J. Spectr. Theory 15 (2025), 279–303
DOI 10.4171/JST/542

Type problem, the first eigenvalue and Hardy inequalities

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Abstract. In this paper, we study the relationship between the type problem and the asymptotic behaviour of the first (Dirichlet) eigenvalues $\lambda_1(B_r)$ of "balls" $B_r := \{\rho < r\}$ on a complete Riemannian manifold M as $r \to +\infty$, where ρ is a Lipschitz continuous exhaustion function with $|\nabla\rho| \leq 1$ a.e. on M. We obtain several sharp results. First, if $r^2\lambda_1(B_r) \geq \gamma > 0$ for all $r > r_0$, we obtain a sharp estimate of the volume growth: $|B_r| \geq c r^{\mu(\gamma)}$. Moreover, when $\gamma > j_0^2 \approx 5.784$, where j_0 denotes the first positive zero of the Bessel function J_0, then M is non-parabolic and we have a Hardy-type inequality. In the case where $r_0 = 0$, a sharp Hardy-type inequality holds. These spectral conditions are satisfied if one assumes that $\Delta\rho^2 \geq 2\mu(\gamma) > 0$. In particular, when $\inf_M \Delta\rho^2 > 4$, M is non-parabolic and we get a sharp Hardy-type inequality. Related results for finite volume case are also studied.

1. Introduction

Let (M, g) be a complete, non-compact Riemannian manifold with $\dim M \geq 2$, and denote by Δ the Laplace operator associated to g. An upper semicontinuous function u on M is called *subharmonic* if $\Delta u \geq 0$ holds in the sense of distributions. If every negative subharmonic function on M has to be a constant, then M is said to be *parabolic*; otherwise M is called *non-parabolic*. It is well known that M is parabolic (resp. non-parabolic) if and only if the Green function $G_M(x, y)$ is infinite (resp. finite) for all $x \neq y$; or the Brownian motion on M is recurrent (resp. transient).

The type problem is how to decide the parabolicity and non-parabolicity through intrinsic geometric conditions. The case of surfaces is classical, for the type of M depends only on the conformal class of g, i.e., the complex structure determined by g. Ahlfors [1] and Nevanlinna [23] first showed that M is parabolic whenever

$$\int_1^{+\infty} \frac{dr}{|\partial B(x_0, r)|} = +\infty, \tag{1.1}$$

Mathematics Subject Classification 2020: 31C12 (primary); 58C40 (secondary).
Keywords: non-parabolic Riemannian manifold, first eigenvalue, Hardy inequality.

where $B(x_0, r)$ is the geodesic ball with center $x_0 \in M$ and radius r. The same conclusion was extended to high dimensional cases by Lyons and Sullivan [21] and Grigor'yan [12, 13]. Moreover, (1.1) can be relaxed to

$$\int_1^{+\infty} \frac{r\,dr}{|B(x_0, r)|} = +\infty$$

(cf. Karp [17], Varopolous [26], and Grigor'yan [12, 13]; see also Cheng and Yau [9]). We refer to the excellent survey [14] of Grigor'yan for other sufficient conditions of parabolicity.

On the other side, it seems more difficult to find sufficient conditions for non-parabolicity. Yet, there is a classical result stating that M is non-parabolic whenever the first (Dirichlet) eigenvalue $\lambda_1(M)$ of M is positive. Recall that

$$\lambda_1(M) := \lim_{j \to +\infty} \lambda_1(\Omega_j)$$

for some/any increasing sequence of precompact open sets $\{\Omega_j\}$ in M, such that $M = \bigcup \Omega_j$. Here, given an open set $\Omega \subset\subset M$, define

$$\lambda_1(\Omega) := \inf\left\{ \frac{\int_\Omega |\nabla \phi|^2 dV}{\int_\Omega \phi^2 dV} : \phi \in \mathrm{Lip}_0(M),\ \mathrm{supp}\,\phi \subset \bar{\Omega},\ \phi \not\equiv 0 \right\},$$

where $\mathrm{Lip}_0(M)$ denotes the set of Lipschitz continuous functions on M with compact supports.

Remark. We use the convention that $\lambda_1(\varnothing) = +\infty$.

The main focus of this paper is to determine the non-parabolicity in the case $\lambda_1(M) = 0$. Grigor'yan showed that M is non-parabolic if the following Faber–Krahn-type inequality holds:

$$\lambda_1(\Omega) \geq f(|\Omega|) \quad \text{for all } \Omega \subset\subset M \text{ such that } |\Omega| \geq v_0 > 0,$$

where f is a positive decreasing function on $(0, +\infty)$ such that

$$\int_{v_0}^{+\infty} \frac{dv}{v^2 f(v)} < +\infty$$

(see, e.g., [14, Theorem 10.3]). We shall use certain quantity measuring the asymptotic behaviour of $\lambda_1(B_r)$ for certain "balls" B_r as $r \to +\infty$, which seems to be easier to analyse. More precisely, let us first fix a nonnegative locally Lipschitz continuous function ρ on M, which is an exhaustion function (i.e., $B_r := \{\rho < r\} \subset\subset M$ for any

$r > 0$), such that $|\nabla\rho| \le 1$ holds a.e. on M. Note that if ρ is the distance $\mathrm{dist}_M(x_0, \cdot)$ from some $x_0 \in M$, then B_r is precisely the geodesic ball $B(x_0, r)$.

To state the main result, we denote by λ_μ the first eigenvalue of the Laplace operator on $[0, 1)$ for the Dirichlet condition at $s = 1$ and with respect to the measure $s^{\mu-1}ds$, i.e.,

$$\lambda_\mu = \inf\left\{ \frac{\int_0^1 |\psi'(s)|^2 s^{\mu-1} ds}{\int_0^1 |\psi(s)|^2 s^{\mu-1} ds} : \psi \in \mathrm{Lip}([0, 1]),\ \psi(1) = 0,\ \psi \not\equiv 0 \right\}.$$

It is known that $\lambda_\mu = j^2_{\mu/2-1}$, where j_ν is the first positive zero of the Bessel function J_ν, with the infimum λ_μ realised by

$$\psi_\mu(s) := s^{1-\mu/2} J_{\mu/2-1}(\sqrt{\lambda_\mu} s). \tag{1.2}$$

We have the following result.

Theorem 1.1. *Suppose that*

$$\lambda_1(B_r) \ge \frac{\lambda_\mu}{r^2}, \tag{1.3}$$

holds for all $r \ge r_0$. Then the following properties hold.

(1) *There is a constant $c > 0$ such that*

$$|B_r| \ge c r^\mu \quad \textit{for all } r \ge r_0. \tag{1.4}$$

Here, the constant c might depend on the geometry of B_{r_0}.

(2) *If $\mu > 2$, then M is non-parabolic.*

(3) *If $\mu > 2$, then the Hardy-type inequality*

$$C \int_M \frac{u^2}{1+\rho^2} dV \le \int_M |\nabla u|^2 dV \tag{1.5}$$

holds for some $C > 0$ and for any $u \in \mathrm{Lip}_0(M)$. If (1.3) holds with $\mu > 2$ for all $r > 0$, then we have the sharp Hardy-type inequality

$$\left(\frac{\mu-2}{2}\right)^2 \int_M \frac{u^2}{\rho^2} dV \le \int_M |\nabla u|^2 dV \quad \textit{for all } u \in \mathrm{Lip}_0(M). \tag{1.6}$$

Remark. (a) It is well known that if $M = \mathbb{R}^n$ and $\rho(x) = |x|$ then

$$\lambda_1(B(0, r)) = \frac{\lambda_n}{r^2} = \frac{j^2_{n/2-1}}{r^2} \quad \text{for all } r > 0.$$

Thus, the volume growth in (1.4) is sharp for $\mu = n$. Moreover, (1.6) is also sharp, since on the Euclidean space $\mathbb{R}^n$, with $n \geq 3$, we have the classical Hardy inequality

$$\Big(\frac{n-2}{2}\Big)^2 \int\limits_{\mathbb{R}^n} \frac{u^2}{|x|^2} dV \leq \int\limits_{\mathbb{R}^n} |\nabla u|^2 dV, \quad \text{for all } u \in \mathrm{Lip}_0(\mathbb{R}^N).$$

and our hypothesis holds with $\mu = n$.

