High frequency forcing of an attracting heteroclinic cycle

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Abstract. This article is concerned with the effect of time-periodic forcing on a vector field exhibiting an attracting heteroclinic network. We show that as the forcing frequency tends to infinity, the dynamics reduces to that of a network under constant forcing, the constant being the average value of the forcing term. We also show that under small constant forcing the network breaks up into an attracting periodic solution that persists for periodic forcing of high frequency.

1. Introduction

Heteroclinic cycles organize the dynamics in a wide range of systems: ecological models of competing species [1, 13, 16], thermal convection [18, 20], game theory [6, 11] and climate science [4]. A paradigmatic example of a robust heteroclinic cycle occurs in Guckenheimer's and Holmes' three-dimensional system [13], also studied by May and Leonard [16] and by Busse and Heikes [5]. Although their initial models have periodic forcing terms, all the theory has been developed for the autonomous case. In this case, the equations are symmetric under permutation of coordinates.

With the non-autonomous forcing terms removed, each of the equilibria on the coordinate axes is of saddle type, and the existence of connecting orbits has been proved. Moreover, attracting heteroclinic networks have been found in an open set in the space of parameters [13]. Other examples from the dissipative category include the equations of Lorenz, Duffing and Lorentz gases acted on by external forces [7].

Symmetry-breaking constant perturbations to robust heteroclinic cycles with one-dimensional connections are well known to result in long-period periodic solutions that lie close to the original cycle. To date there has been very little systematic investigation of the effects of perturbations that are time-periodic, despite being natural for the modelling of many biological effects as, for instance, the effect of seasonality in epidemiological models [10].

From a mathematical point of view, one might expect to make comparisons between the consequences of time-periodic forcing on a heteroclinic cycle and the well-known effects of time-periodic forcing on periodic oscillations, for example, frequency locking (existence of periodic solutions whose period is an integer multiple of the period of the external forcing) and the effects of low/high frequency. These general observations provide double motivations for this work.

In a series of papers, the authors of [9, 19, 23, 24] considered the effect of small-amplitude time-periodic forcing of an attracting heteroclinic network and describe how to reduce the dynamics to a two-dimensional map. In the limit where the heteroclinic cycle is weakly attracting, intervals of frequency locking appear. In the opposite limit, where the heteroclinic cycle becomes strongly stable, no frequency locking is observed. In [21] it is proved that strange attractors are abundant near a heteroclinic cycle when the frequency ω of the forcing term satisfies $\omega \approx 0$, emphasizing the important role of ω on the dynamics. This discussion raised the following question:

(Q) could we identify and isolate the asymptotic effect of ω on the dynamics near a periodically-perturbed robust heteroclinic cycle?

There is no general theory for high-frequency forcing on attracting heteroclinic networks although the analysis of its effect in nonlinear oscillations may be useful in many applications. For instance, a high-frequency signal may block the conduction in the neural system [26].

In the present article, we examine the effect of high frequency periodic forcing on a nonlinear system of differential equations constructed in Aguiar et al. [3] which, in the absence of forcing, exhibits an asymptotically stable network. We prove that the dynamics of the system is equivalent to the averaged one as the frequency of the forcing tends to $+\infty$. Although our computations are done on the example of [3], our results are of far wider interest than the specific problem studied in this paper.

Structure

This article is organized as follows. In Section 2, we describe our object of study and, for the sake of completeness, we review in Section 3 some properties of the unperturbed system. We also motivate the model and describe how the problem under consideration fits in the literature.

The remainder of the article is dedicated to the proof of the main result. In Section 4, we construct suitable cross-sections near the equilibria where the local and global maps will be defined. In Section 5, we obtain the expressions that will be used to compute the first return map to a given cross-section. We also derive auxiliary results that will be helpful to analyse the asymptotic coefficients of the first return map. Section 6 proves the main contribution of this article as well as some dynamical

consequences. The existence of an attracting and hyperbolic solution near the cycle under high frequency forcing is explored in Section 7. We finish the article with a short discussion in Section 8. In Appendix A, we list some notation used throughout this article, as well as their meaning.

2. The model under consideration

Our object of study is the following two-parameter family of ordinary differential equations:

$$\dot{X} = F_{(\nu,\mu)}(X,t)$$

defined in $X = (x, y, z) \in \mathbf{R}^3$ by:

$$\begin{cases} \dot{x} = x(1 - r^2) - \alpha xz + \beta xz^2 + (1 - x)[\nu f(2\omega t) + \mu], \\ \dot{y} = y(1 - r^2) + \alpha yz + \beta yz^2, \\ \dot{z} = z(1 - r^2) - \alpha(y^2 - x^2) - \beta z(x^2 + y^2), \end{cases} \tag{2.1}$$

where

- $r^2 = x^2 + y^2 + z^2$,
- $\omega \in \mathbf{R}^+$,
- $\mu, \nu \in \mathbf{R}_0^+$ are two small and independent parameters, and
- $f : \mathbf{R} \to \mathbf{R}$ is a smooth, non-constant π/ω-periodic map such that

$$\int_0^{\pi/\omega} f(2\omega t)dt = 0.$$

We also assume that

$$\beta < 0 < \alpha, \quad |\beta| < \alpha \quad \Rightarrow \quad \beta^2 < 8\alpha^2. \tag{2.2}$$

Concerning equation (2.1), the amplitude of the autonomous perturbation is governed by the parameter μ whereas ν controls the amplitude of the non-autonomous term. The parameter ω is what we call the *frequency* of the periodic forcing. We refer to $\dot{X} = F_{(0,0)}(X,t)$ as the *unperturbed system*.

3. Motivation and state of the art

In this section, we recall some basic features associated to the system (2.1) when $\nu = \mu = 0$. For the terminology of equivariant differential equations and heteroclinic structures we refer the reader to the book by Golubitsky and Stewart [12].

3.1. The construction of the unperturbed system

Equation (2.1) was originally constructed to obtain a symmetric heteroclinic network associated to two equilibria, using a general construction described in [3], that we proceed to summarise.

Start with the differential equation

$$\dot{X} = (1 - \|X\|^2)X$$

for $X = (x, y, z) \in \mathbf{R}^3$, for which the unit sphere $\mathbf{S}^2$ attracts all points except the origin (i.e., is globally attracting) and all its points are equilibria. Then consider the finite Lie group $\mathscr{G} \subset \mathbb{O}(3)$ generated by the two linear maps

$$\kappa_1(x, y, z) = (-y, x, -z) \quad \text{and} \quad \kappa_2(x, y, z) = (x, -y, z).$$

Add two $\mathscr{G}$-equivariant perturbing terms of order 3, say $\alpha A(X)$ and $\beta B(X)$, to the differential equation. These two terms are chosen to be tangent to $\mathbf{S}^2$, so this sphere is still flow-invariant and attracting. The new equations are then $\mathscr{G}$-equivariant, and therefore, for each subgroup $\mathscr{H} \subset \mathscr{G}$ the fixed point subspace $\text{Fix}(\mathscr{H}) = \{X \in \mathbf{R}^3 : \kappa(X) = X \, \forall \kappa \in \mathscr{H}\}$ is flow-invariant (see [12]). Hence the action of $\mathscr{G}$ on $\mathbf{R}^3$ has the following symmetry flow-invariant planes corresponding to subgroups generated by elements $\kappa \in \mathscr{G}$ such that κ^2 is the identity:

$$\text{Fix}(\kappa_2 \circ \kappa_1^2) = \{X \in \mathbf{R}^3 : \kappa_2 \circ \kappa_1^2(X) = X\} = \{(x, y, z) \in \mathbf{R}^3 : x = 0\},$$
$$\text{Fix}(\kappa_2) = \{X \in \mathbf{R}^3 : \kappa_2(X) = X\} = \{(x, y, z) \in \mathbf{R}^3 : y = 0\},$$

and symmetry axes:

$$\text{Fix}(\kappa_1^2) = \{(x, y, z) \in \mathbf{R}^3 : x = 0 \text{ and } y = 0\},$$
$$\text{Fix}(\kappa_2 \circ \kappa_1^3) = \{(x, y, z) \in \mathbf{R}^3 : x = y \text{ and } z = 0\},$$
$$\text{Fix}(\kappa_2 \circ \kappa_1) = \{(x, y, z) \in \mathbf{R}^3 : x = -y \text{ and } z = 0\}.$$

