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Assaf Hasson · Dmitry Sustretov

Incidence systems on Cartesian powers of algebraic curves

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Abstract. We show that a non-locally modular reduct of the full Zariski structure of an algebraic curve interprets an infinite field.

Keywords: Zilber trichotomy, strongly minimal sets, reducts of ACF, non-local modularity.

Contents

1. Introduction

In [1, §2], Artin describes the basic problem of classifying abstract plane geometries (viewed as incidence systems of points and lines) as follows: "Given a plane geometry [...] assume that certain axioms of geometric nature are true [...] is it possible to find a field k such that the points of our geometry can be described by coordinates from k

Assaf Hasson: Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, 8410501 Be'er Sheva, Israel; hassonas@math.bgu.ac.il

Dmitry Sustretov: Department of Mathematics, KU Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium; dsustretov.math@gmail.com

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and lines by linear equations?" Zilber's trichotomy principle (to be described in more detail in the next section) can be viewed as an abstraction of the above problem, replacing the "axioms of geometric nature" with a well-behaved theory of dimension (see, e.g., [39, §1]).

Conjectured in various forms by Zilber throughout the late 1970s, essentially every aspect of Zilber's trichotomy, in its full generality, was refuted by Hrushovski [20, 21] in the late 1980s. Due to Hrushovski's cornucopia of counterexamples, the conjecture has never been reformulated. Yet, Zilber's principle remains a central and powerful theme in model theory: it has been proved to hold in many natural examples such as differentially closed fields of characteristic 0, algebraically closed fields with a generic automorphism, o-minimal theories and more (see [10, 11, 26, 36, 38]). Many of these special cases of Zilber's trichotomy have had striking applications in algebra and geometry (see [22, 23, 41]). More recently, in [46], Zilber outlines a model theoretic framework for studying far-reaching extensions of the Mordell–Lang conjecture. One of the key features of Zilber's strategy is the trichotomy theorem for Zariski geometries [26].

The key to the classification of Desarguesian plane geometries (the fundamental theorem of projective geometry) is the reconstruction of the underlying field k as the ring of direction preserving endomorphisms of the group of translations. The reconstruction of a field out of abstract geometric data is also the essence of Zilber's trichotomy, and is the engine in many of its applications. A relatively recent application of one such result is Zilber's model theoretic proof [45] of a significant strengthening of a theorem of Bogomolov, Korotiaev, and Tschinkel [3]. The model theoretic heart of Zilber's proof is Rabinovich's trichotomy theorem for *reducts* of algebraically closed fields [40]. In the concluding paragraph of the introduction to [45], Zilber writes: "It is therefore natural to aim for a new proof of Rabinovich's theorem, or even a full proof of the restricted trichotomy along the lines of the classification theorem of Hrushovski and Zilber [26], or by other modern methods [...]. This is a challenge for the model-theoretic community."

The conjecture referred to in Zilber's text above can be formulated as follows.[1,2]

Conjecture A. *Let $\mathcal{M}$ be a strongly minimal non-locally modular reduct of the full Zariski structure on an algebraic curve M over an algebraically closed field K. Then there exist $\mathcal{M}$-definable sets L, E such that $E \subseteq L \times L$ is an equivalence relation with finite classes and L/E with the $\mathcal{M}$-induced structure is a field K-definably isomorphic to K.*

Rabinovich [40] proved Conjecture A in the special case where $M = \mathbb{A}^1$, and her result can be extended by general principles to any rational curve. In the present paper,

[1] The content of Conjecture A is explained for non-experts in Section 2.

[2] This conjecture is a specialisation of Zilber's trichotomy to strongly minimal sets interpretable in algebraically closed fields (though by "the full conjecture" Zilber is referring also to such strongly minimal sets whose universe of the interpretation could, a priori, be a constructible set of dimension greater than one). As will be discussed in the concluding paragraphs of this paper, Conjecture A can be further generalised, and is equivalent to the statement that every non-1-based structure interpretable in an algebraically closed field, K interprets a copy of K, provided the interpretation is rank preserving.

we prove Conjecture A. Our approach to the problem follows the well-known strategy introduced by Zilber and Rabinovich and owes to [30]. Using a standard model theoretic technique, Hrushovski's field configuration (see Section 4.1 for details), the problem is reduced to showing that tangency is (up to a finite correction) $\mathcal{M}$-definable in families.

To achieve this goal, we proceed in two steps. In the first step (carried out in Section 3), we study slopes of families of branches (at a given point) and their behaviour under composition of curves and, in the case where the ambient structure is an expansion of a group, under pointwise addition. This culminates in Proposition 3.15, which is the key to the definability of tangency, and in Lemma 4.19, providing us with the (algebraic) group which is the template allowing us to construct the group configuration.

Section 4, where Conjecture A is proved (Theorem 4.36), is dedicated, mainly, to verifying that the assumptions of the technical result of the previous section can be met in the reduct. In Section 4.3, we show that our definition of slope is meaningful in suitably chosen $\mathcal{M}$-definable families of curves (in positive characteristic). In Section 4.4, where the main step towards proving Theorem 4.36 is carried out, the key difficulty to overcome is in the application of Proposition 3.15. Specifically, Proposition 3.15 is, in essence, a delicate combinatorial result allowing to detect tangency by studying the intersection of pure-dimensional algebraic curves varying in geometrically nice families. In $\mathcal{M}$, we do not have direct access to the algebro-geometric objects appearing in the assumptions of the proposition, yet we want to use the conclusions in order to detect tangency. As it turns out, the main obstacle to achieving this goal is due to isolated points belonging to generic curves in our $\mathcal{M}$-definable families, that are not (a priori) detectable as such by $\mathcal{M}$. If such points conspired to interfere with the intersection of tangent curves, masking the drop in the number of geometric intersection points caused by tangency, our strategy would collapse. Many of the technicalities arising in the proof are dedicated to showing that such situations can always be avoided.

1.1. Relation to earlier and later work

As already mentioned, Rabinovich's work – though widely accepted as one of the first major breakthroughs in the study of Zilber's trichotomy (restricted to geometric settings) – is hard to penetrate, and we know no one who claims to have understood it. For that reason, it is hard to compare Rabinovich's work to the present one, and our main source of inspiration was the result of Marker and Pillay [30] on additive reducts of the complex field.

Having said that, it is clear that the overall strategy in Rabinovich's work, as in the work of Marker–Pillay and the present paper owes much to ideas originating in Zilber's school in Kemerovo. So are some of the geometric techniques and tools applied.[3] It should

[3]The approach in [30] is analytic rather than algebro-geometric, our proof was motivated by those ideas and translated into the language of schemes to accommodate the positive characteristic case.

be stressed, however, that we – as well as several experts we have contacted in this regard – were unable to understand significant parts of Rabinovich's argument. For that reason, we cannot pinpoint the reason for the present work being more general, shorter, and technically simpler.

Let us, however, dwell on one, seemingly, minor technical issue, that is – in reality – rather significant. The problem of isolated points, discussed in the paragraph concluding the previous section, was a serious obstacle to overcome here as well as in several other geometric settings where the reduct gives no direct access to the topology allowing the detection of such phenomena (e.g., [7, 17]). This is also one of the key problems in extending the results of the current work to strongly minimal sets whose universe is a constructible set of dimension greater than one (see [7, §1] for more details). Anyone looking into this problem will soon come to realise that it is tightly connected with the problem of, in Rabinovich's terminology, common points (i.e., points in the plane that cannot be effectively separated by a given family of plane curves). Some of the longest and most technical parts in Rabinovich's monograph are dedicated to controlling the behaviour of common points. Though we could not find an explicit explanation in her text for carrying out these highly technical computations, there is good reason to believe (see, e.g., [40, Proposition 5.1] and cross-references in the proof) that those are needed to handle the *superfluous* (isolated, in our terminology) points.

We point out that Rabinovich's work is a continuation of an earlier, more restrictive, version (appearing as [9] in her paper), where the problem of common points does not arise (see [40, §1, especially pp. 2–3]). This suggests (though the earlier text of Rabinovich is in Russian and, to the best of our knowledge, not publicly available) that the main novelty in [40] is the treatment of the general case, where common points may cause problems. It seems that our approach to the problem (showing that isolated points can be avoided in computations) is more geometric than Rabinovich's rather combinatorial treatment, and as such allows for shorter, less technical proofs. Specifically, we apply algebro-geometric techniques to families of pure-dimensional curves (Lemma 3.13), culminating in the geometric proof of Proposition 3.15. This result is then invoked twice (in the proof of Theorem 4.25). First we use it to show that choosing our $\mathcal{M}$-definable families of curves carefully enough, the intersection-theoretic properties we need are the same for our $\mathcal{M}$-definable families and the associated pure-dimensional families. Then we apply Proposition 3.15 to the associated pure-dimensional families, to recover tangency in $\mathcal{M}$.

From another angle, it is natural to ask (especially, in view of the relevance of the results to anabelian geometry as in [45]) whether the reconstruction of the field from the reduct can be obtained, in some sense, canonically. Since it is not clear what is the right category for considering such a problem, we cannot provide a satisfactory geometric answer to this question. On the model theoretic side, it has recently been shown by Castle and the first author, [8, Theorem 4.19], that the field interpreted in $\mathcal{M}$ can be taken $\emptyset$-definable (in $\mathcal{M}$). This is not too hard to show, a posteriori, once we have obtained a definable family of infinite fields, but seems non-obvious in the process of constructing (as we do here) some infinite definable field. Thus, our rather free usage of parameters, local coordinates and possibly other non-canonical choices does not affect the canonicity of the ultimate result – at least to the extent to which we know how to formulate it.

Added in proof. After the submission of this paper for publication, Castle [7] announced the solution of the restricted trichotomy in characteristic 0. Castle's proof uses a different method for identifying tangency in families than the one presented here. He isolates, however, a technical notion of "the reduct identifying tangency" and shows that this automatically implies the interpretability of an algebraically closed field in the reduct. Castle's result gives a new proof of the results presented in this paper in characteristic 0.

In a recent preprint [9], Castle, Ye and the first author extend these techniques to algebraically closed valued fields of all characteristics (using ideas appearing in the present paper for dealing, in the reduct, with the Frobenius automorphism). Combined with the Ph.D. thesis of Pinzon culminating in [18], this should give, in particular, a complete proof of Zilber's restricted trichotomy for algebraically closed fields in all characteristics.

2. Model-theoretic background

For readers unfamiliar with model theory, we give a self-contained exposition of Conjecture A. In order to keep this introduction as short as possible, we specialise our definitions to the setting in which they will be applied. We refer interested readers to [43, Chapter 1, §(1.2)] for a more detailed discussion of structures and definable sets. Readers familiar with the basics of model theory are advised to only skim through the remainder of the present section, to keep track of notational and other conventions.

Given an algebraic curve M over an algebraically closed field k, *reduced, but not necessarily irreducible, smooth or projective*, the *full Zariski structure on* M, denoted by $\mathfrak{M}$, is the set of k-rational points, $M(k)$ equipped with the collection of all Boolean algebras of constructible sets[4] on the Cartesian powers $M^n(k)$. The full Zariski structure on a curve M is an example of the model theoretic notion of a structure.

A *first-order structure* or simply a *structure*, $\mathcal{N}$, is a non-empty set N (called the universe of $\mathcal{N}$) equipped with a collection $\mathrm{Def}(\mathcal{N})$ of Boolean algebras $\mathrm{Def}_l(\mathcal{N}) \subseteq \mathcal{P}(N^l)$ for all $l > 0$, such that $\mathrm{Def}_l(\mathcal{N})$ contains all diagonals $\Delta^l_{i,j} := \{(x_1, \dots, x_l) : x_i = x_j\}$, and such that $\mathrm{Def}(\mathcal{N})$ is closed under finite Cartesian products and projections of the form $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$. Somewhat analogously to geometric terminology, the tuples $(x_1, \dots, x_n) \in S \subset M^l$ are called *points of the definable set* S. If $A \subseteq N$ is any set, a subset $X \subseteq N^l$ is *definable with parameters in* A (or A*-definable*) if there exists a definable set $Y \subseteq N^{n+m}$ (some $m \geq 0$) such that $Y = Y_a := \{x \in N^l : (x, a) \in Y\}$ for some $a \in A^n$ (some $n \in \mathbb{N}$). As a rule, if A is a set of parameters, we will use the model theoretic convention of writing $a \in A$ as shorthand for "a is a finite tuple from A". Also,

[4] Formally, putting a structure on M requires that a language be specified. Toward that end, a canonical choice would be to take the constructible sets over the field of definition of M. This choice will not change the class of definable sets, affecting only the use of parameters.

when no confusion can arise, if A, a and b are parameters, we will write Aa or ab instead of $A \cup \{a\}$ and $\{a, b\}$, respectively (in such situations, if a is a tuple, we will not distinguish – when such a distinction is unimportant – between the tuple $a := (a_1, \dots, a_n)$ and its image $\{a_1, \dots, a_n\}$).

While the above description of first-order structures is very concise, avoiding many technical details, it does not provide much intuition into what definable sets actually are. In practice, structures are usually obtained by specifying a universe M and some distinguished “basic” (or “atomic”) subsets (of Cartesian powers of M) letting the resulting structure $\mathcal{M}$ be the system of Boolean algebras generated by the atomic sets. A $\emptyset$-definable set in M^n is then the collection of all n-tuples realising a first-order formula (without parameters) in the language consisting of symbols for the atomic sets. Interested readers are referred to any basic textbook in logic for precise definitions (e.g., [29, §1]).

Over an algebraically closed field k, the Chevalley theorem (see, e.g., [29, Corollary 3.2.8]) asserts that the collection of constructible sets on Cartesian powers of (the k-rational points of) an algebraic curve M is closed under projections, and therefore the full Zariski structure $\mathfrak{M}$ on M is, indeed, a structure in the above sense. It is a well-known fact (e.g., [26]) that the field k can be reconstructed from $\mathfrak{M}$. Let us now explain more precisely what is meant by that.

A (partial) function $f\colon N^l \to N$ is definable if its graph is. Thus, for example, we say that a group is definable in $\mathcal{N}$, if there exist a definable set $G \subseteq N^l$ and a definable function $p\colon G \times G \to G$ such that (G, p) is a group (note that the function $x \mapsto x^{-1}$ is automatically definable if (G, p) is a group). The definability of a field in a structure $\mathcal{N}$ is defined analogously. It is not hard to check (and it follows from the main result of [26]) that if $\mathfrak{M}$ is the full Zariski structure on an algebraic curve M over an algebraically closed field k, then a field F is definable in $\mathfrak{M}$ (and F is isomorphic, definably in the standard field structure on k, to k).

But we need a somewhat subtler notion than definability. Consider, as a simple example, the structure $\mathcal{C}$ with universe $\mathbb{C} \times \{0, 1\}$, whose definable sets are all those of the form $\{((x_1, i_1), \dots, (x_n, i_n)) : (x_1, \dots, x_n) \in D\}$, where D is a constructible subset of $\mathbb{C}^n$ and $i_j \in \{0, 1\}$ for all $1 \leq j \leq n$. It is easy to verify that all functions definable in $\mathcal{C}$ are locally constant, and therefore there is no definable field in $\mathcal{C}$. Consider, however, the equivalence relation $x \sim y$ (in $\mathcal{C}$) defined by $y \in (1, 0) \cdot x$ (recalling the interpretation of multiplication in $\mathcal{C}$, this is a $\mathcal{C}$-definable way of saying that x and y have the same first coordinate). Then $\sim$ is a $\mathcal{C}$-definable equivalence relation, and $\mathcal{C}/{\sim}$ is naturally isomorphic to the full Zariski structure on $\mathbb{C}$.

In model-theoretic terms, the structure $\mathcal{C}$ in the previous example *interprets* a field definably isomorphic to $\mathbb{C}$. In general, if $\mathcal{N}$ is a structure, E is a definable equivalence relation on N^l and $\pi\colon N^l \to N^l/E$ is the natural projection, the induced structure on N^l/E is the push-forward of the Boolean algebras on powers of N^l via π. We say that $\mathcal{N}$ *interprets* a field if a field is definable in the structure induced on N^l/E for some l and $\mathcal{N}$-definable equivalence relation E on N^l.

In the above example, the universe $\mathbb{C} \times \{0, 1\}$ of $\mathcal{C}$ is definable in the full Zariski structure on $\mathbb{C}$, and every definable set in $\mathcal{C}$ is definable in $\mathbb{C}$. But, as we have seen, $\mathcal{C}$

is not the full Zariski structure on $\mathbb{C}$. The structure $\mathcal{C}$ is an example of a *reduct* of the full Zariski structure on $\mathbb{C}$. Generally, if $\mathcal{M}$ is a structure whose universe is an algebraic curve M and every $\mathcal{M}$-definable set is $\mathfrak{M}$-definable, then $\mathcal{M}$ is a *reduct* of $\mathfrak{M}$.

Zilber's conjecture is concerned with the question of interpreting a field in a reduct $\mathcal{M}$ of the full Zariski structure $\mathfrak{M}$ on an algebraic curve M over an algebraically closed field k. Assume that an infinite field $\mathbb{F}$ is interpretable in $\mathcal{M}$. Then by [29, Theorem 3.2.20] the universe F of $\mathbb{F}$ can be identified with a constructible subset of k^l for some l, and by [39, Theorem 4.15] k is definably isomorphic to $\mathbb{F}$. Thus, there is a definable finite-to-finite correspondence $\Psi \subseteq F \times M$. It is easy to check that Ψ can be taken to be $\mathcal{M}$-interpretable (e.g., if F is definable in $\mathcal{M}$, then Ψ can be taken to be the graph of a projection function, the general case is slightly more delicate and we skip the details). If we push the family of affine lines in F^2 via Ψ, we obtain a two-dimensional constructible subset U of M such that for any $p, q \in U$, there is a curve $C := \Psi(L)$ – for L an affine line in F^2 – with $p, q \in C$. We have thus verified that for $\mathcal{M}$ to interpret a field, it is necessary that there exist a two-dimensional constructible $U \subseteq M^2$ and a definable set $X \subseteq M^{2+l}$ such that $X_t := \{(x, y) : (x, y, t) \in X\}$ is one-dimensional (or empty) for all $t \in M^l$ and such that for all $p, q \in U$, there exists $t \in M^l$ such that $p, q \in X_t$. The main result of the present work, Theorem 4.36, states that this condition is, in fact, sufficient.

Definition 2.1. Let $\mathcal{M}$ be a reduct of the full Zariski structure $\mathfrak{M}$ on an algebraic curve M over an algebraically closed field k. An $\mathcal{M}$-definable *ample family of curves in* M^2 is a set $X \subseteq M^{2+l}$ such that

- $\dim(X_t) = 1$ for all $t \in M^l$ such that $X_t \neq \emptyset$ and
- there exists a two-dimensional $U \subseteq M^2$ such that for all $p, q \in U$, there exists $t \in M^l$ with $p, q \in X_t$.

In model-theoretic terms, the existence of an ample family as above is equivalent, [29, Lemma 8.1.13], to *non-local modularity* of the structure $\mathcal{M}$. The interested reader is referred to [37, §2] for a more detailed discussion of local modularity and related notions.

Keeping the above notation, if X is an ample family in M^2, we denote by (M, X) the smallest reduct of $\mathfrak{M}$ containing X. We can thus reformulate Conjecture A.

Conjecture B (Zilber's restricted trichotomy in dimension 1). *Let M be an algebraic curve over an algebraically closed field k. Let $X \subseteq M^2 \times T$ be the total space of an ample family in M^2 and some $\mathcal{M}$-definable set $T \subseteq M^l$. Then a field K, k-definably isomorphic to k, is interpretable in $\mathcal{M} = (M, X)$.*

Clearly, if $\mathcal{M}$ is an arbitrary reduct of the full Zariski structure $\mathfrak{M}$ on an algebraic curve M over an algebraically closed field k, and $\mathcal{M}$ admits an $\mathcal{M}$-definable ample family of curves X, then any infinite field interpretable in (M, X) is also interpretable in $\mathcal{M}$.

In [1, §2.4] not only is the field recovered from the affine geometry, but also the geometry is recovered as the affine plane over that field. In the present setting, there are examples due to Hrushovski (see, e.g., [31]) showing that the full Zariski structure of the curve M cannot be recovered from $\mathcal{M}$. This can probably be achieved if X is *very ample* in the sense of [26] (namely, if the set X in Definition 2.1 the separates points in M^2), but we do not study this question here.[5]

3. Tangency

The reconstruction of the field is obtained in two steps. First, we reconstruct a one-dimensional algebraic group, and then – using the group structure to sharpen the same arguments – we reconstruct the field. Roughly, the reconstruction of a group is obtained in three stages: first, we identify a reduct definable family $X \to T$ of algebraic curves whose associated family of slopes at some point $P = (a, a) \in M^2$ is a one-dimensional algebraic group under composition. The second and most crucial part of the proof is commonly dubbed *definability of tangency.* In its cleanest form this consists in showing that, given families $X \to T$ and $Y \to S$ as above, the set of all $(t, s) \in T \times S$ such that X_t is tangent (in an appropriate sense) to Y_s at P is $\mathcal{M}$-definable. Finally, the group is reconstructed by invoking *the group configuration theorem*, a well-known model theoretic technique (to be described in more detail in the next section), using the results of the previous stages. In the next two subsections, we take care of the two first stages of this strategy.

Before providing the technicalities, let us discuss some of the challenges that motivated the definitions to be shortly presented. In the implementation of the strategy outlined above, two difficulties arise.

