $\mathrm{Exp}(X)$ with a colimit topology.

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(2009) In *Derived algebraic geometry VI* [17, Def. 3.3.2], Lurie endows $\mathrm{Exp}(X)$ with a topology reminiscent of the metric topology introduced by Hausdorff on the space of compact subsets of a metric space.

(2016) In *Factorization algebras in quantum field theory* [6, Sect. 1.4.1], Costello and Gwilliam use yet another topology to define factorization algebras, this time using coverings inspired from Weiss.

For a given separated X, the three topologies above are given from finest to coarsest and one then obtains three different levels of strength for the continuity requirement of a factorization algebra. It has been conjectured that the three different definitions agree in the special case of *locally constant* factorization algebras which are, roughly speaking, those factorization algebras $\mathcal{A}$ for which $\mathcal{A}_x$ is "homotopic" to $\mathcal{A}_y$ when $x, y \in X$ both belong to the same contractible open subset.

The set of all finite subsets of X is called the exponential of X because its algebraic properties resemble that of exponential functions. This is the subject of the first section, where we define *exponential functors* in general and give some general properties.

In the second section, we introduce the three different topologies giving rise to the topological exponential (B&D version), the metric exponential (Lurie version) and the minimal exponential (C&G version). We show how each exponential listed above is an exponential functor in the sense of the definition given in the first section. From this we deduce the weak homotopy type of each exponential in the case where X is not necessarily connected, extending contractibility results of Handel [10, Cor. 4.3] and Curtis & Nhu [7].

Finally, the last two sections are dedicated to the study of the stratification of the metric exponential. The goal is to show that it is conically stratified (in the sense of Lurie) for a general class of spaces. For this we need to solve an optimization problem: finding the smallest enclosing ball of a finite number of points in a general normed space; this is the content of the third section. Using a companion article from the second author [13], one can then deduce from the conical stratification of the metric exponential that the notions of locally constant factorization algebras from Beilinson & Drinfeld and Lurie coincide.

1. Exponentials

1.1. Exponential functors

Any continuous function $f\colon \mathbf{R} \to \mathbf{R}$ satisfying $f(x + y) = f(x)f(y)$ must be an exponential $x \mapsto a^x$ with base $a = f(1)$. *The* exponential is traditionally the one with

base e defined
$$\mathrm{e}^x := \sum_{n\in\mathbf{N}} \frac{x^n}{n!}$$
using power series.

An analogous theory can be described in the realm of categories. The set $\mathbf{R}$ can be replaced by any category $\mathcal{C}$, functions can be replaced by functors, sums can be replaced by coproducts and products can be replaced by categorical products.

However, there shall be two main differences between exponential functors and exponential functions. First, what was a *property* of a function in the realm of set theory shall become a *structure* on a functor in category theory. An exponential functor shall be a *symmetric monoidal functor*

$$\mathcal{C}^{\sqcup} \xrightarrow{E} \mathcal{C}^{\times}$$

between $\mathcal{C}$ endowed with the coproduct symmetric monoidal structure and $\mathcal{C}$ endowed with the product symmetric monoidal structure, provided $\mathcal{C}$ admits all finite products and coproducts.

Let us see some of the first obvious consequences. First, since each object X admits a map $\emptyset_{\mathcal{C}} \to X$ with source the initial object of $\mathcal{C}$, one gets a map

$$\bigstar_{\mathcal{C}} \cong E(\emptyset_{\mathcal{C}}) \to E(X)$$

with source the terminal object of $\mathcal{C}$, so each $E(X)$ is a *pointed object* of $\mathcal{C}$. Since every object $X \in \mathcal{C}$ is a commutative monoid with product map $X \amalg X \to X$ the fold map, it follows that $E(X)$ is also a commutative monoid with composition

$$E(X) \times E(X) \cong E(X \amalg X) \to E(X)$$

and with unit the pointing already described.

Second, one needs to replace continuity with an equivalent notion. There is already a notion of continuity for functors in category theory: a functor is (co)continuous if it preserves small (co)limits. This is unreasonable to ask. Instead, let us rewrite an equivalent definition for the continuity of an exponential function: a function $f\colon\mathbf{R} \to \mathbf{R}$ for which $f(x+y) = f(x)f(y)$ for every $x, y \in \mathbf{R}$ is continuous if and only if for every converging series $\sum x_n = x$, the sequence with general term $\prod_{i\leq n} f(x_i)$ converges to $f(x)$. In category theory, convergence is replaced by existence and “limit of a sequence” can be replaced with the “colimit of a filtered diagram”. To make this precise, we first recall the notion of the *finite product* of an infinite family.

Definition 1.1 (Finite product). Let $\mathcal{C}$ be a category with finite products and filtered colimits. Given a small family of pointed objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, let

$$\prod_{i\in I}^{\mathrm{f}} X_i := \varinjlim_{\substack{J\subset I\\ J \text{ finite}}} \prod_{j\in J} X_j$$

denote the finite product (or weak product, or restricted product) of the family obtained by taking the colimit over all finite subsets $J \subset I$.

Example 1.2. In the category of vector spaces (seen as pointed via their zero vector), the finite product of a small family $\{V_i\}_{i\in I}$,

$$\prod_{i\in I}^{\mathrm{f}} V_i = \bigoplus_{i\in I} V_i$$

coincides with their direct sum.

If $\{X_i\}_{i\in I}$ is a small family of pointed topological spaces, then one has

$$\prod_{i\in I}^{\mathrm{f}} X_i \to \prod_{i\in I} X_i^{\mathrm{box}},$$

a continuous injection from the finite product to the product, endowed with the box topology. When the points $* \to X_i$ are all open, this map becomes an open embedding. In this case, a basis of opens of the finite product is given by the images of the products $\prod_{j\in J} U_j$ with $J \subset I$ finite and $U_j \subset X_j$ open.

Construction 1.3. Let $E\colon \mathcal{C}^{\sqcup} \to \mathcal{C}^{\times}$ be a symmetric monoidal functor with $\mathcal{C}$ having enough limits and colimits. Let $\{X_i\}_{i\in I}$ be a small family of objects of $\mathcal{C}$, then for each finite subset $J \subset I$, one gets a map

$$\prod_{i\in J} E(X_i) \cong E\Big(\coprod_{i\in J} X_i\Big) \to E\Big(\coprod_{i\in I} X_i\Big),$$

using the functoriality of E and its monoidal structure. Moreover, for any $K \subset J$, the diagram

$$\begin{array}{ccccc}
\prod_{i\in J} E(X_i) & \xrightarrow{\cong} & E\big(\coprod_{i\in J} X_i\big) & & \\
\uparrow & & \uparrow & \searrow & \\
 & & & & E\big(\coprod_{i\in I} X_i\big) \\
 & & & \nearrow & \\
\prod_{i\in K} E(X_i) & \xrightarrow{\cong} & E\big(\coprod_{i\in K} X_i\big) & &
\end{array}$$

commutes by functoriality of E and its monoidal structure. One then obtains a canonically defined map

$$\prod_{i\in I}^{\mathrm{f}} E(X_i) \to E\Big(\coprod_{i\in I} X_i\Big)$$

from the finite product of $\{E(X_i)\}_{i\in I}$.

Definition 1.4 (Exponential functor). Let $\mathcal{C}$ be a category with finite products, small filtered colimits and small coproducts. An exponential functor of the category $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{C}^{\sqcup} \xrightarrow{E} \mathcal{C}^{\times}$$

between $\mathcal{C}$ endowed with the coproduct symmetric monoidal structure and $\mathcal{C}$ endowed with the product symmetric monoidal structure, for which, in addition, the canonical map

$$\prod_{i\in I}^{\mathrm{f}} E(X_i) \to E\Big(\coprod_{i\in I} X_i\Big)$$

is an isomorphism for every small family $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$. A morphism between two exponential functors is the data of a monoidal natural transformation.

Remark 1.5. One could define exponential functors in the following more abstract way. The coproduct is an *infinitary* symmetric monoidal structure: the operations $\{X_i\}_{i\in I} \mapsto \coprod_{i\in I} X_i$ are associative, symmetric and unital in an obvious way. Similarly, the finite product endows the category of pointed objects $\mathcal{C}_{\bullet}$ with another infinitary symmetric monoidal structure.

An exponential functor can then be defined as an infinitary symmetric monoidal functor $\mathcal{C}^{\sqcup} \to \mathcal{C}_{\bullet}^{\times^{\mathrm{f}}}$. Here it happens that any (finitary) lax monoidal functor $\mathcal{C}^{\sqcup} \to \mathcal{C}_{\bullet}^{\times}$ gives rise to an infinitary lax monoidal functor $\mathcal{C}^{\sqcup} \to \mathcal{C}_{\bullet}^{\times^{\mathrm{f}}}$ which allows us to define exponential functors without first having to develop the theory of infinitary monoidal categories.

The classification result for exponential functions on $\mathbf{R}$ has an equivalent form in the case of exponential functors on the category of sets: these are classified by their base.

Definition 1.6 (Exponential of base A). Let A be a commutative monoid. The exponential of base A is the endofunctor of the category of sets defined by

$$\mathrm{Exp}_A(X) := \{\phi\colon X \to A \mid \phi^{-1}(0) \text{ is cofinite}\}$$

for any set X. If $f\colon X \to Y$ is a function and $\phi\colon X \to A$ is almost null, its image by $\mathrm{Exp}_A(f)$ is the function $\psi\colon Y \to A$, where

$$\psi(y) := \sum_{x \in f^{-1}(y)} \phi(x)$$

for any $y \in Y$. The function ψ is well defined and almost null since A is commutative and ϕ is almost null. The exponential structure of Exp_A is straightforward.

Theorem 1.7. *The assignment $A \mapsto \mathrm{Exp}_A$ induces an equivalence*

$$\mathsf{Commutative\ monoids} = \mathsf{Set\ exponentials}$$

between the category of commutative monoids and the category of exponential functors of the category of sets.

