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Remarks on complete Weingarten hypersurfaces in the Euclidean space

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Abstract. In this paper, first we provide a classification result for a large class of complete rotational Weingarten hypersurfaces in the Euclidean space. Second, we prove sharp inequalities for the norm of the second fundamental form of a class of Weingarten hypersurfaces. We show that sharpness is attained by a cylinder of the Euclidean space.

1. Introduction and statement of the main results

In their 1983 very nice paper, do Carmo and Dajczer [9] introduced and studied the notion of rotational hypersurfaces in spaces of constant sectional curvature, by extending the classical definition of rotational surfaces of the 3-dimensional Euclidean space $\mathbb{R}^3$. In particular, they exhibited explicit parametrizations and computed the principal curvatures of such hypersurfaces, gave sufficient conditions for a hypersurface to be rotational and also proved characterizations of complete rotational hypersurfaces with constant mean curvature, among other results.

Then it has become a natural and interesting question to classify complete rotational hypersurfaces under some mild geometric restrictions, mainly on the constancy of the scalar curvature or, more generally, the constancy of some higher order mean curvature. In fact, a few years after do Carmo and Dajczer's results this was done in a series of papers due to Leite [12], Mori [15], Palmas [17], Hounie and Leite [11] and Colares and Palmas [5]. When the ambient space is the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, we collect the following classification result from above mentioned papers.

Theorem 1.1. *We can classify the complete rotational hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ with constant r-th mean curvature H_r as follows:*

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(i) *There is only one monoparametric family of embedded complete rotational hypersurfaces with constant r-th mean curvature H_r for any $H_r > 0$, $1 \leq r < n$. These hypersurfaces are periodic and cylindrically bounded and they converge, on one side, to a sequence of spheres, which are pairwise vertically, and on the other, to the cylinder $\mathbb{R} \times \mathbb{S}^n(u_0)$, for some $u_0 > 0$.*

(ii) *There is only one monoparametric family of embedded complete rotational hypersurfaces with constant r-th mean curvature $H_r = 0$, $1 \leq r < n$, which converges to a hyperplane.*

(iii) *There is no complete rotational hypersurface with constant r-th mean curvature $H_r < 0$, $1 \leq r < n$, for r even.*

Let us recall that the proof of this theorem relies on an approach introduced by Leite [12] and afterwards generalized by Palmas [17], which is based on a careful analysis of the level sets of a suitable function which gives rise to all rotational hypersurfaces of the ambient space.

In this setting, our first aim in this paper is to extend this approach to a large class of complete rotational Weingarten hypersurfaces immersed into the Euclidean space $\mathbb{R}^{n+1}$ and to prove a classification result in the same spirit of [12, 17]. To this end, given a hypersurface of $\mathbb{R}^{n+1}$, let us recall that it is said to be a *Weingarten hypersurface* (or a W-hypersurface) if there exists a smooth function W of the principal curvatures $\lambda_1, \ldots, \lambda_n$ so that $W(\lambda_1, \ldots, \lambda_n)$ is constant (Chern [4], Hartman [10]). In particular, hypersurfaces with some constant higher order mean curvature are examples of Weingarten hypersurfaces.

We consider here the class of Weingarten hypersurfaces immersed into the Euclidean space whose higher order mean curvatures H_r and H_k, $1 \leq k < r < n$, are linearly related, meaning that the following relation holds:

$$H_r = aH_k + b, \tag{1.1}$$

for certain constants $a, b \in \mathbb{R}$.

Then we shall prove the following result.

Theorem 1.2. *We can classify the complete rotational Weingarten hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ with $H_r = aH_k + b$ as follows:*

(i) *There is only one monoparametric family of embedded complete rotational Weingarten hypersurfaces with $H_r = aH_k + b$, $1 \leq k < r < n$, $a \in \mathbb{R}$ and $b > 0$. These hypersurfaces are periodic and cylindrically bounded and they converge, on one side, to a sequence of spheres, which are pairwise vertically, and on the other, to the cylinder $\mathbb{R} \times \mathbb{S}^n(u_0)$, for some $u_0 > 0$.*

(ii) *There are only two monoparametric families of embedded complete rotational Weingarten hypersurfaces with $H_r = aH_k$, $1 \leq k < r < n$, $a \in \mathbb{R}$*

(and $b = 0$). One of them is periodic and cylindrically bounded and converges, on one side, to a sequence of spheres, which are pairwise vertically, and on the other, to the cylinder $\mathbb{R} \times \mathbb{S}^n(u_0)$, for some $u_0 > 0$. The other family is not cylindrically bounded and converges to a hyperplane.

