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The Boltzmann equation for plane Couette flow

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Abstract. In the paper, we study the plane Couette flow of a rarefied gas between two parallel infinite plates at $y = \pm L$ moving relative to each other with opposite velocities $(\pm\alpha L, 0, 0)$ along the x-direction. Assuming that the stationary state takes the specific form of $F(y, v_x - \alpha y, v_y, v_z)$ with the x-component of the molecular velocity sheared linearly along the y-direction, such steady flow is governed by a boundary value problem for a steady nonlinear Boltzmann equation driven by an external shear force under the homogeneous nonmoving diffuse reflection boundary condition. In the case of the Maxwell molecule collisions, we establish the existence of spatially inhomogeneous nonequilibrium stationary solutions to the steady problem for any small enough shear rate $\alpha > 0$ via an elaborate perturbation approach using Caflisch's decomposition together with Guo's $L^\infty \cap L^2$ theory. The result indicates a polynomial tail at large velocities for the stationary distribution. Moreover, the large time asymptotic stability of the stationary solution with exponential convergence is also obtained and as a consequence the nonnegativity of the steady profile is justified.

Keywords. Boltzmann equation, plane Couette flow, existence, dynamical stability

Contents

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1. Intoduction

The steady state of a rarefied gas between two parallel plates with the same temperatures and opposite velocities is one of the most fundamental boundary value problems in kinetic theory; see the books of Kogan [28], Cercignani [11], Garzó–Santos [22], and Sone [33]. In particular, numerical analysis of the plane Couette flow of rarefied gas on the basis of the nonlinear Boltzmann equation has been extensively conducted in the physics literature; see [29–31, 34, 35]. On the other hand, the mathematical study of this problem, even in the case when there is a temperature gap between two plates and a constant external force parallel to the boundaries, has been carried out by Esposito–Lebowitz–Marra [18, 19], proving the hydrodynamic description of the steady rarefied gas flow via approximation of the corresponding compressible Navier–Stokes equations with no-slip boundary condition. The result in [19] for the hard sphere model was later extended in [13] to the case of hard intermolecular potentials with Grad's angular cutoff as well as to the Maxwell molecule case for which only the polynomial decay of the stationary solution for large velocities is obtained compared to the exponential decay for the hard sphere model. In addition, closely related to the plane Couette flow, the stationary Boltzmann equation for rarefied gas in a Couette flow setting between two coaxial rotating cylinders was also extensively studied by Arkeryd–Nouri [3,4] in the fluid dynamic regime; see also a recent work [1] for further investigation of the ghost effect induced by curvature.

The current study of the plane Couette flow with boundaries is motivated by the previous work [15] by the first two authors on uniform shear flow via the Boltzmann equation without boundaries. We refer the readers to [9, 12, 21, 27, 36, 37] and references therein for more details on uniform shear flow. In particular, in recent significant progress [9], Bobylev–Nota–Velázquez studied the self-similar asymptotics of large time solutions for the Boltzmann equation with a general deformation of small strength and also showed that the self-similar profile can have finite polynomial moments of higher order as long as the deformation strength is small. In this paper, we will take into account the effect of shear force induced by the relative motion of the boundaries. We hope that the current study can shed some light on the relation between the Couette flow with boundary and the uniform shear flow without boundary. A rigorous justification of the behavior of solutions as $L \to \infty$ is left for future research.

To specify the problem, we consider the rarefied gas between two parallel infinite plates with the same uniform temperature $T_0 > 0$; the plate at $y = +L$ is moving with velocity $(U_+, 0, 0)$ and $U_+ = \alpha L$ and the other at $y = -L$ is moving with velocity $(U_-, 0, 0)$ and $U_- = -\alpha L$, where $\alpha > 0$ is a parameter for the shear rate; see Figure 1

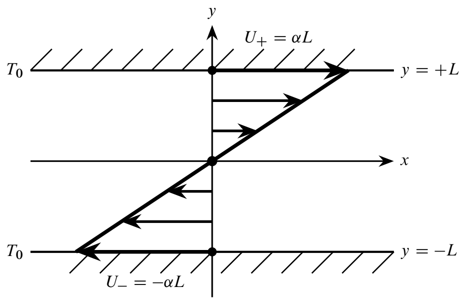

Fig. 1. Plane Couette flow

below. Moreover, we assume that the gas molecules are of Maxwellian type and reflected diffusively on the plates $y = \pm L$.

Let $F = F(y, v) \geq 0$ be the unknown time-independent density distribution function of gas particles with velocity $v = (v_x, v_y, v_z) \in \mathbb{R}^3$ located at $y \in (-L, L)$ along the vertical direction with slab symmetry in the horizontal (x, z)-plane in space. The motion of such rarefied gas can be governed by the steady Boltzmann equation

$$v_y \partial_y F = \frac{1}{\mathrm{Kn}} Q(F, F) \tag{1.1}$$

subject to the diffuse reflection boundary conditions at $y = \pm L$, i.e.,

$$F(\pm L, v) = \mathcal{M}_{T_0}(v_x - U_\pm, v_y, v_z) \int_{v_y \lessgtr 0} F(\pm L, v)|v_2|\, dv \quad \text{for } v_y \gtrless 0, \tag{1.2}$$

and with a given total mass

$$\frac{1}{2L} \int_{-L}^{L} \int_{\mathbb{R}^3} F(y, v)\, dv\, dy = M \tag{1.3}$$

for some positive constant $M > 0$. Here, the dimension parameter $\mathrm{Kn} > 0$ is the Knudsen number given by the ratio of the mean free path length to the typical length and $\mathcal{M}_{T_0} = \mathcal{M}_{T_0}(v)$ associated with the uniform wall temperature T_0 at $y = \pm L$ is a global Maxwellian of the form

$$\mathcal{M}_{T_0}(v) = \frac{1}{2\pi T_0^2} e^{-\frac{|v_x|^2 + |v_y|^2 + |v_z|^2}{2T_0}}, \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3.$$

For the Maxwell molecule model, the collision operator Q, which is bilinear and acts only on the velocity variable, takes the form

$$Q(F_1, F_2)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\cos\theta)[F_1(v_*')F_2(v') - F_1(v_*)F_2(v)]\, d\omega\, dv_*, \tag{1.4}$$

where the velocity pairs (v_*, v) and (v_*', v') satisfy the relation

$$v_*' = v_* - [(v_* - v) \cdot \omega]\omega, \quad v' = v + [(v_* - v) \cdot \omega]\omega, \tag{1.5}$$

denoting the ω-representation according to conservation of momentum and energy in the elastic collision, i.e., $v_* + v = v'_* + v'$ and $|v_*|^2 + |v|^2 = |v'_*|^2 + |v'|^2$. Throughout the paper, we assume that the collision kernel $B_0(\cos\theta)$ with $\cos\theta = (v - v_*)\cdot\omega/|v - v_*|$, depending only on the angle θ between the relative velocity $v - v_*$ and ω, satisfies Grad's angular cutoff assumption

$$0 \le B_0(\cos\theta) \le C|\cos\theta| \tag{1.6}$$

for a generic constant $C > 0$.

