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Quantitative propagation of smallness and spectral estimates for the Schrödinger operator

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Abstract. In this paper, we investigate the quantitative propagation of smallness properties for the Schrödinger operator on a bounded domain in $\mathbb{R}^d$. We extend Logunov and Malinnikova's results concerning the propagation of smallness for A-harmonic functions to solutions of divergence elliptic equations perturbed by real-valued Lipschitz first-order terms and a real-valued bounded zero-order term. We also prove similar results for the gradients of solutions to certain particular equations. This later result enables us to follow the recent strategy of Burq and Moyano for obtaining spectral estimates on rough sets for the Schrödinger operator. Applications to observability estimates and to the null-controllability of associated parabolic equations posed on compact manifolds or the entire Euclidean space are then considered.

1. Introduction

This paper presents several quantitative results on the propagation of smallness for solutions of elliptic partial differential equations and their applications to spectral estimates.

More precisely, our goal is to derive three spheres theorems for solutions and their gradients to divergence elliptic equations perturbed by a bounded zero-order term, i.e., Schrödinger-type equations

$$-\mathrm{div}(A(x)\nabla u) + V(x)u = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is a smooth bounded open connected set of $\mathbb{R}^d$, with $d \geq 1$, $A = A(x)$ is a symmetric uniformly elliptic matrix with Lipschitz entries, and $V = V(x)$ is a bounded real-valued function. From Carleman estimates, it is now classical that the following interpolation inequality holds. For $B \subset \mathcal{K} \subset\subset \Omega$, with B open and $\mathcal{K}$ compact, there exist $C > 0$ and $\alpha \in (0, 1)$ such that, for every solution u to (1.1), we

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have

$$\sup_{\mathcal{K}} |u| \leq C(\sup_{B} |u|)^{\alpha}(\sup_{\Omega} |u|)^{1-\alpha}. \tag{1.2}$$

See for instance [20, Theorem 5.1] or [26, Corollary 2.3] and the references therein.

In [25], in relation to the applications of Yau's conjecture on the volume of the nodal sets for Laplace eigenfunctions (see [23, 24]), the generalization of the three spheres theorems (1.2) for wild sets for solutions u to

$$-\mathrm{div}(A(x)\nabla u) = 0, \quad x \in \Omega, \tag{1.3}$$

was considered. More precisely, Logunov and Malinnikova proved in [25, Theorem 2.1] that, given $E \subset \mathcal{K} \subset\subset \Omega$, with E of positive d-dimensional Lebesgue measure, there exist $C > 0$ and $\alpha \in (0, 1)$ such that, for every solution u to (1.3), one has

$$\sup_{\mathcal{K}} |u| \leq C(\sup_{E} |u|)^{\alpha}(\sup_{\Omega} |u|)^{1-\alpha}.$$

One can even assume that E has positive $(d - 1 + \delta)$-Hausdorff content for every $\delta > 0$. Note that this later result is sharp in the sense that the zeros of the harmonic functions in $\mathbb{R}^d$ for $d \geq 2$ may have positive $(d - 1)$-Hausdorff content. The propagation of smallness for gradients from sets of positive $(d - 1 - \delta)$-Hausdorff content for some (small) δ was also obtained in [25, Theorem 5.1]. As demonstrated in [29], the zeros of $|\nabla u|$ have finite $(d - 2)$-Hausdorff measure. It was conjectured in [25] that the result for $|\nabla u|$ should be expected to hold for sets of positive $(d - 2 + \delta)$-Hausdorff content for any $\delta > 0$. Up to now, this conjecture is still open.

The first goal of this paper is to extend Logunov and Malinnikova's results to the Schrödinger-type equation (1.1). Propagation of smallness for solutions is obtained in full generality in the same setting as [25]. On the other hand, propagation of smallness for gradients is only derived in a particular setting. Indeed, one cannot expect to derive propagation of smallness for gradients of solutions to (1.1) in full generality; see Theorem 2.3 below. Nevertheless, our particular result is sufficient for the applications to spectral estimates that we next describe.

Let M be a compact connected Riemannian manifold of dimension d, possibly with boundary, equipped with a Riemannian metric g. We assume that M is a $\mathscr{C}^1 \cap W^{2,\infty}$, in the sense that the changes of charts are $\mathscr{C}^1$ with Lipschitz derivatives. We consider the following associated elliptic operator:

$$H_{g,V} u = -\Delta_g u + V(x)u, \quad x \in M, \tag{1.4}$$

where g is assumed to be a Lipschitz positive definite metric, $\Delta_g = \mathrm{div}_g \circ \nabla_g$ is the Laplace–Beltrami operator and $V = V(x)$ is a bounded real-valued function. The operator $H_{g,V}$ eventually completed with Dirichlet or Neumann boundary conditions, is an unbounded self-adjoint operator with compact resolvent in $L^2(M)$.

Consequently, it admits an orthonormal basis in $L^2(M)$ of eigenfunctions denoted by $(\varphi_k)_{k\geq 1}$, associated to the sequence of real eigenvalues $(\lambda_k)_{k\geq 1}$ which satisfies

$$\lambda_k \xrightarrow[k\to+\infty]{} +\infty \quad \text{and} \quad \lambda_k \geq -\|V\|_{L^\infty} \quad \text{for all } k \in \mathbb{N}.$$

Given $\Lambda > 0$, we introduce the spectral projector Π_Λ as follows:

$$\Pi_\Lambda u = \sum_{\lambda_k \leq \Lambda} \langle u, \varphi_k \rangle_{L^2} \varphi_k \quad \text{for all } u \in L^2(M).$$

Given a nonempty open subset ω of M, Jerison and Lebeau obtained in [18] through Carleman estimates the following spectral inequality:

$$\|\Pi_\Lambda u\|_{L^2(M)} \leq C e^{C\sqrt{\Lambda}} \|\Pi_\Lambda u\|_{L^2(\omega)} \quad \text{for all } u \in L^2(M). \tag{1.5}$$

This type of estimate is a generalization to linear combination of eigenfunctions of the well-known doubling inequality of Donnelly and Fefferman [10] valid for one eigenfunction. While the constant in the doubling inequality (1.5) is sharp in general according to the vanishing order of the spherical harmonics, it is worth mentioning that it can be made independent of Λ for one single Laplace eigenfunction in surfaces with negative curvature, see [13]. Moreover, (1.5) combining with the so-called Lebeau–Robbiano method [22] leads to the small-time null-controllability of the associated parabolic equations with a control localized in ω. In [3] (see also [2]), Apraiz, Escauriaza, Wang, and Zhang generalize (1.5) to ω of positive d-dimensional Lebesgue measure by assuming that both g and V are analytic. In the very recent work [7], Burq and Moyano replaced the analyticity on the metric g with the sharp Lipschitz assumption, assuming $V = 0$, and obtain (1.5) for ω of positive $(d-\delta)$-Hausdorff content using the new results on propagation of smallness from [25].

The second goal of the paper is to follow Burq and Moyano's strategy starting from our new results on propagation of smallness for gradients of solutions of (1.1), to get new spectral estimates for the Schrödinger operator (1.4) in the compact setting.

On the Euclidean space, we are interested in the following Schrödinger operator:

$$H_{g,V,\kappa} u = -\frac{1}{\kappa(x)} \operatorname{div}(g^{-1}(x)\kappa(x)\nabla u) + V(x)u, \quad x \in \mathbb{R}^d, \tag{1.6}$$

where $g = g(x)$ is a symmetric uniformly elliptic matrix with Lipschitz entries, $\kappa = \kappa(x)$ is a positive bounded Lipschitz function, and $V = V(x)$ is a bounded real-valued function. Notice that $H_{g,V,\kappa}$ is an unbounded self-adjoint operator on $L^2(\mathbb{R}^d, \kappa dx)$. As a consequence, one can still define spectral projectors by

$$\Pi_\Lambda u = 1_{H_{g,V,\kappa}} u = \int_{-\|V\|_\infty}^{\Lambda} dm \quad \text{for all } u \in L^2(\mathbb{R}^d, \kappa dx),$$

where dm is the spectral measure of $H_{g,V,\kappa}$. Contrary to the case of compact manifolds, spectral inequalities of the form

$$\forall \Lambda > 0,\ \exists C_\Lambda > 0,\ \forall u \in L^2(\mathbb{R}^d), \quad \|\Pi_\Lambda u\|_{L^2(\mathbb{R}^d)} \leq C_\Lambda \|\Pi_\Lambda u\|_{L^2(\omega)} \tag{1.7}$$

may require some geometric condition on ω to hold. When $g = I_d$, $\kappa = 1$ and $V = 0$, the Logvinenko–Sereda theorem [27] shows that (1.7) holds if and only if the measurable subset ω is thick. We say that ω is a *thick* subset of $\mathbb{R}^d$ if there exist $R, \gamma > 0$ such that

$$|\omega \cap B(x, R)| \geq \gamma |B(x, R)|, \quad \text{for all } x \in \mathbb{R}^d. \tag{1.8}$$

Under this assumption and still in the case where $g = I_d$, $\kappa = 1$ and $V = 0$, Kovrijkine established in [19] a quantitative version of the Logvinenko–Sereda theorem and showed the following inequality: for every $\Lambda > 0$,

$$\|\Pi_\Lambda u\|_{L^2(\mathbb{R}^d)} \leq \Big(\frac{C_d}{\gamma}\Big)^{C_d(1+R\sqrt{\Lambda})} \|\Pi_\Lambda u\|_{L^2(\omega)} \quad \text{for all } u \in L^2(\mathbb{R}^d), \tag{1.9}$$

where $C_d > 0$ is a positive constant depending only on the dimension. Thanks to estimate (1.9), Egidi and Veselic [14] and Wang, Wang, Zhang, and Zhang [32] established that (1.8) is actually a necessary and sufficient condition for the null-controllability of the associated parabolic equation. These results were generalized by Lebeau and Moyano in [21] under the analyticity assumption on g, V, and κ. Very recently, [6] extended these results to the case of a Lipschitz metric g, a Lipschitz density κ but without potential ($V = 0$), again starting from the results on propagation of smallness by Logunov and Mlinnikova. They were even able to deduce some spectral estimates under weaker assumptions on ω, allowing it to have Lebesgue measure zero.

The third goal of the paper is to follow Burq and Moyano's strategy starting from our new results on propagation of smallness for (1.1) to get new spectral estimates for the Schrödinger operator (1.6) in the non-compact setting.

2. Main results

The goal of this part is to state the main results of the paper, which are quantitative propagation of smallness results for solutions to Schrödinger-type equations in a bounded domain of $\mathbb{R}^d$ and their applications to spectral estimates for Schrödinger operators on compact Riemannian manifolds and the entire Euclidean space.

Recall that, for $s \geq 0$, the s-Hausdorff content (or measure) of a set $E \subset \mathbb{R}^n$ is

$$\mathcal{C}^s_{\mathcal{H}}(E) = \inf\Big\{\sum_j r_j^s \ ;\ E \subset \bigcup_j B(x_j, r_j)\Big\},$$

and the Hausdorff dimension of E is defined as

$$\dim_{\mathcal{H}}(E) = \inf\{s \geq 0 \,;\, \mathcal{C}^s_{\mathcal{H}}(E) = 0\}.$$

We shall denote by $|E|$ the Lebesgue measure of the set E. Let us recall that the Hausdorf content of order d is equivalent to the Lebesgue measure,

$$\exists C_n, c_n > 0,\ \forall A \text{ borelian set, } c_n|A| \leq \mathcal{C}^n_{\mathcal{H}}(A) \leq C_n|A|,$$

and we also have the following relation:

$$\mathcal{C}^s_{\mathcal{H}}(E) > 0 \implies \forall s' \in (0,s),\ \mathcal{C}^{s'}_{\mathcal{H}}(E) \geq \inf(1, \mathcal{C}^s_{\mathcal{H}}(E)). \tag{2.1}$$

2.1. Propagation of smallness for the Schrödinger operator

Let Ω be a bounded domain of $\mathbb{R}^d$. Let us consider the second order elliptic operator

$$H_{A,V}u = -\operatorname{div}(A(x)\nabla u) + V(x)u, \quad x \in \Omega, \tag{2.2}$$

where $A = (a_{ij}(x))_{1,\leq i,j \leq d}$ is a symmetric uniformly elliptic matrix with Lipschitz entries

$$\Lambda_1^{-1}|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda_1|\xi|^2, \quad |a_{ij}(x) - a_{ij}(y)| \leq \Lambda_2|x-y|, \quad x, y \in \Omega, \xi \in \mathbb{R}^d, \tag{2.3}$$

for some $\Lambda_1, \Lambda_2 > 0$, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$, and $V = V(x)$ is a real-valued bounded function, i.e.,

$$V \in L^\infty(\Omega; \mathbb{R}). \tag{2.4}$$

The first main result concerns the propagation of smallness for solutions to Schrödinger-type equations.