3.2. Dynamics of the unperturbed system

The intersection of the flow-invariant sphere $\mathbf{S}^2$ and $\text{Fix}(\langle\kappa\rangle)$, $\kappa \in \mathscr{G}$, is a flow-invariant set. Then, as illustrated in Figure 1, the intersection of this sphere with $\text{Fix}(\kappa_1^2)$ gives rise to two saddle type equilibria

$$\mathbf{v}_+ = (0, 0, 1), \quad \text{and} \quad \mathbf{v}_- = (0, 0, -1),$$

where the derivative of $F_{(0,0)}$ is

$$DF_{(0,0)}(0, 0, \sigma) = \begin{pmatrix} \beta - \sigma\alpha & 0 & 0 \\ 0 & \beta + \sigma\alpha & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{where } \sigma = \pm 1.$$
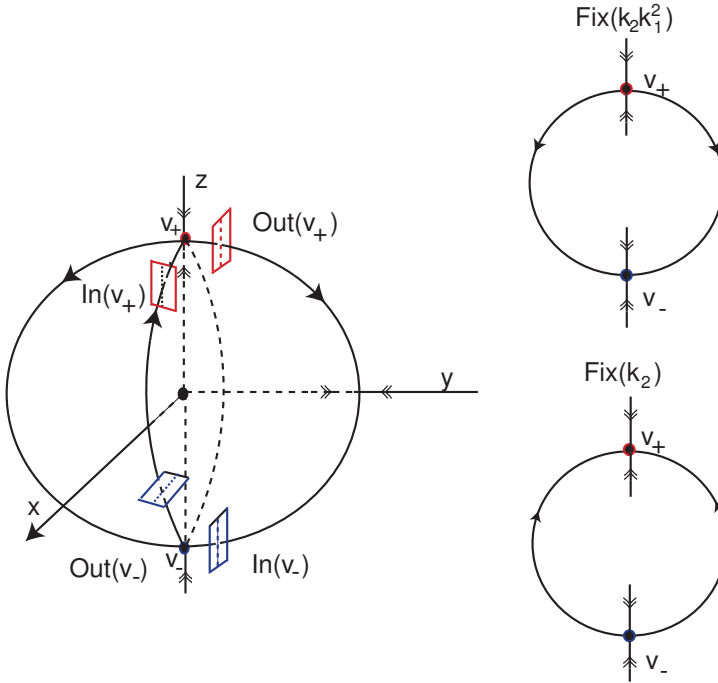
Figure 1. Sketch of the heteroclinic connections when $\mu = \nu = 0$.

On $\mathbf{S}^2$ there are also four unstable foci $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, 0)$ on the lines $\mathrm{Fix}(\kappa_2 \circ \kappa_1^3)$ and $\mathrm{Fix}(\kappa_2 \circ \kappa_1)$. The intersections of the sphere $\mathbf{S}^2$ with the planes $\mathrm{Fix}(\kappa_2 \circ \kappa_1^2)$ and $\mathrm{Fix}(\kappa_2)$ generate two pairs of one-dimensional heteroclinic connections linking the equilibria $\mathbf{v}_+$ and $\mathbf{v}_-$, as depicted in Figure 1.

The union of these equilibria and connections forms four heteroclinic cycles and, taken together, a heteroclinic network denoted Γ. The two-dimensional coordinate subspaces $\mathrm{Fix}(\kappa_2 \circ \kappa_1^2) = \{x = 0\}$ and $\mathrm{Fix}(\kappa_2) = \{y = 0\}$ are flow-invariant and separate the space $\mathbf{R}^3$, hence trajectories starting on a connected component of $\mathbf{R}^3 \backslash (\{x = 0\} \cup \{y = 0\})$ cannot visit another component. Thus, nearby trajectories only visit a neighbourhood of one cycle in the network, hence there is no *switching* in the sense of [2].

Within each of the invariant planes defined by $x = 0$ and $y = 0$, each connecting orbit is a saddle-sink connection. Therefore, the network Γ is robust in the class of $\mathcal{G}$-symmetric vector fields.

Define the map $g : \mathbf{R}^3 \to \mathbf{R}^3$ such that $g(x, y, z) = (x - y)^2 + z^2$. If $\beta = 0$, then the Lie derivative of $F_{(0,0)}$ with respect to g is identically zero on $\mathbf{S}^2$, which means that trajectories of the flow of (2.1) on $\mathbf{S}^2$ move along closed trajectories that are the intersection of $\mathbf{S}^2$ and the level surfaces of g. Therefore, the perturbation $\beta B(X)$ and

condition (2.2) force the foci to be *unstable* when restricted to $\mathbf{S}^2$, and the network Γ to be *globally asymptotically stable* (cf. [3]). Typical trajectories starting near Γ accumulate on one of the cycles in the network and remain near the equilibria for increasing periods of time. These trajectories make fast transitions from one equilibrium point to the next. In particular, there are no periodic solutions near Γ.

The constant $\delta = \frac{\alpha - \beta}{\alpha + \beta} > 1$ measures the *strength of attraction* of each equilibrium because it is the ratio between the contracting (negative tangent to $\mathbf{S}^2$) and the expanding (positive) eigenvalues of $DF_{(0,0)}(0, 0, \sigma)$. It is related with the ratio of consecutive times of sojourn near each equilibrium. Analogously, the constant δ^2 measures the *strength of attraction* of each cycle of Γ.

3.3. The perturbing terms

We add to (2.1) two perturbing terms of different nature, one governed by μ and the other by ν. The term governed by μ is an autonomous perturbation; the other is non-autonomous. If either $\nu > 0$ or $\mu > 0$, only the symmetry κ_2 remains and, in general, the network Γ is destroyed.

Our choice of perturbing term $(1 - x)[\nu f(2\omega t) + \mu]$ is made for two reasons: first, it simplifies the computations and allows comparison with previous work by other authors [3, 9, 14, 15, 19, 24]. Secondly, it simplifies the quantitative reduction of the differential equations to a map on a cylinder. The dynamics of the unperturbed equations will transfer the effect of the perturbation to the other coordinates.

We may split the family of perturbations as follows:

$$(1 - x)[\nu f(2\omega t) + \mu] = \underbrace{\nu (1 - x) f(2\omega t)}_{\text{non-autonomous forcing}} + \underbrace{\mu (1 - x)}_{\text{autonomous}}$$

where

$$\int_0^{\pi/\omega} f(2\omega t)dt = 0.$$

This splitting will be clearer in the sequel.

3.4. State of the art

When either $\mu \neq 0$ or $\nu \neq 0$ the plane defined by $x = 0$ is no longer invariant due to the $(1 - x)$ factor in the perturbing term, hence the connection from $\mathbf{v}_+$ to $\mathbf{v}_-$ is broken. The autonomous case $\mu > \nu = 0$ is the simplest one and may be explored as in [9, §5.2.1] to conclude that the dynamics is governed by an attracting periodic solution.

The case $\nu > \mu = 0$ and $\delta^2 \gtrsim 1 (\Leftrightarrow \delta \gtrsim 1)$ has been studied in [14], where the authors derived the first return map near a heteroclinic cycle for small amplitude of

the perturbing term and reduce the analysis of the non-autonomous system to that of a two-dimensional map on a cylinder.

They found rich dynamical features arising from a discrete-time Bogdanov–Takens bifurcation. When the perturbation strength is small, the first return map has an attracting invariant closed curve that is not contractible on the cylinder. Furthermore, the authors pointed out the existence of two distinct dynamical regimes corresponding to the existence or non-existence of intervals of frequency locking as ω varies. These findings are consistent with those of [23,24] who analysed a similar phenomenon near the May–Leonard cycle.