In this paper, for the boundary value problem (1.1)–(1.3) with finite Knudsen number, we look for stationary solutions of the specific form

$$F_{\mathrm{st}}(y, v_x - \alpha y, v_y, v_z), \tag{1.7}$$

where the horizontal molecular velocity $v_x - \alpha y$ is sheared linearly along the y-direction. After plugging (1.7) into (1.1)–(1.3) and normalizing L, M and T_0 to be 1 for simplicity, the stationary distribution function F_{st} is determined by the following boundary value problem:

$$\begin{cases} v_y \partial_y F_{\mathrm{st}} - \alpha v_y \partial_{v_x} F_{\mathrm{st}} = Q(F_{\mathrm{st}}, F_{\mathrm{st}}), \quad y \in (-1,1),\ v = (v_x, v_y, v_z) \in \mathbb{R}^3, \\ F_{\mathrm{st}}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi}\,\mu \displaystyle\int_{v_y \gtrless 0} F_{\mathrm{st}}(\pm 1, v)|v_y|\,dv, \quad v \in \mathbb{R}^3, \\ \dfrac{1}{2}\displaystyle\int_{-1}^{1}\int_{\mathbb{R}^3} F_{\mathrm{st}}(y, v)\,dv\,dy = 1, \end{cases} \tag{1.8}$$

with the global Maxwellian $\mu = (2\pi)^{-3/2} e^{-|v|^2/2}$. This paper aims to establish the existence of solutions to the above boundary value problem (1.8) for any small enough shear rate $\alpha > 0$, as well as its large time asymptotic stability.

To solve (1.8), we will apply the perturbation approach by taking the shear rate as a small parameter. If $\alpha = 0$, then $F_{\mathrm{st}} = \mu$ is the unique equilibrium solution to the boundary value problem (1.8). However, for $\alpha > 0$, the external shear force drives the rarefied gas far from the equilibrium. Precisely, we set

$$F_{\mathrm{st}} = \mu + \sqrt{\mu}\,\{\alpha G_1 + \alpha^2 G_R\} \tag{1.9}$$

with

$$\int_{-1}^{1}\int_{\mathbb{R}^3} \sqrt{\mu}\, G_1\,dv\,dy = \int_{-1}^{1}\int_{\mathbb{R}^3} \sqrt{\mu}\, G_R\,dv\,dy = 0. \tag{1.10}$$

By plugging (1.9) into (1.8) and comparing coefficients of the equation in the order of α, we obtain the equation for G_1:

$$v_y \partial_y G_1 + L G_1 = -v_x v_y \sqrt{\mu} \tag{1.11}$$

with boundary condition

$$G_1(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu}\int_{v_y \gtrless 0} \sqrt{\mu}\, G_1(\pm 1, v)|v_y|\,dv, \tag{1.12}$$

and the equation for the remainder G_R:

$$
\begin{aligned}
v_y\partial_y G_R - \alpha v_y\partial_{v_x} G_R + \tfrac{1}{2}\alpha v_x v_y G_R + LG_R \\
= v_y\partial_{v_x} G_1 - \tfrac{1}{2} v_x v_y G_1 \\
+ \Gamma(G_1, G_1) + \alpha\{\Gamma(G_R, G_1) + \Gamma(G_1, G_R)\} + \alpha^2\Gamma(G_R, G_R)
\end{aligned}
\tag{1.13}
$$

with boundary condition

$$
G_R(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu}\int_{v_y \gtrless 0} \sqrt{\mu}\, G_R(\pm 1, v)|v_y|\, dv. \tag{1.14}
$$

Here, the linear and nonlinear collision operators L and Γ are given by

$$
Lf = -\mu^{-1/2}\{Q(\mu, \sqrt{\mu}\, f) + Q(\sqrt{\mu}\, f, \mu)\},
$$

and

$$
\Gamma(f, g) = \mu^{-1/2}\{Q(\sqrt{\mu}\, f, \sqrt{\mu}\, g) + Q(\sqrt{\mu}\, g, \sqrt{\mu}\, f)\}.
$$

Properties of these two operators will be presented in Section 2. Note that to solve G_1, both (1.11) and (1.12) with the restriction $\int_{-1}^{1}\int_{\mathbb{R}^3} \sqrt{\mu}\, G_1\, dv\, dy = 0$ are invariant under the transformation $G_1(y, v) \to -G_1(y, -v_x, v_y, v_z)$. Thus, if the solution is unique, then G_1 is odd in v_x, i.e.

$$
G_1(y, v) = -G_1(y, -v_x, v_y, v_z), \quad -1 \le y \le 1,\ v = (v_x, v_y, v_z) \in \mathbb{R}^3. \tag{1.15}
$$

Hence, the diffuse reflection boundary condition (1.12) for G_1 can be reduced to the homogeneous inflow boundary condition

$$
G_1(\pm 1, v)|_{v_y \lessgtr 0} = 0. \tag{1.16}
$$

The first existence result for the Couette flow problem is stated as follows; we use a velocity weight function

$$
w_q = w_q(v) := (1 + |v|^2)^q \tag{1.17}
$$

with an integer $q > 0$.

Theorem 1.1. *Assume that the Boltzmann collision kernel is of Maxwell molecule type* (1.6)*. Then the boundary value problem* (1.8) *admits a unique steady solution* $F_{\mathrm{st}} = F_{\mathrm{st}}(y, v) \ge 0$ *of the form* (1.9) *satisfying* (1.10) *and the following estimates on* G_1 *and* G_R*, respectively.*

(i) *The first order correction* $G_1 = G_1(y, v)$*, uniquely solving the boundary value problem* (1.11)*,* (1.16)*, satisfies* (1.15)*, and for any integers* $m \ge 0$ *and* $q \ge 0$*,*

$$
\|w_q \partial_{v_x}^m G_1\|_{L^\infty} \le \tilde{C}_1, \tag{1.18}
$$

where $\tilde{C}_1 > 0$ *is a constant depending only on* m *and* q*.*

(ii) *The remainder* $G_R = G_R(y, v)$*, uniquely solving the boundary value problem* (1.13)–(1.14)*, has the property that there is an integer* $q_0 > 0$ *such that for any integer* $q \geq q_0$*, there is* $\alpha_0 = \alpha_0(q) > 0$ *depending on* q *such that for any* $\alpha \in (0, \alpha_0)$ *and any integer* $m \geq 0$*,* $\widetilde{G}_R := \sqrt{\mu}\, G_R$ *satisfies*

$$\|w_q \partial_{v_x}^m \widetilde{G}_R\|_{L^\infty} \leq \tilde{C}_{m,q}, \tag{1.19}$$

where $\tilde{C}_{m,q} > 0$ *is a constant depending only on* m *and* q *but independent of* α*.*

Here are some remarks on Theorem 1.1.

