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The Friedrichs extension of a class of discrete symplectic systems

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Abstract. The Friedrichs extension of minimal linear relation being bounded below and associated with the discrete symplectic system with a special linear dependence on the spectral parameter is characterized by using recessive solutions. This generalizes a similar result obtained by Došlý and Hasil for linear operators defined by infinite banded matrices corresponding to even-order Sturm–Liouville difference equations and, in a certain sense, also results of Marletta and Zettl or Šimon Hilscher and Zemánek for singular differential operators.

1. Introduction

Qualitative properties of operators or (more generally) linear relations can be investigated in various ways, including a structure of their spectrum, boundary triplets, or a description of their self-adjoint extensions with a focus on some particular cases. Especially the Friedrichs extension belongs to the very traditional topics, and it has attracted more attention again in recent years, see e.g. [2–4, 30, 40–42, 44, 45, 51–53]. Therefore, in the present paper, we aim to characterize the (domain of the) Friedrichs extension of the minimal linear relation determined by the linear mapping

$$\mathcal{L}(z)_k := \mathcal{J}(z_k - \mathcal{S}_k\, z_{k+1})$$

acting on a weighted space ℓ^2_Ψ consisting of $2n$-vector valued square summable sequences with respect to the weight matrices Ψ_k on the unbounded discrete interval $[0,\infty)_\mathbb{Z} := [0,\infty) \cap \mathbb{Z}$, where the coefficients are $2n \times 2n$ complex-valued matrices satisfying

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{and} \quad \Psi_k^*\, \mathcal{J}\, \Psi_k = \Psi_k\, \mathcal{J}\, \Psi_k = 0 \quad \text{for all } k \in [0,\infty)_\mathbb{Z} \tag{1.1}$$

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with the superscript $*$ denoting the conjugate transpose and $\mathcal{J}$ standing for the $2n \times 2n$ orthogonal and skew-symmetric matrix

$$\mathcal{J} = \mathcal{J}_{2n} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.2}$$

The first equality in (1.1) means that $\mathcal{S}_k$ is symplectic for all k and the mapping $\mathcal{L}$ is closely related to the (nonhomogeneous) *time-reversed discrete symplectic system* on the half-line, because the relation $\mathcal{L}(z) = \lambda \Psi z + \Psi f$ with arbitrary $\lambda \in \mathbb{C}$ is equivalent to

$$z_k(\lambda) = (\mathcal{S}_k + \lambda \mathcal{V}_k)\, z_{k+1}(\lambda) - \mathcal{J} \Psi_k f_k, \quad k \in [0, \infty)_{\mathbb{Z}}, \tag{S_λ^f}$$

where $\mathcal{V}_k = -\mathcal{J}\, \Psi_k\, \mathcal{S}_k$ is such that

$$\mathcal{V}_k^* \mathcal{J} \mathcal{S}_k \text{ is Hermitian} \quad \text{and} \quad \mathcal{V}_k^* \mathcal{J}\, \mathcal{V}_k = 0 \quad \text{for all } k \in [0, \infty)_{\mathbb{Z}}.$$

The associated *minimal linear relation* can be written as

$$T_{\min} = \{\{[z], [f]\} \in T_{\max} \mid z_0 = 0 = \lim_{k\to\infty} z_k^* \mathcal{J} w_k \text{ for all } [w] \in \operatorname{dom} T_{\max}\}, \tag{1.3}$$

which is a restriction of the *maximal linear relation* given by

$$\begin{aligned} T_{\max} := \{\{[z], [f]\} \mid \text{there exists } u \in [z] \text{ such that} \\ \mathcal{L}(u)_k = \Psi_k\, f_k \text{ for all } k \in [0, \infty)_{\mathbb{Z}}\}, \end{aligned} \tag{1.4}$$

where $[z], [f]$ stand for equivalence classes in ℓ_Ψ^2. Actually, the proper definition of the minimal linear relation guarantees that its adjoint relation is $T_{\max}$, i.e., it holds $T_{\min}^* = T_{\max}$ as shown in [10, Theorem 5.10]. These relations and square summability of solutions of discrete symplectic systems were thoroughly studied by the author and his collaborators in [9, 10, 35–38, 48, 49]. Now, we turn our attention to the Friedrichs extension T_F of $T_{\min}$, which is defined as a self-adjoint extension of $T_{\min}$ being bounded below by the same lower bound as $T_{\min}$. However, even though we speak of an extension, this linear relation T_F will be expressed as a *restriction* of $T_{\max}$, which consists of pairs satisfying a zero boundary condition at $k = 0$ and a specific limit condition at ∞ determined by recessive solutions of (S_λ^f) with $f \equiv 0$, i.e., of the (homogeneous) system

$$z_k(\lambda) = (\mathcal{S}_k + \lambda \mathcal{V}_k)\, z_{k+1}(\lambda), \quad k \in [0, \infty)_{\mathbb{Z}}, \tag{S_λ}$$

see Theorem 3.4 for a precise formulation. Our main result relies on several facts from the theory of discrete symplectic systems. The first is a connection between the boundedness from below of $T_{\min}$ and the existence of recessive solutions of (S_λ). The second crucial ingredient is the recessive solution of (S_λ) *per se*, because its properties imply the square integrability and, roughly speaking, presence in the domain of T_F. Here we should emphasize that recessive solutions are defined through the behavior of their first n components of the $2n$-vector-valued solutions, which naturally leads to the restriction that we consider only the case when the weight matrices Ψ_k have for all $k \in [0,\infty)_{\mathbb{Z}}$ the very special block structure

$$\Psi_k := \begin{pmatrix} \mathcal{W}_k & 0 \\ 0 & 0 \end{pmatrix}$$

with $\mathcal{W}_k = \mathcal{W}_k^* > 0$ being $n \times n$ matrices. Finally, we utilize the characterization of all self-adjoint extensions of $T_{\min}$ established in [48, Theorem 3.3 and Remark 3.4] and give precisely d boundary conditions determining the Friedrichs extension, see also Theorem 2.10 and Remark 3.5 (i).

The origin of the study of the concept nowadays known as the Friedrichs extension can be traced back to von Neumann. He showed that for any Hermitian linear operator with a lower bound C there exists its self-adjoint extension, which is also semibounded with a lower bound C' for an arbitrary $C' < C$, see [43, Satz 43]. In addition, as a footnote, von Neumann conjectured that it is even possible to take $C = C'$, i.e., to get a self-adjoint extension with the same lower bound. Subsequently, Friedrichs proved the existence of such an extension in [15, Satz 9], and he was even able to specify, under certain specific assumptions on the coefficients, the domain of this extension for the second-order Sturm–Liouville differential operator, see [16]. This made the extension to be somehow exceptional, and Friedrichs called it as *ausgezeichnete Fortzetzung* (an excellent extension). His approach was based on an associated quadratic form and "boundary terms," which is not, in principle, too far from our treatment. The notion of *Friedrichs extension* appears probably for the first time in Freudenthal's work [17], where a limit characterization of this extension was derived for any lower semibounded Hermitian linear operator. Actually, this technique turns out to be crucial for many subsequent results (including ours). Rellich in [29] provided two alternative characterizations of the Friedrichs extension for the second-order Sturm–Liouville differential operator without the conditions imposed on the coefficients by Friedrichs instead of which he assumed explicitly that the operator is bounded from below. In particular, he showed that the elements of the domain of the Friedrichs extensions behave like a principal solution near the boundary. A similar result can also be found in Kalf's paper [22] but this time utilizing Freudenthal's characterization.

Simultaneously, from the Glazman–Krein–Naimark theorem we know that the domain of any self-adjoint extension of an operator can be described by suitable d "boundary" conditions, where d is equal to positive and negative deficiency indices of the operator, see, e.g., [25, Theorem 4 in Section 18]. Zettl and his co-authors showed that these are the Dirichlet boundary conditions in the case of the Friedrichs extension of regular ordinary differential operators with locally integrable coefficients or in a more general setting, see [3, 26]. On the other hand, in the singular case the Dirichlet boundary condition at one endpoint (or eventually at both endpoints) is not well defined, so another condition is needed in this situation, which was treated in [1,2,4,5,23,27] including the description of the Friedrichs extension as a true extension of the minimal operator in [45]. Since, in almost all these cases, a connection between the differential expressions and linear Hamiltonian differential systems is used, it is not very surprising that later the Friedrichs extension was solely investigated for operators or linear relations associated with these systems itself, see [24,33,44,52].