(b) (1.4) also follows from the sharp Hardy-type inequality (cf. Carron [7, Proposition 2.26]). Indeed, (1.6) implies the following reverse doubling property of order μ (cf. Lansade [18, Proposition 5.2]):

$$\frac{|B_R|}{|B_r|} \geq c_\mu \frac{R^\mu}{r^\mu} \quad \text{for all } R > r > 0.$$

Define

$$\Lambda_* := \liminf_{r \to +\infty} \{r^2 \lambda_1(B_r)\}.$$

Note that $\mu \mapsto \lambda_\mu$ is a strict increasing continuous function when $\mu \geq 2$ (cf. [11]). Thus, a direct consequence of Theorem 1.1 is the following.

Corollary 1.2. *If $\Lambda_* > \lambda_2 = j_0^2 \approx 5.784$, then M is non-parabolic. In other words, if M is parabolic, then $\Lambda_* \leq j_0^2$.*

Remark. We have already said that if $M = \mathbb{R}^2$ and $\rho(x) = |x|$, then

$$\lambda_1(B(0,r)) = \frac{\lambda_2}{r^2} = \frac{j_0^2}{r^2},$$

so that $\Lambda_* = j_0^2$. Since $\mathbb{R}^2$ is parabolic, so the above result turns out to be the best possible.

Conversely, it is natural to ask whether there exists a universal constant c_0 such that M is parabolic whenever $\Lambda_* \leq c_0$. The answer is, however, negative (see Example 5.5 in Section 5).

Theorem 1.1 (1) allows us to estimate Λ_* through volume growth conditions. Cheng and Yau [9] showed that $\lambda_1(M) = 0$ if M has polynomial volume growth. This was extended by Brooks [3], who showed that if the volume $|M|$ of M is infinite, then

$$\lambda_1(M) \leq \frac{\mu^{*2}}{4}, \quad \mu^* := \limsup_{r \to +\infty} \frac{\log |B(x_0, r)|}{r}.$$

Refined results for ends of complete Riemannian manifolds are obtained by Li and Wang [20] (see also Carron [7, Section 2.4]). The following consequence of Theorem 1.1 may be viewed as a quantitative version of the theorem of Cheng and Yau.

Corollary 1.3. *If* $\nu_* := \liminf_{r\to+\infty} \log|B_r|/\log r$, *then* $\Lambda_* \le \lambda_{\nu_*} = j^2_{\nu_*/2-1}$. *In particular, we have*

(1) $\Lambda_* = 0$ *if* $\nu_* = 0$;

(2) $\Lambda_* \lesssim \nu_*$ *if* $0 < \nu_* \ll 1$;

(3) $\Lambda_* \lesssim \nu_*^2$ *if* $\nu_* \gg 1$.

When $|M| < +\infty$, we have $\nu_* = 0$, so that $\lambda_1(B_r)$ decays faster than quadratically as $r \to +\infty$. More precisely, we have the following.

Proposition 1.4. *If* $|M| < \infty$, *then*

$$\widetilde{\Lambda}_* := \liminf_{r\to+\infty} \frac{-\log\lambda_1(B_r)}{r} \ge \alpha_* := \liminf_{r\to+\infty} \frac{-\log|M\setminus B_r|}{r}. \tag{1.7}$$

It follows from Proposition 1.4 that $\lambda_1(B_r)$ decays exponentially if $|M \setminus B_r|$ decays exponentially as $r \to +\infty$. On the other hand, the relationship of $\lambda_1(M \setminus B_r)$ and $|M \setminus B_r|$ is studied by Brooks [4], who proved that if $|M| < \infty$, then

$$\lambda_1(M\setminus B_r) \le \frac{\alpha^{*2}}{4} \quad \text{for all } r > 0,$$

where

$$\alpha^* := \limsup_{r\to+\infty} \frac{-\log|M\setminus B_r|}{r}.$$

A more precise version of Brooks' result is also proved by Li and Wang [20].

Motivated by a result of Dodziuk, Pignataro, Randol, and Sullivan [10], we present an example in Section 5 showing that the estimate in Proposition 1.4 is sharp. Some other examples such that $\lambda_1(B_r)$ have various decaying behaviours are also given in Section 5. In particular, Example 5.5 shows that $\widetilde{\Lambda}_* > 0$ does not necessarily imply $|M| < \infty$, i.e., the assumption that M has finite volume in Proposition 1.4 cannot be removed.

We also show that (1.3) holds under suitable condition on ρ.

Proposition 1.5. *Suppose that* ρ *is a nonnegative locally Lipschitz continuous exhaustion function on* M *such that* $|\nabla\rho| \le 1$ *a.e. and* $\Delta\rho^2 \ge 2\mu$ *in the sense of distributions. Then*

$$\lambda_1(B_r) \ge \frac{\lambda_\mu}{r^2} \quad \textit{for all } r > 0.$$

Proposition 1.5 and Theorem 1.1, immediately yield the following.

Corollary 1.6 (cf. [5]). *If* $\Delta\rho^2 \ge 2\mu > 4$, *then* M *is non-parabolic and the sharp Hardy-type inequality* (1.6) *holds.*

Proposition 1.5 implies several well-known results. First, if M is an n-dimensional Cartan–Hadamard manifold and ρ is the geodesic distance function, then Proposition 1.5, together with the Hessian comparison theorem, yields $\Lambda_* \geq \lambda_n$. In particular, it follows from Theorem 1.1 that M is non-parabolic when $n \geq 3$ (cf. Ichihara [15,16], see also Grigor'yan [14, Theorem 15.3]). Next, if M is a complete n-dimensional minimal submanifold in $\mathbb{R}^N$ and ρ is the restriction of Euclidean distance to a given point $x_0 \in \mathbb{R}^N$, then $\Delta \rho^2 \geq 2n$, so that

$$\lambda_1(B^N(x_0, r) \cap M) \geq \frac{\lambda_n}{r^2}, \tag{1.8}$$

in view of Proposition 1.5. Here $B^N(x_0, r)$ is a Euclidean ball in $\mathbb{R}^N$. (1.8) was first proved by Cheng, Li, and Yau [8] by heat kernel method. In particular, M is non-parabolic for $n \geq 3$, which can also be proved by the isoperimetric inequality of Michael and Simon [22] and [14, Theorem 8.2].

Comments

In an early version of this paper, the last two authors obtained the conclusion of Corollary 1.2 under a worse condition $\Lambda_* > 18.624\ldots$. Shortly afterwards, the first author suggested some ideas for improving certain results. We then decided to write a joint paper on the subject, and the improvements found in this paper are the result of this collaboration.

2. Proof of Theorem 1.1

2.1. Bessel's functions

Assume that $\nu > -1$ and $\mu > 0$. The Bessel function J_ν is given by

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{t}{2}\right)^{2m+\nu},$$

which is a solution of the ODE

$$t^2 J_\nu''(t) + t J_\nu'(t) + (t^2 - \nu^2) J_\nu(t) = 0.$$

Thus, $\psi_\mu(s) = s^{1-\mu/2} J_{\mu/2-1}(\sqrt{\lambda_\mu} s)$ satisfies

$$\psi_\mu''(s) + \frac{\mu - 1}{s} \psi_\mu'(s) + \lambda_\mu \psi_\mu(s) = 0, \tag{2.1}$$

where $\sqrt{\lambda_\mu} = j_{\mu/2-1}$ is the first positive zero of $J_{\mu/2-1}$. In particular, $\psi_\mu(1) = 0$. Moreover, we have

$$\psi_\mu(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma\big(m+\frac{\mu}{2}\big)} \Big(\frac{\sqrt{\lambda_\mu} s}{2}\Big)^{2m},$$

so that $\psi_\mu'(0) = 0$.

Let us first verify the following.

