



Purely magnetic tunneling between radial magnetic wells

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Abstract. This article is devoted to the semiclassical spectral analysis of the magnetic Laplacian in two dimensions. Assuming that the magnetic field is positive and has two symmetric radial wells, we establish an accurate tunneling formula, that is a one-term estimate of the spectral gap between the lowest two eigenvalues. This gap is exponentially small when the semiclassical parameter goes to zero, but positive.

1. Introduction

1.1. Motivation

This article deals with the spectral analysis of the purely magnetic Laplacian in two dimensions, especially with the effect of a symmetry of the magnetic field on the spectrum. More precisely, we are interested in this effect in the semiclassical limit, which is equivalent to the large magnetic field limit. In such a regime, much has been done in the last two decades to describe the bound states of the magnetic Schrödinger operators, see the books [9] and [21] (for earlier references, see the classical texts [2, 7] and references therein). In [21], it is underlined that a motivation is to understand tunneling effects induced by magnetic fields and to extend the Helffer–Sjöstrand theory to magnetic fields. This theory was originally developed in the context of purely electric Schrödinger operators, see [24] and [17, 18]. Helffer and Sjöstrand also added a (rather small) magnetic field in [19], and were able to establish a tunneling formula. This problem was reconsidered recently in [8] (see also [23], that served as a motivation for [8]), where the authors were able to go beyond the smallness assumptions on the magnetic field, but without establishing an accurate expansion of the spectral gap. It is notable that [8] succeeds in proving a lower bound on the gap – thereby establishing that it does not vanish. This motivated another work [14], where the authors obtained a first order asymptotics on the spectral gap. Finally, the full tunneling formula was proved in [20]. In fact, even though the context of the present article is different, its analysis follows similar lines as [20], and provide the reader with a very concise strategy close to the original spirit developed in [17, 18] by establishing a tractable expression of the interaction term.

It turns out that the adaptation of the Helffer–Sjöstrand theory to the *purely* magnetic situation was more complex than expected. Even establishing the semiclassical expansions of the eigenvalues in cases without symmetries was already a challenge (see, for instance, [22] where Birkhoff normal forms are used). A key point in the study of the tunneling effect is to have an accurate description of the eigenfunctions. Such a description was given rather recently in [3] (in a multiscale context) and in [6, 11] in the case of a magnetic well¹ in two dimensions. To prove that these WKB constructions are good approximations in general is still a widely open problem (see [11], which gives a first result in this direction). Concerning the tunneling estimates in the case of pure magnetic fields in two dimensions, we have to mention [5], where the Neumann Laplacian with constant magnetic field is considered in a symmetric domain and the spectral gap accurately described by means of a microlocal dimensional reduction to the boundary. This work inspired similar results such as [1] in the case of magnetic fields vanishing along curves, and [10] in the case of discontinuous magnetic fields. The case of multiple wells in the context of [8, 14] and [5, 10] is considered in [15].

The present article tackles a situation with symmetric magnetic wells. In this case, we know that the eigenfunctions are localized near the bottoms of the wells (see [12]), and not along curves as in [5] (what invites a priori to use another strategy). Until now, except for the WKB constructions of [6, 11], nothing was known in the direction of tunneling results for double-well magnetic fields.

1.2. The purely magnetic tunneling result

Let us now describe the framework of this article and our main result. Choose lengths $a, L > 0$, with $a < L/2$, and magnetic field strengths $0 < b_0 < b_1$. Define the two points (with distance L)

$$x_\ell = \left(-\frac{L}{2}, 0\right) \quad \text{and} \quad x_r = \left(\frac{L}{2}, 0\right).$$

Let the left magnetic field $B_\ell \in C^\infty(\mathbb{R}^2, \mathbb{R})$, have minimal value $b_0 > 0$ attained uniquely and non-degenerately at x_ℓ and be constant equal to $b_1 > b_0$ on $\{|x - x_\ell| > a\}$. Furthermore, we assume that $B_\ell(x) \in [b_0, b_1]$ for all x . We will define the right magnetic field B_r by symmetry, and assume the wells to be radial, symmetric and disjoint. More precisely,

$$B_\ell \text{ is a smooth function of } |x - x_\ell|^2, \text{ and } B_r(x_1, x_2) = B_\ell(-x_1, x_2).$$

See Assumption 1 on B_ℓ below. We consider a double well magnetic field $B \in C^\infty(\mathbb{R}^2, \mathbb{R})$, with minimal value $b_0 > 0$ attained (non-degenerately) at exactly the two points x_ℓ and x_r . The magnetic field will be constant outside the disks of radius a , through the definition (recall that $a < L/2$)

$$B(x) = \begin{cases} B_\ell(x), & \text{if } |x - x_\ell| \leq a, \\ B_r(x), & \text{if } |x - x_r| \leq a, \\ b_1, & \text{otherwise.} \end{cases}$$

¹By this, we mean that the magnetic field has a unique minimum, which is non-degenerate.

Two important quantities of the problem are the flux difference

$$(1.1) \quad M := \frac{1}{4\pi} \int_{\mathbb{R}^2} (B(x) - b_1) dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} (B_\ell(x) - b_1) dx \leq 0,$$

and the relative flux (through the disc of radius $L/2$)

$$(1.2) \quad 0 < N := \frac{8|M|}{b_1 L^2} < \frac{a^2}{(L/2)^2} < 1.$$

We consider a smooth vector potential $\mathbf{A} = (0, A_2)$ associated with B , and the magnetic Laplacian $\mathcal{L}_h = (-ih\nabla - \mathbf{A})^2$. We denote by $\lambda_1(h) \leq \lambda_2(h)$ the two smallest eigenvalues of $\mathcal{L}_h$. In this article, we prove the following estimate on the spectral gap.

Theorem 1.1. *Assume that $L > (2 + \sqrt{6})a$. Then, there exists a positive constant c such that the following holds. We have*

$$\lambda_2(h) - \lambda_1(h) \underset{h \rightarrow 0}{\sim} c e^{-S/h} h^{1/2+b_0/(2b_1)},$$

where

$$S = \int_0^{L/2} \frac{1}{\pi r^2} \left(\int_{D(x_\ell, r)} B(y) dy \right) r dr + I,$$

and

$$I = \frac{b_1 L^2}{4} \left(\frac{N-1}{2} + \sqrt{1-N} - N \ln(1 + \sqrt{1-N}) \right) > 0.$$

As far as the authors know, Theorem 1.1 is the only known result of its kind in the case of pure magnetic wells. Note that:

- (1) The first integral term in S is an Agmon distance between the two wells, see Proposition 2.2 below. Contrary to the case of electric wells, we have here an additional positive term I inside the exponential. This term is a purely magnetic effect, and vanishes when $N = 1$.
- (2) The variations of the field also appears in the power b_0/b_1 of h in the prefactor.
- (3) The constant c can be computed explicitly in terms of B , however it has no simple interpretation, see (5.1).
- (4) It is crucial for our approach that the wells are disjoint. However, our result has the stronger assumption that $L > (2 + \sqrt{6})a$. This can be slightly lightened, see inequality (5.15), but the assumption is here to ensure that the remainder term in Proposition 4.1 is small enough.
- (5) An alternative way of writing S , which could also be of interest, is the following formula:

$$S - I = \frac{-1}{\pi} \int_{\mathbb{R}^2} B(x) \mathbb{1}_{\{|x-x_\ell| \leq L/2\}} \ln \left(\frac{2|x-x_\ell|}{L} \right) dx.$$

1.3. Organization and strategy

The paper is organized as follows. In Section 2, we consider the one-well operator (on the left) called $\mathcal{L}_{h,\ell}$, and we recall the asymptotic expansions of its eigenvalues $(\mu_n(h))_{n \geq 1}$

obtained by Helffer and Kordyukov in [16] (or in [22] with a more geometric point of view), see Proposition 2.1. Then, we describe the groundstate eigenfunction ϕ_ℓ by recalling the results of [11]. This eigenfunction is radial and can be approximated by a WKB Ansatz, see Proposition 2.2. This accurate description of the groundstate of the one-well operator is known to be a corner stone in the analysis of the tunneling effect.

In Section 3, we recall a standard (and rough) localization result: the eigenfunctions of the double-well operator are exponentially localized near x_ℓ and x_r . This allows to decouple the wells and we get the first naive estimate $\lambda_2(h) - \lambda_1(h) = \mathcal{O}(h^\infty)$. Then, we use ϕ_ℓ and ϕ_r (up to a change of gauge) as quasimodes for the double-well operator $\mathcal{L}_h$. This gives an upper bound, see Proposition 3.4.

Section 4 is devoted to the interaction term w_h governing the behavior of the spectral gap $\lambda_2(h) - \lambda_1(h)$, see Propositions 4.1 and 4.2. The integral formula (4.1) for w_h is similar to that obtained in the electric case in [17]. This formula was used in [20] to simplify the analysis in the context of [8, 14]. Note that, contrary to the situation in [8, 14, 20], the interaction phase $\theta(0, x_2)$ is not linear, and the non-linear term is directly related to the fact that the magnetic field is not constant. In Section 5, we establish the one-term asymptotics of w_h , see Proposition 5.1. To estimate the integral giving (4.1), we use the following observation (done in [8] with a constant magnetic field): the eigenfunction ϕ_ℓ has an explicit expression in the interaction zone (this comes from the fact that the magnetic field is radial and constant between the wells), see Lemma 5.2. Combining this with the WKB approximation of Proposition 2.2, we get an explicit integral to estimate, see (5.6) and Lemma 5.3. In this integral (with three variables s_1, s_2 and y), the phase is complex. That is why we look for the complex critical points in the variable y (the vertical variable), see Sections 5.2.2 and 5.2.3. Then, we can use a complex translation of the integral in y and apply the Laplace method, see Section 5.2.5. This reduces the analysis to an integral in two variables (5.12), which can be estimated by the usual Laplace method, see the description of the real phase $F(s)$ in Section 5.2.4.