Theorem 2.1. *Let $\rho, m, \delta > 0$ and $\mathcal{K}, E \subset \Omega$ be measurable subsets such that*

$$\operatorname{dist}(\mathcal{K}, \partial\Omega) \geq \rho, \quad \operatorname{dist}(E, \partial\Omega) \geq \rho, \quad \text{and} \quad \mathcal{C}^{d-1+\delta}_{\mathcal{H}}(E) \geq m. \tag{2.5}$$

There exist

$$C = C(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \rho, m, \delta) > 0$$

and

$$\alpha = \alpha(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \rho, m, \delta) \in (0,1)$$

such that, for every weak solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the elliptic equation

$$-\operatorname{div}(A(x)\nabla u) + V(x)u = 0 \text{ in } \Omega, \tag{2.6}$$

we have

$$\sup_{\mathcal{K}} |u| \leq C(\sup_E |u|)^\alpha (\sup_\Omega |u|)^{1-\alpha}. \tag{2.7}$$

Let $\kappa = \kappa(x) \in W^{1,\infty}(\Omega)$ satisfying

$$\Lambda_1^{-1} \leq \kappa(x) \leq \Lambda_1 \quad \text{and} \quad |\kappa(x) - \kappa(y)| \leq \Lambda_2 |x - y| \quad \text{for all } x, y \in \Omega. \tag{2.8}$$

The second main result concerns the propagation of smallness for the gradients of particular solutions to Schrödinger-type equations.

Theorem 2.2. *There exists $\delta_d \in (0,1)$, depending only on the dimension d, such that the following holds. Let $\rho, m > 0$, $\delta \in [0, \delta_d]$, and $\mathcal{K}, E \subset \Omega$ be measurable subsets such that*

$$\operatorname{dist}(\mathcal{K}, \partial\Omega) \geq \rho, \quad \operatorname{dist}(E, \partial\Omega) \geq \rho, \quad \text{and} \quad \mathcal{C}_{\mathcal{H}}^{d-\delta}(E) \geq m. \tag{2.9}$$

There exist

$$C = C(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \rho, m, \delta) > 0$$

and

$$\alpha = \alpha(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \rho, m, \delta) \in (0,1)$$

such that, for every weak solution $\hat{u}(x,t) \in W^{1,\infty}(\Omega \times (-1,1))$ of the elliptic equation

$$\begin{cases} -\operatorname{div}_x \cdot (A(x)\nabla_x \hat{u}) - \kappa(x)\partial_{tt}\hat{u} + V(x)\hat{u} = 0 & \text{in } \Omega \times (-1,1), \\ \hat{u}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.10}$$

we have

$$\sup_{x \in \mathcal{K}} |\partial_t \hat{u}(x,0)| \leq C \big(\sup_{x \in E} |\partial_t \hat{u}(x,0)|\big)^{\alpha} \|\hat{u}\|_{W^{1,\infty}(\Omega \times (-1,1))}^{1-\alpha}. \tag{2.11}$$

The new difficulty in proving Theorem 2.1 and Theorem 2.2 is that the results of [25] are actually proved for divergence elliptic operators, and their extensions to operators as in (2.19) are not straightforward. One approach could be to adapt all the steps of their proof to a more general elliptic operator as considered here. However, it is worth mentioning that [25] is not self-contained, as recalled by the authors, and it uses some new deep results from [23, 24]. Moreover, the following remark holds.

Remark 2.3. One cannot expect to derive propagation of smallness for the gradients of solutions to (2.6) in full generality as in [25, Theorem 5.1]. Indeed, as noted in [17, Remark, p. 362], let $v \in C^\infty(\mathbb{R}^n)$, $|v| < 1$, then set $u = v^2 + 1$, which solves $-\Delta u + V(x)u = 0$ in $\mathbb{R}^n$ with $V(x) = (\Delta v^2)/(v^2 + 1)$ and we have $v^{-1}(\{0\}) \subset |\nabla u|^{-1}(\{0\})$. Moreover, every closed set in $\mathbb{R}^n$ is the zero set of a C^∞ function, see [34]. In particular, the set $v^{-1}(\{0\})$ could be an arbitrary closed set of $\mathbb{R}^n$, from which there is no hope to obtain a propagation of smallness result for ∇u.

These are the main reasons why we decided to follow an easier path that uses [25, Theorems 2.1 and 5.1] as a black box.

We now present the main steps for obtaining Theorem 2.1 and Theorem 2.2. Without loss of generality, we first reduce to the case $V \geq 0$. This reduction enables us to construct a positive multiplier that converts the Schrödinger-type equation into a divergence elliptic equation.

Reduction to the case $V \geq 0$. We note that one can reduce the proof of Theorem 2.1 and Theorem 2.2 to the case

$$V \geq 0. \tag{2.12}$$

For Theorem 2.1, by looking at the function $\hat{u}(x,t) = u(x)\exp(\lambda t)$ that solves

$$-\mathrm{div}_x \cdot (A(x)\nabla_x \hat{u}) - \partial_{tt}\hat{u} + (V(x) + \lambda^2)\hat{u} = 0 \text{ in } \Omega \times (-1,+1). \tag{2.13}$$

For $\lambda \geq \|V\|_\infty^{1/2}$, we have that

$$\widehat{V}(t,x) = V(x) + \lambda^2 \geq 0;$$

then one can apply Theorem 2.1 with

$$\begin{aligned}
\widehat{\Omega} &= \Omega \times (-1,+1),\\
\widehat{K} &= K \times \Big(-\frac{1}{2}, +\frac{1}{2}\Big) \subset\subset \widehat{\Omega},\\
\widehat{E} &= E \times \Big(-\frac{1}{2}, +\frac{1}{2}\Big) \subset\subset \widehat{\Omega}
\end{aligned}$$

to $\hat{u}$ satisfying (2.13). For Theorem 2.2, the argument is in the same spirit by adding a ghost variable considering $\hat{v}(x,y,t) = \hat{u}(x,t)\exp(\lambda y)$ for some $\lambda \geq \|V\|_\infty^{1/2}$ that solves

$$\begin{cases}
-\mathrm{div}_x \cdot (A(x)\nabla_x \hat{v}) - \partial_{yy}\hat{v} - \kappa(x)\partial_{tt}\hat{v} + (V(x)+\lambda^2)\hat{v} = 0 \\
\qquad\qquad (x,y,t) \in \Omega \times (-1,1) \times (-1,+1), \\
\hat{v}(x,y,0) = 0 \quad \text{in } \Omega \times (-1,1),
\end{cases} \tag{2.14}$$

and applying Theorem 2.2 with

$$\widehat{\Omega} = \Omega \times (-1,1), \quad \widehat{K} = K \times \Big(-\frac{1}{2}, \frac{1}{2}\Big) \subset\subset \widehat{\Omega}, \quad \widehat{E} = E \times \Big(-\frac{1}{2}, \frac{1}{2}\Big) \subset\subset \widehat{\Omega}$$

to $\hat{v}$ that solves (2.14). Therefore, in all the following, we will only consider the case (2.12).

Reduction to a divergence elliptic equation. The key ingredient in the proofs of Theorems 2.1 and 2.2 consists in the construction of a suitable positive multiplier to the equation $H_{A,V}\varphi = 0$, in the case when $V \geq 0$, which shows the existence of $\varphi \in W^{1,\infty}(\Omega_0)$ satisfying

$$-\mathrm{div}(A\nabla\varphi) + V\varphi = 0 \quad \text{in } \Omega_0 \qquad \text{and} \qquad \varphi > 0 \quad \text{in } \Omega_0,$$

where Ω_0 is a smooth domain satisfying

$$\mathrm{dist}(\mathcal{K}, \partial\Omega_0) \geq \frac{\rho}{2}, \quad \mathrm{dist}(E, \partial\Omega_0) \geq \frac{\rho}{2}, \quad \text{and} \quad \Omega_0 \subset\subset \Omega.$$

This enables us to reduce the task of obtaining propagation of smallness for solutions to (2.6) to the application of propagation of smallness for solutions to divergence elliptic equations for $v = u/\varphi$. Indeed, v now satisfies

$$-\mathrm{div}(\varphi^2 A\nabla v) = 0 \quad \text{in } \Omega_0.$$

Thanks to suitable lower and upper bounds on φ, these propagation of smallness estimates obtained on v provide estimates on u. For the case when u satisfies (2.10), the same strategy works but now the equation satisfied by $\hat{v} = \hat{u}/\varphi$ is

$$-\mathrm{div}_x \cdot (\varphi^2 A(x)\nabla_x \hat{v}) - \partial_t(\varphi^2\kappa(x)\partial_t\hat{v}) = 0 \quad \text{in } \Omega_0 \times (-1, 1),$$

which is a divergence elliptic equation with the extra condition $\hat{v}(x, 0) = 0$. Propagation of smallness estimates on $|\nabla_{t,x} v|$ on particular sets will then provide the expected result (2.11).

The following remarks are in order.

Remark 2.4. The main advantage of our proof is that we obtain propagation of smallness for solutions to Schrödinger-type equations in a general setting without redoing all the arguments of Logunov and Malinnikova. However, the main drawback of such a strategy is that, for the gradient of solutions, the application of propagation of smallness for gradients [25, Theorem 5.1] is applied to $v = u/\varphi$, making it difficult to deduce estimates on $|\nabla u|$. Actually, this is not only a technical difficulty, since, as recalled in Theorem 2.3, one cannot expect to obtain propagation of smallness for gradients in full generality. Nevertheless, this method allows us to deal with the particular setting of (2.10) and to obtain Theorem 2.2. Fortunately, the propagation of smallness estimates (2.11) is sufficient for our applications to spectral estimates.

Remark 2.5. Finally, we would like to highlight the very recent preprint [35], from which one can also obtain Theorems 2.1 and 2.2, assuming that $V \in W^{1,\infty}(\Omega; \mathbb{R})$, starting from [25, Theorem 2.1 and Theorem 5.1] as a black box. Its strategy was rather different from ours because it consists in putting the zero-order term V in the principal part of the operator by adding a ghost variable. The Lipschitz assumption on V seemed to be difficult to remove with such a method. Nevertheless, this strategy inspires the next two results treating real-valued first-order terms, see below. Note that the next versions of [35] actually remove the Lipschitz assumption on V by using a similar strategy to ours.

Last but not least, the parameter $\delta_d \in (0,1)$ appearing in Theorem 2.2 is small a priori and actually comes from [25, Theorem 5.1]. The extension to an arbitrary $\delta_d \in (0,1)$ is an open and very likely difficult open problem. However, it is worth mentioning that, for $d = 1$, we can take an arbitrary $\delta \in (0,1)$ by using the propagation of smallness result for gradients [36, Theorem 1.2] in dimension $d+1 = 2$ instead of [25, Theorem 5.1], together with our strategy of reduction to a divergence elliptic equation. The same remark will apply next for spectral estimates and applications. See also [31] for a similar strategy when $d = 1$ but with the use of [36, Theorem 1.1].

Let us end this section by presenting two results on the propagation of smallness for Schrödinger operators with a drift term. Let $W \in W^{1,\infty}(\Omega, \mathbb{R}^d)$.

