A note on Mal'tsev objects

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Abstract. The aim of this work is to compare the distinct notions of *Mal'tsev object* in the sense of Weighill and in the sense of Montoli–Rodelo–Van der Linden.

1. Introduction

A variety of universal algebras $\mathbb{V}$ is called a *Mal'tsev variety* [22, 26] when its theory admits a ternary operation p satisfying the equations

$$p(x, y, y) = x \quad \text{and} \quad p(x, x, y) = y.$$

Such varieties were characterised in [22] by the fact that any pair of congruences R and S on a same algebra X is *2-permutable*, i.e., $RS = SR$. It was shown in [19] that such varieties are also characterised by the fact that any homomorphic relation D from an algebra X to an algebra Z is *difunctional*: $(x_1 D z_2 \wedge x_2 D z_2 \wedge x_2 D z_1) \Rightarrow x_1 D z_1$.

The notion of Mal'tsev variety was generalised to a categorical context in [8] (see also [3, 7, 9]). This was achieved by translating the above properties on homomorphic relations for a variety into similar properties on (internal) relations in a category. A regular category [2] $\mathbb{C}$ is called a *Mal'tsev category* when any pair of equivalence relations R and S in $\mathbb{C}$ on a same object X is such that $RS = SR$. Mal'tsev categories can also be characterised by the difunctionality of relations.

Theorem 1.1 ([7–9]). *A finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if any relation $D \to X \times Z$ in $\mathbb{C}$ is difunctional.*

The main examples of Mal'tsev varieties are Grp of groups, Ab of abelian groups, R-Mod of modules over a commutative ring R, Rng of rings, and Heyt of Heyting algebras. More generally, any variety whose theory contains a group operation is a Mal'tsev variety; an example of a non-Mal'tsev variety is the variety Mon of monoids. As examples of Mal'tsev categories (that are not varietal) we have the category

Grp(Top) of topological groups, any abelian category or the dual of an elementary topos. Also, if $\mathbb{C}$ is a Mal'tsev category, then so are the (co)slice categories $\mathbb{C}/X$ and $X/\mathbb{C}$, for any object X of $\mathbb{C}$ (see [3], for example).

There are also several other well-known characterisations of Mal'tsev varieties through nice properties on relations, such as the fact that every reflexive relation is necessarily a congruence. All of these characteristic properties have been generalised to the Mal'tsev categorical context (see [3, 6–9]). This wide range of nice properties together with the long list of examples has contributed to the great amount of research developed on Mal'tsev varieties and categories over the past 70 years. They are also just the first instance of the family of *n-permutable* varieties [15], for $n \geqslant 2$. An n-permutable variety is such that any pair of congruences R and S on a same algebra X is n-permutable, i.e., $(R, S)_n = (S, R)_n$, where $(R, S) = RSR \cdots$ denotes the composite of R and S, n times; they have been generalised to n-permutable categories in [7]. Concerning n-permutability, the reader may be interested in [18], to see how varietal proofs translate into categorical ones, and [24] for further properties on relations.

Weighill in [27] used the characterisation of a Mal'tsev category obtained through the difunctionality of all relations to introduce a definition of Mal'tsev object. To fully understand this definition, we must develop further on relations in $\mathbb{C}$.

A (binary) *relation R* in a category $\mathbb{C}$ from an object X to an object Z is a span $X \xleftarrow{r_1} R \xrightarrow{r_2} Z$ such that (r_1, r_2) is jointly monomorphic. We identify two relations, $X \leftarrow R \rightarrow Z$ and $X \leftarrow R' \rightarrow Z$, when R factors through R' and vice-versa. When $\mathbb{C}$ admits binary products, a relation R from X to Z is a subobject of $X \times Z$, denoted $\langle r_1, r_2 \rangle \colon R \rightarrow X \times Z$. A relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Z$ in $\mathbb{C}$ is difunctional when the relation

$$\mathbb{C}(Y, X) \xleftarrow{\mathbb{C}(Y, r_1)} \mathbb{C}(Y, R) \xrightarrow{\mathbb{C}(Y, r_2)} \mathbb{C}(Y, Z) \tag{1.i}$$

in Set is difunctional, for every object Y of $\mathbb{C}$. The definition of a reflexive, symmetric, transitive, and equivalence relation in $\mathbb{C}$ is obtained similarly.

Definition 1.2 ([27]). An object Y of $\mathbb{C}$ is called a *W-Mal'tsev object*[1] when, for every relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Z$ in $\mathbb{C}$, the Set-relation (1.i) is difunctional.

It follows from Theorem 1.1 that a finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if all of its objects are W-Mal'tsev objects.

It is well known that the category $(\mathsf{Set})^{\mathrm{op}}$ is a Mal'tsev category [7], hence every set is a W-Mal'tsev object in $(\mathsf{Set})^{\mathrm{op}}$. Weighill's study of W-Mal'tsev objects led to the identification of interesting Mal'tsev subcategories of duals of categories of topological flavour. We will recall and generalise these results in Section 3.

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[1]We added the prefix "W-" to distinguish these from the Mal'tsev objects of Definition 1.4.

Using a completely different approach, Bourn in [5] classified several categorical notions (including that of Mal'tsev category) through the fibration of points. A *point* $(f\colon A \to B, s\colon B \to A)$ in a category $\mathbb{C}$ is a split epimorphism f with a chosen splitting s. We may define the category of points in $\mathbb{C}$, denoted by $\mathsf{Pt}(\mathbb{C})$: a morphism between points is a pair $(x, y)\colon (f, s) \to (f', s')$ of morphisms in $\mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccccc}
B & \xrightarrow{\;s\;} & A & \xrightarrow{\;f\;} & B \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\
B' & \xrightarrow{\;s'\;} & A' & \xrightarrow{\;f'\;} & B'.
\end{array}$$

When $\mathbb{C}$ has pullbacks of split epimorphisms, the forgetful functor $\mathrm{cod}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$, which associates with every split epimorphism its codomain, is a fibration called the *fibration of points* [4].

From the several classifying properties of the fibration of points cod studied in [5], we emphasize the following one.