Remark 1.1. The steady solution $F_{\rm st}$ to the boundary value problem (1.8) is essentially constructed in the regime where the collision is dominated and the shearing effect is weak. By (1.18) and (1.19), the steady solution takes the form of

$$F_{\rm st} = \mu + \alpha\sqrt{\mu}\, G_1 + O(1)\alpha^2 \tag{1.20}$$

with the second order remainder decaying in large velocities only polynomially. The order of the polynomial decay can be arbitrarily large as long as the shear rate is sufficiently small. One generally has $\alpha_0(q) \to 0$ as $q \to \infty$, and in particular one may take

$$\alpha_0(q) = \frac{\nu_0}{8q}$$

as shown in the proof. The result is consistent with the one in [15] for uniform shear flow without boundaries in the spatially homogeneous setting.

Remark 1.2. Without using the odd-in-v_x property as in (1.15), the existence of a solution $G_1(y, v)$ to the BVP (1.11) under the diffuse reflection boundary condition (1.12) can also be established by the same approach as for the remainder G_R. Here, we take this formulation only for brevity of presentation because the proof for the homogeneous inflow boundary is relatively easier than that for the diffuse reflection boundary.

Remark 1.3. We notice that it is necessary to deal with the v_x-derivative estimates due to the appearance of the shear force term $v_y \partial_{v_x} F_{\rm st}$; in particular, the term $v_y \partial_{v_x} G_1$ becomes a source term in equation (1.13) for G_R. We emphasize that although one can obtain the derivative estimates as in (1.18) and (1.19) in v_x, it is impossible to obtain a similar estimate on the v_y-derivative because $G_1(y, v)$ is discontinuous at $v_y = 0$; see (4.25) for an explicit form of G_1 when the nonlocal collision term is omitted.

To establish the nonnegativity of the stationary profile $F_{\rm st}(y, v)$, we further study the following initial boundary value problem for the Boltzmann equation with a shear force:

$$\begin{cases} \partial_t F + v_y \partial_y F - \alpha v_y \partial_{v_x} F = Q(F, F), \quad t > 0,\ y \in (-1, 1),\ v = (v_x, v_y, v_z) \in \mathbb{R}^3, \\ F(0, y, v) = F_0(y, v), \quad y \in (-1, 1),\ v \in \mathbb{R}^3, \\ F(t, \pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi}\, \mu \displaystyle\int_{v_y \gtrless 0} F(t, \pm 1, v)|v_y|\, dv, \quad t \geq 0,\ v \in \mathbb{R}^3. \end{cases} \tag{1.21}$$

One may expect that the solution of the time-dependent problem (1.21) tends in large time toward that of the steady problem (1.8). In this connection, the second result is concerned with the large time asymptotic stability of the stationary solution $F_{\rm st}$ which gives the nonnegativity of $F_{\rm st}$.

Theorem 1.2. *Let $F_{\rm st}(y,v)$ be the steady state obtained in Theorem 1.1 corresponding to a shear rate $\alpha \in (0,\alpha_0)$. There are constants $\varepsilon_0 > 0$, $\lambda_0 > 0$ and $C > 0$, independent of α, such that if the initial data $F_0(y,v) \geq 0$ satisfy*

$$\|w_q[F_0(y,v) - F_{\rm st}(y,v))]\|_{L^\infty} \leq \varepsilon_0$$

with

$$\int_{-1}^{1}\int_{\mathbb{R}^3}[F_0(y,v) - F_{\rm st}(y,v)]\,dv\,dy = 0, \tag{1.22}$$

then the initial boundary value problem (1.21) admits a unique solution $F(t,y,v) \geq 0$ satisfying the following decay estimate:

$$\|w_q[F(t,y,v) - F_{\rm st}(y,v)]\|_{L^\infty} \leq Ce^{-\lambda_0 t}\|w_q[F_0(y,v) - F_{\rm st}(y,v)]\|_{L^\infty} \tag{1.23}$$

for any $t \geq 0$.

Remark 1.4. Thanks to Theorem 1.1, the expansion (1.20) for the steady state $F_{\rm st}(y,v)$ is uniform in $\alpha \in (0,\alpha_0)$ when the large enough integer q is chosen and hence $\alpha_0 = \alpha_0(q) > 0$ is fixed. Thus, the exponential time decay estimate (1.23) also holds uniformly in $\alpha \in (0,\alpha_0)$, in particular, C and λ_0 are independent of α. As $\alpha \to 0$, we are able to recover the exponential convergence of the solution $F(t,y,v)$ to the global Maxwellian μ in L^∞ norm weighted by the polynomial velocity weight $w_q(v)$.

In what follows we present the key points and strategy of the proof of the main results stated above. As pointed out in a recent nice survey by Esposito–Marra [20], stationary non-equilibrium solutions to the Boltzmann equation, despite their relevance in applications, are much less studied than time-dependent solutions, and no general existence theory is available, due to technical difficulties. The readers may refer to [20] and references therein for a thorough review of this subject. As for the Boltzmann equation for the plane Couette flow, [13, 19] mentioned before seem to be the only mathematical works on the fluid dynamic approximation solutions in the steady case for small Knudsen number. But it remains unsolved how to justify the large time asymptotics toward the stationary solution for the time-dependent problem in the same setting of the fluid limit. In this paper, motivated by [15], instead of constructing the fluid dynamic approximation solutions, we focus on the existence and dynamical stability of the plane Couette flow with finite Knudsen number for both the steady and unsteady problems.