Corollary 2.6. *Let $\rho, m, \delta > 0$ and $\mathcal{K}, E \subset \Omega$ be measurable subsets such that*

$$\operatorname{dist}(\mathcal{K}, \partial\Omega) \geq \rho, \quad \operatorname{dist}(E, \partial\Omega) \geq \rho, \quad \textit{and} \quad \mathcal{C}_{\mathcal{H}}^{d-1+\delta}(E) \geq m.$$

There exist positive constant $C = C(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \|W\|_{W^{1,\infty}}, \rho, m, \delta) > 0$ and $\alpha = \alpha(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \|W\|_{W^{1,\infty}}, \rho, m, \delta) \in (0,1)$ such that for every weak solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the elliptic equation

$$-\operatorname{div}(A(x)\nabla u) + W(x) \cdot \nabla u + V(x)u = 0 \text{ in } \Omega, \tag{2.15}$$

we have

$$\sup_{\mathcal{K}} |u| \leq C(\sup_E |u|)^\alpha (\sup_\Omega |u|)^{1-\alpha}. \tag{2.16}$$

Corollary 2.7. *There exists $\delta_d \in (0,1)$ depending only on the dimension d such that the following holds. Let $\rho, m > 0$, $\delta \in [0, \delta_d]$ and $\mathcal{K}, E \subset \Omega$ be measurable subsets such that*

$$\operatorname{dist}(\mathcal{K}, \partial\Omega) \geq \rho, \quad \operatorname{dist}(E, \partial\Omega) \geq \rho, \quad \textit{and} \quad \mathcal{C}_{\mathcal{H}}^{d-\delta}(E) \geq m.$$

There exist positive constants $C = C(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \|W\|_{W^{1,\infty}}, \rho, m, \delta) > 0$ and $\alpha = \alpha(\Omega, \Lambda_1, \Lambda_2, \|V\|_\infty, \|W\|_{W^{1,\infty}}, \rho, m, \delta) \in (0,1)$ such that, for every weak solution $\hat{u}(x,t) \in W^{1,\infty}(\Omega \times (-1,1))$ of the elliptic equation

$$\begin{cases} -\operatorname{div}_x(A(x)\nabla_x \hat{u}) - \kappa(x)\partial_{tt}\hat{u} + W(x) \cdot \nabla_x \hat{u} + V(x)\hat{u} = 0 & \text{in } \Omega \times (-1,1), \\ \hat{u}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.17}$$

we have

$$\sup_{x\in\mathcal{K}} |\partial_t \hat{u}(x,0)| \le C(\sup_{x\in E} |\partial_t \hat{u}(x,0)|)^{\alpha} \|\hat{u}\|^{1-\alpha}_{W^{1,\infty}(\Omega\times(-1,1))}. \tag{2.18}$$

The proofs of Corollaries 2.6 and 2.7 respectively follow from Theorems 2.1 and 2.2 and are given in Section 3.3. The Lipschitz assumption on W could probably be removed and seems to be purely technical.

2.2. Spectral estimates on compact manifolds and applications

Let M be a $C^1 \cap W^{2,\infty}$, connected, compact manifold of dimension $d \geq 1$, possibly with boundary, equipped with a Riemannian metric g. In this section, we fix an atlas $\mathcal{A} = (\mathcal{V}_\sigma, \Psi_\sigma)_{\sigma\in\mathcal{J}}$ containing a finite number of charts with $(W^{2,\infty} \cap C^1)$-diffeomorphisms $\Psi_\sigma\colon \mathcal{V}_\sigma \to \Psi_\sigma(\mathcal{V}_\sigma) \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$ such that there exists a family of open sets $(\mathcal{U}_\sigma)_{\sigma\in\mathcal{J}}$ satisfying

$$M = \bigcup_{\sigma\in\mathcal{J}} \mathcal{U}_\sigma,$$

and such that $\mathcal{U}_\sigma$ is compactly included in the open set $\mathcal{V}_\sigma$ in M, for all $\sigma \in \mathcal{J}$. In the case when M is assumed to be without boundary, then, for any $\sigma \in \mathcal{J}$, $\Psi_\sigma(\mathcal{V}_\sigma)$ is an open set of $\mathbb{R}^d$.

Let us consider the second order elliptic operator

$$H_{g,V} u = -\Delta_g u + V(x)u, \quad x \in M, \tag{2.19}$$

where $V = V(x)$ is a real-valued bounded function, i.e.,

$$V \in L^\infty(M;\mathbb{R}),$$

and g is assumed to be Λ_1-elliptic and Λ_2-Lipschitz, in the sense that if $(g^\sigma_{i,j})_{1\le i,j\le d}$ are the local coordinates of g in a local chart $(\mathcal{V}_\sigma, \Psi_\sigma)$,

$$\Lambda_1^{-1}|\xi|^2 \le \sum_{i,j} g^\sigma_{i,j}(\Psi_\sigma^{-1}(x))\xi_i\xi_j \le \Lambda_1|\xi|^2 \quad \text{for all } x \in \Psi_\sigma(\mathcal{V}_\sigma), \xi \in \mathbb{R}^d,$$

and

$$|g^\sigma_{ij} \circ \Psi_\sigma^{-1}(x) - g^\sigma_{ij} \circ \Psi_\sigma^{-1}(y)| \le \Lambda_2|x-y| \quad \text{for all } x, y \in \Psi_\sigma(\mathcal{V}_\sigma),$$

for some $\Lambda_1 > 0$ and $\Lambda_2 > 0$.

Let us define

$$\mathrm{Dom}(H_{g,V}) = \{u \in H^2(M) : u = 0 \text{ on } \partial M \text{ or } \partial_\nu u = 0 \text{ on } \partial M\}.$$

Note that if $\partial M = \emptyset$, then $\mathrm{Dom}(H_{g,V}) = H^2(M)$. Under these assumptions, it is well known that $H_{g,V}$ admits an orthonormal basis in $L^2(M)$ of eigenfunctions, denoted by $(\varphi_k)_{k\geq 1}$, associated to the sequence of real eigenvalues $(\lambda_k)_{k\geq 1}$. Given $\Lambda > 0$, we introduce the spectral projector Π_Λ as follows:

$$\Pi_\Lambda u = \sum_{\lambda_k \leq \Lambda} \langle u, \varphi_k \rangle_{L^2} \varphi_k \quad \text{for all } u \in L^2(M).$$

Our first main result states the following spectral estimates for the Schrödinger operator (2.19).

Theorem 2.8. *There exists $\delta_d \in (0,1)$ such that, for all $\delta \in [0, \delta_d]$, for every observation set $\omega \subset M$ satisfying $\mathcal{C}^{d-\delta}_{\mathcal{H}}(\omega) \geq m > 0$, there exists $C = C(M, g, V, \mathcal{A}, \delta, m) > 0$ such that for every $\Lambda > 0$, we have*

$$\|\Pi_\Lambda u\|_{L^\infty(M)} \leq C e^{C\sqrt{\Lambda}} \sup_{x\in\omega} |(\Pi_\Lambda u)(x)| \quad \textit{for all } u \in L^2(M). \tag{2.20}$$

In particular, for every measurable set $\omega \subset M$ satisfying $|\omega| \geq m > 0$, there exists $C = C(M, g, V, \mathcal{A}, m) > 0$ such that, for every $\Lambda > 0$, we have

$$\|\Pi_\Lambda u\|_{L^\infty(M)} \leq C e^{C\sqrt{\Lambda}} \|\Pi_\Lambda u\|_{L^1(\omega)} \quad \textit{for all } u \in L^2(M). \tag{2.21}$$

The following comments are in order. First, Theorem 2.8 generalizes [7, Theorem 1] to the case of the Schrödinger operator as in (2.19). Secondly, inequality (2.21) is an L^∞–L^1 spectral estimate from which one can easily deduce the more standard L^2–L^2 spectral estimate as recalled in (1.5) by using the continuous embeddings $L^\infty(M) \hookrightarrow L^2(M)$ and $L^2(\omega) \hookrightarrow L^1(\omega)$

$$\|\Pi_\Lambda u\|_{L^2(M)} \leq C e^{C\sqrt{\Lambda}} \|\Pi_\Lambda u\|_{L^2(\omega)} \quad \text{for all } u \in L^2(M). \tag{2.22}$$

On the other hand, inequality (2.20) is an L^∞–L^∞ spectral estimate from which one can only deduce an L^2–L^∞ spectral estimate

$$\|\Pi_\Lambda u\|_{L^2(M)} \leq C e^{C\sqrt{\Lambda}} \sup_{x\in\omega} |(\Pi_\Lambda u)(x)| \quad \text{for all } u \in L^2(M). \tag{2.23}$$

Moreover, without extra assumption on ω here, there is no hope to transform (2.23) into an L^2–L^2 estimate because one can have $|\omega| = 0$. Last but not least, the parameter $\delta \in (0,1)$ is small a priori and actually comes from Theorem 2.2, so from [25, Theorem 5.1]. The extension to an arbitrary $\delta \in (0,1)$ is an open and, very likely, difficult open problem.

The strategy of the proof of Theorem 2.8 will follow the one in [7, Theorem 1] that uses propagation of smallness for the gradient of solutions to elliptic equations from [25, Theorem 5.1]. The new difficulty here is that the results of [25] are actually proved for divergence elliptic operators, and their extensions to operators as in (2.19) are not straightforward. This is why we will actually use our new Theorem 2.2.

We now focus on the time evolution equation

$$\begin{cases} \partial_t u + H_{g,V} u = 0 & \text{in } (0,+\infty) \times M, \\ u(0,\cdot) = u_0 & \text{in } M, \end{cases} \tag{2.24}$$

completed with homogeneous Dirichlet or Neumann boundary conditions if $\partial M \neq \emptyset$.

Our second main result is the establishment of the following observability inequalities.

Theorem 2.9. *There exists $\delta_d \in (0,1)$ such that, for all $\delta \in [0,\delta_d]$, $m > 0$ and for every measurable set $\omega \subset M$ satisfying $\mathcal{C}_{\mathcal{H}}^{d-\delta}(\omega) \geq m > 0$, there exists a positive constant $C = C(M, g, V, \mathcal{A}, \delta, m) > 0$ such that, for every $T \in (0,1)$ and $u_0 \in L^2(M)$, the solution $u \in C([0,T]; L^2(M))$ of (2.24) satisfies*

$$\|u(T,\cdot)\|_{L^2(M)}^2 \leq Ce^{C/T} \int_0^T (\sup_{x\in\omega} |u(t,x)|)^2 dt. \tag{2.25}$$

For every measurable set $\omega \subset M$ satisfying $|\omega| \geq m > 0$, there exists a positive constant $C = C(M, g, V, \mathcal{A}, m) > 0$ such that, for every $T \in (0,1)$ and $u_0 \in L^2(M)$, the solution $u \in C([0,T]; L^2(M))$ of (2.24) satisfies

$$\|u(T,\cdot)\|_{L^2(M)}^2 \leq Ce^{C/T} \int_0^T \|u(t,\cdot)\|_{L^2(\omega)}^2 dt. \tag{2.26}$$

By using the spectral estimates (2.22), the proof of (2.26) is now classical and originally comes from the Lebeau–Robbiano method for obtaining the null controllability of the heat equation starting from a spectral estimate, see [22] and [20, Section 6]. This strategy was later extended by Miller in [28]. Therefore, (2.26) directly comes from [28, Theorem 2.2] or [5, Theorem 2.1]. The proof of (2.25) is in the same spirit but adapted to the particular functional setting of the $L^2 - L^\infty$ spectral estimate (2.23), see [7, Section 4] for details. The restriction $T \in (0,1)$ that we will keep in the following is simply for obtaining the constant of observability in small time of the form $Ce^{C/T}$.

For a measurable $\omega \subset M$, we finally focus on the controlled system

$$\begin{cases} \partial_t y + H_{g,V} y = h 1_\omega & \text{in } (0,+\infty) \times M, \\ y(0,\cdot) = y_0 & \text{in } M, \end{cases} \tag{2.27}$$

completed with homogeneous Dirichlet or Neumann boundary conditions if $\partial M \neq \emptyset$. In (2.27), at time $t \in [0, +\infty)$, $y(t, \cdot): M \to \mathbb{R}$ is the state and $h(t, \cdot): \omega \to \mathbb{R}$ is the control. In the following, we denote by $\mathcal{M}(M)$ the space of Borel measure on M.

Our last main result of this section provides null-controllability results for (2.27).

Theorem 2.10. *There exists $\delta_d \in (0, 1)$ such that, for all $\delta \in [0, \delta_d]$, $m > 0$, and for every closed measurable set $\omega \subset M$ satisfying $\mathcal{C}^{d-\delta}_{\mathcal{H}}(\omega) \geq m > 0$, there exists $C = C(M, g, V, \mathcal{A}, \delta, m) > 0$ such that, for every $T \in (0, 1)$ and $y_0 \in L^2(M)$, there exists $h \in L^2(0, T; \mathcal{M}(M))$ supported in $(0, T) \times \omega$ satisfying*

$$\int_0^T \|h(t)\|^2_{\mathcal{M}(M)} dt \leq C e^{C/T} \|y_0\|^2_{L^2(M)},$$

such that the solution y of (2.27) satisfies $y(T, \cdot) = 0$.