Theorem 1.3 ([5]). *A finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if* cod *is unital.*

Unitality of cod means that the category $\mathsf{Pt}_Y(\mathbb{C})$ of points in $\mathbb{C}$ over Y is a unital category, for every object Y of $\mathbb{C}$. Recall from [5] that a pointed and finitely complete category is called a *unital category* when, for all objects A, C of $\mathbb{C}$, the pair of morphisms $(\langle 1_A, 0\rangle\colon A \to A \times C, \langle 0, 1_C\rangle\colon C \to A \times C)$ is jointly strongly epimorphic. The fact that $\mathsf{Pt}_Y(\mathbb{C})$ is unital means: for every pullback of points over Y,

$$\begin{array}{ccc}
A \times_Y C & \xleftarrow{\langle sg, 1_C\rangle \atop \xrightarrow{\;\pi_C\;}} & C \\
{\scriptstyle \pi_A}\upharpoonleft\downharpoonright{\scriptstyle \langle 1_A, tf\rangle} & & {\scriptstyle g}\upharpoonleft\downharpoonright{\scriptstyle t} \\
A & \xleftarrow{\;s\;} \atop \xrightarrow{\;f\;} & Y
\end{array} \qquad (1.\mathrm{ii})$$

(which corresponds to a binary product in $\mathsf{Pt}_Y(\mathbb{C})$), the pair of morphisms $(\langle 1_A, tf\rangle, \langle sg, 1_C\rangle)$ is jointly strongly epimorphic.

Since $\mathsf{Pt}_1(\mathbb{C}) \cong \mathbb{C}$, every pointed Mal'tsev category $\mathbb{C}$ is necessarily unital. The converse is false. Indeed, Mon and SRng are examples of unital categories that are not Mal'tsev categories.

Inspired on the classification properties of the fibration of points cod studied in [5], the authors of [25] explored several algebraic categorical notions, such as those of (strongly) unital [5], subtractive [17], Mal'tsev [7–9] and protomodular categories [4],

at an object-wise level. This led to the corresponding notions of (strongly) unital, subtractive, Mal'tsev and protomodular objects. This approach allows one to distinguish "good" objects, i.e., with stronger algebraic properties, in a setting with weaker algebraic properties. The goal of this work was to obtain a categorical-algebraic characterisation of groups amongst monoids and of rings amongst semirings. It was shown in [25] that a monoid Y is a group if and only if Y is a *Mal'tsev object* in Mon if and only if Y is a *protomodular object* in Mon.

Definition 1.4. An object Y of a finitely complete category $\mathbb{C}$ is called a *Mal'tsev object* if the category $\mathrm{Pt}_Y(\mathbb{C})$ of points over Y is a unital category.

It follows from Theorem 1.3 that a finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if all of its objects are Mal'tsev objects.

The notions of W-Mal'tsev object, studied in [27], and that of Mal'tsev object, in the sense of [25], were obtained independently and developed with different goals in mind. Although their definitions are very different in nature, there is an obvious common property to both: a finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if all of its objects are W-Mal'tsev objects if and only if all of its objects are Mal'tsev objects. There are also several other properties shared by both notions, which we will recall in Section 2. Altogether, these observations led us to the natural question:

> *To what extent are W-Mal'tsev and Mal'tsev objects comparable?*

2. W-Mal'tsev object vs. Mal'tsev objects

In this section, we shall compare the notion of W-Mal'tsev object [27] and that of Mal'tsev object [25] in a base category $\mathbb{C}$, which is finitely complete and may, eventually, admit some extra structure. We fix a finitely complete category $\mathbb{C}$ and denote by $\mathrm{W}(\mathbb{C})$ (resp. $\mathrm{M}(\mathbb{C})$), the full subcategory of $\mathbb{C}$ determined by the W-Mal'tsev (resp. Mal'tsev) objects of $\mathbb{C}$.

We begin by combining Theorems 1.1 and 1.3 into one.

Theorem 2.1. *Let $\mathbb{C}$ be a finitely complete category. The following statements are equivalent:*

(i) $\mathbb{C}$ *is a Mal'tsev category;*

(ii)   *diagram (1.i) is a difunctional relation in* Set, *for any object Y in $\mathbb{C}$ and any relation $X \xleftarrow{r_1} R \xrightarrow{r_2} Z$ in $\mathbb{C}$;*

(iii) *for any pullback of points over an arbitrary object Y in $\mathbb{C}$ as in (1.ii), the pair of morphisms $(\langle 1_A, tf \rangle, \langle sg, 1_C \rangle)$ is jointly strongly epimorphic.*
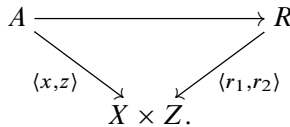
An immediate consequence of Definitions 1.2 and 1.4 is the following one (see (W1) and (M1) below).

Proposition 2.2. *Let $\mathbb{C}$ be a finitely complete category. The following statements are equivalent:*

(i) $\mathbb{C}$ *is a Mal'tsev category;*

(ii)   *all objects of $\mathbb{C}$ are W-Mal'tsev objects;*

(iii) *all objects of $\mathbb{C}$ are Mal'tsev objects.*

The challenge of the comparison process is when the category $\mathbb{C}$ is not a Mal'tsev category, but admits (W-)Mal'tsev objects. Note that both definitions depend heavily on the surrounding category $\mathbb{C}$. To have a Mal'tsev object one must check a property for all pullbacks of points over that object; to have a W-Mal'tsev object one must check a property concerning all relations in $\mathbb{C}$. However, when $\mathbb{C}$ is a regular category [2] with binary coproducts, the result stated below in (W5) gives an independent (of all relations in $\mathbb{C}$) way to check that an object Y is a W-Mal'tsev object. Consequently, one would expect that the notion of a Mal'tsev object is stronger than that of a W-Mal'tsev object. Indeed, that is the case as stated in Proposition 2.4. Additionally, the greater demand on Mal'tsev objects may lead to "trivial" cases. This happens when $\mathbb{C} = V$-Cat (and V is not a cartesian quantale), where the only Mal'tsev object is the empty set $\emptyset$, while a W-Mal'tsev object is precisely a symmetric $V_\wedge$-category (Theorem 3.4). See also Example 2.6 concerning (W-)Mal'tsev objects in OrdGrp, the category of preordered groups. In other contexts, such as Mon, the notions of W-Mal'tsev object and that of Mal'tsev object both coincide with a group (see Proposition 2.5 and [25, Theorem 6.14]).