Firstly, if we consider only the first-order slopes, then due to inseparability issues in positive characteristic it becomes hard to find a one-dimensional family of curves definable in the reduct such that its associated slopes at some point range in a one-dimensional set – such a family is needed to construct the first group configuration (Section 4.4). The solution is to consider tangency information up to any order n and pick the order so that there are enough slopes. Interestingly – and this was apparent already in [30] – in the presence of a group structure, the problem does not arise, which is a good coincidence, since the second group configuration (Section 4.5) has to be built using the first-order tangency information.

Secondly, we cannot work only with smooth points to define the slope, since the operations of composition and pointwise addition that are used in the construction of the group configurations do not preserve smoothness (as smooth points may be mapped onto branch points). Our approach to this is to track a particular branch of a curve at a particular point as the operations of composition and pointwise addition are applied: one can then have control over the slope of a branch, appropriately defined. Note that we use the term *branch* (Definition 3.3) in a more restrictive sense than what is usually understood by it: in a way, our branches are "always smooth" (or, more precisely, "always étale over the first factor M of M^2"), so that the notion of a slope always makes sense for them. For any curve in M^2, the projection either on the first or on the second factor M is going to be

[5] After the submission of the paper this was, indeed, proved by Castle and the first author in [8].

generically étale (Lemma 4.14), even in positive characteristic, and so there is going to be a unique branch at any general enough point on this curve (Lemma 3.4). This statement generalises appropriately to families of curves too. By virtue of Propositions 3.7 and 3.9, the slopes of relevant branches can be tracked as the curves are composed and pointwise added. All curves and branches that we work with in Section 4 are obtained this way.

3.1. Slopes and operations on correspondences

Our main objects of interest are definable subsets in a reduct of the full Zariski structure on a fixed curve M (over a fixed algebraically closed field k), which we intend to study using algebro-geometric tools. In the present section, we produce these tools. Much of the work in the next section is dedicated to fitting our model theoretic setting into the scope of the tools developed here.

Throughout, by varieties we mean schemes of finite type over an algebraically closed field k. We do not assume varieties to be reduced, though algebraic curves (as already mentioned), families of algebraic curves, and their parameter spaces can be assumed reduced. Non-reduced scheme will, however, play a crucial role in our study of the family of scheme theoretic intersection of families of algebraic curves.

We adopt the following non-standard terminology: we call a constructible subset $Z \subseteq M^n$ (for some $n > 1$) of dimension 1 a *curve*, even if it is reducible and or has connected components of dimension 0. If a curve Z does not contain connected components of dimension 0, we call Z a *pure-dimensional curve*. Clearly, every curve contains a maximal pure-dimensional curve. Throughout, we restrict our attention solely to curves whose projections on all coordinates are finite to one (see also Section 4.2 for a more detailed discussion of this point).

In the few situations when we refer to abstract algebraic curves (that is, algebraic varieties of pure dimension 1 over a fixed algebraically closed field), we will use the term *algebraic curve*. We will not distinguish notationally between subsets of M^n definable in a reduct of the full Zariski structure on M and constructible subsets of the varieties (or even schemes) M^n, and in particular between definable curves and their algebro-geometric counterparts.

Recall that any algebraic variety over a perfect field admits a dense Zariski open subset that is smooth (see, e.g., [33, corollary to Theorem 30.5]). Let $Z \subset M^2$ be a pure-dimensional curve and $a = (a_1, a_2) \in Z$ be a smooth point of M^2. Since the completion of the local ring of a smooth point of a variety is a formal power series ring [33, Theorem 29.7], we can pick isomorphisms

$$\widehat{\mathcal{O}_{M,a_1}} \cong k[\![x]\!], \quad \widehat{\mathcal{O}_{M,a_2}} \cong k[\![y]\!]$$

inducing an isomorphism $\widehat{\mathcal{O}_{M^2,a}} \cong k[\![x, y]\!]$, and then $\widehat{\mathcal{O}_{Z,a}} = k[\![x, y]\!]/(f)$ for some $f \in k[\![x, y]\!]$. We call *branches of Z at a* the factors in the prime decomposition of f of the form $y - g_\alpha$, $g_\alpha \in k[\![x]\!]$ (note that this is different from the standard use of the term, but since we will never use the term in its standard meaning in this article, no confusion will occur). In particular, if the projection of Z onto the first factor M in M^2 is étale in a neighbourhood of a, by Hensel's lemma (stated as in [34, Chapter I, §4, Theorem 4.2 (d)]), the natural morphism $k[\![x]\!] \to k[\![x, y]\!]/(f)$ is an isomorphism, and therefore f can be written uniquely as $u(y - g)$, where $u \in k[\![x]\!]^\times$, $g \in k[\![x]\!]$. We call the truncation to the n-th order of the series g_α the *n-th order slope of a branch α of Z*. Naturally, the slope of a branch of a pure-dimensional curve depends on the choice of the isomorphism $\widehat{\mathcal{O}_{M^2,a}} \cong k[\![x, y]\!]$, but this choice does not affect any of the properties of slopes we will be interested in.

Let us note that branches of pure-dimensional curves, introduced formally in Definition 3.3, are formal schemes, but since we need local coordinates for the definition of slopes, it seemed more transparent, in the present somewhat informal overview, to introduce local coordinates right from the start, and to use them also for a quick definition of branches.

We view curves in M^2 as finite-to-finite correspondences between the two factors M. The purpose of the present section is to study the behaviour of slopes of branches with respect to two natural operations on correspondences: composition and "pointwise addition" (see Definition 3.8) when M has a structure of an algebraic group. We will show that if Z, W are two curves and α, β are two branches of Z, W at $a = (a_1, a_2)$, $b = (b_1, b_2) \in M^2$, respectively, and $a_2 = b_1$, then the composition $W \circ Z$ has a branch $\beta \circ \alpha$ at (a_1, b_2) whose slope is the composition of the n-th order slopes of α and β (as truncated polynomials) whenever the latter are defined. A similar statement can be made about the slopes of branches of curves that are "pointwise added" if M has a structure of a group. Later we will construct a group configuration starting from a family of curves definable in a reduct of a full Zariski structure on M such that the set of its n-th order slopes at a given point coincides, up to a finite set, with a one-dimensional algebraic subgroup of $\mathrm{Aut}(k[\![x]\!]/(x^{n+1}))$ (a truncated polynomial f corresponds naturally to the automorphism of $k[\![x]\!]/(x^{n+1})$ sending x to f).

Since we will have to work with families of curves, we will also introduce notions of families of branches and slopes. When the characteristic of the base field is positive, we will often have to work with curves and families of curves in $M \times M^{(p^n)}$, where $M^{(p^n)}$ is the pull-back of M by the Frobenius endomorphism on k (see Section 4.3). For that reason, in the definition below, we do not assume that factors of the ambient product variety are isomorphic.

Definition 3.1 (Families of curves). If X_1, X_2 are two algebraic curves, then by a *family of pure-dimensional curves in $X_1 \times X_2$* we will understand a finite union Z of pure codimension 1 locally closed subsets $Z_i \subset X_1 \times X_2 \times T$, where T is a variety, such that Z_t is a pure-dimensional curve for all $t \in T$. By a *family of curves*, we understand a constructible subset $Z \subset X_1 \times X_2 \times T$, where T is a constructible subset of a variety and such that Z_t is a curve for all $t \in T$. If $X_1 = X_2 = M$ and $T \subset M^l$ for some l and Z is definable in a reduct of the full Zariski structure on M, we call it a *definable family of curves*.

While families of curves arise naturally in the definable context, in order to apply the machinery of slopes we need to work with families of pure-dimensional curves. As long as T is a variety, a family of curves $Z \subset X_1 \times X_2 \times T$ contains a unique maximal family of pure-dimensional curves. The total space Z of a family of pure-dimensional curves is not necessarily a variety; while this is a desirable property that will be important in Section 3.2, we do not include it in the definition so that it can be readily seen that the operations of composition and pointwise addition preserve the class of families of pure-dimensional curves. However, as is shown in the next lemma, one can easily ensure that the total space is a variety at the cost of shrinking the parameter space. The reader is warned, however, that – as a rule – this can only be achieved in the full Zariski structure.

Lemma 3.2. *Let T be a constructible subset of a variety, $W \subset X_1 \times X_2 \times T$ a family of curves. Then there exists $T' \subset T$ Zariski dense in T that is a variety, and there exists a maximal locally closed $W' \subset W \times_T T'$ which is a family of pure-dimensional curves. In particular, W' is a variety.*

Proof. Let us assume, for ease of notation, that $T \subseteq M^n$. It is easily checked using Noetherian induction that any constructible subset of M^n contains a Zariski dense subset that is locally closed in the ambient variety M^n. In particular, there exists a dense $T_0 \subset T$ that is a locally closed subvariety of M^n. Without loss of generality, we may assume T_0 connected. Then $W \times_T T_0$ is a union of locally closed sets of the form $W_i \setminus Z_i$, $i \in I$, where W_i and $Z_i \subset W_i$ are Zariski closed and distinct, and the index set I is finite. Let $I' \subset I$ be the set of those indices i for which W_i has codimension 1 in $M^2 \times T$. Further shrinking T_0, we may assume that $\dim(W_i)_t = 1$, $\dim(Z_i)_t = 0$ for all $t \in T_0$ and all $i \in I'$ (in particular, $Z_i \neq \emptyset$). We put $W' = \bigcup_{i \in I'} W_i \setminus Z_i$. It now suffices to show that if $W_1 \setminus Z_1$ and $W_2 \setminus Z_2$ are as above, then there exists a dense open $T' \subset T_0$ such that $(W_1 \setminus Z_1) \times_T T' \cup (W_2 \setminus Z_2) \times_T T'$ is locally closed; the statement of the lemma then follows by induction on the size of I'.

We have

$$(W_1 \setminus Z_1) \cup (W_2 \setminus Z_2) \\ = (W_1 \cup W_2) \setminus ((Z_1 \cap Z_2) \cup (Z_1 \cap (W_2 \setminus Z_2)) \cup (Z_2 \cap (W_1 \setminus Z_1))).$$

It follows from an easy dimension computation that

$$\dim(Z_1 \cap (W_2 \setminus Z_2)) < \dim Z_1, \quad \dim(Z_2 \cap (W_1 \setminus Z_1)) < \dim Z_2,$$

so in particular the projections of $Z_1 \cap (W_2 \setminus Z_2)$, $Z_2 \cap (W_1 \setminus Z_1)$ to T_0 are not dense. If T' is a dense open set in the complement of the projections, then

$$(W_1 \setminus Z_1) \times_T T' \cup (W_2 \setminus Z_2) \times_T T' = (W_1 \cup W_2) \times_T T' \setminus (Z_1 \cap Z_2) \times_T T'$$

which is locally closed. ■

Given a family of pure-dimensional curves Z as above, we would like to be able to pick branches of the curves Z_t depending algebraically on the parameter $t \in T$. In this case, the local equation of Z in a formal neighbourhood of $\{a\} \times T$ may only exist locally on T, and in order to capture this idea, we have to phrase the definition in terms of formal schemes (we refer the reader to [16, Chapter II, Section 9] or any other standard algebraic geometry reference for the definition of formal schemes).

Definition 3.3 (Branches and families of branches). Let $Z \subset V := X_1 \times X_2 \times T$ be a family of pure-dimensional curves, $a \in M^2$, and assume that $a \in Z_t$ for all $t \in T$. Let $\widehat{X}_1$ be the formal completion of $X_1 \times T$ along $\{a_1\} \times T$, and let $\widehat{Z}$ be the formal completion of Z along $\{a\} \times T$. A *family of branches of Z at a* is a closed formal subscheme $\widehat{Z}_\alpha$ such that the natural projection $\widehat{Z}_\alpha \to \widehat{X}_1$ is an isomorphism. We will call local generators of the ideal sheaf that defines $\widehat{Z}_\alpha$ *local equations of* α. When $Z \subset X_1 \times X_2$ is a single curve, we regard it as a family parametrized by a single point, and we call families of branches of Z just branches. Given a family of branches α, we will denote by α_t the branch given by the fibres $\widehat{Z}_{\alpha_t}$ for all $t \in T$.

Remark. If $Z \subset X_1 \times X_2 \times T$ is a family of curves and T is a variety, then in order to simplify the exposition, we will refer to branches of Z meaning branches of a family of pure-dimensional curves $Z_0 \subset Z$.

Let X_i be algebraic curves, and let $a = (a_1, \dots, a_n) \in X = X_1 \times \cdots \times X_n$ be a smooth point. We say that a *local coordinate system at a* is picked when an isomorphism $\widehat{\mathcal{O}_{X_i,a_i}} \cong k[\![x_i]\!]$ is picked for each a_i; in this case, we understand that there exists an isomorphism $\mathcal{O}_{X,a} \cong k[\![x_1, \dots, x_n]\!]$ induced by these isomorphisms. If local coordinate systems are picked at $a = (a_1, a_2) \in X_1 \times X_2$, $b = (b_1, b_2) \in X_2 \times X_3$, we understand without explicit mention that local coordinate systems are automatically picked at the points $(a_1, b_2) \in X_1 \times X_3$, $(b_2, a_1) \in X_3 \times X_1$ which will be of interest to us later on. Similarly, if X_1 has a group structure and a local coordinate system is picked at a point $a \in X_1$, then we assume it picked at any point $a' \in X_1$ via translation. The next lemma gives a sufficient condition for the existence of a family of branches at a point.

Recall that a morphism of schemes $f: X \to Y$ is called *quasi-finite* if the fibres $f^{-1}(y)$ are finite for all $y \in Y$. A quasi-finite morphism $f: X \to Y$ of locally Noetherian schemes is *unramified* if $\Omega_{X/Y} = 0$ (see [28, Chapter 6, Corollary 2.3]), where $\Omega_{X/Y}$ is the module of Kähler differentials of the morphism f. A morphism f locally of finite type is called *étale* if it is flat and unramified. Basic properties of these notions will be recalled in detail and with references in Section 3.2.

Lemma 3.4. *If $Z \subset V = X_1 \times X_2 \times T$ is a family of pure-dimensional curves and the projection $Z \to X_1$ is étale in a neighbourhood of $\{a\} \times T$ for some $a \in X_1 \times X_2$, then there exists a unique family of branches of Z at a.*

Proof. For any affine open Spec $R \subset X_1 \times T$, let Spec $R' \subset Z$ be an affine open étale over Spec R, let I, I' be the ideals vanishing on $\{a_1\} \times T$, $\{a\} \times T$, respectively, $\widehat{R}$ and $\widehat{R}'$ their respective completions. Then by [42, Tag 0ALJ], $(\widehat{R}, I)$ is a Henselian pair and by [42, Tag 09XI], there exists a unique isomorphism $\widehat{R}' \to \widehat{R}$ that defines the unique family of branches. ∎

Definition 3.5 (Slope). Let X_1, X_2 be algebraic curves, $Z \subset V := X_1 \times X_2$ a pure-dimensional curve, $a \in V$ a smooth point, $a \in Z$, and $I, \mathfrak{m}_a \subset \mathcal{O}_{V,a}$ the ideals of functions that vanish on Z, $\{a\}$, respectively. Assume that a local coordinate system is chosen at a, so that $\varprojlim \mathcal{O}_{V,a}/\mathfrak{m}_a^n \cong k[\![x, y]\!]$. A branch α of Z is therefore defined by a principal ideal J with the property that the composition $k[\![x]\!] \to k[\![x, y]\!] \to k[\![x, y]\!]/J$ is an isomorphism. The inverse of this isomorphism sends y to $f \in xk[\![x]\!]$, and $y - f \in J$. We call $f \bmod x^{n+1} \in k[x]/(x^{n+1})$ the *n-th order slope of Z at α*, denoted by $\tau_n(Z, \alpha)$.

Note that $\tau_n(Z, \alpha)$ depends on the choice of the local coordinate system at a, and that if an n-th order slope of Z at α is defined, then the slopes of all orders of Z at α are defined.

Remark. (i) Let f, g be functions on the formal neighbourhood of a point a common to the pure-dimensional curves Z_1, Z_2, respectively, and whose respective graphs are branches α, β of Z_1, Z_2. If $\tau_n(Z_1, \alpha) = \tau_n(Z_2, \beta)$ but $\tau_{n+1}(Z_1, \alpha) \neq \tau_{n+1}(Z_2, \beta)$, then $f \equiv g \bmod x^{n+1}$ and therefore $f - g = x^{n+1} \cdot r$ for some unit $r \in k[\![x]\!]$, and so

$$k[\![x, y]\!]/(y - f, y - g) \cong k[\![x]\!]/(x^{n+1}).$$

In particular, if Z_1, Z_2 are smooth at a and α, β are their unique respective branches at a, then the intersection multiplicity of Z_1 and Z_2 at a (as defined in, for example, [16, Chapter I, Exercise 5.4, p. 36]) is n.

(ii) Let $X \subset M^2 \times T$, $Y \subset M^2 \times S$ be families of pure-dimensional curves such that $a \in X_t \cap Y_s$ for all $t \in T$, $s \in S$ and $X_t \cap Y_s$ is zero-dimensional for generic t, s. Let α, β be some families of branches of X and Y at a. Then it follows from Krull's maximal ideal theorem that there exists a maximal integer n such that

$$\tau_n(X_t, \alpha_t) = \tau_n(Y_s, \beta_s)$$

for all $t \in T$, $s \in S$.

For the benefit of the reader, we explain what data in the Definitions 3.3 and 3.5 specifies families of branches and slopes, specialising the description in the language of formal schemes to an affine situation. Take Zariski open subsets $U \subset X_1 \times X_2$, $W \subset T$ such that $a \in U$. Let S, R be the rings of regular functions on U, W, and let $J_a, J \subset R \otimes S$ be the ideals of regular functions that vanish on $\{a\} \times W$, $Z \cap U \times W$, respectively. We fix a local coordinate system at a which gives an isomorphism $\varprojlim (R \otimes S)/J_a^n \cong R[\![x, y]\!]$. A choice of a family of branches α is a choice of an element $f_\alpha \in R[\![x]\!]$ such that $y - f_\alpha \in R[\![x, y]\!]$ generates an ideal (necessarily prime) containing $JR[\![x, y]\!]$. The slope $\tau_n(Z_t, \alpha_t)$ is the truncated polynomial $f_\alpha \otimes k(t) \bmod x^{n+1} \in k[\![x]\!]/(x^{n+1})$. From this description, it is clear that if we regard the n-th order slope of Z at α_t as a tuple of coefficients of $f_\alpha \otimes k(t)$, then $t \mapsto \tau_n(Z_t, \alpha_t)$ is a regular function from W to $\mathbb{A}^n$.

Note that the notion of slope is invariant under extensions of the base field. Assume that all objects in the previous paragraph are defined over k, and let $k' \supset k$ be a field extension. Then there exists a family of branches $\alpha_{k'}$ of

$$Z_{k'} = Z \otimes k' \subset (X_1 \otimes k') \times (X_2 \otimes k')$$

and there exists a local coordinate system at a in $(X_1 \otimes k') \times (X_2 \otimes k')$ such that the regular function

$$t \mapsto \tau_n((Z_{k'})_t, (\alpha_{k'})_t)$$

is defined by the polynomials with the same coefficients as the function

$$t \mapsto \tau_n(Z_t, \alpha_t).$$

In model-theoretic terms, this observation implies that once a point $a \in M^2$, a local coordinate system at a, and a family of branches α of Z are fixed, the slope $\tau_n(Z_t, \alpha_t)$ is definable in the language of fields over t.

If $X_1 \times \cdots \times X_n$ is a product of k-varieties, we denote

$$p_{i_1 \dots i_k} \colon X_1 \times \cdots \times X_n \to X_{i_1} \times \cdots \times X_{i_k}$$

the natural projections. Although the notion of the composition of correspondences is standard, we reintroduce it here to fix conventions.

Definition 3.6 (Composition of curves). Let $Z \subset X_1 \times X_2 \times T$, $W \subset X_2 \times X_3 \times S$ be families of curves, and let $p_{i_1 \dots i_k}$ denote projections on products of the factors of the space $X_1 \times X_2 \times X_3 \times T \times S$. Define the *family $W \circ Z$ of compositions of curves from the families W and Z* to be

$$p_{1345}(p_{124}^{-1}(Z) \cap p_{235}^{-1}(W))$$

in $X_1 \times X_3 \times T \times S$. Clearly, if Z, W are definable, then so is $Z \circ W$; on the level of points:

$$W \circ Z = \{(x, z, t, s) \in M^2 \times T \times S : \exists u((x, u, t) \in Z \wedge (u, z, s) \in W)\}.$$

If for all $t \in T$, $s \in S$, all irreducible components of Z_t, W_s project dominantly on X_1, X_2, respectively, then $W \circ Z$ is a family of curves parametrized by $T \times S$.

We denote by Z^{-1} the image of Z under the morphism

$$X_1 \times X_2 \times T \to X_2 \times X_1 \times T$$

permuting the factors X_1 and X_2, in both geometric and definable contexts. We regard the above definitions as applicable to individual curves Z, W by putting $T = S$ to be a point.

Remark. If Z, W are families of pure-dimensional curves such that for all $t \in T$, $s \in S$ all irreducible components of Z_t, W_s project dominantly on X_1, X_2, respectively, then $W \circ Z$ is a family of pure-dimensional curves.

