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Corrigendum to "Pauli Hamiltonians with an Aharonov–Bohm flux"

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Abstract. We correct a mistake in [J. Spectr. Theory 14 (2024), 1147–1193] in the computation of the square of a generic self-adjoint realization of the Dirac operator with an Aharonov–Bohm flux. We prove that only two self-adjoint realizations of the Dirac operator square to self-adjoint realizations of the Pauli operator with Aharonov–Bohm potential.

In [1, Propositition 2.24], it is incorrectly stated that all self-adjoint realizations of the Dirac operator H_{D} square to the Friedrichs realization $H_{\mathrm{P}}^{(\mathrm{F})}$ of the Pauli operator, namely, $(H_{\mathrm{D}}^{\gamma})^2 = H_{\mathrm{P}}^{(\mathrm{F})}$ for all $\gamma \in [0, 2\pi)$. As a matter of fact, only for specific values of the parameter γ and for specific choices of $\beta = (\beta_{ss'}^{(\ell\ell')}) \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$, there actually holds $(H_{\mathrm{D}}^{\gamma})^2 = H_{\mathrm{P}}^{(\beta)}$. The amended version of [1, Proposition 2.24] is as follows.

Proposition 1. *The identity*

$$(H_{\mathrm{D}}^{\gamma})^2 = H_{\mathrm{P}}^{(\beta)}$$

holds if and only if one of the following two alternatives is realised:

(a) $\gamma = 0$, $\beta_{\uparrow\uparrow}^{(00)} = \beta_{\downarrow\downarrow}^{(00)} = \beta_{\downarrow\downarrow}^{(-1-1)} = \infty$, *and* $\beta_{\uparrow\uparrow}^{(-1-1)} = 0$ *(*$\beta_{ss'}^{(\ell\ell')}$ *arbitrarily chosen otherwise), i.e.,* $H_{\mathrm{P}}^{(\beta)}$ *coincides with the Krein extension in the* $(\uparrow, -1)$ *channel and with the Friedrichs extension in all the other channels;*

(b) $\gamma = \pi$, $\beta_{\uparrow\uparrow}^{(00)} = \beta_{\uparrow\uparrow}^{(-1-1)} = \beta_{\downarrow\downarrow}^{(-1-1)} = \infty$, *and* $\beta_{\downarrow\downarrow}^{(00)} = 0$ *(*$\beta_{ss'}^{(\ell\ell')}$ *arbitrarily chosen otherwise), i.e.,* $H_{\mathrm{P}}^{(\beta)}$ *coincides with the Krein extension in the* $(\downarrow, 0)$ *channel and with the Friedrichs extension in all the other channels.*

Remark 1. The Dirac extensions corresponding to $\gamma = 0$ and $\gamma = \pi$ are two distinguished ones. They are indeeed the only scale covariant realizations, i.e., homogeneous of degree -1 under scaling. This facts has implications at the dynamical level.

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Indeed, some dispersive estimates have been recently proved in [2], where it is also mentioned that for other extensions such estimates seem to fail.

Proof of Proposition 1. The proof of the thesis will be achieved in various steps. Without loss of generality, in what follows we fix $\lambda = 1$, omitting the dependence on the spectral parameter for simplicity of notation. We will repeatedly use the expansion as $r \to 0^+$ given in [1, equation (3.2)], alongside with the basic identities (see [1, equations (2.6), (2.7), and (2.33)])

$$\begin{cases} \boldsymbol{G}_{\uparrow}^{(-1)} = \dfrac{1}{\sqrt{2\pi}} \dfrac{\boldsymbol{\xi}_+ + \boldsymbol{\xi}_-}{2}, \\ \boldsymbol{G}_{\downarrow}^{(0)} = \dfrac{1}{\sqrt{2\pi}} \dfrac{\boldsymbol{\xi}_+ - \boldsymbol{\xi}_-}{2}, \end{cases} \iff \begin{cases} \boldsymbol{\xi}_+ = \sqrt{2\pi}(\boldsymbol{G}_{\uparrow}^{(-1)} + \boldsymbol{G}_{\downarrow}^{(0)}), \\ \boldsymbol{\xi}_- = \sqrt{2\pi}(\boldsymbol{G}_{\uparrow}^{(-1)} - \boldsymbol{G}_{\downarrow}^{(0)}). \end{cases} \tag{1}$$