For every measurable set $\omega \subset M$ satisfying $|\omega| \geq m > 0$, there exists a positive constant $C = C(M, g, V, \mathcal{A}, m) > 0$ such that, for every $T \in (0, 1)$, $y_0 \in L^2(M)$, there exists $h \in L^2(0, T; L^2(\omega))$ satisfying

$$\|h\|_{L^2(0,T;L^2(\omega))} \leq C e^{C/T} \|y_0\|_{L^2(M)},$$

such that the solution $y \in C([0, T]; L^2(M))$ of (2.27) satisfies $y(T, \cdot) = 0$.

Recall that the norm of the space of Borel measures on the metric space ω, denoted by $\mathcal{M}(\omega)$, is defined as

$$\|\mu\|_{\mathcal{M}(M)} = \sup_{f \in C^0(M)} \frac{|\int_M f d\mu|}{\|f\|_\infty}.$$

The second part of Theorem 2.10 comes from a classical duality argument together with the use of the observability estimate (2.26), see for instance [8, Theorem 2.44]. The first part is less standard due to the functional setting, but details can be found in [7, Section 5].

2.3. Spectral estimates on the Euclidean space and applications

Let us consider the second order elliptic operator

$$H_{g,V,\kappa} u = -\frac{1}{\kappa(x)} \operatorname{div}(\kappa(x) g^{-1}(x) \nabla u) + V(x) u, \quad x \in \mathbb{R}^d, \tag{2.28}$$

where

$$g(x) = (g_{ij}(x))_{1, \leq i,j \leq d}$$

is a symmetric uniformly elliptic matrix with Lipschitz entries

$$\Lambda_1^{-1}|\xi|^2 \leq \langle g(x)\xi, \xi\rangle \leq \Lambda_1|\xi|^2, \quad |g_{ij}(x) - g_{ij}(y)| \leq \Lambda_2|x-y|, \quad x, y \in \mathbb{R}^d, \xi \in \mathbb{R}^d,$$

for some $\Lambda_1, \Lambda_2 > 0$, $V = V(x)$ is a real-valued bounded function, i.e.,

$$V \in L^\infty(\mathbb{R}^d; \mathbb{R}),$$

and $\kappa \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}_+^*)$ is a positive bounded Lipschitz density satisfying

$$\Lambda_1^{-1} \leq \kappa \leq \Lambda_1. \tag{2.29}$$

The operator $H_{g,V,\kappa}$ is an unbounded self-adjoint operator on $L^2(\mathbb{R}^d, \kappa dx)$ with domain $H^2(\mathbb{R}^d)$. Notice that, under the assumption (2.29), we have

$$\Lambda_1^{-\frac{1}{2}} \|\cdot\|_{L^2(\mathbb{R}^d)} \leq \|\cdot\|_{L^2(\mathbb{R}^d, \kappa dx)} \leq \Lambda_1^{\frac{1}{2}} \|\cdot\|_{L^2(\mathbb{R}^d)},$$

and $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \kappa dx)$. However, let us insist on the fact that the self-adjointness of $H_{g,V,\kappa}$ is related to the scalar product

$$\langle f, g\rangle_{L^2(\mathbb{R}^d, \kappa dx)} = \int_{\mathbb{R}^d} f(x)g(x)\kappa(x)dx.$$

Given $\Lambda > 0$, we introduce the spectral projector as follows:

$$\Pi_\Lambda u = 1_{H_{g,V,\kappa}} u = \int_{-\|V\|_\infty}^{\Lambda} dm \quad \text{for all } u \in L^2(\mathbb{R}^d, \kappa dx),$$

where dm is the spectral measure of $H_{g,V,\kappa}$.

In the sequel, we need the following definition.

Definition 2.11. Let $R > 0$ and $0 < \gamma \leq 1$. A measurable subset $\omega \subset \mathbb{R}^d$ is *γ-thick at scale R* if

$$|\omega \cap B(x, R)| \geq \gamma |B(x, R)|, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.30}$$

A subset $\omega \subset \mathbb{R}^d$ is *thick* if it is γ-thick at scale R for some $R > 0$ and $0 < \gamma \leq 1$.

Let $R > 0$, $m > 0$, and $0 < d' \leq d$. A measurable set $\omega \subset \mathbb{R}^d$ is said *(m, d')-uniformly distributed at scale R* if

$$\mathcal{C}_{\mathcal{H}}^{d'}(\omega \cap B(x, R)) \geq m, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.31}$$

A subset $\omega \subset \mathbb{R}^d$ is *d'-uniformly* distributed if it is (m, d')-uniformly distributed at scale R for some $R, m > 0$.

Notice that, since the d-dimensional Haussdorf measure is equivalent to the Lebesgue measure of $\mathbb{R}^d$, a subset $\omega \subset \mathbb{R}^d$ is d-uniformly distributed if and only if it is thick. In (2.30) and (2.31), the parameter $R > 0$ will be always chosen such that

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}^d} B(k, R). \tag{2.32}$$

Our main result of this section states the following spectral estimates for the Schrödinger operator (2.28) with particular observation sets.

Theorem 2.12. *Let $R > 0$ be such that (2.32) holds, $0 < \gamma \le 1$, and $m > 0$. There exists $\delta_d \in (0, 1)$ such that, for all $\delta \in [0, \delta_d]$ and for every $(m, d - \delta)$-uniformly distributed set $\omega \subset \mathbb{R}^d$ at scale R, there exists*

$$C = C(\Lambda_1, \Lambda_2, \|V\|_{L^\infty}, R, m, \delta) > 0$$

such that, for every $\Lambda > 0$, we have

$$\|\Pi_\Lambda u\|_{L^2(\mathbb{R}^d)} \le Ce^{C\sqrt{\Lambda}} \sum_{k\in\mathbb{Z}^d} \sup_{x\in\omega\cap B(k,R)} |(\Pi_\Lambda u)(x)| \quad \text{for all } u \in L^2(\mathbb{R}^d). \tag{2.33}$$

For every γ-thick set $\omega \subset \mathbb{R}^d$ at scale R, there exists

$$C = C(\Lambda_1, \Lambda_2, \|V\|_{L^\infty}, R, \gamma) > 0$$

such that, for every $\Lambda > 0$, we have

$$\|\Pi_\Lambda u\|_{L^2(\mathbb{R}^d)} \le Ce^{C\sqrt{\Lambda}} \|\Pi_\Lambda u\|_{L^2(\omega)} \quad \text{for all } u \in L^2(\mathbb{R}^d). \tag{2.34}$$

The following remarks are in order. First, Theorem 2.12 generalizes [6, Theorem 1] to the case of the Schrödinger operator as in (2.28). Note that (2.34) was previously established in [21], assuming some analyticity condition on the potential V, and in [31] for the one-dimensional case by exploiting the recent improvement of propagation of smallness for solutions to elliptic equations in dimension $d = 2$ in [36]. See also [1, 33, 35] for the case of unbounded potentials V. The same remark as in the compact setting applies to the parameter $\delta \in (0, 1)$ appearing in the proof of (2.33).

The strategy for the proof of Theorem 2.12 will follow that of [6, Theorem 1], and again the new difficulty arises from the fact that the results of [25] are actually proved for divergence elliptic operators.

We now focus on the evolution equation

$$\begin{cases} \partial_t u + H_{g,V,\kappa} u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{2.35}$$

Our second main result is the establishment of the following observability inequalities.

Theorem 2.13. *Let $R > 0$ be such that (2.32) holds, $0 < \gamma \le 1$, and $m > 0$. There exists $\delta_d \in (0,1)$ such that, for all $\delta \in [0, \delta_d]$ and for every $(m, d-\delta)$-uniformly distributed set $\omega \subset \mathbb{R}^d$ at scale R, there exists*

$$C = C(\Lambda_1, \Lambda_2, \|V\|_{L^\infty}, R, m, \delta) > 0$$

such that, for every $T \in (0,1)$ and $u_0 \in L^2(\mathbb{R}^d)$, the mild solution $u \in C([0,T]; L^2(\mathbb{R}^d))$ of (2.35) satisfies

$$\|u(T,\cdot)\|^2_{L^2(\mathbb{R}^d)} \le C e^{C/T} \sum_{k \in \mathbb{Z}^d} \int_0^T \sup_{x \in \omega \cap B(k,R)} |u|^2(t,x)\,dt.$$

For every γ-thick set $\omega \subset \mathbb{R}^d$ at scale R, there exists

$$C = C(\Lambda_1, \Lambda_2, \|V\|_{L^\infty}, R, \gamma) > 0$$

such that, for every $T \in (0,1)$ and $u_0 \in L^2(\mathbb{R}^d)$, the solution $u \in C([0,T]; L^2(\mathbb{R}^d))$ of (2.35) satisfies

$$\|u(T,\cdot)\|^2_{L^2(\mathbb{R}^d)} \le C e^{C/T} \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)}\,dt. \tag{2.36}$$

This result is, in particular, a generalization of [6, 12] by the addition of the zero-order term V. This also constitutes an improvement of [11, 30], which consider the case where ω is a union of disjoint open balls. Note that the thickness condition turns out to be necessary for (2.36) for $g = I_d$ and $V = 0$, as noted in [14, 32]. Other necessary and sufficient conditions were derived in [4] when looking at controls of the form $h 1_{\omega(t)}$. For a complete proof of Theorem 2.13 from the spectral estimates of Theorem 2.12, see [6] and the references therein.

For a measurable $\omega \subset \mathbb{R}^d$, we finally focus on the controlled system

$$\begin{cases} \partial_t y + H_{g,V,\kappa} y = h 1_\omega & \text{in } (0,+\infty) \times \mathbb{R}^d, \\ y(0,\cdot) = y_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2.37}$$

In (2.37), at time $t \in [0,+\infty)$, $y(t,\cdot) \colon \mathbb{R}^d \to \mathbb{R}$ is the state and $h(t,\cdot) \colon \omega \to \mathbb{R}$ is the control.

Let us denote by $\mathcal{M}(\mathbb{R}^d)$ the space of Borel measures on $\mathbb{R}^d$. Our last main result provides null-controllability results for (2.37).

Theorem 2.14. *Let $R > 0$ be such that (2.32) holds, $0 < \gamma \leq 1$, and $m > 0$. There exists $\delta_d \in (0, 1)$ such that, for all $\delta \in [0, \delta_d]$ and for every $(m, d - \delta)$-uniformly distributed set $\omega \subset \mathbb{R}^d$ at scale R, there exists $C = C(g, V, \kappa, R, m, \delta) > 0$ such that, for every $T \in (0, 1)$ and $y_0 \in L^2(\mathbb{R}^d)$, there exists $h \in L^2(0, T; \mathcal{M}(\mathbb{R}^d))$ supported in $(0, T) \times \omega$ satisfying*

$$\sum_{k\in\mathbb{Z}^d} \int_0^T \|h(t)\|^2_{\mathcal{M}(B(k,R))} dt \leq Ce^{C/T} \|y_0\|^2_{L^2(\mathbb{R}^d)} \tag{2.38}$$

such that the solution y of (2.37) satisfies $y(T, \cdot) = 0$.

For every γ-thick set $\omega \subset \mathbb{R}^d$ at scale R, there exists $C = C(g, V, \kappa, R, \gamma) > 0$ such that, for every $T \in (0, 1)$ and $y_0 \in L^2(\mathbb{R}^d)$, there exists $h \in L^2(0, T; L^2(\omega))$ satisfying

$$\|h\|_{L^2(0,T;L^2(\omega))} \leq Ce^{C/T} \|y_0\|_{L^2(\mathbb{R}^d)}$$

such that the solution $y \in C([0, T]; L^2(\mathbb{R}^d))$ of (2.37) satisfies $y(T, \cdot) = 0$.

In (2.38), $\|h(t)\|_{\mathcal{M}(B(k,R))}$ is the norm of the restriction of $h(t)$ on $B(k, R)$, which is therefore supported on $\omega \cap B(k, R)$, defined as

$$\|h(t)\|_{\mathcal{M}(B(k,R))} = \sup_{f\in C^0(B(k,R))} \frac{|\int_{B(k,R)} f dh(t)|}{\|f\|_{L^\infty(B(k,R))}}.$$

For a complete proof of Theorem 2.14 from the observability inequalities of Theorem 2.13, see [6] and the references therein.