The next proposition relates the n-th order slope of the composition of curves with the n-th order slopes of the original curves. It is key in producing (at this stage, only on the level of the full Zariski structure) a connection between composition of curves and a group operation.

Proposition 3.7. *Let $Z \subset X_1 \times X_2 \times T$ and $W \subset X_2 \times X_3 \times S$ be families of pure-dimensional curves, let α, β be families of branches of Z, W at $a = (a_1, a_2) \in Z$, $b = (b_1, b_2) \in W$, respectively, $a_2 = b_1$. Then there exists a family of branches $\beta \circ \alpha$ of $W \circ Z$ at (a_1, b_2) such that for all $t \in T$, $s \in S$ and for all $n > 0$,*

$$\tau_n(W_s \circ Z_t, (\beta \circ \alpha)_{(t,s)}) = \tau_n(W_s, \beta_s) \circ \tau_n(Z_t, \alpha_t)$$

where the operation "$\circ$" on the right-hand side is composition of truncated polynomials.

Proof. The proof consists essentially in unravelling the definitions. The choice of coordinate systems induces the isomorphisms

$$\widehat{\mathcal{O}_{X_1\times X_2,a}} \cong k[\![x,y]\!], \quad \widehat{\mathcal{O}_{X_2\times X_3,b}} \cong k[\![y,z]\!].$$

If the family of branches α is given Zariski locally around $t \in T$ by an equation $y - f$, $f \in x\mathcal{O}_{T,t}[\![x]\!]$, and β is given by $z - g$, $g \in y\mathcal{O}_{S,s}[\![y]\!]$, then let the family of branches $\beta \circ \alpha$ be given by $z - g \circ f$, $g \circ f \in (\mathcal{O}_{T,t} \otimes \mathcal{O}_{S,s})[\![x]\!]$. Note that the composition $g \circ f$ of the formal power series makes sense and has a zero constant term since both f and g have this property.

Now let h_Z, h_W be generators of the kernels of the maps

$$\mathcal{O}_{X_1\times X_2\times T,(a,t)} \to \mathcal{O}_{Z,(a,t)}, \quad \mathcal{O}_{X_2\times X_3,(b,s)} \to \mathcal{O}_{W,(b,s)},$$

respectively, then $y - f$ divides h_X and $z - g$ divides h_Y. The germ of $W \circ Z$ around (a_1, b_2, t, s) by Definition 3.6 is cut out by the ideal $(h_X, h_Y) \cap k[\![x,z]\!]$, and in order to show that $\beta \circ \alpha$ is a family of branches of $Y \circ X$ at this point, we need to check that $(z - g \circ f)$ contains $(h_X, h_Y) \cap (\mathcal{O}_{T,t} \otimes \mathcal{O}_{S,s})[\![x,z]\!]$, and for that it would suffice to check that

$$(z - g \circ f) = I := (y - f, z - g) \cap (\mathcal{O}_{T,t} \otimes \mathcal{O}_{S,s})[\![x,z]\!].$$

Indeed, it is straightforward to check that for any $n > 0$,

$$(z - g \circ f) = I_n := I/(x^n, z^n)$$

and since I is the inverse limit of I_n, it follows that $(z - g \circ f) = I$. ∎

Definition 3.8 (Pointwise addition of curves). Let G be a one-dimensional algebraic group, and let $X \subset G^2 \times T$, $Y \subset G^2 \times S$ be families of curves. Let $a\colon G \times G \to G$ be the group law, let $\Gamma_a \subset G^3$ be its graph, and denote by $p_{i_1\ldots i_k}$ projections of $G \times G \times G \times G \times T \times S$ on the products of factors. We define the family of curves $X + Y$ *of sums of elements of the families X and Y* to be

$$p_{1456}(p_{234}^{-1}(\Gamma_a) \cap p_{124}^{-1}(X) \cap p_{135}^{-1}(Y))$$

in $G^2 \times T \times S$. Clearly, if X, Y are definable, then so is the family $X + Y$; on the level of points:

$$X + Y := \{(a, b + c, t, s) : (a, b, t) \in X, (a, c, s) \in Y\}.$$

Remark. If X and Y are families of pure-dimensional curves, then so is $X + Y$. It may seem from the notation above that G is supposed to be commutative, even though the definition applies even if this is not the case. In this paper, we will only consider the operation "+" for curves inside groups whose connected component of the identity is commutative.

Let G be a one-dimensional algebraic group, then the *formal group law of* G is defined as the image of the topological generator of $k[\![x]\!] \cong \widehat{\mathcal{O}_{G,e}}$ under the morphism $\widehat{\mathcal{O}_{G,e}} \to \widehat{\mathcal{O}_{G,e} \otimes \mathcal{O}_{G,e}} \cong k[\![x, y]\!]$ induced by the group operation morphism. The truncation to first order of a one-dimensional formal group law is $x + y$ (see, for example, [27, Part I, Section 2.4]).

Proposition 3.9. *Let G be a one-dimensional algebraic group over an algebraically closed field k. Let F be the formal group law of G, and let F_n be its n-th order truncation. Let $X \subset G \times G \times T$, $Y \subset G \times G \times S$ be families of pure-dimensional curves, and let α, β be families of branches at $a = (a_0, a_1)$, $b = (b_0, b_1)$, where $b_0 = a_0$, respectively. Then there exists a family of branches $\alpha + \beta$ of $X + Y$ at $(a_0, a_1 + b_1)$ such that*

$$\tau_n(X_t + Y_s, \alpha_t + \beta_s)(x) = F_n(\tau_n(X_t, \alpha_t)(x), \tau_n(Y_s, \beta_s)(x))$$

if $\tau_n(X_t, \alpha_t)$ and $\tau_n(Y_s, \beta_s)$ are defined. In particular, if $n = 1$,

$$\tau_1(X_t + Y_s, \alpha_t + \beta_s)(x) = \tau_1(X_t, \alpha_t) + \tau_1(Y_s, \beta_s).$$

Proof. As in the proof of Proposition 3.7, this statement follows from the unfolding of the definitions. Reasoning locally, assume that α is cut out by the equation $y - f$, $f \in k[\![x]\!] \otimes \mathcal{O}_{T,t}$ near t, β is cut out by $z - g$, $g \in k[\![x]\!] \otimes \mathcal{O}_{S,s}$ near s. Then $\alpha + \beta$ is cut out by $y - F(f(x), g(x))$ near $(t, s) \in T \times S$. Checking that the latter power series indeed define a family of branches of $X + Y$ is straightforward, and we leave it to the reader. ∎

3.2. *Flat families and definability of tangency*

In the present section, we prove the main technical result of the paper, Proposition 3.15, identifying tangency of two generic elements of two families of curves in terms of properties of the families definable in the reduct $\mathcal{M}$. While we do not give a full definable characterization of tangency, we prove a standard weakening of this result, which, as we will see in the concluding section of the paper, is sufficient for our needs.

The key preliminary step is the observation that if $X \subset M^2 \times T$, $Y \subset M^2 \times S$ are families of pure-dimensional curves and M, T, S are smooth, then the "family of scheme theoretic intersections" $X \times_{M^2} Y \to T \times S$ is flat if restricted to the open subset of $T \times S$ over which it has finite fibres. We refer the reader to any standard exposition of flatness, such as [34, Chapter I, Section 2], for details, and quickly recall some of the key facts. All schemes in this section are assumed Noetherian, and by varieties we mean schemes of finite type over an algebraically closed field k. We identify closed scheme-theoretic points of varieties with geometric points (that is, morphisms $\operatorname{Spec} k \to X$ that are sections of the structure map $X \to \operatorname{Spec} k$).

First, recall that a morphism $f\colon X \to Y$ is flat if all local rings $\mathcal{O}_{X,x}$ are flat $\mathcal{O}_{Y,f(x)}$-modules. In particular, flatness can be checked Zariski locally on the source: if $X = \bigcup_{i=1}^{n} O_i$ is an open cover and O_i is flat over Y for all i, then X is flat over Y [42, Tag 01U5].

Fact 3.10 (Generic flatness, [15, Exposé IV, Théorème 6.10, Corollaire 6.11]). Let Y be an integral scheme, and let $f\colon X \to Y$ be a dominant morphism of finite type. Then there exist open subsets $O \subset X$, $U \subset Y$ such that

$$f|_O\colon O \to Y, \quad f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$$

are flat.

Fact 3.11 ([34, Chapter I, Propositions 2.4 and 2.5], [42, Tag 05BC], [42, Tag 02KB]). We have that

(i) an open immersion is flat;

(ii) a composition of flat morphisms is flat;

(iii) let $X \to Y$ be a flat morphism and let $Z \to Y$ be a morphism. Then $X \times_Y Z \to Z$ is flat;

(iv) let B be a flat A-algebra and consider $b \in B$. If the image of b in $B/\mathfrak{m}B$ is not a zero divisor for any maximal ideal $\mathfrak{m}$ of A, then $B/(b)$ is a flat A-algebra;

(v) a finite morphism $f\colon X \to Y$ is flat if and only if f is a locally free morphism, that is, if $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module;

(vi) if A is an algebra and $I \subset A$ is an ideal, then the completion $\varprojlim A/I^n$ is flat over A.

Lemma 3.12. *Assume that the total space X of a family of pure-dimensional curves $X \subset M^2 \times T$ is a variety and that $M^2 \times T$ is smooth. Then X is flat over T.*

Proof. By definition of a family of pure-dimensional curves, X is open in $\bar{X}$, so by Fact 3.11 (i) and (ii) suffices to show flatness of $\bar{X}$ over T. Passing to a cover of $M^2 \times T$ by affine opens $O_i = \operatorname{Spec} B_i$ suffices to show flatness of $\bar{X} \cap O_i$ over T for all i. But this immediately follows from the definition of a family of pure-dimensional curves and Fact 3.11 (iv). Specifically, since $M^2 \times T$ is smooth, then any Weil divisor is Cartier, i.e., it is given, locally, by vanishing of a function. X is of pure codimension-one, hence a Weil divisor, and hence Cartier. So we may apply Fact 3.11 (iv). ∎

Lemma 3.13. *Let M be a smooth algebraic curve, T, S smooth varieties, $X \subset M^2 \times T$, $Y \subset M^2 \times S$ families of pure-dimensional curves, and assume that X, Y are varieties. Let $U \subset T \times S$ be the set of points u such that $\dim(X \times_{M^2} Y)_u = 0$, let $Z = X \times_{M^2} Y \cap p^{-1}(U)$ be the scheme theoretic intersection, where p is the projection onto $T \times S$. Then the restriction $p\colon Z \to U$ is flat.*

Proof. By Lemma 3.12, X is flat over T. Since M, T, S are smooth and since regular local rings are unique factorization domains, and Y is pure-dimensional, Y is cut out in $M^2 \times S$ by a principal ideal sheaf (see, for example, [32, §19, Theorems 47, 48]). Since

$X \to T$ is flat, by Fact 3.11 (iii) $X \times S \cong X \times_T (T \times S) \to T \times S$ is flat too. Since X is pure-dimensional, the natural closed embedding $X \times_{M^2} Y \to X \times S$ is also cut out by a principal ideal sheaf $\mathcal{J}$, passing by virtue of Fact 3.11 (i) to a cover of $X \times S$ by affine opens $O_i = \operatorname{Spec} B_i$ and applying Fact 3.11 (iv) to the algebras B_i, this closed subscheme is flat precisely over the complement of the subvariety of $T \times S$ consisting of the points u such that the local generator of $\mathcal{J}$ does not vanish on an irreducible component of the fibre $(X \times S)_u$. In other words, it is flat over the open subset of points $u \in T \times S$ such that $(X \times_{M^2} Y)_u$ is zero-dimensional. ∎

Recall now that a morphism $f: X \to Y$ is called *finite* if for any affine open $U = \operatorname{Spec} R \subset Y$, the inverse $f^{-1}(U) = \operatorname{Spec} S$ is affine and S is a finite R-module. A morphism is finite if and only if it is quasi-finite and proper [14, Exposé IV, Corollaire 18.12.4].

Lemma 3.14. *Let $f: X \to Y$ be a quasi-finite morphism of schemes over an algebraically closed k. Consider the functions*

$$m\colon Y(k) \to \mathbb{Z}, \quad y \mapsto \#(f^{-1}(y)),$$
$$w\colon X(k) \to \mathbb{Z}, \quad x \mapsto \dim_k \mathcal{O}_{X,x} \otimes k(f(x)).$$

Then

(i) $w(x) = \dim_{k(x)} \widehat{\mathcal{O}_{X,x}} \otimes k(f(x))$, *where $\widehat{\mathcal{O}_{X,x}}$ is the completion $\varprojlim \mathcal{O}_{X,x}/I^n$ for any ideal $I \subset \mathcal{O}_{X,x}$;*

(ii) *assume that f is flat. Then m is lower semi-continuous and w is upper semi-continuous, that is, the lower level sets of m and the upper level sets of w*

$$\{y \in Y : m(y) \leq n\} \quad \text{and} \quad \{x \in X : w(x) \geq n\}$$

are closed; $y \mapsto \sum_{x \in f^{-1}(y)} w(x)$ is lower semi-continuous and is locally constant if f is finite.

Proof. Let J (resp. $\hat{J}$) be the ideal of $\mathcal{O}_{X,x}$ (resp. $\widehat{\mathcal{O}_{X,x}}$), generated by the image of the maximal ideal of $\mathcal{O}_{Y,f(x)}$, then $\mathcal{O}_{X,x} \otimes k(f(x)) \cong \mathcal{O}_{X,x}/J$. We have a sequence of isomorphisms

$$\mathcal{O}_{X,x} \otimes k(f(x)) \cong \mathcal{O}_{X,x}/J \cong \widehat{\mathcal{O}_{X,x}}/\hat{J} \cong \widehat{\mathcal{O}_{X,x}} \otimes k(f(x)),$$

where the first and the third one are tautological, and the second morphism is an isomorphism because $\hat{J}$ is the completion of J in the I-adic topology. This proves claim (i).

That m is lower semi-continuous follows from [14, Exposé IV, Proposition 15.5.1 (i)] and the fact that flat morphisms of finite type are universally open [14, Exposé IV, Théorème 2.4.6]. Upper semi-continuity of w follows from [42, Tag 0F3D3] and [42, Tag 0F3I] (note that the definition of w from the statement of the lemma and one from [42] coincide on the closed scheme theoretic points of a variety over an algebraically closed field). Lower semi-continuity of $\sum_{x \in f^{-1}(u)} w(u)$ follows from [42, Tag 0F3J], that it is locally constant if f is finite follows from the definition of a weighting [42, Tag 0F3A]. ∎

We can now formulate our main technical result. Roughly, it states that, in suitably chosen families of curves tangency of two curves is witnessed by a lower number of intersection points.

Proposition 3.15. *We keep the notation and assumptions of Lemma 3.13 and assume further that there exists $a \in M^2$ such that X_t, Y_s pass through a for all $t \in T$, $s \in S$. Let α, β be families of branches at a of X, Y, respectively, such that for all $t \in T$, $s \in S$ the slopes of α_t, β_s are defined. Define*

$$n_{\max} = \max\{n : \forall (t,s) \in U(k),\ \tau_n(X_t, \alpha_t) = \tau_n(Y_s, \beta_s)\},$$
$$m_{\max} = \max_{(t,s)\in U(k)} \#(X_t \cap Y_s).$$

Then

$$\{(t,s) \in U(k) : \tau_{n_{\max}+1}(X_t, \alpha_t) = \tau_{n_{\max}+1}(Y_s, \beta_s)\}$$
$$\subseteq \{(t,s) \in U(k) : \#(X_t \cap Y_s) < m_{\max}\}.$$

Proof. Consider $Z = X \times_{M^2} Y \cap M^2 \times U$, let $q\colon Z \to M^2$ be the natural projection, and let $Z = \bigcup_{i=0}^{n} Z_i$ be the decomposition into irreducible components, where $Z_0 = q^{-1}(a)$. We will first show that whenever $p^{-1}(u) \cap Z_0 \cap Z_i \neq \emptyset$ for some $i \neq 0$, we have $\#p^{-1}(u) < \max_{u\in U} \#p^{-1}(u)$. In order to do that, we will show that the function $u \mapsto \#p^{-1}(u) \cap (\bigcup_{i\neq 0} Z_i)$ is lower semi-continuous.

The projection $p\colon Z \to U$ is flat by Lemma 3.13. By [42, Tag 04PW], the closed embedding $Z_{\mathrm{red}} \to Z$, where Z_{red} is Z endowed with the canonical reduced structure, is flat. Since the invariant we are interested in does not depend on the scheme structure, by Fact 3.11 (ii) we may assume Z to be reduced. Furthermore, there exists an open embedding $Z \hookrightarrow \bar{Z}$, where $\bar{Z}$ is flat and finite over U. Indeed, let $\bar{M}$ be a smooth proper algebraic curve that contains M as a dense subset, and let $\bar{X}$, $\bar{Y}$ be the closures of X, Y in $\bar{M}^2 \times T$ and $\bar{M}^2 \times S$. Let $\bar{Z} = \bar{X} \times_{\bar{M}^2} \bar{Y} \cap \bar{M}^2 \times U$, let $\bar{p}$ be the natural projection on U, and denote by $\hat{p}$ its restriction to $\hat{Z} = \bigcup_{i\neq 0} \bar{Z}_i$, where $\bar{Z}_i$ is the irreducible component of $\bar{Z}$ that contains Z_i for each i. By Lemma 3.13, $\bar{p}$ is flat.

By Fact 3.11 (v), the morphism $\bar{p}$ is locally free. It is readily seen that $(\bar{p})_*\mathcal{O}_{\hat{Z}}$ is locally free of rank one less than the rank of $(\bar{p})_*\mathcal{O}_{\bar{Z}}$. Indeed, if $W = \operatorname{Spec} A \subset U$ is an affine open such that $(\bar{p})^{-1}(W) = \operatorname{Spec} B$ and B is a free A-module, we have that $B \cong B/\mathfrak{p} \oplus B/\mathfrak{q}$ as A-module, where $\mathfrak{p}, \mathfrak{q} \subset B$ are ideals cutting out $Z_0 \cap (\bar{p})^{-1}(W)$, $\hat{Z} \cap (\bar{p})^{-1}(W)$, respectively. Since $Z_0 \cong U$, in particular $Z_0 \cap (\bar{p})^{-1}(W) \cong W$, and we have that $B/\mathfrak{p} \cong A$, so $B/\mathfrak{q}$ is free. By Fact 3.11 (v) again, $\hat{p}$ is flat, and by Fact 3.11 (i), its restriction to $\hat{Z} \cap Z = \bigcup_{i\neq 0} Z_i$ is flat. We deduce by Lemma 3.14 (ii) that the function $u \mapsto \#p^{-1}(u) \cap (\bigcup_{i\neq 0} Z_i)$ is lower semi-continuous.

Note that while Z_0 may have non-trivial scheme-theoretic structure, the restriction $p|_{Z_0}\colon (Z_0)_{\mathrm{red}} \to U$ is a homeomorphism, so denote $r\colon U \to Z_0$ its set-theoretic inverse. Let $w\colon Z \to \mathbb{Z}$, $w(z) = \dim_k \mathcal{O}_{Z,z} \otimes k(p(z))$. We claim that w is constant on the open set $Z' = Z_0 \setminus \bigcup_{i\neq 0} Z_i$. By Fact 3.11 (i), Z' is flat over U, and by Fact 3.11 (iii), it is flat over the open $p(Z') \subset U$. The restriction $p|_{Z'}: Z' \to p(Z')$ is still a homeomorphism; we will show that it is a finite morphism.

Note that since U is dense in $T \times S$, it is integral, and since Z' is dense in U, it is integral too. The scheme Z' is of finite type over a field, so clearly quasi-compact, and p is clearly separated (for example, because it is affine), so Zariski's main theorem (see [34, Chapter I, Theorem 1.8]) can be applied to p. Therefore, p factors into a composition of an open embedding $i: Z' \to Z''$ and a finite morphism $p': Z'' \to p(Z')$. Since $\widetilde{p} = p_{\mathrm{red}}|_{Z'}: Z'_{\mathrm{red}} \to p(Z')$ is an isomorphism, the morphism $p'' = i_{\mathrm{red}} \circ \widetilde{p}^{-1} \circ p'_{\mathrm{red}}: Z''_{\mathrm{red}} \to Z''_{\mathrm{red}}$ restricts to the identity morphism on Z'_{red}. If $p|_{Z'}: Z' \to p(Z')$ is not finite, then the open embedding i is an isomorphism and p'' is not an isomorphism. Since passing to a closed subscheme preserves finiteness, we may assume Z' to be dense in Z''. The subset of Z''_{red}, where p'' and the identity morphism coincide, is closed and contains Z', so must be the whole of Z'', which in turn contradicts p'' not being an isomorphism.