Lemma 2.1. *The following properties hold.*

(1) *For any* $[a,b] \subset [0,1]$, *we have*

$$\int_a^b (\lambda_\mu \psi_\mu(s)^2 - \psi_\mu'(s)^2) s^{\mu-1} ds = -\psi_\mu'(b)\psi_\mu(b) b^{\mu-1} + \psi_\mu'(a)\psi_\mu(a) a^{\mu-1}. \tag{2.2}$$

(2) $\psi_\mu'(s) < 0$ *for all* $s \in (0,1]$. *More precisely,*

$$\psi_\mu'(s) s^{\mu-1} = -\lambda_\mu \int_0^s \psi_\mu(t) t^{\mu-1} dt. \tag{2.3}$$

Proof. By (2.1), we have

$$\begin{aligned}(\lambda_\mu \psi_\mu(s)^2 - \psi_\mu'(s)^2) s^{\mu-1} &= \Big(-\psi_\mu''(s)\psi_\mu(s) - \frac{\mu-1}{s}\psi_\mu'(s)\psi_\mu(s) - \psi_\mu'(s)^2\Big) s^{\mu-1} \\ &= -(\psi_\mu'(s)\psi_\mu(s))' s^{\mu-1} - (\mu-1)\psi_\mu'(s)\psi_\mu(s) s^{\mu-2} \\ &= -(\psi_\mu'(s)\psi_\mu(s) s^{\mu-1})'\end{aligned}$$

and

$$-\lambda_\mu \psi_\mu(s) s^{\mu-1} = \psi_\mu''(s) s^{\mu-1} + (\mu-1)\psi_\mu'(s) s^{\mu-2} = (\psi_\mu'(s) s^{\mu-1})',$$

from which (2.2) and (2.3) follow immediately. ■

2.2. The volume growth

Proof of Theorem 1.1 (1). We first assume that $\rho > 0$. Set $\phi = \psi_\mu(\rho/r)$, where ψ_μ is given as (1.2). Then the variational characterization of eigenvalue gives

$$\lambda_\mu \int_{B_r} \psi_\mu\Big(\frac{\rho}{r}\Big)^2 dV \le r^2 \lambda_1(B_r) \int_{B_r} \phi^2 dV \le r^2 \int_{B_r} |\nabla\phi|^2 dV \le \int_{B_r} \psi_\mu'\Big(\frac{\rho}{r}\Big)^2 dV.$$

By using the co-area formula, this can be rewritten as

$$\lambda_\mu \int_0^r \psi_\mu\Big(\frac{t}{r}\Big)^2 d\sigma(t) \le \int_0^r \psi_\mu'\Big(\frac{t}{r}\Big)^2 d\sigma(t),$$

where $d\sigma(t) := (\rho)_\#(dV)$ is a Lebesgue–Stieltjes measure on $(0, +\infty)$. Divide this inequality by $r^{\mu+1}$ and integrate on $r \in [r_0, \bar{r}]$, we obtain

$$\lambda_\mu \int_0^{\bar{r}} \int_{\max\{r_0,t\}}^{\bar{r}} \psi_\mu\Big(\frac{t}{r}\Big)^2 \frac{dr}{r^{\mu+1}}\, d\sigma(t) \le \int_0^{\bar{r}} \int_{\max\{r_0,t\}}^{\bar{r}} \psi_\mu'\Big(\frac{t}{r}\Big)^2 \frac{dr}{r^{\mu+1}} d\sigma(t),$$

in view of Fubini's theorem. By using the new variable $s = t/r$, we get

$$\lambda_\mu \int_0^{\bar{r}} \Bigg(\int_{t/\bar{r}}^{\min\{1,t/r_0\}} \psi_\mu(s)^2 s^{\mu-1} ds \Bigg) \frac{d\sigma(t)}{t^\mu} \le \int_0^{\bar{r}} \Bigg(\int_{t/\bar{r}}^{\min\{1,t/r_0\}} \psi_\mu'(s)^2 s^{\mu-1} ds \Bigg) \frac{d\sigma(t)}{t^\mu}. \tag{2.4}$$

Take $a = t/\bar{r}$ and $b = \min\{1, t/r_0\}$ when $t \le r_0$ in (2.2); we infer from (2.4) and the facts $\psi_\mu(1) = 0$ and $\psi_\mu' \le 0$ that

$$\begin{aligned}
&\int_0^{\bar{r}} -\psi_\mu'\Big(\frac{t}{\bar{r}}\Big)\psi_\mu\Big(\frac{t}{\bar{r}}\Big)\Big(\frac{t}{\bar{r}}\Big)^{\mu-1} \frac{d\sigma(t)}{t^\mu} \\
&\ge \int_0^{\bar{r}} -\psi_\mu'\Big(\min\Big\{1, \frac{t}{r_0}\Big\}\Big)\psi_\mu\Big(\min\Big\{1, \frac{t}{r_0}\Big\}\Big)\Big(\min\Big\{1, \frac{t}{r_0}\Big\}\Big)^{\mu-1} \frac{d\sigma(t)}{t^\mu} \\
&\ge \int_0^{r_0} -\psi_\mu'\Big(\frac{t}{r_0}\Big)\psi_\mu\Big(\frac{t}{r_0}\Big)\Big(\frac{t}{r_0}\Big)^{\mu-1} \frac{d\sigma(t)}{t^\mu},
\end{aligned}$$

i.e.,

$$\frac{1}{\bar{r}^\mu} \int_{B_{\bar{r}}} -\frac{\psi_\mu'\big(\frac{\rho}{\bar{r}}\big)\psi_\mu\big(\frac{\rho}{\bar{r}}\big)}{\big(\frac{\rho}{\bar{r}}\big)} dV \ge \frac{1}{r_0^\mu} \int_0^{r_0} -\frac{\psi_\mu'\big(\frac{\rho}{r_0}\big)\psi_\mu\big(\frac{\rho}{r_0}\big)}{\big(\frac{\rho}{r_0}\big)} dV. \tag{2.5}$$

If we merely have $\rho \ge 0$, then we may also apply the above argument to

$$\rho_\varepsilon := \sqrt{\rho^2 + \varepsilon^2}.$$

Note that we still have

$$|\nabla \rho_\varepsilon| = \frac{\rho|\nabla\rho|}{(\rho^2 + \varepsilon^2)^{1/2}} \le 1.$$

Since

$$B_r^\varepsilon := \{\rho_\varepsilon < r\} = B_{\sqrt{r^2-\varepsilon^2}},$$

it follows that

$$\lambda_1(B_r^\varepsilon) = \lambda_1(B_{\sqrt{r^2-\varepsilon^2}}) \geq \frac{\lambda_\mu}{r^2-\varepsilon^2} \geq \frac{\lambda_\mu}{r^2} \quad \text{for all } r \geq r_{0,\varepsilon} := \sqrt{r_0^2+\varepsilon^2}, \tag{2.6}$$

so that (2.5) still holds with ρ and r_0 replaced by ρ_ε and $r_{0,\varepsilon}$, respectively. Moreover, since ψ_μ is a C^2 function on $[0,1]$ with $\psi_\mu'(0)=0$, there exists a constant $A>0$ such that

$$-\psi_\mu'(s)\psi_\mu(s) \leq As \quad \text{for all } s \in [0,1].$$

Letting $\varepsilon \to 0+$, we see that (2.5) remains valid for ρ and r_0 in view of the dominated convergence theorem, which yields

$$\frac{|B_{\bar r}|}{\bar r^\mu} \geq \frac{1}{Ar_0^\mu}\int_0^{r_0} -\frac{\psi_\mu'\big(\frac{t}{r_0}\big)\psi_\mu\big(\frac{t}{r_0}\big)}{\big(\frac{t}{r_0}\big)}d\sigma(t) =: c \quad \text{for all } \bar r \geq r_0. \qquad \blacksquare$$

Proof of Corollary 1.3. Suppose on the contrary that $\Lambda_* > \lambda_{\nu_*}$. Since the function $\mu \mapsto \lambda_\mu$ is continuous, we have $\Lambda_* > \lambda_{\nu_*+\varepsilon}$ for $\varepsilon \ll 1$, so that $\lambda_1(B_r) \geq \lambda_{\nu_*+\varepsilon}/r^2$ for $r \gg 1$. By Theorem 1.1 (1), we conclude that $|B_r| \gtrsim r^{\nu_*+\varepsilon}$. But this implies $\nu_* \geq \nu_* + \varepsilon$, which is impossible.