In what follows it will be useful to consider properties on relations in a category using generalised elements (see [7], for example). Let R be a relation from X to Z given by the subobject $\langle r_1, r_2 \rangle \colon R \to X \times Z$. Let $x \colon A \to X$ and $z \colon A \to Z$ be morphisms, which can be considered as *generalised elements* of X and Z, respectively. We write $(x, z) \in_A R$ when the morphism $\langle x, z \rangle$ factors through $\langle r_1, r_2 \rangle$,

$$
\begin{array}{ccc}
A & \longrightarrow & R \\
& \searrow{\scriptstyle \langle x,z \rangle} \quad \swarrow{\scriptstyle \langle r_1,r_2 \rangle} & \\
& X \times Z. &
\end{array}
$$

Using this notation, an object Y is a W-Mal'tsev object in $\mathbb{C}$ when: for any relation $\langle r_1, r_2 \rangle \colon R \to X \times Z \in \mathbb{C}$ and morphisms $x_1, x_2 \colon Y \to X$ and $z_1, z_2 \colon Y \to Z$,

$$((x_1, z_2) \in_Y R, (x_2, z_2) \in_Y R, (x_2, z_1) \in_Y R) \Rightarrow (x_1, z_1) \in_Y R. \qquad (2.\text{i})$$

We extract from [27] the main results which are used in the comparison process. Note that in [27] a Mal'tsev category needs not to be finitely complete by definition. Consequently, we adapted the result stated in (W4) to our finitely complete request.

(W1) $W(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is a Mal'tsev category.

(W2) $W(\mathbb{C})$ is closed under colimits and (regular) quotients in $\mathbb{C}$ ([27, Proposition 2.1]).

(W3) When $\mathbb{C}$ is a well-powered regular category with coproducts, then $W(\mathbb{C})$ is a coreflective subcategory of $\mathbb{C}$; thus, $W(\mathbb{C})$ is (finitely) complete whenever $\mathbb{C}$ is ([27, Corollary 2.2]).

(W4) Let $\mathbb{C}$ be a well-powered regular category with binary coproducts. Suppose that any relation in $W(\mathbb{C})$ is also a relation in $\mathbb{C}$. Then $W(\mathbb{C})$ is the largest full subcategory of $\mathbb{C}$ which is a Mal'tsev category and which is closed under binary coproducts and quotients in $\mathbb{C}$ ([27, Corollary 2.5]).

(W5) Let $\mathbb{C}$ be a regular category with binary coproducts. An object Y is a W-Mal'tsev object in $\mathbb{C}$ if and only if, given the (regular epimorphism, monomorphism) factorisation in $\mathbb{C}$

$$
\begin{array}{ccc}
3Y & & \\
\Big\downarrow{\scriptstyle\begin{pmatrix}\iota_1 & \iota_2\\ \iota_2 & \iota_2\\ \iota_2 & \iota_1\end{pmatrix}} & \xrightarrow{\ e\ } & R \\
& \swarrow{\scriptstyle\langle r_1,r_2\rangle} & \\
2Y \times 2Y & &
\end{array}
\qquad (2.\text{ii})
$$

(which guarantees that $(\iota_1, \iota_2) \in_Y R$, $(\iota_2, \iota_2) \in_Y R$, $(\iota_2, \iota_1) \in_Y R$), we have $(\iota_1, \iota_1) \in_Y R$. As usual, we write $Y + Y = 2Y$, $Y + Y + Y = 3Y$, and $\iota_j: Y \to kY$ for the j-th coproduct coprojection ([27, Proposition 2.3]).

The main results we wish to emphasize from [25] are the following ones:

(M1) $M(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is a Mal'tsev category ([25, Proposition 6.10]).

(M2) If $M(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$, then $M(\mathbb{C})$ is a Mal'tsev category ([25, Corollary 6.11]).

(M3) When $\mathbb{C}$ is a pointed category, then $\mathbb{C}$ is a unital category if and only if 0 is a Mal'tsev object ([25, Proposition 6.4]).

(M4) When $\mathbb{C}$ is a regular category, then $M(\mathbb{C})$ is closed under quotients in $\mathbb{C}$ ([25, Proposition 6.12]).

(M5) When $\mathbb{C}$ is a regular category, an object Y in $\mathbb{C}$ is a Mal'tsev object if and only if every double split epimorphism over Y

$$
\begin{array}{ccc}
D & \underset{f'}{\overset{s'}{\leftrightarrows}} & C \\
{\scriptstyle g'}\big\uparrow\big\downarrow{\scriptstyle t'} & & {\scriptstyle g}\big\uparrow\big\downarrow{\scriptstyle t} \\
A & \underset{f}{\overset{s}{\leftrightarrows}} & Y
\end{array}
\qquad (2.\text{iii})
$$

(meaning that the four "obvious" squares commute) is a *regular pushout*, i.e., the comparison morphism $\langle g', f' \rangle : D \to A \times_Y C$ is a regular epimorphism.

As a direct consequence of (M3), we have a trivial comparison result in a pointed context.

Proposition 2.3. *Let $\mathbb{C}$ be a pointed finitely complete category. Then the zero object is a W-Mal'tsev object, but it is not necessarily a Mal'tsev object.*

Proof. The zero object 0 is always a W-Mal'tsev object in any pointed category. Indeed, given any relation $R \to X \times Z$, the only generalised elements we can use are the zero morphisms $0_X : 0 \to X$ and $0_Z : 0 \to Z$. So, the implication in (2.i) obviously holds when $Y = 0$. On the other hand, 0 is only a Mal'tsev object when $\mathbb{C}$ is a unital category by (M3). ∎

When $\mathbb{C}$ is a regular category, then both $\mathsf{W}(\mathbb{C})$ and $\mathsf{M}(\mathbb{C})$ are closed under quotients in $\mathbb{C}$ – (M4) and (W2), where the later actually holds for any finitely complete $\mathbb{C}$. If $\mathbb{C}$ also has binary coproducts, we may use (W5) to conclude that the notion of Mal'tsev object is stronger than that of W-Mal'tsev object.

Proposition 2.4. *Let $\mathbb{C}$ be a regular category with binary coproducts. If Y is a Mal'tsev object in $\mathbb{C}$, then it is a W-Mal'tsev object in $\mathbb{C}$.*

Proof. Consider the double split epimorphism over Y

$$
\begin{array}{ccc}
3Y & \xleftarrow{\iota_{32}=\binom{\iota_3}{\iota_2}} & \\
 & \xrightarrow{\binom{\iota_2}{\iota_2}{\iota_1}} & 2Y \\
{\scriptstyle \binom{\iota_1}{\iota_2}{\iota_2}}\big\uparrow\big\downarrow{\scriptstyle \iota_{12}=\binom{\iota_1}{\iota_2}} & & {\scriptstyle \nabla}\big\uparrow\big\downarrow{\scriptstyle \iota_2} \\
2Y & \xleftarrow{\iota_2} & \\
 & \xrightarrow{\nabla=\binom{1_Y}{1_Y}} & Y,
\end{array}
$$

which is a regular pushout by assumption. We then get a (regular epimorphism, mono-morphism) factorisation as in (2.ii), where $R = \mathsf{Eq}(\nabla)$ is the kernel pair of ∇. It easily follows that $(\iota_1, \iota_1) \in_Y R$, since $R = \mathsf{Eq}(\nabla)$ and $\nabla \iota_1 = \nabla \iota_1$. This shows that Y is a W-Mal'tsev object by (W5). ∎