Proof. The assignment $A \mapsto \mathrm{Exp}_A$ is functorial, its inverse takes an exponential E and extracts the commutative monoid $E(*)$. By construction one has a canonical isomorphism $\mathrm{Exp}_A(*) = A$ and the maps $\mathrm{Exp}_A(\emptyset) \to \mathrm{Exp}_A(*)$ and $\mathrm{Exp}_A(*) \times \mathrm{Exp}_A(*) = \mathrm{Exp}_A(* \amalg *) \to \mathrm{Exp}_A(*)$ recover the commutative monoid structure on A.

Conversely, if E is an exponential, let A denote the commutative monoid $E(*)$. Then one has $E(X) \cong \prod^{\mathrm{f}}_X A = \mathrm{Exp}_A(X)$ for every set X. Lastly, let us show that $E(f) \cong \mathrm{Exp}_A(f)$ for every function $f\colon X \to Y$. The case where Y is a singleton and X is finite is true by construction and corresponds to the monoid structure $\prod_X A \to A$ of A. Taking the colimit over finite subsets gives us the case $\prod^{\mathrm{f}}_X A \to A$ where X is infinite. Finally, the general case is obtained by writing a function $f\colon X \to Y$ as a disjoint union $f_y\colon X_y \to \{y\}$ with $y \in Y$,

$$E(f) \cong \prod^{\mathrm{f}}_{y \in Y} E(f_y) \cong \mathrm{Exp}_A(f_y) \cong \mathrm{Exp}_A(f).$$

The natural isomorphism $\mathrm{Exp}_A \cong E$ we have just described, is monoidal by construction.

It is straightforward to check that $\mathrm{Exp}_A(*) = A$ is natural in A and $\mathrm{Exp}_{E(*)} \cong E$ is natural in E. ∎

Example 1.8. The exponential of base $\mathbf{N}$

$$\mathrm{Exp}_{\mathbf{N}}(X) = X^0 \amalg X \amalg X^2_{\mathbf{S}_2} \amalg X^3_{\mathbf{S}_3} \amalg \cdots$$

is the exponential functor corresponding to the analyst exponential of base e. Indeed, every $\phi\colon X \to \mathbf{N}$ can be interpreted as a multiplicity function giving the recipe to cook up a tuple out of the complement of $\phi^{-1}(0) \subset X$.

Let $\mathbf{I}_2$ be the idempotent commutative monoid on two elements. Then the exponentials of base $\mathbf{I}_2$ and $\mathbf{Z}_2$ have identical sets

$$\mathrm{Exp}_{\mathbf{I}_2}(X) = \mathrm{Exp}_{\mathbf{Z}_2}(X) = \{S \subset X \mid S \text{ is finite}\}$$

but their monoid structures are different: for the exponential of base $\mathbf{I}_2$, the pair $(\{x\}, \{x\})$ is sent to $\{x\}$ whereas for that of base $\mathbf{Z}_2$, it is sent to $\emptyset$.

Remark 1.9. One can extend the definition of an exponential functor to accommodate any monoidal structure on the target. For example, the classification theorem above also holds for exponential functors $\mathsf{Vect}_{\mathbf{R}}^{\oplus} \to \mathsf{Vect}_{\mathbf{R}}^{\otimes}$: they are equivalent to unital commutative $\mathbf{R}$-algebras.

The exponential of base $\mathbf{R}[\mathrm{X}]$ is the symmetric algebra functor. The exponential of base $\mathbf{R}[\mathbf{Z}_2]$ is the antisymmetric algebra functor.

1.2. *The* exponential functor

As is apparent from the definition of the exponential functors with bases, there is a preferred exponential, the exponential of base $\mathbf{I}_2$, which we shall refer to as *the* exponential functor, and denote it simply by Exp.

For any set X, $\mathrm{Exp}(X)$ can be identified with the set of all finite subsets of X. For each function $f\colon X \to Y$, the associated function $\mathrm{Exp}(f)\colon \mathrm{Exp}(X) \to \mathrm{Exp}(Y)$ sends a finite subset $S \subset X$ to $f(S) \subset Y$.

The exponential can also be described as a particular colimit. This is the definition one can use to define the exponential in a general category.

Definition 1.10. Let $\mathcal{C}$ be a category admitting finite products and small colimits. Let $\mathsf{Fin}_{\twoheadrightarrow}$ denote the category of finite sets and surjections. Given an object $X \in \mathcal{C}$ and a surjection $\phi\colon I \twoheadrightarrow J$ between two finite sets, one gets a split monomorphism

$$X^{\phi}\colon X^J \hookrightarrow X^I$$

which means that X defines a functor $\mathsf{Fin}_{\twoheadrightarrow}^{\mathrm{op}} \to \mathcal{C}$.

The exponential functor on $\mathcal{C}$ is the colimit

$$\mathrm{Exp}(X) := \varinjlim_{I \in \mathsf{Fin}_{\twoheadrightarrow}^{\mathrm{op}}} X^I$$

of the functor $X\colon \mathsf{Fin}_{\twoheadrightarrow}^{\mathrm{op}} \to \mathcal{C}$, with the convention that $X^{\emptyset}$ is the terminal object of $\mathcal{C}$, for every $X \in \mathcal{C}$.

We shall first make a remark about the structure of this colimit and then show its universal property.

Definition 1.11. Let ω_* denote the poset

$$\omega_* := \{0\} \amalg \{1 < 2 < \cdots < n < \cdots\}.$$

The opposite category of the category of finite sets and surjections admits a canonical functor to ω_* sending a finite set I to its cardinal. Hence the colimit defining Exp can be computed in two steps.

Notation 1.12. For $n \in \omega_*$, let

$$\mathrm{Exp}^{\leq_* n}(X) := \varinjlim_{0<|I|\leq n} X^I$$

if $n \neq 0$, and let $\mathrm{Exp}^{\leq_* 0}(X)$ be $\bigstar_{\mathcal{C}}$ the terminal object of $\mathcal{C}$.

For example, if $X \in \mathsf{Set}$, $\mathrm{Exp}^{\leq_* n}(X)$ is the set of all non-empty finite subsets $S \subset X$ having at most n elements.

Since ω_* has an isolated point, we shall let

$$\mathrm{Exp}^*(X) := \varinjlim_{0<|I|} X^I$$

be the subobject called the Ran space of X in some parts of the literature.

Remark 1.13. By construction

$$\mathrm{Exp}(X) = \bigstar_{\mathcal{C}} \amalg \mathrm{Exp}^*(X) \quad \text{and} \quad \mathrm{Exp}^*(X) = \varinjlim_{n>0} \mathrm{Exp}^{\leq_* n}(X).$$

Theorem 1.14 (The exponential is an exponential). *Let $\mathcal{C}$ be a category with finite products and small colimits. Assume moreover that $Y \mapsto X \times Y$ commutes with all small colimits for every $X \in \mathcal{C}$. Then* the *exponential* $\mathrm{Exp}\colon \mathcal{C} \to \mathcal{C}$ *has the canonical structure of an exponential functor.*

Proof. For each finite set I, the functor $X \mapsto X^I$ commutes with filtered colimits. Hence we have

$$\varinjlim_{\substack{J\subset K \\ J \text{ finite}}} \mathrm{Exp}\Big(\coprod_{j\in J} X_j\Big) = \mathrm{Exp}\Big(\coprod_{k\in K} X_k\Big).$$

So it shall be enough to show that Exp is a symmetric monoidal functor.

Let $\{X_k\}_{k\in K}$ be a finite family of elements of $\mathcal{C}$. One has a sequence of canonical isomorphisms

$$\begin{aligned}
\mathrm{Exp}\Big(\coprod_{k\in K} X_k\Big) &= \varinjlim_{I} \Big(\coprod_{k\in K} X_k\Big)^I \\
&= \varinjlim_{I} \coprod_{(I\to K)} \prod_{k\in K} X_k^{I_k} \qquad \text{(by distributivity)}
\end{aligned}$$

$$
\begin{aligned}
&= \varinjlim_{(I\to K)} \prod_{k\in K} X_k^{I_k} && \text{(by cofinality, see (3) below)}\\
&= \varinjlim_{\{I_k\}_{k\in K}} \prod_{k\in K} X_k^{I_k} && \text{(by isomorphy, see (2) below)}\\
&= \prod_{k\in K} \varinjlim_{I_k} X_k^{I_k} && \text{(by distributivity)}\\
&= \prod_{k\in K} \mathrm{Exp}(X_k)
\end{aligned}
$$

where:

(1) given a map of finite sets $\phi\colon I \to K$, we let $I_k := \phi^{-1}(k)$ for each $k \in K$;

(2) the coproduct induces an isomorphism of categories

$$\mathsf{Fin}^K = \mathsf{Fin}_{/K}$$

sending $\{I_k\}_{k\in K}$ to $\coprod_{k\in K} I_k \to K$;

(3) given a small category $\mathcal{C}$ and an object $K \in \mathcal{C}$, let $p\colon \mathcal{C}_{K/} \to \mathcal{C}$ denote the forgetful functor. Then for every $x \in \mathcal{C}$, the canonical map

$$p^{-1}(x) \to p_{/x}$$

is cofinal. Moreover, the fiber $p^{-1}(x)$ is discrete. Thus for any functor F with source the coslice $\mathcal{C}_{K/}$, its colimit can be computed as

$$\varinjlim_{(K\to x)} F = \varinjlim_{x} \varinjlim_{p_{/x}} F = \varinjlim_{x} \varinjlim_{p^{-1}(x)} F = \varinjlim_{x} \coprod_{(K\to x)} F.$$

It is straightforward to check that these canonical isomorphisms endow Exp with the structure of an exponential functor. ∎

Remark 1.15. If $Y \mapsto X \times Y$ only commutes with coproducts, then one can show that we still get structural maps $\mathrm{Exp}(\coprod_{i\in I} X_i) \to \prod^{\mathrm{f}}_{i\in I} \mathrm{Exp}(X_i)$ turning the exponential into what one would call an oplax infinitary symmetric monoidal functor.