Increasing the perturbation strength there are periodic solutions that bifurcate into a closed contractible invariant curve and into a region where the dynamics is conjugate to a full shift on two symbols ($\Rightarrow$ chaos and strange attractors emerge).

For the case $\nu > \mu = 0$ and $\delta^2 \gg 1$, the bistability described for the case $\delta^2 \gtrsim 1$ disappears. Authors of [23, 24] have shown the equivalence of the dynamics to that of a circle map, and discuss whether the circle map is likely to be invertible or non-invertible. The existence of regular and chaotic dynamics largely depends on the order of ω as one may check in [21, Table 3].

3.5. Novelty

In the present paper, the forcing frequency is our principal bifurcation parameter and we are interested in the case ω large. Pursuing an answer to question (Q) in the context of strongly attracting systems ($\delta^2 \gg 1$), this work may be seen as the natural continuation of [14].

Our main result states that, when the forcing frequency $\omega \to +\infty$, the effect of the perturbation governed by (ν, μ) is reduced to the effect of the autonomous term. In other words, we show that the asymptotic dynamics associated to $\lim_{\omega \to +\infty} F_{(\nu,\mu)}$ is qualitatively the same as that of the averaged system $F_{(0,\mu)}$. The proof is performed via the construction of a first return map to a suitable cross-section.

As a consequence, when $\omega \to +\infty$, the non-autonomous equation (2.1) has become autonomous, and the system dimension decreases as the non-autonomous component for "fast motion" governed by ν has been dropped completely.

3.6. Insight into the reasoning

In order to improve the readability of the article, we sketch the route of the proof of the main result of the paper.

(1) We transform (2.1) into an autonomous equation in $\mathbf{R}^4$ by adding a new coordinate s for the forcing time. We call the new equation the *suspension of* (2.1).

(2) We obtain isolating blocks and cross-sections near each equilibrium for the unperturbed differential equation. After the addition of non-autonomous terms to (2.1) the equilibria may no longer be equilibria, but the cross-sections remain transverse to the flow of the suspension.

(3) We derive the expressions for the linearization of (2.1) near $\mathbf{v}_+$ and $\mathbf{v}_-$, say $\Phi_{\mathbf{v}_+}$ and $\Phi_{\mathbf{v}_-}$. The computation of these maps is a bit more tricky than in the autonomous hyperbolic case because some of the times of flight depend on f. These maps depend on the phase space coordinates (x_2, w_2) and also on the suspension time s.

(4) The local maps $\Phi_{\mathbf{v}_+}$ and $\Phi_{\mathbf{v}_-}$ depend on the two expressions $K_1(v, \mu)$ and $K_2(v, \mu)$, where

$$K_1(0, 0) = K_2(0, 0) = 0.$$

(5) In Section 5.5, we assume that $f(t) = \sin t$ in order to make analytic progresses in the explicit computation of $K_1(v, \mu)$ and $K_2(v, \mu)$ although we emphasise that the result is valid for any f satisfying the conditions of Section 2.

(6) After defining suitable global maps (natural in our scenario), we define the first return map $\mathcal{R}_{(v,\mu)}$ as the composition of local and global maps, and we prove the main result of this paper, the content of Theorem 3,

$$\lim_{\omega \to \infty} \mathcal{R}_{(v,\mu)} = \mathcal{R}_{(0,\mu)}.$$

(7) Finally, for the sake of completeness, we prove that the dynamics of $\mathcal{R}_{(0,\mu)}$ is governed by a sink whose period goes to $+\infty$ as μ goes to 0. When $\mu = 0$, the sink collapses into Γ.

4. Cross-sections

Our results will be obtained analysing the first return map to a suitable cross-section to the flow of (2.1), obtained from the transitions between four cross-sections for the unperturbed equation. In this section, we construct suitable cross-sections near the equilibria where the local and global maps will be defined. To do this, consider cubic neighbourhoods $V_\pm$ in $\mathbf{R}^3$ of $\mathbf{v}_\pm$,

$$V_\sigma = \{(x, y, w) : |x| < \varepsilon, |y| < \varepsilon, |w| < \varepsilon\}, \quad w = z - \sigma, \quad \sigma = \pm 1,$$

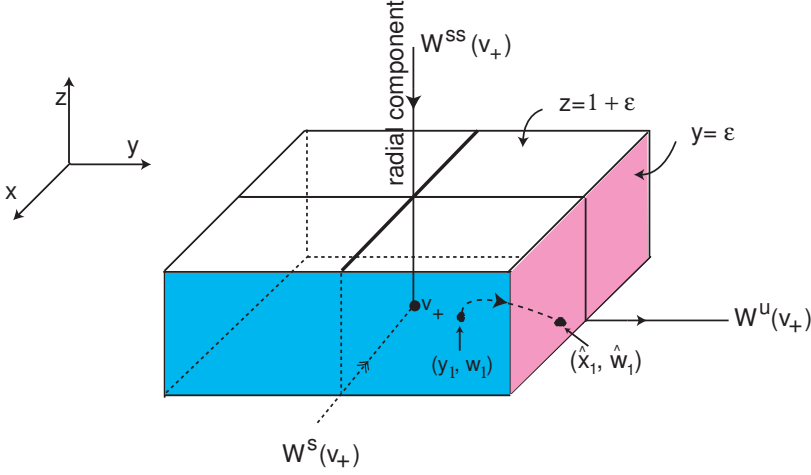for $\varepsilon > 0$ small. As suggested by Figure 2, we use the following cross-sections contained in the boundary of V_+.

Figure 2. Scheme of the cross-sections In($\mathbf{v}_+$) and Out($\mathbf{v}_+$) in the neighbourhood V_+ of $\mathbf{v}_+$. The set $W^{ss}(\mathbf{v}_+)$, the local strong stable manifold of $\mathbf{v}_+$, corresponds to the radial direction in the invariant sphere $\mathbf{S}^2$.

- In($\mathbf{v}_+$) = $\{(\varepsilon, y, w) : |y| < \varepsilon, |w| < \varepsilon\}$ with coordinates (y_1, w_1). It consists of points whose trajectories go into V_+ in small positive time.

- Out($\mathbf{v}_+$) = $\{(x, \varepsilon, w) : |x| < \varepsilon, |w| < \varepsilon\}$ with coordinates $(\widehat{x}_1, \widehat{w}_1)$. It consists of points whose trajectories go out of V_+ in small positive time.

The cross-sections contained in the boundary of V_- are

- In($\mathbf{v}_-$) = $\{(x, \varepsilon, w) : |x| < \varepsilon, |w| < \varepsilon\}$ with coordinates (x_2, w_2) with points whose trajectories go into V_- in small positive time,

- Out($\mathbf{v}_-$) = $\{(\varepsilon, y, w) : |y| < \varepsilon, |w| < \varepsilon\}$ with coordinates $(\widehat{y}_2, \widehat{w}_2)$ containing points whose trajectories go out of V_- in small positive time.

The local stable manifolds of $\mathbf{v}_+$ and $\mathbf{v}_-$ in the cross-sections In($\mathbf{v}_+$) and In($\mathbf{v}_-$) are given by

$$W^s(\mathbf{v}_+) \cap \text{In}(\mathbf{v}_+) = \{(0, w_1) : |w_1| < \varepsilon\}, \quad W^s(\mathbf{v}_-) \cap \text{In}(\mathbf{v}_-) = \{(0, w_2) : |w_2| < \varepsilon\}.$$

From now on we restrict our attention to the $y > 0$ component of In($\mathbf{v}_+$)\$W^s(\mathbf{v}_+)$ (respectively the $x > 0$ component of In($\mathbf{v}_-$)\$W^s(\mathbf{v}_-)$) and we abuse notation by calling it In($\mathbf{v}_+$) (respectively In($\mathbf{v}_-$)). All the results hold on the other component, but follow a different cycle in the heteroclinic network Γ. By rescaling the variables (x, y, z) and the parameters μ and ν we may take $\varepsilon = 1$ in the cross-sections defined above.