2. The one-well operator and its groundstate

For the one-well problem, we change gauge to profit from the radially. Notice that since a gauge change is just a multiplication by a complex phase, estimates on decay of eigenfunctions trivially translate from one gauge to another. However, our main result on the tunneling is very strongly dependent on the complex phases appearing in integrals, so when combining the two one-well problems it is crucial to keep track of the involved gauge changes.

We consider the vector potential

$$(2.1) \quad \mathbf{A}_\ell(x) = \int_0^1 B_\ell(x_\ell + t(x - x_\ell)) t(x - x_\ell)^\perp dt,$$

with $\text{curl } \mathbf{A}_\ell = B_\ell$, and we consider the corresponding magnetic Laplacian

$$\mathcal{L}_{h,\ell} = (-ih\nabla - \mathbf{A}_\ell)^2.$$

We denote by $(\mu_n(h))_{n \geq 1}$ the non-decreasing sequence of its eigenvalues. Note that the assumptions on B_ℓ can be rewritten as follows.

Assumption 1. *The magnetic field B_ℓ has the form*

$$B_\ell(x) = \beta\left(\frac{|x - x_\ell|^2}{2}\right),$$

where $\beta: [0, \infty) \rightarrow [b_0, b_1]$ is a smooth function such that

$$\beta(u) > \beta(0) = b_0 \quad \text{for } u > 0, \quad \text{and} \quad \beta'(0) > 0.$$

Moreover, $\beta(r^2/2) = b_1$ for $r > a$.

Under this assumption, the following proposition is well known (see Theorem 1.2 in [16], Corollary 1.7 in [22], and also Section 2.2 of [11], where the radial case is specifically considered).

Proposition 2.1. *Let $n \geq 1$. For h small enough, $\mu_n(h)$ belongs to the discrete spectrum of $\mathcal{L}_{h,\ell}$ and*

$$\mu_n(h) = b_0 h + ((2n d_0 + d_1) h^2 + o(h^2)),$$

where

$$d_0 = \frac{\sqrt{\det H}}{b_0}, \quad d_1 = \frac{(\text{Tr } H^{1/2})^2}{2b_0} \quad \text{and} \quad H = \frac{1}{2} \text{Hess } B_\ell(x_\ell).$$

In the following, we will denote $\mu_h = \mu_1(h)$. Let us now describe the groundstate ϕ_ℓ of $\mathcal{L}_{h,\ell}$ and its WKB approximation. It is already known that ϕ_ℓ is radial in the variable $r = |x - x_\ell|$ and does not vanish if h is small enough, see Proposition 4.7 in [11]. In particular, it solves the ODE (see Section 4 of [11])

$$(2.2) \quad \left(-h^2 r^{-1} \partial_r r \partial_r + \left(\frac{\alpha(r)}{r} \right)^2 \right) \phi_\ell = \mu_h \phi_\ell, \quad \text{with} \quad \alpha(r) = \int_0^{r^2/2} \beta(u) \, du.$$

Equation (2.2) can be interpreted as the eigenvalue equation of a radial *electric* Schrödinger operator. This observation, combined with the WKB construction in Theorem 2.5 of [11], implies that ϕ_ℓ is approximated by a WKB Ansatz (see for instance [13, 17, 18], or the pedagogical paper in one dimension [4], Lemma 2.1 and Proposition 2.7). The decay of this Ansatz is given by the function

$$\Phi_\ell(x) = \frac{1}{2} \int_0^{|x-x_\ell|^2/2} \frac{1}{v} \left(\int_0^v \beta(u) \, du \right) dv,$$

which is solution to the eikonal equation

$$|\nabla \Phi_\ell|^2 = \left(\frac{\alpha(r)}{r} \right)^2, \quad \text{with } r = |x - x_\ell|.$$

Proposition 2.2. *Under Assumption 1, there exists a sequence of smooth radial (with center at x_ℓ) functions $(a_j)_{j \geq 0}$ on $\mathbb{R}^2$, with $a_0 > 0$, such that the following holds. For any given $p \in \mathbb{N}$, letting $a_{h,\ell}(x) = \sum_{j=0}^p h^j a_j(x)$, and considering*

$$\phi_\ell^{\text{WKB}}(x) = h^{-1/2} a_{h,\ell}(x) e^{-\Phi_\ell(x)/h},$$

we have

$$e^{\Phi_\ell(x)/h} (\mathcal{L}_{h,\ell} - \mu_h) \phi_\ell^{\text{WKB}} = \mathcal{O}(h^{p+1}).$$

Moreover, for all $\varepsilon \in (0, 1)$, there exist $C, h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$(2.3) \quad \|(-ih\nabla - \mathbf{A}_\ell)(e^{(1-\varepsilon)\Phi_\ell} \phi_\ell/h)\| \leq Ch\|\phi_\ell\|.$$

We also have the approximation

$$e^{\Phi_\ell(x)/h} (\phi_\ell(x) - \phi_\ell^{\text{WKB}}(x)) = \mathcal{O}(h^{p+1}),$$

locally uniformly.

Remark 2.3. Let us make a couple of remarks about Φ_ℓ and a_0 .

(1) We can rewrite Φ_ℓ in terms of B_ℓ . In particular, we have

$$\Phi_\ell(0) = \frac{1}{2} \int_0^{L/2} \frac{1}{\pi r^2} \left(\int_{D(x_\ell, r)} B(y) dy \right) r dr,$$

which is the quantity appearing in the coefficient S from Theorem 1.1. It is the flux of an averaged version of B . Note also the approximation near x_ℓ :

$$\Phi_\ell(x) = \frac{b_0}{4h} |x - x_\ell|^2 + o(|x - x_\ell|^2).$$

(2) From Section 4.5.2 in [11], we have an explicit description of a_0 :

$$a_0(x) = A_0 \exp \left(\int_0^{|x-x_\ell|^2/2} \frac{\beta(0) - \beta(s)}{2\alpha(s)} ds \right),$$

where $A_0 > 0$ is the unique positive number such that, for all $R > 0$,

$$\lim_{h \rightarrow 0} \|h^{-1/2} a_0(x) e^{-\Phi_\ell(x)/h}\|_{L^2(D(x_\ell, R))}^2 = 1,$$

which follows from the Laplace method. Indeed, from the approximation of the phase Φ_ℓ and the L^2 -normalization, we get

$$1 = h^{-1} \int_{D(x_\ell, R)} |a_0(x)|^2 \exp \left(-\frac{b_0}{2h} |x|^2 \right) dx + o(1) = 2\pi |a_0(x_\ell)|^2 b_0^{-1} + o(1),$$

so that $A_0 = \sqrt{b_0/(2\pi)}$.

Remark 2.4. By symmetry, we have a similar WKB approximation in the right well. Considering the vector potential

$$\mathbf{A}_r(x) = \int_0^1 B_r(x_r + t(x - x_r)) t(x - x_r)^\perp dt,$$

we denote by $\mathcal{L}_{h,r}$ the corresponding magnetic Laplacian. Its positive normalized ground-state is denoted by ϕ_r .

3. First estimates on the tunneling effect

3.1. A naive localization estimate

The following propositions are already known, see, for instance, the discussion in Section 1.4 of [12].

Proposition 3.1. *Let $C_0 > 0$. There exist $\alpha, C, h_0 > 0$ such that for all $h \in (0, h_0)$ and all eigenpair (λ, ψ) with $\lambda \leq b_0 h + C_0 h^2$, we have*

$$\int_{\mathbb{R}^2} e^{2\alpha \min(|x-x_r|, |x-x_\ell|)/h^{1/4}} |\psi|^2 dx \leq C \|\psi\|^2.$$

This rather rough exponential decay of the eigenfunctions is enough to prove that the spectrum of the double-well operator is the superposition of the spectra of the one-well operators, modulo $\mathcal{O}(h^\infty)$.

Proposition 3.2. *There exist $c, h_0 > 0$ such that, for all $h \in (0, h_0)$, we have*

$$|\lambda_2(h) - \lambda_1(h)| = \mathcal{O}(h^\infty),$$

$$\text{and } \lambda_3(h) - \lambda_1(h) \geq ch^2.$$

3.2. A better upper bound of the splitting

We now use the WKB approximation, Proposition 2.2, to prove a better upper bound on the spectral gap. This bound is still not good enough to prove Theorem 1.1, but will be used later in the proof of Proposition 4.1. Let

$$(3.1) \quad S_0 = \Phi_\ell\left(\frac{L}{2} - a, 0\right) = \frac{1}{2} \int_0^{(L-a)^2/2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv,$$

which can be interpreted as an Agmon distance from the left well to the border of the right well.

Notation 3.3. In the following, we use the classical notation (from [17, 18]) that $f(h) = \tilde{\mathcal{O}}(e^{-S_0/h})$ when, for all $\varepsilon > 0$, we have $f(h) = \mathcal{O}(e^{-(S_0-\varepsilon)/h})$.

Proposition 3.4. *We have*

$$|\lambda_2(h) - \lambda_1(h)| = \tilde{\mathcal{O}}(e^{-S_0/h}) \quad \text{and} \quad \lambda_1(h) = \mu(h) + \tilde{\mathcal{O}}(e^{-S_0/h}).$$

Before giving a proof of Proposition 3.4, let us introduce a pair of quasimodes $\tilde{\phi}_\ell$ and $\tilde{\phi}_r$. Given any $\eta \in (0, L/2 - a)$, let χ_ℓ be a smooth cutoff function equal to 1 on $\{x_1 < L/2 - a - \eta\}$ and 0 on $\{x_1 > L/2 - a - \eta/2\}$. In the end, we will choose η as small as necessary. We let

$$(3.2) \quad \tilde{\phi}_\ell(x) = \chi_\ell(x) \hat{\phi}_\ell(x) \quad \text{and} \quad \hat{\phi}_\ell(x) = e^{i\sigma_\ell(x)/h} \phi_\ell(x),$$

where

$$\sigma_\ell(x) = \int_{[0,x]} (\mathbf{A} - \mathbf{A}_\ell) \cdot ds$$

is a gauge change from $\mathbf{A}_\ell$ to $\mathbf{A}$.