Since $j_\nu \sim 2\sqrt{\nu+1}$ as $\nu \to -1+$ (cf. Piessens [25]), we have

$$\lambda_{\nu_*} = j_{\nu_*/2-1}^2 \sim 2\nu_*, \quad \nu_* \to 0+,$$

from which assertions (1) and (2) immediately follow. On the other hand, since $j_\nu \sim \nu$ as $\nu \to +\infty$ (cf. Watson [27, pp. 521], see also Elbert [11, Section 1.4]), we have

$$\lambda_{\nu_*} = j_{\nu_*/2-1}^2 \sim \nu_*^2/4, \quad \nu_* \to +\infty,$$

which implies (3). $\blacksquare$

2.3. The Hardy-type inequalities

Recall that the capacity $\mathrm{cap}(K)$ of a compact set $K \subset M$ is given by

$$\mathrm{cap}(K) := \inf \int_M |\nabla\psi|^2 dV,$$

where the infimum is taken over all $\psi \in \mathrm{Lip}_0(M)$ with $0 \leq \psi \leq 1$ and $\psi|_K = 1$. The following criterion is of fundamental importance (cf. Grigor'yan [14, Theorem 5.1], and Ancona [2, pp. 46–47], see also Carron [6, Definition 2.13]).

Theorem 2.2. *Let (M, g) be a complete Riemannian manifold. Then the following properties are equivalent:*

(1) *M is non-parabolic;*

(2) $\operatorname{cap}(K) > 0$ *for some/any compact set $K \subset M$ with non-empty interior;*

(3) *given some/any pen subset $U \subset\subset M$, there exists a constant $C = C(U)$ such that*

$$\int_U u^2 dV \leq C(U) \int_M |\nabla u|^2 dV$$

for any $u \in \operatorname{Lip}(M)$ with a compact support.

By Theorem 2.2, Theorem 1.1 (2) is a direct consequence of the Hardy-type inequality (1.5). We shall take a unified approach to proving the Hardy inequalities (1.5) and (1.6). To begin with, set

$$\Phi(s) = J_{\mu/2-1}(\sqrt{\lambda_\mu} s).$$

The function Φ satisfies $\Phi(0) = \Phi(1) = 0$ and

$$\Phi''(s) + \frac{1}{s}\Phi'(s) + \Big(\lambda_\mu - \frac{\big(\frac{\mu}{2} - 1\big)^2}{s^2}\Big)\Phi(s) = 0. \tag{2.7}$$

We shall make use the following property of Φ.

Lemma 2.3. *For any $s \in [0, 1]$, we have*

$$\int_0^x \Phi'(s)^2 s ds = \lambda_\mu \int_0^x \Phi(s)^2 s ds - \Big(\frac{\mu-2}{2}\Big)^2 \int_0^x \Phi(s)^2 \frac{ds}{s} + \Phi'(x)\Phi(x)x. \tag{2.8}$$

In particular,

$$\int_0^1 \Phi'(s)^2 s ds = \lambda_\mu \int_0^1 \Phi(s)^2 s ds - \Big(\frac{\mu-2}{2}\Big)^2 \int_0^1 \Phi(s)^2 \frac{ds}{s}. \tag{2.9}$$

Proof. By (2.7), we have

$$\begin{aligned} \lambda_\mu \Phi(s)^2 s - \nu^2 \Phi(s)^2 \frac{1}{s} &= \Phi''(s)\Phi(s)s + \Phi'(s)\Phi(s) \\ &= (\Phi'(s)\Phi(s))' - \Phi'(s)^2 s, \end{aligned}$$

from which (2.8) and (2.9) follow immediately. ∎

Proof of Theorem 1.1 (3). As in the proof of (1), we first consider the case $\rho > 0$. Given $u \in \mathrm{Lip}_0(M)$, define

$$\phi(x) := u(x)\Phi\Big(\frac{\rho(x)}{r}\Big), \quad x \in M.$$

Since $|\nabla\rho| \le 1$, we have

$$|\nabla\phi|^2 \le \Phi\Big(\frac{\rho}{r}\Big)^2|\nabla u|^2 + \frac{1}{r^2}u^2\Phi'\Big(\frac{\rho}{r}\Big)^2 + 2\frac{u}{r}\langle\nabla u, \nabla\rho\rangle\Phi'\Big(\frac{\rho}{r}\Big)\Phi\Big(\frac{\rho}{r}\Big).$$

The variational characterization of eigenvalue gives

$$\begin{aligned}\frac{\lambda_\mu}{r^2}\int_{B_r}\phi^2 dV &\le \lambda_1(B_r)\int_{B_r}\phi^2 dV \le \int_{B_r}|\nabla\phi|^2 dV \\ &\le \int_{B_r}|\nabla u|^2\Phi\Big(\frac{\rho}{r}\Big)^2 dV + \frac{1}{r^2}\int_{B_r}u^2\Phi'\Big(\frac{\rho}{r}\Big)^2 dV \\ &\quad + 2\int_{B_r}\frac{u}{r}\langle\nabla u, \nabla\rho\rangle\Phi'\Big(\frac{\rho}{r}\Big)\Phi\Big(\frac{\rho}{r}\Big)dV \quad \text{for all } r > r_0. \end{aligned} \tag{2.10}$$

We then divide (2.10) by r and integrate for $r \in (r_0, +\infty)$. (Under the condition of (3) we set $r_0 = \inf_M \rho > 0$.) First,

$$\begin{aligned}&\int_{r_0}^{+\infty}\int_{B_r}u(x)^2\Phi\Big(\frac{\rho(x)}{r}\Big)^2 dV(x)\frac{dr}{r^3} \\ &= \int_M\Bigg(\int_{\max\{r_0,\rho(x)\}}^{+\infty}\Phi\Big(\frac{\rho(x)}{r}\Big)^2\frac{dr}{r^3}\Bigg)u(x)^2 dV(x) \\ &= \int_M\Bigg(\int_0^{\min\{1,\rho(x)/r_0\}}\Phi(s)^2 s\,ds\Bigg)\frac{u(x)^2}{\rho(x)^2}dV(x) \ge \Bigg(\int_0^1\Phi(s)^2 s\,ds\Bigg)\int_{M\setminus B_{r_0}}\frac{u^2}{\rho^2}dV,\end{aligned}$$

where we used the new variable $s = \rho(x)/r$ in the second step. Analogously, we have

$$\begin{aligned}&\int_{r_0}^{+\infty}\int_{B_r}|\nabla u(x)|^2\Phi\Big(\frac{\rho(x)}{r}\Big)^2 dV(x)\frac{dr}{r} \\ &= \int_M\Bigg(\int_{\max\{r_0,\rho(x)\}}^{+\infty}\Phi^2\Big(\frac{\rho(x)}{r}\Big)\frac{dr}{r}\Bigg)|\nabla u(x)|^2 dV(x) \\ &= \int_M\Bigg(\int_0^{\min\{1,\rho(x)/r_0\}}\Phi^2(s)\frac{ds}{s}\Bigg)|\nabla u(x)|^2 dV(x) \le \Bigg(\int_0^1\Phi^2(s)\frac{ds}{s}\Bigg)\int_M|\nabla u|^2 dV\end{aligned}$$

and

$$\begin{aligned}
&\int_{r_0}^{+\infty}\int_{B_r} u(x)^2\Phi'\Big(\frac{\rho(x)}{r}\Big)^2 dV(x)\frac{dr}{r^3}\\
&=\int_M\Big(\int_{\max\{r_0,\rho(x)\}}^{+\infty}\Phi'\Big(\frac{\rho(x)}{r}\Big)^2\frac{dr}{r^3}\Big)u(x)^2dV(x)\\
&=\int_M\Big(\int_0^{\min\{1,\rho(x)/r_0\}}\Phi'(s)^2sds\Big)\frac{u^2(x)}{\rho(x)^2}dV(x)\\
&\le\Big(\int_0^1\Phi'(s)^2sds\Big)\int_{M\setminus B_{r_0}}\frac{u^2}{\rho^2}dV+\int_{B_{r_0}}\Big(\int_0^{\rho(x)/r_0}\Phi'(s)^2sds\Big)\frac{u^2}{\rho^2}dV\\
&=:\Big(\int_0^1\Phi'(s)^2sds\Big)\int_{M\setminus B_{r_0}}\frac{u^2}{\rho^2}dV+I_1(r_0),
\end{aligned}$$