5. Local maps

The aim of this section is to obtain expressions that will be used to compute the first return map to the cross-section $\text{In}(\mathbf{v}_+)$ in the flow of (2.1). The computations are similar to those of [14], so we give only an overview. The expression for the first return map to this cross-section to Γ is obtained as the composition of two types of maps: *local maps* between the neighbourhood walls of each $V_\pm$, and *global maps* from one neighbourhood to the other. Here, we obtain the local maps (at leading order in ν, μ) by computing the point where a solution hits each cross-section and the time the solution takes to move between cross-sections.

Suspension

For $(\nu, \mu) \neq (0, 0)$ we consider the suspension of (2.1) given by

$$\begin{cases} \dot{X} = F_{(\nu,\mu)}(X, s), \\ \dot{s} = 1, \end{cases} \qquad \text{where } X = (x, y, z) \text{ and } s \in \mathbf{S}^1 \qquad (5.1)$$

and the augmented cross-sections $\mathbf{S}^1 \times \text{In}(\mathbf{v}_\pm)$ and $\mathbf{S}^1 \times \text{Out}(\mathbf{v}_\pm)$ to the suspended flow. Although $\mathbf{v}_+$ and $\mathbf{v}_-$ are no longer constant solutions of (2.1) for $(\nu, \mu) \neq (0, 0)$, for small (ν, μ) the augmented cross-sections defined above are still crossed transversely by trajectories.

5.1. Linearisation

The linearisation of (2.1) near $\mathbf{v}_\pm$ is

$$\begin{cases} \dot{x} = (\beta - \sigma\alpha)x - \nu f(2\omega t) - \mu, \\ \dot{y} = (\alpha + \sigma\beta)y, \\ \dot{w} = -2(w + \sigma), \end{cases} \qquad w = z - \sigma, \quad \sigma = \pm 1. \qquad (5.2)$$

Equation (2.1) may be written in the form

$$\dot{X} = \mathcal{M}X + R(X) - [\nu f(2\omega t) + \mu](1, 0, 0)^T$$

for $X^T = (x, y, z)$ where $\dot{X} = \mathcal{M}X - [\nu f(2\omega t) + \mu](1, 0, 0)^T$ is any of the equations (5.2) for $\sigma = \pm 1$ and

$$\mathcal{M} = \begin{pmatrix} \beta - \sigma\alpha & 0 & 0 \\ 0 & \beta + \sigma\alpha & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The constant matrix $\mathcal{M}$ has no eigenvalues with zero real part, both the perturbation $\nu f(2\omega t) + \mu$ and the nonlinear part $R(X)$ are bounded and the nonlinear part $R(X)$ is uniformly Lipschitz in a compact neighbourhood of $\mathbf{S}^2$. Under these conditions, Palmer's theorem [17, pp. 754] implies that there are neighbourhoods of $\mathbf{v}_+$ and $\mathbf{v}_-$, where the vector field is C^1 conjugate to its linearisation.

5.2. Local map near $\mathbf{v}_+$

The calculation of the first return map will use the form of the general solution of (5.2). For $z = 1 + w$ we get $\dot{w} = -2(1 + w)$. Using the Lagrange method of variation of parameters as described in [22, pp. 842], the solution of the linearised system (5.2) near $\mathbf{v}_+$, with initial condition $(x, y, w)(s) = (1, y_1, w_1) \in \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_+)\backslash W^s(\mathbf{v}_+)$ at time s, is

$$
\begin{cases}
x(t, s) = \mathrm{e}^{(\beta-\alpha)(t-s)}\left(1 - \int_s^t \mathrm{e}^{-(\beta-\alpha)(\tau-s)}(\nu f(2\omega\tau) + \mu)d\tau\right), \\
y(t, s) = y_1 \mathrm{e}^{(\alpha+\beta)(t-s)}, \\
w(t, s) = (w_1 + 1)\mathrm{e}^{-2(t-s)} - 1.
\end{cases} \tag{5.3}
$$

The *time of flight* T_1 from $\mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_+)\backslash W^s(\mathbf{v}_+)$ to $\mathbf{S}^1 \times \mathrm{Out}(\mathbf{v}_+)$ is defined as the minimum non-negative time T_1 such that a trajectory starting at $\mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_+)\backslash W^s(\mathbf{v}_+)$ hits $\mathbf{S}^1 \times \mathrm{Out}(\mathbf{v}_+)$. If $(1, y_1, w_1) \in \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_+)\backslash W^s(\mathbf{v}_+)$ then $T_1(y_1, w_1)$ is the solution $T_1 = t$ of $y(t) = 1$. Hence

$$
y(T_1) = 1 \quad \Leftrightarrow \quad y_1 \mathrm{e}^{(\alpha+\beta)(T_1-s)} = 1 \quad \Leftrightarrow \quad \ln\left(\frac{1}{y_1}\right) = (\alpha + \beta)(T_1 - s).
$$

In this case, T_1 does not depend on ν or on μ. These solutions arrive at $\mathrm{Out}(\mathbf{v}_+)$ at a time

$$
T_1 = s + \ln\left(\frac{1}{y_1}\right)^{\frac{1}{\alpha+\beta}} = s - \frac{1}{\alpha+\beta}\ln y_1.
$$

Replacing t by T_1 in the first and third equation of (5.3), we get the map

$$
\Phi_{\mathbf{v}_+} : \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_+) \rightarrow \mathbf{S}^1 \times \mathrm{Out}(\mathbf{v}_+),
$$

$$
\Phi_{\mathbf{v}_+}(s, y_1, w_1) = \begin{pmatrix} s - \frac{1}{\alpha+\beta}\ln y_1 \\ y_1^\delta\left(1 - \int_s^{T_1} \mathrm{e}^{-(\beta-\alpha)(\tau-s)}(\nu f(2\omega\tau) + \mu)d\tau\right) \\ (w_1 + 1)y_1^{\frac{2}{\alpha+\beta}} - 1 \end{pmatrix} = (T_1, \hat{x}_1, \hat{w}_1). \tag{5.4}
$$

5.3. Local map near $\mathbf{v}_-$

The treatment of (5.2) for $\sigma = -1$ is similar to Section 5.2, although the computations involve more steps. The solution of (5.2), with initial condition

$$(x, y, w)(s) = (x_2, 1, w_2) \in \mathbf{S}^1 \times \text{In}(\mathbf{v}_-)$$

at time s, is

$$
\begin{cases}
x(t) = x_2 e^{(\alpha+\beta)(t-s)} \left(1 - \dfrac{1}{x_2} \displaystyle\int_s^t e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau) + \mu) d\tau \right), \\[2mm]
y(t) = e^{(\beta-\alpha)(t-s)}, \\[2mm]
w(t) = (w_2 - 1)e^{-2(t-s)} + 1.
\end{cases}
\tag{5.5}
$$

The time T_2 of arrival at $\text{Out}(\mathbf{v}_-)$, starting at $\text{In}(\mathbf{v}_-)$ is more difficult to compute than T_1. This is why we use its Taylor expansion at $(0, 0)$ truncated at second order of ν and μ. We write $T_2(\nu, \mu)$ to stress its dependence on the bifurcation parameters.

Lemma 1. *The time of flight T_2 inside V_- of $(x_2, 1, w_2) \in \mathbf{S}^1 \times \text{In}(\mathbf{v}_-)$ only depends on x_2 and may be written as*

$$
T_2(\nu, \mu) = s - \frac{1}{\alpha+\beta} \ln x_2 + \left[\frac{1}{(\alpha+\beta)} \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau) + \mu) d\tau \right]
$$
$$
+ \mathcal{O}(\|(\nu, \mu)\|^2),
$$

where $\mathcal{O}(\|(\nu, \mu)\|^2)$ denotes the usual Landau notation and $T_2(0, 0) = s - \frac{\ln x_2}{\alpha+\beta}$.