Corollary 1.16. *Under the same assumptions, for every $X \in \mathcal{C}$, $\mathrm{Exp}(X)$ is the free idempotent commutative monoid on X.*

Proof. Since Exp is an exponential functor, $\mathrm{Exp}(X)$ is a commutative monoid as explained earlier.

It is idempotent for the following reason: for every finite set I, the diagram

$$
\begin{array}{ccccccc}
X^I & \xrightarrow{\text{diagonal}} & X^I \times X^I & \longrightarrow & (X \amalg X)^{I\sqcup I} & \xrightarrow{\text{fold}} & X^{I\sqcup I}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathrm{Exp}(X) & \longrightarrow & \mathrm{Exp}(X)\times\mathrm{Exp}(X) & \xrightarrow{\cong} & \mathrm{Exp}(X \amalg X) & \longrightarrow & \mathrm{Exp}(X),
\end{array}
$$

where all vertical maps are canonical maps, commutes. Moreover, the top composite map equals the map $X^I \to X^{I \sqcup I}$ induced by the fold map $I \sqcup I \to I$. Hence the full top right composite map is again the canonical map $X^I \to \varinjlim_J X^J$. Since this is true for every finite I, this shows that the bottom composition is the identity of $\mathrm{Exp}(X)$.

Let (M, μ) be a commutative and idempotent monoid in $\mathcal{C}$. Assume a given map $\psi \colon X \to M$. Then for every finite I, one has a well-defined map

$$X^I \xrightarrow{\psi^I} M^I \xrightarrow{\mu} M$$

because μ is associative and commutative. Moreover, because μ is idempotent, the diagram

$$\begin{array}{ccccc} X^I & \xrightarrow{\psi^I} & M^I & & \\ {\scriptstyle X^\phi}\big\downarrow & & {\scriptstyle M^\phi}\big\downarrow & \searrow^{\mu} & \\ X^J & \xrightarrow{\psi^J} & M^J & \xrightarrow{\mu} & M \end{array}$$

commutes for every $\phi \colon I \twoheadrightarrow J$. One thus gets a map $\widetilde{\psi} \colon \mathrm{Exp}(X) \to M$ extending ψ. The map $\widetilde{\psi}$ is obviously unital.

To show that $\widetilde{\psi}$ is compatible with μ is to show that

$$\begin{array}{ccc} \mathrm{Exp}(X) \times \mathrm{Exp}(X) & \xrightarrow{\widetilde{\psi} \times \widetilde{\psi}} & M \times M \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ \mathrm{Exp}(X) & \xrightarrow{\widetilde{\psi}} & M \end{array}$$

commutes, which can be done by precomposing with $X^I \times X^J$ for all I, J finite sets. We then only need to show that

$$\begin{array}{ccccc} X^I \times X^J & \xrightarrow{\psi^I \times \psi^J} & M^I \times M^J & \xrightarrow{\mu \times \mu} & M \times M \\ \big\downarrow & & \big\downarrow & & \\ (X \amalg X)^{I \sqcup J} & \xrightarrow{(\psi \amalg \psi)^{I \sqcup J}} & (M \amalg M)^{I \sqcup J} & & \big\downarrow{\scriptstyle \mu} \\ {\scriptstyle \mathrm{fold}}\big\downarrow & & {\scriptstyle \mathrm{fold}}\big\downarrow & & \\ X^{I \sqcup J} & \xrightarrow{\psi^{I \sqcup J}} & M^{I \sqcup J} & \xrightarrow{\mu} & M \end{array}$$

commutes. The left two squares commute by functorality. The right square commutes because μ is unital, commutative and associative. ■

2. Topologies on the exponential

In this section, we shall review three different topologies on the set $\mathrm{Exp}(X)$ of finite subsets $S \subset X$ of a topological space X, with the goal of transforming Exp into an exponential endofunctor of the category of topological spaces.

2.1. The topological exponential

As explained in the previous section, the exponential can be computed with the help of a colimit ranging over the opposite category of the category of finite sets and surjections. Computing the colimit in the category of topological spaces, one obtains the *topological exponential*, of which we give a simpler definition.

Definition 2.1 (Topological exponential). The topological exponential $\mathrm{Exp}_{\mathrm{T}}(X)$ of a topological space X is the topological space with set of points $\mathrm{Exp}(X)$, the set of finite subsets $S \subset X$, endowed with the finest topology such that the canonical maps

$$X^n \to \mathrm{Exp}(X)$$

given by sending each tuple $(x_1, \dots, x_n)$ to the subset $[x_1, \dots, x_n] \subset X$ it represents, be continuous for every $n \geq 0$.

The topological exponential is used by Beilinson and Drinfeld [4, Sect. 3.4.1] to define factorization algebras on a topological space. As we shall soon see, the topological exponential suffers one drawback: it is not an exponential functor because the functor $Y \mapsto X \times Y$ commutes with colimits only when X is core-compact.

2.1.1. First steps towards exponentiability.

Proposition 2.2. *Let $\{X_i\}_{i \in I}$ be a small family of topological spaces. The bijection*

$$\mathrm{Exp}_{\mathrm{T}}\Big(\coprod_{i \in I} X_i\Big) \to \prod_{i \in I}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}(X_i)$$

is continuous.

Proof. To show that this map is continuous it is enough to check that its composition with the projections $(\coprod_{i \in I} X_i)^K \to \mathrm{Exp}_{\mathrm{T}}(\coprod_{i \in I} X_i)$ is continuous for every finite set K. For such a K, the space $(\coprod_{i \in I} X_i)^K$ is a disjoint union of spaces of the form $\prod_{j \in J} X_j^{K_j}$ with $J \subset I$ finite, and each projection map $\prod_{j \in J} X_j^{K_j} \to \prod_{j \in J} \mathrm{Exp}_{\mathrm{T}}(X_j) \to \prod_{i \in I}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}(X_i)$ is continuous. ∎

Lemma 2.3 ([10, Props. 2.4 & 2.5]). *Given a separated space X, the projection map*

$$X^n \to \mathrm{Exp}_{\mathrm{T}}(X)$$

factors as a composite

$$X^n \to \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(X) \subset \mathrm{Exp}_{\mathrm{T}}(X)$$

of a closed quotient map followed by a closed embedding, for every $n \geq 0$.

Lemma 2.4. *For any small family of separated topological spaces $\{X_i\}_{i\in I}$ and every natural n, the canonical map*

$$\prod_{i\in I}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(X_i) \to \mathrm{Exp}_{\mathrm{T}}\Big(\coprod_{i\in I} X_i\Big)$$

is continuous.

Proof. By definition of the finite product, it is enough to show it for I finite. Since each X_i is separated, the quotient map $X_i^n \twoheadrightarrow \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(X_i)$ is perfect (Lemma 2.3), and hence a perfect map. Thus the product $\prod_{i\in I} X_i^n \to \prod_{i\in I} \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(X_i)$ is again a perfect map, and, in particular, a quotient map. Then, the map

$$\prod_{i\in I} X_i^n \to \Big(\coprod_{i\in I} X_i\Big)^{I\sqcup\cdots\sqcup I} \to \mathrm{Exp}_{\mathrm{T}}\Big(\coprod_{i\in I} X_i\Big)$$

is continuous and by the previous observation, factors as a continuous map through the quotient $\prod_{i\in I} \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(X_i)$. ■

2.1.2. The topological exponential is not an exponential.

Lemma 2.5. *The topological exponential* $\mathrm{Exp}_{\mathrm{T}}(\mathbf{S}^1)$ *contains a copy of the infinite bouquet of circles* $\bigvee^{\omega} \mathbf{S}^1$.

Proof. We shall build a sequence of closed embeddings

$$\begin{array}{ccccccc}
\mathbf{S}^1 & \hookrightarrow & \cdots & \hookrightarrow & \bigvee_{i=1}^{n} \mathbf{S}^1 & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Exp}_{\mathrm{T}}^{\leq * 1}(\mathbf{S}^1) & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(\mathbf{S}^1) & \hookrightarrow & \cdots
\end{array}$$

which shall lead to a closed embedding $\bigvee^{\omega} \mathbf{S}^1 \hookrightarrow \mathrm{Exp}_{\mathrm{T}}(\mathbf{S}^1)$. For this, we embed two circles into the torus $\mathbf{T}^2$ via the vectors $(0,1)$ and $(1,1)$, three circles in $\mathbf{T}^3$ via the vectors $(0,0,1)$, $(0,1,1)$ and $(1,1,1)$ etc.. This defines a compatible family of closed embeddings $\bigvee_{i=1}^{n} \mathbf{S}^1 \hookrightarrow \mathbf{T}^n$. Since moreover the projection map $\mathbf{T}^n \to \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(\mathbf{S}^1)$ is closed, we get continuous closed maps $\bigvee_{i=1}^{n} \mathbf{S}^1 \to \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(\mathbf{S}^1)$. By construction, they are injective and fit as expected in the diagram of closed embeddings above. ■

Theorem 2.6 (The topological exponential is not an exponential). *The canonical continuous bijection*

$$\mathrm{Exp}_{\mathrm{T}}(\mathbf{Q} \amalg \mathbf{S}^1) \to \mathrm{Exp}_{\mathrm{T}}(\mathbf{Q}) \times \mathrm{Exp}_{\mathrm{T}}(\mathbf{S}^1)$$

is not a homeomorphism.