Now by Lemma 3.14 (ii), w is upper semi-continuous on $Z(k)$ and in particular on Z_0, but since Z' is flat and finite over U, w is constant on $Z'(k)$. Therefore, w takes the value $w_{\min,0} = \min_{x \in Z_0} w(x)$ on the latter, and if $w(r(u)) > w_{\min,0}$ for some $u \in U$, then $r(u) \in Z_i \cap Z_0$ for some $i \neq 0$ and therefore $\#p^{-1}(u)$ is not maximal. It follows that

$$\{(t,s) \in U(k) : w(r(u)) > w_{\min,0}\} \subseteq \{(t,s) \in U(k) : \#(X_t \cap Y_s) < m_{\max}\}.$$

It is left to prove that

$$w(r(t,s)) > w_{\min,0} \quad \text{for all } t, s \text{ such that } \tau_{n_{\max}+1}(X_t, \alpha_t) = \tau_{n_{\max}+1}(Y_s, \beta_s).$$

To establish this, it is enough to prove the statement on an affine Zariski open subset $\operatorname{Spec} R \subset U \times M^2$ intersecting Z_0 non-trivially. Let $f, g \in R$ be the equations of $X \times S \cap \operatorname{Spec} R$, $Y \times T \cap \operatorname{Spec} R$, respectively. Let $I \subset R$ be the ideal of functions that vanish on $q^{-1}(a)$, and let $\widehat{R} = \varprojlim R/I^n$.

Let $f = f_1 \cdots f_N$, $g = g_1 \cdots g_K$ be decompositions into pairwise coprime factors in $\widehat{R}$, and let f_α and g_β be those factors that are local equations of α, β. Apply the Chinese remainder theorem (see [2, Chapter 9, p. 99, Exercise 9]) twice: first, to $\widehat{R}/(f,g)$ to get the decomposition

$$\widehat{R}/(f,g) = \bigoplus_{i=1}^{N} \widehat{R}/(f_i, g),$$

second, to each direct summand $\widehat{R}/(f_i, g)$ to get

$$\widehat{R}/(f,g) = \bigoplus_{i=1}^{N} \bigoplus_{j=1}^{K} \widehat{R}/(f_i, g_j).$$

Both applications are justified: since f_i are pairwise coprime in $\widehat{R}$ (that is, $(f_i) + (f_k) = \widehat{R}$ for $i \neq k$), the ideals (f_i, g) are pairwise coprime in $\widehat{R}/(g)$, similarly, for each i, the ideals (f_i, g_j) are pairwise coprime in $\widehat{R}/(f_i, g)$.

Tensoring with $k(u)$ and applying Lemma 3.14 (i), we get

$$w(r(u)) = \sum_{i=1}^{N}\sum_{j=1}^{K} \dim_{k(u)} \widehat{R}/(f_i, g_j) \otimes k(u).$$

Therefore, if $u = (t, s) \in U(k)$ and $\tau_{n_{\max}+1}(X_t, \alpha_t) = \tau_{n_{\max}+1}(Y_s, \beta_s)$, then by the remark after Definition 3.5,

$$\dim_k \widehat{R}/(f_\alpha, g_\beta) \otimes k(u)$$

takes a value strictly greater than the minimum it achieves on $U(k)$. By Fact 3.11 (vi), $\widehat{R}$ is a flat R-algebra, and by applying Fact 3.11 (iv) twice, as in the proof of Lemma 3.13, Spec $\widehat{R}/(f_i, g_j)$ is flat over U for all i, j. Since by Lemma 3.14 (ii) for each pair of prime factors f_i, g_j, the value $\dim_k \widehat{R}/(f_i, g_j) \otimes k(u)$ is upper semi-continuous in u, it follows that $w(r(t, s)) > w_{\min,0}$ as soon as slopes of order $(n_{\max} + 1)$ of α_t and β_s coincide. ■

4. Interpretation of the field

In the present section, we tie together the results obtained above to produce the main result of the paper. We start with some additional technicalities and reductions.

4.1. *The group configuration*

In stable theories – a model theoretic framework encompassing all structures considered in the present work – certain combinatorial configurations of elements are known to exist only in the presence of an interpretable group or – in a more restrictive setting – an interpretable field. It is by constructing such configurations – using the "definable intersection theory" developed in the previous sections – that the main theorem of the present paper is proved.

Before describing these configurations in more detail, we need some model theoretic preliminaries. As in Section 2, we will specialise the definitions to the setting in which they will be used. As above, we will be working in the full Zariski structure $\mathfrak{M}$ on an algebraic curve M over an algebraically closed field k. In order to keep the definitions as simple as possible, we further assume that k is of infinite transcendence degree. In Lemma 4.8, we explain why this assumption is harmless. We will be mostly concerned with a structure $\mathcal{M} := (M, X)$, where $X \subseteq M^2 \times T \subseteq M^{2+l}$ is the total space of an ample family (recall that (M, X) denotes the smallest structure on $M(k)$ containing $X(k)$). Throughout the text, unless explicitly stated otherwise, by *definable* we mean "definable with parameters". When this is not clear from the context, we will write $\mathcal{M}$-definable or $\mathfrak{M}$-definable to stress in which structure we are working.

Definition 4.1. (1) If D is an $\mathcal{M}$-definable set, we let $\dim(D) := \dim(\mathrm{cl}(D))$, where cl denotes the Zariski closure of D.

(2) If A is a set of parameters and $a \in M^l$, we denote $\dim(a/A) := \min\{\dim(D) : a \in D\}$, where D ranges over all subsets of D^l $\mathcal{M}$-definable over A.

(3) We say that $a \in M$ is $\mathcal{M}$-algebraic over A if $\dim(a/A) = 0$. We denote $\mathrm{acl}_{\mathcal{M}}(A) := \{a \in M : \dim(a/A) = 0\}$.

(4) We say that a is $\mathcal{M}$-generic in D over A if $\dim(a/A) = \dim(D)$.

(5) We say that a is $\mathcal{M}$-independent of B over A if $\dim(a/A) = \dim(a/AB)$.

Remark. Note that $\dim(D)$ is (by definition) the same as the algebro-geometric dimension of D. This implies that $\dim_{\mathcal{M}}(a/A) \geq \dim_{\mathfrak{M}}(a/A)$, but there is no need for equality to hold. In particular, $\mathrm{acl}_{\mathcal{M}}(A) \subseteq \mathrm{acl}(A)$, where on the right-hand side acl is the field-theoretic algebraic closure in k. It follows that $\mathfrak{M}$-independence (which coincides with the field-theoretic notion of algebraic independence) implies $\mathcal{M}$-independence, but not necessarily the other way around.

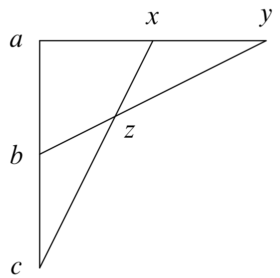
Definition 4.2. An infinite set D definable (or interpretable) in $\mathcal{M}$ is *strongly minimal* if every $\mathcal{M}$-definable subset of D is finite or cofinite.

It is an easy exercise to verify that if D is an $\mathcal{M}$-definable set and $\dim(D) > 1$, then there are a projection $\pi\colon D \to M^{\dim(D)-1}$ and an open $U \subseteq M^{\dim(D)-1}$ such that $\pi^{-1}(u) \cap D$ is infinite for all $u \in U$. In particular, D is strongly minimal only if $\dim(D) = 1$. Thus, D is strongly minimal if and only if it is one-dimensional and cannot be written as the disjoint union of two one-dimensional $\mathcal{M}$-definable subsets. We say that $\mathcal{M}$ is strongly minimal if M is (as an $\mathcal{M}$-definable set).

Remark. An $\mathcal{M}$-definable set D may be strongly minimal with respect to the structure $\mathcal{M}$ but not with respect to the structure $\mathfrak{M}$.

As we will see below, we can easily reduce the proof of our main result to the case where $\mathcal{M}$ is strongly minimal. Under this additional assumption, we can finally introduce the group configuration:

Definition 4.3 (Group configuration). Let $\mathcal{M}$ be as above, and assume that it is strongly minimal. The set $\{a, b, c, x, y, z\}$ of tuples



is a *group configuration* if there exists an integer n such that

- all elements of the diagram are pairwise independent and $\dim(a, b, c, x, y, z) = 2n + 1$;

- $\dim a = \dim b = \dim c = n$, $\dim x = \dim y = \dim z = 1$;
- all triples of tuples lying on the same line are dependent, and moreover, $\dim(a, b, c) = 2n$, $\dim(a, x, y) = \dim(b, z, y) = \dim(c, x, z) = n + 1$;

Two group configurations G_1, G_2 are *inter-algebraic* if for any pair of tuples $a \in G_1$, $a' \in G_2$ in the corresponding vertices, $\mathrm{acl}_{\mathcal{M}}(a) = \mathrm{acl}_{\mathcal{M}}(a')$.

Assume that G is a connected $\mathcal{M}$-definable group acting transitively on a strongly minimal definable set X, then one can construct a group configuration as follows: let g, h be independent generics in G, and let u be a generic of X (we will justify the assumption that such generics exist later on), then $(g, h, g \cdot h, u, g \cdot u, g \cdot h \cdot u)$ is a standard group configuration (associated with the action of G on X). Below (Lemma 4.19), we show that for a suitably constructed $\mathcal{M}$-definable family of curves passing through a fixed point, the set of n-th order slopes of curves in the family coincides for some n with a one-dimensional algebraic group H (viewed as acting on itself by multiplication). Proposition 3.15 will then allow us to "pull back" a group configuration (in $\mathfrak{M}$) associated with this group H into a group configuration in $\mathcal{M}$. This will, essentially, finish the proof, using the following well-known fact.

Fact 4.4 (Hrushovski). Assume that M is a strongly minimal structure, and let $G_1 = (a, b, c, x, y, z)$ be a group configuration. Then there exists a definable group G acting transitively on a strongly minimal set X.

This follows from the main theorem of [4] and the fact that infinitely definable groups in stable theories are intersections of definable groups (see, for example, [39, Theorem 5.18]) and the fact that any group definable in an algebraically closed field is (definably isomorphic to) an algebraic group (see [39, Theorem 4.13]). The original proofs of these statements are contained in [19].

To construct a field, we will have to work a little harder. First, we want to relate the group configuration we started with, with a standard group configuration associated with the group provided by Fact 4.4. Toward that end, we need to assure that the group configuration we started with contained only relevant information. This is captured in the following technical definition.

Definition 4.5. A group configuration (a_1, a_2, a_3, x, y, z) is *minimal* if

$$\mathrm{acl}_{\mathcal{M}}(\mathrm{Cb}(x, y)/a_1) = \mathrm{acl}_{\mathcal{M}}(a_1),$$
$$\mathrm{acl}_{\mathcal{M}}(\mathrm{Cb}(y, z)/a_2) = \mathrm{acl}_{\mathcal{M}}(a_2),$$
$$\mathrm{acl}_{\mathcal{M}}(\mathrm{Cb}(x, z)/a_3) = \mathrm{acl}_{\mathcal{M}}(a_3).$$

Remark. We will not go into the definition of canonical bases (see, e.g., [37, p. 19]), but for the benefit of readers unfamiliar with this model theoretic notion, we mention that

(1) The minimality condition is readily checked to be equivalent to the condition that whenever there are $a_i' \in \mathrm{acl}_{\mathcal{M}}(a_i)$ such that $(a_1', a_2', a_3', x, y, z)$ is still a group configuration, then $a_i \in \mathrm{acl}_{\mathcal{M}}(a_i')$ for all $i = 1, 2, 3$.

(2) By dimension considerations, it follows from the previous remark that any group configuration (a_1, a_2, a_3, x, y, z) gives rise to a minimal group configuration $(a'_1, a'_2, a'_3, x, y, z)$ with $a'_i \in \operatorname{acl}(a_i)$ for all i. In particular, if $\dim(a_i) = 1$ for all i, then (a_1, a_2, a_3, x, y, z) is a minimal configuration.

(3) Roughly, $\operatorname{Cb}((x, y)/a)$ is the model theoretic analogue of the field of definition of the locus of (x, y) over $\operatorname{acl}(a)$. In the full Zariski structure, this is, in fact, the definition. To define this in the reduct, we proceed as follows: for $\mathcal{M}$-definable sets, $X, Y \subseteq M^n$ (some n) define $X \sim Y$ if $\dim(X) = \dim(Y)$ and $\dim(X \triangle Y) < \dim(X)$. Let X be $\mathcal{M}$-definable over $\operatorname{acl}_{\mathcal{M}}(a)$ and minimal containing (x, y) (i.e., $X \sim (Y \cap X)$) for all Y $\mathcal{M}$-definable over $\operatorname{acl}_{\mathcal{M}}(a)$ and containing (x, y). Then $\operatorname{Cb}((x, y)/a)$ is the set fixed pointwise under all $\mathcal{M}$-automorphisms fixing $X/{\sim}$.

(4) For most purposes in the current paper, it will suffice to know that if $X \to T$ is a nearly faithful family of curves (see below), then t is (up to inter-algebraicity) a canonical base for x over t for any generic point x of X_t, and if through $x_1, \ldots, x_k$ there is only one curve X_t in X, then t is (up to inter-algebraicity) a canonical base of $(x_1, \ldots, x_k)$. For more details on canonical bases in a similar context, we refer the interested reader to [29, §8.2].

For minimal group configurations, we have the following.

Fact 4.6 ([37, Theorem V.4.5]). If the group configuration in the statement of Fact 4.4 is, additionally, assumed to be minimal, then the action of the group G on X as provided above can be taken to be faithful, and this group action has an associated group configuration $G_2 = (g, h, g \cdot h, u, g \cdot u, g \cdot h \cdot u)$ inter-algebraic with G_1. In particular, $\dim G = \dim a$.

This, finally, allows obtaining a field as follows.

Fact 4.7 (Hrushovski [19]). Let G be an $\mathcal{M}$-definable group acting transitively and faithfully on a strongly minimal set X. Then either $\dim(G) = 1$ or there exists a definable field structure on X and either $\dim(G) = 2$ and $G \cong \mathbb{G}_a \rtimes \mathbb{G}_m$, or $\dim(G) = 3$ and $G \cong \mathrm{PSL}_2$.

An exposition of the above fact can be found in [39, Theorem 3.27]. Establishing that G is isomorphic to $\mathbb{G}_a \rtimes \mathbb{G}_m$ or to PSL_2 is the crucial point in the proof of Fact 4.7. In the present context, where G and X are definable in an algebraically closed field (rather, the full Zariski structure on an algebraic curve), this statement can be established using a simpler direct algebraic proof.

4.2. *Some standard reductions*

We make some standard simple reductions that will allow us to more easily use the results obtained in the previous sections as well as the group and field configurations described above. In model theoretic terms, Chevalley's theorem and Hilbert's Nullstellensatz are quantifier elimination and model completeness (for the theory of algebraically closed fields). With those, the next lemma is standard.

Lemma 4.8. *We may assume that k is of infinite transcendence degree (over the prime field).*

Proof. Let $K \geq k$ be an algebraically closed field extension of infinite transcendence degree. We let $M' := M(K)$, and for any D $\mathcal{M}$-definable without parameters, we let $D' := D(K)$. We obtain a structure $\mathcal{M}' := (M', X')$. By model completeness and quantifier elimination (Hilbert's Nullstellensatz and Chevalley's theorem, respectively, see, e.g., [29, Corollary 3.2.8]), X' is ample (if $U \subseteq M$ is an open set witnessing the fact that X is ample, then U' witnesses that X' is). Note also that any set S $\mathfrak{M}'$-definable without parameters is of the form D' for some $\mathcal{M}$-definable set D.

Assume that a field is interpretable in $\mathcal{M}'$. This means that there are D, E $\mathcal{M}'$-definable (without parameters) and parameters $\overline{a} \in K^l$ and $\overline{b} \in K^n$ such that $E_{\overline{b}}$ is an equivalence relation (of the correct arity) and such that $D_{\overline{a}}/E_{\overline{b}}$ is an infinite field. Let $L_{\overline{c}}$ and $A_{\overline{d}}$ be the graphs of multiplication and addition, respectively, for L, A $\mathcal{M}'$-definable without parameters.

Consider the set S of all parameters $(\overline{x}, \overline{y}, \overline{z}, \overline{w})$ such that $E_{\overline{y}}$ is an equivalence relation on $D_{\overline{x}}$ and $L_{\overline{z}}$, $A_{\overline{w}}$ turn $D_{\overline{x}}/E_{\overline{y}}$ into an infinite field. We claim that S is $\mathcal{M}'$-definable without parameters. This is easy since if $C \subseteq M^{r+s}$ is any constructible set, then the set $\{|C_v| : v \in M^s, |C_v| < \infty\}$ is uniformly bounded, say, by N, and thus the set $\{v \in M^s : |C_v| < \infty\}$ is definable since C_v is infinite if and only if $|C_v| > N$, which is a definable property (of v). By Hilbert's Nullstellensatz and Chevalley's theorem again, S has a point in k, meaning that an infinite field is interpretable already in k. ∎

In model-theoretic terms, the above lemma only means that interpretability of an (infinite) field is a first order property, and therefore preserved under the passage to elementary substructures. The most useful – for our purposes – property of fields of infinite transcendence degree is the following consequence of Chevalley's theorem and the compactness theorem of first-order logic.

Fact 4.9. If k is of infinite transcendence degree, then any $\mathcal{M}$-definable set D has generic points over any finite set of parameters A.

We need the following (weak) version of [44, Theorem B.1.43].

Fact 4.10. If $\mathcal{M}$ is strongly minimal and not locally modular, then there exists an ample definable family of curves $X \subseteq M^2 \times M^l$ with the property that for any $t \in M^l$, the set

$$E_t := \{s \in M^l : |X_t \cap X_s| = \infty\}$$

is finite.

For the purposes of the present paper, we call an ample definable family of curves as above a *nearly faithful family of curves*. Combined with the (easy) fact that local modularity is preserved under naming parameters [37, Remark IV.1.8], this gives the following.

Fact 4.11. If $\mathcal{M}$ is strongly minimal and not locally modular, then there exists a nearly faithful ample family whose generic members are strongly minimal subsets of M^2.

It follows immediately from uniform finiteness that if a strongly minimal set $\mathcal{M}$ admits a definable nearly faithful family of curves $X \subseteq M^2 \times T$, then it admits a nearly faithful family of plane curves (defined over the same parameters) $X' \subseteq M^2 \times T$ with the property that for all $t \in T$, the curve X_t has finite-to-one projects on both coordinates.

Finally, we obtain the following lemma.

Lemma 4.12. *We may assume that M is a smooth curve and that $\mathcal{M}$ is strongly minimal.*

The proof is well known (see, e.g., [37, Lemma IV.1.7] for a much more general statement). In the present setting, this can easily be shown directly. We leave the details as an exercise.

Summing up the above discussion, we have shown the following.

Corollary 4.13. *To prove Conjecture A, it suffices to prove: Let $\mathcal{M}$ be a strongly minimal reduct of the full Zariski structure $\mathfrak{M}$ on a smooth algebraic curve M over an algebraically closed field K of infinite transcendence degree. Then either $\mathcal{M}$ is locally modular or $\mathcal{M}$ interprets a field K-definably isomorphic to K. Moreover, we may assume that the lack of local modularity of $\mathcal{M}$ is witnessed by a nearly faithful family of curves whose generic members are strongly minimal.*

4.3. Generically unramified projections

In order to apply the machinery of slopes and tangency discussed in Section 3.1, we need to produce, definably in $\mathcal{M}$, large enough families of curves where these notions are defined and carry information. Lemma 4.14 below guarantees the former requirement, namely that for any curve $X \subset M^2$, the slope is defined on a dense open subset of either X or X^{-1} (uniformly in parameters). The second requirement is more delicate, as pointed out, for example, in the concluding remarks of [30]. In more technical terms, the problem pointed out by Marker and Pillay is that if the projection $p_2: Z \to M$ is everywhere ramified for a curve $Z \subset M^2$ (e.g., the curve cut out by the equation $y = x^p$ in $\mathbb{A}^1 \times \mathbb{A}^1$), then even if p_2 is dominant, $\tau_1(Z, \alpha) = 0$ for any branch α at any point of Z. In Lemmas 4.16 and 4.15, we develop the tools allowing us to construct, definably in $\mathcal{M}$, curves in M^2 whose projections on both factors M are generically unramified.

The following lemma ensures that at least one of the projections on a factor M of a family of curves is generically étale for a general element of the family, which by Lemma 3.4 implies existence of slopes for a generic element of the family. The fact that the support of the module of Kähler differentials is closed and Fact 3.10 imply that being étale and being unramified are open on the source. In particular, in order to check whether a dominant morphism $f: X \to Y$ is étale on a dense open subset of X it suffices to check if $\Omega_{k(X)/k(Y)} = 0$, or equivalently (see [28, Exercise 6.2.9], also [28, Lemma 6.1.13]), if $k(X) \supset k(Y)$ is a separable extension. We refer the reader to any standard algebraic geometry reference (e.g., [28, Section 6], [16, Chapter II, Section 8 and Chapter IV, Section 2]) for the details on Kähler differentials and ramification.

Lemma 4.14. *Let M be an irreducible algebraic curve over a field of any characteristic. Let $X \subset M^2 \times T$ be a family of pure-dimensional curves, and assume that X and T are irreducible. Then there exists a dense open subset $T' \subseteq T$ such that either $p_1\colon X_t \to M$ or $p_2\colon X_t \to M$ is generically étale for all $t \in T'$.*

Proof. Let ξ be the generic point of T in the scheme-theoretic sense. Denote $M_\xi = M \otimes k(\xi)$, $X_\xi = X \otimes k(\xi)$. By slightly abusing notation, denote by $p_1, p_2\colon X_\xi \to M_\xi$ the natural projections.