(iii) *There is no complete rotational Weingarten hypersurface with $H_r = aH_k + b$, $1 \le k < r < n$, for r even, $a \le 0$ and $b < 0$.*

In the case $a = 0$, Theorem 1.2 coincides with the previous classification obtained by Leite [12] and Palmas [17]. Thus, Theorem 1.2 is a generalization of results previously cited. The case of rotational Weingarten hypersurfaces with $H_n = aH_k + b$, $1 \le k < n$ has been treated by the author in [7].

Our second aim is to get a sharp lower (upper) bound for the supremum (infimum) of the norm of the second fundamental form of Weingarten hypersurfaces with two distinct principal curvatures satisfying (1.1). Regarding this theme, many works have been done nowadays to treat hypersurfaces with some constant higher order mean curvature immersed into $\mathbb{R}^{n+1}$, by using different approaches (for instance, Cheng–Yau's ideas [3], Ôtsuki's ideas [16], Omori–Yau maximum principle, rotational hypersurfaces [9], principal curvature theorem [19], among others) and under various assumptions (for instance, positive sectional curvature, two distinct principal curvatures, upper or lower bounds of the squared norm of the second fundamental form, among others).

Let us highlight two recent approaches. In [1], Alías and García-Martinez, and in [2], Alías and Meléndez, used the so-called principal curvature theorem due to Smyth and Xavier [19], and in [14], Meléndez and Palmas introduced tools related to rotational hypersurfaces, and proved a sharp lower (upper) estimate for the supremum (infimum) of the norm of the second fundamental form of hypersurfaces in $\mathbb{R}^{n+1}$ with some constant higher order mean curvature and two distinct principal curvatures (in [14] the authors also were able to apply the method when the ambient space is the Euclidean sphere or the hyperbolic space). In particular, we can state the following.

Theorem 1.3 ([2, Theorems 2 and 3] and [14, Corollary 1.8]). *Let M^n be a complete oriented hypersurface of $\mathbb{R}^{n+1}$, $n \ge 3$ and $1 < r < n$, with constant r-th mean curvature $H_r > 0$ and two distinct principal curvatures, one of them being simple. Then*

$$\min_{\Sigma} |A|^2 \le (n-1)\left(\frac{nH_r}{n-r}\right)^{2/r} \le \max_{\Sigma} |A|^2, \tag{1.2}$$

where $|A|$ stands for the norm of the second fundamental form A of M^n. Moreover, equality holds in the left-hand side (or right-hand side) of (1.2) if and only if M^n is isometric to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(u_0)$ with $u_0 = (\frac{n-r}{nH_r})^{1/r}$.

Here, by using both approaches aforementioned, we extend the above result to the class of Weingarten hypersurfaces given by (1.1) as follows.

Theorem 1.4. *Let M^n be a complete oriented Weingarten hypersurface of $\mathbb{R}^{n+1}$, $n \geq 3$, with $H_r = aH_k + b$, where $1 \leq k < r < n$, $a \in \mathbb{R}$ and $b > 0$. Suppose that M^n has two distinct principal curvatures λ and μ, with μ being simple and $\lambda > 0$. Then,*

$$\min_{\Sigma}|A|^2 \leq |A_0|^2 \leq \max_{\Sigma}|A|^2, \tag{1.3}$$

where $|A_0|$ is the norm of the second fundamental form of the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(u_0)$, for some $u_0 > 0$. Moreover, equality holds in the left-hand side (or right-hand side) of (1.3) if and only if M^n is isometric to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(u_0)$.

Here, in the case $a = 0$, Theorem 1.4 agrees with the results due to Alías and Meléndez ([2, Theorems 2 and 3]) and Meléndez and Palmas ([14, Corollary 1.8]). When $k = 1$ we recover the very recent theorem due to the author (see [6, Theorem 1]). If $r = 2$, we also recover [8, Theorem 1.1] (in the case $p = 1$ and $c = 0$ there).

Regarding the condition $\lambda > 0$ on M^n in Theorem 1.4, we observe that it is very mild and implied by various others conditions as shown by the author in [7, Remark 1].

The paper is organized as follows. In Section 2, we recall some basic facts and notations that will appear in the paper and some key lemmas that will be essential in the proofs of the main results. Finally, Sections 3 and 4 are dedicated to exhibiting the proofs of Theorems 1.2 and 1.4.