Proof. Using the same tori embeddings as in the previous lemma, one can fit a copy of $\mathbf{Q} \times \bigvee_{i=1}^{n} \mathbf{S}^1$ in $\mathrm{Exp}_{\mathrm{T}}^{\leq * n+1}(\mathbf{Q} \amalg \mathbf{S}^1)$ for every $n \geq 1$. One can then check that the continuous bijection $\mathrm{Exp}_{\mathrm{T}}(\mathbf{Q} \amalg \mathbf{S}^1) \to \mathrm{Exp}_{\mathrm{T}}(\mathbf{Q}) \times \mathrm{Exp}_{\mathrm{T}}(\mathbf{S}^1)$ restricts to the continuous bijection $\varinjlim_{0<n<\omega} \mathbf{Q} \times \bigvee_{i=1}^{n} \mathbf{S}^1 \to \mathbf{Q} \times \bigvee^{\omega} \mathbf{S}^1$ which is not a homeomorphism [14, Sect. 3.2]. ∎

2.1.3. The topological exponential is almost an exponential. As we have just seen, the canonical continuous bijection $\mathrm{Exp}_{\mathrm{T}}(X \amalg Y) \to \mathrm{Exp}_{\mathrm{T}}(X) \times \mathrm{Exp}_{\mathrm{T}}(Y)$ is not always a homeomorphism. However, when X and Y are separated, its inverse is still sequentially continuous.

Remark 2.7 (Converging sequences in a colimit topology). Let $Z_0 \hookrightarrow \cdots \hookrightarrow Z_p \hookrightarrow \cdots$ be a sequence of closed embeddings between T_1 topological spaces and let Z denote its colimit. Then every morphism $K \to Z$ with K compact factors through one $Z_p \subset Z$ [11, Prop. 2.4.2]. More generally this is true if Z is the colimit of an ordinal sequence of closed embeddings.

As a consequence, if X is separated, a sequence $(S_n)_{n \in \mathbf{N}}$ in $\mathrm{Exp}_{\mathrm{T}}(X)$, converges only if the sequence of cardinals $|S_n|_{n<\omega}$ is bounded.

Proposition 2.8. *Let $\{X_i\}_{i \in I}$ be a small family of separated spaces, the canonical bijection*

$$\prod_{i \in I}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}(X_i) \to \mathrm{Exp}_{\mathrm{T}}\Big(\coprod_{i \in I} X_i\Big)$$

is sequentially continuous.

Proof. Let S be a sequence in $\mathrm{Exp}_{\mathrm{T}}(\coprod_{i \in I} X_i)$. Since S_n is a finite subset of $\coprod_{i \in I} X_i$ for each natural number n, the union $\bigcup_{n<\omega} S_n$ intersects only a countable number of X_i, and we can thus reduce to the case where I is countable. Given a sequence $X_0, X_1, \ldots$ of separated spaces, the sequence

$$\mathrm{Exp}_{\mathrm{T}}(X_0) \hookrightarrow \mathrm{Exp}_{\mathrm{T}}(X_0) \times \mathrm{Exp}_{\mathrm{T}}(X_1) \hookrightarrow \mathrm{Exp}_{\mathrm{T}}(X_0) \times \mathrm{Exp}_{\mathrm{T}}(X_1) \times \mathrm{Exp}_{\mathrm{T}}(X_2) \hookrightarrow \cdots$$

is made of closed embeddings between separated spaces. Thus $\bigcup_{n<\omega} S_n$ intersects only a finite number of X_i (Remark 2.7).

Then, we only need to consider the case of two separated spaces X and Y. In that case, the sequence S is then comprised of a pair of two sequences $S(X)$ and

$S(Y)$. Because $\mathrm{Exp}_{\mathrm{T}}(X)$ is a union of closed embeddings between separated spaces, $S(X)$ is bounded in cardinality (Remark 2.7). The same is true for $S(Y)$. We conclude using that $\mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X) \times \mathrm{Exp}_{\mathrm{T}}^{\leq *n}(Y) \to \mathrm{Exp}_{\mathrm{T}}(X \amalg Y)$ is continuous for every n (Lemma 2.4). ■

2.1.4. The topological exponential is a restricted exponential. As we have explained earlier, the topological exponential is not an exponential functor because the functor $Y \mapsto X \times Y$ does not commute with colimits in general, for a given X. One might ask whether the exponential property could still hold if restricted to core-compact spaces, i.e., the spaces X for which $Y \mapsto X \times Y$ commutes with colimits. The answer to this question is non-obvious as $\mathrm{Exp}_{\mathrm{T}}(X)$ is usually not going to be core-compact, even when X is.

Proposition 2.9. *Let $\{X_i\}_{i \in I}$ be a small family of separated and core-compact topological spaces. The canonical bijection*

$$\prod_{i \in I}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}(X_i) \to \mathrm{Exp}_{\mathrm{T}}\Big(\coprod_{i \in I} X_i\Big)$$

is a homeomorphism.

Proof. We only need to show that the above map is continuous (Proposition 2.2). By definition of the finite product, we can reduce to the case of a finite I. Since each X_i is separated, the projection map $X_i^n \to \mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X_i)$ is a perfect map and thus $\mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X_i)$ is core-compact. Because sequential unions of core-compact spaces commute with finite products [14], one has canonical homeomorphisms

$$\prod_{i \in I} \mathrm{Exp}_{\mathrm{T}}(X_i) = \prod_{i \in I} \varinjlim_{n \in \omega_*} \mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X_i) = \varinjlim_{n \in \omega_*} \prod_{i \in I} \mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X_i)$$

and the map

$$\prod_{i \in I} \mathrm{Exp}_{\mathrm{T}}(X_i) = \varinjlim_{n \in \omega_*} \prod_{i \in I} \mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X_i) \to \mathrm{Exp}_{\mathrm{T}}\Big(\coprod_{i \in I} X_i\Big)$$

is continuous, as a colimit of continuous maps (Lemma 2.4). ■

Corollary 2.10. *Let X be a separated and core-compact topological space. Then* $\mathrm{Exp}_{\mathrm{T}}(X)$ *is the free idempotent commutative topological monoid on X.*

2.2. The metric exponential

In addition to not being an exponential functor, the topological exponential also does not preserve metrizability. In fact, $\mathrm{Exp}_{\mathrm{T}}(X)$ is almost never metrizable.

Proposition 2.11. *Let X be a metrizable topological space. If X has an accumulation point, then $\mathrm{Exp}_{\mathrm{T}}(X)$ is not metrizable.*

Proof. Pick a metric D inducing the topology on $\mathrm{Exp}_{\mathrm{T}}(X)$. Let $x \in X$ be an accumulation point. For every $n \geq 0$, using the quotient map $X^n \to \mathrm{Exp}_{\mathrm{T}}^{\leq * n}(X)$, one can find a subset $S_n \subset X$ made of exactly n elements such that $D(S_n, \{x\}) \leq 1/n$. In other words, $S_n \to_{n \to +\infty} \{x\}$ and $|S_n| \to_{n\to+\infty} +\infty$ which is forbidden (Remark 2.7). ■

One way to remedy this is to compute the colimit defining the exponential not in the category of topological spaces but rather in the category $\mathtt{Met}_\infty$ of generalized metric spaces.

A generalized metric space is a metric space whose distance function is allowed to have the value $+\infty$. Morphisms in $\mathtt{Met}_\infty$ are the metric maps: the maps $f\colon (M, d_M) \to (N, d_N)$ such that $d_N(f(x), f(y)) \leq d_M(x, y)$ for every $x, y \in M$.

The main advantage of the category of generalized metric spaces is that it admits all small limits and colimits [15, Exm. 4.5 (3)]. Computing the colimit defining the exponential in $\mathtt{Met}_\infty$, one obtains the *metric exponential*, of which we give a concrete definition.

Definition 2.12 (Metric exponential of a metric space). Given a (generalized) metric space (X, d), its metric exponential is the generalized metric space $(\mathrm{Exp}(X), D)$, where

$$D(S, T) := \max \begin{cases} \max_{s\in S} \min_{t\in T} d(s,t), \\ \max_{t\in T} \min_{s\in S} d(s,t). \end{cases}$$

We shall denote the metric exponential by $\mathrm{Exp}_{\mathrm{M}}(X)$. In particular, one has $D([\emptyset], T) = D(T, [\emptyset]) = +\infty$ when T is not empty.

Remark 2.13. The metric subspace $\mathrm{Exp}^*_{\mathrm{M}}(X) \subset \mathrm{Exp}_{\mathrm{M}}(X)$ is used by Lurie as an intermediate tool to deal with locally constant non-unital factorization algebras which are locally constant cosheaves on $\mathrm{Exp}^*_{\mathrm{T}}(X)$ [17, Def. 3.3.2]. In *Higher Algebra* [18, Rem. 5.5.4.12] he suggests using a variant of the exponential $\mathrm{Exp}_{\mathrm{M}}(X)$ where $D([\emptyset], T) = D(T, [\emptyset]) = 0$ for every $T \subset X$, to deal with unital factorization algebras. This topology on the exponential differs from all of the topologies discussed in this paper and we are not aware of any further use of it.

The metric exponential has also been used by Knudsen [12] in his work extending the constructions of Francis and Gaitsgory [9] to the topological setup.

The topology of the metric exponential admits a basis given by opens of the form $[U_i]_{i\in I}$ where

$$S \in [U_i]_{i\in I} \quad \Leftrightarrow \quad \forall i \in I, \quad S \cap U_i \neq \emptyset.$$

This allows us to define the metric exponential $\mathrm{Exp}_{\mathrm{M}}(X)$ when X is only a topological space.

Definition 2.14 (Metric exponential of a topological space). For a topological space X, its metric exponential $\mathrm{Exp}_{\mathrm{M}}(X)$ consists of the set $\mathrm{Exp}(X)$ endowed with the coarsest topology including all $[U_i]_{i\in I}$ for every finite set I of open subsets $U_i \subset X$.

This definition is functorial: if $f\colon X \to Y$ is a continuous map, the preimage of $[U_i]_{i\in I}$ by $\mathrm{Exp}(f)$ equals $[f^{-1}(U_i)]_{i\in I}$.

Before looking at the exponential property of $\mathrm{Exp}_{\mathrm{M}}$, we shall discuss how some limits and colimits are computed in Met_∞. Given a small family $\{(X_i, d_i)\}_{i\in I}$ of (pointed) metric spaces, their

- *coproduct* is the disjoint union of sets $\coprod_{i\in I} X_i$ endowed with the distance d for which
$$d(x_i, y_j) = \begin{cases} d_i(x_i, y_i), & \text{if } i = j, \\ +\infty, & \text{if } i \neq j, \end{cases}$$
- *product* is the product set $\prod_{i\in I} X_i$ endowed with the sup metric
$$d(\{x_i\}, \{y_i\}) := \sup_{i\in I} d_i(x_i, y_i),$$
- *finite product* is the finite product set endowed with the sup metric. In other words, in that case, the natural map
$$\prod_{i\in I}^{\mathrm{f}} X_i \to \prod_{i\in I} X_i$$
is an isometric embedding.