3. Proof of the propagation of smallness for Schrödinger operators

The goal of this section is to establish the quantitative propagation of smallness for solutions to Schrödinger operators.

Let Ω be a bounded domain of $\mathbb{R}^d$. We consider the elliptic operator $H_{A,V}$ defined in (2.2), with the Lipschitz assumption on A, i.e., (2.3) and the boundedness assumption on V, i.e., (2.4). Moreover, as explained in Theorem 2.2, one can assume that $V \geq 0$.

Reduction to a smooth bounded domain Ω_0. Given $\mathcal{K}$ and E such that (2.5) or (2.9) hold then from [9, Proposition 8.2.1], one can find two C^∞-domains Ω_0 and Ω_1 such that

$$\operatorname{dist}(\mathcal{K}, \partial\Omega_0) \geq \frac{\rho}{2}, \quad \operatorname{dist}(E, \partial\Omega_0) \geq \frac{\rho}{2}, \quad \text{and} \quad \Omega_0 \subset\subset \Omega_1 \subset\subset \Omega. \tag{3.1}$$

3.1. Existence of a positive multiplier and reduction to the divergence form

In this part, we will construct the positive multiplier φ in Ω_0, and then reduce the Schrödinger-type equation to a divergence elliptic equation in Ω_0 with the help of the multiplier.

The following result is quite standard.

Lemma 3.1. *There exists $\varphi \in W^{1,\infty}(\Omega_0)$ satisfying*

$$-\mathrm{div}(A(x)\nabla\varphi) + V(x)\varphi = 0 \text{ in } \Omega_0. \tag{3.2}$$

Moreover, there exist

$$c = c(\Omega, \Omega_0, \Omega_1, \Lambda_1, \Lambda_2, \|V\|_\infty) > 0 \quad \text{and} \quad C = C(\Omega, \Omega_0, \Omega_1, \Lambda_1, \Lambda_2, \|V\|_\infty) > 0$$

such that

$$\varphi \geq c \quad \text{in } \Omega_0, \tag{3.3}$$

and

$$\|\varphi\|_{W^{1,\infty}(\Omega_0)} \leq C. \tag{3.4}$$

Proof. We first solve the boundary elliptic problem. By [15, Theorem 8.3], there exists $\varphi \in W^{1,2}(\Omega_1)$ such that

$$-\mathrm{div}(A(x)\nabla\varphi) + V(x)\varphi = 0 \quad \text{in } \Omega_1 \qquad \text{and} \qquad \varphi = 1 \quad \text{on } \partial\Omega_1. \tag{3.5}$$

Let us take $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ such that, for every $x = (x_1, \dots, x_d) \in \Omega_1$, we have $x_1 - y_1 \geq 0$. Let $\lambda \geq 0$ and let us define

$$\varphi_-(x) = \exp(-\lambda(x_1 - y_1)), \ \varphi_+(x) = 1 \quad \text{for all } x = (x_1, \dots, x_d) \in \Omega_1.$$

Then, φ_-, respectively φ_+, is a subsolution, respectively a supersolution, to (3.5) for some $\lambda > 0$ depending on Λ_1, Λ_2, and $\|V\|_\infty$, that is

$$-\mathrm{div}(A(x)\nabla\varphi_-) + V(x)\varphi_- \leq 0 \quad \text{in } \Omega_1, \qquad \varphi_- \leq 1 \quad \text{on } \partial\Omega_1,$$

and

$$-\mathrm{div}(A(x)\nabla\varphi_+) + V(x)\varphi_+ \geq 0 \quad \text{in } \Omega_1, \qquad \varphi_+ \geq 1 \quad \text{on } \partial\Omega_1.$$

So, by the weak maximum principle stated in [15, Theorem 8.1], we have that

$$\varphi_-(x) \leq \varphi(x) \leq \varphi_+(x) \quad \text{for all } x \in \Omega_1.$$

This proves that there exists $c = c(\Omega, \Omega_0, \Omega_1, \Lambda_1, \Lambda_2, \|V\|_\infty) > 0$ such that

$$c \leq \varphi(x) \leq 1 \quad \text{for all } x \in \Omega_1. \tag{3.6}$$

In particular, (3.6) implies (3.3).

Moreover, from the existence and uniqueness theorem for the Dirichlet problem (3.5) for strong solutions stated in [15, Theorem 9.15], we actually have that $\varphi \in W^{2,p}(\Omega_1)$ for every $1 < p < +\infty$. By now applying the $W^{2,p}$-regularity estimate from [15, Theorem 9.11] and (3.6), we have that, for $1 < p < +\infty$, there exists $C = C(\Omega, \Omega_0, \Omega_1, \Lambda_1, \Lambda_2, \|V\|_\infty, p) > 0$ such that

$$\|\varphi\|_{W^{2,p}(\Omega_0)} \leq C\|\varphi\|_{L^p(\Omega_1)} \leq C|\Omega_1|^{1/p}\|\varphi\|_{L^\infty(\Omega_1)} \leq C|\Omega|^{1/p+1}.$$

By taking p sufficiently large, and by using Sobolev embeddings [15, Theorem 7.26] to guarantee that $W^{2,p}(\Omega_0) \hookrightarrow W^{1,\infty}(\Omega_0)$, we therefore deduce (3.4) from the previous estimate. ∎

We have the following result.

Lemma 3.2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to*

$$-\operatorname{div}(A(x)\nabla u) + V(x)u = 0 \quad \text{in } \Omega.$$

Let φ be as in Lemma 3.1. Then, $v = u/\varphi \in W^{1,2}(\Omega_0)$ satisfies

$$-\operatorname{div}(\varphi^2 A\nabla v) = 0 \quad \text{in } \Omega_0. \tag{3.7}$$

Moreover, the symmetric matrix $\widehat{A} = \varphi^2 A$ is uniformly elliptic and has Lipschitz entries, and there exist

$$C_1 = C(\Omega, \Omega_0, \Omega_1, \Lambda_1, \Lambda_2, \|V\|_\infty) > 0$$

and

$$C_2 = C(\Omega, \Omega_0, \Omega_1, \Lambda_1, \Lambda_2, \|V\|_\infty) > 0$$

such that

$$C_1^{-1}|\xi|^2 \leq \langle \widehat{A}(x)\xi, \xi\rangle \leq C_1|\xi|^2, \ |\hat{a}_{ij}(x) - \hat{a}_{ij}(y)| \leq C_2 \quad \text{for all } x, y \in \Omega_0, \xi \in \mathbb{R}^d. \tag{3.8}$$

Proof. The proof is a straightforward computation at the variational formulation level so we omit it. ∎

3.2. Propagation of smallness

In this part, we suppose that the Lipschitz assumption on A (i.e., (2.3)) and the boundedness assumption on V (i.e., (2.4)) still hold, together with $V \geq 0$.

We first deal with the propagation of smallness for solutions to elliptic equations. We have the following quantitative propagation of smallness for solutions to divergence elliptic equations from [25].

Theorem 3.3 ([25, Theorem 2.1]). *Let $\rho, m, \delta > 0$ and $\mathcal{K}, E \subset \Omega$ be measurable subsets such that*

$$\operatorname{dist}(\mathcal{K}, \partial\Omega) \geq \rho, \quad \operatorname{dist}(E, \partial\Omega) \geq \rho, \quad \textit{and} \quad \mathcal{C}_{\mathcal{H}}^{d-1+\delta}(E) \geq m.$$

There exist $C = C(\Omega, \Lambda_1, \Lambda_2, \rho, m, \delta) > 0$ and $\alpha = \alpha(\Omega, \Lambda_1, \Lambda_2, \rho, m, \delta) \in (0, 1)$ such that, for every weak solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the elliptic equation

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \textit{in } \Omega,$$

we have

$$\sup_{\mathcal{K}} |u| \leq C(\sup_E |u|)^\alpha (\sup_\Omega |u|)^{1-\alpha}.$$

As a consequence, the proof of Theorem 2.1 is as follows.

Proof of Theorem 2.1. From Lemma 3.1, there exists $\varphi \in W^{1,\infty}(\Omega_0)$ satisfying the elliptic equation (3.2) and the lower bound (3.3). Then, from Theorem 3.2, defining $v = u/\varphi$, v satisfies (3.7). As a consequence, one can apply Theorem 3.3 to v and to the Lipschitz diffusion matrix $\widehat{A} = \varphi^2 A$ that satisfies (3.8) in Ω_0, note that we use (3.1). We deduce the propagation of smallness for v that leads to the propagation of smallness for u, i.e., (2.7) by using the lower bound (3.3), the $W^{1,\infty}$-bound (3.4), and again (3.1) ensuring that

$$\|u\|_{L^\infty(\Omega_0)} \leq \|u\|_{L^\infty(\Omega)}.$$

This concludes the proof. ∎

We now present results on propagation of smallness for gradient of solutions to elliptic equations.

Theorem 3.4 ([25, Theorem 5.1]). *There exists $\delta_d \in (0, 1)$, depending only on the dimension d, such that the following holds. Let $\rho, m > 0$, $\delta \in [0, \delta_d]$, and $\mathcal{K}, E \subset \Omega$, be measurable subsets such that*

$$\operatorname{dist}(\mathcal{K}, \partial\Omega) \geq \rho, \quad \operatorname{dist}(E, \partial\Omega) \geq \rho, \quad \textit{and} \quad \mathcal{C}_{\mathcal{H}}^{d-1-\delta}(E) \geq m.$$

There exist $C = C(\Omega, \Lambda_1, \Lambda_2, \rho, m, \delta) > 0$ and $\alpha = \alpha(\Omega, \Lambda_1, \Lambda_2, \rho, m, \delta) \in (0, 1)$ such that, for every weak solution $u \in W^{1,\infty}(\Omega)$ of the elliptic equation

$$-\nabla \cdot (A(x)\nabla u) = 0 \quad \textit{in } \Omega,$$

we have

$$\sup_{\mathcal{K}} |\nabla u| \leq C(\sup_E |\nabla u|)^\alpha (\sup_\Omega |\nabla u|)^{1-\alpha}.$$

By now, we aim at studying elliptic equations in non-divergence form. Actually, we are only able to deal with some particular cases, that are (2.10) from Theorem 2.2.

Proof of Theorem 2.2. Let $\varphi = \varphi(x)$ be as in Lemma 3.1. Then $\hat{v} = \hat{u}(x,t)/\varphi(x)$ is a solution to

$$\begin{cases} -\nabla_x \cdot (\varphi^2 A(x)\nabla_x \hat{v}) - \kappa(x)\partial_{tt}(\varphi^2 \hat{v}) = 0 & \text{in } \Omega_0 \times (-1,1), \\ \hat{v}(x,0) = 0 & \text{in } \Omega_0. \end{cases}$$

One can then apply Theorem 3.4 to $\widehat{\Omega}_0 = \Omega_0 \times (-1,+1)$, $\widehat{\mathcal{K}} = \mathcal{K} \times \{0\}$, and $\widehat{E} = E \times \{0\}$. Note that

$$\operatorname{dist}(\widehat{\mathcal{K}}, \partial\widehat{\Omega}_0) \geq \frac{\rho}{2}, \quad \operatorname{dist}(\widehat{E}, \partial\widehat{\Omega}_0) \geq \frac{\rho}{2}, \quad \text{and} \quad \mathcal{C}_{\mathcal{H}}^{d+1-1-\delta}(\widehat{E}) \geq m.$$

We then have

$$\sup_{x\in\mathcal{K}} |\partial_t \hat{v}(x,0)| \leq C(\sup_{x\in E} |\partial_t \hat{v}(x,0)|)^{\alpha}(\sup_{(x,t)\in\Omega_0\times(-1,1)} |\nabla_{x,t}\hat{v}(x,t)|)^{1-\alpha}.$$

By using (3.3), the $W^{1,\infty}$-bound on φ, i.e., (3.4) in Ω_0, and also

$$\|\hat{v}\|_{W^{1,\infty}(\hat{\Omega}_0)} \leq \|\hat{u}\|_{W^{1,\infty}(\Omega_0\times(-1,1))},$$

we then deduce (2.11). ∎

3.3. Propagation of smallness for drifted Schrödinger operators

We aim to establish the propagation of smallness for drifted Schrödinger operators. The proofs follow from Theorems 2.1 and 2.2. Both of them involve adding a new variable in order to remove the drift term.