Proof of Proposition 3.4. Note that, on $\{x_1 < L/2 - a\}$,

$$\nabla \sigma_\ell = \mathbf{A} - \mathbf{A}_\ell.$$

Therefore we have, on $\{x_1 < L/2 - a - \eta\}$,

$$(\mathcal{L}_h - \mu_h)\tilde{\phi}_\ell = 0.$$

More precisely, we have

$$(\mathcal{L}_h - \mu_h)\tilde{\phi}_\ell = e^{-i\sigma_\ell/h}(\mathcal{L}_{h,\ell} - \mu_h)(\chi_\ell \phi_\ell) = e^{-i\sigma_\ell/h}[\mathcal{L}_{h,\ell}, \chi_\ell]\phi_\ell.$$

Thus, thanks to (2.3),

$$\|(\mathcal{L}_h - \mu_h)\tilde{\phi}_\ell\| = \tilde{\mathcal{O}}(e^{-S_\eta/h}),$$

where

$$(3.3) \quad S_\eta = \Phi_\ell\left(\frac{L}{2} - a - \eta, 0\right) = \Phi_r\left(-\frac{L}{2} + a + \eta, 0\right).$$

In the same way, and using a similar notation for the right well, we have

$$\|(\mathcal{L}_h - \mu_h)\tilde{\phi}_r\| = \tilde{\mathcal{O}}(e^{-S_\eta/h}).$$

Since $\tilde{\phi}_\ell$ and $\tilde{\phi}_r$ are linearly independent (for h small enough), the spectral theorem implies that there are at least two eigenvalues of $\mathcal{L}_h$ close to μ_h modulo $\tilde{\mathcal{O}}(e^{-S_\eta/h})$. These are necessarily the first two eigenvalues since μ_h is the groundstate energy of the one-well operator. We conclude by noting that $S_\eta \rightarrow S_0$ as $\eta \rightarrow 0$. ■

4. The interaction term

The aim of this section is to prove the following propositions, which follow from the celebrated strategy developed by Helffer and Sjöstrand in [17, 18] (see also [13]).

Proposition 4.1. *Assume $\eta > 0$ is small enough. Then we have*

$$\lambda_2(h) - \lambda_1(h) = 2|w_h| + \tilde{\mathcal{O}}(e^{-2S_\eta/h}) + \mathcal{O}(|t_h|^2),$$

where $w_h = \langle (\mathcal{L}_h - \mu)\tilde{\phi}_\ell, \tilde{\phi}_r \rangle$, $t_h = \langle \tilde{\phi}_\ell, \tilde{\phi}_r \rangle$, S_η is given in (3.3), and $\tilde{\phi}_\ell, \tilde{\phi}_r$ in (3.2).

Hence, the splitting is given by the coefficient w_h . Of course, we have to prove that the remainders are smaller than w_h , i.e., $2S_0 > S$, which is done in Section 5, and requires $L > (2 + \sqrt{6})a$. Similarly to the standard strategy, we can rewrite the coefficient w_h as an integral on the line $x_1 = 0$ separating the two wells. The novelty lies in the emergence of complex coefficients and phases.

Proposition 4.2. *Letting*

$$\sigma_\star(x) = \int_{[0,x]} (\mathbf{A} - \mathbf{A}_\star) \cdot ds \quad \text{and} \quad \theta(x) = \sigma_r(x) - \sigma_\ell(x) = \int_{[0,x]} (\mathbf{A}_\ell - \mathbf{A}_r) \cdot ds,$$

we have

$$(4.1) \quad w_h = h^2 \int_{\mathbb{R}} \left(2\partial_1 \phi_\ell(0, x_2) \phi_\ell(0, x_2) + i \frac{k(0, x_2)}{h} \phi_\ell(0, x_2)^2 \right) e^{-i\theta(0, x_2)/h} dx_2,$$

with $k(x) = \partial_1 \sigma_\ell(x) + \partial_1 \sigma_r(x)$. Moreover, we have the explicit formulas

$$\theta(0, x_2) = \frac{b_1 L x_2}{2} + 2M \arctan(2x_2/L)$$

and

$$k(0, x_2) = b_1 x_2 + \frac{2M x_2}{L^2/4 + x_2^2},$$

where M is given in (1.1).

Note that the quantity M can be rewritten in terms of β :

$$(4.2) \quad M = \int_0^\infty (\beta(u) - b_1) du.$$

For the convenience of the reader, we will sum up the strategy in the following sections and try to make it short and transparent.

4.1. Proof of Proposition 4.1

We let

$$F = \ker(\mathcal{L}_h - \lambda_1(h)) \oplus \ker(\mathcal{L}_h - \lambda_2(h))$$

and let Π_F be the associated orthogonal projection. Let us consider the projections of our one-well quasimodes $g_\star = \Pi_F \tilde{\phi}_\star$. We let

$$G = \begin{pmatrix} \langle g_\ell, g_\ell \rangle & \langle g_\ell, g_r \rangle \\ \langle g_r, g_\ell \rangle & \langle g_r, g_r \rangle \end{pmatrix} = \begin{pmatrix} g_\ell \\ g_r \end{pmatrix} \cdot \begin{pmatrix} g_\ell & g_r \end{pmatrix} \geq 0.$$

We consider the matrix of the quadratic form associated with $\mathcal{L}_h$ in the basis (g_ℓ, g_r) ,

$$L = \begin{pmatrix} \langle \mathcal{L}_h g_\ell, g_\ell \rangle & \langle \mathcal{L}_h g_\ell, g_r \rangle \\ \langle \mathcal{L}_h g_r, g_\ell \rangle & \langle \mathcal{L}_h g_r, g_r \rangle \end{pmatrix}.$$

The following lemma describes the asymptotic behavior of the matrix L , which essentially follows from the classical estimates in [17] (see also [4]).

Lemma 4.3. *We have*

$$(4.3) \quad \|g_\star - \tilde{\phi}_\star\| = \tilde{O}(e^{-S_\eta/h}) \quad \text{and} \quad Q_h(g_\star - \tilde{\phi}_\star) = \tilde{O}(e^{-2S_\eta/h}),$$

where Q_h denotes the quadratic form of $\mathcal{L}_h$. Moreover, the family (g_r, g_ℓ) is asymptotically an orthonormal basis of F . More precisely, we have

$$(4.4) \quad G = \text{Id} + T + \tilde{O}(e^{-2S_\eta/h}), \quad \text{where} \quad T = \begin{pmatrix} 0 & \langle \tilde{\phi}_\ell, \tilde{\phi}_r \rangle \\ \langle \tilde{\phi}_r, \tilde{\phi}_\ell \rangle & 0 \end{pmatrix}.$$

In addition, we have

$$(4.5) \quad L = \begin{pmatrix} \mu_h & w_h \\ \bar{w}_h & \mu_h \end{pmatrix} + \mu_h T + \tilde{O}(e^{-2S_\eta/h}), \quad \text{where} \quad w_h = \langle (\mathcal{L}_h - \mu) \tilde{\phi}_\ell, \tilde{\phi}_r \rangle.$$

Proof. We recall that

$$(\mathcal{L}_h - \mu_h)\tilde{\phi}_\star = \tilde{\mathcal{O}}(e^{-S_\eta/h}),$$

and notice that, by definition of F and Proposition 3.4,

$$(\mathcal{L}_h - \mu_h)g_\star = \tilde{\mathcal{O}}(e^{-S_0/h}).$$

We infer that

$$(4.6) \quad (\mathcal{L}_h - \mu_h)(g_\star - \tilde{\phi}_\star) = \tilde{\mathcal{O}}(e^{-S_\eta/h}).$$

The first L^2 -estimate in (4.3) follows by using the spectral theorem and the spectral gap given in Proposition 3.2. Then, we get the second estimate by using (4.6) and taking the scalar product with $g_\star - \tilde{\phi}_\star$.

Let us now discuss the asymptotic behavior of G . By the Pythagorean theorem, we have

$$\langle \tilde{\phi}_\star, \tilde{\phi}_\star \rangle = \langle g_\star, g_\star \rangle + \langle \tilde{\phi}_\star - g_\star, \tilde{\phi}_\star - g_\star \rangle.$$

By using the support of $\chi_\star$, the definition of $\phi_\star$ and its exponential decay, we have

$$(4.7) \quad \langle \tilde{\phi}_\star, \tilde{\phi}_\star \rangle = 1 + \tilde{\mathcal{O}}(e^{-2S_\eta/h}).$$

With (4.3), we deduce that

$$\langle g_\star, g_\star \rangle = 1 + \tilde{\mathcal{O}}(e^{-2S_\eta/h}).$$

In the same way, we write

$$\langle \tilde{\phi}_\ell, \tilde{\phi}_r \rangle = \langle g_\ell, g_r \rangle + \langle \tilde{\phi}_\ell - g_\ell, \tilde{\phi}_r - g_r \rangle.$$

We deduce (4.4).