The literature on a discrete analogue of this problem is, however, humbler. The Friedrichs extension of the Jacobi operator or the second order Sturm–Liouville difference expression was studied in [7, 8, 18] and for higher order expressions through the banded symmetric matrices in [14]. Furthermore, very recently, Friedrichs extension in the setting of a linear Hamiltonian difference system was investigated in [28, 51]. As it is well known that this system can be written as a discrete symplectic system, but not vice versa in general, we provide a generalization of the latter results. For completeness, we mention that the first attempts of a unification of these results for any even order Sturm–Liouville differential and difference expressions through the calculus on time scales were presented in [46, 50]. Our main result (given in Theorem 3.4) should not be anyhow surprising as it yields the same conclusion as in the continuous case or for Jacobi operators, the latter of which is, in fact, a special case of (S_λ^f). Nonetheless, it completes this direction by a nontrivial generalization of the above-mentioned results, including the linear Hamiltonian difference system. We also note that principal or recessive solutions still remain the main tool in the characterization of the Friedrichs extension; although we can find various approaches to this characterization in the general theory of linear relations, their application to the specific cases mentioned above still seems to be somehow restricted.

The paper is organized as follows. In the next section we introduce the notation used and the basic setting of system (S_λ^f), recall a general characterization of the Friedrichs extension in the theory of linear relations and the notion of the recessive solution of discrete symplectic systems, and derive several preliminary results. The main result is established in Section 3.

2. Preliminaries

Throughout the paper, all matrices are considered over the field of complex numbers $\mathbb{C}$. For $r, s \in \mathbb{N}$ we denote by $\mathbb{C}^{r\times s}$ the space of all complex-valued $r \times s$ matrices and $\mathbb{C}^{r\times 1}$ will be abbreviated as $\mathbb{C}^r$. In particular, the $r \times r$ *identity* and *zero matrices* are written as I_r and 0_r, where the subscript is omitted whenever it is not misleading (for simplicity, the zero vector is also written as 0). By e_i for $i \in \{1, \dots, n\}$ or $i \in \{1, \dots, 2n\}$ we mean the elements of the canonical basis of $\mathbb{R}^n$ or $\mathbb{R}^{2n}$, i.e., the columns of I_n or I_{2n}. For a given matrix $M \in \mathbb{C}^{r\times s}$ we indicate by M^*, $\ker M$, $\operatorname{rank} M$, $M^\dagger$, $M \geq 0$, and $M > 0$ respectively, its conjugate transpose, kernel, rank, the Moore–Penrose generalized inverse, positive semidefiniteness, and positive definiteness. Furthermore, we denote by $\mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times s}$ the space of sequences defined on $[0,\infty)_{\mathbb{Z}}$ of complex $r \times s$ matrices, where typically $r \in \{n, 2n\}$ and $1 \leq s \leq 2n$. In particular, we write only $\mathbb{C}([0,\infty)_{\mathbb{Z}})^r$ in the case $s = 1$. If $M \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times s}$, then $M(k) := M_k$ for $k \in [0,\infty)_{\mathbb{Z}}$ and if $M(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times s}$, then $M(\lambda, k) := M_k(\lambda)$ for $k \in [0,\infty)_{\mathbb{Z}}$ with $M_k^*(\lambda) := [M_k(\lambda)]^*$. If $M \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times s}$ and $N \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{s\times p}$, then $MN \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times p}$, where $(MN)_k := M_k N_k$ for $k \in [0,\infty)_{\mathbb{Z}}$. We put $[z_k]_{k=m}^n := z_n - z_m$. We also adopt a common notation that $2n$-vector-valued sequences or solutions of (S_λ^f) are denoted by small letters, typically $z = \binom{x}{u}$ with n-vector valued components, while $2n \times m$ matrix-valued solutions are denoted by capital letters, typically $Z = \binom{X}{U}$ with $n \times m$ matrix-valued components. For completeness, we note that any solution of (S_λ^f) can be easily seen as a solution of (S_0^f).

Finally, the square summability is defined via the semi-inner product

$$\langle z, u\rangle_\Psi := \sum_{k=0}^{\infty} z_k^*\, \Psi_k\, u_k$$

and the induced semi-norm

$$\|z\|_\Psi := \sqrt{\langle z, z\rangle_\Psi}$$

with respect to the weight matrices Ψ_k specified in (1.1), i.e., we restrict our attention to the space

$$\ell_\Psi^2 = \ell_\Psi^2([0,\infty)_{\mathbb{Z}}) := \{z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n} \mid \|z\|_\Psi < \infty\}$$

and, subsequently, to the corresponding Hilbert space

$$\tilde{\ell}_\Psi^2 = \tilde{\ell}_\Psi^2([0,\infty)_{\mathbb{Z}}) := \ell_\Psi^2/\{z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n} \mid \|z\|_\Psi = 0\}$$

consisting of equivalence classes, which are denoted by $[z]$.

In the most general setting, a *linear relation* $\mathcal{T}$ is defined as a linear subspace of the Cartesian product of two vector spaces $\mathcal{X}$ and $\mathcal{Y}$. We focus only on the case when $\mathcal{X} = \mathcal{Y}$ and it is a Hilbert space $\mathcal{H}$, which provides suitable tools to study nondensely defined operators via their graphs. For a deeper insight in this theory, we refer to [6] and in a connection with discrete symplectic systems to [10,48,49]. We recall that the *domain* of a linear relation $\mathcal{T} \subseteq \mathcal{H} \times \mathcal{H}$ is defined as

$$\operatorname{dom} \mathcal{T} := \{z \in \mathcal{H} \mid \text{there exists } f \in \mathcal{H} \text{ such that } \{z, f\} \in \mathcal{T}\}$$

and the *adjoint* relation of $\mathcal{T}$ as

$$\mathcal{T}^* := \{\{y, g\} \in \mathcal{H}^2 \mid \langle z, g\rangle - \langle f, y\rangle = 0 \text{ for all } \{z, f\} \in \mathcal{T}\}.$$

The linear relation $\mathcal{T}$ is said to be *(semi)bounded from below by* $c \in \mathbb{R}$, i.e., $\mathcal{T} \geq c$, if

$$\langle f, z\rangle \geq c\,\langle z, z\rangle \quad \text{for all } \{z, f\} \in \mathcal{T} \tag{2.1}$$

and, in particular, *nonnegative*, i.e., $\mathcal{T} \geq 0$, if

$$\langle f, z\rangle \geq 0 \quad \text{for all } \{z, f\} \in \mathcal{T}.$$

The largest c satisfying (2.1) is said to be the *lower bound* of $\mathcal{T}$. In that case, the linear relation is necessarily symmetric and it has equal defect numbers, which guarantees the existence of its self-adjoint extension(s). In particular, in the case of equal deficiency indices there exists the Friedrichs extension $\mathcal{T}_F$ of $\mathcal{T}$ as defined in [6, Section 5.3], which can be characterized as follows, see [6, Corollary 5.3.4]

Theorem 2.1. *Let $\mathcal{T}$ be a semibounded linear relation in $\mathcal{H}^2$. Then $\{z, f\} \in \mathcal{T}_F$ if and only if $\{z, f\} \in \mathcal{T}^*$ and there exists a sequence $\{\{z_n, f_n\}\}_{n=1}^\infty \in \mathcal{T}$ such that*

$$z_n \to z \quad \text{and} \quad \langle z_n, f_n\rangle \to \langle z, f\rangle \quad \text{as } n \to \infty.$$

Similarly to Freudenthal's characterization in the operator case, it can be shown that $\{z, f\} \in \mathcal{T}_F$ if and only if $\{z, f\} \in \mathcal{T}^*$ and there exist a sequence

$$\{\{z_n, f_n\}\}_{n=1}^\infty \in \mathcal{T}$$

such that

$$z_n \to z \quad \text{and} \quad \langle z_n - z_m, f_n - f_m\rangle \to 0 \quad \text{as } n, m \to \infty, \tag{2.2}$$

see also [11, 19–21].