where

$$I_1(r_0)\le\frac{\|\Phi'\|^2_{L^\infty([0,1])}}{2r_0^2}\int_{B_{r_0}}u^2dV\quad\text{for all } r_0\ge\inf_M\rho, \tag{2.11}$$

so that $I_1(r_0)=0$ if $r_0=\inf_M\rho$. Eventually,

$$\begin{aligned}
&\int_{r_0}^{+\infty}\int_{B_r}\frac{u}{r}\langle\nabla u,\nabla\rho\rangle\Phi'\Big(\frac{\rho}{r}\Big)\Phi\Big(\frac{\rho}{r}\Big)dV\frac{dr}{r}\\
&=\int_M\Big(\int_{\max\{r_0,\rho(x)\}}^{+\infty}\Phi'\Big(\frac{\rho(x)}{r}\Big)\Phi\Big(\frac{\rho(x)}{r}\Big)\frac{dr}{r^2}\Big)u(x)\langle\nabla u(x),\nabla\rho(x)\rangle dV(x)\\
&=\int_M\Big(\int_0^{\min\{1,\rho(x)/r_0\}}\Phi'(s)\Phi(s)ds\Big)\frac{u(x)}{\rho(x)}\langle\nabla u(x),\nabla\rho(x)\rangle dV(x)\\
&=\int_{B_{r_0}}\Phi\Big(\frac{\rho}{r_0}\Big)^2\frac{u}{\rho}\langle\nabla u,\nabla\rho\rangle dV=:I_2(r_0),
\end{aligned}$$

where we have used the fact

$$2\int_0^1\Phi'(s)\Phi(s)ds=\Phi^2(1)-\Phi^2(0)=0.$$

We also have $I_2(r_0) = 0$ if $r_0 = \inf_M \rho$. Moreover, since $\Phi(0) = 0$, we get

$$\Phi^2(s) \leq \|\Phi'\|^2_{L^\infty([0,1])} s^2$$

for all $s \geq 0$, so that,

$$I_2(r_0) \leq \frac{\|\Phi'\|^2_{L^\infty([0,1])}}{r_0} \int_{B_{r_0}} |\nabla u||u| dV \quad \text{for all } r_0 \geq \inf_M \rho \tag{2.12}$$

These together with (2.10) yield

$$\begin{aligned}
&\lambda_\mu \Big(\int_0^1 \Phi(s)^2 s ds \Big) \int_{M \setminus B_{r_0}} \frac{u^2}{\rho^2} dV \\
&\leq \Big(\int_0^1 \Phi^2(s) \frac{ds}{s} \Big) \int_M |\nabla u|^2 dV + \Big(\int_0^1 \Phi'(s)^2 s ds \Big) \int_{M \setminus B_{r_0}} \frac{u^2}{\rho^2} dV \\
&\quad + I_1(r_0) + I_2(r_0).
\end{aligned}$$

Thus,

$$\Big(\frac{\mu - 2}{2}\Big)^2 \int_{M \setminus B_{r_0}} \frac{u^2}{\rho^2} dV \leq \int_M |\nabla u|^2 dV + \frac{I_1(r_0) + I_2(r_0)}{A} \tag{2.13}$$

in view of (2.9), where $A = \int_0^1 \Phi^2(s) ds/s$. If (1.3) holds for all $r > 0$, we may take $r_0 = \inf_M \rho$, so that (1.6) follows immediately from (2.13).

Under the condition of (2), we infer from (2.11)–(2.13) that

$$\Big(\frac{\mu - 2}{2}\Big)^2 \int_M \frac{u^2}{1 + \rho^2} dV \leq (1 + \delta) \int_M |\nabla u|^2 dV + C_\delta \int_{B_{r_0}} u^2 dV \tag{2.14}$$

holds for any $u \in \mathrm{Lip}_{\mathrm{loc}}(M)$ with a compact support and $\delta > 0$, where

$$C_\delta = \Big(\frac{\mu - 2}{2}\Big)^2 + \frac{\|\Phi'\|^2_{L^\infty([0,1])}}{A r_0^2} + \frac{\|\Phi'\|^4_{L^\infty([0,1])}}{A^2 r_0^2 \delta}. \tag{2.15}$$

Note that C_δ only depends on μ and r_0 if δ is fixed.

The above proofs of inequalities (1.6) and (2.14) require an additional condition $\rho > 0$. In general, if $\inf_M \rho = 0$, we consider $\rho_\varepsilon := \sqrt{\rho^2 + \varepsilon^2}$. Analogously to the proof of Theorem 1.1 (1), we have (2.6), so that (2.14) becomes

$$\Big(\frac{\mu - 2}{2}\Big)^2 \int_M \frac{u^2}{1 + \varepsilon^2 + \rho^2} dV \leq (1 + \delta) \int_M |\nabla u|^2 dV + C_{\delta,\varepsilon} \int_{B_{r_0^\varepsilon}} u^2 dV,$$

where $C_{\delta,\varepsilon}$ is given by (2.15) with r_0^2 replaced by $r_0^2+\varepsilon^2$. Thus, (2.14) follows by letting $\varepsilon\to 0$. Moreover, if (1.3) holds for all $r>0$, then $\lambda_1(B_r^\varepsilon)\ge \lambda_\mu/r^2$ when $r>\varepsilon$. Take $r_0=\varepsilon=\inf_M \rho_\varepsilon$ in (2.13) with ρ replaced by ρ_ε; we have

$$\left(\frac{\mu-2}{2}\right)^2\int_M \frac{u^2}{\rho_\varepsilon^2}dV\le \int_M |\nabla u|^2 dV.$$

We obtain (1.6) immediately by letting $\varepsilon\to 0$, which completes the proof of (3).

To prove (2), we shall first derive the non-parabolicity of M from (2.14) when $\mu>2$. By Theorem 1.1 (1), there exist some constants $c>0$ and $\mu>2$ such that $|B_r|\ge cr^\mu$ for $r\gg 1$. Thus,

$$\int_M \frac{dV}{1+\rho^2}\ge \limsup_{r\to+\infty}\frac{|B_r|}{1+r^2}=+\infty.$$

Choose $\bar r\gg r_0$ so that

$$\left(\frac{\mu-2}{2}\right)^2\int_{B_{\bar r}}\frac{dV}{1+\rho^2}\ge C_\delta|B_{r_0}|+1+\delta.$$

Then

$$\int_M |\nabla u|^2 dV\ge 1$$

whenever $u\in \mathrm{Lip}_0(M)$ and $u=1$ on $\overline{B}_{\bar r}$. Thus, $\mathrm{cap}(\overline{B}_{\bar r})\ge 1$ and M is non-parabolic in view of Theorem 2.2 (2).