Let $\Omega_{M_\xi/k(\xi)}, \Omega_{X_\xi/k(\xi)}$ be the sheaves of modules of Kähler differentials on the generic fibres

$$M_\xi = M \otimes_k k(\xi) \quad \text{and} \quad X_\xi = X \otimes_k k(\xi),$$

respectively. Since $\iota\colon X_\xi \to M_\xi^2$ is a closed embedding, the pull-back

$$\iota^*\colon\ p_1^*\Omega_{M_\xi/k(\xi)} \oplus p_2^*\Omega_{M_\xi/k(\xi)} \to \Omega_{X_\xi/k(\xi)}$$

is surjective. Taking stalks at the generic point χ of X_ξ, we get a surjective map of vector spaces over the field $k(\chi) = k(X)$

$$\iota^*\colon\ p_1^*\Omega_{M_\xi/k(\xi)} \otimes k(\chi) \oplus p_2^*\Omega_{M_\xi/k(\xi)} \otimes k(\chi) \to \Omega_{X_\xi/k(\xi)} \otimes k(\chi).$$

Each summand on the left is either trivial or one-dimensional. Since i^* is surjective, it follows that at least one of the summands is mapped surjectively on the destination. Therefore, the stalk at $k(\chi)$ of either $\Omega_{X_\xi/k(\xi)}/p_1^*\Omega_{M_\xi/k(\xi)}$ or $\Omega_{X_\xi/k(\xi)}/p_2^*\Omega_{M_\xi/k(\xi)}$ vanishes, and we conclude. ∎

Suppose we have a family of pure-dimensional curves $X \subset M^2 \times T$ such that for some a, $a \in X_t$ for all $t \in T$, and assume that for all $t \in T$ the morphism $p_1\colon X_t \to M$ is étale in some neighbourhood of a. Then by Lemma 3.4, there exists a unique branch α of X at a. It might be the case, though, that $\tau_n(X_t, \alpha)$ vanishes for all n, for all $t \in T$, if p_2 is everywhere ramified on the component of X_t that contains a. The simple, but crucial, observation below is that in this case one can consider the family $X \circ X^{-1}$ which does not have this pathology, and p_1, p_2 are both generically unramified for any of its members.

Recall that if $f\colon X \to Y$ is a morphism of schemes over a field of characteristic p, then $\mathrm{Fr}_f\colon X \to X^{(p/Y)} = X \times_{f,Y,\mathrm{Fr}_Y} Y$, the *relative Frobenius morphism*, is defined to be $\mathrm{Fr}_X \times f$, where $\mathrm{Fr}_X, \mathrm{Fr}_Y$ are the absolute Frobenius endomorphisms of X, Y, respectively. If Y is the spectrum of a field, then $X^{(p/Y)}$ is denoted just $X^{(p)}$. If $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, $S = R[r_1, \dots, r_n]/I$, then $X^{(p/Y)} = R[r_1', \dots, r_n']/I^{(p)}$, where $I^{(p)} = \{f^{(p)} = \sum_J a_J^p (r')^J : f = \sum_J a_J r^J \in I\}$ (J is a multi-index), and $\mathrm{Fr}_{X/Y}^*(r_i') = r_i^p$. On the level of points, if $X \hookrightarrow Y \times \mathbb{A}^n$, then

$$\mathrm{Fr}_{X/Y}(y, x_1, \dots, x_n) = (y, x_1^p, \dots, x_n^p).$$

The natural projection $\mathrm{Fr}_{X/Y}(X) \to Y$ is given by $(y, x_1, \dots, x_n) \mapsto y$.

We start with a standard lemma (we give the details for clarity and completeness).

Lemma 4.15. *Let $f: X \to Y$ be a finite morphism of irreducible varieties over a field of characteristic $p > 0$, and let $F = \mathrm{Fr}_f$ be the relative Frobenius morphism. Assume that f is everywhere ramified. Then there exists an $n > 0$ such that the natural projection $F^n(X) = X \times_{f,Y,F^n} Y \to Y$ is generically unramified.*

Proof. Since f is everywhere ramified, the field extension $k(Y) \subset k(X)$ is inseparable. Let L be the separable closure of $k(Y)$ in $k(X)$, then $k(Y) \subset L$ is a separable extension and $L \subset k(X)$ is a purely inseparable extension. Since $L \subset k(X)$ is a finite extension, there exists a smallest number n such that $h^{p^n} \in L$ for any $h \in k(X)$. We claim that $k(F^n(X)) \subset L$, which will conclude the proof, as this shows that $k(F^n(X))$ is a separable extension of $k(Y)$.

To prove the above claim, let $X_0 \subset X$, $Y_0 \subset Y$ be dense open affine subvarieties such that X_0 is finite over Y_0. Then

$$k[X_0] = k[Y_0][h_1, \dots, h_n]/I \quad \text{and} \quad k[F^n(X_0)] = k[Y_0][g_1, \dots, g_n]/I^{(p^n)},$$

and there is an embedding of rings $k[X_0] \subset k(X_0)$. It is immediate from the definition of the relative Frobenius morphism that there exists an injection $k[F^n(X_0)] \hookrightarrow L$ sending g_i to $h_i^{p^n}$, so $F^n(X_0)$ is unramified over Y_0, and we conclude. ∎

Finally, the key lemma is the following.

Lemma 4.16. *Let $X \subset M^2 \times T$, $Y \subset M^2 \times S$ be two families of pure-dimensional curves. Let us denote projections of $M \times M \times T$ (resp. $M \times M \times S$) on products of factors by q (resp. q') with subscripts. Let $m > 1$ be an integer, and let $X' = F^m_{q_{23}}(X)$, $Y' = F^m_{q'_{23}}(Y)$. Then*

$$X \circ Y^{-1} = X' \circ (Y')^{-1}.$$

Proof. Let us denote projections from $M \times M \times M \times T \times S$ (resp. $M \times M^{(p^m)} \times M \times T \times S$) onto products of factors by r (resp. r') with subscripts. After unravelling the definitions, one observes that

$$X \circ Y^{-1} = r_{1345}(Z), \quad X' \circ (Y')^{-1} = r'_{1345}(\mathrm{Fr}_{r_{1345}}(Z))$$

for

$$Z = r^{-1}_{1245}(X \times S) \cap r^{-1}_{2345}(Y^{-1} \times T) \subset M^3 \times T \times S.$$

These projections coincide, since by the definition of the relative Frobenius morphism $r'_{1345} \circ \mathrm{Fr}_{p_{1345}} = r_{1345}$. ∎

For the benefit of the reader, let us consider the situation in Lemma 4.16 at the level of points. Denote by $F: M \to M^{(p)}$ the Frobenius morphism and assume M is affine and cut out by the equation $f(x_1, \dots, x_n)$ in $\mathbb{A}^n$, then $M^{(p)}$ is cut out by $f(x_1^p, \dots, x_n^p) = f^p$, and a point $(x_1, \dots, x_n)$ is sent by F to $(x_1^p, \dots, x_n^p)$. The map $\mathrm{Fr}_{q_{23}}$ in Lemma 4.16 sends a tuple $(x, y, t) \in M^2 \times T$ to $(F(x), y, t)$ and similarly for $\mathrm{Fr}_{q'_{23}}$. By definition,

$$(b, a, t) \in Y \text{ if and only if } (F(b), a, t) \in Y',$$
$$(b, c, s) \in X \text{ if and only if } (F(b), c, s) \in X'.$$

Consider $Z = \{(a,b,c,t,s) : (b,a,t) \in X, (b,c,s) \in Y\}$. Then

$$X \circ Y^{-1} = \{(a,c,t,s) \in M^2 \times T \times S : \exists b\, ((b,a,t) \in Y,\\ (b,c,s) \in X = p_{1345}(Z))\}.$$

Also, $\mathrm{Fr}_{r_{1345}}(a,b,c,t,s) = (a, F(b), c, t, s)$ and

$$X' \circ (Y')^{-1} = \{(a,c,t,s) \in M^2 \times T \times S : \exists b\, ((F(b),a,t) \in Y',\\ (F(b),c,s) \in X = p_{1345}(\mathrm{Fr}_{r_{1345}}(Z)))\}.$$

4.4. Interpretation of a one-dimensional group

In the present section, we construct a group interpretable in $\mathcal{M}$. As already explained, this will be done by constructing a group configuration in $\mathcal{M}$. In order to construct this group configuration, a one-dimensional algebraic group (Lemma 4.19) G associated with slopes is "lifted", using Proposition 3.15, to a group configuration in $\mathcal{M}$.

Remark. Throughout this section and until the end of this paper, we fix a smooth algebraic curve M over an algebraically closed field K of infinite transcendence degree, and a reduct $\mathcal{M}$ of the full Zariski structure $\mathfrak{M}$ on M. We assume that the reduct is not locally modular. By default, the term *definable* will refer to definability in $\mathcal{M}$. Unless explicitly stated otherwise, by definable families we mean *stationary* nearly faithful ample families of curves, where a family $X \to T$ is *stationary* if every definable non-empty open subset of T is dense.

The reader should be advised that, at least a priori, stationarity in the sense of the full Zariski structure on M need not be the same as stationarity in the reduct, $\mathcal{M}$.

Before proceeding, we need a couple of easy observations.

Lemma 4.17. *Let $r : \mathrm{End}_k(k[\varepsilon]/(\varepsilon^{n+1})) \to \mathrm{End}_k(k[\varepsilon]/(\varepsilon^2))$ be the map sending an endomorphism φ to the unique endomorphism mapping ε to $\varphi(\varepsilon) \bmod \varepsilon^2$. Then*

$$\mathrm{Aut}(k[\varepsilon]/(\varepsilon^{n+1})) = r^{-1}(\mathrm{Aut}(k[\varepsilon]/(\varepsilon^2))).$$

Proof. Straightforward (see a similar statement for formal power series, for example, in [12, Corollary 7.17]). ∎

The following lemma should be well known to experts.

Lemma 4.18. *Let $X \to T$ and $Y \to S$ be one-dimensional nearly faithful definable families of one-dimensional subsets of M^2. Assume that for all $t \in T$, $s \in S$, all projections $p_i : X_t \to M$, $p_i : Y_s \to M$ are dominant. If for generic (t,s) the set $\{(t',s') : |X_{t'} \circ Y_{s'} \cap X_t \circ Y_s| = \infty\}$ is infinite, then $\mathcal{M}$ interprets a one-dimensional group.*

Proof. Fix (t,s) generic. Since any curve in $X \circ Y$ intersecting $X_t \circ Y_s$ in an infinite set must contain (up to a finite set) a strongly minimal component of $X_t \circ Y_s$, and since only finitely many such components exist, it will suffice to show that, unless any such component is contained in finitely many curves of the form $X_s \circ Y_t$ of the composition family, $\mathcal{M}$-interprets an infinite group.

Let $E \subseteq X_t \circ Y_s$ be strongly minimal. By [13, proof of Lemma 3.20], either $\mathcal{M}$ interprets a one-dimensional group or $\dim_{\mathcal{M}}(\mathrm{Cb}_{\mathcal{M}}(E/\emptyset)) = 2$ (the latter notation can be interpreted, equivalently, as: there exists an $\mathcal{M}$-definable nearly faithful family of curves defined over a two-dimensional parameter set, and E is generic in that family). For the sake of clarity and completeness, we give some details. Our aim is, in the case when $\dim_{\mathcal{M}}(\mathrm{Cb}_{\mathcal{M}}(E/\emptyset)) < 2$, to construct a group configuration.

For ease of notation, let $e := \mathrm{Cb}_{\mathcal{M}}(E/\emptyset)$. Since $X_t \circ Y_s$ has finitely many strongly minimal components, we see that $e \in \mathrm{acl}_{\mathcal{M}}(s, t)$. Also, $(E \circ Y_s^{-1}) \cap X_t$ is infinite (and one-dimensional), so by almost faithfulness of X, we get that $t \in \mathrm{acl}_{\mathcal{M}}(s, e)$, and by symmetry $s \in \mathrm{acl}_{\mathcal{M}}(t, e)$. It follows, since s, t are independent generics that $\dim_{\mathcal{M}}(e) > 0$. If $\dim_{\mathcal{M}}(e) = 1$, let $(x, y) \in X_t$ be generic and z such that $(y, z) \in Y_s$ and $(x, z) \in E$, then (s, t, e, x, y, z) is readily verified to be a group configuration.

If $\dim_{\mathcal{M}}(e) = 2$, then, as $e \in \mathrm{acl}_{\mathcal{M}}(s, t)$ and $\dim(s, t, e) = 2$, we get by additivity of dimension that $s, t \in \mathrm{acl}_{\mathcal{M}}(e)$. So there are only finitely many (t', s') such that $E \subseteq X_{t'} \circ Y_{s'}$, which is what we had to show. ∎

Remark. Recall that our aim in this section is to interpret in $\mathcal{M}$ a strongly minimal group G. It follows from the previous lemma that one way of achieving this is to find $X \to T$ and $Y \to S$ one-dimensional definable families of strongly minimal subsets of M^2 with the property that $\{(t', s') : |X_{t'} \circ Y_{s'} \cap X_t \circ Y_s| = \infty\}$ is infinite. In order not to overload the formulation of the sequel, we will tacitly assume that, whenever Lemma 4.18 is invoked, this is not the case – as otherwise we have found our group, and we can move on to the next section.

We now proceed to finding the one-dimensional algebraic group of slopes needed for the construction of the group configuration.

Lemma 4.19. *There exist a nearly faithful definable family $Y \subset M^2 \times S$ with S strongly minimal, an $\mathfrak{M}$-definable locally closed irreducible set $S_0 \subset S$, a point $a = (a_1, a_2) \in M^2$, $a_1 = a_2$, such that $a \in Y_s$ for all $s \in S_0$, and a family of branches β of $Y \times_S S_0$ at a such that for some $n > 0$ the locally closed set $\{\tau_n(Y_s, \beta_s) : s \in S_0\}$ almost coincides with a one-dimensional connected subgroup $H \subset \mathrm{Aut}(k[x]/(x^{n+1}))$.*

Proof. Fix some nearly faithful definable family $X \subseteq M^2 \times T$ witnessing non-local modularity of $\mathcal{M}$ and such that X_t is strongly minimal for generic $t \in T$, as provided by Fact 4.11. We may further require that the fibres $\pi_i^{-1}(a)$ for both projections of X_t on the factors M are finite for all $a \in M$.

Pick an irreducible component X' of X dominant over an irreducible component $T_0 \subset T$ of maximal dimension, and such that X' is a family of curves. Let M_0 be the connected component of M such that $M_0^2 \times T_0$ contains X'. By Lemma 4.14 applied to the closure of X', without loss of generality, we may assume that the restriction of p_1 to X'_t is dominant and generically étale for t in a dense subset $T_1 \subset T_0$. By Lemma 4.15, there exists a number m such that the restriction of p_{23} to $X'' = \mathrm{Fr}^m_{p_{23}}(X') \cap M_0^2 \times T_1$ is generically unramified, and since X' is nearly faithful, the projection is also dominant. In particular, for any $t \in T_1$ the projection $p_2 \colon X''_t \to M_0$ is generically unramified.

For each $a \in M_0^2$, consider the set $S^a \subset T_1$ of $t \in T_1$ such that $a \in X''_t$ and denote $X^a = X \cap M^2 \times S^a$. Let $U \subset X''$ be the complement of the ramification locus of the restriction of p_{23} to X''. It follows from dimension considerations that there exist $a \in M_0^2$ and an irreducible locally closed subset $S_0 \subset S^a$ such that $\dim S_0 = 1$, $\{a\} \times S_0 \cap U$ is dense in $\{a\} \times S_0$, and $a \in X''_t$ is smooth for $t \in S_0$. Because $a \in X''_t$ is smooth for any $t \in S_0$, there exists by Lemma 3.4 a unique family of branches α of $X'' \cap M_0^2 \times S_0$ at a. Selecting a generic enough, it follows that $\tau_1(X''_t, \alpha_t) \neq 0$ for t in a dense open subset of S_0 by the choice of S_0. Indeed, since the projection of each X''_t on both coordinates is finite to one, and since the projection of X''_t onto the second factor M_0 is generically unramified, it follows that $\tau_1(X''_t, a) \neq 0$ for generic a. Thus, by Lemma 4.17 $\tau_n(X''_t, \alpha_t) \in \mathrm{Aut}(k[x]/(x^{n+1}))$ for all $n \geq 1$ for all $t \in S_0$. Pick some $t_0 \in S_0$ generic over all the data and let $Y = X^a \circ X_{t_0}^{-1}$. Then by Lemma 4.16, $X \circ X_{t_0}^{-1} \cap M_0^2 \times S_0 = X'' \circ (X''_{t_0})^{-1}$ and $\tau_1(Y_t, \alpha_t \circ \alpha_{t_0}^{-1}) = \tau_1(X''_t \circ (X''_{t_0})^{-1}, \alpha_t \circ \alpha_{t_0}^{-1}) \in \mathrm{Aut}(k[x]/(x^{n+1}))$ for t in a dense open subset of S_0. Clearly, $\alpha \circ \alpha_{t_0}^{-1}$ is a family of branches at a point $(a_1, a_2) \in M_0^2$ such that $a_1 = a_2$.

By Krull's intersection theorem and since S_0 has non-zero dimension, there exists a smallest number n such that $|\{\tau_n(X_t, \alpha_t) : t \in S_0\}| > 1$. If $n = 1$, then $\{\tau_n(X_t, \alpha_t) : t \in S_0\}$ coincides with a one-dimensional subgroup of $\mathrm{Aut}(k[x]/(x^2)) \cong k^\times$ up to a finite set. If $n > 1$, then the slope $\tau_{n-1}(X'_t, \alpha_t)$ as t ranges in S_0 is constant, and therefore $\tau_{n-1}(Y_t, \alpha_t \circ \alpha_{t_0}^{-1}) = 1$. Then it follows that $\tau_n(Y_t, \alpha_t \circ \alpha_{t_0}^{-1})$ almost coincides with $\mathrm{Ker}(\mathrm{Aut}(k[x]/(x^{n+1})) \to \mathrm{Aut}(k[x]/(x^n)))$. In either case, the family Y satisfies the main part of the lemma over the irreducible component S_0. Near faithfulness of Y follows from Lemma 4.18 applied to X^a and $(X^a)^{-1}$, observing that Y is a subfamily of $X^a \circ (X^a)^{-1}$, and that a generic member of Y is generic (over a, not over a, t_0) in $X^a \circ (X^a)^{-1}$. ∎

Before proceeding to the construction of a group in $\mathcal{M}$, we need some more preliminary work. First, we fix some *ad hoc* terminology and notation that will simplify the discussion.

Notation. Let $X \to T$ be a definable family of curves in M^2. We denote:

(1) For $a \in M^2$, let $T^a := \{t \in T : a \in X_t\}$ and $X^a := \{X_t : t \in T^a\}$, the definable subfamily of all curves incident to the point a.

(2) $X^0 := \{X_t^0 : t \in T\}$, where X_t^0 is the union of algebro-geometric 0-dimensional components of X_t.

(3) $X^1 \subseteq X \times_T T'$ is a family of pure-dimensional curves for a dense $T' \subseteq T$, as provided by Lemma 3.2.

Note that X_t^0 is $\mathfrak{M}$-definable over t, and thus X^0 is $\mathfrak{M}$-definable. A priori, X_t^0 is not necessarily $\mathcal{M}$-definable over t, and it is, therefore, not clear whether X^0 is $\mathcal{M}$-definable. We will, of course, not assume this. We merely point out that had we known that X^0 were $\mathcal{M}$-definable, the proof of our main result would have been considerably simplified (allowing us to apply Proposition 3.15 for $\mathcal{M}$-definable families of curves). Point (4) of the definition below will be the key, ultimately allowing us to circumvent this problem.

Definition 4.20. We say that a nearly faithful $\mathcal{M}$-definable family of curves $X \to T$ satisfies *property* (a, n) for $a \in M^2$ and a positive integer n if

(1) $a \in X_t$ for all t.

(2) There exists a family β of branches of X at a such that $\{\tau_n(X_t, \beta_t) : t \in T\}$ is one-dimensional and contains, up to a finite set, a one-dimensional connected algebraic group, H.

(3) For all $a' \in M^2$, if $a' \neq a$, then $\dim(T^{a'}) = 0$.

(4) If p belongs to a zero-dimensional component of $X_s \circ X_t$ for $s, t \in T$ $\mathfrak{M}$-independent generics, then p is $\mathfrak{M}$-generic in M^2 (over $\emptyset$).

The group H is *the group of slopes of* X *at* a (associated with the family of branches β).

To show the existence of families that satisfy property (a, n) for some a, n, we need to prove the following lemma.

Lemma 4.21. *Let* $X \to T$, $Y \to S$ *be stationary families. Then there exists* $X' \to T$, $Y' \to S$ *$\mathcal{M}$-definable over* $\mathrm{acl}_{\mathfrak{M}}(\emptyset)$ *such that*

(1) $X_t = X'_t$, $Y_s = Y'_s$ *up to a finite set, for all* $t \in T$, $s \in S$.

(2) *If* $t \in T$, $s \in S$ *are* $\mathfrak{M}$*-independent generics and* $a \in X_t \circ Y_s$ *is such that* $\{a\}$ *is contained in a zero-dimensional component, then* a *is* $\mathfrak{M}$*-generic over* $\emptyset$.

Proof. We may assume that for $t \in T$ generic, if a is contained in a 0-dimensional component of X_t, then $a \notin \mathrm{acl}_{\mathfrak{M}}(\emptyset)$. Otherwise, note that by stationarity (and genericity of t) we get that $a \in X_{t'}$ for all generic $t' \in T$. So $a \in \mathrm{acl}_{\mathfrak{M}}(\emptyset)$. Since there are at most finitely many b incident to all but finitely many $X_{t'}$, we may simply set $X' := (X \setminus \{a\}) \times T$, eliminating the problem in finitely many similar steps.