First of all, for the original Couette flow problem (1.1)–(1.3), we note that a direct perturbation approach by linearization of the boundary condition in α with the techniques of [13, 16, 18, 19] can be applied to prove the existence of stationary solutions, because the inhomogeneous data appear only on the tangent (x,z)-plane. The solution thus obtained has the structure around global Maxwellians of the form

$$F(y,v)=\mu(v)+\sqrt{\mu(v)}\,(\alpha g_1+\alpha^2 g_2+\cdots)$$

corresponding to the linearization of the wall Maxwellians at $y=\pm L$,

$$\mu(v_x\pm\alpha L,v_y,v_z)=\mu(v)+(\pm\alpha L)\mu_1(v)+(\pm\alpha L)^2\mu_2(v)+\cdots.$$

On the other hand, in the formulation used in this paper, we rather look for the solution of the specific structure (1.7), and hence the problem can be reduced to solving (1.8) for the Boltzmann equation driven by an external shear force under the homogeneous non-moving diffuse reflection boundary condition. This means that the solution to the Couette flow problem (1.1)–(1.3) is established around the local Maxwellian $\mu(v_x-\alpha y,v_y,v_z)$ instead of the global Maxwellian μ such that the kinetic diffusive reflection boundary condition (1.2) is satisfied for the background solution $\mu(v_x-\alpha y,v_y,v_z)$. In addition, as mentioned before, it seems more convenient to use the formulation with shear forces than the original one driven by the relative motion of boundaries in order to understand the asymptotic behavior of solutions as $L\to\infty$, that is, how the Couette flow with boundaries converges to a shear flow without boundary; that is closely related to what has been studied in the previous works [15] for uniform shear flow in the spatially homogeneous setting.

We also comment on the boundary value problem (1.11)–(1.12) for the first order correction term $G_1(y,v)$. Notice that the inhomogeneous source term $-v_xv_y\sqrt{\mu}$ in (1.11) does not satisfy the boundary condition (1.12), so a space-dependent nontrivial solution is induced. If the boundary condition is omitted and only the spatially homogeneous equation is considered, the corresponding solution can be written as

$$L^{-1}(-v_xv_y\sqrt{\mu})=-\frac{1}{2b_0}v_xv_y\sqrt{\mu} \tag{1.24}$$

with the positive constant $b_0:=3\pi\int_{-1}^{1}B_0(z)z^2(1-z^2)\,dz$. The form (1.24) is then consistent with the uniform shear flow in [15]. To solve the boundary value problem (1.11)–(1.12), the same approach as for the remainder G_R can be applied. However, in order to simplify the proof, we have made use of an additional property (1.15) to reduce the diffusive reflection boundary condition (1.12) to the homogeneous inflow boundary condition (1.16). To treat (1.11) and (1.16), we develop a direct L^∞-L^2 method without using the stochastic cycles as in [25]. In particular, thanks to the splitting $L=\nu_0-K$, if the nonlocal term KG_1 is omitted, the solution to the boundary value problem

$$v_y\partial_yG_1+\nu_0G_1=\mathfrak{F},\quad G_1(\pm1,v)|_{v_y\lessgtr0}=0,$$

can be explicitly expressed as

$$G_1(y,v)=\mathbf{1}_{v_y>0}\int_{-1}^{y}e^{-\frac{\nu_0(y-y')}{v_y}}v_y^{-1}\mathfrak{F}(y',v)\,dy'+\mathbf{1}_{v_y<0}\int_{y}^{1}e^{-\frac{\nu_0(y-y')}{v_y}}v_y^{-1}\mathfrak{F}(y',v)\,dy'.$$

Moreover, we use the bootstrap argument as in [14] to treat the following problem with a parameter $\sigma \in [0, 1]$:

$$v_y \partial_y G_1 + \nu_0 G_1 = \sigma K G_1 + \mathfrak{F}, \quad G_1(\pm 1, v)|_{v_y \lessgtr 0} = 0.$$

With the solvability starting from $\sigma = 0$, we are able to iteratively solve the above boundary value problem for σ over the intervals $[0, \sigma_*]$, $[\sigma_*, 2\sigma_*]$, and so on, where $\sigma_* > 0$ is small enough such that $\sigma_* K G_1$ can be regarded as a source term in the L^∞ estimation. Therefore, in the end, the original problem corresponding to $\sigma = 1$ can be solved. In this procedure, the uniform L^∞ estimate can be obtained through the interplay with the L^2 estimates by using Guo's technique [25]. Here, we have omitted the discussion of the mass conservation (1.10) for G_1. In fact, inspired by [25], an extra damping term ϵG_1 with the vanishing parameter $\epsilon > 0$ has to be used; cf. Section 4 for details.

We now discuss some key points about estimating the remainder G_R solving the boundary value problem (1.13)–(1.14). The direct L^∞-L^2 approach is no longer available because the linear term $\frac{1}{2}\alpha v_x v_y G_R$ cannot be controlled in the large velocity regime. Notice that this term arises from the action of the shear force on the exponential weight function $\sqrt{\mu}$ in the perturbation. To overcome it, as in [15], we apply Caflisch's decomposition

$$\sqrt{\mu}\, G_R = G_{R,1} + \sqrt{\mu}\, G_{R,2},$$

where $G_{R,1}$ and $G_{R,2}$ satisfy the coupled boundary value problems

$$\begin{cases} v_y \partial_y G_{R,1} - \alpha v_y \partial_{v_x} G_{R,1} + \nu_0 G_{R,1} = \chi_M \mathcal{K} G_{R,1} - \frac{1}{2}\alpha \sqrt{\mu}\, v_x v_y G_{R,2} + \mathcal{F}_1, \\ G_{R,1}(\pm 1, v)|_{v_y \lessgtr 0} = 0, \end{cases} \tag{1.25}$$

and

$$\begin{cases} v_y \partial_y G_{R,2} - \alpha v_y \partial_{v_x} G_{R,2} + L G_{R,2} = (1 - \chi_M)\mu^{-1/2} \mathcal{K} G_{R,1} + \mathcal{F}_2, \\ G_{R,2}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \displaystyle\int_{v_y \gtrless 0} \sqrt{\mu}\, G_R(\pm 1, v) |v_y|\, dv, \end{cases} \tag{1.26}$$

respectively. Then, in (1.25), the term $-\frac{1}{2}\alpha \sqrt{\mu}\, v_x v_y G_{R,2}$ can be controlled due to the appearance of $\sqrt{\mu}$. Here, since the operator norm of $\mathcal{K}$ may not be small, the term $\chi_M \mathcal{K} G_{R,1}$ over the large velocity regime can be viewed as a source in (1.25) for $G_{R,1}$, while the complementary term $(1 - \chi_M)\mu^{-1/2} \mathcal{K} G_{R,1}$ is taken as a source in (1.26) for $G_{R,2}$. A crucial observation inspired by [2] in estimating $G_{R,1}$ is that the norm of the weighted operator $w_q \chi_M \mathcal{K}$ on L^∞_v with the polynomial velocity weight $w_q = (1 + |v|^2)^q$ can be arbitrarily small as long as M and q are chosen sufficiently large; see Lemma 2.4. Notice that Lemma 2.4 holds only for the Maxwell molecule potential, as shown in the proof. Compared to the previous work [15] for uniform shear flow, it is more complicated to solve the coupling steady boundary value problems (1.25) and (1.26) in a bounded domain. We now list the main steps in the proof.