Step 1. We show that $\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \subseteq \mathcal{D}(H_{\mathrm{D}}^{\gamma})$ if and only if one of the following alternatives holds:

(a.1) $\gamma = 0$, $\beta_{\uparrow\uparrow}^{(00)} = \beta_{\downarrow\downarrow}^{(00)} = \beta_{\downarrow\downarrow}^{(-1-1)} = \infty$ and $\beta_{ss'}^{(\ell\ell')}$ arbitrary otherwise;

(b.1) $\gamma = \pi$, $\beta_{\uparrow\uparrow}^{(00)} = \beta_{\uparrow\uparrow}^{(-1-1)} = \beta_{\downarrow\downarrow}^{(-1-1)} = \infty$ and $\beta_{ss'}^{(\ell\ell')}$ arbitrary otherwise;

(c.1) $\gamma \in [0, 2\pi)$ arbitrary, $\beta_{\uparrow\uparrow}^{(00)} = \beta_{\uparrow\uparrow}^{(-1-1)} = \beta_{\downarrow\downarrow}^{(00)} = \beta_{\downarrow\downarrow}^{(-1-1)} = \infty$ and $\beta_{ss'}^{(\ell\ell')}$ arbitrary otherwise.

Let us firstly notice that [1, equation (2.4)] entails $\boldsymbol{\phi}_\lambda(r, \theta) = o(1)$ as $r \to 0^+$, for all $\boldsymbol{\phi}_\lambda \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$. From here and from [1, equations (2.6), (2.7), and (3.2)] it follows that, for any $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{P}}^{(\beta)})$, there holds

$$\begin{aligned} \boldsymbol{\psi} &= \boldsymbol{\phi}_\lambda + \sum_{s,\ell} q_s^{(\ell)} \boldsymbol{G}_{\lambda,s}^{(\ell)} \\ &= \begin{pmatrix} q_{\uparrow}^{(0)} \frac{\Gamma(\alpha)}{2^{1-\alpha}} \frac{1}{\sqrt{2\pi}} \frac{1}{r^\alpha} + q_{\uparrow}^{(-1)} \frac{\Gamma(1-\alpha)}{2^\alpha} \frac{e^{-i\theta}}{\sqrt{2\pi}} \frac{1}{r^{1-\alpha}} \\ q_{\downarrow}^{(0)} \frac{\Gamma(\alpha)}{2^{1-\alpha}} \frac{1}{\sqrt{2\pi}} \frac{1}{r^\alpha} + q_{\downarrow}^{(-1)} \frac{\Gamma(1-\alpha)}{2^\alpha} \frac{e^{-i\theta}}{\sqrt{2\pi}} \frac{1}{r^{1-\alpha}} \end{pmatrix} + o(1), \qquad \text{as } r \to 0^+. \end{aligned}$$

On the other hand, assuming that $\psi \in \mathcal{D}(H_{\mathrm{D}}^{\gamma})$, by similar arguments and (1), we infer

$$\boldsymbol{\psi} = \boldsymbol{\phi} + \mu(\boldsymbol{\xi}_+ + e^{i\gamma}\boldsymbol{\xi}_-) = \begin{pmatrix} \mu(1 + e^{i\gamma}) \frac{\Gamma(1-\alpha)}{2^\alpha} \frac{e^{-i\theta}}{\sqrt{2\pi}} \frac{1}{r^{1-\alpha}} \\ \mu(1 - e^{i\gamma}) \frac{\Gamma(\alpha)}{2^{1-\alpha}} \frac{1}{\sqrt{2\pi}} \frac{1}{r^\alpha} \end{pmatrix} + o(1), \qquad \text{as } r \to 0^+.$$

Matching the above asymptotics, we get $q_{\uparrow}^{(0)} = 0$, $q_{\downarrow}^{(-1)} = 0$, $q_{\uparrow}^{(-1)} = \mu(1 + e^{i\gamma})$ and $q_{\downarrow}^{(0)} = \mu(1 - e^{i\gamma})$. In particular, $\mu(1 - e^{i\gamma})q_{\uparrow}^{(-1)} = \mu(1 + e^{i\gamma})q_{\downarrow}^{(0)}$. Given that $q_{\uparrow}^{(-1)}, q_{\downarrow}^{(0)} \in \mathbb{C}$ are independent parameters, it appears that the only admissible alternatives are the following:

(a.1) $\gamma = 0$, $q_{\uparrow}^{(0)} = q_{\downarrow}^{(-1)} = q_{\downarrow}^{(0)} = 0$ and $q_{\uparrow}^{(-1)} \in \mathbb{C}$ arbitrary;

(b.1) $\gamma = \pi$, $q_{\uparrow}^{(0)} = q_{\downarrow}^{(-1)} = q_{\uparrow}^{(-1)} = 0$ and $q_{\downarrow}^{(0)} \in \mathbb{C}$ arbitrary;

(c.1) $\mu = 0$ and $q_{\uparrow}^{(0)} = q_{\downarrow}^{(-1)} = q_{\uparrow}^{(-1)} = q_{\downarrow}^{(0)} = 0$.

It is easy to check that these are indeed equivalent to the conditions (a.1), (b.1), and (c.1), respectively.

Step 2. Let us now prove that $H_{\mathrm{D}}^{\gamma}(\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})) = H_{\mathrm{D}}(\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})) \not\subseteq \mathcal{D}(H_{D}^{\gamma'})$ for all $\gamma, \gamma' \in [0, 2\pi)$. We first remark that $\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$ consists of functions which are indeed regular at the origin. So, any self-ajoint realization H_{D}^{γ} of the Dirac operator acts on $\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$ as the basic differential operator H_{D}, which accounts for the first identity in the claim. Taking this into account, we now proceed to show that $H_{\mathrm{D}}\boldsymbol{\phi} \notin \mathcal{D}(H_{D}^{\gamma'})$ for any given $\boldsymbol{\phi} \in \mathcal{D}(H_{\mathrm{P}}^{(F)})$ and for any $\gamma' \in [0, 2\pi)$, ultimately proving the thesis.

Recalling that $H_{\mathrm{P}}^{(F)} = H_{\mathrm{S}}^{(F)} \oplus H_{\mathrm{S}}^{(F)}$, where $H_{\mathrm{S}}^{(F)}$ is the Friedrichs extension of the Schrödinger operator $H_{\mathrm{S}} = (-i\nabla + \mathbf{A})^2$ (see [1, Remark 2.2]), we deduce that any $\boldsymbol{\phi} \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$ can be represented as

$$\boldsymbol{\phi} = R_{\mathrm{S}}^{(\mathrm{F})}(-1) \oplus R_{\mathrm{S}}^{(\mathrm{F})}(-1)\boldsymbol{f}, \quad \text{for some } \boldsymbol{f} = \begin{pmatrix} f_{\uparrow} \\ f_{\downarrow} \end{pmatrix} \in L^2(\mathbb{R}^2;\mathbb{C}^2). \tag{2}$$

Here $R_{\mathrm{S}}^{(\mathrm{F})}(-1)$ is the resolvent of $H_{\mathrm{S}}^{(\mathrm{F})}$, acting by convolution with the integral kernel [1, equation (2.13)]

$$R_{\mathrm{S}}^{(\mathrm{F})}(-1;\mathbf{x},\mathbf{x}') = \sum_{\ell\in\mathbb{Z}} I_{|\ell+\alpha|}(r \wedge r') K_{|\ell+\alpha|}(r \vee r') \frac{e^{i\ell(\theta-\theta')}}{2\pi}. \tag{3}$$

Keeping in mind that, in polar coordinates, we have

$$H_{\mathrm{D}} = \begin{pmatrix} 0 & e^{-i\theta}(-i\partial_r - \frac{\partial_\theta + i\alpha}{r}) \\ e^{i\theta}(-i\partial_r + \frac{\partial_\theta + i\alpha}{r}) & 0 \end{pmatrix},$$

in view of (2) and (3), we get

$$\begin{aligned}
&(H_{\mathrm{D}}\boldsymbol{\phi})_{\uparrow} \\
&=: \sum_{k\in\mathbb{Z}} \Big[\Big(-i\partial_r - i\frac{(k+\alpha)}{r}\Big) K_{|k+\alpha|}(r)\Big] \int_0^r \mathrm{d}r'\, r' I_{|k+\alpha|}(r') f_{\downarrow}^{k}(r')\, \frac{e^{i(k-1)\theta}}{\sqrt{2\pi}} \\
&\quad + \sum_{k\in\mathbb{Z}} \Big[\Big(-i\partial_r - i\frac{(k+\alpha)}{r}\Big) I_{|k+\alpha|}(r)\Big] \int_r^{+\infty} \mathrm{d}r'\, r' K_{|k+\alpha|}(r') f_{\downarrow}^{k}(r')\, \frac{e^{i(k-1)\theta}}{\sqrt{2\pi}},
\end{aligned}$$