2. Preliminaries

Let M^n be an oriented (connected) rotational hypersurface immersed into the Euclidean space $\mathbb{R}^{n+1}$. Following [9], M^n is constructed by taking the orbit of a curve α, called the profile curve, under the orthogonal transformations of $\mathbb{R}^{n+1}$ leaving fixed an axis, called the rotation axis, that does not meet the curve. Without loss of generality we can take the curve α generating M^n parametrized by arc length as

$$\alpha(s) = (x(s), 0, \ldots, 0, z(s)) : I \subset \mathbb{R} \to \mathbb{R}^{n+1}, \quad x(s) > 0,$$

and the rotation axis as being the x_{n+1}-axis. In particular,

$$\dot{x}^2 + \dot{z}^2 = 1,$$

where the dot denotes the derivative relative to arc length s. Then a parametrization for M^n is

$$\varphi : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}, \quad \varphi(s,p) = (x(s)\psi(p), z(s)),$$

where $\psi(p)$ is an orthogonal parametrization of the unit sphere $\mathbb{S}^{n-1}$. It follows from [9, Proposition 3.2] that there are at most two different principal curvatures on M^n, given by

$$\lambda_1 = \cdots = \lambda_{n-1} = \frac{\sqrt{1-\dot{x}^2}}{x}, \tag{2.1}$$

and

$$\mu = -\frac{\ddot{x}}{\sqrt{1-\dot{x}^2}}.$$

The *r-th mean curvature* H_r of the hypersurface M^n is defined as the mean of the r-th symmetric function of the principal curvatures:

$$\binom{n}{r} H_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}, \quad 1 \le r \le n.$$

From the discussion above it is not difficult to check that a rotational hypersurface M^n has prescribed r-th mean curvature H_r if and only if

$$nH_r x^r = (n-r)(1-\dot{x}^2)^{\frac{r}{2}} - r(1-\dot{x}^2)^{\frac{r-2}{2}} \ddot{x} x. \tag{2.2}$$

In what follows, we provide two auxiliary lemmas, which have been obtained by the author in [7], and for the sake of completeness we also exhibit its proof here. The first one gives a necessary and sufficient condition, in terms of a differential equation, for a rotational hypersurface M^n being Weingarten satisfying the relation

$$H_r = aH_k + b, \tag{2.3}$$

for constants $a, b \in \mathbb{R}$.

Lemma 2.1. *Let M^n be a rotational hypersurface in $\mathbb{R}^{n+1}$. Then (2.3) holds if and only if x satisfies the differential equation*

$$\begin{aligned} nbx^{n-1} &= (n-r)(1-\dot{x}^2)^{\frac{r}{2}} x^{n-r-1} - r(1-\dot{x}^2)^{\frac{r-2}{2}} \ddot{x} x^{n-r} \\ &\quad - a\big((n-k)(1-\dot{x}^2)^{\frac{k}{2}} x^{n-k-1} - k(1-\dot{x}^2)^{\frac{k-2}{2}} \ddot{x} x^{n-k}\big). \end{aligned} \tag{2.4}$$

Equation (2.4) is equivalent to its first integral

$$x^{n-r}(1-\dot{x}^2)^{\frac{r}{2}} - ax^{n-k}(1-\dot{x}^2)^{\frac{k}{2}} - bx^n = C, \tag{2.5}$$

where $C \in \mathbb{R}$ is a constant.

Proof. From (2.2) we find

$$nH_r x^{n-1} = (n-r)(1-\dot{x}^2)^{\frac{r}{2}} x^{n-r-1} - r(1-\dot{x}^2)^{\frac{r-2}{2}} \ddot{x} x^{n-r}$$

and

$$nH_k x^{n-1} = (n-k)(1-\dot{x}^2)^{\frac{k}{2}} x^{n-k-1} - k(1-\dot{x}^2)^{\frac{k-2}{2}} \ddot{x} x^{n-k}.$$

Then it follows immediately that M^n satisfies (2.3) if and only if it also satisfies (2.4).

Finally, by noting that

$$\frac{d}{ds}\left(x^{n-r}(1-\dot{x}^2)^{\frac{r}{2}} - a x^{n-k}(1-\dot{x}^2)^{\frac{k}{2}}\right) = nbx^{n-1}\dot{x}$$

and taking the integral of both sides, we obtain (2.5), as desired. ∎

Having Lemma 2.1 in mind, it becomes clear that the set of the points in $\mathbb{R}^2$, $(x(s), \dot{x}(s))$, where $x(s)$ is a local solution of (2.4), is a level curve of the function

$$G_{r,k,a,b}(u,v) = u^{n-r}(1-v^2)^{\frac{r}{2}} - a u^{n-k}(1-v^2)^{\frac{k}{2}} - bu^n, \quad \text{with } u > 0,\ v^2 \leq 1. \tag{2.6}$$

Following Leite [12] and Palmas [17], we deduce that the sets $(x, \dot{x})$, where x is a local solution of (2.4), are the connected components of the level sets of $G_{r,k,a,b}$ contained in the region $\{u > 0 \text{ and } v^2 \leq 1\}$. We also consider the definition below.