Similarly, we may assume that if $a = (a_1, a_2)$ is contained in a zero-dimensional component of X_t, then $a_1, a_2 \notin \mathrm{acl}_{\mathfrak{M}}(\emptyset)$. Thus, we may assume that both (a_1, a_2) are $\mathfrak{M}$-generic over $\emptyset$. The same is, of course, true of Y.

Denoting by X_t^0, Y_s^0 the unions of zero-dimensional components of X_t, Y_s and noting that $(X_t \circ Y_s)^0 \supseteq X_t^0 \circ Y_s \cup X^t \circ Y_s^0$, we get that for $s, t \in T$ independent generics any isolated point of $X_t \circ Y_s$ is generic over $\emptyset$. ∎

We have thus shown the following assertion.

Corollary 4.22. *There exist* $a_1 \in M$, *a natural number* $n > 0$ *and a one-dimensional definable family of curves that satisfy property* (a, n) *for* $a = (a_1, a_1)$.

Proof. Clauses (1) and (2) of the definition of property (a, n) are achieved by taking a family as provided by Lemma 4.19. Condition (4) is provided by Lemma 4.18, and condition (3) is obtained by removing finitely many points common to all generic independent curves in the resulting family. ■

The same proofs give also the following.

Corollary 4.23. *If $X \to T$ is a family that satisfies property (a, n), then up to – possibly – finitely many corrections, $X \circ X$ and $X \circ X^{-1}$ also satisfy property (a, n).*

Note however that in the above corollary if X is one-dimensional, then the families $X \circ X$ and $X \circ X^{-1}$ will not be one-dimensional. It follows, however, that if $t \in T$ is generic, then the one-dimensional families $X \circ X_t$ and $X \circ X_t^{-1}$ will satisfy property (a, n). The following is a strengthening of the above observation that we will need later on for technical reasons.

Lemma 4.24. *Let $X \to T$ be a family that satisfies property (a, n). Let H be the group of slopes of X at a (associated with some family of branches). Then there exists an infinite nearly faithful family of generically strongly minimal sets $Z \to L$ such that $a \in Z_l$ for all l, and there exists a family of branches γ at Q such that $\tau_n(Z_l, \beta_l) = 1 \in H$ for some $\mathfrak{M}$-generic $l \in L$.*

Proof. By Proposition 3.7, the n-th order slopes of $Y := X \circ X$ at a (associated with the family of branches $\widetilde{\beta} := \beta \circ \beta$) also almost coincides with H. Let R parameterise Y, then there exists an $\mathfrak{M}$-irreducible component $W \subseteq R$ such that $\tau_n(Y_r, \widetilde{\beta}_r) \in H$ for all $r \in W$ (and, in particular, the slope is defined). Let $r_0 \in W$ be generic. So there exists some r_1 such that $\tau_n(Y_{r_1}, \widetilde{\beta}_1) = \tau_n(Y_{r_0}, \widetilde{\beta}_r)^{-1}$. In fact, by dimension considerations, since (by the remark following Lemma 4.18) $\dim(R) = 2$, the set R_1 of r_1 with the above property is one-dimensional. So we chose such $r_1 \in R_1$ generic over r_0. Let $Z_{l_0} \subseteq X_{t_1} \circ X_{t_0}$ be the $\mathcal{M}$-definable, strongly minimal component containing the branch $\beta_{t_1} \circ \beta_{t_0}$. It follows that $\dim_{\mathfrak{M}}(\mathrm{Cb}(Z)_{l_0}) \geq 1$, and so $\dim_{\mathcal{M}}(\mathrm{Cb}(Z)_{l_0}) \geq 1$. Let $Z \to L$ be the $\mathcal{M}$-definable family whose generic member is Z_{l_0}. So there is an $\mathfrak{M}$-generic subfamily of $Z \to L$ with the property that $\tau_n(Z_{l'}, \gamma_{l'}) = 1$ (see Proposition 3.7) for a family γ of branches of Z at a and for all l' in that subfamily. ■

We are finally ready to prove the main result of this section.

Theorem 4.25. *Let $\mathcal{M}$ be a non-locally modular reduct of an algebraic curve M over an algebraically closed field. Then $\mathcal{M}$ interprets a one-dimensional group.*

Proof. We prove the theorem by constructing a group configuration. By Corollary 4.13, we may assume that M is smooth, and we identify M with $M(K)$ for some algebraically closed field K of infinite transcendence degree. We will freely use the remark after Definition 3.3 and Lemma 3.2, referring to branches of suitable pure-dimensional subfamilies of definable families of curves when we speak about branches of definable families of curves.

Let $X \to T$ be a one-dimensional definable family of curves that satisfies property (a, n) for some point $a = (a_1, a_1)$ as provided by Corollary 4.22, H the associated group of slopes for the family of branches β. Absorbing into the language the parameters needed to define X, we may assume that it is $\emptyset$-definable.

We fix a standard group configuration

$$\mathcal{H} := \{g, h, k, gh, gk, h^{-1}k\}$$

associated with the action of H on itself by multiplication.

According to Lemma 4.19, there exists an irreducible component $W \subseteq T$ such that $\tau_n(X_t, \beta_t) \in H$ for all generic $t \in W$. Identifying (up to a finite set) $\{\tau_n(X_t, \beta_t) : t \in W\}$ with elements of $H \leq \mathrm{Aut}(k[\varepsilon]/(\varepsilon^{n+1}))$, we get that $\tau_n(X_t, \beta_t)$ is $\mathfrak{M}$-inter-algebraic with t. At the price of replacing W with a (dense) open subset, we may assume that W is smooth.

Any $\mathfrak{M}$-independent points $s, t \in W$ generic over all the data are, in particular, generic and independent in the sense of the reduct $\mathcal{M}$. Let $u \in T$ be such that

$$\tau_n(X_u, \beta_u) = \tau_n(X_s, \beta_s)\tau_n(X_t, \beta_t) = \tau_n(X_s \circ X_t, \beta_s \circ \beta_t).$$

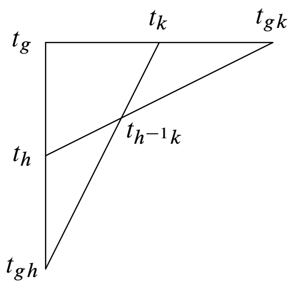
Such a u exists since the relative slopes of X_t and X_s are generic in H, which is one-dimensional. Since the product of two independent generic elements of H is again generic in H, we can find such a u.

Getting back to our group configuration $\mathcal{H}$, the above construction gives us a subset of T,

$$\mathcal{T}_{\mathcal{H}} := \{t_g, t_h, t_k, t_{gh}, t_{gk}, t_{h^{-1}k}\},$$

such that for every $s \in \mathcal{H}$ we have $\tau_n(X_{t_s}, \beta_{t_s}) = s$. Our goal is to show that $\mathcal{T}_{\mathcal{H}}$ is a group configuration in the sense of $\mathcal{M}$.

We have to verify the three sets of conditions appearing in Definition 4.3. That the elements of $\mathcal{T}_{\mathcal{H}}$ are pairwise $\mathcal{M}$-independent follows from the fact that for all $s \in \mathcal{H}$ also $s \in \mathrm{acl}_{\mathfrak{M}}(t_s)$ and the elements of $\mathcal{H}$ are $\mathfrak{M}$-independent. That all elements in $\mathcal{T}_{\mathcal{H}}$ have dimension 1 follows from the fact that T is strongly minimal and the elements of $\mathcal{T}_{\mathcal{H}}$ are generic in T. So it remains only to verify the third set of conditions, namely, that every collinear triple of elements in the following diagram is $\mathcal{M}$-dependent:



The rest of the proof will be dedicated to that end. Since the situation is symmetric, it will suffice to show that if $s, t \in W$ are generic independent $\tau(X_u, \beta_u) = \tau_n(X_s, \beta_s)\tau(X_t, \beta_t)$, then $u \in \mathrm{acl}_{\mathcal{M}}(s, t)$. Note that since W is $\mathfrak{M}$-strongly minimal, $u \in \mathrm{acl}_{\mathfrak{M}}(s, t)$.

To achieve our goal, we would like to apply Proposition 3.15 to the family $\widetilde{E} \to R$ given by $X \circ X$ and the family $X \to T$, in order to show that the curve X_u intersects the curve $X_s \circ X_t$ in a smaller than generic number of points. The problem is that neither $\widetilde{E} \to R$ nor $X \to T$ can be assumed to be pure-dimensional families of curves, which is a crucial assumption in the statement of the proposition. To circumvent this problem, we will show that $X_u \cap (X_s \circ X_t)$ meets no zero-dimensional components of either curve, allowing us to apply the proposition with the pure-dimensional families $\widetilde{E}^1 \to R$ and $X^1 \to T$ without changing the number of intersection points.

For technical reasons that will be made clear later on, we need to slightly twist the family $\widetilde{E} \to R$ that we are working with. Let $Z \to L$ be a nearly faithful family of curves that satisfies property (a, m) for $m > n$, let γ be a family of branches at a, all as provided by Lemma 4.24. We let $E \to R$ be $\widetilde{E} \circ Z_{l_0}$ for some $l_0 \in L$ $\mathfrak{M}$-generic and independent over all the data. Note that, by Proposition 3.7 and the choice of the family $Z \to L$, we have that

$$\tau_n(X_s \circ X_t, \beta_s \circ \beta_t) = \tau_n(X_s \circ X_t \circ Z_{l_0}, \beta_s \circ \beta_t \circ \gamma_{l_0})$$

whenever both sides of the equations are defined. For the sake of clarity, we let α be the family of branches of E at a. Namely, $\alpha = \beta \circ \beta \circ \gamma_{l_0}$.

It will be convenient to already note at this stage the following slight strengthening of Lemma 4.18.

Claim 1. *We may assume that if $r \in R$ is generic, then $|\{r' : \widetilde{E}_r \cap \widetilde{E}_{r'}\}| = \infty$ is finite.*

Proof. The claim would follow from Lemma 4.18 if the members of X were strongly minimal. In the general case, if $r \in R$ is generic and $\widetilde{E}_r = X_s \circ X_t$, then any strongly minimal $F_r \subseteq \widetilde{E}_r$ is contained in $C_s \circ D_t$ for some strongly minimal $C_s \subseteq X_s$ and $D_t \subseteq X_t$. By Lemma 4.18 applied to the families $\{D_t : t \in T\}$ and $\{C_s : s \in S\}$, we get that $s, t \in \mathrm{acl}_{\mathcal{M}}(\mathrm{Cb}(F_r))$. Since $\mathrm{Cb}(F_r) \in \mathrm{acl}(\mathrm{Cb}(E_r))$, we conclude that $s, t \in \mathrm{acl}(\mathrm{Cb}(E_r))$, which is what we needed. ∎

Note that the fact that $\widetilde{E}$ is the composition of two copies of X did not play any role in the proof above, and we could invoke Lemma 4.18 with X and $(X \circ Z_{l_0})$ to get the same conclusion for the family $E := X \circ (X \circ Z_{l_0})$.

Let us fix some additional notation. We have $s, t \in W$ independent generics, and $u \in W$ such that $X_s \circ X_t$ is n-tangent to X_u at a. We let $R(u)$ be the set of all $r \in R$ such that E_r is n-tangent to X_u at a, i.e., $\tau_n(E_r, \alpha_r) = \tau_n(X_u, \beta_u)$. Let $E(u) := \{E_r : r \in R(u)\}$. In other words, $R(u)$ is the parameter set of all curves in the family $E \to R$ n-tangent to $X_s \circ X_t$ at a and $E(u)$ is the subfamily of E over the parameter set $R(u)$. So $E(u) \to R(u)$ is an $\mathfrak{M}$-definable subfamily of E of dimension 1. We denote $r = (s, t) \in R$, so $r \in R(u)$ and it is $\mathfrak{M}$-generic as such (indeed, $u \in \mathrm{acl}_{\mathfrak{M}}(s', t')$ for all $(s', t') \in R(u)$, so the result follows by genericity of (s, t) and additivity of dimension). Replacing, if needed, $R(u)$ with the $\mathfrak{M}$-definable strongly minimal component of $R(u)$ containing r we may assume that $R(u)$ is strongly minimal.

The following is the main step in the proof.

Claim 2. *Assume that* $u \notin \mathrm{acl}_{\mathcal{M}}(s,t)$. *Let* $\{x_1,\dots,x_k\} = (X_u \cap E_r) \setminus \{a\}$. *Then* x_i *is* $\mathfrak{M}$*-generic in* X_u *for all* i.

Proof. First, note that $x_i \notin \mathrm{acl}_{\mathcal{M}}(\emptyset)$, because otherwise, since $u \in T$ is generic, we would get that $\dim(T^{x_i}) = 1$, contradicting clause (3) of Definition 4.20. Note that the exact same argument shows that $x_i \notin \mathrm{acl}_{\mathfrak{M}}(\emptyset)$. Next, as $r \in \mathrm{acl}_{\mathcal{M}}(s,t)$ and since X is an $\mathcal{M}$-strongly minimal family, our assumption that $u \notin \mathrm{acl}_{\mathcal{M}}(s,t)$ implies that u is $\mathcal{M}$-generic over r, and by Lemma 4.18 (and the remark following it) applied to the acl($\emptyset$)-definable strongly minimal subsets of E_r, we get that

$$\dim(\mathrm{Cb}(E_r)/\emptyset) = \dim(r/\emptyset) = 2.$$

Now assume that x_1 is not $\mathfrak{M}$-generic in X_u. Since r is $\mathfrak{M}$-generic in $R(u)$ (and, in R), it follows that $\dim(R(u)^{x_1}) = 1$. Indeed, since $\dim_{\mathfrak{m}}(x_1/u) = 0$ (by assumption), it follows that $\dim_{\mathfrak{M}}(r/ux_1) = \dim_{\mathfrak{M}}(r/u)$, so r is generic in $R(u)$ over x_1, and the strong minimality of $R(u)$ implies that $x_1 \in E_{r'}$ for all generic $r' \in R(u)$. Thus, in fact, $R(u)$ is a generic subset of R^{x_1}. Recall, moreover, that there exists a family α of branches of all curves in $E(u)$ at a such that $\tau_n(E(u),\alpha_r) = \tau(X_u,\beta_u)$.

We will show that this leads to a contradiction. We split the argument into cases according to $\dim_{\mathcal{M}}(x_1/\emptyset)$. The case $x_1 \in \mathrm{acl}_{\mathcal{M}}(\emptyset)$ has already been discarded. If x_1 is non-$\mathcal{M}$-generic in M^2, then there exists a curve F, $\mathcal{M}$-definable over $\emptyset$ such that x_1 is generic in F. So u is contained in the set of all u' such that $F \cap X_{u'} \cap E_r \neq \emptyset$. Because F is $\emptyset$-definable in $\mathcal{M}$, necessarily $|E_r \cap F| < \infty$ (otherwise, since r is generic in R, it would follow that all curves in E have a component contained in F, which is impossible). Thus, by condition (3) of Definition 4.20, there are only finitely many such u'. So u is $\mathcal{M}$-algebraic over r – contradicting our assumption.

Thus, we may assume that x_1 is $\mathcal{M}$-generic in M^2. We will now focus on the family $E \to R$. Since x_1 is $\mathcal{M}$-generic in M^2, for any $r_1, r_2 \in R^{x_1}$ independent generics, $m := |E_{r_1} \cap E_{r_2}|$ is obtained on an $\mathfrak{M}$-generic subset of parameters of $R \times R$. Consider the $\mathfrak{M}$-definable family $E^1 \to R$ of pure-dimensional curves associated with $E \to R$. Note that for $\mathfrak{M}$-generic independent $u, w \in R$, we have $|E_u \cap E_w| = |E^1_u \cap E^1_w| = m$.

On the other hand, Lemma 3.13, and hence Proposition 3.15, are applicable to two copies of the family $E^1 \to R$, possibly after shrinking R, so as to ensure, using Fact 3.10 that $E^1 \to R$ is flat and that R is smooth.

Let $W_0 \subseteq R$ be the dense open set obtained by shrinking R as in the previous paragraph, we may further assume that W_0 and $E \cap p^{-1}(W_0)$ are varieties (see, e.g., Lemma 3.2), so Lemma 3.13 and Proposition 3.15 apply for all $(v,w) \in W := W_0 \times W_0$, if $\tau_n(E^1_v,\beta_v) = \tau_n(E^1_w,\beta_w)$, then either $\dim(E^1_w \cap E^1_v) = 1$ or $\#(E^1_w \cap E^1_v) < m$.

For generic v, the set of w such that $(v,w) \in W$ and $\dim(E^1_w \cap E^1_v) = 1$ is finite by Claim 1. So, for generic $(v,w) \in W$, we see that $\dim(E^1_w \cap E^1_v) = 0$, so necessarily $\#(E^1_w \cap E^1_v) < m$. We now show that this must imply that $E^0_w \cap E_r \neq \emptyset$ for generic w. Indeed, since W_0 is dense in R and $\emptyset$-definable, $\mathfrak{M}$-genericity of r in R implies that it is also generic in W_0. Since $R(u)^{x_1}$ is generic in R^{x_1} (in the sense that it contains an open subset of R^{x_1}), we can find some $w \in R(u) \cap W_0$ $\mathfrak{M}$-generic and $\mathfrak{M}$-independent of r (over all the data gathered so far) so that (r, w) is $\mathfrak{M}$-generic in W. Moreover, by definition of $R(u)$ we know that $\tau_n(E_r^1, \beta_r) = \tau_n(E_w^1, \beta_w)$, and by what we have just said, this must imply that $\#(E_w^1 \cap E_r^1) < m$. Because x_1 is $\mathcal{M}$-generic in M^2 and $w, r \in R^{x_1}$ are $\mathcal{M}$-independent generics, they are, in fact, independent generic in R^2 over $\emptyset$. So $\#(E_w \cap E_r) = m$, implying that $E_w \cap E_r^0 \neq \emptyset$.

Finally, as w is $\mathfrak{M}$-independent of r and $\mathfrak{M}$-generic in $R(u)$, and since $E_r^0 \subseteq \mathrm{acl}_{\mathfrak{M}}(r)$, we get, precisely as above, that there is some $c \in E_r^0$ such that R^c contains $R(u)$, up to a finite set. This implies that $\dim_{\mathfrak{M}}(c/u) = 0$, and hence $\dim_{\mathfrak{M}}(c/\emptyset) \leq \dim_{\mathfrak{M}}(u/\emptyset) = 1$. This contradicts Corollary 4.23 (specifically, clause (4) of Definition 4.20). ■

It follows from Claim 2 that $X_u^0 \cap E_r = \emptyset$. We also need to show that X_u does not meet E_r in an isolated point of the latter. It is here that the twist of the family $\widetilde{E} \to R$ by a generic curve from $Z \to L$ plays its role.

Claim 3. *If $u \notin \mathrm{acl}_{\mathcal{M}}(s, t)$, then $X_u \cap E_r^0 = \emptyset$.*

Proof. Recall that $E_r = X_s \circ X_t \circ Z_{l_0}$. Assume that there exists some $x_i \in X_u \cap (X_s \circ X_t \circ Z_{l_0})^0$. By Lemma 4.21 applied to $\widetilde{E}(u) \to R(u)$ and $Z \to L$, if $r' \in R(u)$ is generic and $l \in L$ is generic independent of r', then any $x_i \in (E_{r'} \circ Z_l)^0$ is either $\mathfrak{M}$-generic over $\emptyset$ or contained in one of finitely many sets of the form $\{a\} \times M$ and $M \times \{a\}$ for $a \in \mathrm{acl}_{\mathfrak{M}}(\emptyset)$. But $X_u \cap (M \times \{a\} \cup \{a\} \times M) \subseteq \mathrm{acl}_{\mathfrak{M}}(u)$, so $x_i \in \mathrm{acl}_{\mathfrak{M}}(u)$, contradicting the previous claim. ■

The conclusion of the discussion, up to this point, is that if $u \notin \mathrm{acl}_{\mathcal{M}}(s, t)$, then $E_r \cap X_u = E_r^1 \cap X_u^1$. This allows us to conclude the following.