Let us finally deal with L . The analysis is similar. First, we write

$$\langle \mathcal{L}_h \tilde{\phi}_\star, \tilde{\phi}_\star \rangle = \langle \mathcal{L}_h g_\star, g_\star \rangle + \langle \mathcal{L}_h(\tilde{\phi}_\star - g_\star), \tilde{\phi}_\star - g_\star \rangle.$$

With the localization formula (sometimes called the “IMS formula”, see Proposition 4.2 in [21]) and using again (4.7), we get

$$\langle \mathcal{L}_h \tilde{\phi}_\star, \tilde{\phi}_\star \rangle = \mu_h + \tilde{\mathcal{O}}(e^{-2S_\eta/h}),$$

so that, with (4.3),

$$\langle \mathcal{L}_h g_\star, g_\star \rangle = \mu_h + \tilde{\mathcal{O}}(e^{-2S_\eta/h}).$$

Then, we write

$$\langle \mathcal{L}_h \tilde{\phi}_\ell, \tilde{\phi}_r \rangle = \langle \mathcal{L}_h g_\ell, g_r \rangle + \langle \mathcal{L}_h(\tilde{\phi}_\ell - g_\ell), \tilde{\phi}_r - g_r \rangle,$$

and we get

$$\langle \mathcal{L}_h g_\ell, g_r \rangle = \langle \mathcal{L}_h \tilde{\phi}_\ell, \tilde{\phi}_r \rangle + \tilde{\mathcal{O}}(e^{-2S_\eta/h}),$$

where we used (4.3) and the Cauchy–Schwarz inequality to control the remainder. We deduce that

$$\langle \mathcal{L}_h g_\ell, g_r \rangle = \mu_h \langle \tilde{\phi}_\ell, \tilde{\phi}_r \rangle + \langle (\mathcal{L}_h - \mu_h) \tilde{\phi}_\ell, \tilde{\phi}_r \rangle + \tilde{\mathcal{O}}(e^{-2S_\eta/h}),$$

and (4.5) follows. ■

We have all the elements at hand to prove Proposition 4.1. The matrix L is the matrix of $\mathcal{L}_h$ in a non-orthonormal basis of F . That is why we consider the new family

$$\begin{pmatrix} \mathfrak{g}_\ell \\ \mathfrak{g}_r \end{pmatrix} = G^{-1/2} \begin{pmatrix} g_\ell \\ g_r \end{pmatrix},$$

which is orthonormal since

$$\begin{pmatrix} \mathfrak{g}_\ell \\ \mathfrak{g}_r \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{g}_\ell & \mathfrak{g}_r \end{pmatrix} = G^{-1/2} \begin{pmatrix} g_\ell \\ g_r \end{pmatrix} \cdot \begin{pmatrix} g_\ell & g_r \end{pmatrix} G^{-1/2} = \text{Id}.$$

Now, the matrix of $\mathcal{L}_h$ in the basis $(\mathfrak{g}_\ell, \mathfrak{g}_r)$ is $G^{-1/2} L G^{-1/2}$, and we have

$$\begin{aligned} G^{-1/2} L G^{-1/2} &= \left(1 - \frac{T}{2}\right) \left(\begin{pmatrix} \mu_h & w_h \\ \bar{w}_h & \mu_h \end{pmatrix} + \mu_h T \right) \left(1 - \frac{T}{2}\right) + \tilde{\mathcal{O}}(e^{-2S_\eta/h}) \\ &= \begin{pmatrix} \mu_h & w_h \\ \bar{w}_h & \mu_h \end{pmatrix} + \tilde{\mathcal{O}}(e^{-2S_\eta/h}) + \mathcal{O}(|t_h|^2). \end{aligned}$$

The proposition follows, since the eigenvalues of this matrix are $\mu_h \pm |w_h|$.

4.2. Proof of Proposition 4.2

Let us first establish the integral formula (4.1) for the interaction term w_h . Recalling the definition (3.2) of $\tilde{\phi}_\ell$ and $\tilde{\phi}_r$, we have

$$w_h = \langle (\mathcal{L}_h - \mu_h) \chi_\ell \hat{\phi}_\ell, \chi_r \hat{\phi}_r \rangle = \langle [\mathcal{L}_h, \chi_\ell] \hat{\phi}_\ell, \chi_r \hat{\phi}_r \rangle.$$

Thus, by letting $P = -i\hbar \nabla - \mathbf{A}$, we get

$$w_h = \langle (P \cdot [P, \chi_\ell] + [P, \chi_\ell] P) \hat{\phi}_\ell, \chi_r \hat{\phi}_r \rangle = \langle (P \cdot [P, \chi_\ell] + [P, \chi_\ell] P) \hat{\phi}_\ell, \hat{\phi}_r \rangle_{L^2(\mathbb{R}_+^2)}.$$

Integrating by parts, we deduce

$$\begin{aligned} w_h &= \langle [P, \chi_\ell] \hat{\phi}_\ell, P \hat{\phi}_r \rangle_{L^2(\mathbb{R}_+^2)} + \langle [P, \chi_\ell] P \hat{\phi}_\ell, \hat{\phi}_r \rangle_{L^2(\mathbb{R}_+^2)} \\ &= -i\hbar \int_{x_1 > 0} \chi'_\ell(x_1) (\hat{\phi}_\ell \overline{(-i\hbar \partial_1 - A_1) \hat{\phi}_r} + (-i\hbar \partial_1 - A_1) \hat{\phi}_\ell \overline{\hat{\phi}_r}) dx. \end{aligned}$$

Our choice of gauge is such that $A_1 = 0$, and therefore

$$w_h = \hbar^2 \int_{x_1 > 0} (\partial_1 \chi_\ell) (\hat{\phi}_\ell \overline{\partial_1 \hat{\phi}_r} - \partial_1 \hat{\phi}_\ell \overline{\hat{\phi}_r}) dx.$$

Then, integrating by part with respect to x_1 ,

$$\begin{aligned} (4.8) \quad w_h &= \hbar^2 \int_{\mathbb{R}} (\partial_1 \hat{\phi}_\ell \overline{\hat{\phi}_r} - \hat{\phi}_\ell \overline{\partial_1 \hat{\phi}_r})(0, x_2) dx_2 \\ &\quad + \hbar^2 \int_{x_1 > 0} \chi_\ell (\partial_1 (\partial_1 \hat{\phi}_\ell \overline{\hat{\phi}_r}) - \partial_1 (\hat{\phi}_\ell \overline{\partial_1 \hat{\phi}_r})) dx. \end{aligned}$$

We notice that

$$\begin{aligned}
& h^2 \int_{x_1 > 0} \chi_\ell (\partial_1 (\partial_1 \hat{\phi}_\ell \overline{\hat{\phi}_r}) - \partial_1 (\hat{\phi}_\ell \partial_1 \overline{\hat{\phi}_r})) \, dx \\
&= h^2 \int_{x_1 > 0} \chi_\ell (\partial_1^2 \hat{\phi}_\ell \overline{\hat{\phi}_r} - \hat{\phi}_\ell \overline{\partial_1^2 \hat{\phi}_r}) \, dx, \\
&= - \int_{x_1 > 0} \chi_\ell (\mathcal{L}_h \hat{\phi}_\ell \overline{\hat{\phi}_r} - (-ih\partial_2 - A_2)^2 \hat{\phi}_\ell \overline{\hat{\phi}_r} \\
&\quad - \hat{\phi}_\ell \overline{\mathcal{L}_h \hat{\phi}_r} + \hat{\phi}_\ell \overline{(-ih\partial_2 - A_2)^2 \hat{\phi}_r}) \, dx, \\
&= \int_{x_1 > 0} \chi_\ell ((-ih\partial_2 - A_2)^2 \hat{\phi}_\ell \overline{\hat{\phi}_r} - \hat{\phi}_\ell \overline{(-ih\partial_2 - A_2)^2 \hat{\phi}_r}) \, dx = 0,
\end{aligned}$$

where we used that, on the support of χ_ℓ , $\mathcal{L}_h \hat{\phi}_r = \mu_h \hat{\phi}_r$ and $\mathcal{L}_h \hat{\phi}_\ell = \mu_h \hat{\phi}_\ell$; and in the last equality we integrated by part with respect to x_2 . Thus, equation (4.8) reads

$$w_h = h^2 \int_{\mathbb{R}} (\partial_1 \hat{\phi}_\ell \overline{\hat{\phi}_r} - \hat{\phi}_\ell \overline{\partial_1 \hat{\phi}_r})(0, x_2) \, dx_2.$$

There remains to notice that $\hat{\phi}_\star = e^{i\sigma_\star/h} \phi_\star$, and therefore we get (4.1) due to the symmetry $\phi_\ell(x_1, x_2) = \phi_r(-x_1, x_2)$ around $x_1 = 0$.

We end this section by giving explicit formulas for θ and k , which concludes the proof of Proposition 4.2.

Lemma 4.4. *We have*

$$(4.9) \quad \theta(x_1, x_2) = \frac{b_1 L x_2}{2} + M \left(\arctan \left(\frac{x_2}{x_1 + L/2} \right) + \arctan \left(\frac{x_2}{-x_1 + L/2} \right) \right),$$

and

$$(4.10) \quad k(0, x_2) = b_1 x_2 + \frac{2M x_2}{L^2/4 + x_2^2}.$$

Proof. For all x such that $x_1 \in (-L/2 + a, L/2 - a)$, we have $\nabla \times (\mathbf{A} - \mathbf{A}_\star) = 0$, and thus

$$\begin{aligned}
\sigma_\star(x) &= \int_{[0, x]} (\mathbf{A} - \mathbf{A}_\star) \cdot ds \\
&= \int_0^{x_1} (\mathbf{A} - \mathbf{A}_\star)_1(u, 0) \, du + \int_0^{x_2} (\mathbf{A} - \mathbf{A}_\star)_2(x_1, v) \, dv = \int_0^{x_2} (\mathbf{A} - \mathbf{A}_\star)_2(x_1, v) \, dv,
\end{aligned}$$

where we used the explicit expression (2.1) of $\mathbf{A}_\star$ and the fact that $A_1 = 0$. We have

$$\begin{aligned}
A_\ell(x_1, x_2) &= \int_0^1 \beta \left(\frac{t^2 |x - x_\ell|^2}{2} \right) t \, dt \begin{pmatrix} -x_2 \\ x_1 + L/2 \end{pmatrix} \\
&= \frac{1}{|x - x_\ell|^2} \int_0^{|x - x_\ell|^2/2} \beta(u) \, du \begin{pmatrix} -x_2 \\ x_1 + L/2 \end{pmatrix}
\end{aligned}$$

and, similarly,

$$A_r(x_1, x_2) = \frac{1}{|x - x_r|^2} \int_0^{|x-x_r|^2/2} \beta(u) du \begin{pmatrix} -x_2 \\ x_1 - L/2 \end{pmatrix}.$$