Definition 2.2. A solution of (2.4) is *complete* if either x is defined for every $s \in \mathbb{R}$ or if the pair $(x, \dot{x})$ only admits $(0, \pm 1)$ as limit values.

Thus only the level sets

$$G_{r,k,a,b}(x, \dot{x}) = C, \tag{2.7}$$

where x is a complete solution of (2.4), totally contained in the region $\{u > 0$ and $v^2 \leq 1\}$, give rise to complete rotational Weingarten hypersurfaces satisfying (2.3).

Before stating our next auxiliary result and for the sake of clarity, let us compute first the partial derivatives of $G_{r,k,a,b}$. Since $1 \leq k \leq r < n$ we obtain

$$\frac{\partial G_{r,k,a,b}}{\partial u} = u^{n-r-1}\left((n-r)(1-v^2)^{\frac{r}{2}} - (n-k)au^{r-k}(1-v^2)^{\frac{k}{2}} - nbu^r\right),$$
$$\frac{\partial G_{r,k,a,b}}{\partial v} = u^{n-r}v\left(-r(1-v^2)^{\frac{r-2}{2}} + kau^{r-k}(1-v^2)^{\frac{k-2}{2}}\right).$$

In particular, when $b \neq 0$, any level set of $G_{r,k,a,b}$ intersecting one of the horizontal lines $v^2 = 1$ must necessarily leave the region $\{u > 0$ and $v^2 \leq 1\}$, except if it is one of the horizontal lines $v^2 = 1$.

Lemma 2.3. *For $a \in \mathbb{R}$, $b > 0$ or $a > 0$, $b = 0$, and $1 \le k < r < n$, the function $G_{r,k,a,b}$ has a unique critical point given by $(u_0, 0)$, where $u_0 > 0$ is the unique positive root of*

$$g(u) = nbu^r + (n-k)au^{r-k} - (n-r) = 0. \tag{2.8}$$

Moreover, $(u_0, 0)$ is a local maximum and non-degenerate, $C_0 := G_{r,k,a,b}(u_0, 0) > 0$ and, for every $0 < C < C_0$, the level set $G_{r,k,a,b}(u, v) = C$ is a closed curve surrounding $(u_0, 0)$ and contained in the region interior to the zero level set of $G_{r,k,a,b}$.

Proof. The case $a = 0$ was proved in [14, Lemma 3.1 and Proposition 3.2]. Now, let us assume that $a \neq 0$. Then $\frac{\partial G_{r,k,a,b}}{\partial v} = 0$ implies that

$$v = 0 \quad \text{or} \quad -r(1-v^2)^{\frac{r-2}{2}} + kau^{r-k}(1-v^2)^{\frac{k-2}{2}} = 0.$$

In the first case, $v = 0$, we deduce from $\frac{\partial G_{r,k,a,b}}{\partial u} = 0$ that $g(u) = 0$, where $g(u)$ is given by (2.8). Then it is not difficult to see that there exists a unique positive root $u_0 > 0$ of equation (2.8). The corresponding critical point $(u_0, 0)$ satisfies, by (2.6),

$$G_{r,k,a,b}(u_0, 0) = u_0^{n-r}(1 - au_0^{r-k} - bu_0^r).$$

From $g(u_0) = 0$ we deduce that

$$G_{r,k,a,b}(u_0, 0) = u_0^{n-r}(r - aku_0^{r-k}). \tag{2.9}$$

If $a < 0$, it follows immediately that $G_{r,k,a,b}(u_0, 0) > 0$. In the case $a > 0$, it is straightforward to check that

$$g\left(\left(\frac{r}{ak}\right)^{\frac{1}{r-k}}\right) > 0,$$

so that $u_0 < (\frac{r}{ak})^{\frac{1}{r-k}}$ and, consequently, from (2.9) we find $G_{r,k,a,b}(u_0, 0) > 0$. To classify the critical point $(u_0, 0)$, we calculate the second partial derivatives to obtain

$$\frac{\partial^2 G_{r,k,a,b}}{\partial u^2}(u_0, 0) = -u_0^{n-r-2}((n-r)(r-k) + nkbu_0^r) < 0,$$
$$\frac{\partial^2 G_{r,k,a,b}}{\partial v^2}(u_0, 0) = -u_0^{n-r}(r - aku_0^{r-k}) < 0,$$
$$\frac{\partial^2 G_{r,k,a,b}}{\partial u \partial v}(u_0, 0) = 0.$$

Therefore, $(u_0, 0)$ is a local maximum and non-degenerate.

In the second case,

$$-r(1-v^2)^{\frac{r-2}{2}} + kau^{r-k}(1-v^2)^{\frac{k-2}{2}} = 0,$$

a simple computation shows that $\frac{\partial G_{r,k,a,b}}{\partial u} = 0$ has no solution.