The following hypothesis summarizes the basic assumptions concerning the coefficients of system (S_λ) or (S_λ^f) with the special linear dependence on the spectral parameter. These systems can be determined either by the pair of coefficient matrices $\{\mathcal{S}, \mathcal{V}\}$ or by the pair $\{\mathcal{S}, \Psi\}$ and there is no difference between these two approaches as the matrices Ψ_k and $\mathcal{V}_k$ are mutually connected via the equalities $\Psi_k = \mathcal{J}\mathcal{S}_k\mathcal{J}\mathcal{V}_k^*\mathcal{J}$ and $\mathcal{V}_k = -\mathcal{J}\,\Psi_k\,\mathcal{S}_k$. Furthermore, Hypothesis 2.2 yields that system (S_λ^f) can be written by using the matrices $\mathbb{S}_k(\lambda) := \mathcal{S}_k + \lambda\mathcal{V}_k$, which satisfy the symplectic-type equality $\mathbb{S}_k^*(\bar{\lambda})\mathcal{J}\mathbb{S}_k(\lambda) = \mathcal{J}$. This guarantees the existence of a unique solution of any initial value problem associated with (S_λ^f).

Hypothesis 2.2. A number $n \in \mathbb{N}$ and matrix-valued sequences $\mathcal{S} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ and $\mathcal{W} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{n\times n}$ are given such that

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{and} \quad \mathcal{W}_k^* = \mathcal{W}_k > 0 \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}.$$

The matrices $\mathcal{S}_k$ admit the $n \times n$ block decomposition

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}$$

for all $k \in [0,\infty)_{\mathbb{Z}}$ and the matrix-valued sequences $\Psi, \mathcal{V} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ are defined as

$$\Psi_k := \begin{pmatrix} \mathcal{W}_k & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V}_k := -\mathcal{J}\, \Psi_k\, \mathcal{S}_k \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}.$$

Our main result combines tools from the spectral theory of linear relations and from the oscillation theory of (S_λ) with $\lambda \in \mathbb{R}$, where an important role is played by a "special" type of matrix-valued solutions, for which we need the following notions.

Definition 2.3. Let $\nu \in \mathbb{R}$ be fixed and Hypothesis 2.2 be satisfied. A $2n \times n$ matrix-valued solution $Z(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ of (S_ν) is said to be a *conjoined basis* if it satisfies $\operatorname{rank} Z_k(\nu) = n$ and $Z_k^*(\nu)\mathcal{J} Z_k(\nu) = 0$ for some (and hence for any) $k \in [0,\infty)_{\mathbb{Z}}$. Two conjoined bases $Z(\nu), \tilde{Z}(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ of (S_ν) are said to be *normalized* if $Z_k^*(\nu)\mathcal{J}\tilde{Z}_k(\nu) = I$ for some (and hence for any) $k \in [0,\infty)_{\mathbb{Z}}$.

A comprehensive treatise on the qualitative theory of discrete symplectic systems can be found in the recent book [13]. Our notion of recessive solutions in Definition 2.4 follows the traditional concept introduced in [12] and its generalization was studied in [31, 32].

Definition 2.4. Let $\nu \in \mathbb{R}$ be fixed and Hypothesis 2.2 be satisfied. A conjoined basis $\tilde{Z}(\nu) = \begin{pmatrix} \tilde{X}(\nu) \\ \tilde{U}(\nu) \end{pmatrix}$ of system (S_ν) is said to be a *recessive solution* if, for large $k \in [0,\infty)_{\mathbb{Z}}$, the matrix $\tilde{X}_k(\nu)$ is nonsingular, it holds $-\tilde{X}_k^{-1}(\nu)\mathcal{B}_k\tilde{X}_{k+1}^{*-1}(\nu) \geq 0$ and simultaneously $\lim_{k\to\infty} X_k^{-1}(\nu)\tilde{X}_k(\nu) = 0$ for any conjoined basis $Z(\nu)$ normalized with $\tilde{Z}(\nu)$, i.e., such that $Z_k^*(\nu)\mathcal{J}\tilde{Z}_k(\nu) \equiv I$.

Note that the recessive solution is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix. However, not every system (S_λ) possesses a recessive solution. Its existence can by guaranteed by two additional assumptions as we show in Theorem 2.5. Let $\nu \in \mathbb{R}$ be fixed. System (S_ν) is said to be *nonoscillatory* if there exists $M \in [0,\infty)_{\mathbb{Z}}$ such that it is *disconjugate* on $[M, N+1]_{\mathbb{Z}}$ for every $N \in [M,\infty)_{\mathbb{Z}}$, i.e., the matrix-valued solution $Z(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ determined by the initial condition $Z_{N+1}(\nu) = \binom{0}{-I}$ satisfies

$$\ker X_k(\nu) \subseteq \ker X_{k+1}(\nu) \quad \text{and} \quad -X_{k+1}(\nu)X_k^{\dagger}(\nu)\mathcal{B}_k \geq 0 \tag{2.3}$$

for all $k \in [M, N]_{\mathbb{Z}}$, see [13, Theorem 2.41]. In the opposite case, system (S_ν) is called *oscillatory*. The nonoscillatory behavior implies that every conjoined basis $Z(\nu)$ of (S_ν) satisfies condition (2.3) for all $k \in [0, \infty)_{\mathbb{Z}}$ large enough and, consequently, the kernel of $X_k(\nu)$ is eventually constant. In addition, we say that system (S_ν) is *disconjugate on* $[0, \infty)_{\mathbb{Z}}$ if it is nonoscillatory with $M = 0$.

System (S_λ) is (*completely*) *controllable* on a discrete interval $[N, \infty)_{\mathbb{Z}}$ if for any nontrivial finite discrete subinterval $[K, M]_{\mathbb{Z}} \subset [N, \infty)_{\mathbb{Z}}$ the trivial solution $z(\lambda) \equiv 0$ is the only solution of (S_λ) with $x_k(\lambda) = 0$ for all $k \in [K, M]_{\mathbb{Z}}$, i.e., the subsystem

$$0 = \mathcal{B}_k u_{k+1} \quad \text{and} \quad u_k = \mathcal{D}_k u_{k+1}, \quad k \in [K, M-1]_{\mathbb{Z}}, \tag{2.4}$$

has only the trivial solution, see also equations (2.5)–(2.6) below. This happens, e.g., when $\mathcal{B}_k$ is invertible for all $k \in [N, \infty)_{\mathbb{Z}}$. Note that the subsystem in (2.4) does not involve λ, so the controllability can be seen as a global property of system (S_λ) independent of λ. System (S_λ) is *eventually controllable* if there exists $N \in [0, \infty)_{\mathbb{Z}}$ such that it is completely controllable on $[N, \infty)_{\mathbb{Z}}$. This property together with the eventually constant kernel $X_k(\nu)$ mentioned above implies the invertibility of $X_k(\nu)$ for all $k \in [0, \infty)_{\mathbb{Z}}$ large enough, i.e., if system (S_ν) is nonoscillatory and eventually controllable, then for every conjoined basis $Z(\nu)$, there exists $N \in [0, \infty)_{\mathbb{Z}}$ such that $X_k(\nu)$ is invertible and $-X_{k+1}(\nu)X_k^{-1}(\nu)\mathcal{B}_k \geq 0$ for all $k \in [N, \infty)_{\mathbb{Z}}$.

The following result is a time-reversed analogue of [12, Theorem 3.1] and [13, Theorem 2.66]. We omit its proof because it can be done in the same way as in the mentioned references.

Theorem 2.5. *Let Hypothesis 2.2 hold and $\nu \in \mathbb{R}$ be such that system (S_ν) is nonoscillatory and eventually controllable. Then (S_ν) possesses a recessive solution $\widetilde{Z} = \binom{\widetilde{X}}{\widetilde{U}} \in \mathbb{C}([0, \infty)_{\mathbb{Z}})^{2n \times n}$, which can be equivalently characterized by the condition*

$$\lim_{k \to \infty} \lambda_{\min}\Big(-\sum_{j=k_0}^{k} \widetilde{X}_j^{-1}(\nu)\mathcal{B}_j \widetilde{X}_{j+1}^{*-1}(\nu)\Big) = \infty,$$

where $\lambda_{\min}$ stands for the smallest eigenvalue of the matrix indicated and $k_0 \in [0, \infty)_{\mathbb{Z}}$ is large enough.