Finally, it follows from Theorem 2.2 (3) that

$$\int_{B_{r_0}} u^2 dV\le C(B_{r_0})\int_M |\nabla u|^2 dV$$

for any $u\in \mathrm{Lip}_0(M)$. This, together with (2.14), gives (1.5) with

$$C=\left(\frac{\mu-2}{2}\right)^2\frac{1}{1+\delta+C_\delta C(B_{r_0})}.$$

If we fix some $\delta>0$, then C only depends on μ, r_0 and the geometry of M. ■

3. Proof of Proposition 1.4

By definition, there exists a sequence $\{r_k\}$ with $\lim_{k\to+\infty} r_k=+\infty$, such that

$$\lambda_1(B_{r_k})>e^{-(\widetilde{\Lambda}_*+\varepsilon)r_k}$$

for some $0 < \varepsilon \ll 1$. Again, for $k \geq 1$ and $0 < \delta < 1$, we take a cut-off function $\phi_k : M \to [0, 1]$ such that $\phi_k|_{B_{\delta r_k}} = 1$, $\phi_k|_{M \setminus B_{r_k}} = 0$ and $|\nabla \phi_k| \leq (1-\delta)^{-1} r_k^{-1}$. Then

$$e^{-(\widetilde{\Lambda}_*+\varepsilon)r_k} |B_{\delta r_k}| \leq \lambda_1(B_{r_k}) \int_M \phi_k^2 dV \leq \int_M |\nabla \phi_k|^2 dV$$
$$\leq \frac{1}{(1-\delta)^2 r_k^2} |B_{r_k} \setminus B_{\delta r_k}|$$
$$\leq \frac{1}{(1-\delta)^2 r_k^2} |M \setminus B_{\delta r_k}|.$$

That is,

$$|M| \geq (1 + e^{-(\widetilde{\Lambda}_*+\varepsilon)r_k}(1-\delta)^2 r_k^2)|B_{\delta r_k}|,$$

which is equivalent to

$$|M \setminus B_{\delta r_k}| \geq \frac{e^{-(\widetilde{\Lambda}_*+\varepsilon)r_k}(1-\delta)^2 r_k^2}{1 + e^{-(\widetilde{\Lambda}_*+\varepsilon)r_k}(1-\delta)^2 r_k^2} |M|.$$

Thus,

$$\alpha_* \leq \lim_{k\to\infty} \frac{-\log |M \setminus B_{\delta r_k}|}{\delta r_k} \leq \frac{\widetilde{\Lambda}_* + \varepsilon}{\delta}.$$

Letting $\delta \to 1-$ and $\varepsilon \to 0+$, we conclude that $\widetilde{\Lambda}_* \geq \alpha_*$.

4. Proof of Proposition 1.5

Let $w \in \mathrm{Lip}_{\mathrm{loc}}(M)$ and $v \in L^1_{\mathrm{loc}}(M)$; by $\Delta w \geq v$ in the sense of distributions (or weakly) for some locally integrable function v, we mean

$$\int_M \langle \nabla w, \nabla \varphi \rangle dV \leq -\int_M v\varphi dV \tag{4.1}$$

for any nonnegative function $\varphi \in C_0^\infty(M)$. Since $C_0^\infty(M)$ is dense in $\mathrm{Lip}_0(M)$ with respect to the Sobolev norm $\|\cdot\|_{W^{1,2}}$, we see that $\Delta w \geq v$ weakly if and only if (4.1) holds for all nonnegative $\varphi \in \mathrm{Lip}_0(M)$. One can define $\Delta w \leq v$ in a similar way.

We first prove the following technical lemma.

Lemma 4.1. *Suppose that $\Delta w \geq v$ weakly and $w > 0$. Let $f : (0, +\infty) \to (0, +\infty)$ be a smooth function. Then the following properties hold:*

(1) *if f is increasing, then $\Delta f(w) \geq f'(w)v + f''(w)|\nabla w|^2$ weakly;*

(2) *if f is decreasing, then $\Delta f(w) \leq f'(w)v + f''(w)|\nabla w|^2$ weakly.*

Proof. (1) Let $\varphi \in \mathrm{Lip}_0(M)$ and $\varphi \geq 0$. If f is increasing, then we also have $f'(w)\varphi \in \mathrm{Lip}_0(M)$ and $f'(w)\varphi \geq 0$. Thus,

$$\begin{aligned}\int_M \langle \nabla f(w), \nabla\varphi\rangle dV &= \int_M \langle f'(w)\nabla w, \nabla\varphi\rangle dV \\ &= \int_M \langle \nabla w, \nabla(f'(w)\varphi)\rangle dV - \int_M f''(w)|\nabla w|^2\varphi dV \\ &\leq -\int_M (f'(w)v + f''(w)|\nabla w|^2)\varphi dV,\end{aligned}$$

which proves the first assertion.

(2) Analogously, we have

$$\begin{aligned}\int_M \langle \nabla f(w), \nabla\varphi\rangle dV &= \int_M \langle f'(w)\nabla w, \nabla\varphi\rangle dV \\ &= -\int_M \langle \nabla w, \nabla(-f'(w)\varphi)\rangle dV - \int_M f''(w)|\nabla w|^2\varphi dV \\ &\geq -\int_M (f'(w)v + f''(w)|\nabla w|^2)\varphi dV.\end{aligned}$$

■

Proof of Proposition 1.5. We follow the ideas in [5]. Assume for a moment that $\rho > 0$. Apply Lemma 4.1 (1) with $w = \rho^2$ and $f(t) = t^{1/2}$; we get

$$\Delta\rho \geq \frac{\mu - |\nabla\rho|^2}{\rho} \tag{4.2}$$

weakly.

Let ϕ be a Lipschitz continuous function which is positive on B_r and fix $u \in \mathrm{Lip}_0(B_r)$. With $v := u/\phi$, we have

$$\begin{aligned}\int_M |\nabla u|^2 dV &= \int_M \phi^2|\nabla v|^2 dV + \int_M v^2|\nabla\phi|^2 dV + 2\int_M \phi v\langle\nabla\phi, \nabla v\rangle dV \\ &= \int_M \phi^2|\nabla v|^2 dV + \int_M \langle\nabla(\phi v^2), \nabla\phi\rangle dV.\end{aligned} \tag{4.3}$$

Let us choose $\phi = \psi_\mu(\rho/r)$, where

$$\psi_\mu(s) = s^{1-\mu/2} J_{\mu/2-1}(\sqrt{\lambda_\mu} s).$$

Recall that ψ_μ is a solution of the ODE (2.1) and $\psi'_\mu \le 0$. Thus, (4.2) and (2.1) together with Lemma 4.1 (2) yield

$$\begin{aligned}\Delta\phi &\le \psi'_\mu\Big(\frac{\rho}{r}\Big)\frac{\mu-|\nabla\rho|^2}{\rho r}+\psi''_\mu\Big(\frac{\rho}{r}\Big)\frac{|\nabla\rho|^2}{r^2}\\ &=\psi'_\mu\Big(\frac{\rho}{r}\Big)\frac{\mu-1}{\rho r}+\psi''_\mu\Big(\frac{\rho}{r}\Big)\frac{1}{r^2}+\psi'_\mu\Big(\frac{\rho}{r}\Big)\frac{1-|\nabla\rho|^2}{\rho r}-\psi''_\mu\Big(\frac{\rho}{r}\Big)\frac{1-|\nabla\rho|^2}{r^2}\\ &=-\frac{\lambda_\mu}{r^2}\phi+\frac{1-|\nabla\rho|^2}{r^2}\Big(\frac{\psi'_\mu(\frac{\rho}{r})}{\rho/r}-\psi''_\mu\Big(\frac{\rho}{r}\Big)\Big).\end{aligned} \tag{4.4}$$

We have (cf. Lebedev [19, formula (5.3.5)])

$$\psi'_\mu(s)=-s\psi_{\mu+2}(s),$$

from which it follows that

$$\frac{\psi'_\mu(s)}{s}-\psi''_\mu(s)=-\psi_{\mu+2}(s)-(-s\psi_{\mu+2}(s))'=-s^2\psi_{\mu+4}(s)\le 0.$$

Then (4.4) implies

$$\Delta\phi\le-\frac{\lambda_\mu}{r^2}\phi,$$

which gives

$$\int_M\langle\nabla(\phi v^2),\nabla\phi\rangle dV\ge-\int_M\phi v^2\Big(-\frac{\lambda_\mu}{r^2}\phi\Big)dV=\frac{\lambda_\mu}{r^2}\int_M u^2dV.$$

This, together with (4.3), yields

$$\frac{\lambda_\mu}{r^2}\int_{B_r}u^2dV\le\int_{B_r}|\nabla u|^2dV\quad\text{for all } u\in \mathrm{Lip}_{\mathrm{loc}}(B_r) \tag{4.5}$$

under the additional condition $\rho>0$.