Finally, let us observe that the zero level set $G_{r,k,a,b}(u,v) = 0$ is given by a portion of the ellipse

$$\alpha^2 u^2 + v^2 = 1, \quad u > 0,$$

where $\alpha > 0$ is the unique positive root of $\alpha^r - a\alpha^k - b = 0$. By the unicity of the critical point, $(u_0, 0)$ belongs to the region interior to the zero level set. Moreover, every $0 < C < C_0$ is a regular value of $G_{r,k,a,b}$, so that the level set $G_{r,k,a,b}(u,v) = C$ is a smooth curve and a compact set. This jointly with the classification of 1-manifolds concludes the proof of the lemma. ■

At this point and for future reference, let us recall that a rotational hypersurface M^n is said to be *cylindrically bounded* if there exists a complete cylinder $\Sigma_0 = \mathbb{R} \times \mathbb{S}^{n-1}(r)$ with the same axis of rotation of M^n such that M^n is contained in the closure of the component of $\mathbb{R}^{n+1} - \Sigma_0$ containing the rotation axis.

To close this section, for the sake of completeness and to establish the notation that we will use in the proofs of our theorems, we state below two well-known results that will be crucial ingredients in the next sections. The former one due to do Carmo and Dajczer [9] gives sufficient conditions for hypersurfaces of the Euclidean space to be rotational.

Theorem 2.4 ([9, Theorem 4.2]). *Let M^n be an arbitrary hypersurface in $\mathbb{R}^{n+1}$, $n \geq 3$. Assume that the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M^n satisfy $\lambda_1 = \cdots = \lambda_{n-1} = \lambda \neq 0$, $\lambda_n = \mu = \mu(\lambda)$, and $\lambda - \mu \neq 0$. Then M^n is contained in a rotational hypersurface.*

The second one is the so-called principal curvature theorem due to Smyth and Xavier [19].

Theorem 2.5 (The principal curvature theorem, see [19, §1]). *Let M^n be a complete immersed orientable hypersurface in $\mathbb{R}^{n+1}$, which is not a hyperplane, and let A denote its second fundamental form with respect to a global unit normal field. Let $\Lambda \subset \mathbb{R}$ be the set of nonzero values assumed by the eigenvalues of A and let $\Lambda^{\pm} = \Lambda \cap \mathbb{R}^{\pm}$.*

(i) *If Λ^+ and Λ^- are both nonempty,* $\inf \Lambda^+ = \sup \Lambda^- = 0$.

(ii) *If Λ^+ or Λ^- is empty then the closure $\overline{\Lambda}$ of Λ is connected.*

3. Proof of Theorem 1.2

As said above, to prove Theorem 1.2 we must analyze the admissible values for C in (2.7), those that give rise to complete rotational Weingarten hypersurfaces. Since the case $a = 0$ was proved by Palmas [17] (see [17, Section 1.2]), here we always assume that $a \neq 0$.

Case (i). When $C = 0$, we note that the zero level set $G_{r,k,a,b}(x, \dot{x}) = 0$ consists of the portion of the ellipse

$$\alpha^2 x^2 + \dot{x}^2 = 1, \quad \text{with } x > 0,$$

where $\alpha > 0$ is the unique positive root of the equation $\alpha^r - a\alpha^k - b = 0$. Then by integration we see that the corresponding hypersurface is a sphere parametrized by

$$x(s) = \frac{\sin(\alpha s)}{\alpha} \quad \text{and} \quad z(s) = \frac{\cos(\alpha s)}{\alpha}.$$

The translation of this sphere along the x_{n+1}-axis gives a sequence of spheres which are pairwise vertically.

When $C = C_0$, since $(u_0, 0)$ is an isolated critical point, then it follows that the level set $G_{r,k,a,b}(x, \dot{x}) = C_0$ is a unitary set $\{(u_0, 0)\}$. Thus $x(s) = u_0$ must be constant and the principal curvatures of the corresponding hypersurface are also constant and given by

$$\lambda_1 = \cdots = \lambda_{n-1} = \frac{1}{u_0} \quad \text{and} \quad \mu = 0.$$

Hence, by the classical results due to Levi-Civita [13] and Segre [18] on isoparametric hypersurfaces of $\mathbb{R}^{n+1}$, we have that M^n is isometric to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(u_0)$, $u_0 > 0$.

Next, for every $0 < C < C_0$, we obtain from the proof of [12, Theorem 3.4] that all level curves $G_{r,k,a,b}(x, \dot{x}) = C$ correspond to embedded, complete, periodic and cylindrically bounded hypersurfaces.