This shows that

$$A_\ell(x) = \left(\frac{M}{|x - x_\ell|^2} + \frac{b_1}{2} \right) \begin{pmatrix} -x_2 \\ x_1 + L/2 \end{pmatrix}, \quad A_r(x) = \left(\frac{M}{|x - x_r|^2} + \frac{b_1}{2} \right) \begin{pmatrix} -x_2 \\ x_1 - L/2 \end{pmatrix}.$$

Then, θ is given by

$$\theta(x_1, x_2) = (\sigma_r - \sigma_\ell)(x_1, x_2) = \int_0^{x_2} (A_{\ell,2}(x_1, v) - A_{r,2}(x_1, v)) dv,$$

and we find (4.9). For $k(0, x_2)$, we use

$$\nabla(\sigma_\ell + \sigma_r) = 2\mathbf{A} - (\mathbf{A}_\ell + \mathbf{A}_r),$$

to get

$$k(0, x_2) = \partial_1(\sigma_\ell + \sigma_r)(0, x_2) = -(\mathbf{A}_\ell + \mathbf{A}_r)_1(0, x_2) = -2A_{\ell,1}(0, x_2),$$

and (4.10) follows. ■

Remark 4.5. Note that the term involving M is here due to the variations of the magnetic field. This term is absent in [8, 14].

5. Estimate of the interaction integral

This section is mainly devoted to the proof of the following proposition, which implies Theorem 1.1 (by combining Proposition 4.1 and Lemmas 5.9 and 5.10), with the constant c given by

$$(5.1) \quad c = \frac{4\pi a_0^2(0)}{\Gamma(\delta_0)} \left(\frac{b_1 L^2}{8} \right)^{\delta_0} \left(\frac{1-N}{1+\sqrt{1-N}} \right)^{2\delta_0} (1-N)^{-1/4},$$

where a_0 is the function given in Remark 2.3.

Proposition 5.1. *The following estimate holds:*

$$w_h \underset{h \rightarrow 0}{\sim} - \frac{2\pi a_0^2(0)}{\Gamma(\delta_0)} \left(\frac{b_1 L^2}{8} \right)^{\delta_0} \left(\frac{1-N}{1+\sqrt{1-N}} \right)^{2\delta_0} (1-N)^{-1/4} h^{1-\delta_0} e^{-S/h},$$

where

$$S = 2\Phi_\ell(0) + I = \int_0^{L^2/8} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv + I,$$

with

$$I = \frac{b_1 L^2}{4} \left(\frac{N-1}{2} + \sqrt{1-N} - N \ln(1 + \sqrt{1-N}) \right) > 0,$$

where M and N are given in (4.2) and (1.2) respectively, and

$$(5.2) \quad \delta_0 = \frac{b_1 - b_0}{2b_1}.$$

5.1. Description of ϕ_ℓ

The aim of this section is to give the explicit expression of ϕ_ℓ in the interaction zone. This can be done by integrating a classical differential equation known as the Kummer differential equation, which appeared in [8] and [14] but with different parameters. Here, μ_h is of order h , and the parameter M is not zero since the magnetic field is not constant.

Lemma 5.2. For $|x - x_\ell| \geq a$,

$$\phi_\ell(x) = C(h) |x - x_\ell|^\gamma \int_0^{+\infty} e^{-\frac{b_1}{4h}(1+2t)|x-x_\ell|^2} m_h(t) dt,$$

where

$$(5.3) \quad \gamma = \frac{|M|}{h}, \quad \delta = \frac{b_1 h - \mu_h}{2h b_1}, \quad m_h(t) = t^{\delta-1} (1+t)^{\gamma-\delta},$$

and

$$(5.4) \quad C(h) = \frac{\phi_\ell(0)}{|x_\ell|^\gamma} \left(\int_0^{+\infty} e^{-\frac{b_1}{4h}(1+2t)|x_\ell|^2} m_h(t) dt \right)^{-1}.$$

Note that $\delta \sim \delta_0$ (with δ_0 as defined in (5.2)) as $h \rightarrow 0$.

Proof. Since the function ϕ_ℓ is radial (with center at the point x_ℓ), we can write $\phi_\ell(x) = \phi(|x - x_\ell|)$. Then, ϕ solves the radial equation

$$-h^2 r^{-1} \partial_r r \partial_r \phi + \frac{\alpha(r)^2}{r^2} \phi = \mu_h \phi, \quad \text{with } \alpha(r) = \int_0^{r^2/2} \beta(u) du.$$

We are interested in the region $r \geq a$, where $\beta = b_1$. There, we have

$$\alpha(r) = M + \frac{b_1 r^2}{2},$$

where we recall that $M = \int_0^{a^2/2} (\beta - b_1) du$. The equation reads

$$-h^2 r^{-1} \partial_r r \partial_r \phi + \left(\frac{M}{r} + \frac{b_1 r}{2} \right)^2 \phi = \mu_h \phi,$$

so that

$$-h^2 r^{-1} \partial_r r \partial_r \phi + \left(\frac{M^2}{r^2} + \frac{b_1^2 r^2}{4} \right) \phi = (\mu_h - b_1 M) \phi.$$

Then, we let $\phi = r^\gamma \psi$ and we have

$$-h^2 r^{-1} (r^{-\gamma} \partial_r r^\gamma) r (r^{-\gamma} \partial_r r^\gamma) \psi + \left(\frac{M^2}{r^2} + \frac{b_1^2 r^2}{4} \right) \psi = (\mu_h - b_1 M) \psi,$$

so that

$$-h^2 r^{-1} (\partial_r + \gamma r^{-1}) r (\partial_r + \gamma r^{-1}) \psi + \left(\frac{M^2}{r^2} + \frac{b_1^2 r^2}{4} \right) \psi = (\mu_h - b_1 M) \psi.$$

We deduce that

$$-h^2 r^{-1} (\partial_r + \gamma r^{-1}) \psi - h^2 (\partial_r + \gamma r^{-1})^2 \psi + \left(\frac{M^2}{r^2} + \frac{b_1^2 r^2}{4} \right) \psi = (\mu_h - b_1 M) \psi.$$

Then,

$$-h^2 \partial_r^2 \psi - h^2 (1 + 2\gamma) r^{-1} \partial_r \psi + \left(\frac{M^2 - h^2 \gamma^2}{r^2} + \frac{b_1^2 r^2}{4} \right) \psi = (\mu_h - b_1 M) \psi.$$

We choose γ so that $\gamma^2 = M^2 h^{-2}$. Thus,

$$-h^2 \partial_r^2 \psi - h^2 (1 + 2\gamma) r^{-1} \partial_r \psi + \frac{b_1^2 r^2}{4} \psi = (\mu_h - b_1 M) \psi.$$

We let $\rho = cr^2/2$ and we have $c \partial_\rho = r^{-1} \partial_r$. Note that

$$\partial_r^2 = \partial_r r r^{-1} \partial_r = r^{-1} \partial_r + r^2 (r^{-1} \partial_r)^2 = c \partial_\rho + 2c\rho \partial_\rho^2.$$

We let $\psi(r) = \Psi(cr^2/2)$ and we find that

$$\rho \partial_\rho^2 \Psi + (1 + \gamma) \partial_\rho \Psi - \frac{b_1^2 \rho}{4h^2 c^2} \Psi = \frac{b_1 M - \mu_h}{2h^2 c} \Psi,$$

and then we let

$$\Psi(\rho) = F(\rho) e^{-\frac{b_1}{2hc} \rho}.$$

We get

$$\rho \partial_\rho^2 F + \left(1 + \gamma - \rho \frac{b_1}{hc} \right) \partial_\rho F = \left(\frac{b_1 M - \mu_h}{2h^2 c} + \frac{b_1 h (1 + \gamma)}{2h^2 c} \right) F.$$

This leads to choose $\gamma = |M|h^{-1}$ and $c = b_1 h^{-1}$ to get

$$\rho \partial_\rho^2 F + (1 + \gamma - \rho) \partial_\rho F = \frac{b_1 h - \mu_h}{2hb_1} F,$$

which is the well-known Kummer equation (and solvable by means of the Fourier or Laplace transform). Therefore, since we are looking for a decaying solution, we have, for some normalizing constant $C(h)$,

$$F(\rho) = C(h) \int_0^{+\infty} e^{-t\rho} t^{\delta-1} (1+t)^{\gamma-\delta} dt, \quad \text{with } \delta = \frac{b_1 h - \mu_h}{2hb_1}.$$

Therefore,

$$\begin{aligned} \phi(r) &= r^\gamma e^{-b_1 r^2/(4h)} F(b_1 r^2/(2h)) \\ &= C(h) r^\gamma e^{-b_1 r^2/(4h)} \int_0^{+\infty} e^{-b_1 r^2 t/(2h)} t^{\delta-1} (1+t)^{\gamma-\delta} dt. \end{aligned} \quad \blacksquare$$

It is possible to estimate the normalization constant $C(h)$ by means of the WKB-approximation of Proposition 2.2 and a direct estimate of the integral. This idea was used in Lemma 3.6 of [14]. However note that, contrary to [14], Lemma 5.3 is the only instance where we use WKB approximations.

Lemma 5.3. *The normalization constant satisfies*

$$C(h) = h^{-1/2} a_0(0) e^{-\Phi_\ell(0)/h} e^{b_1 L^2/16h} \frac{\Gamma(\delta_0)^{-1} h^{-\delta_0}}{|x_\ell|^{||M||/h}} \left(\frac{b_1 L^2}{8} - |M| \right)^{\delta_0} (1 + o(1)).$$

where a_0 and Φ_ℓ were defined in Proposition 2.2.