Claim 4. *The parameter u is $\mathcal{M}$-algebraic over t, s.*

Proof. Let us assume that this is not the case. By Proposition 3.7, $\tau_n(X_t \circ X_s \circ Z_{l_0}, \beta_t \circ \beta_s \circ \gamma_{l_0}) = \tau_n(X_t, \beta_t) \circ \tau_n(X_s, \beta_s)$. Let $m = \max_{\bar{t},\bar{s},\bar{u} \in T} \#(X_{\bar{t}} \circ X_{\bar{s}} \circ Z_{l_0} \cap X_{\bar{u}})$, then by Lemma 3.14, $m = \#((X_t \circ X_s \circ Z_{l_0}) \cap X_u)$ for (t, s, u) generic in $T \times T \times T$. Let $\widetilde{T} \subseteq T$ be as provided by Lemma 4.19. By Lemma 3.13 and Proposition 3.15, the set of parameters $w \in \widetilde{T}$ such that $\tau_n(X_w^1, \beta_w) = \tau_n(X_t^1, \beta_t) \circ \tau_n(X_s^1, \alpha_s)$ is contained in

$$W_1 := \{w \in T : \dim(X_t \circ X_s \cap X_w) = 1 \text{ or } \#((X_t \circ X_s \circ Z_{l_0})^1 \cap X_w^1) < m\}.$$

By strong minimality of T, the set $\{w : \#(X_t \circ X_s \circ Z_{l_0} \cap X_w) < m\}$ is finite. Also, for $\mathfrak{M}$-generic w, we have $(X_t \circ X_s \circ Z_{l_0})^1 \cap X_w^1 = X_t \circ X_s \circ Z_{l_0} \cap X_w$. So the set of w such that $\#((X_t \circ X_s \circ Z_{l_0})^1 \cap X_w^1) < m$ is finite. Since X satisfies property (a, n), by Lemma 4.18 the set $\{w : \dim(X_t \circ X_s \circ Z_{l_0} \cap X_w) = 1\}$ is finite. So W_1 is finite. Similarly,

$$W := \{w \in T : \dim(X_t \circ X_s \circ Z_{l_0} \cap X_w) = 1 \text{ or } \#(X_t \circ X_s \circ Z_{l_0} \cap X_w) < m\}$$

is finite, and moreover, W is $\mathcal{M}$-definable. Our assumption that $u \notin \mathrm{acl}_{\mathcal{M}}(s, t)$ allows us to apply Claim 2 combined with Claim 3 to get that $X_s \circ X_t \circ Z_{l_0} \cap X_u = (X_s \circ X_t \circ Z_{l_0})^1 \cap X_u^1$. Since $u \in W_1$, it follows that $u \in W$, proving that in fact $u \in \mathrm{acl}(s, t)$. ■

Claim 4 shows that, indeed, $\mathcal{H}_{\mathcal{T}}$ is an $\mathcal{M}$-group configuration, and the desired conclusion is obtained by applying Fact 4.4. ■

The following proposition can be proved in greater generality (and follows, essentially, from [25, Section 3]), but we only need a simpler result. We thank B. Castle for suggesting the following simplification of the Hrushovski–Pillay argument suitable for our needs.

Proposition 4.26. *In the notation of the previous proof, assume that the group H almost coinciding with $\{\tau_n(Y_t, \alpha_t) : t \in \widetilde{S}\}$ is isomorphic to $\mathbb{G}_a$. Then the connected component of the identity of the group provided by Theorem 4.25 is not $\mathfrak{M}$-isomorphic to $\mathbb{G}_m$.*

Proof. First, we point out that there are no one-dimensional subgroups of $\mathbb{G}_a \times G_m$ projecting dominantly on both factors. Indeed, if such a group G existed, then the kernel H of its projection to $\mathbb{G}_a$ would be finite (since G is one-dimensional), say $|H| = n$. Replacing G by $\{(x, y^n) : (x, y) \in G\}$, we may assume that, in fact, H is trivial. So the projection of G onto $\mathbb{G}_m$ is a definable group homomorphism from $\mathbb{G}_a$ onto $\mathbb{G}_m$, but such cannot exist (e.g., because $\mathbb{G}_m$ has unbounded torsion).

Thus, it will suffice to show that any inter-algebraic group configurations for the groups $\mathbb{G}_a$ and $\mathbb{G}_m$ would give rise to a non-trivial subgroup of $\mathbb{G}_a \times \mathbb{G}_m$. So assume towards a contradiction that

$$G_1 = \{a, b, a+b, x, x+a, x+b\} \quad \text{and} \quad G_2 = \{e, f, e\cdot f, y, e\cdot y, f\cdot y\}$$

are group configurations for the groups $\mathbb{G}_a$ and $\mathbb{G}_m$, respectively, and

$$\operatorname{acl}(a) = \operatorname{acl}(e), \quad \operatorname{acl}(b) = \operatorname{acl}(f), \quad \operatorname{acl}(a+b) = \operatorname{acl}(ef).$$

Note that

$$\dim(b, a+b, ef) = 2 > \dim(a+b, ef) = 1,$$

so b is generic independent over $(a+b, ef)$.

Consider the types

$$p := \operatorname{tp}(a, e), \quad q := \operatorname{tp}(b, f) \quad \text{and} \quad r := \operatorname{tp}(a+b, ef).$$

By what we have just said, for any realisation (b', f') of q generic over $(a+b, ef)$, there is a field automorphism fixing $(a+b, ef)$ and mapping (b, f) to (b', f') and (a, e) to some (a', e'). It follows that

$$a' = a + b - b' \quad \text{and} \quad e' = ef/f'.$$

So $(b - b', f/f')$ is in the stabiliser of p (under the action of $K \times K^*$ on itself). Clearly, as we vary b', $(b - b', f/f')$ varies in an infinite set, and similarly when varying f'. Thus, the stabiliser of p is an infinite non-trivial definable (necessarily one-dimensional) subgroup of $K \times K^*$, a contradiction. ■

4.5. Interpretation of the field and proof of the main theorem

In this section, we interpret the field K in the reduct $\mathcal{M}$, concluding the proof of the main theorem of this paper. The results of the previous subsection allow us to replace $\mathcal{M}$ by an algebraic group G, interpretable in $\mathcal{M}$ (we only have to verify that the induced structure is non-locally modular). As in the previous subsection, the interpretation of the field boils down to the construction of a field configuration. The construction of the field configuration will depend on whether the (connected component of the) group G is isomorphic (in K) to $\mathbb{G}_a$, $\mathbb{G}_m$ or to an elliptic curve. The question to address is how to find an $\mathcal{M}$-definable strongly minimal $Z \subseteq G^2$ whose set of 1-slopes $\{\tau_1(Z,z) : x \in Z\}$ (see below) is infinite. The easiest is the case of an elliptic curve.

Lemma 4.27. *Let E be an elliptic curve and Z a closed one-dimensional irreducible subset of $G = E^2$. Identify T_gG with T_0G via the isomorphism $d\lambda_g\colon T_0G \to T_gG$ for $\lambda_g(x) = g \cdot x$. Suppose that for any $z \in Z$, the tangent space $T_zZ \subset T_0G$ is constant. Then Z is a coset of a closed subgroup of G.*

Proof. Since Z is a projective curve with a trivial tangent bundle, it is an elliptic curve itself by the Riemann–Roch formula. Since any morphism between Abelian varieties with finite fibres, which preserves the identity, automatically preserves the group structure by the rigidity lemma (see [35, p. 43]), Z is a coset of an Abelian subvariety of G. ∎

To obtain the analogous results for $\mathbb{G}_a$, we first need a couple of technical lemmas. We thank S. Pinzon for pointing out a mistake in an earlier version of the proof. The current proof builds on Pinzon's proof of an analogous statement in the theory of algebraically closed valued fields of positive characteristic p (ACVF_p).

We start with some notation and terminology. Let M be an algebraic curve, and consider a curve $Z \subset M^2$. For every point $z \in Z$ such that p_1 is étale in a neighbourhood of z, there exists by Lemma 3.4 a unique branch at z, call it α_z. We will use the notation

$$\tau_n(Z,z) := \tau_n(Z,\alpha_z).$$

For any group $(G,\cdot)$ with identity $e \in G$, for any $a = (a_1,a_2) \in G^2$, define the maps $t_a\colon G^2 \to G^2$

$$t_a(x_1,x_2) = (a_1^{-1} \cdot x_1, a_2^{-1} \cdot x_2),$$

and for any one-dimensional locally closed subset $Z \subset G^2$, define

$$s_n(Z) = \{\tau_n(t_z(Z),(e,e)) : z \in Z\} \subset k[x]/(x^{n+1}),$$

where z ranges in those points of Z for which the n-th order slope is well defined. Identifying $k[x]/(x^{n+1})$ with the affine n-space, we observe that the set $s_n(Z)$ is constructible.

If $Z \subset \mathbb{G}_a^2$ is an irreducible locally closed one-dimensional set that is not a dense subset of a coset of a subgroup, then $s_1(Z)$ is clearly a one-dimensional set when the characteristic of the ground field is 0. In positive characteristic, $s_1(Z)$ might be 0-dimensional for various reasons.

If Z is a coset of a subgroup of G, then all of its translates coincide on an open dense set and therefore slopes of all orders at (e, e) are constant. In particular, $\dim s_n = 0$ for all $n \geq 1$. If, however, Z is not a coset, $t_z(Z)$ are distinct for different $z \in Z$ and therefore by Krull's maximal ideal theorem for some n, $\tau_n(t_z(Z), (e, e))$ will vary depending on $z \in Z$. The following quantity is then well defined,

$$N(Z) = \min\{n \geq 1 : \dim s_n(Z) = 1\}.$$

Lemma 4.28. *Let $X \subset \mathbb{G}_a^2 \times T$ be a family of curves parametrized by a curve T, and assume that $\widetilde{a} = (a_1, a_2) \in X_t$ for all t in some neighbourhood of $a_3 \in T$. Let x_1, x_2, t be a choice of local coordinates of $\mathbb{G}_a^2 \times T$ at $a = (a_1, a_2, a_3) \in X$ such that x_1, x_2 are local coordinates of $\mathbb{G}_a^2$ at $\widetilde{a}$. Let $f \in k[\![x_1, x_2, t]\!]$ be the local equation of X in $\mathbb{G}_a^2 \times T$ at a. Then $\tau_n(X_t, \widetilde{a})$ is constant for t in a neighbourhood of a_3 if and only if $f \equiv f_0 \bmod x_1^{n+1}$ for some $f_0 \in k[\![x_1, x_2]\!]$.*

Proof. Let $\mathcal{I}$ be the ideal generated by x_1, x_2 in the ring $R := [x_1, x_2] \otimes \mathcal{O}_{T,a_3}$, and let $\mathcal{J}$ be the ideal of the same ring consisting of functions vanishing at the point a. Then

$$\widehat{R} := \varprojlim R/\mathcal{I}^n = k[\![x_1, x_2]\!] \otimes \mathcal{O}_{T,a_3} \quad \text{and} \quad \varprojlim R/\mathcal{J}^n \cong k[\![x_1, x_2, t]\!]$$

for some formal parameter t, and the ideal of functions vanishing on X in R generates an ideal $(\overline{f}) \subset \widehat{R}$. Since R is a Noetherian local ring, the natural maps $R \to \widehat{R} \to k[\![x_1, x_2, t]\!]$ are injective. Let $f \in k[\![x_1, x_2, t]\!]$ be the image of $\overline{f}$. By the definition of slopes, $\tau_n(X_t, \widetilde{a})$ depends only on $\overline{f}(t)$ modulo x_1^{n+1}. In particular, $\tau_n(X_t, \widetilde{a})$ is constant for t in a neighbourhood of a_3 if and only if $\overline{f} \equiv \overline{f_0} \bmod x_1^{n+1}$ for some $\overline{f_0} \in k[\![x_1, x_2]\!] \subset \widehat{R}$. Let f_0 be the image of $\overline{f_0}$ in $k[\![x_1, x_2, t]\!]$, then clearly, $f_0 \in k[\![x_1, x_2]\!]$. Since the map $\widehat{R} \to k[\![x_1, x_t, t]\!]$ is injective, $f \equiv f_0 \bmod x_1^{n+1}$ if and only if $\overline{f} \equiv \overline{f_0} \bmod x_1^{n+1}$, and the statement follows. ∎

Lemma 4.29. *Assume that the ground field is of positive characteristic p. Let $Z \subset \mathbb{G}_a^2$ be an irreducible pure-dimensional curve that is not contained in a coset of a subgroup of $\mathbb{G}_a^2$. Assume further that $(0, 0) \in Z$, and that the projection of a neighbourhood of $(0, 0) \in Z$ on the first $\mathbb{G}_a$ is étale, so Z has a local equation $y - f$, where*

$$f = \sum_{j>0} b_j x^j \in k[\![x]\!].$$

Then we have

(1) *$N(Z) = \min\{p^i : \exists b_j \neq 0,\ j = m \cdot p^i,\ m \neq 1, p \nmid m\}$;*

(2) *there exists an open subset $U \subset Z$ such that*

$$\tau_{N(Z)-1}(Z - t_z(Z), (0, 0)) = 0, \quad N(Z - t_z(Z)) = N(Z)$$

for all $z \in U$ and

$$\dim\{\tau_{N(Z)}(Z - t_z(Z), (0, 0)) : z \in U\} = 1.$$

Proof. Let the closure of Z be cut out in $\mathbb{G}_a^2$ by a polynomial equation $P(x, y)$. Pick coordinates x, y, u, v on each of the factors of $\mathbb{G}_a^4$. Let $Z' \subset \mathbb{G}_a^4$ be the set cut-out by the equation $P(u, v)$, and let $Z'' \subset Z'$ be cut out in Z' by the equation $P(x-u, y-v)$. A fibre of Z'' over $(u_0, v_0) \in \mathbb{G}_a^2$ is thus the closure of $t_{(u_0,v_0)}Z$. We identify the completion of the local ring of Z' at $(0, 0, 0, 0)$ with $k[\![x, y, u]\!]$ via the projection on the first three coordinates. One readily sees that the local equation of Z'' in Z' at $(0, 0, 0, 0)$ is

$$g = y - f(x-u) + f(u) \in k[\![y, x, u]\!].$$

By Lemma 4.28, if $\dim s_n(Z) = 1$ and $\dim s_m(Z) = 0$ for all $m < n$, then a monomial of the form $u^l x^n$ for some $l > 0$ occurs with a non-zero coefficient in g. Substituting the expression for f into the above expression for g, we get

$$g = y - \sum_{j>0} b_j \Bigg(\sum_{0<k\leq j} \binom{j}{k} (-1)^{j-k} u^{j-k} x^k \Bigg),$$

If f contains a term ax^n, then its contribution to g are the terms

$$-a\binom{n}{1} u x^{n-1},\ a\binom{n}{2} u^2 x^{n-2},\ \ldots,\ a\binom{n}{n-1} (-1)^{n-1} u^{n-1} x,$$

and if $n = p^j$, then the coefficients vanish in a characteristic p field since $\binom{p^j}{k} \equiv 0 \bmod p$ for $k < p^j$. Raising to p^j-th power is additive modulo p, so if $p \nmid m$, then $(x+u)^{mp^j} \equiv ((x+u)^m)^{p^j} \bmod p$, and since $\binom{mp^j}{lp^j}$ is the coefficient of $u^l x^{m-l}$ in $(x+u)^m$ raised to the power p^j, $\binom{mp^j}{lp^j} = \binom{m}{l}^{p^j}$.

In particular, if $n = mp^j$ and $p \nmid m$, then $\binom{mp^j}{p^j} = m^{p^j} \not\equiv 0 \bmod p$, so the term $u^{p^j} x^{(m-1)p^j}$ has a non-zero coefficient in g, which proves the first claim (and, more generally, $\binom{mp^j}{lp^j} \not\equiv 0 \bmod p$ if $p \nmid m, l$).

Now let W be the closed subvariety of $\mathbb{G}_a^2 \times Z \subset \mathbb{G}_a^4$ such that the fibre of W over a point $z \in Z$ is $Z - t_z(Z)$. Applying the reasoning from the proof of Proposition 3.9, we get that the local equation of W is

$$h = y - f(x-u) + f(u) + f(x),$$

Arguing as above, we see that by Lemma 4.28 to prove the second claim it suffices to show that a monomial of the form $u^j x^{N(Z)}$, $j > 0$, occurs in h with a non-zero coefficient while no monomials of the form $u^j x^n$, $j \geq 0$, occur in h with non-zero coefficients for $n < N(Z)$ (so that $h \equiv y \bmod x^{N(Z)}$) and for $n = mp^j$, $p \nmid m$, $p^j < N(Z)$ (so that $N(Z - t_z(Z)) = N(Z)$).

Since the term $x^{mN(Z)}$ occurs in f with a non-zero coefficient, its contribution to $f(x-u) - f(u) - f(x)$ consists of the terms $u^{lN(Z)} x^{N(Z)}$, $0 < l < m$ with non-zero coefficients. And since by definition of $N(Z)$, x^n occurs in f with a non-zero coefficient if and only if $n = p^j$ or $n = mp^j$ and $p^j \geq N(Z)$, the statement from the previous paragraph is fulfilled. ∎

Lemma 4.30. *Assume that the ground field is of positive characteristic p. Let G be an algebraic group such that the connected component of the identity G_0 is isomorphic to $\mathbb{G}_a$. Let $Z \subset G^2$ be a curve that is not a Boolean combination of cosets of subgroups of G^2. Then there exist a family of pure-dimensional curves $W \subset G^2 \times Z$ definable in (G, Z), and an irreducible component $Z_0 \subset Z \cap G_0^2$ of dimension 1, such that*

$$\dim\{\tau_1(W_z, (0,0)) : z \in Z_0\} = 1.$$

Proof. Replacing Z by a shift and swapping coordinates, if needed, we may assume that there exists a pure-dimensional irreducible curve $Z_0 \subset Z$ such that $(0,0) \in Z_0$, Z_0 is not contained in a coset, and such that the projection on the first coordinate in $\mathbb{G}_a^2$ is étale in a neighbourhood of $(0,0) \in Z$. Consider the definable family of curves $Y \subset G^2 \times Z$ with the fibres

$$Y_z = Z - t_z(Z), \quad z \in Z.$$

For $z \in Z_0$, the slope of Y_z at $(0,0)$ is well defined. By Lemma 4.29,

$$\dim\{\tau_{N(Z)}(Y_z, (0,0)) : z \in Z_0\} = 1,$$

and $\tau_{N(Z)-1}(Y_z, (0,0)) = 0$ for z in a dense subset of Z_0. It follows that there exists $z' \in Z_0$ such that $\tau_{N(Z)}(Y_{z'}, (0,0)) \neq 0$. It also follows from the characterization of $N(Z)$ in Lemma 4.29 that the local equation of Y_z is an $N(Z)$-th power of a power series, and therefore $p_2 \colon \mathbb{G}_a^4 \to \mathbb{G}_a$ restricted to $Y' = \mathrm{Fr}_{p_2}^m(Y)$ is generically unramified, where m is such that $p^m = N(Z)$.

In particular, for generic $z \in Z_0$, the restriction of the projection on the second coordinate of $\mathbb{G}_a^2$ to Y_z' is generically unramified and

$$\dim\{\tau_1(Y_z', (0,0)) : z \in Z_0\} = 1.$$

Define the family W by putting $W_z = Y_z \circ Y_{z'}^{-1}$. Then by Lemma 4.16, $W_z = Y_z' \circ (Y_{z'}')^{-1}$ for z in a dense subset of Z_0 and

$$\tau_1(W_z, (a_2, b_2)) = \tau_1(Y_z', a) \circ (\tau_1(Y_z', b))^{-1}$$

for all points $a = (a_1, a_2), b = (b_1, b_2) \in Y_z$ such that $a_1 = b_1$ and such that the right-hand side makes sense. The statement of the lemma follows. ∎

Lemma 4.31. *Let G be a one-dimensional algebraic group whose connected component of the identity G_0 is isomorphic to $\mathbb{G}_m$. Let $Z \subset G^2$ be a curve that is not a Boolean combination of cosets of subgroups of G^2. Then either there exists an irreducible pure-dimensional curve $Z_0 \subset Z$ such that $\dim s_1(Z_0) = 1$, or there exists a group definable in (G, Z) whose connected component of the identity is not isomorphic to $\mathbb{G}_m$.*

Proof. Pick a local coordinate systems on $\mathbb{G}_m$, uniformly, as in the proof of Lemma 4.30. Assume that $\dim s_1(Z) = 0$, and so $s_1(Z_i)$ is a singleton for each one-dimensional irreducible component $Z_i \subset Z$. Let Z_0 be one of the irreducible components of Z that is not contained in a coset, then there exists a smallest $n > 1$ such that $\dim s_n(Z_0) = 1$.

Then, by the same reasoning as in the proof of Lemma 4.19, we may consider the family $Y \subset G^2 \times Z$ by putting $Y_z = t_z(Z) \circ (t_{z_0}(Z))^{-1}$ for some $z_0 \in Z_0$, so that $\{\tau_1(Y_z, (0,0)) : z \in Z_0\}$ almost coincides with

$$\mathrm{Ker}(\mathrm{Aut}(k[x]/(x^{n+1})) \to \mathrm{Aut}(k[x]/(x^n)) \cong \mathbb{G}_a.$$

The definable family Y can be used to construct a group configuration as in the proof of Theorem 4.25, and therefore a group is interpretable in (G, Z). By Proposition 4.26, the connected component of the identity of this group is not isomorphic to $\mathbb{G}_m$. ∎

We can finally interpret the field.

Theorem 4.32. *Let G be a one-dimensional algebraic group over an algebraically closed field, $Z \subset G^2$ a one-dimensional constructible subset that is not a Boolean combination of cosets. Then $(G, \cdot, Z)$ interprets a field.*

Proof. Let G_0 be the connected component of the identity e of G. If $G_0 = \mathbb{G}_a$ or G_0 is an elliptic curve, then by Lemmas 4.27, 4.30, there exist a definable family $Y \subset G^2 \times S$ of curves, S strongly minimal, and an irreducible locally closed set $S_0 \subset S$ such that there is a unique family of branches α of $Y_0 = Y \cap S_0$ at $(e, e) \in G^2$, and such that $\tau_1(Y_s, \alpha_s)$ is not constant as s ranges in S_0. If $G = \mathbb{G}_m$, by Lemma 4.31, either such a family exists, or a definable one-dimensional group G' with the connected component of the identity not isomorphic to $\mathbb{G}_m$ (and therefore isomorphic to either $\mathbb{G}_a$ or to an elliptic curve) is interpretable in $(G, \cdot, Z)$, and we may prove the theorem for the structure induced on G'. We, therefore, may continue with the assumption that such a family exists. Clearly, $Y_0 \to S_0$, and we may assume that S_0 is smooth at the price of possibly shrinking S_0. Further shrinking S_0, we can ensure $Y_0 \to S_0$ to be flat (by Fact 3.10). Let K be a field of infinite transcendence degree over the base field k. We identify first order slopes, which are truncated polynomials in $K[\varepsilon]/(\varepsilon^2)$ divisible by ε, with K, and we will use multiplicative notation for composition. We will freely use the remark after Definition 3.3 and Lemma 3.2, referring to branches of suitable pure-dimensional subfamilies of definable families of curves when we speak about branches of definable families of curves.