Proposition 2.15 (Exponential property). *The metric exponential* $\mathrm{Exp}_{\mathrm{M}}$ *is an exponential functor for both* Met_∞ *and* Top.

Proof. Starting with the metric case: let $\{(X_i, d_i)\}_{i\in I}$ be a small family of metric spaces. We only need to show that the bijections in the exponential structure of Exp are isometric. Let S and T be two finite subsets of the union in $(X, d) := \coprod_{i\in I}(X_i, d_i)$ and write $S_i := S \cap X_i$ and $T_i := T \cap X_i$ for every $i \in I$. By construction of the disjoint union, if $s \in S$ and $t \in T$ do not belong to the same component X_i, their distance $d(s,t)$ in X is infinite. As a consequence the distance $D(S, T)$ in $\mathrm{Exp}_{\mathrm{M}}(X)$ becomes
$$D(S,T) = \max \begin{cases} \max_{s\in S} \min_{t\in T} d(s,t) = \sup_{i\in I} \max_{s\in S_i} \min_{t\in T_i} d_i(s,t), \\ \max_{t\in T} \min_{s\in S} d(s,t) = \sup_{i\in I} \max_{t\in T_i} \min_{s\in S_i} d_i(s,t), \end{cases}$$
and thus $D(S,T) = \sup_{i\in I} D_i(S_i, T_i)$.

Let $\{X_i\}_{i\in I}$ be a small family of topological spaces. Because finite sets can only intersect a finite number of connected components, the open sets of the form $[U_{j,k}]_{j\in J, k\in K_j}$ where $J \subset I$ and each K_j are finite, and where each $U_{j,k} \subset X_j$ is open, form a basis of the topology of $\mathrm{Exp}_{\mathrm{M}}(\coprod_{i\in I} X_i)$. It corresponds bijectively to the base open set inside $\prod^{\mathrm{f}}_{i\in I} \mathrm{Exp}_{\mathrm{M}}(X_i)$ given by $\prod_{j\in J}[U_{j,k}]_{k\in K_j}$. Thus the bijection $\mathrm{Exp}_{\mathrm{M}}(\coprod_{i\in I} X_i) \to \prod^{\mathrm{f}}_{i\in I} \mathrm{Exp}_{\mathrm{M}}(X_i)$ is a homeomorphism. ■

Proposition 2.16. *Let X be a (generalized) metric space. Then $\mathrm{Exp}_{\mathrm{M}}(X)$ is the free idempotent commutative metric monoid on X.*

Proof. The canonical map $X \to \mathrm{Exp}_{\mathrm{M}}(X)$ is an isometry by construction.

So the only thing to show is that for (A, d) an idempotent and commutative metric monoid, the map $\mathrm{Exp}_{\mathrm{M}}(A) \to A$ sending $S \subset A$ to $\prod_{s\in S} s \in A$ – which is well defined because A is commutative and is a monoid map because A is idempotent – is a metric map.

Given two finite subsets $S, T \subset A$, we need to show that $d(\prod_{s\in S} s, \prod_{t\in T} t) \leq D(S, T)$. If S or T is empty, it is immediate. Because A is an idempotent metric monoid $d(a, bc) = d(aa, bc) \leq \max(d(a, b), d(a, c))$ for every $a, b, c \in A$. By straightforward induction, one gets the case where either S or T has a unique element. Let n be an integer and assume that the inequality has been shown for every S, T with $|S| + |T| \leq n$. Let $S, T \subset A$ with $|S| + |T| = n + 1$. Without loss of generality, we can assume that there exists $x \in S$ such that $D(S, T) = d(x, T)$. Let S_0 denote the complement of x in S. Then

$$\begin{aligned}
d\Big(\prod_{s\in S} s, \prod_{t\in T} t\Big) &= d\Big(x \times \prod_{s\in S_0} s, \prod_{t\in T} t\Big) \\
&\leq \max\Big(d\Big(x, \prod_{t\in T} t\Big), d\Big(\prod_{s\in S_0} s, \prod_{t\in T} t\Big)\Big) && (A \text{ is metric}) \\
&\leq \max(d(x, T), D(S_0, T)) && (\text{by hypothesis}) \\
&= D(S, T) && (\text{by definition of } x)
\end{aligned}$$

ending showing that $\mathrm{Exp}_{\mathrm{M}}(A) \to A$ is a metric map. ■

Proposition 2.17. *For every topological space X, the identity*

$$\mathrm{Exp}_{\mathrm{T}}(X) \to \mathrm{Exp}_{\mathrm{M}}(X)$$

is a continuous map, which restricts to homeomorphisms

$$\mathrm{Exp}^{\leq *n}_{\mathrm{T}}(X) = \mathrm{Exp}^{\leq *n}_{\mathrm{M}}(X)$$

for every $n \in \omega_*$, whenever X is separated.

Proof. Let $U \subset X$ be an open subset. Let $n \geq 1$ be an integer and let σ denote the permutation $(1 \cdots n)$. Then the preimage along $X^n \to \mathrm{Exp}(X)$ of $[U]$ is the set $\bigcup_{i\leq n} \sigma^i(U \times X^{n-1})$ which is open. It follows that $[U]$ is open in $\mathrm{Exp}_{\mathrm{T}}(X)$. For a finite I, $[U_i]_{i\in I} = \bigcap_{i\in I}[U_i]$ is then also open in $\mathrm{Exp}_{\mathrm{T}}(X)$. ■

2.3. The minimal exponential

Definition 2.18. Given a topological space, the *minimal exponential* $\mathsf{Exp}_\vee(X)$ is the set $\mathrm{Exp}(X)$ endowed with the coarsest topology containing the subsets $\mathrm{Exp}(U) \subset \mathrm{Exp}(X)$ for all open subsets $U \subset X$.

Remark 2.19. One distinctive feature of the minimal exponential is that the point presenting the empty configuration $[\emptyset]$ is dense.

Families of open subsets $\{U_i \subset V\}_{i\in I}$ for which $\{\mathrm{Exp}(U_i) \subset \mathrm{Exp}(V)\}_{i\in I}$ is a cover in the minimal exponential, were introduced by Weiss in his work on the embedding calculus [22]. This notion of covering is used by Costello and Gwilliam to define factorization algebras in general [6, Sect. 1.4.1]. It is also used by Ayala and Francis in their study of factorization homology [2, Sect. 2.6].

Proposition 2.20 (Exponential property). *The minimal exponential is an exponential functor on the category of topological spaces.*

Proof. Let $\{X_i\}_{i\in I}$ be a small family of spaces. Since $[\emptyset]$ is open in the minimal topology, the finite product of the $\mathsf{Exp}_\vee(X_i)$ is a subspace of the product endowed with the box topology.

Given a family of open subsets $\{U_i \subset X_i\}_{i\in I}$, one has bijections

$$\mathrm{Exp}\Big(\coprod_{i\in I} U_i\Big) = \prod_{i\in I}^{\mathrm{f}} \mathrm{Exp}(U_i) = \Big(\prod_{i\in I} \mathrm{Exp}(U_i)\Big) \cap \Big(\prod_{i\in I}^{\mathrm{f}} \mathrm{Exp}(X_i)\Big)$$

showing the correspondence between the two bases of open sets $\mathsf{Exp}_\vee(\coprod_{i\in I} X_i)$ and $\prod_{i\in I}^{\mathrm{f}} \mathsf{Exp}_\vee(X_i)$. ∎

Remark 2.21 (Minimality). The functors $\mathrm{Exp}_{\mathrm{T}}$, $\mathrm{Exp}_{\mathrm{M}}$ and $\mathsf{Exp}_\vee$ preserve open embeddings between topological spaces. In the category of exponential functors of Top with base $\mathbf{I}_2$ having this preservation property, $\mathsf{Exp}_\vee$ is a final object.

2.4. Weak homotopy type of the exponentials

The functors $X \mapsto \mathrm{Exp}_{\mathrm{T}}^{\leq *n}(X)$ have interesting homotopy properties as shown by Handel. In particular, he showed that for X a separated and path connected space, $\mathrm{Exp}_{\mathrm{T}}^{*}(X)$ is weakly contractible [10, Cor. 4.3]. In the meantime, Lurie has also shown that $\mathrm{Exp}_{\mathrm{M}}^{*}(M)$ is weakly contractible when M is a connected manifold [17, Thm. 3.3.6].

Here we shall enhance these results by describing the weak homotopy type of each exponential for any separated and locally path connected space. Since $[\emptyset]$ is dense in $\mathsf{Exp}_\vee(X)$, it follows that $\mathsf{Exp}_\vee(X)$ is contractible for any space X. Hence, we shall focus on the metric and the topological exponentials.

We start with a lemma due to Beilinson and Drinfeld.