An interesting question is whether $\mathsf{W}(\mathbb{C})$ and $\mathsf{M}(\mathbb{C})$ are themselves Mal'tsev cat-egories. From (W3) we know that, if $\mathbb{C}$ is a well-powered regular category with co-products, then $\mathsf{W}(\mathbb{C})$ is finitely complete. Also, if every relation in $\mathsf{W}(\mathbb{C})$ is also a relation in $\mathbb{C}$, then the full subcategory $\mathsf{W}(\mathbb{C})$ is a Mal'tsev category; actually, it is the largest one which is closed under binary coproducts and quotients in $\mathbb{C}$ (see (W4)). On the other hand, there are no "obvious" conditions on $\mathbb{C}$ from which we could deduce finite completeness for $\mathsf{M}(\mathbb{C})$. So, we can only conclude that $\mathsf{M}(\mathbb{C})$ is a Mal'tsev cat-egory when it is closed under finite limits in $\mathbb{C}$; this is (M2). When this is the case, we still do not know whether $\mathsf{M}(\mathbb{C})$ is the largest full subcategory of $\mathbb{C}$ which is a Mal'tsev category.

In [25] is was shown that the Mal'tsev objects in Mon are precisely the groups ([25, Theorem 6.14]). This is also the case with respect to W-Mal'tsev objects in Mon. It follows from the above and the next proposition that Grp is the largest full subcat-egory of Mon which is a Mal'tsev category and is closed under binary coproducts and quotients in Mon.

Proposition 2.5. *A monoid Y is a W-Mal'tsev object in* Mon *if and only if Y is a group.*

Proof. If Y is a group, then it is a Mal'tsev object in Mon; thus, it is a W-Mal'tsev object in Mon by Proposition 2.4. For the converse, suppose that $(Y, +, 0)$ is a W-Mal'tsev object in Mon. We use additive notation although Y is not necessarily an abelian monoid. Since Mon is a regular category with binary coproducts, we can apply (W5) to conclude that $(\iota_1, \iota_1) \in_Y R$, where R is as in diagram (2.ii). So, for any $x \in Y$, $x \neq 0$, we have $([\underline{x}], [\underline{x}]) \in R$. (We use the notations $\iota_1(x) = [\underline{x}]$, $\iota_2(x) = [\overline{x}]$, for any $x \in Y$, for the coprojections $\iota_1, \iota_2 \colon Y \to 2Y$, and $\iota_1(x) = [\underline{x}]$, $\iota_2(x) = [\overline{x}]$, $\iota_3(x) = [\widetilde{x}]$, for any $x \in Y$, for the coprojections $\iota_1, \iota_2, \iota_3 \colon Y \to 3Y$.) Since $e \colon 3Y \to R$ is surjective, there exists an element $[\underline{u_1}\,\overline{v_1}\,\widetilde{w_1} \cdots \underline{u_k}\,\overline{v_k}\,\widetilde{w_k}] \in 3Y$ such that

$$e([\underline{u_1}\,\overline{v_1}\,\widetilde{w_1} \cdots \underline{u_k}\,\overline{v_k}\,\widetilde{w_k}]) = ([\underline{x}], [\underline{x}]).$$

We deduce that

$$\begin{cases} [\underline{u_1}\,\overline{v_1 + w_1} \cdots \underline{u_k}\,\overline{v_k + w_k}] = [\underline{x}], \\ [\overline{u_1 + v_1}\,\underline{w_1} \cdots \overline{u_k + v_k}\,\underline{w_k}] = [\underline{x}]; \end{cases}$$

consequently

$$\begin{cases} v_1 + w_1 = 0, \ldots, v_k + w_k = 0, & \text{and} \quad u_1 + \cdots + u_k = x, \\ u_1 + v_1 = 0, \ldots, u_k + v_k = 0, & \text{and} \quad w_1 + \cdots + w_k = x. \end{cases}$$

We may define the element $y = v_k + \cdots + v_1$ of Y which is the inverse of x,

$$\begin{aligned} x + y &= u_1 + \cdots + u_{k-1} + (u_k + v_k) + v_{k-1} + \cdots + v_1 \\ &= u_1 + \cdots + (u_{k-1} + v_{k-1}) + \cdots + v_1 = \cdots = 0, \\ y + x &= v_k + \cdots + v_2 + (v_1 + w_1) + w_2 + \cdots + w_k \\ &= v_k + \cdots + (v_2 + w_2) + \cdots + w_k = \cdots = 0. \quad \blacksquare \end{aligned}$$

We finish this section with the example of (W-)Mal'tsev objects in OrdGrp, the category of preordered groups. We denote a preordered group by $(Y, +, \leqslant)$, even though the associated group $(Y, +, 0)$ is not necessarily abelian. Recall from [13] that the *positive cone* of Y, $P_Y = \{x \in Y : 0 \leqslant x\}$, is always a submonoid of Y which is closed under conjugation in Y. It was shown in [13] that a preordered group $(Y, +, \leqslant)$ is a Mal'tsev object if and only if the preorder relation $\leqslant$ is an equivalence relation if and only if P_Y is a group. We follow an argument similar to that of the proof of Proposition 2.5 to analyse, in the next example, properties of the positive cone of a W-Mal'tsev object in OrdGrp.

Example 2.6. Let $(Y, +, \leqslant)$ be a W-Mal'tsev object in OrdGrp. Since OrdGrp is a regular category with binary coproducts, we can apply (W5) to conclude that $(\iota_1, \iota_1) \in_Y R$, where R is as in diagram (2.ii). So, for any $x \in P_Y$, $x \neq 0$, we have $([x], [x]) \in P_R$. Since e is a regular epimorphism, there exists a positive element $[\underline{u_1}\overline{v_1}\widetilde{w_1} \cdots \underline{u_k}\overline{v_k}\widetilde{w_k}]$ $\in P_{3Y}$ such that $e([\underline{u_1}\overline{v_1}\widetilde{w_1} \cdots \underline{u_k}\overline{v_k}\widetilde{w_k}]) = ([x], [x])$. We deduce that

$$\begin{cases} [\underline{u_1}\overline{v_1 + w_1} \cdots \underline{u_k}\overline{v_k + w_k}] = [x], \\ [\overline{u_1 + v_1}\underline{w_1} \cdots \overline{u_k + v_k}\underline{w_k}] = [x]; \end{cases}$$

consequently

$$\begin{cases} v_1 + w_1 = 0, \ldots, v_k + w_k = 0, & \text{and} \quad u_1 + \cdots + u_k = x, \\ u_1 + v_1 = 0, \ldots, u_k + v_k = 0, & \text{and} \quad w_1 + \cdots + w_k = x. \end{cases}$$

We conclude that $u_1 = w_1, \ldots, u_k = w_k$ and the above positive element of $3Y$ has the shape $[\underline{u_1}\overline{-u_1}\widetilde{u_1} \cdots \underline{u_k}\overline{-u_k}\widetilde{u_k}]$. The monotone morphism

$$\begin{pmatrix} 0 \\ 1_Y \\ 0 \end{pmatrix} : 3Y \qquad\qquad\qquad \to Y,$$
$$[\underline{u_1}\overline{-u_1}\widetilde{u_1} \cdots \underline{u_k}\overline{-u_k}\widetilde{u_k}] \mapsto -u_1 - \cdots - u_k$$

takes positive elements of $3Y$ to positive elements of Y; thus $-u_1 - \cdots - u_k \in P_Y$.