and

$$
\begin{aligned}
&(H_{\mathrm{D}}\boldsymbol{\phi})_{\downarrow} \\
&= \sum_{k\in\mathbb{Z}} \Big[\Big(-i\,\partial_r + i\,\frac{(k+\alpha)}{r}\Big) K_{|k+\alpha|}(r)\Big] \int_0^r \mathrm{d}r'\, r' I_{|k+\alpha|}(r') f_{\uparrow}^{k}(r')\, \frac{e^{i(k+1)\theta}}{\sqrt{2\pi}} \\
&\quad + \sum_{k\in\mathbb{Z}} \Big[\Big(-i\,\partial_r + i\,\frac{(k+\alpha)}{r}\Big) I_{|k+\alpha|}(r)\Big] \int_r^{+\infty} \mathrm{d}r'\, r' K_{|k+\alpha|}(r') f_{\uparrow}^{k}(r')\, \frac{e^{i(k+1)\theta}}{\sqrt{2\pi}},
\end{aligned}
$$

where the f_s^k are the Fourier coefficients in the angular wave expansion of f_s.

Now, recall that the boundary conditions encoded in $\mathcal{D}(H_{\mathrm{D}}^{\gamma})$ entail the evaluation of the trace maps $c_{-\alpha}^{s}, c_{\alpha-1}^{s} \colon \mathcal{D}(H_{\mathrm{D}}^{\gamma}) \to \mathbb{C}$, see [1, Proposition 2.23 and equations (2.34) and (2.35)]. Using the known identities $\big(-\partial_r \pm \frac{\nu}{r}\big) K_\nu(r) = K_{\nu\pm1}(r)$ and $\big(-\partial_r \pm \frac{\nu}{r}\big) I_\nu(r) = -I_{\nu\pm1}(r)$ [3, equation 10.29.2] and $K_{-\nu}(z) = K_\nu(z)$ [3, equation 10.27.3], we infer

$$
\begin{aligned}
\langle (H_{\mathrm{D}}\boldsymbol{\phi})_{\uparrow}\rangle(r) &= \frac{i}{\sqrt{2\pi}} \bigg(K_\alpha(r) \int_0^r \mathrm{d}r'\, r' I_{1+\alpha}(r') f_{\downarrow}^{1}(r') \\
&\qquad - I_\alpha(r) \int_r^{\infty} \mathrm{d}r'\, r' K_{1+\alpha}(r') f_{\downarrow}^{1}(r') \bigg), \\
\langle (H_{\mathrm{D}}\boldsymbol{\phi})_{\uparrow} e^{i\theta}\rangle(r) &= \frac{i}{\sqrt{2\pi}} \bigg(K_{1-\alpha}(r) \int_0^r \mathrm{d}r'\, r' I_{\alpha}(r') f_{\downarrow}^{0}(r') \\
&\qquad - I_{-(1-\alpha)}(r) \int_r^{\infty} \mathrm{d}r'\, r' K_{\alpha}(r') f_{\downarrow}^{0}(r') \bigg), \\
\langle (H_{\mathrm{D}}\boldsymbol{\phi})_{\downarrow}\rangle(r) &= \frac{i}{\sqrt{2\pi}} \bigg(K_\alpha(r) \int_0^r \mathrm{d}r'\, r' I_{1-\alpha}(r') f_{\uparrow}^{-1}(r') \\
&\qquad - I_{-\alpha}(r) \int_r^{\infty} \mathrm{d}r'\, r' K_{1-\alpha}(r') f_{\uparrow}^{-1}(r') \bigg), \\
\langle (H_{\mathrm{D}}\boldsymbol{\phi})_{\downarrow} e^{i\theta}\rangle(r) &= \frac{i}{\sqrt{2\pi}} \bigg(K_{1-\alpha}(r) \int_0^r \mathrm{d}r'\, r' I_{2-\alpha}(r') f_{\uparrow}^{-2}(r') \\
&\qquad - I_{1-\alpha}(r) \int_r^{\infty} \mathrm{d}r'\, r' K_{2-\alpha}(r') f_{\uparrow}^{-2}(r') \bigg).
\end{aligned}
$$