- *Step 1.* We first modify the coupled boundary value problems with two parameters, $\epsilon > 0$ small enough and $0 \le \sigma \le 1$ (see (5.10)), and obtain the a priori estimates uniform in ϵ and σ in the L^∞ framework; see Lemma 5.1 and the proof of Proposition 5.1. For the proof of Lemma 5.1, we apply Guo's approach [25] to the shear flow problem in a slab. In particular, we introduce the mild formulation (5.23) to treat the diffuse boundary condition with the help of Lemma 8.1, and re-prove Ukai's trace theorem in Lemma 3.1 for the L^2 estimates.

- *Step 2.* Similar to finding the first order correction term G_1, we design an explicit procedure to solve the parameterized boundary value problem (5.10) iteratively for $\sigma \in [0, 1]$ from $\sigma = 0$ to $\sigma = 1$ for any fixed $\epsilon > 0$; see Lemma 5.2. Notice that the problem for $\sigma = 0$ is reduced to the one without the nonlocal collision terms under the homogeneous inflow boundary condition so that the method of characteristics can be directly applied.
- *Step 3.* We study the limit $\epsilon \to 0$ to obtain the desired solution; see Section 5.4 for details. The key point is to obtain the macroscopic estimates in order to bound the L^2 norm of $G_{R,2}$ in terms of the L^∞ norm of $G_{R,1}$. We apply the dual argument developed first in [16]. Note that it is delicate to make these estimates uniform for any small parameter $\epsilon > 0$.

With the existence of a stationary solution F_{st}, the asymptotic stability of the perturbation $F = F_{st} + \sqrt{\mu} f$ as in (6.1) is considered in the reformulated IBVP (6.2). Technically, we follow the same strategy as for the steady problem. More precisely, we also use the decomposition $\sqrt{\mu} f = f_1 + \sqrt{\mu} f_2$ with f_1, f_2 satisfying the coupled IBVPs (6.4)–(6.5) and (6.6)–(6.7), respectively. In order to treat initial data with only the polynomial velocity weight, we set $f_2(0, y, v) \equiv 0$ and the boundary conditions on f_1 and f_2 both as diffuse reflections which are slightly different from (1.25) and (1.26) in the steady problem. Moreover, in contrast with the steady case, we need to construct suitable temporal energy functionals so as to close the a priori estimates. In particular, the energy functional for the second component f_2 in Caflisch's decomposition is complicated, because there is a subtle interplay with f_1. For this, we make use of a linear combination of estimates for the two functionals, where the smallness of the shear rate α and finiteness of the domain play an important role. Specifically, we obtain estimates (7.2) and (7.3) for the weighted L^∞ norms. To treat the L^2 estimates on the right hand side of (7.3), we construct another functional $\mathcal{E}_{\text{int}}(t)$ in Lemma 7.2 (see (7.29)), to capture the macroscopic dissipation, and conclude the desired estimates (7.33) and hence (7.36).

Finally, we remark that there have been extensive studies on the stability of shear flow in the multi-dimensional space domain in the context of fluid dynamic equations [32]; in particular, we mention the important mathematical contributions by Bedrossian et al. [5,7,8] for either an infinite 2D channel domain $\mathbb{T}_x \times \mathbb{R}_y$ or an infinite 3D channel domain $\mathbb{T}_x \times \mathbb{R}_y \times \mathbb{T}_z$, and an interesting work by Ionescu–Jia [26] on the asymptotic stability of the Couette flow for the 2D Euler equations in the 2D finite channel domain $\mathbb{T}_x \times [0, 1]$ with the zero normal velocities on the two boundary planes $y = 0, 1$; see also the nice survey [6] and references therein. In fact, in comparison with the 1D problem (1.8) under consideration, it would be more interesting to study the existence and asymptotic stability of stationary solutions in the multi-dimensional setting corresponding to those works on fluid dynamic equations. Moreover, it is also challenging to study the fluid dynamic limit for these problems as in [13, 18, 19] when the vanishing Knudsen number is taken into account. We expect that this paper together with [15] can shed some light on the above problems.

The rest of this paper is organized as follows. In Section 2, we give some basic estimates on the linearized and nonlinear collision operators. In particular, we obtain Lemma 2.4 which is crucially used to obtain the smallness of the nonlocal operator $\mathcal{K}$ for large velocity. In Section 3, we revisit Ukai's trace theorem in both the steady and time-dependent cases for the transport operator with shear force in the 1D setting under consideration. In Sections 4 and 5, we establish estimates on the first order correction G_1 and the remainder G_R, respectively, and hence complete the proof of Theorem 1.1 without showing nonnegativity of the stationary solution. Then we study the time-dependent problem for local-in-time existence in Section 6 and the exponential time asymptotic stability of the stationary solution in Section 7 so that the nonnegativity of the stationary solution follows. The appendix Section 8 includes some estimates on the boundary product measure when there are multiple bounces induced by the diffuse boundary condition.

Notations. Throughout this paper, C denotes some generic positive (generally large) constant and λ denotes a generic positive (generally small) constant. $D \lesssim E$ means that there is a generic constant $C > 0$ such that $D \le CE$; $D \sim E$ means $D \lesssim E$ and $E \lesssim D$; and $\mathbf{1}_A$ indicates the characteristic function of the set A. We denote by $\|\cdot\|$ the $L^2((-1,1)\times\mathbb{R}^3)$ or $L^2(-1,1)$ or $L^2(\mathbb{R}^3)$ norm. Sometimes, without any confusion, we use $\|\cdot\|_{L^\infty}$ to denote either the $L^\infty([-1,1]\times\mathbb{R}^3)$ norm or the $L^\infty(\mathbb{R}^3)$ norm. Moreover, $(\cdot,\cdot)$ denotes the L^2 inner product in $(-1,1)\times\mathbb{R}^3$ with the L^2 norm $\|\cdot\|$ and $\langle\cdot\rangle$ denotes the L^2 inner product in $\mathbb{R}^3_v$. We denote by $\gamma_+ = \{(1,v) \mid v\in\mathbb{R}^3,\ v_y>0\}\cup\{(-1,v)\mid v\in\mathbb{R}^3,\ v_y<0\}$ the outgoing set, by $\gamma_- = \{(1,v)\mid v\in\mathbb{R}^3,\ v_y<0\}\cup\{(-1,v)\mid v\in\mathbb{R}^3,\ v_y>0\}$ the incoming set, and by $\gamma_0 = \{(\pm1,v)\mid v\in\mathbb{R}^3,\ v_y=0\}$ the grazing set. Furthermore $|f|_{2,\pm} = |f\mathbf{1}_{\gamma_\pm}|_2$ represents the L^2 norm of $f(y,v)$ on the boundary $y=\pm1$. Finally, we define

$$P_\gamma f(\pm1,v) = \sqrt{\mu(v)}\int_{n(\pm1)\cdot v'>0} f(x,v')\sqrt{\mu(v')}\,(n(\pm1)\cdot v')\,dv',$$

where $n(\pm1) = (0,\pm1,0)$. One sees that $P_\gamma f$ defined on $\{\pm1\}\times\mathbb{R}^3$ is an L^2_v-projection with respect to the measure $|v_y|\sqrt{\mu(v)}\,dv$ for any function f defined on γ_+.