Proof. Let us derive the Taylor expression of $T_2(\nu, \mu)$ of degree 1,

$$
T_2(\nu, \mu) = T_2(0, 0) + \frac{\partial T_2}{\partial \nu}(0, 0) + \frac{\partial T_2}{\partial \mu}(0, 0) + \mathcal{O}(\|(\nu, \mu)\|^2).
\tag{5.6}
$$

By definition of time of flight in V_-, we may write

$$
\begin{aligned}
1 &\equiv x(T_2(\nu, \mu), \nu, \mu) \\
&\overset{(5.5)}{=} e^{(\alpha+\beta)(T_2(\nu,\mu)-s)} \left(x_2 - \nu \int_s^{T_2(\nu,\mu)} e^{-(\alpha+\beta)(\tau-s)} f(2\omega\tau) d\tau \right.\\
&\qquad\qquad\qquad\qquad \left. - \mu \int_s^{T_2(\nu,\mu)} e^{-(\alpha+\beta)(\tau-s)} d\tau \right).
\end{aligned}
$$

For $\nu = \mu = 0$, we get

$$
x_2 e^{(\alpha+\beta)(T_2(0,0)-s)} = 1 \quad \Rightarrow \quad T_2(0, 0) = s - \frac{\ln x_2}{\alpha+\beta}.
\tag{5.7}
$$

Using the chain rule applied to the equality $x(T_2(\nu, \mu), \nu, \mu) = 1$ at $(\nu, \mu) = (0, 0)$, we get

$$\frac{\partial x}{\partial t}(T_2(0, 0), 0, 0)\frac{\partial T_2}{\partial \nu}(0, 0) + \frac{\partial x}{\partial \nu}(T_2(0, 0), 0, 0) = 0$$

and thus

$$(\alpha + \beta)x_2 e^{(\alpha+\beta)(T_2(0,0)-s)}\frac{\partial T_2}{\partial \nu}(0, 0) - \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)} f(2\omega\tau)d\tau = 0.$$

According to the left-hand side of (5.7), the previous equality may be simplified as

$$(\alpha + \beta)\frac{\partial T_2}{\partial \nu}(0, 0) - \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)} f(2\omega\tau)d\tau = 0,$$

$$\Leftrightarrow \frac{\partial T_2}{\partial \nu}(0, 0) = \frac{1}{(\alpha + \beta)} \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)} f(2\omega\tau)d\tau.$$

Analogously, we have

$$(\alpha + \beta)\frac{\partial T_2}{\partial \mu}(0, 0) - \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}d\tau = 0$$

and hence

$$\frac{\partial T_2}{\partial \mu}(0, 0) = \frac{1}{(\alpha + \beta)} \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}d\tau.$$

Replacing $T_2(0, 0)$, $\frac{\partial T_2}{\partial \nu}(0, 0)$ and $\frac{\partial T_2}{\partial \mu}(0, 0)$ in (5.6), it yields

$$T_2(\nu, \mu) = s - \frac{1}{\alpha + \beta} \ln x_2 + \left[\frac{1}{(\alpha + \beta)} \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau) + \mu)d\tau\right]$$
$$+ \mathcal{O}(\|(\nu, \mu)\|^2)$$

and the result follows. ∎

From now on, we omit the remainder $\mathcal{O}(\|(\nu, \mu)\|^2)$ of T_2 in the computations. Using the expression of T_2 obtained in Lemma 1 in (5.5), we may deduce that

$$y(T_2) = e^{(\beta-\alpha)(T_2-s)}$$
$$= \exp\left(-\frac{(\beta-\alpha)}{\alpha+\beta} \ln x_2\right.$$
$$\left. + \left[\frac{(\beta-\alpha)}{(\alpha+\beta)} \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau) + \mu)d\tau\right]\right)$$
$$= x_2^\delta \exp\left(-\delta \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau) + \mu)d\tau\right).$$

Analogously, we may write

$$w(T_2) = 1 + (w_2 - 1)e^{-2(T_2-s)}$$

$$= 1 + (w_2 - 1)\exp\left(\frac{2}{\alpha+\beta}\ln x_2\right.$$

$$+ \left[\frac{-2}{(\alpha+\beta)}\int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right]\right)$$

$$= 1 + (w_2 - 1)x_2^{2/(\alpha+\beta)}$$

$$\cdot \exp\left(\frac{-2}{(\alpha+\beta)}\int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right).$$

Therefore, we define the local map as

$$\Phi_{\mathbf{v}_-} : \mathbf{S}^1 \times \text{In}(\mathbf{v}_-) \to \mathbf{S}^1 \times \text{Out}(\mathbf{v}_-),$$

$$\Phi_{\mathbf{v}_-} = \begin{pmatrix} s - \frac{1}{\alpha+\beta}\ln x_2 + \frac{1}{(\alpha+\beta)}\int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau \\ x_2^\delta \exp\left(-\delta\int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right) \\ 1 + (w_2 - 1)x_2^{2/(\alpha+\beta)}\exp\left(\frac{-2}{\alpha+\beta}\int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)}(\nu f(2\omega\tau)+\mu)d\tau\right) \end{pmatrix}$$

$$= (T_2, \hat{y}_2, \hat{w}_2). \tag{5.8}$$

It follows that if $x_2 > 0$ then $\hat{y}_2 > 0$. To establish a similar statement for $\Phi_{\mathbf{v}_+}$ we will need the information of Lemma 2 below.

5.4. Summary

The expressions for $\Phi_{\mathbf{v}_+}$ (cf. (5.4)) and $\Phi_{\mathbf{v}_-}$ (cf. (5.8)) may be written as

$$\Phi_{\mathbf{v}_+}(s, y_1, w_1) = \begin{pmatrix} s - \frac{1}{\alpha+\beta}\ln y_1 \\ y_1^\delta(1 - K_1) \\ (w_1 + 1)y_1^{\frac{2}{\alpha+\beta}} \end{pmatrix} = (T_1, \hat{x}_1, \hat{w}_1)$$

and

$$\Phi_{\mathbf{v}_-}(s, x_2, w_2) = \begin{pmatrix} s - \frac{1}{\alpha+\beta}\ln x_2 + \frac{K_2}{(\alpha+\beta)} \\ x_2^\delta \exp(-\delta K_2) \\ 1 + (w_2 - 1)\left(x_2^{2/(\alpha+\beta)}\exp\left(\frac{-2K_2}{\alpha+\beta}\right)\right) \end{pmatrix} = (T_2, \hat{y}_2, \hat{w}_2)$$

where

$$\delta = \frac{\alpha - \beta}{\alpha + \beta} > 1,$$

$$K_1 = \int_s^{T_1} e^{-(\beta-\alpha)(\tau-s)} (\nu f(2\omega\tau) + \mu) d\tau,$$

$$K_2 = \int_s^{T_2(0,0)} e^{-(\alpha+\beta)(\tau-s)} (\nu f(2\omega\tau) + \mu) d\tau.$$

Both K_1 and K_2 depend on s, ν and μ. Furthermore, when $\mu = \nu = 0$, we get

$$(T_1, \hat{x}_1, \hat{w}_1) = \left(s - \frac{1}{\alpha + \beta} \ln y_1, y_1^\delta, (w_1 + 1)y_1^{\frac{2}{\alpha+\beta}} \right),$$

$$(T_2, \hat{y}_2, \hat{w}_2) = \left(s - \frac{1}{\alpha + \beta} \ln x_2, x_2^\delta, 1 + (w_2 - 1)x_2^{2/(\alpha+\beta)} \right),$$

corresponding to the expressions of the local maps for the unperturbed case. Note that, although the second and third coordinates of $\Phi_{\mathbf{v}_+}$ are well defined and equal to zero at $(s, y_1, w_1) = (s, 0, 0)$, the first coordinate tends to infinity as y_1 goes to zero, since this point corresponds to the heteroclinic connection from $\mathbf{v}_+$ to $\mathbf{v}_-$. In other words, $(s, 0, 0) \in \text{In}(\mathbf{v}_+)$ is a point that never returns. The same remark applies to $\Phi_{\mathbf{v}_-}$.