Exploiting the asymptotics of the Bessel functions I_ν, K_ν [3, equations 10.30.1 and 10.30.2], it can be checked that

$$
\begin{aligned}
\left|\int_0^r \mathrm{d}r' r' I_{1+\alpha}(r') f_\downarrow^1(r')\right| &\leqslant \|f_\downarrow\|_2 \left(\int_0^r \mathrm{d}r' r' |I_{1+\alpha}(r')|^2\right)^{1/2} \leqslant C r^{2+\alpha},\\
\left|\int_r^\infty \mathrm{d}r' r' K_{1+\alpha}(r') f_\downarrow^1(r')\right| &\leqslant \|f_\downarrow\|_2 \left(\int_r^\infty \mathrm{d}r' r' |K_{1+\alpha}(r')|^2\right)^{1/2} \leqslant C r^{-\alpha};\\
\left|\int_0^r \mathrm{d}r' r' I_{\alpha}(r') f_\downarrow^0(r')\right| &\leqslant \|f_\downarrow\|_2 \left(\int_0^r \mathrm{d}r' r' |I_{\alpha}(r')|^2\right)^{1/2} \leqslant C r^{1+\alpha},\\
\left|\int_0^r \mathrm{d}r' r' I_{1-\alpha}(r') f_\uparrow^{-1}(r')\right| &\leqslant \|f_\uparrow\|_2 \left(\int_0^r \mathrm{d}r' r' |I_{1-\alpha}(r')|^2\right)^{1/2} \leqslant C r^{2-\alpha},\\
\left|\int_0^r \mathrm{d}r' r' I_{2-\alpha}(r') f_\uparrow^{-2}(r')\right| &\leqslant \|f_\uparrow\|_2 \left(\int_0^r \mathrm{d}r' r' |I_{2-\alpha}(r')|^2\right)^{1/2} \leqslant C r^{3-\alpha},\\
\left|\int_r^\infty \mathrm{d}r' r' K_{2-\alpha}(r') f_\uparrow^{-2}(r')\right| &\leqslant \|f_\uparrow\|_2 \left(\int_r^\infty \mathrm{d}r' r' |K_{2-\alpha}(r')|^2\right)^{1/2} \leqslant C r^{-(1-\alpha)}.
\end{aligned}
$$

On the other hand, keeping in mind that $\boldsymbol{f} = (H_P + 1)\boldsymbol{\phi}$, see (2), recalling the definition of the trace maps $\tau_s^{(\ell)}\colon \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}) \to \mathbb{C}$, see [1, equation (2.14)], integrating by parts and using the asymtptotics [1, equation (3.2)], we obtain

$$
\begin{aligned}
\int_r^\infty \mathrm{d}r' r' K_\alpha(r') f_\downarrow^0(r') &= \langle G_\downarrow^{(0)} \mid \boldsymbol{f} \rangle_{L^2(\mathbb{R}^2 \setminus B_r(\mathbf{0}))}\\
&= \sum_{s' \in \{\uparrow,\downarrow\}} \int_{\partial B_r(\mathbf{0})} \mathrm{d}\Sigma_r \, [(G_\downarrow^{(0)})_{s'} \partial_r \phi_{s'} - \partial_r (G_\downarrow^{(0)})_{s'} \phi_{s'}]\\
&= \int_0^{2\pi} r \mathrm{d}\theta \, [g^{(0)} \partial_r \phi_\downarrow - \partial_r g^{(0)} \phi_\downarrow]\\
&= \frac{\Gamma(\alpha)}{2^{1-\alpha}} \lim_{r \to 0^+} \frac{1}{r^\alpha} \int_0^{2\pi} \mathrm{d}\theta \, (\alpha + r\partial_r)\phi_\downarrow \frac{1}{\sqrt{2\pi}} + o(1)\\
&= \tau_\downarrow^{(0)} \boldsymbol{\phi} + o(1), \quad \text{as } r \to 0^+.
\end{aligned}
$$

A similar computation yields

$$\int_r^\infty \mathrm{d}r' r' K_{1-\alpha}(r') f_\uparrow^{-1}(r') = \tau_\uparrow^{(-1)}\boldsymbol{\phi} + o(1).$$