Proof. Letting $\varphi(t) = \frac{b_1 L^2}{8} t - |M| \ln(1+t)$, we have

$$\int_0^{+\infty} e^{-\frac{b_1}{4h}(1+2t)|x_\ell|^2} m_h(t) dt = e^{-\frac{b_1 L^2}{16h}} \int_0^{+\infty} e^{-\varphi(t)/h} t^{\delta-1} (1+t)^{-\delta} dt.$$

Since $b_1 L^2/8 - |M| > 0$ (by (1.2)), we have $\varphi' > 0$. Note also that φ is strictly convex. By means of the change of variable $u = \varphi(t)$, we get

$$\int_0^{+\infty} e^{-\varphi(t)/h} t^{\delta-1} (1+t)^{-\delta} dt = (1 + o(1)) \frac{\Gamma(\delta_0) h^{\delta_0}}{(b_1 L^2/8 - |M|)^{\delta_0}}.$$

By inserting in (5.4), we get

$$C(h) = (1 + o(1)) h^{-\delta_0} \Gamma(\delta_0)^{-1} e^{b_1 L^2/16h} \frac{\phi_\ell(0)}{|x_\ell|^{||M||/h}} \left(\frac{b_1 L^2}{8} - |M| \right)^{\delta_0}.$$

The result now follows by inserting the approximation $\phi_\ell(0) \approx \phi_\ell^{\text{WKB}}(0)$, i.e., the WKB-approximation from Proposition 2.2. $\blacksquare$

5.2. Consequence of the explicit expression

Given Lemma 5.2, the function w_h is an explicit integral. Let us analyze its asymptotic behavior and establish Proposition 5.1. First note, with Lemma 5.2, that

$$\phi_\ell(0, x_2) = C(h) \int_0^\infty \left(\frac{L^2}{4} + x_2^2 \right)^{\gamma/2} e^{-\frac{b_1}{4h}(1+2t)(L^2/4+x_2^2)} m_h(t) dt,$$

and

$$\begin{aligned} \partial_1 \phi_\ell(0, x_2) &= C(h) \int_0^\infty \left(\frac{\gamma L/2}{L^2/4 + x_2^2} - \frac{b_1 L}{4h} (1+2t) \right) \left(\frac{L^2}{4} + x_2^2 \right)^{\gamma/2} \\ &\quad \cdot e^{-\frac{b_1}{4h}(1+2t)(L^2/4+x_2^2)} m_h(t) dt. \end{aligned}$$

5.2.1. Rescaling of the interaction integral. In (4.1), we let $x_2 = Ly/2$ and we have

$$w_h = \frac{L}{2} h^2 \int_{\mathbb{R}} \left(2\partial_1 \phi_\ell\left(0, \frac{Ly}{2}\right) \phi_\ell\left(0, \frac{Ly}{2}\right) + i \frac{k(0, Ly/2)}{h} \phi_\ell\left(0, \frac{Ly}{2}\right)^2 \right) e^{-i\theta(0, \frac{Ly}{2})/h} dy.$$

Note that Lemma 4.4 gives

$$\theta\left(0, \frac{Ly}{2}\right) = \frac{b_1 L^2}{4} y + 2M \arctan y \quad \text{and} \quad k(0, Ly/2) = \frac{b_1 L}{2} \left(y - \frac{Ny}{1+y^2}\right),$$

with N from (1.2). Moreover, with $\langle y \rangle^2 = 1 + y^2$,

$$\phi_\ell(0, Ly/2) = C(h) \left(\frac{L}{2}\right)^\gamma \langle y \rangle^\gamma \int_0^{+\infty} e^{-\frac{b_1 L^2}{16h}(1+2t)\langle y \rangle^2} m_h(t) dt,$$

and

$$\partial_1 \phi_\ell\left(0, \frac{Ly}{2}\right) = C(h) \left(\frac{L}{2}\right)^\gamma \int_0^{+\infty} \left(\frac{2\gamma/L}{1+y^2} - \frac{b_1 L}{4h}(1+2t)\right) \langle y \rangle^\gamma e^{-\frac{b_1 L^2}{16h}(1+2t)\langle y \rangle^2} m_h(t) dt.$$

Therefore, recalling the definitions of M, N from (1.1) and (1.2), we can write

$$(5.5) \quad w_h = hC(h)^2 \left(\frac{L}{2}\right)^{2\gamma} \frac{b_1 L^2}{4} W_h,$$

where (with $s = (s_1, s_2) \in \mathbb{R}^2$)

$$(5.6) \quad W_h = \int_{\mathbb{R}} \int_{\mathbb{R}_+ \times \mathbb{R}_+} \omega(s, y) \langle y \rangle^{2\gamma} m_h(s_1) m_h(s_2) e^{-\frac{b_1 L^2}{8h}(1+s_1+s_2)\langle y \rangle^2 - \frac{i}{h} \theta(0, Ly/2)} ds dy$$

and

$$(5.7) \quad \omega(s, y) = \left(\frac{N}{1+y^2} - 1\right)(1 - iy) - 2s_1.$$

Now we need to estimate W_h , which is simpler than w_h since written in the rescaled unitless variable y .

5.2.2. The phase. In view of (5.6), it is natural to introduce the following complex phase:

$$\psi(s, y) = \frac{b_1 L^2}{8} (1 + s_1 + s_2) \langle y \rangle^2 + i \frac{b_1 L^2}{4} y - 2i|M| \arctan y - |M| \ln(1 + y^2),$$

where the logarithmic term is extracted from $\langle y \rangle^\gamma$. This may also be written as

$$(5.8) \quad \psi(s, y) = \frac{b_1 L^2}{8} c(s) \langle y \rangle^2 + i \frac{b_1 L^2}{4} y - 2|M| \ln(1 + iy),$$

with

$$c(s) := 1 + s_1 + s_2,$$

and where we use the standard complex logarithm, analytic on $\mathbb{C} \setminus \mathbb{R}_-$. Therefore, (5.6) becomes

$$(5.9) \quad W_h = \int_{\mathbb{R}} \int_{\mathbb{R}_+ \times \mathbb{R}_+} \omega(s, y) m_h(s_1) m_h(s_2) e^{-\psi(s, y)/h} ds dy,$$

where ω is given in (5.7) and m_h in (5.3). In the remainder of this section, we will calculate the leading term in W_h as $h \rightarrow 0$. To do so, we study the critical points of the phase in Lemmas 5.4, 5.5, and 5.6 (successively in the s and y variables). Once this is done, a complex deformation of the contour of integration of the y -integral (Lemma 5.7) allows to extract the desired result in Lemma 5.8.

5.2.3. Critical points in y . Let us consider the complex critical points of ψ and focus on the one with negative imaginary part.

Lemma 5.4. *Letting*

$$z_- = z_-(s) = \frac{c(s) - 1}{2c(s)} - \frac{1}{2} \sqrt{\left(\frac{c(s) - 1}{c(s)}\right)^2 + \frac{4}{c(s)}(1 - N)} < 0,$$

we have, for all $y \in \mathbb{R}$, and all $s \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\partial_y \psi(s, iz_-(s)) = 0, \quad \Re \partial_y^2 \psi(s, y + iz_-) \geq \frac{b_1 L^2}{4}(1 - N) \quad \text{and} \quad \Im \partial_y^2 \psi(s, iz_-) = 0.$$

In particular, $\Re \psi(s, \cdot + iz_-)$ is strongly convex uniformly in s . We also have $|z_-(s)| \leq 1/c(s) \leq 1$ and $z_-(0) = -\sqrt{1 - N}$.

Proof. The complex critical points of $y \mapsto \psi(s, y)$ solve the equation

$$c(s)y + i - \frac{iN}{1 + iy} = 0.$$

With $y = iz$, we get

$$z^2 - \frac{c - 1}{c} z + \frac{N - 1}{c} = 0.$$

We recall that $N \in (0, 1)$. The discriminant is equal to

$$\Delta = \left(\frac{c - 1}{c}\right)^2 + \frac{4}{c}(1 - N),$$

and it is positive. In this case, there are two real roots:

$$z_{\pm} = \frac{c - 1}{2c} \pm \frac{1}{2} \sqrt{\left(\frac{c - 1}{c}\right)^2 + \frac{4}{c}(1 - N)}.$$

We let $y_{\pm} = iz_{\pm}$. Since $N \in (0, 1)$, we have $z_- \in (-c^{-1}, 0)$. It is interesting to notice that

$$\partial_y^2 \psi(s, y) = \frac{b_1 L^2}{4} \left(c - \frac{N}{(1 + iy)^2} \right),$$

so that, for all $y \in \mathbb{R}$,

$$\partial_y^2 \psi(s, y + iz_-) = \frac{b_1 L^2}{4} \left(c(s) - N \frac{(1 - z_- - iy)^2}{((1 - z_-)^2 + y^2)^2} \right),$$

and thus

$$\partial_y^2 \psi(s, y + iz_-) = \frac{b_1 L^2}{4} \left(c(s) - N \frac{(1 - z_-)^2 - y^2}{((1 - z_-)^2 + y^2)^2} + 2iN \frac{(1 - z_-)y}{((1 - z_-)^2 + y^2)^2} \right).$$

The imaginary part vanishes at $y = 0$, and the real part is bounded from below, which concludes the proof. $\blacksquare$

We also have the following useful property of the critical point $iz_-(s)$.