The maximal linear relation is defined as in (1.4), while the minimal linear relation displayed in (1.3) is defined as the closure of the *pre-minimal linear relation* T_0, which consists of $\{[z], [f]\} \in T_{\max}$ such that $\hat{z}_0 = 0$ and $\hat{z}_k = 0$ for all $k \in [0, \infty)_{\mathbb{Z}}$

large enough and a suitable representative $\hat{z} \in [z]$. The following hypothesis guarantees that the minimal linear relation can be written as in (1.3). It is called the *strong Atkinson condition* or *definiteness condition* and it is a classical assumption in the Weyl–Titchmarsh theory for differential or difference equations. In addition, it is equivalent to the fact that for any $\{[z],[f]\} \in T_{\max}$ there is a unique $\hat{z} \in [z]$ such that $\mathcal{L}(\hat{z})_k = \Psi_k f_k$ for all $k \in [0,\infty)_{\mathbb{Z}}$, see [10, Theorem 5.2]. Since the latter equality is also independent of the choice of a representative of $f \in [f]$, we may write only $\{z, f\} \in T_{\max}$ whenever the hypothesis is satisfied.

Hypothesis 2.6 (Strong Atkinson condition). Hypothesis 2.2 is satisfied, and a number $\nu \in \mathbb{C}$ and a finite interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} := [a,b]_{\mathbb{Z}} \subseteq [0,\infty)_{\mathbb{Z}}$ exist such that every nontrivial solution $z(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ of system (S_ν) satisfies $\sum_{k\in\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}} z_k^*(\lambda)\Psi_k z_k(\lambda) > 0$.

Note that the strong Atkinson condition is independent of the choice of $\lambda \in \mathbb{C}$, i.e., Hypothesis 2.6 means that $\sum_{k\in\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}} z_k^*(\lambda)\Psi_k z_k(\lambda) > 0$ is satisfied for all nontrivial solutions of (S_λ) for any $\lambda \in \mathbb{C}$, see e.g., [47, Lemma 2.1]. Alternatively, this condition is equivalent to the fact that the trivial solution is the only solution of (S_λ) such that $\sum_{k\in\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}} z_k^*(\lambda)\Psi_k z_k(\lambda) = 0$. Such systems are also said to be *definite* on the discrete interval $[0,\infty)_{\mathbb{Z}}$.

In the next part we aim to connect the disconjugacy of (S_λ) on $[0,\infty)_{\mathbb{Z}}$ and the boundedness from below of $T_{\min}$. Due to the special block structure of $\mathcal{S}_k$ and Ψ_k described in Hypothesis 2.2, system (S_λ) can be written as the pair of equations

$$x_k(\lambda) = \mathcal{A}_k x_{k+1}(\lambda) + \mathcal{B}_k u_{k+1}(\lambda), \tag{2.5}$$

$$u_k(\lambda) = \mathcal{C}_k x_{k+1}(\lambda) + \mathcal{D}_k u_{k+1}(\lambda) + \lambda \mathcal{W}_k x_k(\lambda). \tag{2.6}$$

Then, a sequence $z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ is said to be *admissible* if it satisfies equation (2.5), which does not involve the parameter λ explicitly, i.e., the space of all admissible sequences of (S_λ) is independent of λ. In the next lemma, we introduce an quadratic functional associated with (S_λ^f) and describe its connection to the inner product $\langle\cdot,\cdot\rangle_\Psi$, from which we will derive the dependence of the disconjugacy on λ.

Lemma 2.7. *Let $\lambda \in \mathbb{C}$ be arbitrary, Hypothesis 2.2 be satisfied, and for any $z = \binom{x}{u} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ define the quadratic functional*

$$\begin{aligned}\mathcal{F}_\lambda(z) := -\sum_{k\in[0,\infty)_{\mathbb{Z}}} \{ x_{k+1}^* \mathcal{C}_k^*(\lambda) \mathcal{A}_k x_{k+1} + 2\,\mathrm{Re}(x_{k+1}^* \mathcal{C}_k^*(\lambda) \mathcal{B}_k u_{k+1}) \\ + u_{k+1}^* \mathcal{D}_k^*(\lambda) \mathcal{B}_k u_{k+1} \},\end{aligned}$$

where

$$\mathcal{C}_k(\lambda) := \mathcal{C}_k + \lambda \mathcal{W}_k \mathcal{A}_k \quad \text{and} \quad \mathcal{D}_k(\lambda) := \mathcal{D}_k + \lambda \mathcal{W}_k \mathcal{B}_k.$$

If $z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ is an admissible sequence of (S_λ)*, it reduces to*

$$\begin{aligned} \mathcal{F}_\lambda(z) &= \sum_{k\in[0,\infty)_{\mathbb{Z}}} (u_k - \mathcal{C}_k(\lambda)x_{k+1} - \mathcal{D}_k(\lambda)u_{k+1})^* x_k + [u_k^* x_k]_{k=0}^{\infty} \\ &= \sum_{k\in[0,\infty)_{\mathbb{Z}}} (u_k - \mathcal{C}_k x_{k+1} - \mathcal{D}_k u_{k+1})^* x_k + [u_k^* x_k]_{k=0}^{\infty} - \lambda\langle z,z\rangle_\Psi \end{aligned} \tag{2.7}$$

and, furthermore, if z solves (S_ν^f) *for some $\nu \in \mathbb{C}$ then*

$$\mathcal{F}_\lambda(z) = (\bar{\nu} - \lambda)\langle z,z\rangle_\Psi + \langle z,f\rangle_\Psi + [u_k^* x_k]_{k=0}^{\infty}.$$

Especially, when $\{[z],[f]\} \in T_0$, we obtain

$$\mathcal{F}_\lambda(z) = \langle z,f\rangle_\Psi - \lambda\langle z,z\rangle_\Psi = \langle z, f-\lambda z\rangle_\Psi$$

for any $z \in [z]$ and $f \in [f]$.

It was shown in [34, Theorem 3.1] and [13, Theorem 2.41] that the disconjugate property of system (S_ν) on $[M,N+1]_{\mathbb{Z}}$ is equivalent with the positivity of the associated quadratic functional

$$-\sum_{k=M}^{N} \{x_{k+1}^* \mathcal{C}_k^*(\nu)\mathcal{A}_k x_{k+1} + 2\,\mathrm{Re}\,(x_{k+1}^*\mathcal{C}_k^*(\nu)\mathcal{B}_k u_{k+1}) + u_{k+1}^*\mathcal{D}_k^*(\nu)\mathcal{B}_k u_{k+1}\} \tag{2.8}$$

for any admissible $z = \binom{x}{u} \in \mathbb{C}([M,N+1]_{\mathbb{Z}})^{2n}$ with $x_M = 0 = x_{N+1}$ and $x \neq 0$. Recall that the space of all admissible sequences for (S_λ) is independent on λ, so from (2.7) we get that for any admissible $z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ it holds

$$\mathcal{F}_\lambda(z) = \mathcal{F}_\nu(z) + (\nu - \lambda)\langle z,z\rangle_\Psi. \tag{2.9}$$

Therefore, the nonnegativity (or positivity) of $\mathcal{F}_\nu(z)$ for some $\nu \in \mathbb{R}$ implies the same property for all $\lambda < \nu$. Subsequently, upon combining with Hypothesis 2.6, we get the following corollary, whose second part is a simple consequence of Theorem 2.5.

Corollary 2.8. *Let Hypothesis 2.6 be satisfied and $\nu \in \mathbb{R}$ be such that system* (S_ν) *is disconjugate on $[0,\infty)_{\mathbb{Z}}$ and eventually controllable. Then system* (S_λ) *is disconjugate on $[0,\infty)_{\mathbb{Z}}$ and possess a recessive solution for any $\lambda \leq \nu$.*

The last part of Lemma 2.7 shows that the boundedness from below of $T_{\min}$ is closely connected to the nonnegativity of $\mathcal{F}_\lambda(\cdot)$ or, in fact, with the disconjugacy of (S_ν) on $[0,\infty)_{\mathbb{Z}}$ as stated in the next theorem.