2. Basic estimates

In this section we summarize some basic estimates to be used in the following sections. Let us first give some elementary estimates for the linearized collision operator L and the nonlinear collision operator Γ, defined by

$$Lg = -\mu^{-1/2}\{Q(\mu,\sqrt{\mu}\,g) + Q(\sqrt{\mu}\,g,\mu)\} \tag{2.1}$$

and

$$\Gamma(f,g) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} g) \\ = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0 \mu^{1/2}(v_*)[f(v_*')g(v') - f(v_*)g(v)]\, d\omega\, dv_*. \tag{2.2}$$

It is known that

$$Lf = \nu f - Kf$$

with

$$\begin{cases} \nu = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\cos\theta)\mu(v_*)\, d\omega\, dv_* = \nu_0, \\ Kf = \mu^{-1/2}\{Q(\mu^{1/2} f, \mu) + Q_{\text{gain}}(\mu, \mu^{1/2} f)\}, \end{cases} \tag{2.3}$$

where Q_{gain} denotes the positive part of Q in (1.4). Note that ν_0 is a positive constant in the case of Maxwell molecule collision. The kernel of L, denoted as $\ker L$, is a five-dimensional space spanned by

$$\{1, v, |v|^2 - 3\}\sqrt{\mu} := \{\phi_i\}_{i=1}^5.$$

Define a projection from L^2 to $\ker L$ by

$$\mathbf{P}_0 g = \{a_g + \mathbf{b}_g \cdot v + (|v|^2 - 3)c_g\}\sqrt{\mu}$$

for $g \in L^2$, and correspondingly denote the operator $\mathbf{P}_1$ by $\mathbf{P}_1 g = g - \mathbf{P}_0 g$, which is orthogonal to $\mathbf{P}_0$ in L^2.

It is also convenient to define

$$\mathcal{L} f = -\{Q(f,\mu) + Q(\mu, f)\} = \nu f - \mathcal{K} f$$

with

$$\nu f = \nu_0 f, \quad \mathcal{K} f = Q(f,\mu) + Q_{\text{gain}}(\mu, f) = \sqrt{\mu}\, K(f/\sqrt{\mu}), \tag{2.4}$$

according to (2.3).

The following lemma is concerned with the integral operator K given by (2.3), and its proof for the hard sphere model was given in [25, Lemma 3, p. 727]. Recall (1.17) for the polynomial velocity weight w_q.

Lemma 2.1. *Let K be defined as in* (2.3). *Then*

$$Kf(v) = \int_{\mathbb{R}^3} \mathbf{k}(v, v_*) f(v_*)\, dv_*$$

with

$$|\mathbf{k}(v, v_*)| \le C\{1 + |v - v_*|^{-2}\} e^{-\frac{1}{8}|v-v_*|^2 - \frac{1}{8}\frac{\left||v|^2 - |v_*|^2\right|^2}{|v-v_*|^2}}.$$

Moreover, let

$$\mathbf{k}_w(v, v_*) = w_q(v)\mathbf{k}(v, v_*) w_{-q}(v_*) \tag{2.5}$$

with $q \geq 0$. *Then*

$$\int_{\mathbb{R}^3} \mathbf{k}_w(v, v_*) e^{\varepsilon |v-v_*|^2/8}\, dv_* \leq \frac{C}{1+|v|}$$

for any $\varepsilon \geq 0$ *small enough.*

For the velocity weighted v_x-derivative estimates of the nonlinear operator Γ, we have the following lemma.

Lemma 2.2. *In the Maxwell molecular case,*

$$\|w_q \partial_{v_x}^m \Gamma(f,g)\|_{L^2_v} \leq C \sum_{m' \leq m} \|w_q \partial_{v_x}^{m'} f\|_{L^2_v} \|w_q \partial_{v_x}^{m-m'} g\|_{L^2_v}, \tag{2.6}$$

$$\|w_q \partial_{v_x}^m \Gamma(f,g)\|_{L^\infty} \leq C \sum_{m' \leq m} \|w_q \partial_{v_x}^{m'} f\|_{L^\infty} \|w_q \partial_{v_x}^{m-m'} g\|_{L^\infty}, \tag{2.7}$$

for any integers $m \geq 0$ *and* $q \geq 0$. *Moreover, for* $q > 3/2$ *and* $m \geq 0$,

$$\|w_q \partial_{v_x}^m Q(F_1, F_2)\|_{L^\infty} \leq C \sum_{m_1 \leq m} \|w_q \partial_{v_x}^{m-m_1} F_1\|_{L^\infty} \|w_q \partial_{v_x}^{m_1} F_2\|_{L^\infty}. \tag{2.8}$$

Proof. We prove (2.7) only, since the proofs for (2.6) and (2.8) are similar and they follow from the proof of [23, Lemma 2.3, p. 1111] and [2, Proposition 3.1, p. 397] respectively. By definition (2.2), we have

$$\begin{aligned}
\partial_{v_x}^m \Gamma(f,g) &= \partial_{v_x}^m \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \mu^{1/2}(v_*) f(v_*') g(v')\, d\omega\, dv_* \\
&\quad - \partial_{v_x}^m \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \mu^{1/2}(v_*) f(v_*) g(v)\, d\omega\, dv_* \\
&= \partial_{v_x}^m \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \mu^{1/2}(v_*) f(v_*') g(v')\, d\omega\, dv_* \\
&\quad - c_0 \partial_{v_x}^m g(v) \int_{\mathbb{R}^3} \mu^{1/2}(v_*) f(v_*)\, dv_*,
\end{aligned}$$

where we have used $\int_{\mathbb{S}^2} B_0\, d\omega = c_0$ for a constant $c_0 > 0$. Recalling (1.5), by the change of variable $\tilde{u} = v_* - v$ we then have

$$\begin{aligned}
&\partial_{v_x}^m \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \mu^{1/2}(v_*) f(v_*') g(v')\, d\omega\, dv_* \\
&\quad = \partial_{v_x}^m \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 \mu^{1/2}(\tilde{u}+v) f(v+\tilde{u}_\perp) g(v+\tilde{u}_\parallel)\, d\omega\, d\tilde{u} \\
&\quad = \sum_{m_1+m_2 \leq m} C_m^{m_1,m_2} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 (\partial_{v_x}^{m-m_1-m_2} \mu^{1/2})(\tilde{u}+v) \\
&\qquad\qquad \times (\partial_{v_x}^{m_1} f)(v+\tilde{u}_\perp)(\partial_{v_x}^{m_2} g)(v+\tilde{u}_\parallel)\, d\omega\, d\tilde{u},
\end{aligned}$$

where $\tilde{u}_\parallel = (\tilde{u}\cdot\omega)\omega$ and $\tilde{u}_\perp = \tilde{u} - \tilde{u}_\parallel$. Then, by taking directly the L^∞ norm, (2.7) holds because

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 (\partial_{v_x}^m \mu^{1/2})(v_*)\, d\omega\, dv_* < \infty$$

for any integer $m \geq 0$. This completes the proof of Lemma 2.2. ■

The following lemma can be found in [24, Lemmas 3.2 and 3.3, pp. 638–639], where the hard sphere model was considered.