Lemma 5.5. *For all $s \in \mathbb{R}_+ \times \mathbb{R}_+$, we have $\partial_{s_j} z_-(s) > 0$ and*

$$\partial_{s_j} z_-(0) = \frac{z_-(0) + N - 1}{2z_-(0)} = \frac{1}{2}(1 + \sqrt{1 - N}).$$

Proof. We let

$$P(c, z) = z^2 - \frac{c - 1}{c} z + \frac{N - 1}{c},$$

and we notice that $P(c(s), z_-(s)) = 0$. In particular, we have

$$\partial_c P + (\partial_{s_j} z_-) \partial_z P = 0.$$

The result follows, since we have $\partial_c P(c(s), z_-(s)) = -(z_-(s) + N - 1)c(s)^{-2} > 0$, and $\partial_z P(c(s), z_-(s)) = 2z_-(s) - (c(s) - 1)/c(s) < 0$. ■

5.2.4. Critical points in s . Let us look at (5.9) and underline that m_h depends on h (see (5.3)). This suggests to consider the function

$$(5.10) \quad F(s) = \psi(s, iz_-(s)) - |M| \ln[(1 + s_1)(1 + s_2)],$$

which is well defined and smooth in a neighborhood of $\mathbb{R}_+ \times \mathbb{R}_+$.

Lemma 5.6. *The function $[0, +\infty)^2 \ni s \mapsto F(s)$ is real-valued, it tends to $+\infty$ at infinity, and it is strictly convex. It has a unique minimum at $(0, 0)$, whose value is*

$$F(0) = \frac{b_1 L^2}{4} \left(\frac{N}{2} + \sqrt{1 - N} - N \ln(1 + \sqrt{1 - N}) \right) \geq \frac{b_1 L^2}{8} > 0.$$

Moreover, $\nabla F(0) = 0$ and $\nabla^2 F(0) > 0$.

Proof. By (5.8) we have

$$F(s) = \frac{b_1 L^2}{4} \left(\frac{c(s)}{2} (1 - z_-^2(s)) - z_-(s) - N \ln(1 - z_-(s)) \right) - |M| \ln[(1 + s_1)(1 + s_2)],$$

which is real for all $s \in \mathbb{R}_+ \times \mathbb{R}_+$. Then, since iz_- is a critical point in y , we have

$$\partial_s (\psi(s, iz_-(s))) = \partial_s \psi(s, iz_-(s)) + i \partial_y \psi(s, iz_-(s)) \partial_s z_- = (\partial_s \psi)(s, iz_-(s)).$$

Moreover, we have

$$\partial_s \psi(s, y) = \frac{b_1 L^2}{8} \langle y \rangle^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence,

$$\nabla_s F(s) = \frac{b_1 L^2}{8} (1 - z_-^2(s)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - |M| \begin{pmatrix} \frac{1}{1+s_1} \\ \frac{1}{1+s_2} \end{pmatrix}.$$

Note that $z_-(0) = -\sqrt{1 - N}$ and thus $\nabla_s F(0) = 0$. Computing the second derivatives and using that $\partial_{s_1} z_- = \partial_{s_2} z_- > 0$, we see that, for all s ,

$$\partial_{s_1} \partial_{s_2} F = \partial_{s_j}^2 F - \frac{|M|}{(1 + s_j)^2} > 0.$$

Thus, $\nabla^2 F > 0$. This implies that the derivative along any line starting from the origin is increasing, and the result follows. ■

5.2.5. Estimate of W_h . We recall (5.9) and we write

$$W_h = \int_{\mathbb{R}_+ \times \mathbb{R}_+} P(s) e^{-F(s)/h} \int_{\mathbb{R}} e^{-(\psi(s,y) - \psi(s,iz_-(s)))/h} \omega(s, y) dy ds,$$

with $P(s) = (s_1 s_2)^{\delta-1} [(1+s_1)(1+s_2)]^{-\delta}$ and F defined in (5.10).

Lemma 5.7. *For all $s \in \mathbb{R}_+ \times \mathbb{R}_+$, we have*

$$(5.11) \quad \int_{\mathbb{R}} e^{-\psi(s,y)/h} \omega(s, y) dy = \int_{\mathbb{R}} e^{-\psi(s,y+iz_-(s))/h} \omega(s, y + iz_-(s)) dy.$$

Proof. For $R > 0$ very large, we define Γ_R , the positively oriented rectangle in the complex plane,

$$\Gamma_R = [-R, R] \cup [-R + iz_-, R + iz_-] \cup [-R + iz_-, -R] \cup [R, R + iz_-].$$

Since $|z_-(s)| < 1$, the function ω is holomorphic inside Γ_R , and by the Cauchy formula,

$$\int_{\Gamma_R} e^{-\psi(s,y)/h} \omega(s, y) dy = 0.$$

The integral on $[-R, R]$ (respectively, on $[-R + iz_-, R + iz_-]$) converges to the left-hand side (respectively, the right-hand side) of (5.11) as $R \rightarrow \infty$. Hence, we only have to bound the integral on the two remaining parts, namely $\Gamma_R^\pm = [\pm R + iz_-, \pm R]$. For instance,

$$I_R^+ = \int_{\Gamma_R^+} e^{-\psi(s,y)/h} \omega(s, y) dy = \int_0^{z_-(s)} e^{-\psi(s, R+it)/h} \omega(s, R+it) i dt.$$

For all $t \in [-1, 1]$, we can bound

$$\operatorname{Re} \psi(s, R+it) \geq \frac{b_1 L^2}{4} \left(\frac{R^2}{2} - 1 - N \ln \sqrt{4 + R^2} \right),$$

and we deduce that $I_R^+ \rightarrow 0$ as $R \rightarrow \infty$. We proceed similarly for Γ_R^- . ■

Considering Lemma 5.7, we rewrite

$$W_h = \int_{\mathbb{R}_+ \times \mathbb{R}_+} P(s) e^{-F(s)/h} \int_{\mathbb{R}} e^{-(\psi(s,y+iz_-(s)) - \psi(s,iz_-(s)))/h} \omega(s, y + iz_-(s)) dy ds,$$

and we use the following adaptation of the Laplace method for the y -integral.

Lemma 5.8. *For all $s \in \mathbb{R}_+ \times \mathbb{R}_+$, we have*

$$\begin{aligned} \int_{\mathbb{R}} e^{-(\psi(s,y+iz_-(s)) - \psi(s,iz_-(s)))/h} \omega(s, y + iz_-(s)) dy \\ = \sqrt{\frac{2\pi h}{\operatorname{Re} \partial_y^2 \psi(s, iz_-(s))}} \omega(s, iz_-(s)) + \mathcal{O}(h^{3/2}), \end{aligned}$$

and the remainder is uniform with respect to $s \in \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. Let us decompose the phase with its real and imaginary parts:

$$\psi(s, y + iz_-) - \psi(s, iz_-) = \Re(\psi(s, y + iz_-) - \psi(s, iz_-)) + i\Theta(s, y),$$

where $\Theta(s, 0) = \partial_y \Theta(s, 0) = \partial_y^2 \Theta(s, 0) = 0$ (see Lemma 5.4). By the Taylor formula, we have

$$\Re(\psi(s, y + iz_-) - \psi(s, iz_-)) = y^2 \int_0^1 (1-u) \Re \partial_y^2 \psi(s, iz_- + uy) du = \chi_s(y)^2,$$

with

$$\chi_s(y) = y \left(\int_0^1 (1-u) \Re \partial_y^2 \psi(s, iz_- + uy) du \right)^{1/2},$$

where we used the positivity of $\Re \partial_y^2 \psi(s, iz_- + uy)$ ensured by Lemma 5.4. Note that

$$(\chi_s^{-1})'(0) = \left(\frac{2}{\Re \partial_y^2 \psi(s, iz_-(s))} \right)^{1/2}.$$

Thanks to the change of variable $\tilde{y} = \chi_s(y)/\sqrt{h}$, we find

$$\begin{aligned} & \int_{\mathbb{R}} e^{-(\psi(s, y + iz_-(s)) - \psi(s, iz_-(s)))/h} \omega(s, y + iz_-(s)) dy \\ &= \sqrt{h} \int_{\mathbb{R}} e^{-\tilde{y}^2} \omega(s, iz_-(s)) (\chi_s^{-1})'(0) d\tilde{y} \\ &+ \sqrt{h} \int_{\mathbb{R}} e^{-\tilde{y}^2} (1 - e^{-\frac{i}{h} \Theta(s, \chi^{-1}(\sqrt{h}\tilde{y}))}) \omega(s, iz_-(s)) (\chi_s^{-1})'(0) d\tilde{y} \\ &+ h^{3/2} \int_{\mathbb{R}} e^{-\tilde{y}^2} e^{-\frac{i}{h} \Theta(s, \chi^{-1}(\sqrt{h}\tilde{y}))} \tilde{y}^2 f_1(\tilde{y}) d\tilde{y}, \end{aligned}$$

for some function f_1 which is bounded, uniformly with respect to s . Here we used that the linear terms in $\tilde{y}$ give 0 when integrated against the Gaussian. The first integral gives the main contribution. The third integral is of order $h^{3/2}$. Finally, since Θ vanishes at the order 3, we can write $1 - e^{-i\Theta/h}$ as a linear term in $\tilde{y}$ plus a term of order h . The linear term vanishes when integrated against the Gaussian, and the second integral is thus of order $h^{3/2}$. ■

From Lemma 5.8 we see that

$$\begin{aligned} & \left| W_h - \int_{\mathbb{R}_+} \sqrt{\frac{2\pi h}{\Re \partial_y^2 \psi(s, iz_-(s))}} P(s) e^{-F(s)/h} \omega(s, iz_-(s)) ds \right| \\ (5.12) \quad & \leq Ch^{3/2} \int P(s) e^{-F(s)/h} ds \end{aligned}$$

We can use the Laplace method to estimate these integrals. We recall from Lemma 5.6 that F has a unique critical point and global minimum at $s = 0$. With (5.7), we have

$$\omega(s, iz_-(s)) = \left(\frac{N}{1 - z_-(s)^2} - 1 \right) (1 + z_-(s)) - 2s_1.$$