5.5. Auxiliary result

In order to have an explicit expression for K_1 and K_2, we assume that $f(t) = \sin t$. The integrals K_1 and K_2 are linear on ν and μ and, under the previous assumption, they may be computed explicitly.

Lemma 2. *The following equalities hold:*

(1) *If $f(t) = \sin t$ then the expression for K_1 is*

$$K_1 = \frac{\nu y_1^{-\delta}}{(\beta-\alpha)^2 + 4\omega^2} \left[(\beta - \alpha) \sin(2\omega(T_1 - s)) + 2\omega \cos(2\omega(T_1 - s)) \right]$$

$$- \frac{\nu}{(\beta-\alpha)^2 + 4\omega^2} \left[(\beta - \alpha) \sin(2\omega s) + 2\omega \cos(2\omega s) \right] + \mu \frac{y_1^{-\delta} - 1}{\alpha - \beta}.$$

(2) *If $f(t) = \sin t$ then the expression for K_2 is*

$$K_2 = \frac{\nu x_2}{(\alpha+\beta)^2 + 4\omega^2} \left[(\alpha + \beta) \sin(2\omega(T_2(0,0) - s)) \right.$$

$$\left. + 2\omega \cos(2\omega(T_2(0,0) - s)) \right]$$

$$- \frac{\nu}{(\alpha+\beta)^2 + 4\omega^2} \left[(\alpha + \beta) \sin(2\omega s) + 2\omega \cos(2\omega s) \right] + \mu \frac{1 - x_2}{\alpha + \beta}.$$

Proof. (1) Using the linearity of the integral, K_1 may be written as

$$\nu \int_s^{T_1} e^{-(\beta-\alpha)(\tau-s)} f(2\omega\tau) d\tau + \mu \int_s^{T_1} e^{-(\beta-\alpha)(\tau-s)} d\tau.$$

Each summand may be computed explicitly. For the second one we have

$$\int e^{-(\beta-\alpha)(\tau-s)}d\tau = \frac{-e^{-(\beta-\alpha)(\tau-s)}}{(\beta-\alpha)}.$$

From [14, Lemma 6] (integrating by parts twice) and since $f(t) = \sin t$, we get

$$\int e^{-(\beta-\alpha)(\tau-s)} \sin(2\omega\tau)d\tau = \frac{e^{-(\beta-\alpha)(\tau-s)}}{(\beta-\alpha)^2 + 4\omega^2}\big[(\beta-\alpha)\sin(2\omega\tau) + 2\omega\cos(2\omega\tau)\big].$$

To compute K_1 we need $T_1 - s = -\frac{\ln y_1}{\beta+\alpha}$, hence

$$e^{-(\beta-\alpha)(T_1-s)} = y_1^{(\beta-\alpha)/(\beta+\alpha)} = y_1^{-\delta}$$

and the expression for K_1 follows.

(2) Analogous computations. ∎

In particular, it follows from this lemma that $K_1 < 1$ for sufficiently large ω, since in this case the terms with ν are small and the term with μ is negative. Hence, for large ω, if $y_1 > 0$ and $\Phi_{\mathbf{v}_+}(s, y_1, w_1) = (T_1, \hat{x}_1, \hat{w}_1)$ then $\hat{x}_1 > 0$.

5.6. Global and first return maps

The first return map to $\mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_-)$ will be

$$\mathcal{R}_{(\nu,\mu)} := (\Psi_{\mathbf{v}_+\to\mathbf{v}_-}) \circ \Phi_{\mathbf{v}_+} \circ (\Psi_{\mathbf{v}_-\to\mathbf{v}_+}) \circ \Phi_{\mathbf{v}_-} \qquad (5.9)$$

where

$$\Psi_{\mathbf{v}_+\to\mathbf{v}_-} : \mathbf{S}^1 \times \mathrm{Out}(\mathbf{v}_+) \to \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_-)$$

and

$$\Psi_{\mathbf{v}_-\to\mathbf{v}_+} : \mathbf{S}^1 \times \mathrm{Out}(\mathbf{v}_-) \to \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_+)$$

are the global maps whose expressions are given below.

Trajectories that remain close to the network Γ spend long times near the equilibria and make fast transitions from each neighbourhood $V_\pm$ to the next one. Thus the first components of $\Psi_{\mathbf{v}_+\to\mathbf{v}_-}$ and $\Psi_{\mathbf{v}_-\to\mathbf{v}_+}$, representing the time transitions, may be disregarded. Since the symmetry κ_2 remains for $\mu, \nu > 0$, we may assume that $\Psi_{\mathbf{v}_-\to\mathbf{v}_+}$ is the identity. Finally, since the cycle is broken for $\mu > 0$, the map $\Psi_{\mathbf{v}_+\to\mathbf{v}_-}$ depends on μ in an affine way. Therefore, for $a > 0$, we take the transition maps as

$$\Psi_{\mathbf{v}_+\to\mathbf{v}_-}(s, \hat{x}_1, \hat{w}_1) \mapsto (s, \hat{x}_1 + a\mu, \hat{w}_1) = (s, x_2, w_2),$$
$$\Psi_{\mathbf{v}_-\to\mathbf{v}_+}(s, \hat{x}_2, \hat{w}_2) \mapsto (s, \hat{x}_2, \hat{w}_2) = (s, x_1, w_1).$$

6. Main result

We are now in a position to show that when the frequency ω of the forcing tends to infinity, the first return map for $F_{(v,\mu)}$ approaches that of the averaged system $F_{(0,\mu)}$.

Theorem 3. *For initial conditions close to any of the cycles in the network Γ, the first return map*

$$\mathcal{R}_{(v,\mu)} : \mathbf{S}^1 \times \text{In}(\mathbf{v}_-) \to \mathbf{S}^1 \times \text{In}(\mathbf{v}_-)$$

for (2.1) *satisfies*

$$\lim_{\omega \to \infty} \mathcal{R}_{(v,\mu)} = \mathcal{R}_{(0,\mu)}.$$

Proof. To prove the result we do not need to write explicitly the analytical expression of $\mathcal{R}_{(v,\mu)}$. First of all, note that the global maps defined in Section 5.6 do not depend on v. From the expressions (5.4) and (5.8) derived in Section 5.4 it is clear that $\Phi_{\mathbf{v}_+}$ and $\Phi_{\mathbf{v}_-}$ only depend on ω, v and μ through the integrals K_1 and K_2. Their expressions are computed explicitly in Lemma 2 above, where we show them to be of the form

$$K_i(\omega, s, v, \mu) = v H_{i,v}(\omega, s) + \mu H_{i,\mu}(\omega, s), \quad i = 1, 2.$$

In both cases the term $H_{i,v}(\omega, s)$ contains one of the factors

$$1/[(\beta \pm \alpha)^2 + 4\omega^2]$$

multiplying a combination of sines and cosines, the last ones multiplied by ω. Hence, $H_{i,v}(\omega, s)$ consists of a term that goes to zero with ω^2 when $\omega \to +\infty$ multiplying functions that are either bounded or the product of ω by something bounded. Therefore, $\lim_{\omega \to \infty} H_{i,v}(\omega, s) = 0$ and the result follows. ∎

7. Dynamics for ω large

We have proved that the dynamics associated to $\lim_{\omega \to +\infty} F_{(v,\mu)}$ is qualitatively the same as that of the autonomous system $F_{(0,\mu)}$. In this section, we analyse the map $F_{(0,\mu)}$ and obtain some dynamical information. Before going further, note that if $v = 0$ then

$$K_1 = \mu \frac{y_1^{-\delta} - 1}{\alpha - \beta} \quad \text{and} \quad K_2 = \mu \frac{1 - x_2}{\alpha + \beta}.$$

Our first result is technical and provides an explicit expression for the components of $\mathcal{R}_{(0,\mu)}$.