Theorem 2.9. *Let Hypothesis 2.6 be satisfied and $\nu \in \mathbb{R}$ be such that system* (S_ν) *is disconjugate on* $[0,\infty)_{\mathbb{Z}}$. *Then,* $T_{\min}$ *is bounded from below by a lower bound* $c \geq \nu$ *or equivalently* $T_{\min} - \lambda I$ *is bounded from below by* $c - \lambda > 0$ *for all* $\lambda < \nu$. *Consequently, the deficiency indices of* $T_{\min}$ *satisfy* $d_+(T_{\min}) = d_-(T_{\min}) = d_\mu(T_{\min})$ *for all* $\mu \in \mathbb{C}\setminus[\nu,\infty)$ *and any self-adjoint extension of* $T_{\min}$ *is bounded from below.*

Proof. Since $T_{\min} = \overline{T_0}$, it suffices to show that T_0 is bounded from below. From the definition of T_0 and the positivity of (2.8) on any subinterval $[M, N+1]_{\mathbb{Z}} \subset [0,\infty)_{\mathbb{Z}}$ with $M = 0$ we get $\langle z, f - \nu z\rangle_\Psi = \mathcal{F}_\nu(z) > 0$ for all $z \in \operatorname{dom} T_0$. This shows that T_0 is bounded from below by $c \geq \nu$ or equivalently the boundedness of $T_{\min} - \lambda I$ from below by $c - \lambda > 0$ for all $\lambda < \nu$. The second part of the statement follows immediately from [6, Proposition 1.4.6] and [6, Proposition 5.5.8]. ∎

The second part of Theorem 2.9 shows that the disconjugacy of (S_ν) on $[0,\infty)_{\mathbb{Z}}$ guarantees the existence of a self-adjoint extension of $T_{\min}$, which is possible if and only if the positive and negative deficiency indices $d_+(T_{\min})$ and $d_-(T_{\min})$, respectively, are equal, see [11, Corollary, p. 34]. This equality $d_+(T_{\min}) = d_-(T_{\min}) =: d$ can be alternatively interpreted under Hypothesis 2.6 so that systems (S_λ) and ($S_{\bar\lambda}$) possess the same number d of linearly independent square summable solutions for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$, see [10, Corollary 5.12]. If, in addition, there is a number $\nu \in \mathbb{R}$ such that system (S_ν) has the same number d of linearly independent square summable solutions, then all self-adjoint extensions of $T_{\min}$ admit the following characterization, see [48, Theorem 3.3 and Remark 3.4]. This statement turns out to be yet another crucial ingredients in the proof of our main result, see Lemma 3.3.

Theorem 2.10. *Let Hypothesis 2.6 be satisfied and assume that*

(i) *both* (S_i) *and* (S_{-i}) *possess d linearly independent square summable solutions;*

(ii) *there exists $\nu \in \mathbb{R}$ such that also system* (S_ν) *possess d linearly independent square summable solutions, which are denoted as (suppressing the argument ν)* $\Theta^{[1]}, \dots, \Theta^{[d]}$ *and arranged so that the* $2(d-n)\times 2(d-n)$ *leading principal submatrix of the $d \times d$ matrix* $\Upsilon := \Theta_0^* \mathcal{J}\, \Theta_0$ *has a full rank, where* $\Theta_k := (\Theta_k^{[1]}, \dots, \Theta_k^{[d]})$ *for* $k \in [0,\infty)_{\mathbb{Z}}$.

Then, a linear relation $T \subseteq \tilde\ell^2_\Psi \times \tilde\ell^2_\Psi$ *is a self-adjoint extension of the minimal linear relation* $T_{\min}$ *if and only if there exist matrices* $M \in \mathbb{C}^{d\times 2n}$ *and* $L \in \mathbb{C}^{d\times 2(d-n)}$ *such that*

$$\operatorname{rank}(M, L) = d, \quad M\mathcal{J}M^* - L\,\Upsilon_{2(d-n)\times 2(d-n)}\,L^* = 0, \tag{2.10}$$

and

$$T = \left\{ \{z, f\} \in T_{\max} \,\middle|\, M z_0 - L \begin{pmatrix} (\Theta^{[1]}, z)_\infty \\ \vdots \\ (\Theta^{[2(d-n)]}, z)_\infty \end{pmatrix} = 0 \right\}, \tag{2.11}$$

where $\Upsilon_{2(d-n)\times 2(d-n)}$ *is the* $2(d-n)\times 2(d-n)$ *principal leading submatrix of* Υ *and* $(\Theta^{[j]}, z)_\infty := \lim_{k\to\infty} \Theta_k^{[j]*} \mathcal{J} z_k$ *for* $j = 1, \dots, 2(d-n)$ *exist due to the square summability of both sequences.*

3. Main result

At this moment, we have presented all preliminary results needed to establish our main result. Its proof is based on the following three lemmas. In the first lemma, we show that $x_0 = 0$ for every $\{z, f\} \in T_F$. In the second lemma, we prove that the columns of a recessive solution of (S_λ) with $\lambda < \nu$ belong to the domain of the Friedrichs extension of $T_{\min}$, which is denoted as T_F. Thereafter, in the third lemma, we show that a certain linear relation determined by a (part of a) recessive solution is self-adjoint.

Lemma 3.1. *Let Hypothesis 2.6 be satisfied and* $\nu \in \mathbb{R}$ *be such that system* (S_ν) *is disconjugate on* $[0,\infty)_\mathbb{Z}$ *and eventually controllable. Then* $x_0 = 0$ *for any* $\{z, f\} \in T_F$.

Proof. The assumptions guarantee that the Friedrichs extension of the minimal linear relation $T_{\min}$ exists by Theorem 2.9. Condition (2.2) together with Hypothesis 2.2 means that $\{z, f\} \in T_F$ if and only if $\{z, f\} \in T_{\max}$ and there exists a sequence of pairs $\{\{z^{[n]}, f^{[n]}\}\}_{n=1}^\infty \in T_{\min}$ such that

$$\lim_{n\to\infty} \|z^{[n]} - z\|_\Psi = \lim_{n\to\infty} \sum_{k=0}^\infty (x_k^{[n]} - x_k)^* \mathcal{W}_k (x_k^{[n]} - x_k) = 0$$

and, by using Lemma 2.7,

$$\begin{aligned}&\langle z^{[n]} - z^{[m]}, f^{[n]} - f^{[m]}\rangle_\Psi \\ &\quad = \mathcal{F}_0(z^{[n]} - z^{[m]}) - [(u_k^{[n]} - u_k^{[m]})^*(x_k^{[n]} - x_k^{[m]})]_{k=0}^\infty \to 0\end{aligned}$$

as $n, m \to \infty$. Since $\mathcal{W}_k > 0$ for all $k \in [0,\infty)_\mathbb{Z}$ by Hypothesis 2.2, the first condition yields that $\lim_{n\to\infty} \mathcal{W}_0(x_0^{[n]} - x_0) = 0$. Since $x_0^{[n]} = 0$ for all $n \in \mathbb{N}$ by (1.3), it follows that also $x_0 = 0$ for every $\{z, f\} \in T_F$. ∎

Lemma 3.2. *Let Hypothesis 2.6 be satisfied and* $\nu \in \mathbb{R}$ *be such that system* (S_ν) *is disconjugate on* $[0,\infty)_\mathbb{Z}$ *and eventually controllable. Then, for any* $\lambda \le \nu$, *all the columns of the recessive solution* $\tilde{Z}(\lambda)$ *belong to* ℓ^2_Ψ *and their trivializations at* 0 *given by*

$$\tilde{\tilde{z}}^{[j]}(\lambda) = \begin{cases} 0, & k \in [0,a]_\mathbb{Z}, \\ \tilde{z}^{[j]}(\lambda), & k \in [b+1,\infty)_\mathbb{Z} \end{cases}$$

belong to dom T_F *for all* $j = 1, \ldots, n$*, where* a, b *are the endpoints of the discrete interval* $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}$ *from Hypothesis* 2.6.