3.3.1. Proof of Corollary 2.6. Let $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ be a solution of (2.15), i.e.,

$$-\operatorname{div}(A(x)\nabla u) + W(x)\cdot\nabla u + V(x)u = 0 \quad \text{in } \Omega.$$

We define $v\colon (x,y) \in \Omega \times (0,1) \mapsto u(x)e^y \in \mathbb{R}$. We also define a new diffusion matrix map by

$$\widehat{A}\colon x \in \Omega \mapsto \begin{pmatrix} A(x) & -\frac{W(x)}{2} \\ -\frac{{}^tW(x)}{2} & \lambda \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)},$$

where $\lambda > 0$ will be choosen later. By direct computations, we have

$$\begin{aligned} &\operatorname{div}_{x,y}(\widehat{A}(x)\nabla_{x,y} v) \\ &\quad = \operatorname{div}_x(A(x)\nabla_x v) - \frac{1}{2}\partial_y(W(x)\cdot\nabla_x v) - \frac{1}{2}\operatorname{div}_x(W(x)\partial_y v) + \lambda\partial_{yy} v. \end{aligned}$$

Since $\partial_y v = v$, it follows that, for all $(x, y) \in \Omega \times (0,1)$,

$$\begin{aligned} -\mathrm{div}_{x,y}(\widehat{A}(x)\nabla_{x,y} v) &= -\mathrm{div}_x(A(x)\nabla_x v) + \frac{1}{2} W(x) \cdot \nabla_x v + \frac{1}{2}\mathrm{div}_x(W(x)v) - \lambda v \\ &= e^y(-\mathrm{div}_x(A(x)\nabla u) + W(x)\nabla u) + \Big(\frac{1}{2}\mathrm{div}(W(x)) - \lambda\Big)v \\ &= \Big(\frac{1}{2}\mathrm{div}(W(x)) - \lambda - V(x)\Big)v. \end{aligned}$$

We obtain that v is solution to the following elliptic equation:

$$-\mathrm{div}_{x,y}(\widehat{A}(x)\nabla_{x,y} v) + \widehat{V}(x)v = 0 \quad \text{in } \Omega \times (0,1),$$

where

$$\widehat{V}(x) = V(x) + \lambda - \frac{1}{2}\mathrm{div}(W(x)).$$

Let us prove that we can choose $\lambda > 0$ sufficiently large such that $\widehat{A}$ is uniformly elliptic. In the following, $C_d > 0$ is a positive constant, depending only on the dimension d, such that

$$|W(x)| \le C_d \|W\|_{L^\infty} \quad \text{for all } x \in \Omega.$$

For all $V = (V_d, v_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$, we have

$$\begin{aligned} \langle \widehat{A}(x)V, V\rangle &= \langle A(x)V_d, V_d\rangle + \langle W(x), V_d\rangle v_{d+1} + \lambda |v_{d+1}|^2 \\ &\ge \Lambda |V_d|^2 - C_d \|W\|_{L^\infty} |V_d||v_{d+1}| + \lambda |v_{d+1}|^2 \\ &\ge \frac{\Lambda}{2}|V_d|^2 + \Big(\lambda - \frac{C_d}{\Lambda}\|W\|^2_{L^\infty}\Big)|v_{d+1}|^2. \end{aligned}$$

We can therefore fix $\lambda = \lambda(\Lambda, \|W\|_\infty) > 0$ such that $\widehat{A}$ is uniformly elliptic. We apply Theorem 2.1 to obtain

$$\sup_{\mathcal{K} \times (\frac{1}{3}, \frac{2}{3})} |v| \le C(\sup_{E \times (\frac{1}{3}, \frac{2}{3})} |v|)^\alpha (\sup_{\Omega \times (0,1)} |v|)^{1-\alpha},$$

where C and α are positive constants provided by Theorem 2.1. This readily implies that

$$\sup_{\mathcal{K}} |u| \le e^{\frac{2}{3}} C (\sup_E |u|)^\alpha (\sup_\Omega |u|)^{1-\alpha},$$

and this ends the proof of (2.16) and so the proof of Corollary 2.6.

3.3.2. Proof of Corollary 2.7. The proof is very close to the one for Corollary 2.6. For the sake of conciseness, we do not detail all the computations.

Let $\hat{u}(x,t) \in W^{1,\infty}(\Omega \times (-1,1))$ be a solution to the elliptic equation (2.17), i.e.,

$$\begin{cases} -\mathrm{div}_x(A(x)\nabla_x \hat{u}) - \kappa(x)\partial_{tt}\hat{u} + W(x)\cdot\nabla_x \hat{u} + V(x)\hat{u} = 0 & \text{in } \Omega \times (-1,1), \\ \hat{u}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

We define $\hat{v}: (x, y, t) \in \Omega \times (0, 1) \times (-1, 1) \mapsto \hat{u}(x, t)e^y \in \mathbb{R}$. We also define a new diffusion matrix map by

$$\widehat{A}: x \in \Omega \mapsto \begin{pmatrix} A(x) & -\frac{W(x)}{2} \\ -\frac{{}^t W(x)}{2} & \lambda \end{pmatrix} \in M_{d+1}(\mathbb{R}),$$

where $\lambda = \lambda(\Lambda, \|W\|_\infty) > 0$ is such that $\widehat{A}$ is uniformly elliptic. Direct computations show that $\hat{v}$ is a solution to the following elliptic equation:

$$\begin{cases} -\mathrm{div}_{x,y}(\widehat{A}(x)\nabla_{x,y}\hat{v}) - \kappa(x)\partial_{tt}\hat{v} + \widehat{V}(x)\hat{v} = 0 & \text{in } \Omega \times (0,1) \times (-1,1), \\ \hat{v}(x, y, 0) = 0 & \text{in } \Omega \times (0,1), \end{cases}$$

with

$$\widehat{V}(x) = V(x) + \lambda - \frac{1}{2}\mathrm{div}(W(x)).$$

The conclusion (2.18) then follows from Theorem 2.2.

4. Proof of the spectral estimates

This section aims to prove Theorem 2.8 and Theorem 2.12.

4.1. Spectral estimates on compact manifolds

The goal is to prove Theorem 2.8. In the first part, we reduce the task of obtaining the spectral estimates for sets of positive Lebesgue measures (2.21) to obtaining spectral estimates for sets of positive Hausdorff measures (2.20). In the second and third parts, we establish the results of Theorem 2.8 for the manifold M without boundary. Firstly, we prove a local version of (2.20), i.e., replacing the L^∞-bound on M in the left-hand side of (2.20) with an L^∞-bound on a chart of M. Secondly, by using the compactness and connectedness of the manifold M, we propagate these local spectral estimates to the whole manifold. In the fourth part, we complete the proof of Theorem 2.8 and address the case when $\partial M \neq \emptyset$ with Dirichlet or Neumann boundary conditions on ∂M. The proof uses the double manifold trick introduced in [7, Section 3].

4.1.1. Reduction of spectral estimates to sets of positive Hausdorff measures. In this part, we prove that the spectral estimates for sets of positive Hausdorff measures (2.20) imply the spectral estimates for sets of positive Lebesgue measures (2.21).

Let $\omega \subset M$ such that $|\omega| > m > 0$. Let us define $u = \Pi_\Lambda u$ with $\|u\|_{L^2(M)} = 1$. Let us consider

$$\hat{\omega} = \Big\{x \in \omega : |u(x)| \leq \frac{1}{2C}e^{-C\sqrt{\Lambda}}\|u\|_{L^\infty(M)}\Big\}. \tag{4.1}$$

If $|\hat{\omega}| \geq m/2$, then we have, for $\delta = \delta(d) \in (0,1)$, by applying (2.1),

$$\mathcal{C}^d_{\mathcal{H}}(\hat{\omega}) > c_d \frac{m}{2} \implies \mathcal{C}^{d-\delta}_{\mathcal{H}}(\hat{\omega}) \geq \min\Big(1, c_d \frac{m}{2}\Big).$$

So, one can apply (2.20) to $\hat{\omega}$ to get, by definition of (4.1),

$$\|u\|_{L^\infty(M)} \leq Ce^{C\sqrt{\Lambda}} \sup_{x \in \hat{\omega}} |u(x)| \leq \frac{\|u\|_{L^\infty(M)}}{2}.$$

This is impossible because this leads to $u = 0$. Therefore, $|\hat{\omega}| < m/2$ and, consequently,

$$\int_{\omega} |u(x)| dx \geq \int_{\omega \setminus \hat{\omega}} |u(x)| dx \geq \frac{m}{(4C)} e^{-C\sqrt{\Lambda}} \|u\|_{L^\infty(M)},$$

leading to (2.21).

4.1.2. Local spectral estimates. In this part, we assume that M is without boundary, $\partial M = \emptyset$. The purpose is to establish local spectral estimates holding in each charts of the manifold M. We recall that, in Section 2.2, we have fixed an atlas $\mathcal{A} = (\mathcal{V}_\sigma, \Psi_\sigma)_{\sigma \in \mathcal{J}}$ containing a finite number of charts with $W^{2,\infty} \cap C^1$-diffeomorphisms $\Psi_\sigma : \mathcal{V}_\sigma \to \Psi_\sigma(\mathcal{V}_\sigma) \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$ such that there exists a family of open sets $(\mathcal{U}_\sigma)_{\sigma \in \mathcal{J}}$ satisfying

$$M = \bigcup_{\sigma \in \mathcal{J}} \mathcal{U}_\sigma, \tag{4.2}$$

and such that $\mathcal{U}_\sigma$ is compactly included in the open set $\mathcal{V}_\sigma$ in M, for all $\sigma \in \mathcal{J}$. Moreover, since $\partial M = \emptyset$, $\Psi_\sigma(\mathcal{V}_\sigma)$ is an open set of $\mathbb{R}^d$, for any $\sigma \in \mathcal{J}$.

The main result of this part is the following one.

Proposition 4.1. *There exists $\delta_d \in (0,1)$ such that, for all $\delta \in [0, \delta_d]$ and for every $\sigma \in \mathcal{J}$ and $m > 0$, there exist $C = C(M, g, V, \sigma, m, \delta) > 0$ and $\alpha = \alpha(M, g, V, \sigma, m, \delta) \in (0,1)$ such that, for all subsets ω with $C^{d-\delta}_{\mathcal{H}}(\omega \cap \mathcal{U}_\sigma) > m$ and $\Lambda > 0$,*

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_\sigma)} \leq Ce^{C\sqrt{\Lambda}} \big(\sup_{\omega \cap \mathcal{U}_\sigma} |\Pi_\Lambda u|\big)^\alpha \|\Pi_\Lambda u\|^{1-\alpha}_{L^\infty(M)} \quad \text{for all } u \in L^2(M). \tag{4.3}$$

Proof. First, one can assume that $V \geq 0$ just by considering the elliptic operator

$$H_{g,V} + \|V\|_\infty \quad \text{defined in } \mathrm{Dom}(H_{g,V}),$$

that has the same eigenfunctions $(\varphi_k)_{k \geq 1}$ as the elliptic operator $H_{g,V}$ corresponding to the shifted eigenvalues $\lambda_k + \|V\|_\infty$.