If $C < 0$, then the level curves $G_{r,k,a,b}(x, \dot{x}) = C$ are given by

$$(1 - \dot{x}^2)^{\frac{r}{2}} - a x^{r-k} (1 - \dot{x}^2)^{\frac{k}{2}} = \frac{C}{x^{n-r}} + b x^r,$$

and leave the region $\{x > 0 \text{ and } \dot{x}^2 \leq 1\}$ at the point $C + bx^n = 0$ or $\dot{x}^2$ reaches 1 in a finite interval. Thus the corresponding hypersurfaces are not complete.

Case (ii). In this case, for $C = 0$ the connected components of the zero level set $G_{r,k,a,0}(x, \dot{x}) = 0$ are given by

$$\dot{x}^2 = 1 \quad \text{and} \quad a^{\frac{2}{r-k}} x^2 + \dot{x}^2 = 1, \quad x > 0.$$

In the first case, $x(s) = s$ and $z(s) = 0$, so that the corresponding hypersurface is a hyperplane. In the second case, we have a sphere.

Now let us consider $C > 0$. If $a > 0$, the result follows as in Case (i). For $a < 0$ the level sets $G_{r,k,a,0}(x, \dot{x}) = C$ are given by

$$(1 - \dot{x}^2)^{\frac{r}{2}} - a x^{r-k}(1 - \dot{x}^2)^{\frac{k}{2}} = \frac{C}{x^{n-r}}. \tag{3.1}$$

Then $x(s)$ attains a minimum value and is unbounded from above. So $x(s) \to +\infty$ implies $\dot{x}^2 \to 1$. Hence, as in Leite [12], every corresponding hypersurface is not cylindrically bounded and this family converges to a hyperplane.

When $C < 0$, we obtain for $a > 0$ that the level sets $G_{r,k,a,0}(x, \dot{x}) = C$ are given as in (3.1) and the result follows. In the case $a < 0$, not even local solutions $x(s)$ exist.

Case (iii). In this case, for every $C \leq 0$ there are no local solutions of $G_{r,k,a,b}(x, \dot{x}) = C$ in the region $\{x > 0 \text{ and } \dot{x}^2 \leq 1\}$. When $C > 0$ the level curves $G_{r,k,a,b}(x, \dot{x}) = C$ leave the region $\{x > 0 \text{ and } \dot{x}^2 \leq 1\}$ at the point $C + bx^n = 0$ or $\dot{x}^2$ reaches 1 in a finite interval. In any case there are no complete hypersurfaces. This finishes the proof of the theorem.

4. Proof of Theorem 1.4

As already said, let us denote by λ and μ the distinct principal curvatures of M^n with multiplicities $n-1$ and 1, respectively. Then a straightforward computation shows that

$$nH_k = \lambda^{k-1}((n-k)\lambda + k\mu)$$

and

$$nH_r = \lambda^{r-1}((n-r)\lambda + r\mu).$$

By the relation between the mean curvatures (1.1), we deduce that

$$\mu\lambda^{k-1}(r\lambda^{r-k} - ka) = nb + (n-k)a\lambda^k - (n-r)\lambda^r. \tag{4.1}$$

Since λ is assumed to be positive on M^n we have

$$r\lambda^{r-k}(p) - ka \neq 0, \quad \forall p \in \Sigma^n,$$

because, otherwise, by equation (4.1), we would get

$$b = -\frac{r-k}{k}\lambda^r(p_0) < 0,$$

for some $p_0 \in \Sigma^n$, contradicting the hypothesis $b > 0$. Then the following relation holds:

$$\mu = \mu(\lambda) = \frac{nb + (n-k)a\lambda^k - (n-r)\lambda^r}{\lambda^{k-1}(r\lambda^{r-k} - ka)}. \tag{4.2}$$

In particular, letting $\lambda_{rk} := (\frac{ka}{r})^{\frac{1}{r-k}}$, we find that either $a > 0$ and $0 < \lambda < \lambda_{rk}$, or

$$a > 0 \quad \text{and} \quad \lambda > \lambda_{rk},$$

or

$$a \leq 0 \quad \text{and} \quad \lambda > 0.$$

We claim that the first case, $a > 0$ and $0 < \lambda < \lambda_{rk}$, cannot happen. Indeed, if $k = 1$, it is not difficult to see that $\mu(\lambda)$ is strictly decreasing on $0 < \lambda < \lambda_{r1}$ and

$$\mu(\lambda) \to -\frac{nb}{a} < 0, \quad \text{as} \quad \lambda \to 0^+,$$

so that Λ^+ and Λ^- are nonempty with $\sup \Lambda^- < 0$, contradicting the principal curvature theorem. When $k > 1$, we see that μ is also a negative function on $0 < \lambda < \lambda_{rk}$ and

$$\begin{aligned} \mu(\lambda) &\to -\infty, \quad \text{as} \quad \lambda \to 0^+ \\ \mu(\lambda) &\to -\infty, \quad \text{as} \quad \lambda \to \lambda_{rk}^-, \end{aligned}$$

so that $\mu(\lambda)$ attains its maximum at some $\lambda_\mu \in (0, \lambda_{rk})$ with $\mu(\lambda_\mu) < 0$. Thus Λ^+ and Λ^- are nonempty with $\sup \Lambda^- < 0$ and the claim follows.