In general, we use $\rho_\varepsilon := (\rho^2+\varepsilon^2)^{1/2}$ instead of ρ. Note that $|\nabla\rho_\varepsilon|\le 1$ and

$$\Delta\rho_\varepsilon^2=\Delta\rho^2\ge 2\mu.$$

Thus, (4.5) holds if B_r is replaced by B_r^ε in view of (2.6). The proposition follows by letting $\varepsilon\to 0$. ∎

5. Examples

Let $M=\mathbb{R}\times\mathbb{S}^1$ be equipped with the following Riemannian metric:

$$g=dt^2+\eta'(t)^2d\theta^2,\quad t\in\mathbb{R},\ e^{i\theta}\in\mathbb{S}^1,$$

where $\eta\colon \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\eta'(t) > 0$ and $\lim_{t\to-\infty} \eta(t) = 0$. Dodziuk, Pigmataro, Randol, and Sullivan [10, Proposition 3.1] showed that if $\eta(t) = e^t$, then $\lambda_1(M) \geq 1/4$, so that (M, g) is non-parabolic.

In general, we consider

$$h(t) := \int_0^t \frac{ds}{\eta'(s)}, \quad t \in \mathbb{R},$$

which gives a function on M. A straightforward calculation shows

$$\Delta h = \frac{\partial^2 h}{\partial t^2} + \frac{\eta''(t)}{\eta'(t)}\frac{\partial h}{\partial t} + \frac{1}{\eta'(t)^2}\frac{\partial^2 h}{\partial \theta^2} = 0,$$

i.e., h is a harmonic function on M. Moreover, in the new coordinate system $(\tilde{t}, \theta) := (h(t), \theta)$, we may write

$$g = \eta'(t)^2(h'(t)^2 dt^2 + d\theta^2) = \eta'(t)^2(d\tilde{t}^2 + d\theta^2),$$

which indicates that (M, g) is conformally equivalent to the cylinder $(\inf h, \sup h) \times \mathbb{S}^1$ equipped with a flat metric. In conclusion, we have the following result.

Proposition 5.1. *(M, g) is non-parabolic if and only if $\inf h > -\infty$ or $\sup h < +\infty$.*

Let $\rho(t, \theta) = |t|$. Clearly, ρ is an exhaustion function which satisfies $|\nabla \rho|_g \leq 1$. Indeed, ρ is the geodesic distance to the circle $\{0\} \times \mathbb{S}^1$. The goal of this section is to investigate the asymptotic behaviour of $\lambda_1(B_r)$ as $r \to +\infty$ for different choices of η. We start with the following elementary lower estimate.

Proposition 5.2. *We have*

$$\lambda_1(B_r) \geq \frac{1}{4} \inf_{|t|\leq r} \frac{\eta'(t)^2}{\eta(t)^2}.$$

Proof. The idea is essentially implicit in [10]. Since $dV = \eta'(t) dt d\theta$, we have

$$\int_{-r}^{r} \phi^2 \eta'(t) dt = -2 \int_{-r}^{r} \phi \frac{\partial \phi}{\partial t} \eta(t) dt, \quad \text{for all } \phi \in C_0^\infty(B_r),$$

so that

$$\int_{-r}^{r} \phi^2 \eta'(t) dt \leq 4 \int_{-r}^{r} \left|\frac{\partial \phi}{\partial t}\right|^2 \frac{\eta(t)^2}{\eta'(t)} dt \leq 4 \int_{-r}^{r} |\nabla \phi|^2 \frac{\eta(t)^2}{\eta'(t)} dt$$

$$\leq 4 \sup_{|t|\leq r} \frac{\eta(t)^2}{\eta'(t)^2} \int_{-r}^{r} |\nabla \phi|^2 \eta'(t) dt$$

in view of the Cauchy–Schwarz inequality. Thus,

$$\int_{B_r} \phi^2 dV = \int_0^{2\pi}\int_{-r}^{r} \phi^2 \eta'(t)dtd\theta \le 4 \sup_{|t|\le r} \frac{\eta(t)^2}{\eta'(t)^2} \int_{B_r} |\nabla\phi|^2 dV,$$

from which the assertion immediately follows. ∎

We give the following test example for Proposition 5.1 and 5.2.

Example 5.3. Given $\alpha > 0$, take η such that

$$\eta(t) = \begin{cases} (-t)^{-\alpha}, & t < -1, \\ 2t^{\alpha}, & t > 1. \end{cases}$$

We claim that $\Lambda_* \sim \alpha^2/4$ as $\alpha \to +\infty$. To see this, first note that

$$\Lambda_* = \liminf_{r\to+\infty} \{r^2\lambda_1(B_r)\} \ge \frac{\alpha^2}{4}.$$

in view of Proposition 5.2. Then, by using

$$|B_r| = 2\pi \int_{-r}^{r} \eta'(t)dt = 2\pi(\eta(r) - \eta(-r)) = 4\pi r^{\alpha} - 2\pi r^{-\alpha} \quad \text{for all } r \gg 1,$$

we see that

$$\nu_* = \liminf_{r\to+\infty} \frac{\log|B_r|}{\log r} = \alpha.$$

This, together with Corollary 1.3, gives

$$\frac{\alpha^2}{4} \le \Lambda_* \le \lambda_\alpha = j^2_{\alpha/2-1} \sim \frac{\alpha^2}{4},$$

which verifies the claim.

In particular, M is non-parabolic provided $\alpha \gg 1$, in view of Corollary 1.2. More precisely, Proposition 5.1 implies that M is non-parabolic if and only if $\alpha > 2$.

The next example shows that estimate (1.7) in Proposition 1.4 is sharp.

Example 5.4. Given $\alpha > 0$, take η such that

$$\eta'(t) = e^{-\alpha|t|} \quad \text{for all } |t| > 1. \tag{5.1}$$

Then, we have

$$\liminf_{r\to+\infty} \frac{-\log\lambda_1(B_r)}{r} = \alpha = \liminf_{r\to+\infty} \frac{-\log|M\setminus B_r|}{r}. \tag{5.2}$$

Namely, (1.7) is sharp.

Indeed, since

$$|M \setminus B_r| = 4\pi \int_r^{\infty} e^{-\alpha t} dt \asymp e^{-\alpha r}, \quad r \gg 1,$$

we have

$$\liminf_{r\to+\infty} \frac{-\log |M \setminus B_r|}{r} = \alpha, \tag{5.3}$$

which implies

$$\liminf_{r\to+\infty} \frac{-\log \lambda_1(B_r)}{r} \geq \alpha, \tag{5.4}$$

in view of Proposition 1.4. On the other hand, we have the following Hardy-type inequality (cf. Opic and Kufner [24, pp. 100–103]):

$$\int_{-r}^{r} \phi(t)^2 \eta'(t) dt \lesssim e^{\alpha r} \int_{-r}^{r} \phi'(t)^2 \eta'(t) dt \quad \text{for all } \phi \in C_0^\infty((-r,r)), \tag{5.5}$$

where the implicit constant is independent of r. By (5.5), we immediately see that

$$\lambda_1(B_r) \gtrsim e^{-\alpha r},$$

which combined with (5.3) and (5.4) gives (5.2).

For reader's convenience, we include here a rather short proof. Since $\int_{-\infty}^{+\infty} \eta'(t)dt$ is finite in view of (5.1), it follows that

$$\int_{-r}^{r} \phi(t)^2 \eta'(t) dt \leq \sup_{-r<t<r} \phi(t)^2 \int_{-r}^{r} \eta'(t) dt \lesssim \sup_{-r<t<r} \phi(t)^2.$$

On the other hand, by setting $|\phi(t_0)| = \sup_{-r<t<r} |\phi(t)|$, we have

$$\int_{-r}^{r} |\phi'(t)| dt \geq \int_{-r}^{t_0} |\phi'(t)| dt \geq \left| \int_{-r}^{t_0} \phi'(t) dt \right| = |\phi(t_0)| = \sup_{-r<t<r} |\phi(t)|.$$

This, together with the Cauchy–Schwarz inequality, yields

$$\sup_{-r<t<r} \phi(t)^2 \leq \left(\int_{-r}^{r} \phi'(t)^2 \eta'(t) dt \right) \left(\int_{-r}^{r} \frac{1}{\eta'(t)} dt \right) \lesssim e^{\alpha r} \int_{-r}^{r} \phi'(t)^2 \eta'(t) dt.$$

Remark. By Proposition 5.2, we only obtain a weaker conclusion

$$\lambda_1(B_r) \geq \frac{1}{4} \frac{\eta'(r)^2}{\eta(r)^2} \gtrsim e^{-2\alpha r}.$$

The following example shows that the converse of Theorem 1.1 (2) (or Corollary 1.2) does not hold, i.e., M can be non-parabolic with $\lambda_1(B_r)$ decaying so quickly that $\Lambda_* = 0$.