Lemma 2.22. *Let G be a group endowed with an extra operation $\wedge\colon G \times G \to G$ such that $\wedge$ is associative, idempotent and such that $ab \wedge cd = (a \wedge c)(b \wedge d)$ for any $a, b, c, d \in G$. Then G is a trivial group.*

Proof. For every $g \in G$ one has

$$g \wedge g = g \implies (1g) \wedge (1g) = g \implies (1 \wedge g)^2 = g.$$

So for every $h \in G$, letting $g = 1 \wedge h$,

$$(1 \wedge h)^2 = (1 \wedge 1 \wedge h)^2 = 1 \wedge h$$

since G is group, we get $1 \wedge h = 1$ and $h = (1 \wedge h)^2 = 1$ for every $h \in G$. ∎

Lemma 2.23. *If X is path connected, then $\mathrm{Exp}^*_{\mathrm{T}}(X)$ is path connected. As a consequence, $\mathrm{Exp}^*_{\mathrm{M}}(X)$ is also path connected.*

Proof. Given two proper finite subsets $S, T \subset X$, there exists a large enough positive $n \in \mathbf{N}$ and two tuples $(s_1, \dots, s_n)$ and $(t_1, \dots, t_n)$ representing respectively S and T. Since X is path connected, there exists a path between those two tuples in X^n and since the map $X^n \to \mathrm{Exp}_{\mathrm{T}}(X)$ is continuous by construction and factors through $\mathrm{Exp}^*_{\mathrm{T}}(X)$, this gives us a continuous path between S and T in $\mathrm{Exp}^*_{\mathrm{T}}(X)$. ∎

In what follows, let us denote by $\mathbf{I}_2$ the commutative idempotent monoid with two elements $(\{0, 1\}, \vee)$ and endow it with the discrete topology.

Theorem 2.24. *Let X be a locally path connected topological space. The monoid map*

$$\mathrm{Exp}_{\mathrm{M}}(X) \xrightarrow{\exists} \bigoplus_{\pi_0(X)} \mathbf{I}_2$$

sending a finite subset $S \subset X$ to the family $\{\exists_i\}_{i \in \pi_0(X)}$ with $\exists_i = 0$ if and only if no element of S belongs to the connected component $X_i \subset X$, is continuous and a weak homotopy equivalence.

Moreover, if X is also separated, the induced continuous map

$$\mathrm{Exp}_{\mathrm{T}}(X) \xrightarrow{\exists} \bigoplus_{\pi_0(X)} \mathbf{I}_2$$

is also a weak homotopy equivalence.

Proof. Using that $\mathrm{Exp}_{\mathrm{M}}$ is an exponential and the fact that for each connected component $X_i \subset X$, $\mathrm{Exp}_{\mathrm{M}}(X_i)$ is the disjoint union of $[\emptyset]$ and $\mathrm{Exp}^*_{\mathrm{M}}(X_i)$, $\mathrm{Exp}_{\mathrm{M}}(X)$ splits as

$$\mathrm{Exp}_{\mathrm{M}}(X) = \prod_{i \in \pi_0(X)}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{M}}(X_i) = \coprod_{\substack{J \subset \pi_0(X) \\ J \text{ finite}}} \prod_{j \in J} \mathrm{Exp}^*_{\mathrm{M}}(X_j)$$

immediately showing that the map $\exists$ is continuous and that $\pi_0(\exists)$ is a bijection. Let $S \subset X$ be a finite subset. Then, since $S \cup S = S$, the monoid structure of $\mathrm{Exp}_{\mathrm{M}}(X)$ induces an associative and idempotent map

$$\pi_n(\mathrm{Exp}_{\mathrm{M}}(X), S) \times \pi_n(\mathrm{Exp}_{\mathrm{M}}(X), S) \to \pi_n(\mathrm{Exp}_{\mathrm{M}}(X), S)$$

which satisfies the exchange property, for every $n > 0$. As a consequence, each of these groups is trivial (Lemma 2.22).

When X is separated, the canonical bijection

$$\mathrm{Exp}_{\mathrm{T}}(X) \to \prod_{i \in \pi_0(X)}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}(X_i)$$

is sequentially continuous with continuous inverse (see Proposition 2.8) and thus one has

$$\pi_0(\mathrm{Exp}_{\mathrm{T}}(X)) = \pi_0\Big(\prod_{i \in \pi_0(X)}^{\mathrm{f}} \mathrm{Exp}_{\mathrm{T}}(X_i)\Big)$$

because the segment $[0, 1]$ is a sequential space. Since spheres and balls are also sequential spaces, the sequentially continuous map $\mathrm{Exp}_{\mathrm{T}}(X) \times \mathrm{Exp}_{\mathrm{T}}(X) \to \mathrm{Exp}_{\mathrm{T}}(X)$ still induces maps

$$\pi_n(\mathrm{Exp}_{\mathrm{M}}(X), S) \times \pi_n(\mathrm{Exp}_{\mathrm{M}}(X), S) \to \pi_n(\mathrm{Exp}_{\mathrm{M}}(X), S),$$

so using the same proof as for the metric case, we see that $\exists \colon \mathrm{Exp}_{\mathrm{T}}(X) \to \bigoplus_{i \in \pi_0(X)} \mathbf{I}_2$ is a weak equivalence. ∎

Remark 2.25. The above theorem was proven by Handel in the case where X is path-connected and separated [10, Cor. 4.3]. Curtis & Nhu showed that $\mathrm{Exp}_{\mathrm{M}}(X)$ is *homeomorphic* to a linear space, whenever X is a connected, locally path connected metric space, which is a countable union of finite dimensional compact spaces [7]; it is in particular *contractible* in the strong sense.

3. Interlude: Minimal enclosing balls

Given a normed vector space V and a proper finite subset $S \subset V$, a minimal enclosing ball for S is a closed ball $B \subset V$ which contains S and such that any other ball containing S has a bigger radius (Figure 1).

Using classic convex optimization results, one can show the existence and uniqueness of a minimal enclosing ball in the case of rotund reflexive normed vector spaces [19]. One may wonder whether the center c_S and the radius r_S of the minimal enclosing ball of a proper finite subset S vary continuously with S. This question is naturally posed using the metric exponential $\mathrm{Exp}_{\mathrm{M}}(V)$.

For such a general space as a reflexive vector space, one can only show that the center c_S varies continuously with S *for the weak topology* of V. The continuity of the center becomes strong if one instead considers a restricted version of the minimal enclosing ball problem. This is what we shall see here.

Definition 3.1 (Restricted minimal enclosing ball). Let V be a normed vector space and let $S \subset V$ be a proper finite subset of V. A restricted minimal enclosing ball is a closed ball $B \subset V$ containing S and whose center belongs to the convex hull $\mathrm{Conv}(S)$ of S, such that, any other ball with center in $\mathrm{Conv}(S)$ and containing S has a bigger radius.

Remark 3.2. In a Hilbert space H, the restricted minimal enclosing ball of $S \subset H$ coincides with its minimal enclosing ball.

Restricted minimal enclosing balls might not be unique for a given norm. We shall then restrict our attention to spaces that can be endowed with a norm with strictly convex unit ball.

Definition 3.3 (Rotund vector space). We shall say that a topological vector space is rotund if its topology can be induced by a norm for which the closed unit ball is

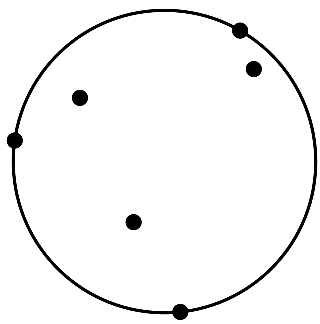

Figure 1. The enclosing circle of a finite set of points in the plane.

strictly convex: the equation

$$\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\|$$

holds only when $x = y$. By extension, we shall say that such a norm *is rotund.*

Example 3.4. Finite dimensional vector spaces are rotund. More generally separable complete normable spaces are rotund [5, Thm. 9]. The space ℓ^∞ is not rotund [8, Thm. 8]. Every reflexive normed vector space is rotund [16, Cor. 1 (i)].

Lemma 3.5. *Let V be a normed vector space. The correspondence sending $S \in \mathrm{Exp}^*_{\mathrm{M}}(V)$ to its convex hull $\mathrm{Conv}(S) \subset V$ is continuous.*

Proof. It is upper hemicontinuous: let $v \in V$ and let $\varepsilon > 0$. One has

$$\mathrm{Conv}^{\mathrm{u}}(\mathrm{B}(v,\varepsilon)) := \{S \mid \mathrm{Conv}(S) \subset \mathrm{B}(v,\varepsilon)\} = \mathrm{B}(\{v\},\varepsilon)$$

showing that the upper inverse image preserves opens.

It is lower hemicontinuous: let

$$S \in \mathrm{Conv}^{\mathrm{l}}(\mathrm{B}(v,\varepsilon)) := \{S \mid \mathrm{Conv}(S) \cap \mathrm{B}(v,\varepsilon) \neq \emptyset\},$$

then there exists $s \in S$ such that $\|s - v\| < \varepsilon$. Let $\delta = \varepsilon - \|s - v\|$, then $\mathrm{B}(S,\delta) \subset \mathrm{Conv}^{\mathrm{l}}(\mathrm{B}(v,\varepsilon))$ showing that the lower inverse image preserves opens. ■

Lemma 3.6. *The function*

$$\begin{aligned} V \times \mathrm{Exp}^*_{\mathrm{M}}(V) &\to \mathbf{R}_+, \\ (v, S) &\mapsto \max_{s \in S} \|v - s\| \end{aligned}$$

is continuous.

Proof. Consider a converging sequence $(v_n, S_n) \to_{n\to\infty} (v, S)$. For $\varepsilon > 0$ small enough, if S_n is at distance less than ε from S, then S_n must have more points than S and for each $x \in S_n$, there is a unique $s_x \in S$ such that $\|x - s_x\| \leq \varepsilon$. This gives us a partition of S_n as $S_n = \bigcup_{s \in S} S_n(s)$. Then for $\|v_n - v\| \leq \varepsilon$ and $D(S_n, S) \leq \varepsilon$ one has

$$\begin{aligned} \left| \max_{s \in S} \|v - s\| - \max_{t \in S_n} \|v_n - t\| \right| &\leq \max_{s \in S} \left| \|v - s\| - \max_{t \in S_n(s)} \|v_n - t\| \right| \\ &\leq \max_{s \in S} \Big(\|v - v_n\| + \max_{t \in S_n(s)} \|s - t\| \Big) \\ &\leq 2\varepsilon, \end{aligned}$$

showing that $(v, S) \mapsto \max_{s \in S} \|v - s\|$ is continuous. ■

Theorem 3.7 (Solution to the restricted minimal enclosing ball problem). *Let V be a vector space endowed with a rotund norm. Then every proper finite subset $S \subset V$ admits a restricted minimal enclosing ball of radius*

$$\mathrm{r}_S := \inf_{v \in \mathrm{Conv}(S)} \max_{s \in S} \|s - v\|$$

and this ball is unique.