By the discussion above we have two cases to consider:

(i) $a > 0$ and $\lambda > \lambda_{rk}$;

(ii) $a \leq 0$ and $\lambda > 0$.

From now on, whenever we write $\lambda > 0$, we let it be implicit that in case (i) we must have $\lambda > \lambda_{rk}$, unless we explicitly specify otherwise.

For the sake of clarity let us proceed by dividing the proof into three steps as follows.

Step 1. Properties of $\mu(\lambda)$

A simple computation by using (4.2) gives the following expression to $\mu(\lambda)$:

$$\mu(\lambda) = \frac{nb}{\lambda^{k-1}(r\lambda^{r-k} - ka)} + \frac{n(r-k)a\lambda}{r(r\lambda^{r-k} - ka)} - \frac{n-r}{r}\lambda. \tag{4.3}$$

In particular,

$$\begin{aligned} \frac{d\mu}{d\lambda}(\lambda) = {} & -nb\frac{r(r-1)\lambda^{r-k} - k(k-1)a}{\lambda^k(r\lambda^{r-k} - ka)^2} \\ & - \frac{nr(r-k)(r-k-1)a\lambda^{r-k} + n(r-k)ka^2}{(r\lambda^{r-k} - ka)^2} - \frac{n-r}{r}, \end{aligned} \tag{4.4}$$

and it is not difficult to verify that the following properties hold:

(μ_0) M^n is a rotational hypersurface.
This follows directly from the result due to do Carmo and Dajczer (Theorem 2.4 above) and equation (4.3).

(μ_1) There exists a unique $\lambda_0 > 0$ such that $\mu(\lambda_0) = 0$.
Of course, by (4.2), λ_0 is the unique positive root of the equation

$$-(n-r)\lambda^r + (n-k)a\lambda^k + nb = 0. \quad (4.5)$$

Further, in case (i), it must be that $\lambda_0 > \lambda_{rk}$. Moreover, denoting by I_{λ_0} the interval $(\lambda_{rk}, \lambda_0]$, in case (i), and $(0, \lambda_0)$, in case (ii), we also deduce that

$$\mu(\lambda) > 0 \quad \text{on } I_{\lambda_0},$$

and

$$\mu(\lambda) < 0 \quad \text{if } \lambda > \lambda_0.$$

(μ_2) $\mu(\lambda)$ is strictly decreasing on I_{λ_0}.
Indeed, when $a \geq 0$, we see from (4.4) that $\mu'(\lambda) < 0$ on $(\lambda_{rk}, +\infty) \supset I_{\lambda_0}$. For $a < 0$, the claim follows directly from (4.2) and μ_1).

(μ_3) There exists a unique $\tilde{\lambda} > 0$ such that $\mu(\tilde{\lambda}) = \tilde{\lambda}$.
Of course, by using equation (4.3), we see that $\tilde{\lambda}$ is the unique positive root of the equation

$$-\lambda^r + a\lambda^k + b = 0.$$

In case (i), it follows that $\lambda_{rk} < \tilde{\lambda}$. Moreover, it is simple to see that $\tilde{\lambda} < \lambda_0$, because of (4.5). Let $I_{\tilde{\lambda}}$ be the interval $(\lambda_r, \tilde{\lambda})$, in case (i), and $(0, \tilde{\lambda})$, in case (ii). Analogously, we define $I^{\tilde{\lambda}} = (\tilde{\lambda}, \lambda_2]$. Then

$$\mu(\lambda) > \lambda \quad \text{on } I_{\tilde{\lambda}},$$

and

$$\mu(\lambda) < \lambda \quad \text{on } I^{\tilde{\lambda}}.$$

(μ_4) Via a somewhat more involved computation we can rewrite equation (4.4) as

$$\frac{d\mu}{d\lambda} = -(n-1) - \frac{n(r(r-1)\lambda^{r-k} - k(k-1)a)(-\lambda^r + a\lambda^k + b)}{\lambda^k (r\lambda^{r-k} - ka)^2}.$$