Proof. For better clarity, the proof is divided into three steps, which concerns with the behavior of the columns of $\tilde{Z}(\lambda)$ in a neighborhood of ∞. More precisely, by using a recessive solution of (S_λ) we construct a sequence of $2n \times n$ matrix-valued solutions of this system, which converges to $\tilde{Z}(\lambda)$ (the first step). These solutions give rise to pairs $\{\tilde{z}^{[j,m]}, \tilde{\tilde{f}}^{[j,m]}\} \in T_0 - \lambda I$ (the second step), which satisfy both limit conditions in (2.2) from the characterization of T_F (the third step).

Step 1. Since $\lambda \le \nu$, the system (S_λ) is disconjugate on $[0,\infty)_{\mathbb{Z}}$ and eventually controllable as well by Corollary 2.8. Thus, it possesses a recessive solution $\tilde{Z}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ by Theorem 2.5 and we denote its columns as $\tilde{z}^{[1]}(\lambda), \ldots, \tilde{z}^{[n]}(\lambda)$. The disconjugacy of ($\mathrm{S}_\lambda$) implies, in addition, that $-\tilde{X}_k^{-1}(\lambda)\mathcal{B}_k\tilde{X}_{k+1}^{*-1}(\lambda) \ge 0$, and so for the matrix-valued sequence $\Lambda_k := \sum_{j=0}^{k-1} -\tilde{X}_j^{-1}(\lambda)\mathcal{B}_j\tilde{X}_{j+1}^{*-1}(\lambda)$ we have $\lim_{k\to\infty}\Lambda_k^{-1} = 0$ by Theorem 2.5. Let us define the so-called associated dominant solution of (S_λ) as

$$\hat{X}_k(\lambda) := \tilde{X}_k(\lambda)\Lambda_k \quad \text{and} \quad \hat{U}_k(\lambda) := \tilde{U}_k(\lambda)\Lambda_k + \tilde{X}_k^{*-1}.$$

Then $\tilde{Z}(\lambda)$ and $\hat{Z}(\lambda)$ form normalized conjoined bases of (S_λ), the matrices $\hat{X}_k(\lambda)$ are eventually nonsingular by the controllability of (S_λ), and $\hat{X}_k^{-1}(\lambda)\tilde{X}_k(\lambda)$ is Hermitian. For a fixed $m \in [0,\infty)_{\mathbb{Z}}$ large enough we define (suppressing the "dependence" on λ)

$$X_k^{[m]} := \tilde{X}_k - \hat{X}_k\hat{X}_m^{-1}\tilde{X}_m \quad \text{and} \quad U_k^{[m]} := \tilde{U}_k - \hat{U}_k\hat{X}_m^{-1}\tilde{X}_m, \quad k \in [0,\infty)_{\mathbb{Z}}.$$

Then $Z^{[m]}$ is a solution of (S_λ) as a linear combination of two solutions of this system with

$$Z_m^{[m]} = \begin{pmatrix} 0 \\ \tilde{U}_m - \hat{U}_m\hat{X}_m^{-1}\tilde{X}_m \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{X}_m^{*-1} \end{pmatrix}.$$

Furthermore, it holds

$$X_k^{[m]} = \tilde{X}_k - \tilde{X}_k\Lambda_k\Lambda_m^{-1}\tilde{X}_m^{-1}\tilde{X}_m = \tilde{X}_k[I - \Lambda_k\Lambda_m^{-1}] \to \tilde{X}_k,$$
$$U_k^{[m]} = \tilde{U}_k - [\tilde{U}_k\Lambda_k + \tilde{X}_k^{*-1}]\Lambda_m^{-1}\tilde{X}_m^{-1}\tilde{X}_m = \tilde{U}_k[I - \Lambda_k\Lambda_m^{-1}] - \tilde{X}_k^{*-1}\Lambda_m^{-1} \to \tilde{U}_k$$

as $m \to \infty$.

Step 2. Let

$$z^{[j,m]} = \begin{pmatrix} x^{[j,m]} \\ u^{[j,m]} \end{pmatrix} := Z^{[m]}e_j$$

and

$$\tilde{z}_k^{[j,m]} = \begin{pmatrix} \tilde{x}_k^{[j,m]} \\ \tilde{u}_k^{[j,m]} \end{pmatrix} := \begin{cases} z_k^{[j,m]}, & k \in [0,m]_{\mathbb{Z}}, \\ 0, & k \in [m+1,\infty)_{\mathbb{Z}}, \end{cases}$$

and

$$\tilde{f}_k^{[j,m]} := \begin{cases} 0, & k \neq m, \\ \begin{pmatrix} -\mathcal{W}_m^{-1} \hat{X}_m^{*-1} e_j \\ 0 \end{pmatrix}, & k = m, \end{cases}$$

with $\mathcal{W}_m$ being the $n \times n$ left upper block of Ψ_m. Then obviously $\tilde{z}^{[j,m]}, \tilde{f}^{[j,m]} \in \ell^2_\Psi$, the sequence $\tilde{z}^{[j,m]}$ is admissible, and by a direct calculation we can verify that $\mathcal{L}(\tilde{z}^{[j,m]})_k = \Psi_k(\lambda \tilde{z}_k^{[j,m]} + \tilde{f}_k^{[j,m]})$ for all $k \in [0,\infty)_{\mathbb{Z}}$, i.e., $\{\tilde{z}^{[j,m]}, \tilde{f}^{[j,m]}\} \in T_{\max} - \lambda I$ for any $m \in [0,\infty)_{\mathbb{Z}}$. In addition, according to the Patching lemma established in [48, Lemma 3.1], we have also $\{\tilde{\tilde{z}}^{[j,m]}, \tilde{\tilde{f}}^{[j,m]}\} \in T_{\max} - \lambda I$ with

$$\tilde{\tilde{z}}_k^{[j,m]} = \begin{cases} 0, & k \in [0,a]_{\mathbb{Z}}, \\ \tilde{z}_k^{[j,m]}, & k \in [b+1,\infty)_{\mathbb{Z}} \end{cases} \quad \text{and} \quad \tilde{\tilde{f}}_k^{[j,m]} = \begin{cases} 0, & k \in [0,a]_{\mathbb{Z}}, \\ \tilde{f}_k^{[j,m]}, & k \in [b+1,\infty)_{\mathbb{Z}}, \end{cases}$$

which yields that $\{\tilde{\tilde{z}}^{[j,m]}, \tilde{\tilde{f}}^{[j,m]}\} \in T_0 - \lambda I$ for all $m > b+1$.

Step 3. Now, we show that $\{\tilde{\tilde{z}}^{[j,m]}, \tilde{\tilde{f}}^{[j,m]}\} \in T_0 - \lambda I \subseteq T_{\max} - \lambda I$ is such that, for all $j \in \{1,\dots,n\}$, it satisfies $\tilde{\tilde{z}}^{[j,m]} \to \tilde{z}^{[j]}$ as $m \to \infty$ and simultaneously

$$\langle \tilde{\tilde{f}}^{[j,m]} - \tilde{\tilde{f}}^{[j,\ell]}, \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]} \rangle \to 0 \quad \text{as } m,\ell \to \infty. \tag{3.1}$$