Lemma 4. *For $\nu = 0$ and $(s, x_2, w_2) \in \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_-)$ writing $\mathcal{R}_{(0,\mu)} = (h_1, h_2, h_3)$ the following equalities hold:*

(1) $h_1(s, x_2, w_2) = s - \frac{1+\delta}{\alpha+\beta} \ln x_2 + \frac{K_2(1+\delta)}{\alpha+\beta}$;

(2) $h_2(s, x_2, w_2) = x_2^{\delta^2} \exp(-\delta^2 K_2)[1 - \frac{\mu}{\alpha-\beta}] - \frac{\mu}{\alpha-\beta} + a\mu$;

(3) $h_3(s, x_2, w_2) = [2 + (w_2 - 1)x_2^{\frac{2}{\alpha+\beta}} \exp(\frac{-2K_2}{\alpha+\beta})][x_2^\delta \exp(-\delta K_2)]^{\frac{2}{\alpha+\beta}}$.

Proof. The proof follows by composing of the local maps (5.4) and (5.8) derived in Section 5.4 and the global maps $\Psi_{\mathbf{v}_+ \to \mathbf{v}_-}$ and $\Psi_{\mathbf{v}_- \to \mathbf{v}_+}$ in the order prescribed by

$$\mathcal{R}_{(0,\mu)} = (\Psi_{\mathbf{v}_+ \to \mathbf{v}_-}) \circ \Phi_{\mathbf{v}_+} \circ (\Psi_{\mathbf{v}_- \to \mathbf{v}_+}) \circ \Phi_{\mathbf{v}_-},$$

cf. (5.9). We start by obtaining for $\nu = 0$ the expressions

$$y_1 = x_2^\delta \exp(-\delta K_2), \quad K_2 = \mu \frac{1 - x_2}{\alpha + \beta}, \quad K_1 = \mu \frac{y_1^{-\delta} - 1}{\alpha - \beta} = \mu \frac{x_2^{-\delta^2} \exp(\delta^2 K_2) - 1}{\alpha - \beta}.$$

Hence, for (2) we get

$$\begin{aligned}
h_2(s, x_2, w_2) &= y_1^\delta (1 - K_1) \\
&= (x_2^\delta \exp(-\delta K_2))^\delta \left(1 - \mu \frac{(x_2^{-\delta} \exp(-\delta K_2))^{-\delta} - 1}{\alpha - \beta}\right) + a\mu \\
&= x_2^{\delta^2} \exp(-\delta^2 K_2) - \frac{\mu}{\alpha - \beta} + \mu \frac{x_2^{\delta^2} \exp(-\delta^2 K_2)}{\alpha - \beta} + a\mu \\
&= x_2^{\delta^2} \exp(-\delta^2 K_2) \left[1 - \frac{\mu}{\alpha - \beta}\right] - \frac{\mu}{\alpha - \beta} + a\mu.
\end{aligned}$$

For (1),

$$\begin{aligned}
h_1(s, x_2, w_2) &= s - \frac{1}{\alpha + \beta} \ln x_2 + \frac{K_2}{\alpha + \beta} - \frac{1}{\alpha + \beta} \ln y_1 \\
&= s - \frac{1 + \delta}{\alpha + \beta} \ln x_2 + \frac{K_2(1 + \delta)}{\alpha + \beta}.
\end{aligned}$$

Finally, for (3),

$$w_1 = 1 + (w_2 - 1)\left(x_2^{2/(\alpha+\beta)} \exp\left(\frac{-2K_2}{\alpha + \beta}\right)\right),$$

$$\begin{aligned}
h_3(s, x_2, w_2) &= (w_1 + 1)y_1^{\frac{2}{\alpha+\beta}} \\
&= \left[2 + (w_2 - 1)x_2^{\frac{2}{\alpha+\beta}} \exp\left(\frac{-2K_2}{\alpha + \beta}\right)\right][x_2^\delta \exp(-\delta K_2)]^{\frac{2}{\alpha+\beta}}. \quad \blacksquare
\end{aligned}$$

From now on, we are interested in the dynamics when $\nu = 0$. The map h_2 just depends on x_2; this is why we may define $h_2 : [0, 1] \to \mathbf{R}$ as

$$h_2(x_2) := h_2(s, x_2, w_2).$$

Lemma 5. *The following assertions are valid for $(s, x_2, w_2) \in \mathbf{S}^1 \times \mathrm{In}(\mathbf{v}_-)$:*

(1) *For $\mu = 0$, $x_2 = 0$ is a hyperbolic attracting fixed point of $h_2(x_2) = x_2^{\delta^2}$.*

(2) *If $\mu > 0$, then:*

  (a) *if $a - \frac{1}{\alpha - \beta} > 0$, then h_2 has a hyperbolic attracting fixed point $x^\star$ of order $\mathcal{O}(\mu)$;*

  (b) *if $\frac{4\alpha}{(\alpha + \beta)^2} > 1$, then for any $s \in \mathbf{S}^1$ and any $x_2 \in [0, 1]$ the map $w_2 \mapsto h_3(s, x_2, w_2)$ is a Lipschitz contraction in the variable w_2.*

Proof. (1) For $\mu = 0$, we have $K_2 = 0$ and then $h_2(x_2) = x_2^{\delta^2}$ (cf. item (2) of Lemma 4), whose fixed point is 0. Since $h_2'(0) = 0$, the hyperbolicity and attractiveness follow.

(2) (a) For $\mu = 0$, the graph of $h_2(x_2)$ crosses transversely the graph of the identity at $x_2 = 0$, then, for small $\mu \neq 0$, the two graphs still cross transversely at a nearby value $x^*(\mu)$ of x_2, the hyperbolic continuation of the fixed point found in (1). Since $h_2(0) = \mu(a - \frac{1}{\alpha - \beta})$ then $x^*(\mu) > 0$ if and only if $h_2(0) > 0$ and in this case the fixed point is a hyperbolic attractor with $\lim_{\mu \to 0} x^\star(\mu) = 0$. The location of this fixed point is sketched in Figure 3.
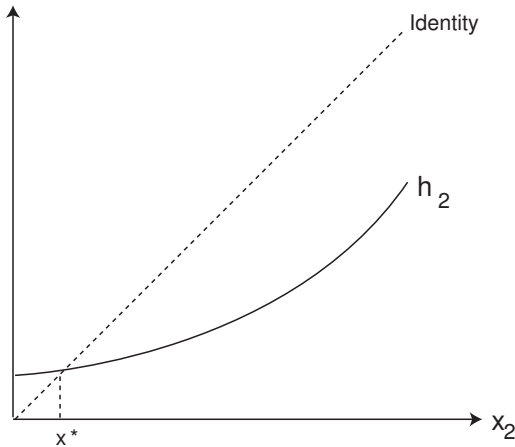

Figure 3. The graph of h_2 intersects the identity map once near the origin.

(b) By item (3) of Lemma 4, the map h_3 does not depend on s and may be written as

$$h_3(s, x_2, w_2) = C_1(x_2) + C_2(x_2)w_2$$

where $C_1, C_2 : [0, 1] \rightarrow \mathbf{R}$ are smooth maps with

$$C_2(x_2) = x_2^{\frac{2+2\delta}{\alpha+\beta}} \exp\left(\frac{-2K_2(1+\delta)}{\alpha+\beta}\right).$$

Since

$$\frac{2+2\delta}{\alpha+\beta} = \frac{4\alpha}{(\alpha+\beta)^2} \overset{\text{Hyp}}{>} 1$$

and $|\exp(\frac{-2K_2(1+\delta)}{\alpha+\beta})| < 1$, then h_3 is a Lipschitz contraction in the variable w_2. ∎

Although $x_2 = 1$ is also a fixed point of h_2 (for $\mu = 0$), we are disregarding this point from the statement of Lemma 5 for two reasons: first it is repelling and second it lies on the boundary of the domain of $\mathcal{R}_{(0,0)}$. Please note that the restriction $\frac{4\alpha}{(\alpha+\beta)^2} > 1$ of Lemma 5 (2b) is also used in [14, §4.6] for the reduced map.