In particular, $\frac{d\mu}{d\lambda}(\lambda) = -(n-1)$ if and only if $\lambda = \tilde{\lambda}$. Moreover,

$$\frac{d\mu}{d\lambda}(\lambda) < -(n-1) \quad \text{on } I_{\tilde{\lambda}},$$

and

$$\frac{d\mu}{d\lambda}(\lambda) > -(n-1) \quad \text{on } I^{\tilde{\lambda}}.$$

Step 2. Analysis of the second fundamental form

The squared norm of the second fundamental form is given by

$$|A|^2(\lambda) = |A|^2 = (n-1)\lambda^2 + \mu^2.$$

Then properties (μ_3) and (μ_4) imply the following:

(A_1) $|A|^2(\lambda)$ is strictly decreasing on $I_{\widetilde{\lambda}}$;

(A_2) $|A|^2(\lambda)$ is strictly increasing on $I^{\widetilde{\lambda}}$.

In particular, $|A|^2(\lambda)$ attains its global minimum on I_{λ_2} at $\widetilde{\lambda}$ (in case (i), $\widetilde{\lambda}$ is the global minimum on $\lambda > \lambda_{rk}$).

Step 3. Conclusion

Property (μ_0) gives that M^n is a rotational hypersurface. Let $G_{r,k,a,b}(x, \dot{x}) = C$ be the level curve associated to M^n. Then

$$x^{n-r}(1-\dot{x}^2)^{\frac{r}{2}} - ax^{n-k}(1-\dot{x}^2)^{\frac{k}{2}} - bx^n = C.$$

Let us observe that the proof of Theorem 1.2 gives $C > 0$, because of the completeness of M^n. Moreover, from the previous equation and (2.1) we find

$$\lambda^r - a\lambda^k = \frac{C}{x^n} + b. \tag{4.6}$$

In particular, we can look at the principal curvature λ as a function of x and C, that is, $\lambda = \lambda(x, C)$. Then it follows from (4.6) that

$$\begin{aligned}(r\lambda^{r-1} - ka\lambda^{k-1})\frac{\partial\lambda}{\partial x} &= -\frac{nC}{x^{n+1}} < 0,\\ (r\lambda^{r-1} - ka\lambda^{k-1})\frac{\partial\lambda}{\partial C} &= \frac{1}{x^n} > 0,\end{aligned}$$

which implies that $\lambda(x, C)$ is decreasing in x and increasing in C. By using (4.6) once more, we still observe that

$$-\lambda^r(x, C) + a\lambda^k(x, C) + b = -\frac{C}{x^n} < 0.$$

Then the definition of $\widetilde{\lambda}$ implies $\lambda(x, C) > \widetilde{\lambda}$. We also claim that $\lambda(x, C) \leq \lambda_0$, because, otherwise, we would get, by (μ_1), that $\mu(\lambda) < 0$ and, consequently, Λ^+ and Λ^- are nonempty with $\inf \Lambda^+ > 0$, arriving at a contradiction to the principal curvature theorem, proving the claim. Then taking into account property (A_2) we obtain that $|A|^2(x, C) = |A|^2(\lambda(x, C))$ is strictly decreasing in x and strictly increasing in C.

From Lemma 2.3 we know that the level curve $G_{r,k,a,b}(x, \dot{x}) = C$ is a closed curve surrounding the critical point $(u_0, 0)$ of $G_{r,k,a,b}$. Thus $x(s)$ has a minimum

$x(s_0) = x_{\min}$ and a maximum $x(s_1) = x_{\max}$ with $x_{\min} \leq u_0 \leq x_{\max}$. Then $|A|^2(x, C)$ attains its minimum value $|A|^2(x_{\max}, C)$ and its maximum value $|A|^2(x_{\min}, C)$. Also, since $\dot{x}(s_0) = \dot{x}(s_1) = 0$, it follows by (2.1) that

$$\lambda(x_{\max}, C) \leq \lambda(u_0, C_0) \leq \lambda(x_{\min}, C).$$

Since $|A|^2(\lambda(x, C))$ is strictly increasing in λ on $I^{\tilde{\lambda}}$ we find

$$|A|^2(x_{\max}, C) \leq |A|^2(u_0, C_0) \leq |A|^2(x_{\min}, C),$$

that is,

$$\min_{\Sigma} |A|^2 \leq |A_0|^2 \leq \max_{\Sigma} |A|^2,$$

where $|A_0|$ is the norm of the second fundamental form of the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(u_0)$.

Finally, if equality on the left-hand side holds in the last inequality, then $x(s) = u_0$ is constant and the level curve reduces to the critical point $(u_0, 0)$. Therefore, by the proof of Theorem 1.2, M^n is isometric to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(u_0)$. The same occurs if equality on the right-hand side holds. This concludes the proof of Theorem 1.4.

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