Lemma 2.3. *In the Maxwell molecular case, there is a constant* $\delta_0 > 0$ *such that*

$$\langle Lf, f\rangle = \langle L\mathbf{P}_1 f, \mathbf{P}_1 f\rangle \geq \delta_0 \|\mathbf{P}_1 f\|^2. \tag{2.9}$$

Moreover, for any integer $m > 0$*, there are constants* $\delta_1 > 0$ *and* $C > 0$ *such that*

$$\langle \partial_{v_x}^m Lf, \partial_{v_x}^m f\rangle \geq \delta_1 \|\partial_{v_x}^m f\|^2 - C\|f\|^2. \tag{2.10}$$

Proof. Since (2.9) is quite elementary, we only show (2.10). As in Lemma 2.2, the key point here is to show that the action of the derivatives $\partial_{v_x}^m$ on the nonlocal operator L does not involve any other partial derivatives such as ∂_{v_y} or ∂_{v_z}. By (2.1) and (2.9), we have

$$\begin{aligned}\langle \partial_{v_x}^m Lf, \partial_{v_x}^m f\rangle &= \langle L\partial_{v_x}^m f, \partial_{v_x}^m f\rangle + \sum_{m_1<m} C_m^{m_1} \langle \partial_{v_x}^{m-m_1} L\partial_{v_x}^{m_1} f, \partial_{v_x}^m f\rangle \\ &\geq \delta_0 \|\mathbf{P}_1[\partial_{v_x}^m f]\|^2 - \sum_{m_1<m} C_m^{m_1} |\langle \partial_{v_x}^{m-m_1} L\partial_{v_x}^{m_1} f, \partial_{v_x}^m f\rangle| \end{aligned} \tag{2.11}$$

with

$$\begin{aligned}&\mathbf{1}_{m_1<m}\partial_{v_x}^{m-m_1} L\partial_{v_x}^{m_1} f \\ &= -\sum_{m_1+m_2<m} C_m^{m_1,m_2} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\partial_{v_x}^{m-m_1-m_2}\mu^{1/2})(\tilde{u}+v) \\ &\qquad\qquad \times (\partial_{v_x}^{m_1} f)(v+\tilde{u}_\perp)(\partial_{v_x}^{m_2}\mu^{1/2})(v+\tilde{u}_\parallel)\,d\omega\,d\tilde{u} \\ &\quad + \partial_{v_x}^m \mu^{1/2}(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0\mu^{1/2}(v_*)f(v_*)\,d\omega dv_* \\ &\quad - \sum_{m_1+m_2<m} C_m^{m_1,m_2} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\partial_{v_x}^{m-m_1-m_2}\mu^{1/2})(\tilde{u}+v) \\ &\qquad\qquad \times (\partial_{v_x}^{m_1} f)(v+\tilde{u}_\parallel)(\partial_{v_x}^{m_2}\mu^{1/2})(v+\tilde{u}_\perp)\,d\omega d\tilde{u},\end{aligned}$$

where we have used the change of variable $\tilde{u} = v_* - v$ again. Consequently, as in (2.6), it follows that

$$\begin{aligned}\sum_{m_1<m} C_m^{m_1} |\langle \partial_{v_x}^{m-m_1} L\partial_{v_x}^{m_1} f, \partial_{v_x}^m f\rangle| &\leq \eta\|\partial_{v_x}^m f\|^2 + C_\eta \sum_{m_1<m} \|\partial_{v_x}^{m_1} f\|^2 \\ &\leq \eta\|\partial_{v_x}^m f\|^2 + C_\eta\eta_1\|\partial_{v_x}^m f\|^2 + C_{\eta,\eta_1}\|f\|^2\end{aligned} \tag{2.12}$$

for small enough constants $\eta > 0$ and $\eta_1 > 0$, where Sobolev's interpolation inequality $\|\partial_{v_x}^{m_1} f\|^2 \leq \eta_1\|\partial_{v_x}^m f\|^2 + C_{\eta_1}\|f\|^2$ has been used.

On the other hand, it can be easily checked that

$$\|\mathbf{P}_1[\partial_{v_x}^m f]\| \geq \|\partial_{v_x}^m f\| - \|\mathbf{P}_0[\partial_{v_x}^m f]\| \geq \|\partial_{v_x}^m f\| - C\|f\|. \tag{2.13}$$

Finally, plugging (2.12) and (2.13) into (2.11) gives (2.10). This completes the proof of Lemma 2.3. ∎

Next, the following lemma which was proved in [15, Proposition 3.1, p. 13] plays a significant role in obtaining L^∞ estimates of the first component in Caflisch's decomposition of solutions.

Lemma 2.4. *Let $\mathcal{K}$ be given by (2.4). Then for any integer $m \geq 0$, there is $C > 0$ such that for any arbitrarily large $q > 0$ we have*

$$\sup_{|v|\geq M} w_q|\partial_{v_x}^m \mathcal{K} f| \leq \frac{C}{q} \sum_{0\leq m'\leq m} \|w_q \partial_{v_x}^{m'} f\|_{L^\infty} \tag{2.14}$$

for some $M = M(q) > 0$. In particular, one can choose $M = q^2$.