Example 5.5. Take η such that

$$\eta'(t) = \begin{cases} n^2, & 2^n < t < 2^n + 1, \\ e^t, & 2^n + 2 < t < 2^{n+1} - 1, \end{cases}$$

and

$$\eta'(t) \geq n^2, \quad 2^n - 1 \leq t \leq 2^n + 2$$

for $n = 2, 3, \ldots$. Then M is non-parabolic but $\lambda_1(B_r)$ shall decay exponentially, so that $\Lambda_* = 0$.

Actually, the non-parabolicity of M is a direct consequence of Proposition 5.1, for $\sup h = \int_0^{+\infty} \frac{dt}{\eta'(t)} < +\infty$. To study the behaviour of $\lambda_1(B_r)$, consider the test function,

$$\phi(t) := \begin{cases} t, & 0 \leq t \leq 1, \\ 1, & 1 \leq t \leq r - 1, \\ r - t, & r - 1 \leq t \leq r, \\ 0, & \text{otherwise}. \end{cases}$$

We obtain

$$\lambda_1(B_r) \leq \frac{\int_{-r}^{r} \phi'(t)^2 \eta'(t) dt}{\int_{-r}^{r} \phi(t)^2 \eta'(t) dt} \leq \frac{\eta(r) - \eta(r-1) + \eta(1) - \eta(0)}{\eta(r-1) - \eta(1)}. \tag{5.6}$$

For $r = 2^n + 1$, where $n \in \mathbb{Z}^+$ and $n \geq 2$, we have $\eta(r) - \eta(r-1) = n^2$ and

$$\eta(r-1) = \eta(2^n) \geq \int_{2^{n-1}+2}^{2^n - 1} e^t dt \asymp e^{2^n},$$

so that (5.6) gives

$$\lambda_1(B_{2^n+1}) \lesssim e^{-2^n} n^2.$$

In general, for $r \gg 1$, take $n \in \mathbb{Z}^+$ such that $2^n + 1 \leq r < 2^{n+1} + 1$. Then

$$\lambda_1(B_r) \leq \lambda_1(B_{2^n+1}) \lesssim e^{-2^n} n^2 \lesssim e^{-r/2} (\log r)^2$$

and hence $\Lambda_* = 0$.

Remark. We have

$$\Lambda^* := \limsup_{r \to +\infty} r^2 \lambda_1(B_r) = 0$$

and

$$\widetilde{\Lambda}_* = \liminf_{r\to+\infty} \frac{-\log \lambda_1(B_r)}{r} \geq \frac{1}{2} > 0.$$

On the other hand, M is non-parabolic with an infinite volume.

Our last example shows that $\lambda_1(B_r)$ can also decay to 0 very slowly.

Example 5.6. Let μ be a positive, smooth and decreasing function on $[1, +\infty)$ satisfying

(1) $\lim_{t\to+\infty} \mu(t) = 0$,

(2) $\int_1^{+\infty} \mu(s)ds = +\infty$,

(3) $t\mu(t)$ is increasing on $[c, +\infty)$ for some $c \gg 1$.

Take η such that

$$\eta(t) = \begin{cases} e^{-\int_1^{-t} \mu(s)ds}, & t < -1, \\ 2e^{\int_1^{t} \mu(s)ds}, & t > 1. \end{cases}$$

We claim that

$$\lambda_1(B_r) \asymp \mu(r)^2. \tag{5.7}$$

To see this, first note that $\eta'(t)/\eta(t) = \mu(-t)$ for $t < -1$ and $\eta'(t)/\eta(t) = \mu(t)$ for $t > 1$, which implies

$$\lambda_1(B_r) \geq \frac{1}{4} \inf_{|t|\leq r} \frac{\eta'(t)^2}{\eta(t)^2} = \frac{\mu(r)^2}{4}. \tag{5.8}$$

in view of Proposition 5.2. On the other hand, we have $r\mu(r) \geq c\mu(c) > 0$ for $r \geq c \gg 1$ in view of the condition (3). Thus, we may take $0 < \varepsilon \leq c\mu(c)/2$ so that

$$r_\varepsilon := r - \varepsilon\mu(r)^{-1} = r(1 - \varepsilon r^{-1}\mu(r)^{-1}) \geq \frac{r}{2} \quad \text{for all } r \geq c. \tag{5.9}$$

Set $I_r := (-r, -r_\varepsilon)$. Since $\eta''(t) = -\mu'(-t)\eta(t) + \mu(-t)\eta'(t) \geq 0$, i.e., $\eta'(t)$ is increasing on $(-\infty, -1]$, it follows that

$$\begin{aligned} \lambda_1(B_r) &\leq \lambda_1(\{(t,\theta) \in M : -r \leq t \leq -r_\varepsilon\}) \\ &\leq \inf_{\phi\in C_0^\infty(I_r)} \left\{ \frac{\int_{I_r} \phi'(t)^2 \eta'(t)dt}{\int_{I_r} \phi(t)^2 \eta'(t)dt} \right\} \\ &\leq \inf_{\phi\in C_0^\infty(I_r)} \left\{ \frac{\int_{I_r} \phi'(t)^2 dt}{\int_{I_r} \phi(t)^2 dt} \right\} \cdot \frac{\eta'(-r_\varepsilon)}{\eta'(-r)} \\ &= \lambda_1(I_r) \cdot \frac{\eta'(-r_\varepsilon)}{\eta'(-r)}. \end{aligned}$$

Since $\lambda_1(I_r) \lesssim |I_r|^{-2} \asymp \mu(r)^2$, we obtain

$$\lambda_1(B_r) \lesssim \mu(r)^2 \cdot \frac{\eta'(-r_\varepsilon)}{\eta'(-r)}. \tag{5.10}$$

We have

$$\frac{\eta'(-r_\varepsilon)}{\eta'(-r)} = \frac{\mu(r_\varepsilon)}{\mu(r)} \exp\Bigg(\int_{r_\varepsilon}^{r} \mu(s)ds\Bigg) \leq \frac{\mu(r_\varepsilon)}{\mu(r)} \exp\Big(\varepsilon \frac{\mu(r_\varepsilon)}{\mu(r)}\Big),$$

for μ is decreasing and $r - r_\varepsilon = \varepsilon\mu(r)^{-1}$. By condition (3) and (5.9), we have

$$\frac{\mu(r_\varepsilon)}{\mu(r)} \leq \frac{r}{r_\varepsilon} \leq 2.$$

Thus,

$$\frac{\eta'(-r_\varepsilon)}{\eta'(-r)} = O(1) \quad \text{as } r \to +\infty.$$

This, together with (5.8) and (5.10), gives (5.7).

Particular choices of μ give the following:

(1) for $\mu(t) = t^{-1}(\log t)^\beta$ with $\beta > 0$, $\lambda_1(B_r) \asymp r^{-2}(\log r)^{2\beta}$;

(2) for $\mu(t) = t^{-\alpha}$ with $0 < \alpha < 1$, $\lambda_1(B_r) \asymp r^{-2\alpha}$;

(3) for $\mu(t) = (\log(t+1))^{-\gamma}$ with $\gamma > 0$, $\lambda_1(B_r) \asymp (\log r)^{-2\gamma}$.

In all three cases, we have

$$\Lambda_* = \liminf_{r\to+\infty}\{r^2\lambda_1(B_r)\} = +\infty.$$

Thus, these Riemannian manifolds (M, g) are non-parabolic in view of Corollary 1.2.

Acknowledgements. G. Carron thanks the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment.

Funding. B.-Y. Chen and Y. Xiong are supported by National Natural Science Foundation of China, No. 12271101.

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Received 11 July 2024; revised 5 December 2024.

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