If Y is an abelian group, then from $x = u_1 + \cdots + u_k \in P_Y$, we deduce $-x = -u_1 - \cdots - u_k \in P_Y$, i.e., P_Y is a group. In this case, W-Mal'tsev objects and Mal'tsev objects coincide in OrdGrp. We do not know whether the two notions coincide for the non-abelian case.

3. V-categories and (W-)Mal'tsev objects

In this section, we will generalise the characterisations of W-Mal'tsev objects in the duals of the category of metric spaces and of the category of topological spaces obtained in [27]. For that we will make use of the concepts of V-category and of (U, V)-category. Here, as in [20], the notion of V-category will play the role of a metric space, while a (U, V)-category (as introduced in [14]) will play the role of a topological space. We start by presenting the basic tools of this approach.

Throughout V is a unital and integral quantale; that is, V is a complete lattice equipped with a tensor product $\otimes$, with unit $k = \top \neq \bot$, that distributes over arbitrary joins. As a category, V is a monoidal closed category.

When more than one tensor product may be considered, we use the notation $V_\otimes$ to indicate that we are using the tensor product $\otimes$ in V.

Definition 3.1. A V-*category* is a set X together with a map $X \times X \to V$, whose image of (x, x') we denote by $X(x, x')$, such that, for each $x, x', x'' \in X$,

(R) $k \leqslant X(x, x)$;

(T) $X(x, x') \otimes X(x', x'') \leqslant X(x, x'')$.

A V-category is said to be *symmetric* if, for every $x, x' \in X$, $X(x, x') = X(x', x)$. A V-*functor* $f : X \to Y$ is a map such that, for all $x, x' \in X$, $X(x, x') \leqslant Y(f(x), f(x'))$.

The two axioms of a V-category express the existence of identities and the categorical composition law, but they may also be seen as a reflexivity and a transitivity condition, or even as two conditions usually imposed to metric structures: if in the complete half real line $[0, \infty]$ we consider the order relation $\geqslant$ and the tensor product $+$, (R) and (T) read as

(R') $0 \geqslant X(x, x)$;

(T') $X(x, x') + X(x', x'') \geqslant X(x, x'')$,

with (R) meaning that the distance from a point to itself is 0, and (T) the usual triangular inequality.

We denote by V-Cat the category of V-categories and V-functors, and by V-Cat$_{\mathrm{sym}}$ its full subcategory of symmetric V-categories.

Remark 3.2. When V is a complete lattice which is a *frame*, so that finite meets distribute over arbitrary joins, then $V_\wedge = (V, \leqslant, \wedge, \top)$ is a (unital and integral) quantale. We call such quantales *cartesian*. It is well known that, if $V_\wedge$ is a cartesian quantale, then the category $V_\wedge$-Cat has special features, like being cartesian closed [20]. Here we will show that it also has a key role in the study of W-Mal'tsev objects of $(V_\wedge\text{-Cat})^{\mathrm{op}}$.

Moreover, even if the quantale $V_\otimes$ is not a frame, the $V_\otimes$-categories X which, in addition, verify

$$X(x, x') \wedge X(x', x'') \leqslant X(x, x'')$$

will be specially relevant, and we will call them also $V_\wedge$-categories. We note that, since $k = \top$, they also satisfy $\top \leqslant X(x, x)$ for every $x \in X$; moreover, $u \otimes v \leqslant u \wedge v$ always holds, so the above condition guarantees immediately condition (T) of the above definition.

Examples 3.3.   (1) When $V = 2 = (\{0 < 1\}, \wedge, 1)$, a V-category is a preordered set and a V-functor is a monotone map. Hence V-Cat is the well-known category Ord of preordered sets and monotone maps.

(2) When $V = [0, \infty]_+ = ([0, \infty], \geqslant, +, 0)$, that is, V is the complete lattice $[0, \infty]$, ordered by $\geqslant$, with tensor product $\otimes = +$, a V-category is a (*generalised*) *Lawvere metric space* [20] (not necessarily separated nor symmetric, with ∞ as a possible distance), since the two conditions above mean that $X(x, x) = 0$ and $X(x, y) + X(y, z) \geqslant X(x, z)$, for $x, y, z \in X$, and a V-functor is a *non-expansive map*.

When V is the cartesian quantale $[0, \infty]_{\max} = ([0, \infty], \geqslant, \max, 0)$, with tensor product $\otimes = \max$, then $V_{\max}$-Cat is the category of (*generalised*) *ultrametric spaces*.

(3) The complete lattice $([0, 1], \leqslant)$ can be equipped with several tensor products – usually called *t-norms* – including the *Lukasiewicz sum*, which lead to interesting instances of categories of the form V-Cat, like Lawvere metric spaces, ultrametric spaces, and *bounded metric spaces*.

(4) The set of *distribution functions*

$$\Delta = \left\{ \varphi \colon [0, \infty] \to [0, 1] : \text{for all } \alpha \in [0, \infty],\ \varphi(\alpha) = \bigvee_{\beta < \alpha} \varphi(\beta) \right\}$$

with the pointwise order is a complete lattice. The tensor product is defined, for each $\varphi, \psi \in \Delta$, by

$$(\varphi \otimes \psi)(\alpha) = \bigvee_{\beta + \gamma \leqslant \alpha} \varphi(\beta) \times \psi(\gamma),$$

having as unit the distribution function $\kappa\colon [0, \infty] \to [0, 1]$, with $\kappa(\alpha) = 0$ if $\alpha = 0$ and $\kappa(\alpha) = 1$ otherwise. Then Δ-Cat is the category of *probabilistic metric spaces* and (*probabilistic*) *non-expansive* maps.

For more details and examples see for instance [12].