Combining the above results, we find

$$c_{\alpha-1}^{\uparrow}(H_{\mathrm{D}}\boldsymbol{\phi}) = \lim_{r\to 0^+} r^{1-\alpha}\langle (H_{\mathrm{D}}\boldsymbol{\phi})_\uparrow e^{i\theta}\rangle = -\frac{i\,2^{1-\alpha}}{\sqrt{2\pi}\,\Gamma(\alpha)}\tau_\downarrow^{(0)}\boldsymbol{\phi},$$

$$c_{-\alpha}^{\downarrow}(H_{\mathrm{D}}\boldsymbol{\phi}) = \lim_{r\to 0^+} r^{\alpha}\langle (H_{\mathrm{D}}\boldsymbol{\phi})_\downarrow\rangle = -\frac{i\,2^{\alpha}}{\sqrt{2\pi}\,\Gamma(1-\alpha)}\tau_\uparrow^{(-1)}\boldsymbol{\phi},$$

which imply, in turn,

$$H_{\mathrm{D}}\boldsymbol{\phi} = \begin{pmatrix} -i\,\frac{2^{1-\alpha}}{\Gamma(\alpha)}[\tau_\downarrow^{(0)}\boldsymbol{\phi}]\frac{e^{-i\theta}}{\sqrt{2\pi}}\frac{1}{r^{1-\alpha}} \\ -i\,\frac{2^{\alpha}}{\Gamma(1-\alpha)}[\tau_\uparrow^{(-1)}\boldsymbol{\phi}]\frac{1}{\sqrt{2\pi}}\frac{1}{r^{\alpha}} \end{pmatrix} + o(1). \tag{4}$$

On the other hand, if $H_{\mathrm{D}}\boldsymbol{\phi}$ were to belong to $\mathcal{D}(H_D^{\gamma'})$, there should exist some $\mu' \in \mathbb{C}$ such that

$$H_{\mathrm{D}}\boldsymbol{\phi} = \begin{pmatrix} \mu'(1+e^{i\gamma'})\frac{\Gamma(1-\alpha)}{2^{\alpha}}\frac{e^{-i\theta}}{\sqrt{2\pi}}\frac{1}{r^{1-\alpha}} \\ \mu'(1-e^{i\gamma'})\frac{\Gamma(\alpha)}{2^{1-\alpha}}\frac{1}{\sqrt{2\pi}}\frac{1}{r^{\alpha}} \end{pmatrix} + o(1).$$

Matching the coefficients in the above expansions, we would obtain $\sin(\gamma'/2)\tau_\downarrow^{(0)}\boldsymbol{\phi} = i\cos(\gamma'/2)\tau_\uparrow^{(-1)}\boldsymbol{\phi}$, which is absurd, given that the traces $\tau_\downarrow^{(0)}\boldsymbol{\phi}, \tau_\uparrow^{(-1)}\boldsymbol{\phi} \in \mathbb{C}$ are independent for a generic $\boldsymbol{\phi} \in \mathcal{D}(H_{\mathrm{P}}^{(F)})$.

Step 3. We now prove that $\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \subseteq \mathcal{D}((H_{\mathrm{D}}^{\gamma})^2)$ if and only if one of the following alternatives holds:

(a.3) condition (a.1) in Step 1 holds and $\beta_{\uparrow\uparrow}^{(-1-1)} = 0$;

(b.3) condition (b.1) in Step 1 holds and $\beta_{\downarrow\downarrow}^{(00)} = 0$.

Let us firstly remark that, considering the basic definition $\mathcal{D}((H_{\mathrm{D}}^{\gamma})^2) = \{\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{\gamma}) \mid H_{\mathrm{D}}^{\gamma}\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{\gamma})\}$, the claim proved in Step 2 ensures that $\mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})})$ is not a subset of $\mathcal{D}((H_{\mathrm{D}}^{\gamma})^2)$ for any $\gamma \in [0, 2\pi)$. This rules out the alternative (c.1) in Step 1. So, we only have to examine alternatives (a.1) and (b.1) therein. As an example, hereafter we discuss case (a.1).