So, without loss of generality, let $\ell > m > b+1$. Then, by a direct calculation, we get

$$\begin{aligned}
&\langle \tilde{\tilde{f}}^{[j,m]} - \tilde{\tilde{f}}^{[j,\ell]}, \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]} \rangle_\Psi \\
&\quad = \sum_{k=0}^{\infty} (\tilde{\tilde{f}}_k^{[j,m]} - \tilde{\tilde{f}}_k^{[j,\ell]})^* \Psi_k (\tilde{\tilde{z}}_k^{[j,m]} - \tilde{\tilde{z}}_k^{[j,\ell]}) \\
&\quad = \sum_{k=b+1}^{\infty} (\tilde{\tilde{f}}_k^{[j,m]*} \Psi_k \tilde{\tilde{z}}_k^{[j,m]} + \tilde{\tilde{f}}_k^{[j,\ell]*} \Psi_k \tilde{\tilde{z}}_k^{[j,\ell]} - \tilde{\tilde{f}}_k^{[j,m]*} \Psi_k \tilde{\tilde{z}}_k^{[j,\ell]} - \tilde{\tilde{f}}_k^{[j,\ell]*} \Psi_k \tilde{\tilde{z}}_k^{[j,m]}) \\
&\quad = \tilde{f}_m^{[j,m]*} \Psi_m \tilde{z}_m^{[j,m]} + \tilde{f}_\ell^{[j,\ell]*} \Psi_\ell \tilde{z}_\ell^{[j,\ell]} - \tilde{f}_m^{[j,m]*} \Psi_m \tilde{z}_m^{[j,\ell]} - \tilde{f}_\ell^{[j,\ell]*} \Psi_\ell \tilde{z}_\ell^{[j,m]} \\
&\quad = 0 + 0 - \begin{pmatrix} -\mathcal{W}_m^{-1} \hat{X}_m^{*-1} e_j \\ 0 \end{pmatrix}^* \begin{pmatrix} \mathcal{W}_m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X}_m - \hat{X}_m \hat{X}_\ell^{-1} \tilde{X}_\ell \\ \tilde{U}_m - \hat{U}_m \hat{X}_\ell^{-1} \tilde{X}_\ell \end{pmatrix} - 0 \\
&\quad = \mathrm{e}_j^* \hat{X}_m^{-1} \mathcal{W}_m^{-1} \mathcal{W}_m (\tilde{X}_m - \hat{X}_m \hat{X}_\ell^{-1} \tilde{X}_\ell) e_j \\
&\quad = \mathrm{e}_j^* (\hat{X}_m^{-1} \tilde{X}_m - \hat{X}_\ell^{-1} \tilde{X}_\ell) e_j,
\end{aligned}$$

where we used the special block structure of Ψ_k and the facts that $\tilde{z}_\ell^{[j,m]} = 0$ and $\tilde{X}_m^{[m]} = 0 = \tilde{X}_\ell^{[\ell]}$. Thus, (3.1) holds due to the definition of the recessive solution. Furthermore, from Theorem 2.9 we know that $T_{\min} - \lambda I$ is bounded from below by a lower bound $c - \lambda > 0$, so

$$\langle \tilde{\tilde{f}}^{[j,m]} - \tilde{\tilde{f}}^{[j,\ell]}, \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]} \rangle_\Psi \geq (c - \lambda) \langle \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]}, \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]} \rangle_\Psi,$$

which together with the previous conclusion implies that also

$$\langle \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]}, \tilde{\tilde{z}}^{[j,m]} - \tilde{\tilde{z}}^{[j,\ell]} \rangle_\Psi \to 0$$

as $m, \ell \to \infty$, i.e., the sequence $\{\tilde{\tilde{z}}^{[j,m]}\}_{m=0}^\infty$ is a Cauchy sequence in ℓ_Ψ^2. The definiteness condition guarantees that each $\tilde{\tilde{z}}^{[j,m]}$ gives rise to a unique equivalence class, so each column $\tilde{z}^{[j]}$ of the recessive solution belongs to ℓ_Ψ^2. Consequently, $\tilde{\tilde{z}}^{[j]}$ belongs to the domain of the Friedrichs extension of $T_{\min} - \lambda I$, which is equal to $T_F - \lambda I$ as shown in [6, identity (5.3.4)]. Therefore, $\tilde{\tilde{z}}^{[j]} \in \operatorname{dom} T_F$ for all $j \in 1, \ldots, n$ and the proof is complete. ■

In the last lemma, we prove that the linear relation

$$U := \{\{z, f\} \in T_{\max} \mid x_0 = 0 \text{ and } \lim_{k\to\infty} z_k^* \mathcal{J} \tilde{z}_k^{[i_j]} = 0 \text{ for all } j = 1, \ldots, d - n\} \tag{3.2}$$

is a self-adjoint extension of $T_{\min}$, where $\tilde{z}^{[i_j]} = \tilde{z}^{[i_j]}(\lambda)$ is the i_j-th column of the recessive solution $\tilde{Z}(\lambda)$ being such that $\tilde{z}_m^{[i_j]} = e_{i_j}$ for a suitable $m \in [0, \infty)_{\mathbb{Z}}$, arbitrary $\lambda < \nu$, and certain indices i_j with $j = 1, \ldots, d - n$ specified in the proof of Lemma 3.3, see equality (3.3). This is done by a construction of the matrix Υ mentioned in Theorem 2.10. Note that we do not emphasize the dependence of U on λ in (3.2), because we will show in Theorem 3.4 that it is only formal in the present context and U represents a Friedrichs extension of $T_{\min}$ for all suitable λ.

Lemma 3.3. *Let assumptions of Lemma 3.2 hold and $\lambda < \nu$ be arbitrary. The linear relation U defined in (3.2) is a self-adjoint extension of $T_{\min}$.*

Proof. Since $\lambda < \nu$, it follows from Theorem 2.9 that system (S_λ) possesses $n \leq d \leq 2n$ linearly independent square summable solutions, so the first assumption of Theorem 2.10 is satisfied and a self-adjoint extension of $T_{\min}$ exists. The proof is therefore completed by, at first, constructing a matrix-valued solution $\Theta(\lambda) \in \mathbb{C}([0, \infty)_{\mathbb{Z}})^{2n\times d}$ of system (S_λ) consisting of d linearly independent square summable solutions satisfying the second assumption of Theorem 2.10, and thereafter constructing matrices M and L satisfying the conditions in (2.10) such that the corresponding self-adjoint extension displayed in (2.11) is equal to U.

From Lemma 3.2, we know that all n columns of a recessive solution $\tilde{Z}(\lambda)$ are square summable and, without loss of generality, we may assume that $\tilde{X}_m(\lambda) = I$ for some $m \in [0,\infty)_{\mathbb{Z}}$. We complete these solutions with the remaining $d-n$ linearly independent square summable solutions of (S_λ), which can be taken as the columns of some $\widehat{Z}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times(d-n)}$. For simplicity, we suppress the dependence on λ in the rest of the proof. If we put

$$\widehat{\widehat{Z}} := \widehat{Z} - \widetilde{Z}\widehat{X}_m \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times(d-n)},$$

then it solves (S_λ) and $\widehat{\widehat{Z}}_m = \begin{pmatrix} 0 \\ \widehat{\widehat{U}}_m \end{pmatrix}$. Since $\widehat{\widehat{X}}_m = 0$, it follows that $\operatorname{rank} \widehat{\widehat{U}}_m = d-n$, and one can easily deduce that, after an appropriate constant multiple of $\widehat{\widehat{Z}}$, there are indices $i_1,\dots,i_{d-n}$ such that

$$\begin{pmatrix} e^*_{i_1} \\ \vdots \\ e^*_{i_{d-n}} \end{pmatrix} \widehat{\widehat{U}}_m = I_{d-n}. \tag{3.3}$$

Then we can build the $2n\times d$ matrix-valued solution $\Theta = (\Theta^{[1]} \quad \Theta^{[2]} \quad \Theta^{[3]})$ mentioned above from $\Theta^{[2]} := \widehat{\widehat{Z}}$ and the blocks

$$\Theta^{[1]} := \widetilde{Z} \begin{pmatrix} e^*_{i_1} \\ \vdots \\ e^*_{i_{d-n}} \end{pmatrix}^* \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times(d-n)}$$

and

$$\Theta^{[3]} := \widetilde{Z} \begin{pmatrix} e^*_{s_1} \\ \vdots \\ e^*_{s_{2n-d}} \end{pmatrix}^* \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times(2n-d)},$$

where the indices $i_1,\dots,i_{d-n} \in \{1,\dots,n\}$ correspond to the choice of rows of $\widehat{\widehat{U}}_m$ described in (3.3) and $s_1,\dots,s_{2n-d} \in \{1,\dots,n\}\backslash\{i_1,\dots,i_{d-n}\}$ are all the remaining indices. To justify this choice, we need to show that the $2(d-n)\times 2(d-n)$ leading principal submatrix of the $d\times d$ matrix $\Upsilon := \Theta_0^* \mathcal{J}\, \Theta_0$ has a full rank. Since $\lambda \in \mathbb{R}$, it holds $\Upsilon = \Theta_m^* \mathcal{J}\, \Theta_m$ by the Wronskian-type identity, see [49, identity (3.4)], and the submatrix can be decomposed as

$$\Upsilon_{2(d-n)\times 2(d-n)} = \begin{pmatrix} \Upsilon^{[1,1]}_{2(d-n)\times 2(d-n)} & \Upsilon^{[1,2]}_{2(d-n)\times 2(d-n)} \\ \Upsilon^{[2,1]}_{2(d-n)\times 2(d-n)} & \Upsilon^{[2,2]}_{2(d-n)\times 2(d-n)} \end{pmatrix}$$

$$= \begin{pmatrix} \Theta_m^{[1]*} \mathcal{J}\, \Theta_m^{[1]} & \Theta_m^{[1]*} \mathcal{J}\, \Theta_m^{[2]} \\ \Theta_m^{[2]*} \mathcal{J}\, \Theta_m^{[1]} & \Theta_m^{[2]*} \mathcal{J}\, \Theta_m^{[2]} \end{pmatrix}.$$