The forgetful functor V-Cat $\to$ Set is topological, hence V-Cat is complete and cocomplete, with limits and colimits formed as in Set and equipped with the corresponding initial and final V-category structures, respectively. In [21, Corollary 8] it is shown that, under suitable conditions, V-Cat is an extensive category. In V-Cat, epimorphisms are pullback-stable, since they are exactly surjective V-functors and pullbacks are formed as in Set. Therefore, as shown in [23, Proposition 3], $(V$-Cat$)^{\mathrm{op}}$ *is a weakly Mal'tsev category*. In addition, in [16, Theorem 4.6] it is shown that $(V$-Cat$)^{\mathrm{op}}$ *is a quasi-variety*, so in particular it is a regular category. Still, we think it is worth to prove here directly that $(V$-Cat$)^{\mathrm{op}}$ *is a regular category*. Indeed, in V-Cat, regular monomorphisms coincide with extremal monomorphisms, and are exactly the injective maps $f\colon X \to Y$ such that $X(x, x') = Y(f(x), f(x'))$, for all $x, x' \in X$. Moreover, V-Cat has the stable orthogonal factorisation system (epimorphism, regular monomorphism), which factors every V-functor $f\colon X \to Y$ as

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
 & {}_{e}\searrow \quad \nearrow_{m} & \\
 & Z = f(X) &
\end{array}
$$

where e is the corestriction of f to $Z = f(X)$ and $Z(y, y') = Y(y, y')$ for all $y, y' \in Z$. Given a pushout in V-Cat

$$
\begin{array}{ccc}
X & \xrightarrow{\ m\ } & Z \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow[\ n\ ]{} & W
\end{array}
$$

with m a regular monomorphism, for simplicity we assume that m is an inclusion. We know that, since pushouts are formed as in Set, we may consider $W = (Y + Z)/{\sim}$, where, for $y \in Y$ and $z \in Z$, $y \sim z$ exactly when $z \in X$ and $f(z) = y$, and n an inclusion. Then, for every $y, y' \in Y$,

$$
\begin{aligned}
W([y], [y']) &= Y(y, y') \vee \bigvee_{x \sim y, x' \sim y'} Z(x, x') \\
&= Y(y, y') \vee \bigvee_{x \sim y, x' \sim y'} X(x, x') \qquad (m \text{ is a regular monomorphism}) \\
&= Y(y, y') \qquad\qquad\qquad\qquad\quad (f \text{ is a } V\text{-functor})
\end{aligned}
$$

and therefore the inclusion $n\colon Y \to W$ is a regular monomorphism.

Hence, since $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ is a regular category with binary coproducts, we know by (W5) that a V-category Y is a W-Mal'tsev object if and only if, given the V-functor $f\colon Y^2 + Y^2 \to Y^3$ defined by
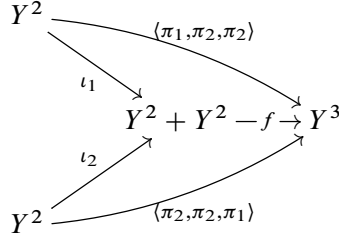
where $\pi_i\colon Y^2 \to Y$, $i = 1, 2$, are the product projections, and its (epimorphism, regular monomorphism)-factorisation in $V\text{-}\mathsf{Cat}$

$$(Y^2 + Y^2 \xrightarrow{f} Y^3) = (Y^2 + Y^2 \xrightarrow{e} X \xrightarrow{m} Y^3),$$

there is a unique V-functor $g\colon X \to Y$ such that the following diagram commutes:

$$\tag{3.i}$$
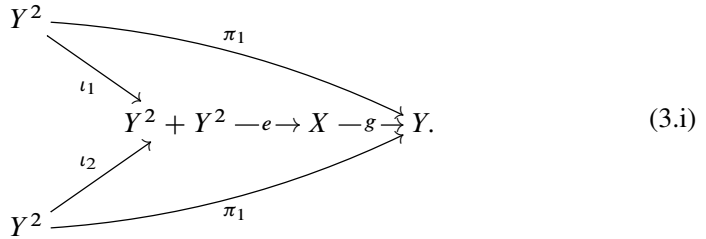
As a set, X is the image of the map f, that is

$$X = X_1 \cup X_2, \quad \text{with} \quad X_1 = \{(x, y, y) : x, y \in Y\}, \quad X_2 = \{(y, y, x) : x, y \in Y\};$$

its V-category structure is inherited from Y^3, that is

$$X((x, y, z), (x', y', z')) = Y(x, x') \wedge Y(y, y') \wedge Y(z, z').$$

The map $g\colon X \to Y$ must assign x both to each (x, y, y) and each (y, y, x).

Theorem 3.4. *Let Y be a V-category.*

(1) *The following conditions are equivalent:*

   (i)   *Y is a W-Mal'tsev object in $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$;*

   (ii)   *for all $x, y, z \in Y$, $Y(z, x) \wedge Y(z, y) \leqslant Y(x, y)$;*

   (iii) *Y is a symmetric $V_\wedge$-category.*

(2) *If V is not a cartesian quantale, then Y is a Mal'tsev object in $(V\text{-}\mathsf{Cat})^{\mathrm{op}}$ if and only if $Y = \emptyset$.*

Proof. (1) (i) $\Rightarrow$ (ii): If the map $g \colon X \to Y$ from (3.i) is a V-functor, then in particular,

$$X((x, z, z), (x, x, y)) = Y(x, x) \wedge Y(z, x) \wedge Y(z, y)$$
$$\leqslant Y(g(x, z, z), g(x, x, y)) = Y(x, y).$$

(ii) $\Rightarrow$ (iii): With $z = y$ in (ii) one gets

$$Y(y, x) = Y(y, x) \wedge Y(y, y) \leqslant Y(x, y),$$

hence Y is symmetric. Then, for any $x, y, z \in Y$,

$$Y(x, z) \wedge Y(z, y) = Y(z, x) \wedge Y(z, y) \leqslant Y(x, y).$$

(iii) $\Rightarrow$ (i): We want to prove that g is a V-functor; that is, for each $(x, y, z), (x', y', z')$ in X,

$$X((x, y, z), (x', y', z')) = Y(x, x') \wedge Y(y, y') \wedge Y(z, z') \leqslant Y(g(x, y, z), (x', y', z')).$$

In the case $x = y$ and $x' = y'$, or in the case $y = z$ and $y' = z'$, the inequality is trivially satisfied; when $x = y$ and $y' = z'$, using (iii), one obtains

$$Y(x, x') \wedge Y(x, y') \wedge Y(z, y') = Y(z, y') \wedge Y(y', x) \wedge Y(x, x') \leqslant Y(z, x'),$$

and in the case $y = z$ and $x' = y'$ one gets

$$Y(x, x') \wedge Y(y, x') \wedge Y(y, z') = Y(x, x') \wedge Y(x', y) \wedge Y(y, z') \leqslant Y(x, z').$$