Assume $\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \subseteq \mathcal{D}((H_{\mathrm{D}}^{0})^2)$, for some suitable $\beta \in \mathrm{M}_{4,\mathrm{Herm}}(\mathbb{C})$ fulfilling condition (a.1) in Step 1. On the one hand, for any $\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{P}}^{(\beta)})$, we have

$$\boldsymbol{\psi} = \boldsymbol{\phi} + q_\uparrow^{(-1)}\boldsymbol{G}_\uparrow^{(-1)} = \boldsymbol{\phi} + q_\uparrow^{(-1)}\frac{\boldsymbol{\xi}_+ + \boldsymbol{\xi}_-}{2\sqrt{2\pi}}\,,$$

where $\boldsymbol{\phi} \in \mathcal{D}(H_{\mathrm{P}}^{(\mathrm{F})}) \subset \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$. Recalling the definition of H_{D}^{0}, see [1, Proposition 2.23], and using the asymptotic expansion (4), we obtain

$$\begin{aligned} H_{\mathrm{D}}^{0}\boldsymbol{\psi} &= H_{\mathrm{D}}\boldsymbol{\phi} + \frac{i}{2} q_{\uparrow}^{(-1)}(\boldsymbol{\xi}_{+} - \boldsymbol{\xi}_{-}) \\ &= \begin{pmatrix} -i\,\frac{2^{1-\alpha}}{\Gamma(\alpha)}(\tau_{\downarrow}^{(0)}\boldsymbol{\phi})\frac{e^{-i\theta}}{\sqrt{2\pi}}\frac{1}{r^{1-\alpha}} \\ \big[-i\,\frac{2^{\alpha}}{\Gamma(1-\alpha)}(\tau_{\uparrow}^{(-1)}\boldsymbol{\phi}) + i\,\frac{\Gamma(\alpha)}{2^{1-\alpha}} q_{\uparrow}^{(-1)}\big]\frac{1}{\sqrt{2\pi}}\frac{1}{r^{\alpha}} \end{pmatrix} + o(1), \quad \text{as } r \to 0^{+}. \end{aligned}$$

On the other hand, since $H_{\mathrm{D}}^{0}\boldsymbol{\psi} \in \mathcal{D}(H_{\mathrm{D}}^{0})$, there must exist some $\tilde{\boldsymbol{\phi}} \in \mathcal{D}[Q_{\mathrm{P}}^{(\mathrm{F})}]$ and $\tilde{\mu} \in \mathbb{C}$ such that

$$H_{\mathrm{D}}^{0}\boldsymbol{\psi} = \tilde{\boldsymbol{\phi}} + \tilde{\mu}(\boldsymbol{\xi}_{+} + \boldsymbol{\xi}_{-}) = \begin{pmatrix} \tilde{\mu}2^{1-\alpha}\Gamma(1-\alpha)\frac{e^{-i\theta}}{\sqrt{2\pi}}\frac{1}{r^{1-\alpha}} \\ 0 \end{pmatrix} + o(1), \quad \text{as } r \to 0^{+}.$$

Matching the $\downarrow$-component in the above asymptotics, we get

$$\tau_{\uparrow}^{(-1)}\boldsymbol{\phi} = \frac{\pi}{2\sin(\pi\alpha)} q_{\uparrow}^{(-1)}.$$

At the same time, the boundary condition in [1, equation (2.10)] entails

$$\tau_{\uparrow}^{(-1)}\boldsymbol{\phi} = [(L(1) + \beta)\mathbf{q}]_{\uparrow}^{(-1)} = \Big[\frac{\pi}{2\sin(\pi\alpha)} + \beta_{\uparrow\uparrow}^{(-1,-1)}\Big] q_{\uparrow}^{(-1)}.$$

The above identities yield $\beta_{\uparrow\uparrow}^{(-1,-1)} = 0$, which ultimately proves claim (b.3).

Step 4. To infer the thesis, we need to show that the inclusion $\mathcal{D}(H_{\mathrm{P}}^{(\beta)}) \subseteq \mathcal{D}((H_{\mathrm{D}}^{\gamma})^2)$, proved in the preceding Step 3 for suitable choices of γ and β, is indeed an equality. As a matter of fact, the operators $H_{\mathrm{P}}^{(\beta)}$ and $(H_{\mathrm{D}}^{\gamma})^2$, with γ and β fixed as in Step 3, have to coincide because they are self-adjoint extensions of the same closable symmetric operator, namely, $(\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}))^2$ on $C_c^{\infty}(\mathbb{R}^2 \setminus \{\mathbf{0}\})$. ■

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