We fix $\sigma \in \mathcal{J}$, we now work in a coordinate patch $\mathcal{U}_\sigma \subset\subset \mathcal{V}_\sigma$, and we define the sets

$$\mathcal{V} = \Psi_\sigma(\mathcal{V}_\sigma), \quad \mathcal{U} = \Psi_\sigma(\mathcal{U}_\sigma), \quad \text{and} \quad E = \Psi_\sigma(\omega \cap \mathcal{U}_\sigma).$$

For $\Lambda > 0$, we then consider

$$u(x) = \sum_{\lambda_k \leq \Lambda} u_k \varphi_k(x), \quad x \in M,$$

and its local push forward version U:

$$U(x) = u \circ \Psi_\sigma^{-1}(x) = \sum_{\lambda_k \leq \Lambda} u_k (\varphi_k \circ \Psi_\sigma^{-1})(x) = \sum_{\lambda_k \leq \Lambda} u_k \Phi_k(x), \quad x \in \mathcal{V}.$$

We then add an extra-variable to u by defining

$$\hat{u}(x,t) = \sum_{\lambda_k \leq \Lambda} u_k \frac{\sinh(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \varphi_k(x), \quad (x,t) \in M \times (-2,+2),$$

and its local push forward version $\widehat{U}$:

$$\widehat{U}(x,t) = \sum_{\lambda_k \leq \Lambda} u_k \frac{\sinh(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \Phi_k(x), \quad (x,t) \in \mathcal{V} \times (-2,+2).$$

In the chart $(\mathcal{V}_\sigma, \Psi_\sigma)$, let us consider $(g_{i,j})_{1\leq i,j\leq d}$, the local coordinates of the metric g. We define, for $x \in \mathcal{V}$, $G(x) = (g_{i,j}(\Psi_\sigma^{-1}(x)))_{1\leq i,j\leq d}$. We observe that $\widehat{U}$ solves

$$\begin{cases} -\nabla_x \cdot (A(x)\nabla_x \widehat{U}) - \kappa(x)\partial_{tt}\widehat{U} + \widehat{V}(x)\widehat{U} = 0 & \text{in } \mathcal{V} \times (-2,2), \\ \widehat{U}(x,0) = 0 & \text{in } \mathcal{V}, \end{cases} \tag{4.4}$$

with $A = G(x)^{-1}\sqrt{\det G(x)}$, $\kappa(x) = \sqrt{\det G(x)}$, and $\widehat{V}(x) = \sqrt{\det G(x)}V(\Psi_\sigma^{-1}(x))$ satisfying the hypotheses (2.3), (2.4), and (2.8). We can then apply Theorem 2.2 to $\widehat{U}$ with $\Omega = \widetilde{\mathcal{V}}$, $\mathcal{K} = \mathcal{U}$, and $E = \Psi_\sigma(\omega \cap \mathcal{U}_\sigma)$ such that $\mathcal{U} \subset\subset \widetilde{\mathcal{V}} \subset\subset \mathcal{V}$ to get

$$\sup_{x\in\mathcal{U}} |\partial_t \widehat{U}(x,0)| \leq C(\sup_{x\in E} |\partial_t \widehat{U}(x,0)|)^\alpha \|\widehat{U}\|^{1-\alpha}_{W^{1,\infty}_{t,x}(\widetilde{\mathcal{V}}\times(-1,1))}. \tag{4.5}$$

The left-hand side of (4.5) exactly gives

$$\sup_{x\in\mathcal{U}} |\partial_t \widehat{U}(x,0)| = \sup_{x\in\mathcal{U}} |U(x)| = \sup_{x\in\mathcal{U}_\sigma} |u(x)|. \tag{4.6}$$

The first right-hand side term of (4.5) exactly gives

$$\sup_{x\in E} |\partial_t \widehat{U}(x,0)| = \sup_{x\in E} |U(x)| = \sup_{x\in\omega\cap\mathcal{U}_\sigma} |u(x)|. \tag{4.7}$$

Moreover, by using the elliptic equation (4.4) satisfied by $\widehat{U}$, we first obtain, from [16, Theorem 3.8 and Corollary 3.2] and a straightforward Caccioppoli's inequality, that $\widehat{U}$ is Hölder-continuous in $\widetilde{\mathcal{V}} \times (-1,1)$, and we have

$$\|\widehat{U}\|_{L^\infty(\widetilde{\mathcal{V}}\times(-1,1))} \leq C\|\widehat{U}\|_{L^2(\mathcal{V}\times(-2,+2))}; \tag{4.8}$$

then, from [16, Theorem 3.13 and Theorem 3.1], that $\nabla \widehat{U}$ is Hölder-continuous in $\widetilde{\mathcal{V}} \times (-1, 1)$, and we get

$$\|\nabla_{t,x} \widehat{U}\|_{L^\infty(\widetilde{\mathcal{V}}\times(-1,1))} \leq C\|\widehat{U}\|_{L^2(\mathcal{V}\times(-2,+2))}. \tag{4.9}$$

So, we deduce from the two previous bounds (4.8) and (4.9) that the second right-hand side term of (4.5) is bounded as follows:

$$\|\widehat{U}\|_{W^{1,\infty}_{t,x}(\widetilde{\mathcal{V}}\times(-1,1))} \leq C\|\widehat{U}\|_{L^2(\mathcal{V}\times(-2,+2))}. \tag{4.10}$$

By using a change of variable, we then obtain that

$$\|\widehat{U}\|_{L^2(\mathcal{V}\times(-2,+2))} \leq C\|\hat{u}\|_{L^2(\mathcal{V}_\sigma\times(-2,+2))} \leq \|\hat{u}\|_{L^2(M\times(-2,+2))}. \tag{4.11}$$

Now, by using the orthogonality of the eigenfunctions $(\varphi_k)_{k\geq 1}$ in $L^2(M)$, we then obtain that

$$\|\hat{u}\|_{L^2(M\times(-2,+2))} \leq C\exp(C\sqrt{\Lambda})\|u\|_{L^2(M)}. \tag{4.12}$$

We now gather (4.5)–(4.7),and (4.10)–(4.12), and we get

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_\sigma)} \leq Ce^{C\sqrt{\Lambda}}\big(\sup_{\omega\cap\mathcal{U}_\sigma} |\Pi_\Lambda u|\big)^\alpha \|\Pi_\Lambda u\|^{1-\alpha}_{L^2(M)} \quad \text{for all } u \in L^2(M). \tag{4.13}$$

Finally, (4.13) leads in particular to (4.3). ∎

4.1.3. Propagation to the whole manifold. In this part, we prove Theorem 2.8 by using the connectedness of the manifold M to propagate the estimates (4.3) to the whole manifold M, which is still assumed to be without boundary.

Proof of Theorem 2.8 in the case $\partial M = \emptyset$. We define the following subset $\mathcal{I} \subset \mathcal{J}$ such that $\sigma \in \mathcal{I}$ if and only if there exist $C_\sigma > 0$ and $\alpha_\sigma \in (0, 1)$ so that

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_\sigma)} \\ \leq C_\sigma e^{C_\sigma\sqrt{\Lambda}}\big(\sup_{x\in\omega} |\Pi_\Lambda u(x)|\big)^{\alpha_\sigma} \|\Pi_\Lambda u\|^{1-\alpha_\sigma}_{L^\infty(M)} \quad \text{for all } u \in L^2(M), \Lambda > 0.$$

Thanks to (4.2), we have

$$M = \bigcup_{\sigma\in\mathcal{I}} \mathcal{U}_\sigma \cup \bigcup_{\sigma\notin\mathcal{I}} \mathcal{U}_\sigma.$$

First of all, $\mathcal{I}$ is not empty. Indeed, since $\mathcal{C}^{d-\delta'}_{\mathcal{H}}(\omega) > m$, there exists $\sigma_0 \in \mathcal{J}$ such that $\mathcal{C}^{d-\delta'}_{\mathcal{H}}(\omega \cap \mathcal{U}_{\sigma_0}) > m/N$, where N denotes the cardinality of the finite set $\mathcal{J}$. It is then sufficient to apply Proposition 4.1 to obtain $\sigma_0 \in \mathcal{I}$.

Let us assume, by contradiction, that $\mathcal{I} \neq \mathcal{J}$. Since M is connected, there exist $\sigma \in \mathcal{I}$ and $\tilde{\sigma} \notin \mathcal{I}$ such that $\mathcal{U}_\sigma \cap \mathcal{U}_{\tilde{\sigma}} \neq \emptyset$. By applying Proposition 4.1 with $j = \tilde{\sigma}$

and $\omega = \mathcal{U}_\sigma \cap \mathcal{U}_{\tilde{\sigma}}$ that is open, there exist $C_{\sigma,\tilde{\sigma}} > 0$ and $0 < \alpha_{\sigma,\tilde{\sigma}} < 1$ such that, for $\Lambda > 0$ and $u \in L^2(M)$,

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_{\tilde{\sigma}})} \le C_{\sigma,\tilde{\sigma}} e^{C_{\sigma,\tilde{\sigma}}\sqrt{\Lambda}} \|\Pi_\Lambda u\|^{\alpha_{\sigma,\tilde{\sigma}}}_{L^\infty(\mathcal{U}_\sigma \cap \mathcal{U}_{\tilde{\sigma}})} \|\Pi_\Lambda u\|^{1-\alpha_{\sigma,\tilde{\sigma}}}_{L^\infty(M)},$$

so that

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_{\tilde{\sigma}})} \le C_{\sigma,\tilde{\sigma}} e^{C_{\sigma,\tilde{\sigma}}\sqrt{\Lambda}} \|\Pi_\Lambda u\|^{\alpha_{\sigma,\tilde{\sigma}}}_{L^\infty(\mathcal{U}_\sigma)} \|\Pi_\Lambda u\|^{1-\alpha_{\sigma,\tilde{\sigma}}}_{L^\infty(M)}. \tag{4.14}$$

Moreover, since $\sigma \in \mathcal{I}$, there exist $C'_\sigma > 0$ and $0 < \alpha'_\sigma < 1$ such that

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_\sigma)} \le C'_\sigma e^{C'_\sigma\sqrt{\Lambda}} (\sup_{x\in\omega} |\Pi_\Lambda u(x)|)^{\alpha'_\sigma} \|\Pi_\Lambda u\|^{1-\alpha'_\sigma}_{L^\infty(M)} \quad \text{for all } u \in L^2(M), \Lambda > 0. \tag{4.15}$$

Let $\Lambda > 0$ and $u \in L^2(M)$ such that $\|\Pi_\Lambda u\|_{L^\infty(M)} = 1$. We deduce from (4.14) and (4.15) that

$$\|\Pi_\Lambda u\|_{L^\infty(\mathcal{U}_{\tilde{\sigma}})} \le C''_{\sigma,\tilde{\sigma}} e^{C''_{\sigma,\tilde{\sigma}}\sqrt{\Lambda}} (\sup_{x\in\omega} |\Pi_\Lambda u(x)|)^{\beta_{\sigma,\tilde{\sigma}}}$$

with $0 < \beta_{\sigma,\tilde{\sigma}} = \alpha'_\sigma \alpha_{\sigma,\tilde{\sigma}} < 1$ and $C''_{\sigma,\tilde{\sigma}} = \max(C'^{\alpha_{\sigma,\tilde{\sigma}}}_\sigma C_{\sigma,\tilde{\sigma}}, \alpha_{\sigma,\tilde{\sigma}} C'_\sigma + C_{\sigma,\tilde{\sigma}})$. It readily follows that, for all $\Lambda > 0$ and for all $u \in L^2(M)$,

$$\|\Pi_\Lambda u\|_{L^\infty(U_k)} \le C''_{\sigma,\tilde{\sigma}} e^{C''_{\sigma,\tilde{\sigma}}\sqrt{\Lambda}} (\sup_{x\in\omega} |\Pi_\Lambda u(x)|)^{\beta_{\sigma,\tilde{\sigma}}} \|\Pi_\Lambda u\|^{1-\beta_{\sigma,\tilde{\sigma}}}_{L^\infty(M)} . \sigma$$

Thus, $\tilde{\sigma} \in \mathcal{I}$, and this provides a contradiction.

To conclude, we have $\mathcal{I} = \mathcal{J}$ and by defining

$$0 < \alpha = \min_{\sigma\in\mathcal{J}} \alpha_\sigma < 1 \quad \text{and} \quad C = \max_{\sigma\in\mathcal{J}} C_\sigma > 0,$$

we have

$$\|\Pi_\Lambda u\|_{L^\infty(M)} \le C e^{C\sqrt{\Lambda}} (\sup_{x\in\omega} |\Pi_\Lambda u(x)|)^{\alpha} \|\Pi_\Lambda u\|^{1-\alpha}_{L^\infty(M)} \quad \text{for all } u \in L^2(M), \Lambda > 0,$$

which readily provides

$$\|\Pi_\Lambda u\|_{L^\infty(M)} \le C^{\frac{1}{\alpha}} e^{\frac{C}{\alpha}\sqrt{\Lambda}} \sup_{x\in\omega} |\Pi_\Lambda u(x)| \quad \text{for all } u \in L^2(M), \Lambda > 0. \qquad \blacksquare$$

4.1.4. The double manifold. In this part, we prove Theorem 2.8 for a manifold with boundary M and Dirichlet or Neumann boundary conditions on ∂M. The idea involves reducing this problem to the case of a manifold without boundary by gluing two copies of M along the boundary, such that the new double manifold $\widetilde{M}$ inherits a Lipschitz metric. This allows us to apply the previous results (without boundary) to the double manifold. This approach is detailed in [7, Section 3]; however, the only point we need to check in our setting is the equation satisfied by the eigenfunctions on the double manifold.