Moreover, the function

$$\mathrm{Exp}^*_{\mathrm{M}}(V) \to V \times \mathbf{R}_+$$

mapping a proper finite subset $S \subset V$ to the pair $(\mathrm{c}_S, \mathrm{r}_S)$ where c_S denotes the center of the restricted minimal enclosing ball of S, is continuous.

Proof. Because S is finite, its convex hull $\mathrm{Conv}(S)$ is compact in V. As a consequence, r_S is finite and S admits a restricted minimal enclosing ball. It is unique: because the norm is rotund, the function $v \mapsto \max_{s \in S} \|s - v\|$ is strictly convex, so its infimum on the convex hull $\mathrm{Conv}(S)$ is attained at a unique point.

The continuity of c_S and r_S can be obtained using the maximum theorem [1, Thm. 17.31]: the correspondence $S \mapsto \mathrm{Conv}(S)$ is continuous with compact values and the function $(v, S) \mapsto \max_{s \in S} \|s - v\|$ is continuous by the previous lemmas. ■

4. Stratification of the exponentials

The exponential of a set X admits a natural counting function $\mathrm{Exp}(X) \to \mathbf{N}$ sending each finite subset $S \subset X$ to its cardinal $|S|$. In this section, we study the exponentials endowed with the stratification given by this counting function. We shall show that, under some conditions on X, the metric exponential $\mathrm{Exp}_{\mathrm{M}}(X)$ is conically stratified. The final result is that the ∞-categories of constructible hypersheaves on $\mathrm{Exp}_{\mathrm{M}}(X)$ and $\mathrm{Exp}_{\mathrm{T}}(X)$ are equivalent, which leads to the following statement: locally constant factorization algebras on X in the sense of Beilinson–Drinfeld are equivalent to locally constant factorization algebras on X in the sense of Lurie.

4.1. The exponentials as stratified spaces

There are several inequivalent definitions of stratified spaces. The following one is a mild one, introduced by Lurie [17, Def. A.5.1].

Definition 4.1 (Stratified space). A stratified space is the data of a poset P, endowed with the topology whose open sets are the upward-closed subsets, and a continuous map $f : X \to P$.

A morphism of stratified spaces is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ P & \longrightarrow & Q \end{array}$$

where the top map is continuous and the bottom map is a poset map.

In our case, we select the poset ω_*. Since the empty configuration is dense in $\mathsf{Exp}_\vee(X)$, the minimal exponential shall never be stratified over ω_* or even ω as soon as X is not empty. We shall thus only have a look at the two other exponentials.

Proposition 4.2. *When X is separated, the canonical maps*

$$\mathrm{Exp}_{\mathrm{T}}(X) \to \mathrm{Exp}_{\mathrm{M}}(X) \to \omega_*$$

are continuous. Moreover, one has homeomorphisms

$$\mathrm{Exp}_{\mathrm{T}}^{\leq_* n}(X) = \mathrm{Exp}_{\mathrm{M}}^{\leq_* n}(X)$$

for every $n \in \omega_*$.

Proof. We need to show that $\mathrm{Exp}_{\mathrm{M}}^{\leq_* n}(X)$ is a closed subset of $\mathrm{Exp}_{\mathrm{M}}(X)$ for every $n \in \omega_*$. For $n = 0$ this is obvious. Let S be a non-trivial finite subset of X. Since X is separated, one can find a disjoint family of open neighborhoods $\{s \in U_s\}_{s \in S}$. Then $[U_s]_{s \in S}$ becomes an open neighborhood of S in $\mathrm{Exp}_{\mathrm{M}}(X)$ which lies in the complement of $\mathrm{Exp}_{\mathrm{M}}^{\leq_* n}(X)$.

When X is separated, Handel has shown that the opens of the form $[U_i]_{i \in I} \cap \mathrm{Exp}_{\mathrm{T}}^{\leq_* n}(X)$ form a basis of the topology of $\mathrm{Exp}_{\mathrm{T}}^{\leq_* n}(X)$ [10, Prop. 2.11], giving us the homeomorphism $\mathrm{Exp}_{\mathrm{T}}^{\leq_* n}(X) = \mathrm{Exp}_{\mathrm{M}}^{\leq_* n}(X)$ for every $n \in \omega_*$. ■

4.2. Cones and joins

Definition 4.3 (Geometric open cone). For a topological space X, the *geometric open cone* of X is the set

$$\mathrm{C}(X) := \{0\} \amalg (\mathbf{R}_+^* \times X)$$

with topology defined as follows: A subset $U \subset \mathrm{C}(X)$ is open if and only if $U \cap (\mathbf{R}_+^* \times X)$ is open, and if $0 \in U$, then $(0, \varepsilon) \times X \subset U$ for some positive real number ε.

If X is stratified over a poset P, then $\mathrm{C}(X)$ is naturally stratified over the poset $P^{\triangleleft}$ obtained from P by adding a new element smaller than every other element of P.

Warning 4.4. One should not confuse the *geometric open cone* on X with the topologist's *open cone* defined as the quotient $\mathbf{R}_+ \times X/\{0\} \times X$. When X is compact and separated, the geometric cone on X and the open cone on X are homeomorphic. This is no longer true in the general case: the geometric cone on the open interval $(0,1)$ can be embedded in $\mathbf{R}^2$, whereas the open cone on $(0,1)$ is not metrizable.

If (X,d) is a metric space, the topology of the geometric open cone $\mathrm{C}(X)$ is metrizable by letting $d((\lambda,x),(\mu,y)) = \max(|\lambda-\mu|, d(x,y))$ and by adding $d(0,(\lambda,x)) = \lambda$.

Definition 4.5 (Geometric join). Given two posets P and Q, their geometric join is the poset

$$P \bowtie Q := P \amalg (P \times Q) \amalg Q$$

where one adds to the disjoint sum the additional relations $p < (p,q)$ and $q < (p,q)$ for every $(p,q) \in P \times Q$.

Let $X \to P$ and $Y \to Q$ be two stratified topological spaces. Their geometric join $X \bowtie Y$ is the set

$$X \bowtie Y := X \amalg (X \times (0,1) \times Y) \amalg Y$$

where a basis of opens is given by the opens $U \subset X \times (0,1) \times Y$ together with opens $X \amalg X \times (0,\varepsilon) \times V$ with $V \subset Y$ open, and opens $W \times (\delta,1) \times Y \amalg Y$ with $W \subset X$ open.

It is naturally stratified over $P \bowtie Q$.

Warning 4.6. Similarly to what we just said about cones, when X and Y are both separated and compact, the *geometric join* of X and Y is homeomorphic to the topologist's *join* $X \times [0,1] \times Y/R$ where R is the relation identifying $X \times \{0\} \times Y \sim X$ and $X \times \{1\} \times Y \sim Y$. In general, this is no longer the case.

Proposition 4.7. *Let $X \to P$ and $Y \to Q$ be two stratified spaces. Then there is a homeomorphism*

$$\mathrm{C}(X) \times \mathrm{C}(Y) \cong \mathrm{C}(X \bowtie Y)$$

over the canonical isomorphism $P^{\triangleleft} \times Q^{\triangleleft} = (P \bowtie Q)^{\triangleleft}$.

Proof. The map sends bijectively tuples $(\lambda,(x,t,y)) \in \mathrm{C}(X \bowtie Y)$ to tuples $((\lambda t, x), (\lambda(1-t), y)) \in \mathrm{C}(X) \times \mathrm{C}(Y)$ and obviously respects the isomorphism $P^{\triangleleft} \times Q^{\triangleleft} = (P \bowtie Q)^{\triangleleft}$. Let us see why it is open: there are four different cases to look at.

Case 1: open neighborhoods of the tip of $\mathrm{C}(X \bowtie Y)$. Let $\varepsilon > 0$, then the open $\{0\} \amalg (0,\varepsilon) \times (X \bowtie Y)$ is mapped to the open $(\{0\} \amalg (0,\varepsilon) \times X) \times (\{0\} \amalg (0,\varepsilon) \times Y)$.

Case 2: open neighborhoods of $\mathrm{C}(X \bowtie Y)$ not containing the tip but including X. An open of the form $(\alpha,\beta) \times (X \amalg X \times (0,\varepsilon) \times V)$ with $0 < \alpha < \beta$ is mapped to the open $(\{0\} \amalg (0,\varepsilon\alpha) \times X) \times ((1-\varepsilon)\alpha, (1-\varepsilon)\beta) \times V$.

Case 3: open neighborhoods of $\mathrm{C}(X \bowtie Y)$ not containing the tip but including Y. Confere supra.

Case 4: a general open of $\mathrm{C}(X \bowtie Y)$. Let $U \subset X$, $V \subset Y$ opens, $0 < \alpha < \beta$, $0 \leq t < s \leq 1$. Then $(\alpha, \beta) \times U \times (t, s) \times V$ is mapped to the open $(t\alpha, s\beta) \times U \times ((1-s)\alpha, (1-t)\beta) \times V$.

Since the image of basis neighborhoods form a basis of neighborhoods of $\mathrm{C}(X) \times \mathrm{C}(Y)$, one can see that it is a homeomorphism. ∎

Remark 4.8. A very similar proposition has been given by Ayala, Francis and Tanaka using the topologist's open cone and join instead of the geometric ones [3, Sect. 3.4.1]. Of course, both propositions agree in the case where both X and Y are compact and separated.

4.3. Conical stratification

There are many inequivalent notions of "goodness" for stratified space. The definition below is a mild one introduced by Lurie [17, Def. A.5.5].

Definition 4.9 (Conically stratified space). Let $f : X \to A$ be a stratified topological space. One says that X is conically stratified whenever for each $p \in A$ and each $x \in X_p$, there exists an open neighborhood $U_p \subset X_p$ of x and a stratified space L over $P_{p<}$ such that $U_p \subset X_p$ can be extended to a stratified space over the poset map $P_{p<}^{\triangleleft} = P_{p\leq} \subset P$.