In particular, since $z_-(0) = -\sqrt{1-N}$,

$$\omega(0, iz_-(0)) = 0.$$

Thus, we need to consider the linear approximation of $\omega(s, iz_-(s))$ near $s = 0$. We have

$$\begin{aligned} \omega(s, iz_-(s)) &= \left(\frac{N}{(1 - z_-(0))^2} - 1 \right) \nabla z_-(0) \cdot s - 2s_1 + \mathcal{O}(|s|^2) \\ &= \left(\frac{N}{(1 + \sqrt{1-N})^2} - 1 \right) \frac{1}{2} (1 + \sqrt{1-N}) (s_1 + s_2) - 2s_1 + \mathcal{O}(|s|^2) \\ &= -\sqrt{1-N} (s_1 + s_2) - 2s_1 + \mathcal{O}(|s|^2), \end{aligned}$$

where we used Lemma 5.5. With the Laplace method, we deduce that

$$\int_{\mathbb{R}_+^2} \sqrt{\frac{2\pi h}{\operatorname{Re} \partial_y^2 \psi(s, iz_-(s))}} P(s) e^{-F(s)/h} \omega(s, iz_-(s)) ds = c_0 h^{1+\delta_0} e^{-F(0)/h} (1 + \mathcal{O}(\sqrt{h})),$$

with

$$c_0 = -\sqrt{\frac{2\pi}{\Re \partial_y^2 \psi(0, iz_-(0))}} \int_{\mathbb{R}_+^2} (s_1 s_2)^{\delta_0-1} (\sqrt{1-N}(s_1 + s_2) + 2s_1) e^{-\frac{1}{2} \operatorname{Hess}_0 F(s,s)} ds.$$

Note that this integral can be computed explicitly (first replace $2s_1$ by $s_1 + s_2$ by symmetry, then on the set $\{s_2 > s_1\}$ use the variables $v = s_2 - s_1$ and $u = s_1 s_2$). We find

$$(5.13) \quad c_0 = -\frac{8\pi\Gamma(\delta_0)}{b_1 L^2} (1-N)^{-1/4} \left(\frac{b_1 L^2}{8} \right)^{-\delta_0} (1 + \sqrt{1-N})^{-2\delta_0}.$$

The second integral in (5.12) can also be calculated using the Laplace method. However, it is easier since the factor $\omega(s, iz_-(s))$ is not present. Since $\omega(s, iz_-(s))$ vanishes linearly at $s = 0$ this means that this second integral becomes a $1/\sqrt{h}$ bigger compared to the first, but since it had an extra factor h , we get the final result

$$W_h = c_0 h^{1+\delta_0} e^{-F(0)/h} (1 + \mathcal{O}(\sqrt{h})),$$

with c_0 given by (5.13).

5.2.6. Estimate of w_h : End of the proof of Proposition 4.2. Recalling (5.5), we get

$$w_h = C(h)^2 \left(\frac{L}{2} \right)^{2\gamma} \frac{b_1 L^2}{4} c_0 h^{2+\delta_0} e^{-F(0)/h} (1 + \mathcal{O}(h^{1/2})),$$

where c_0 was defined in (5.13). Therefore, with Lemma 5.3,

$$w_h \underset{h \rightarrow 0}{\sim} \frac{b_1 L^2}{4} a_0(0)^2 e^{-2\Phi_\ell(0)/h} h^{1-\delta_0} \Gamma(\delta_0)^{-2} \left(\frac{b_1 L^2}{8} - |M| \right)^{2\delta_0} c_0 e^{-I/h},$$

where $I = F(0) - b_1 L^2/8$. With the explicit value of $F(0)$ in Lemma 5.6, we see that $I > 0$. This finishes the proof of Proposition 5.1.

5.2.7. Comparison with the remainders . In view of Proposition 4.1, one needs to check that $2S_0 > S$. We recall that S_0 is given in (3.1).

Lemma 5.9. *The quantity I can be rewritten as*

$$I = \int_{L^2/8}^{\frac{L^2}{8}(1+\sqrt{1-N})^2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv - 2 \int_0^{L^2/8} \beta(u) du.$$

Moreover, if $L > (2 + \sqrt{6})a$, then $2S_0 > S$.

Proof. We have now to ensure that $S < 2S_0$, i.e.,

$$(5.14) \quad \int_0^{L^2/8} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv + I < \int_0^{(L-a)^2/2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv.$$

We can rewrite I using the Agmon distance,

$$I = \int_{L^2/8}^{\frac{L^2}{8}(1+\sqrt{1-N})^2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv - 2 \int_0^{L^2/8} \beta(u) du.$$

Therefore, (5.14) becomes

$$(5.15) \quad \int_{(L-a)^2/2}^{\frac{L^2}{8}(1+\sqrt{1-N})^2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv < 2 \int_0^{L^2/8} \beta(u) du.$$

Note that the left-hand side is positive because

$$\frac{L}{2} (1 + \sqrt{1-N}) \geq \frac{L}{2} (2 - \sqrt{N}) \geq L - a,$$

where we used (1.2).

Let us assume $L > (2 + \sqrt{6})a$, and prove (5.15) in this case. We bound the left-hand side as follows:

$$\int_{(L-a)^2/2}^{\frac{L^2}{8}(1+\sqrt{1-N})^2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv \leq \int_{(L-a)^2/2}^{L^2/2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv,$$

and using $\beta \leq b_1$, we deduce

$$\int_{(L-a)^2/2}^{\frac{L^2}{8}(1+\sqrt{1-N})^2} \frac{1}{v} \left(\int_0^v \beta(u) du \right) dv \leq \frac{b_1}{2} (L^2 - (L-a)^2) = \frac{b_1}{2} (2La - a^2).$$

Moreover, the right-hand side of (5.15) is bounded from below by

$$2 \int_0^{L^2/8} \beta(u) du = \frac{b_1 L^2}{4} - 2|M| \geq \frac{b_1}{2} \left(\frac{L^2}{2} - 2a^2 \right).$$

Thus, the inequality (5.15) is true as soon as $L^2/2 - 2a^2 > 2La - a^2$, which is equivalent to $L > (2 + \sqrt{6})a$. ■

Let us now deal with the remainder involving t_h defined in Proposition 4.1. It can be analyzed similarly to w_h .

Lemma 5.10. *We have*

$$t_h = \tilde{\mathcal{O}}(e^{-S/h}).$$

Proof. We can write, for some cutoff function χ supported in $(-L/2, L/2)$,

$$\begin{aligned} t_h &= C(h)^2 \int_{\mathbb{R}^2} dx \chi(x_1) |x - x_\ell|^\gamma |x - x_r|^\gamma e^{-i\theta(x)/h} \\ &\quad \cdot \int_{\mathbb{R}_+^2} e^{-\frac{b_1}{4h}[(1+2s_1)|x-x_\ell|^2 + (1+2s_2)|x-x_r|^2]} m_h(s_1) m_h(s_2) ds. \end{aligned}$$

We use the rescaling $s = hu$ and get

$$\begin{aligned} t_h &= h^2 C(h)^2 \int_{\mathbb{R}^2} dx \chi(x_1) |x - x_\ell|^\gamma |x - x_r|^\gamma e^{-i\theta(x)/h} \\ &\quad \cdot \int_{\mathbb{R}_+^2} e^{-\frac{b_1}{4h}[(1+2hu_1)|x-x_\ell|^2 + (1+2hu_2)|x-x_r|^2]} m_h(hu_1) m_h(hu_2) du. \end{aligned}$$

This allows to rewrite t_h as

$$t_h = h^{2\delta-2} h^2 C(h)^2 \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_1 \chi(x_1) e^{-\frac{\Psi(x)+i\theta(x)}{h}} \underbrace{D_\ell(x; h) D_r(x; h)}_{=D(x; h)},$$

with

$$\Psi(x) = \frac{b_1}{4} (|x - x_\ell|^2 + |x - x_r|^2) - \frac{|M|}{2} \ln(|x - x_\ell|^2 |x - x_r|^2)$$

and

$$D_\star(x; h) = \int_{\mathbb{R}_+} v^{\delta-1} (1 + hv)^{-\delta} e^{-\frac{b_1 v}{2} |x - x_\star|^2} e^{\frac{|M| \ln(1+hv)}{h}} dv.$$

Note that

$$v \frac{b_1 |x - x_\star|^2}{2} - |M| h^{-1} \ln(1 + hv) \geq v \left(\frac{b_1 a^2}{2} - |M| \right) > 0;$$

thus, $D_\star$ defines a function belonging to the Schwartz class $\mathcal{S}([-L/2 + a, L/2 - a] \times \mathbb{R})$, uniformly in h .

Then, we observe that $x_1 = 0$ is the unique critical point of the (even) function

$$[-L/2 + a, L/2 - a] \ni x_1 \mapsto \Psi(x_1, x_2)$$

and that $\partial_{x_1} \theta(0, x_2) = 0$. Moreover, $\partial_{x_1}^2 \Psi(0, x_2) \geq b_1(1 - N) > 0$. This allows to use the Laplace strategy as we did in the proof of Lemma 5.8. This shows that, for some $d > 0$ and some bounded $(v(h))_{h>0}$,

$$t_h = C(h)^2 d h^{v(h)} \int_{\mathbb{R}} (\partial_{x_1}^2 \Psi(0, x_2))^{-1/2} e^{-\frac{\Psi(0, x_2) + i\theta(0, x_2)}{h}} (D(0, x_2; h) + h R(x_2; h)) dx_2,$$

with $R(\cdot; h)$ a function in the Schwartz class that is analytic. The phase of the latter integral is the one appearing in w_h (see (5.5) and (5.6), with $s = 0$). Therefore, similar considerations as in the proof of Proposition 5.1 give the rough estimate

$$t_h = \tilde{O}(e^{-S/h}). \quad \blacksquare$$

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