From this point on, modulo Proposition 3.15, the argument is standard.

Take $a_1, a_2, b_1, b_2, u \in S_0(K)$ generic and pairwise independent. Let $c_1, c_2 \in S_0(K)$ be such that

$$\tau_1(Y_{c_1}, \alpha_{c_1}) = \tau_1(Y_{a_1}, \alpha_{a_1})\tau_1(Y_{b_1}, \alpha_{b_1}),$$
$$\tau_1(Y_{c_2}, \alpha_{c_2}) = \tau_1(Y_{a_2}, \alpha_{a_2})\tau_1(Y_{b_1}, \alpha_{b_1}) + \tau(Y_{b_2}, \alpha_{b_2}).$$

This is possible since the image of the function $s \mapsto \tau_1(Y_s, \alpha_s)$ for s ranging in S_0 is of dimension 1, and the values of slopes on the right-hand side of the equations above are generic in $\mathrm{End}(k[x]/(x^2))$ for generic pairwise independent values of parameters. Therefore,

$$\tau_1(Y_{a_1}, \alpha_{a_1})\tau_1(Y_{b_1}, \alpha_{b_1}) \quad \text{and} \quad \tau_1(Y_{a_2}, \alpha_{a_2})\tau_1(Y_{b_1}, \alpha_{b_1}) + \tau_1(Y_{b_2}, \alpha_{b_2})$$

are generic, and c_1, c_2 as required can be found in $S_0(K)$. Let z, v be such that

$$\tau_1(Y_z, \alpha_z) = \tau_1(Y_{a_1}, \alpha_{a_1})\tau_1(Y_u, \alpha_u) + \tau_1(Y_{a_2}, \alpha_{a_2}),$$
$$\tau_1(Y_v, \alpha_v) = \tau_1(Y_{b_1}, \alpha_{b_1})^{-1}\tau_1(Y_u, \alpha_u) - \tau_1(Y_{b_2}, \alpha_{b_2}).$$

By a similar reasoning, z, v are generic. It also follows from the way c_1, c_2, z, v were defined that

$$\tau_1(Y_z, \alpha_z) = \tau_1(Y_{c_1}, \alpha_{c_1})\tau_1(Y_v, \alpha_v) + \tau_1(Y_{c_2}, \alpha_{c_2}).$$

We will now show that (c_1, c_2) is algebraic over (a_1, a_2) and (b_1, b_2) in the sense of $(G, \cdot, Z)$. By Propositions 3.7 and 3.9,

$$\tau_1(Y_{a_1} \circ Y_{b_1}, \alpha_{a_1} \circ \alpha_{b_1}) = \tau_1(Y_{a_1}, \alpha_{a_1})\tau_1(Y_{b_1}, \alpha_{b_1}),$$
$$\tau_1(Y_{a_2} \circ Y_{b_1} + Y_{b_2}, \alpha_{a_2} \circ \alpha_{b_1} + \alpha_{b_2}) = \tau_1(Y_{a_2}, \alpha_{a_2})\tau_1(Y_{b_1}, \alpha_{b_1}) + \tau_1(Y_{b_2}, \alpha_{b_2}).$$

Let $l_1 = \max_{c_1, a_1, b_1 \in S_0} \#(Y_{c_1} \cap (Y_{a_1} \circ Y_{b_1}))$ and $l_2 = \#(Y_{c_2} \times_{G^2} (Y_{a_2} \circ Y_{b_1} + Y_{b_2}))$ for $a_1, a_2, b_1, b_2, c_1, c_2 \in S_0$ generic and independent. Since the number of intersection points is a $(G, \cdot, Z)$-definable property, it does not matter what particular parameters a_i, b_i, c_i we take as long as they are generic and independent (in the sense of $(G, \cdot, Z)$). By Lemma 3.13 and Proposition 3.15, the (M, X)-definable set

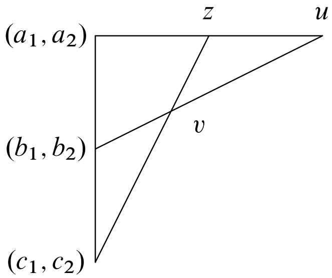
$$\{w \in S_0 : \dim(Y_w \cap (Y_{a_1} \circ Y_{b_1})) = 1 \text{ or } \#(Y_w \cap (Y_{a_1} \circ Y_{b_1})) < l_1\}$$

contains c_1 and by definition of l_1 is finite. By Lemma 3.13 and Proposition 3.15 again, the (M, X)-definable set

$$\{w \in S_0 : \dim(Y_w \cap (Y_{a_2} \circ Y_{b_1} + Y_{b_2})) = 1 \text{ or } \#(Y_w \cap (Y_{a_2} \circ Y_{b_1} + Y_{b_2})) < l_2\}$$

contains c_2 and by definition of l_2 is finite.

Arguing similarly, by application of Lemma 3.13 and Proposition 3.15, we deduce that for all lines in the diagram



each vertex is in the algebraic closure of two other collinear vertices, and so this constitutes a group configuration. Therefore, by Fact 4.4, there exists a two-dimensional group definable in $(G, \cdot, Z)$ that acts transitively on a one-dimensional set.

The conditions of the Fact 4.6 are verified as well: for instance, for the uppermost line, $B = \{\tau_1(Y_{a_1}, \alpha_{a_1}), \tau_1(Y_{a_2}, \alpha_{a_2})\}$ is by construction a canonical base of the type $\text{tp}(\tau_1(Y_z, \alpha_z), \tau_1(Y_u, \alpha_u)/B)$ in the full Zariski structure. Since the natural morphism

$S_0 \to \mathrm{Aut}(k[\varepsilon]/(\varepsilon^2)), s \mapsto \tau_1(Y_s, \alpha_s)$ has finite fibres, a canonical base of $\mathrm{tp}(z, u/a_1, a_2)$ is inter-algebraic with $\{a_1, a_2\}$ in the full Zariski structure. Since passing to the reduct can only enlarge a canonical base, the canonical base of $\mathrm{tp}(z, u/a_1, a_2)$ is inter-algebraic with $\{a_1, a_2\}$. The same argument applies to $\mathrm{tp}(u, v/b_1, b_2)$ and $\mathrm{tp}(z, v/c_1, c_2)$.

By Fact 4.7, the group G is isomorphic to the affine group $\mathbb{G}_a(k) \rtimes \mathbb{G}_m(k)$ of an infinite definable field k. ∎

In order to apply the above results, we need the following, which is a well-known model theoretic folklore (see, [8, Theorem 7.2]). We give a proof specialised to the case where we need it.

Lemma 4.33. *Let G be a strongly minimal group interpretable in $\mathcal{M}$. Then there exists a strongly minimal $Z \subseteq G^2$ that is not a finite Boolean combination of cosets of definable subgroups.*

Proof. To simplify the discussion, let us call, for the purposes of this proof only, subsets of G that are finite Boolean combinations of cosets of G^n (any n) G-linear. By strong minimality, G is in finite-to-finite correspondence with $\mathcal{M}$ (this follows, in general, from the fact that $\mathcal{M}$ is unidimensional. In the present setting, G can be assumed to have been obtained from Theorem 4.4 using a one-dimensional group configuration, so the existence of a finite-to-finite correspondence follows from the statement). It follows that G is not locally modular, as the image of any ample family of one-dimensional subsets of M^2 under this finite-to-finite correspondence is an ample family in G^2.

Since G is not locally modular, it admits by [13, Proposition 3.21] a nearly faithful ample family of generically strongly minimal curves $X \to T \subseteq G^2 \times T$ of dimension 3 (i.e., $\dim(T) = 3$). Let G^0 denote the $\mathfrak{M}$-connected component of G. Let $t \in T$ and $x_0 \in G^0$ be independent $\mathfrak{M}$-generics. Let y_0 be such that $(x_0, y_0) \in X_t$ and assume that $y_0 \in gG^0$ for some $g \in G$ (that we can choose independent of (x_0, y_0)). Then $X_t[g] := \{(x, y) : (x, gy) \in (G^0)^2 \cap X_t\}$ is an $\mathfrak{M}$-definable curve in $(G^0)^2$ and $X[g] := \{X_t[g] : t \in T\}$ is a definable family of curves in $(G^0)^2$. Since G/G^0 is finite and X is nearly faithful, the correspondence $X_t \mapsto X_t[g]$ is finite-to-one and on a generic subset of T. Therefore, $X[g]$ is readily checked to be a three-dimensional, nearly faithful ample family of curves in $(G^0)^2$. Moreover, if X_t is G-linear (for some $t \in T$), then so is $X_t[g]$. So it will suffice to show that X can be chosen so that $X_t[g]$ is not G-linear for generic $t \in T$.

If G^0 is $\mathfrak{M}$-definably isomorphic to either $\mathbb{G}_m$ or to an elliptic curve, $\mathcal{E}$, then for generic $t \in T$, $X_t[g]$ is not G-linear, since there are no definable families of subgroups of $\mathbb{G}_m^2$ or of $\mathcal{E}^2$. Let us elaborate: assume towards a contradiction that for generic $t \in T$ the curve X_t is G-linear. So $X_t[g]$ is also G-linear for all such t. In this setting, there are finitely many $\emptyset$-definable one-dimensional subgroups $H_1, \ldots, H_k$ of $(G^0)^2$ such that $X_t[g]$ coincides, up to a finite set, with a union of cosets of the H_i. Since $(G^0)^2/H_i$ is one-dimensional for all i, near faithfulness of $X[g]$ implies that $X[g]$ is, at most, one-dimensional, a contradiction.

So we are reduced to the case where G^0 is $\mathfrak{M}$-definably isomorphic to $\mathbb{G}_a$. This case is dealt with in Lemma 4.35. ∎

Lemma 4.34. *Let $X \subset \mathbb{G}_a^n$ be a hypersurface, let $a = (a_1, \dots, a_n) \in X$ be a general enough point, and let $x_1, \dots, x_n$ be local coordinates on $\mathbb{G}_a^n$ near a. For a value $b \in \mathbb{G}_a$ and a fixed coordinate i, consider the level set*

$$X_b = \{(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) \in \mathbb{G}_a^{n-1} : (y_1, \dots, b, \dots, y_n) \in X\}.$$

Assume that a belongs to a unique irreducible component of X_{a_i}. If X_b contains an irreducible component that is a coset of a subgroup of $\mathbb{G}_a^{n-1}$ for all b in a neighbourhood of a_i, then the local equation of X at a has the form

$$f = \sum_{j \neq i} \sum_{l > 0} \sum_{m \geq 0} c_{jlm} x_i^l x_j^{p^m}$$

for some coefficients $c_{jlm} \in k$.

Proof. The proof is similar to the proof of Lemma 4.28. Choose a curve $Z \subset X$ containing a. Consider the following commutative diagram:

$$\begin{array}{ccc} X' := X \times_{\mathbb{G}_a} Z & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \mathbb{G}_a^{n-1} \times Z & \longrightarrow & \mathbb{G}_a^n, \end{array}$$

where $\mathbb{G}_a$ is the i-th factor in $\mathbb{G}_a^n$. Consider formal completions of the elements of the diagram along Z:

$$\begin{array}{ccc} \widehat{X}' & \longrightarrow & \widehat{X} \\ \big\downarrow & & \big\downarrow \\ \widehat{Y}' & \longrightarrow & \widehat{Y}. \end{array}$$

Since a is general enough, $\widehat{\mathcal{O}_{Z,a}}$ is isomorphic to a power series ring, and also the image of Z under some power of a relative Frobenius is generically étale over the i-th $\mathbb{G}_a$. Choose local coordinates $x_1, \dots, x_n$ on $\mathbb{G}_a^n$ at a and introduce local coordinates on $\mathbb{G}_a^{n-1} \times Z$ at a as follows: let $\overline{x}_1, \dots, \overline{x}_{i-1}, \overline{x}_{i+1}, \dots, \overline{x}_n$ be obtained from $x_1, \dots, x_{i-1}$, $x_{i+1}, \dots, x_n$ by translation, and let z be such that $z^{p^M} = x_i$, where M is the degree of inseparability of the field extension $k((z)) : k((x_i))$. Then

$$k[\![x_1, \dots, x_n]\!] \subset k[\![\overline{x}_1, \dots, \overline{x}_{i-1}, z, \overline{x}_{i+1}, \dots, \overline{x}_n]\!],$$

and we have

$$\overline{x}_j = x_j - z_j, \quad \text{for some } z_j \in k[\![x_i]\!].$$

The formal scheme $\widehat{X}'$ is cut out in $\mathbb{G}_a^{n-1} \times Z$ by

$$\overline{f} \in k[\![\overline{x}_1, \dots, \widehat{x}_i, \dots, \overline{x}_n]\!] \otimes k[Z].$$

If all level sets X_b for b in a neighbourhood of a_i are cosets of a subgroup of $\mathbb{G}_a^{n-1}$, then the evaluation of $\overline{f}$ at such points b is an additive polynomial. In particular, the image of $\overline{f}$ in $k[\![\overline{x}_1, \dots, \widehat{\overline{x}}_i, \dots, \overline{x}_n]\!] \otimes \mathcal{O}_{Z,a}$ is of the form

$$\sum_{j \neq i} \sum_{m \geq 0} \overline{c}_{jm} \overline{x}_j^{p^m}, \quad \overline{c}_{jm} \in \mathcal{O}_{Z,a}.$$

Then $\widehat{X}$ is cut out in $\widehat{Y}$ by the image f of $(\overline{f})^{p^m}$ in $k[\![x_1, \dots, x_n]\!]$. After the change of coordinates, one has

$$f = \sum_{j \neq i} \sum_{m \geq 0} \overline{c}_{jm}^{\,p^M} (x_j - z_j)^{p^{m+M}},$$

and the statement of the lemma follows. ∎

Lemma 4.35. *Let G be an algebraic group such that the connected component of the identity of G is isomorphic to $\mathbb{G}_a^n$. If $X \subset G^n$ is a constructible set which is not a Boolean combination of cosets of subgroups of G^n, then there exists a one-dimensional set $W \subset G^2$ definable in $\mathcal{M} := (G, X)$ that contains a locally closed irreducible subset W_0 that is not a coset of a subgroup of $\mathbb{G}_a^2$.*

Proof. We proceed by downward induction on n.

We may assume that X is $\mathcal{M}$-stationary. We may further assume that there is no $\mathfrak{M}$-component $X' \subseteq X$ with $\dim(X) = \dim(X')$ whose closure is a coset of a subgroup of $\mathbb{G}_a^n$. Indeed, if $G + a$ almost coincided with X', we would get that the generic stabiliser of X, $\mathrm{Stab}^*(X) := \{g : \dim((X+g) \triangle X) < \dim(X)\}$ (where $\triangle$ denotes the symmetric difference) is $\mathcal{M}$-definable, this group coincides with G and is $\mathcal{M}$-definable. Therefore, $X \setminus G + a$ is $\mathcal{M}$-definable, is not a Boolean combination of cosets of subgroups, and $\dim(X \setminus G + a) < \dim(X)$, so we may conclude by a further induction on dimension.

Let $X_0 \subset X$ be a locally closed subset of maximal dimension that is not a coset of a subgroup. By what we have just said, $\dim(X_0) = \dim(X)$.

Consider the projections of X on various products $\mathbb{G}_a^{n-1}$ of the factors of the direct product $\mathbb{G}_a^n$. If the image of any of these projections is not a coset of a subgroup, we conclude by induction. Otherwise, images of X_0 under all such projections are cosets of some subgroups. Since X_0 is not a coset, the fibres of all these projections restricted to X_0 are finite, so the image is a coset of a subgroup H such that $\dim H = \dim X_0 = m$. Assume for definiteness that H is the image of the projection $\mathbb{G}_a^n \to \mathbb{G}_a^{n-1}$ that forgets the last coordinate. Consider now an $\mathfrak{M}$-definable $\pi\colon H \to \mathbb{G}_a^m$ with finite fibres. By quantifier elimination for algebraically closed fields, π has the form

$$\pi = (\pi_1, \dots, \pi_m)\colon H \to \mathbb{G}_a^m, \quad \pi_i = \mathrm{Fr}^{-a_i} \circ \tau_i,$$

where a_i are positive integers and τ_i are morphisms of algebraic groups (for some choice of identification of H with a closed subgroup of $\mathbb{G}_a^n$). In particular, if $\tau = (\tau_1, \dots, \tau_m)$, then $(\tau \times \mathrm{id})(X)$, where id is applied to the last coordinate, is not a coset, and all the more $(\pi \times \mathrm{id})(X_0)$ is not one. Indeed, if $(\tau \times \mathrm{id})X_0$ were a coset H', then X_0 would have to be one of the irreducible components of the pre-image $(\tau \times \mathrm{id})^{-1} H'$, which are all cosets.

Since X_0 is a top-dimensional irreducible component in X, $\pi(X_0)$ is an irreducible component of the image of X under a definable projection $G^n \to G^m$. Therefore, unless $m = n - 1$, we may also make an induction step.

So we are reduced to the case where X_0 is a hypersurface in $\mathbb{G}_a^n$. Pick a generic point $a = (a_1, \ldots, a_n) \in X_0$. Let $f \in k[\![x_1, \ldots, x_n]\!]$ be the generator of the ideal cutting out the germ of X_0 in the completed local ring $\widehat{\mathcal{O}_{\mathbb{G}_a^n, a}}$. Since X_0 is not a coset, f is not an additive series, that is, it does not consist only of monomials containing only p^j-th, $j \geq 0$, powers of variables. By Lemma 4.34, there exists a coordinate i such that level sets $(X_0)_b$ with respect to this coordinate are not cosets for b in an open neighbourhood of a_i – if it were not the case, f would be an additive series.

Clearly, the level set of X_0 is contained in the corresponding level set of X, which is $\mathcal{M}$-definable and contained in G^{n-1}. This finishes the induction step. ∎

Remark. In the above proof, it is not hard to see that if we obtain a two-dimensional nearly faithful family $X \to T$ such that each X_t contains (up to a finite set) a curve of the form $a_t x + b_t$, then X can be used directly to construct a field configuration.

We can now sum up everything to obtain the main result of this paper.

Theorem 4.36. *Let M be an algebraic curve, and let $X \subset M^2 \times T \subset M^2 \times M^l$ be an ample family of curves. Then $\mathcal{M} = (M, X)$ interprets a field.*

Proof. By Corollary 4.13, we may assume that M is smooth, that k is of infinite transcendence degree and that X is a nearly faithful family of generically strongly minimal sets. Thus we can apply Theorem 4.25, allowing us to conclude that $\mathcal{M}$ interprets a strongly minimal group G. By [39, Theorem 4.13], G is an algebraic group. The group G is in an $\mathcal{M}$-definable finite-to-finite correspondence with M, so it is a one-dimensional algebraic group. Moreover, any $\mathcal{M}$-definable ample family of curves in M^2 maps through this correspondence into an ample family of curves in G^2 of the same dimension. So G is not locally modular. By [24, Theorem 4.1 (b)], there is some definable $Z \subseteq G^n$ that is not a finite Boolean combination of cosets. By Lemma 4.33, we may assume that, in fact, $Z \subseteq G^2$. Therefore, we may apply Theorem 4.32 to get the desired conclusion. ∎

Remark. The field obtained in the conclusion of our main theorem is definably isomorphic to k by [39, Theorem 4.15]. In particular, since k is a pure field, so is the field we reconstruct.

As a corollary, we obtain a standard generalisation.

Corollary 4.37. *Let M be a constructible set in an algebraically closed field K. Let $\mathcal{M}$ be a rank-preserving reduct of the full Zariski structure M, i.e., $\mathrm{RM}_{\mathcal{M}}(X) = \mathrm{RM}_K(X)$ for all $\mathcal{M}$-definable $X \subseteq M^n$ (any n). Then $\mathcal{M}$ interprets a definably isomorphic copy of K if and only if $\mathcal{M}$ is not* 1*-based.*

Proof. Since $\mathcal{M}$ is a rank-preserving reduct, any strongly minimal structure interpretable in $\mathcal{M}$ is one-dimensional. By our main theorem, it is enough, therefore, to show that $\mathcal{M}$ is not 1-based if and only if it interprets a non-locally modular strongly minimal type. This follows readily form Beuchler's dichotomy, [5], combined with [6]. The only point to note is that because $\mathcal{M}$ is interpretable in K, it has finite Morley rank (in fact, $\mathrm{RM}_{\mathcal{M}}(M) \leq \mathrm{RM}_K(M)$, by the very definition of Morley rank), and since K is strongly minimal, also the K-U-rank of M is finite, and thus also the U-rank of $\mathcal{M}$ is finite. ■

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