Theorem 6. *For $v = 0$ and small $\mu > 0$, if $\frac{4\alpha}{(\alpha+\beta)^2} > 1$ and $a > \frac{1}{\alpha-\beta}$, then the flow associated to $F_{(0,\mu)}$ has an attracting periodic solution whose period P satisfies $P = \mathcal{O}(-\ln \mu)$.*

Proof. Let $x^\star$ be the hyperbolic and attracting fixed point of h_2 obtained in Lemma 5. Since the map $w_2 \mapsto h_3(s, x^\star, w_2)$ is a Lipschitz contraction in the variable w_2, it has an attracting fixed point $w^\star$ that depends smoothly on x_2. This means that there exists $P > 0$ such that

$$\mathcal{R}_{(0,\mu)}(s, x^\star, w^\star) = \mathcal{R}_{(0,\mu)}(s + P, x^\star, w^\star),$$

therefore the flow associated to $F_{(0,\mu)}$ has an attracting periodic solution of period P. This periodic solution spends a time $(T_2(s, x^\star, w^\star) - s)$ inside V_-, followed, inside V_+, by a flight time $(T_1(s^+, y^+, w^+) - s^+)$ with

$$(s^+, y^+, w^+) = \Psi_{\mathbf{v}_- \rightarrow \mathbf{v}_+} \circ \Phi_{\mathbf{v}_-}(s, y^\star, w^\star).$$

Using item (1) of Lemma 4, this solution has period given by

$$
\begin{aligned}
P &= s - \frac{1+\delta}{\alpha+\beta}\ln x^\star + \frac{K_2(1+\delta)}{\alpha+\beta} \\
&= s - \frac{1+\delta}{\alpha+\beta}\ln x^\star + \mu\frac{1-x^\star}{\alpha+\beta}\frac{(1+\delta)}{\alpha+\beta} \\
&\overset{x^\star=\mathcal{O}(\mu)}{=} s - \frac{1+\delta}{\alpha+\beta}\ln x^\star + \mathcal{O}(\mu).
\end{aligned}
$$

Since $x^\star = \mathcal{O}(\mu)$ (by Lemma 5), the period P is of the order $\mathcal{O}(-\ln\mu)$ completing the proof. ∎

From Theorems 3 and 6 the next corollary follows immediately.

Corollary 7. *For small $v, \mu > 0$ and very large ω, if $\frac{4\alpha}{(\alpha+\beta)^2} > 1$ and $a > \frac{1}{\alpha-\beta}$, then the flow associated to $F_{(v,\mu)}$ has an attracting periodic solution whose period P satisfies $P = \mathcal{O}(-\ln \mu)$.*

8. Discussion and final remarks

In this work, we investigate the influence of high frequency forcing on a differential equation exhibiting a clean attracting heteroclinic network – *clean* in the sense that the unstable manifolds of all the nodes lie in the network. Our result says that if the frequency of the non-autonomous perturbation goes to infinity, then the dynamics of the vector field is governed by the averaged system: the non-autonomous equation (2.1) behaves like an autonomous one. Our main result has been motivated by Tsai and Dawes [9, 23, 24] in the context of the Guckenheimer and Holmes example. They claim without proof that the time-periodic forcing term has an effect equivalent to that of the time-averaged perturbation term.

Our findings agree well with the theory developed by Cheng-Gui et al. [8]. They considered a system of the form

$$\dot{x} = f(x) + B \cos(\omega t) H \tag{8.1}$$

where $x \in \mathbf{R}^n$ represents the state vector of the nonlinear system, $H = (1, 1, \ldots, 1)^T$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear vector field and $B \cos(\omega t)$ denotes a forcing with frequency ω and amplitude B. The unforced system ($H = \bar{0}$) may exhibit stationary, periodic or chaotic behaviour for different system parameters. They state that a general solution of (8.1), say $x(t)$, may be written as the sum of a slow motion $X(t)$ and a fast motion $\Psi(t)$,

$$x(t) = X(t) + \frac{1}{\omega} \Psi(t, \omega)$$

where $\Psi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is a $2\pi/\omega$ t-periodic function with zero mean. If $\omega \to +\infty$, then the solution is governed by the slow-motion which is the solution of the original unperturbed system. The effect of high-frequency forcing becomes apparent. Stating our result in their terms, as the forcing frequency tends to infinity the equation for the fast motion drops out completely.

Further, in Section 5.5, we have used the map $f(t) = \sin t$ in (2.1) and $H = (1, 0, \ldots, 0)^T$ but our work is still valid for any smooth t-periodic map f with zero average.

Conditions under which (2.1) has a chaotic regime in the form of a suspended horseshoe map have been obtained in [15], where it appears through the destruction of an invariant torus. We may conclude that here, when $\omega = +\infty$, in the extended phase space (equation (5.1)) we cannot write the global map as in [15, Section 4.3].

More specifically, in the limit case $\omega = +\infty$, the set

$$W^u(\mathbf{S}^1 \times \{\mathbf{v}_-\}) \cap \text{In}(\mathbf{S}^1 \times \{\mathbf{v}_+\})$$

is not a non-degenerate graph of a multimodal function. When $\omega = +\infty$ the necessary distortion to obtain chaos does not hold so there is no guarantee of the torus-breakdown effects. This limit case was left open by Wang [25, pp. 4391].

   We conjecture that our result holds for any attracting and clean heteroclinic network where the connections are one-dimensional. Since nonlinear systems driven by high frequency forcing are prevalent in nature and engineering, we expect that these results are valuable and helpful to those applications.

A. Notation

In Table 1, we list the main notation used in this paper in order of appearance with the reference of the section where it is defined.

Notation	Meaning	Section
Γ	Heteroclinic network formed by four cycles	3.2
δ	Saddle-value of $\mathbf{v}_+$ and $\mathbf{v}_-$	3.2
$V_\pm$	Neighbourhoods of $\mathbf{v}_+$ and $\mathbf{v}_-$	4
$\text{In}(\mathbf{v}_\pm)$	Cross-sections near $\mathbf{v}_\pm$ for $F_{(0,0)}$	4
$\text{Out}(\mathbf{v}_\pm)$		
$\mathbf{S}^1 \times \text{In}(\mathbf{v}_\pm)$	Augmented cross-sections for $F_{(\nu,\mu)}$	5
$\mathbf{S}^1 \times \text{Out}(\mathbf{v}_\pm)$		
$\Phi_{\mathbf{v}_+}$	Local map from $\mathbf{S}^1 \times \text{In}(\mathbf{v}_+)$ to $\mathbf{S}^1 \times \text{Out}(\mathbf{v}_+)$	5.2
$\Phi_{\mathbf{v}_-}$	Local map from $\mathbf{S}^1 \times \text{In}(\mathbf{v}_-)$ to $\mathbf{S}^1 \times \text{Out}(\mathbf{v}_-)$	5.3
T_1	Time of flight inside V_+	5.2
T_2	Time of flight inside V_-	5.3
$\Psi_{\mathbf{v}_+ \to \mathbf{v}_-}$	Global map from $\mathbf{S}^1 \times \text{Out}(\mathbf{v}_+)$ to $\mathbf{S}^1 \times \text{In}(\mathbf{v}_-)$	5.6
$\Psi_{\mathbf{v}_- \to \mathbf{v}_+}$	Global map from $\mathbf{S}^1 \times \text{Out}(\mathbf{v}_-)$ to $\mathbf{S}^1 \times \text{In}(\mathbf{v}_+)$	5.6
$\mathcal{R}_{(\nu,\mu)}$	Return map to $\mathbf{S}^1 \times \text{In}(\mathbf{v}_-)$	5.6
(h_1, h_2, h_3)	Components of the map $\mathcal{R}_{(0,\mu)}$	7
$\frac{4\alpha}{(\alpha+\beta)^2} > 1$	Condition forcing h_2 to be dominant in $\mathcal{R}_{(0,\mu)}$	7
$x^\star$	Stable fixed point of h_2	7

Table 1. Notation

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