(2) Since $(V\text{-Cat})^{\mathrm{op}}$ is regular we may use the characterisation of Mal'tsev objects of (M5). First we point out that a double split epimorphism (2.iii) in $(V\text{-Cat})^{\mathrm{op}}$ is also a double split epimorphism in V-Cat, with the role of split epimorphisms and split monomorphisms interchanged,

$$\begin{array}{ccc}
D & \underset{f'}{\overset{s'}{\rightleftarrows}} & C \\
g' \big\Vert \, {\scriptstyle t'} & & g \big\Vert \, {\scriptstyle t} \\
A & \underset{f}{\overset{s}{\rightleftarrows}} & Y
\end{array} \qquad (3.\mathrm{ii})$$

(this diagram is just the dual of (2.iii)).

Let V be a non-cartesian quantale. To show that the only Mal'tsev object in $(V\text{-Cat})^{\mathrm{op}}$ is the empty set it is enough to check that $Y = \{1\}$, with $Y(1, 1) = k$, is not a Mal'tsev object, since by (M4) Mal'tsev objects are closed under quotients and, for any non-empty V-category Z, any map $Y \to Z$ is a split monomorphism in V-Cat.

We will build a double split epimorphism in V-Cat as above so that it is not a *regular pullback*, i.e., the comparison morphism $\langle g', f' \rangle \colon A +_Y C \to D$ is not a regular monomorphism. If V is not cartesian then there exist $u, v \in V$ such that $u \otimes v < u \wedge v$. Consider in (3.ii) the symmetric V-categories $A = \{0, 1\}$, $C = \{1, 2\}$, and $D = \{0, 1, 2\}$, with $A(0, 1) = u$, $C(1, 2) = v$, and $D(0, 2) = u \wedge v$, with the obvious V-functors (with $s'(0) = 1$ and $t'(2) = 1$). Then (3.ii) is a double split epimorphism but the comparison morphism $\langle g', f' \rangle \colon A +_Y C \to D$, which is in fact a bijection, is not a regular monomorphism because in the pushout $(A +_Y C)(0, 2)$ must be $u \otimes v$.   ∎

Remark 3.5. We point out that condition (ii) in case $\otimes = \wedge$ is the separation axiom (R1) studied in [10, Section 2.2].

The two statements of Theorem 3.4 have an immediate consequence, showing how drastically different may be the two notions of Mal'tsev object in the context of $(V\text{-Cat})^{\mathrm{op}}$. For instance, if $V = [0, \infty]_+$, then a Lawvere metric space is a W-Mal'tsev object exactly when it is a symmetric ultrametric space, while it is a Mal'tsev object only if it is empty.

Corollary 3.6.   (1) *If V is a cartesian quantale, then $(V\text{-Cat}_{\mathrm{sym}})^{\mathrm{op}}$ is a Mal'tsev category.*

(2) *For every unital and integral quantale V, the largest Mal'tsev subcategory of $(V\text{-Cat})^{\mathrm{op}}$ closed under binary coproducts and regular epimorphisms is $(V_\wedge\text{-Cat}_{\mathrm{sym}})^{\mathrm{op}}$.*

It is straightforward to check that, if we restrict our study to symmetric $V_\otimes$-categories, in case the tensor product $\otimes$ is commutative, the double split epimorphism built above also works. Hence we may also conclude that Mal'tsev objects trivialise in this more restrictive setting.

Corollary 3.7. *If V is a commutative unital and integral non-cartesian quantale, then the only Mal'tsev object in $(V\text{-Cat}_{\mathrm{sym}})^{\mathrm{op}}$ is the empty V-category.*

In order to generalise Weighill's result for topological spaces we need to introduce an extra ingredient, the *ultrafilter monad* on Set, and its extension to V-Rel (see [14] for details). As Barr showed in [1], a topological space can be described via its ultrafilter convergence, meaning that a topological space can be given by a set X, together with a relation between ultrafilters on X and points of X, $UX \rightarrowtail X$ such that, for every $x \in X$, $\mathfrak{x} \in UX$, and $\mathfrak{X} \in U^2 X$,

(1) $\dot{x} \to x$ (where $\dot{x}$ is the principal ultrafilter defined by x);

(2) $\mathfrak{X} \to \mathfrak{x}$ and $\mathfrak{x} \to x \implies \mu(\mathfrak{X}) \to x$ (where $\mu(\mathfrak{X})$ is the Kowalski sum of $\mathfrak{X}$).

Given topological spaces X and Y, a map $f\colon X \to Y$ is continuous if, for all $\mathfrak{x} \in UX$ and $x \in X$, $Uf(\mathfrak{x}) \to f(x)$ whenever $\mathfrak{x} \to x$.

This approach can be generalised making use of the ultrafilter monad U, a unital and integral quantale V, and a lax extension of U to V-Rel.

Definition 3.8. A (U, V)-*category* is a set X together with a map $a\colon UX \times X \to V$ whose image of $(\mathfrak{x}, x)$ we denote by $X(\mathfrak{x}, x)$, such that, for each $x \in X$, $\mathfrak{x} \in UX$, $\mathfrak{X} \in U^2 X$,

(R) $k \leqslant X(\dot{x}, x)$;

(T) $X(\mathfrak{X}, \mathfrak{x}) \otimes X(\mathfrak{x}, x) \leqslant X(\mu(\mathfrak{X}), x)$,

where by $X(\mathfrak{X}, \mathfrak{x})$ we mean the image, under U, of the V-relation $a\colon UX \nrightarrow X$.

A (U, V)-*functor* $f\colon X \to Y$ is a map such that $X(\mathfrak{x}, x) \leqslant Y(Uf(\mathfrak{x}), f(x))$, for all $x \in X$, $\mathfrak{x} \in UX$.

We denote by (U, V)-Cat the category of (U, V)-categories and (U, V)-functors.

Examples 3.9.   (1) As shown by Barr [1], if $V = 2$, then (U, V)-Cat is isomorphic to the category Top of topological spaces and continuous maps.

(2) As shown in [11], if $V = [0, \infty]_+$, then (U, V)-Cat is isomorphic to the category App of Lowen's *approach spaces* and non-expansive maps.