We have already seen that the minimal exponential $\mathsf{Exp}_{\vee}(X)$ is never stratified. Even though the topological exponential is always a stratified space over ω_* when X is separated, it is usually impossible for the topological exponential $\mathrm{Exp}_{\mathrm{T}}(X)$ to be conically stratified; conical opens would allow sequences with unbounded cardinality to converge (Remark 2.7) [13, Thm. 2.14]. We shall then restrict our attention to the metric exponential $\mathrm{Exp}_{\mathrm{M}}(X)$ and show that it is conically stratified for a large class of spaces X.

Lemma 4.10. *Let V be a normed vector space, then the function*

$$\mathbf{R}_+ \times V \times \mathrm{Exp}_{\mathrm{M}}(V) \to \mathrm{Exp}_{\mathrm{M}}(V)$$

sending a triple (λ, v, S) to the configuration

$$\lambda S + v := \{\lambda s + v \mid s \in S\}$$

is continuous.

Proof. One has

$$D(\lambda S + v, \mu T + w) \leq \|v - w\| + \lambda D(S, T) + |\lambda - \mu| D(0, T)$$

which shows that the function is continuous. ∎

Proposition 4.11. *Let V be a rotund vector space and let us denote by $\mathrm{S_M}(V) \subset \mathrm{Exp}^*_{\mathrm{M}}(V)$ the subspace of configurations whose minimal enclosing ball has center 0 and radius 1. Since such a configuration must have at least two points, $\mathrm{S_M}(V)$ is naturally stratified over the poset $\omega_{2\leq}$. Then, one has a canonical homeomorphism*

$$\mathrm{Exp}^*_{\mathrm{M}}(V) = V \times \mathrm{C}(\mathrm{S_M}(V))$$

over the isomorphism $\omega_{1\leq} = \omega_{2\leq}^{\triangleleft}$.

Proof. In both cases, the map sends one point configurations $v \in V$ to the tuple $(v, 0)$ where 0 represents the tip of the cone, and sends multiple point configurations $S \subset V$ to the tuple $(\mathrm{c}_S, (\mathrm{r}_S, \mathrm{r}_S^{-1}(S - \mathrm{c}_S)))$. The inverse map simply sends tuples $(v, (\lambda, S))$ to $\lambda S + v$.

By the previous lemma and since $S \mapsto \mathrm{c}_S$ and $S \mapsto \mathrm{r}_S$ are continuous, it is clear that the bijection restricts to a homeomorphism between the open subspace of non-punctual configurations on one side and the product of V with the interior of the cone on the other side.

Finally, if $S_n \to v$ is a converging sequence with limit a punctual configuration, then by continuity $\mathrm{c}_{S_n} \to v$ and $\mathrm{r}_{S_n} \to 0$, which means that the image of S_n converges to $(v, 0)$ by definition of the topology of the cone. Conversely, if $(v_n, (\lambda_n, S_n))$ is a sequence converging to $(v, 0)$, this means by definition of the topology of the cone that $\lambda_n \to 0$ and since S_n is bounded, then $\lambda_n S_n \to 0$ so that $v_n + \lambda_n S_n \to v$ in $\mathrm{Exp}^*_{\mathrm{M}}(V)$. ∎

Theorem 4.12. *When X is a separated topological space locally homeomorphic to a rotund vector space, then $\mathrm{Exp_M}(X)$ is conically stratified.*

Proof. Since the empty set is a disjoint point from the rest of the space, it emits a conical neighborhood trivially. Let $S \in \mathrm{Exp}^*_{\mathrm{M}}(M)$ so that $|S| > 0$. By assumption, for each $s \in S$, one can find an open embedding $V_s \hookrightarrow X$ carrying the origin of a rotund vector space V_s to $s \in X$. Moreover, these can be chosen to be disjoint in X.

Since $\mathrm{Exp_M}$ is an exponential which also preserves open embeddings, one can build a stratified open embedding

$$\begin{array}{ccccccc}
\prod_{s\in S} \mathrm{Exp}^*_{\mathrm{M}}(V_s) & \hookrightarrow & \prod_{s\in S} \mathrm{Exp_M}(V_s) & \xrightarrow{\cong} & \mathrm{Exp_M}\left(\coprod_{s\in S} V_s\right) & \hookrightarrow & \mathrm{Exp_M}(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{s\in S} \omega_{1\leq} & \xrightarrow{=} & \prod_{s\in S} \omega_{1\leq} & \xrightarrow{+} & \omega_* & \xrightarrow{=} & \omega_*
\end{array}$$

whose image contains S.

Since a finite product of cones is again homeomorphic to a cone as a stratified space (Proposition 4.7) and since $\mathrm{Exp}^*_{\mathrm{M}}(V_s)$ is homeomorphic to $V_s \times \mathrm{C}(\mathrm{S_M}(V_s))$ for every $s \in S$ (Proposition 4.11), one gets stratified homeomorphisms

$$\begin{array}{ccccc}
\prod_{s\in S} \mathrm{Exp}^*_{\mathrm{M}}(V_s) & \xrightarrow{\cong} & \prod_{s\in S} V_s \times \mathrm{C}(\mathrm{S}_{\mathrm{M}}(V_s)) & \xrightarrow{\cong} & \prod_{s\in S} V_s \times \mathrm{C}(\bowtie_{s\in S} \mathrm{S}_{\mathrm{M}}(V_s)) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{s\in S} \omega_{1\leq} & \xrightarrow{\cong} & \prod_{s\in S} (\omega_{2\leq})^{\triangleleft} & \xrightarrow{\cong} & (\bowtie_{s\in S} \omega_{2\leq})^{\triangleleft}
\end{array}$$

which concludes the proof. ∎

Example 4.13. Since Fréchet manifolds are locally homeomorphic to Hilbert spaces [21, Thm. 6.1] and Hilbert spaces are rotund, $\mathrm{Exp}_{\mathrm{M}}(V)$ is conically stratified when V is a Fréchet manifold.

Remark 4.14. Since $\mathrm{Exp}_{\mathrm{M}}(X)$ is conically stratified, it follows that each truncated version $\mathrm{Exp}^{\leq * n}_{\mathrm{M}}(X)$ is also conically stratified. This truncated result was obtained by Ayala, Francis and Tanaka for X a manifold [3, Prop. 3.7.5].

Corollary 4.15. *Let X be a metrizable space, locally homeomorphic to a rotund topological vector space. Then, the ∞-categories of ω_*-constructible hypersheaves of spaces on $\mathrm{Exp}_{\mathrm{T}}(X)$ and $\mathrm{Exp}_{\mathrm{M}}(X)$ are canonically equivalent.*

Moreover, both can be represented as the ∞-category of functors from the exit path ∞-category $\mathsf{Exit}_{\omega_*}(\mathrm{Exp}_{\mathrm{T}}(X)) = \mathsf{Exit}_{\omega_*}(\mathrm{Exp}_{\mathrm{M}}(X))$ to the ∞-category of spaces.

Proof. Since we know that $\mathrm{Exp}_{\mathrm{M}}(X)$ is conically stratified, we only need to check the other axioms of the main theorem of *Constructible hypersheaves via exit paths* [13, Cor. 3.13]. Since X is metrizable, $\mathrm{Exp}_{\mathrm{M}}(X)$ is also metrizable and thus paracompact.

We now prove that each stratum is locally of singular shape. Being a local property, we can reduce to the case where X is homeomorphic to a separated locally convex topological vector space V [17, Rem. A.4.16]. The stratum $0 \in \omega_*$ amounts to a single point so there is nothing to prove.

Assume $n \geq 1$. By assumption the convex open sets form a basis of the topology of V which is stable under finite intersections. As a consequence, the opens of the form $[C_s]_{s\in S} \cap \mathrm{Exp}^n_{\mathrm{M}}(V)$ where $\{C_s\}_S$ is a family of $|S| = n$ disjoint convex open subsets of V, form a basis of the topology of $\mathrm{Exp}^n_{\mathrm{M}}(V)$ which is stable under intersection. It is then enough to see that $[C_s]_{s\in S} \cap \mathrm{Exp}^n_{\mathrm{M}}(V)$ has singular shape [17, Lem. A.4.14], which immediately follows from the fact that $[C_s]_{s\in S} \cap \mathrm{Exp}^n_{\mathrm{M}}(V)$ is homeomorphic to $\prod_{s\in S} C_s$ and is thus contractible. ∎

In short, this corollary says that the definition of locally constant factorization algebras from Beilinson–Drinfeld agrees with that of Lurie. The metrizability axiom above cannot be easily removed as shown by the following proposition.

Proposition 4.16. *There exists a paracompact topological space X for which neither $\mathrm{Exp}_{\mathrm{T}}(X)$ nor $\mathrm{Exp}_{\mathrm{M}}(X)$ is paracompact.*

Proof. Let X be the set of real numbers $\mathbf{R}$ endowed with the lower limit topology: the topology whose basis of opens is made of the half open intervals $[a, b)$. This space is paracompact but the product X^2 is not even normal [20]. Since the quotient map $X^2 \to X^2_{\mathbf{S}_2}$ is closed ($\mathbf{S}_2$ being finite), $X^2_{\mathbf{S}_2}$ is also not normal. As it is a closed subset of both $\mathrm{Exp}_{\mathrm{T}}(X)$ and $\mathrm{Exp}_{\mathrm{M}}(X)$ by the previous lemma, neither can be paracompact. ■

5. Open questions

Open question 5.1. The metric exponential of a Fréchet manifold is conically stratified (Theorem 4.12). Can this result be extended to conically stratified manifolds?

Open question 5.2. Can one relate factorization algebras in the sense of Costello–Gwilliam with some factorizable sheaves on the minimal exponential?

Open question 5.3. Is there a way to relate (locally constant) factorization algebras in the sense of Costello–Gwilliam with the ones from Beilinson–Drinfeld and Lurie?

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