Let $\widetilde{M} = \bar{M} \times \{-1, 1\}/\partial M$, the double space made of two copies of $\bar{M}$ where we identified the points on the boundary, $(x, -1)$ and $(x, 1)$, $x \in \partial M$.

Theorem 4.2 (The double manifold). *There exist a C^∞ structure on the double manifold $\widetilde{M}$, a metric $\tilde{g} \in W^{1,\infty}$ on $\widetilde{M}$, and a potential $\widetilde{V} \in L^\infty(\widetilde{M})$, such that the following holds.*

- *The maps*
$$i^\pm : x \in M \to (x, \pm 1) \in \widetilde{M} = M \times \{\pm 1\}/\partial M$$
are isometric embeddings.
- *The potential $\widetilde{V}$ is such that*
$$\widetilde{V}(x, \pm 1) = V(x), \quad x \in M.$$
- *For any eigenfunction φ_λ with eigenvalue λ of the operator $H_{g,V}$ with Dirichlet or Neumann boundary conditions, there exists an eigenfunction $\widetilde{\varphi_\lambda}$ with the same eigenvalue λ of the operator $H_{\tilde{g},\widetilde{V}}$ on $\widetilde{M}$ such that*

$$\widetilde{\varphi_\lambda}\,|_{M\times\{1\}} = \varphi_\lambda, \quad \widetilde{\varphi_\lambda}\,|_{M\times\{-1\}} = \begin{cases} -\varphi_\lambda & \text{(Dirichlet boundary conditions),} \\ \varphi_\lambda & \text{(Neumann boundary conditions).} \end{cases}$$

The proof exactly follows the same lines as [7, Theorem 7]. The main difference comes from the fact that we need to deal with a potential $V \in L^\infty$. One of the main difficulties in the proof of [7, Theorem 7] consists in computing the Laplacian of $\tilde{\varphi}_\lambda$ on the new manifold $\widetilde{M}$, thanks to the jump formula. In particular, there is no new difficulty in adding this potential to the proof of [7, Theorem 7]. For the sake of conciseness, we omit the proof of Theorem 4.2.

The results of Theorem 2.8 are readily implied by Theorem 4.2.

4.2. Spectral estimates on the Euclidean space

The goal is to prove Theorem 2.12. In the first part, we show that (2.34) can be deduced from (2.33) in the case when $\delta = 0$. In the second part, we prove the spectral estimates (2.33).

4.2.1. Reduction of spectral estimates to uniformly distributed sets. In this first part, we explain how (2.34) can be deduced from (2.33). Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda > 0$ such that $g = \Pi_\Lambda g$. If $\omega \subset \mathbb{R}^d$ is a thick subset, i.e., satisfying (2.30), we define an auxiliary subset $\tilde{\omega} = \bigcup_{k \in \mathbb{Z}^d} \tilde{\omega}_k$, where for all $k \in \mathbb{Z}^d$,

$$\tilde{\omega}_k = \left\{x \in \omega \cap B(k,R), |g(x)|^2 \leq \frac{2}{|\omega \cap B(k,R)|} \int_{\omega \cap B(k,R)} |g(y)|^2 dy\right\} \subset B(k,R).$$

By definition, we have, for all $k \in \mathbb{Z}^d$,

$$\int_{\omega \cap B(k,R)} |g(x)|^2 dx \geq \int_{(\omega \cap B(k,R)) \setminus \tilde{\omega}_k} |g(x)|^2 dx \geq \frac{2|(\omega \cap B(k,R)) \setminus \tilde{\omega}_k|}{|\omega \cap B(k,R)|} \int_{\omega \cap B(k,R)} |g(y)|^2 dy.$$

Thus, if

$$\int_{\omega \cap B(k,R)} |g(y)|^2 dy > 0,$$

then

$$|(\omega \cap B(k,R)) \setminus \tilde{\omega}_k| \leq \frac{|\omega \cap B(k,R)|}{2},$$

which implies

$$|\tilde{\omega}_k| \geq \frac{\gamma}{2} |B(k,R)|,$$

thanks to the thickness property satisfied by ω i.e., (2.30). Otherwise, if

$$\int_{\omega \cap B(k,R)} |g(y)|^2 dy = 0,$$

then $g \equiv 0$ in $\omega \cap B(k,R)$ so $\tilde{\omega}_k = \omega \cap B(k,R)$, therefore

$$|\tilde{\omega}_k| = |\omega \cap B(k,R)| \geq \gamma |B(k,R)| > \frac{\gamma}{2} |B(k,R)|.$$

Finally, $\tilde{\omega}$ is still a thick subset of $\mathbb{R}^d$, and it follows from the spectral estimate (2.33) that

$$\begin{aligned}
\|g\|^2_{L^2(\mathbb{R}^d)} &\leq Ce^{C\sqrt{\Lambda}} \sum_{k \in \mathbb{Z}^d} \sup_{x \in \tilde{\omega}_k} |g(x)|^2 \\
&\leq Ce^{C\sqrt{\Lambda}} \sum_{k \in \mathbb{Z}^d} \frac{2}{|\omega \cap B(k,R)|} \int_{\omega \cap B(k,R)} |g(x)|^2 dx \\
&\leq \frac{2}{\gamma |B(0,R)|} Ce^{C\sqrt{\Lambda}} \sum_{k \in \mathbb{Z}^d} \int_{\omega \cap B(k,R)} |g(x)|^2 dx \\
&\leq Ce^{C\sqrt{\Lambda}} \int_\omega |g(x)|^2 dx,
\end{aligned}$$

since

$$1 \le \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{B(k,R)} \le C(d).$$

This concludes the proof of (2.34).

4.2.2. Spectral estimates. In this part, we prove the spectral estimates (2.33). We can assume $V \ge 0$ since spectral estimates for $H_{g,V,\kappa} + \|V\|_{L^\infty}$ readily imply spectral estimates for $H_{g,V,\kappa}$. Let $m, R > 0$ and $\omega \subset \mathbb{R}^d$ satisfying

$$\mathcal{C}_{\mathcal{H}}^{d-\delta}(\omega \cap B(x,R)) \ge m \quad \text{for all } x \in \mathbb{R}^d,$$

for $\delta \in [0, \delta_d]$ with $0 < \delta_d < 1$ provided by Theorem 2.2.

Let us fix $\lambda > 0$ and $f = \Pi_\lambda f$. The strategy, inspired by the works [6,7], consists in adding a ghost dimension and defining the following $(d+1)$-dimensional function:

$$F_\lambda(x,y) = \frac{\sinh(\sqrt{H_{g,V,\kappa}}\, y)}{\sqrt{H_{g,V,\kappa}}} \Pi_\lambda f(x), \quad (x,y) \in \mathbb{R}^d \times (-5R, 5R).$$

Notice that $F_\lambda \in H^2(\mathbb{R}^d \times (-5R,5R))$ and that F_λ satisfies the following elliptic equation

$$-\nabla_x \cdot (\kappa(x) g^{-1}(x) \nabla_x F_\lambda) - \kappa(x)\partial_y^2 F_\lambda + \kappa(x)V(x)F_\lambda = 0 \quad \text{in } \mathbb{R}^d \times (-5R,5R). \tag{4.16}$$

Moreover, we have

$$F_\lambda(\cdot,0) = 0 \quad \text{and} \quad \partial_y F(\cdot,0) = \Pi_\lambda f \quad \text{on } \mathbb{R}^d.$$

In the following, the constants will be of the form

$$C = C(\Lambda_1, \Lambda_2, \|V\|_\infty, R, m, \delta) > 0$$

and can change from one line to another.

Since the conclusion of Corollary 2.2 is invariant by translations and because of the uniform bounds on g and V and κ, we have that there exist positive constants $C > 0$ and $0 < \alpha < 1$ such that, for all $k \in \mathbb{Z}^d$,

$$\begin{aligned}&\sup_{x \in B(k,R)} |\partial_y F_\lambda(x,0)| \\ &\quad \le C\Big(\sup_{x \in \omega \cap B(k,R)} |\partial_y F_\lambda(x,0)|\Big)^\alpha \|F_\lambda(x,y)\|^{1-\alpha}_{W^{1,\infty}_{x,y}(B(k,2R)\times(-R,R))},\end{aligned}$$

which implies

$$\sup_{x \in B(k,R)} |\Pi_\lambda f(x)| \le C\Big(\sup_{x \in \omega \cap B(k,R)} |\Pi_\lambda f(x)|\Big)^\alpha \|F_\lambda(x,y)\|^{1-\alpha}_{W^{1,\infty}_{x,y}(B(k,2R)\times(-R,R))}. \tag{4.17}$$

Moreover, by using the elliptic equation (4.16) satisfied by F_λ, we deduce from [16, Theorem 3.8 and Corollary 3.2] and [16, Theorem 3.13 and Theorem 3.1] that we have

$$\|F_\lambda(x,y)\|_{W^{1,\infty}_{x,y}(B(k,2R)\times(-R,R))} \leq C\|F_\lambda\|_{L^2(B_{d+1}((k,0),5R))}.$$

This implies, together with (4.17), that

$$\begin{aligned}\|\Pi_\lambda f\|^2_{L^2(B(k,R))} &\leq \sup_{x\in B(k,R)} |\Pi_\lambda f(x)|^2 \\ &\leq C\big(\sup_{x\in\omega\cap B(k,R)} |\Pi_\lambda f(x)|\big)^{2\alpha}\|F_\lambda\|^{2(1-\alpha)}_{L^2(B_{d+1}((k,0),5R))}.\end{aligned}$$

It therefore follows from Young's inequality that there exists $\beta > 0$ such that, for all $\varepsilon > 0$,

$$\|\Pi_\lambda f\|^2_{L^2(B(k,R))} \leq C\varepsilon^{-\beta} \sup_{x\in\omega\cap B(k,R)} |\Pi_\lambda f(x)|^2 + \varepsilon\|F_\lambda\|^2_{L^2(B_{d+1}((k,0),5R))}.$$

By summing over all the integers $k \in \mathbb{Z}^d$ and using the facts that

$$1 \leq \sum_{k\in\mathbb{Z}^d} \mathbb{1}_{B(k,R)} \quad \text{and} \quad \sum_{k\in\mathbb{Z}^d} \mathbb{1}_{B_{d+1}((k,0),5R)} \leq C(d)\mathbb{1}_{\mathbb{R}^d\times(-5R,5R)},$$

for some positive constant $C(d) \geq 1$ depending only on the dimension, we have, for all $\varepsilon > 0$,

$$\begin{aligned}\|\Pi_\lambda f\|^2_{L^2(\mathbb{R}^d)} &\leq \sum_{k\in\mathbb{Z}^d} \|\Pi_\lambda f\|^2_{L^2(B(k,R))} \\ &\leq C\varepsilon^{-\beta} \sum_{k\in\mathbb{Z}^d} \sup_{x\in\omega\cap B(k,R)} |\Pi_\lambda f(x)|^2 + \varepsilon C\|F_\lambda\|^2_{L^2(\mathbb{R}^d\times(-5R,5R))}.\end{aligned} \tag{4.18}$$

Note that

$$\frac{\sinh(yt)^2}{t^2} \leq Ce^{C\sqrt{\lambda}} \quad \text{for all } y \in (-5R,5R), \forall t \in (0,\sqrt{\lambda}),$$

so we have

$$\begin{aligned}\|F_\lambda\|^2_{L^2(\mathbb{R}^d\times(-5R,5R))} &\leq \int_{-5R}^{5R}\int_{\mathbb{R}^d} \Big|\frac{\sinh(y\sqrt{H_{g,V,\kappa}})}{\sqrt{H_{g,V,\kappa}}}\Pi_\lambda f(x)\Big|^2 dx dy \\ &\leq Ce^{C\sqrt{\lambda}}\|\Pi_\lambda f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{4.19}$$

Finally, we have shown that (4.19) holds for some constant

$$C = C(\Lambda_1, \Lambda_2, \|V\|_\infty, R, m, \delta) > 0$$

independent of $\varepsilon > 0$. It follows from (4.18) with $\varepsilon = e^{-C\sqrt{\lambda}}/C$ and (4.19) that

$$\|\Pi_\lambda f\|^2_{L^2(\mathbb{R}^d)} \leq Ce^{C\sqrt{\lambda}} \sum_{k\in\mathbb{Z}^d} \sup_{x\in\omega\cap B(k,R)} |\Pi_\lambda f(x)|^2.$$

This ends the proof of Theorem 2.12.

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