Proposition 3.10. *The category $((U, V)\text{-Cat})^{\mathrm{op}}$ is regular.*

Proof. As in V-Cat, in (U, V)-Cat (epimorphisms, regular monomorphisms) form a stable factorisation system, and a morphism $m\colon X \to Z$ is a regular monomorphism if and only if it is an injective map and $X(\mathfrak{x}, x) = Z(Um(\mathfrak{x}), m(x))$, for all $\mathfrak{x} \in UX$ and $x \in X$. Given a pushout in (U, V)-Cat

$$
\begin{array}{ccc}
X & \xrightarrow{\;m\;} & Z \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
Y & \xrightarrow[\;n\;]{} & W
\end{array}
$$

with m a regular monomorphism, which we assume, for simplicity, to be an inclusion, as for V-Cat we may consider $W = (Y + Z)/\sim$, where, for $y \in Y$ and $z \in Z$, $y \sim z$ exactly when $z \in X$ and $f(z) = y$. Then n, and therefore also Un, is an injective map, and, for every $\mathfrak{y} \in UY$ and $y \in Y$,

$$
W(Un(\mathfrak{y}), [y]) = Y(\mathfrak{y}, y) \vee \bigvee_{\substack{Ug(\mathfrak{z}) = Un(\mathfrak{y}), \\ x \sim y}} Z(\mathfrak{z}, x).
$$

Any $\mathfrak{z} \in UZ$ with $Ug(\mathfrak{z}) = Un(\mathfrak{y})$ must contain X, and consequently it is the image under Um of an ultrafilter $\mathfrak{x}$ in X; hence $Z(\mathfrak{z}, x) = X(\mathfrak{x}, x) \leqslant Y(Uf(\mathfrak{x}), f(x)) =$

$Y(y, y)$. This implies that $Y(y, y) = W(Un(y), [y])$, and therefore $n \colon Y \to W$ is a regular monomorphism.   ∎

Proposition 3.11.   (1) *For a (U, V)-category Y the following conditions are equivalent:*

    (i) *Y is a W-Mal'tsev object in $((U, V)\text{-Cat})^{\mathrm{op}}$;*

    (ii) *for all $z \in UY$, $x, y \in Y$,*

        (a) $Y(z, x) \wedge Y(z, y) \leqslant Y(\dot{x}, y)$;

        (b) $Y(z, x) \wedge Y(\dot{x}, y) \leqslant Y(z, y)$.

  (2) *If V is not a cartesian quantale, then Y is Mal'tsev object in $((U, V)\text{-Cat})^{\mathrm{op}}$ if and only if $Y = \emptyset$.*

Proof. (1) The map $g \colon X \to Y$ is a (U, V)-functor if and only if for every $w \in UX$ and $(x, y, z) \in X$,

$$X(w, (x, y, z)) = Y(U\pi_1(w), x) \wedge Y(U\pi_2(w), y) \wedge Y(U\pi_3(w), z)$$
$$\leqslant Y(Ug(w), g(x, y, z)).$$

For simplicity, we will denote the i-th projection $Y^k \to Y$ ($k = 2, 3$) by π_i and $U\pi_i(w)$ by w_i.

Let $h \colon Y \times Y \to X$ with $h(x, y) = h(x, y, y)$.

(i) $\Rightarrow$ (ii): Fix $z \in UY$ and $x \in Y$. Since the canonical map $U(Y^2) \to U(Y)^2$ is surjective, there exists $\widetilde{z} \in U(Y^2)$ such that $\widetilde{z}_1 = \dot{x}$ and $\widetilde{z}_2 = z$. Then $w = Uh(\widetilde{z}) \in UX$ and, since $g \cdot h = \pi_1$, $w_1 = \dot{x}$, $w_2 = z = w_3$, and $Ug(w) = \dot{x}$. Then we have

$$X(w, (x, x, y)) \leqslant Y(\dot{x}, y) \Rightarrow Y(\dot{x}, x) \wedge Y(z, x) \wedge Y(z, y) \leqslant Y(\dot{x}, y),$$

that is, (a) holds.

Analogously, let $\widehat{z} \in U(Y^2)$ be such that $\widehat{z}_1 = z$ and $\widehat{z}_2 = \dot{x}$, and let $\mathfrak{x} = Uh(\widehat{z})$, so that $Ug(\mathfrak{x}) = z$. Then, for every $y \in Y$,

$$X(\mathfrak{x}, (x, x, y)) \leqslant Y(z, y) \Rightarrow Y(z, x) \wedge Y(\dot{x}, x) \wedge Y(\dot{x}, y) \leqslant Y(z, y),$$

that is, (b) holds.

(ii) $\Rightarrow$ (i): Conversely, assume (a) and (b). Let $w \in UX$ and assume that $X_1 \in w$ (for the case $X_2 \in w$ the proof is analogous). Then $w = Uh(y)$ for some $y \in U(Y^2)$. Then

$$X(w, (x, y, y)) = Y(y_1, x) \wedge Y(y_2, y) \wedge Y(y_2, y) \leqslant Y(y_1, x) = Y(Ug(w), x),$$
$$X(w, (x, x, y)) = Y(y_1, x) \wedge Y(y_2, x) \wedge Y(y_2, y) \leqslant Y(y_1, x) \wedge Y(\dot{x}, y)$$
$$\leqslant Y(y_1, y);$$

that is, g is a (U, V)-functor.

(2) We may argue as in the proof of Theorem 3.4, and use (U, V)-categories $Y = \{1\}$, $A = \{0, 1\}$, $C = \{1, 2\}$ and $D = \{0, 1, 2\}$ with the structures given by $A(\dot{0}, 1) = A(\dot{1}, 0) = u$, $C(\dot{1}, 2) = C(\dot{2}, 1) = v$, and $D(\dot{0}, 2) = D(\dot{2}, 0) = u \otimes v$. ∎

Remark 3.12. The result of the proposition above applies immediately to Top and App, and it may be extended to other suitable Set-monads. For simplicity we chose to restrict ourselves to the ultrafilter monad, recovering the result in Top and obtaining a characterisation of W-Mal'tsev objects in $(\mathsf{App})^{\mathrm{op}}$.

At a first glance it may seem that here we need an extra condition to recover the result in Top. Indeed, in $(\mathsf{Top})^{\mathrm{op}}$ condition (a) is the well-known separation axiom (R1) we have mentioned in Remark 3.5, while condition (b) follows from axiom (T) of the definition of (U, V)-category in case V is cartesian, which is the case of Top. When $V = [0, \infty]_+$ condition (a) is stronger than (R1), which reads as, for every $\mathfrak{z} \in UY$, $x, y \in Y$, $Y(\mathfrak{z}, x) + Y(\mathfrak{z}, y) \geqslant Y(\dot{x}, y)$, and (b) does not follow from axiom (T).

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