Then $\Upsilon^{[1,1]}_{2(d-n)\times 2(d-n)} = 0$ as it is a submatrix of $\widetilde{Z}^*_m \mathcal{J} \widetilde{Z}_m$, which is zero by the definition of the recessive solution (it has to be a conjoined basis), while $\widetilde{X}_m = I$ and $\hat{\hat{X}}_m = 0$ yield $\widetilde{Z}^*_m \mathcal{J} \hat{\hat{Z}}_m = \hat{\hat{U}}_m$, so

$$\Upsilon^{[1,2]}_{2(d-n)\times 2(d-n)} = -\Upsilon^{[2,1]}_{2(d-n)\times 2(d-n)} = \begin{pmatrix} e^*_{i_1} \\ \vdots \\ e^*_{i_{d-n}} \end{pmatrix} \hat{\hat{U}}_m = I_{d-n}$$

by (3.3) and (1.2). In addition, $\hat{\hat{X}}_m = 0$ implies also $\Upsilon^{[2,2]}_{2(d-n)\times 2(d-n)} = \hat{\hat{Z}}^*_m \mathcal{J} \hat{\hat{Z}}_m = 0$. Therefore,

$$\operatorname{rank} \Upsilon_{2(d-n)\times 2(d-n)} = \operatorname{rank} \begin{pmatrix} 0 & I_{d-n} \\ -I_{d-n} & 0 \end{pmatrix} = 2(d-n), \tag{3.4}$$

i.e., the matrix-valued solution Θ satisfies the assumption (ii) of Theorem 2.10.

It remains to express the linear relation U from (3.2) as in (2.11). If we put

$$M := \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L := \begin{pmatrix} 0 & 0 \\ I_{d-n} & 0 \end{pmatrix},$$

then we can verify by a simple calculation that these matrices satisfy the conditions in (2.10). Simultaneously, the equality

$$0 = M z_0 - L \begin{pmatrix} (\Theta e_1, z)_\infty \\ \vdots \\ (\Theta e_{2(d-n)}, z)_\infty \end{pmatrix} = \begin{pmatrix} x_0 \\ 0_{d-n} \end{pmatrix} - \begin{pmatrix} 0_n \\ (\Theta e_1, z)_\infty \\ \vdots \\ (\Theta e_{d-n}, z)_\infty \end{pmatrix} = \begin{pmatrix} x_0 \\ (\Theta e_1, z)_\infty \\ \vdots \\ (\Theta e_{d-n}, z)_\infty \end{pmatrix}$$

utilized in (2.11) is equivalent to the pair of conditions $x_0 = 0$ and $\lim_{k\to\infty} z^*_k \mathcal{J} \tilde{z}^{[i_j]}_k = 0$, because $\Theta e_j = \tilde{z}^{[i_j]}$ for all $j \in \{1, \dots, d-n\}$. Therefore, the linear relation U is a self-adjoint extension of $T_{\min}$. ■

Now, upon combining the preceding lemmas and the self-adjointness of the Friedrichs extension of $T_{\min}$ and of the linear relation U, we obtain the main result showing that $T_F = U$.

Theorem 3.4. *Let Hypothesis 2.6 be satisfied and $\nu \in \mathbb{R}$ be such that system* (S_ν) *is disconjugate on $[0,\infty)_{\mathbb{Z}}$ and eventually controllable. Then, for any $\lambda < \nu$, the linear relation U defined in* (3.2) *is the Friedrichs extension of $T_{\min}$, i.e.,*

$$T_F = \{\{z, f\} \in T_{\max} \mid x_0 = 0 \text{ and } \lim_{k\to\infty} z^*_k \mathcal{J} \tilde{z}^{[i_j]}_k(\lambda) = 0 \text{ for all } j = 1, \dots, d-n\}.$$

In particular, if system (S_λ) *is in the limit point case (i.e., $d = n$), then*

$$T_F = \{\{z, f\} \in T_{\max} \mid x_0 = 0\},$$

while in the limit circle case (i.e., $d = 2n$), we have

$$T_F := \{\{z, f\} \in T_{\max} \mid x_0 = 0 \text{ and } \lim_{k\to\infty} z_k^* \mathcal{J} \tilde{z}_k^{[j]}(\lambda) = 0 \text{ for all } j = 1, \dots, n\}.$$

Proof. We recall that the given assumptions guarantee the existence of the Friedrichs extension of $T_{\min}$ by Theorem 2.9. We already know that the linear relation U is a self-adjoint extension of $T_{\min}$ by Lemma 3.3. Now, let $\{z, f\} \in T_F$ be arbitrary. Then also $\{z, f\} \in T_F^*$ and $x_0 = 0$ by Lemma 3.1. Let $\tilde{\tilde{z}}^{[i_1]}, \dots, \tilde{\tilde{z}}^{[i_{d-n}]}$ be as in Lemma 3.2, in particular, $\tilde{\tilde{z}}^{[i_1]}, \dots, \tilde{\tilde{z}}^{[i_{d-n}]} \in \operatorname{dom} T_F$ and they coincide with $\tilde{z}^{[i_1]}, \dots, \tilde{z}^{[i_{d-n}]}$ for all k large enough. The self-adjointness of T_F yields

$$0 = \langle f^{[j]}, z\rangle_\Psi - \langle \tilde{\tilde{z}}^{[i_j]}, f\rangle_\Psi = [(\tilde{\tilde{z}}^{[i_j]}, z)_k]_{k=0}^\infty = \lim_{k\to\infty} z_k^* \mathcal{J} \tilde{\tilde{z}}_k^{[i_j]}$$

for all $j \in \{1, \dots, d-n\}$, where $f^{[1]}, \dots, f^{[d-n]}$ are such that $\{\tilde{\tilde{z}}^{[i_1]}, f^{[1]}\}, \dots, \{\tilde{\tilde{z}}^{[i_{d-n}]}, f^{[d-n]}\} \in T_F$. This shows that $\{z, f\} \in U$, which means $T_F \subseteq U$. However, the self-adjointness of U implies also the opposite inclusion $U = U^* \subseteq T_F^* = T_F$ and, therefore, $T_F = U$. The rest of the statement is a simple consequence of this result in the case when $d = n$ and $d = 2n$, respectively. ∎

Remark 3.5. (i) Besides the more general setting in our treatise, which could be easily applied also in the continuous case, Theorem 3.4 is a discrete analogue of [52, Theorem 4.2], [33, Theorem 3.1], and [24, Theorem 13] concerning the Friedrichs extension of *operators* associated with the linear Hamiltonian differential system. It is worth noting that, in contrast to the mentioned results, we neither require any particular number of linearly independent square summable solutions (e.g., the limit circle case as in [33, Theorem 3.1]) nor formally overdetermine T_F as in [24, Theorem 13], because we explicitly deal only with d "boundary" conditions in accordance with Theorem 2.10; n of them come from $x_0 = 0$ by the first part of the proof of Lemma 3.2, while the remaining $d - n$ specify the behavior at infinity and they are derived from (3.3) so that (3.4) holds, compare with [24, Remark after Theorem 12].

(ii) The square summability of the columns of a recessive solution $\tilde{Z}(\lambda)$ established in Lemma 3.2 will be an object of our further research because it seems to be an essential property rooted in the definition of this solution similarly to the continuous case, see [39].

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