Proof. Since the general case

$$\sup_{|v|\geq M} w_q|\partial_{v}^m \mathcal{K} f| \leq \frac{C}{q} \sum_{0\leq m'\leq m} \|w_q \partial_{v}^{m'} f\|_{L^\infty}$$

was proved in [15, Proposition 3.1, p. 13], as in Lemma 2.3 we only point out that the derivative $\partial_{v_x}^m$ acting on the nonlocal operator $\mathcal{K}$ does not involve other derivatives such as ∂_{v_y} or ∂_{v_z}. Indeed, in view of (2.4), similar to the proof of Lemma 2.3, we have

$$\begin{aligned}
\partial_{v_x}^m \mathcal{K} f = & \sum_{m_1\leq m} C_m^{m_1} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\partial_{v_x}^{m_1} f)(v+\tilde{u}_\perp)(\partial_{v_x}^{m-m_1}\mu)(v+\tilde{u}_\parallel)\,d\omega\,d\tilde{u} \\
& - \partial_{v_x}^m \mu(v) \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 f(v_*)\,d\omega\,dv_* \\
& + \sum_{m_1\leq m} C_m^{m_1} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\partial_{v_x}^{m_1} f)(v+\tilde{u}_\parallel)(\partial_{v_x}^{m-m_1}\mu)(v+\tilde{u}_\perp)\,d\omega\,d\tilde{u} \\
= & \sum_{m_1\leq m} C_m^{m_1} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\partial_{v_x}^{m_1} f)(v_*')(\partial_{v_x}^{m-m_1}\mu)(v')\,d\omega\,dv_* \\
& - (\partial_{v_x}^m \mu)(v) \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0 f(v_*)\,d\omega\,dv_* \\
& + \sum_{m_1\leq m} C_m^{m_1} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_0(\partial_{v_x}^{m_1} f)(v')(\partial_{v_x}^{m-m_1}\mu)(v_*')\,d\omega\,dv_*.
\end{aligned}$$

Then a similar calculation to estimating $\mathcal{J}_1$ and $\mathcal{J}_2$ in [15, Proposition 3.1, p. 13] yields (2.14). This completes the proof of Lemma 2.4. ∎

3. A trace theorem

In this section, we present the following version of Ukai's trace theorem; see also [17, Lemma 2.3, p. 22] and [17, Lemma 3.2, p. 56], respectively.

Lemma 3.1. *Let $\varepsilon > 0$ and $y \in [-h, h]$ with $0 < h < \infty$, and denote the near-grazing set of γ_+ or γ_- as*

$$\gamma_\pm^\varepsilon \equiv \{(y, v) \in \gamma_\pm : |v_y| \leq \varepsilon \text{ or } |v_y| \geq 1/\varepsilon,\ v = (v_x, v_y, v_z)\}.$$

Then there exists a constant $C_{\varepsilon,h} > 0$ depending on ε and h such that

$$|f\mathbf{1}_{\gamma_\pm\setminus\gamma_\pm^\varepsilon}|_{L^1} \le C_{\varepsilon,h}\big\{\|f\|_{L^1} + \|\{v_y\partial_y - \alpha v_y\partial_{v_x}\}f\|_{L^1}\big\}. \tag{3.1}$$

Moreover,

$$\int_0^T |f\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}(t)|_{L^1}\,dt \le C_{\varepsilon,h}\left\{\|f(0)\|_{L^1} + \int_0^T [\|f(t)\|_{L^1} + \|\{\partial_t + v_y\partial_y - \alpha v_y\partial_{v_x}\}f(t)\|_{L^1}]\,dt\right\} \tag{3.2}$$

for any $T \ge 0$.

Proof. To prove (3.1), we only consider the case that the boundary phase is outgoing, because the incoming case can be treated similarly. We introduce a parameter $t \in \mathbb{R}$ and treat (y, v) as functions of t. Consider the characteristic line $[s, Y(s;t, y, v), V(s;t, y, v)]$ passing through $(y, v) = (t, y(t), v(t))$ such that

$$\frac{dY}{ds} = v_y, \quad \frac{dV_x}{ds} = -\alpha v_y. \tag{3.3}$$

Then

$$Y(s;t, y, v) = y - (t - s)v_y, \quad V(s;t, y, v) = (v_x + \alpha(t - s)v_y, v_y, v_z), \tag{3.4}$$

for $(y, v) \in \gamma_+\setminus\gamma_+^\varepsilon$. Along this trajectory, one has the identity

$$f(y, v) = f(Y(s;t, y, v), V(s;t, y, v)) + \int_s^t \frac{d}{d\tau} f(Y(\tau;t, y, v), V(\tau;t, y, v))\,d\tau. \tag{3.5}$$

On the other hand, $(y, v) \in \gamma_+\setminus\gamma_+^\varepsilon$ also implies $h\varepsilon \le t_{\mathbf{b}}(y, v) \le h/\varepsilon$, where $t_{\mathbf{b}}$ is as in (5.17) below. Therefore, by taking $s \in [t - t_{\mathbf{b}}(y, v), t]$, we infer from (3.5) that

$$\begin{aligned}
&\int_{\gamma_+\setminus\gamma_+^\varepsilon} |f(y, v)|\,|v_y|\,dv \\
&\le C_{\varepsilon,h}\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_{t-\mathbf{b}(y,v)}^t |f(Y(s;t, y, v), V(s;t, y, v))|\,|v_y|\,ds\,dv \\
&\quad + C_{\varepsilon,h}\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_{t-\mathbf{b}(y,v)}^t \left|\frac{d}{ds} f(Y(s;t, y, v), V(s;t, y, v))\right| |v_y|\,ds\,dv \\
&= C_{\varepsilon,h}\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_{t-\mathbf{b}(y,v)}^t |f(Y(s;t, y, v), V(s;t, y, v))|\,|v_y|\,ds\,dv \\
&\quad + C_{\varepsilon,h}\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_{t-\mathbf{b}(y,v)}^t |[v_y\partial_Y - \alpha v_y\partial_{V_x}]f(Y(s;t, y, v), V(s;t, y, v))|\,|v_y|\,ds\,dv.
\end{aligned} \tag{3.6}$$

Next, in light of the Jacobian

$$\frac{\partial(Y(s),V(s))}{\partial(s,v)}=\begin{vmatrix} v_y & 0 & s & 0\\ -\alpha v_y & 1 & -\alpha s & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}=v_y, \tag{3.7}$$

and by the change of variable

$$[\tilde{y},u]=[Y(s;t,y,v),V(s;t,y,v)]=[y-(t-s)v_y,v_x+\alpha(t-s)v_y,v_y,v_z],$$

one gets

$$\int_{\gamma_+\backslash\gamma_+^\varepsilon}\int_{t-t_b(y,v)}^{t}|f(Y(s;t,y,v),V(s;t,y,v))|\,|v_y|\,ds\,dv\le\int_{\mathbb{R}^3}\int_{-h}^{h}|f(\tilde{y},u)|\,d\tilde{y}\,du. \tag{3.8}$$

Similarly, by noticing that $\partial_Y f(Y,V)=\partial_{\tilde{y}}f(\tilde{y},u)$, $\partial_{V_x}f(Y,V)=\partial_{u_x}f(\tilde{y},u)$ and $v_y=u_y$, one has

$$\begin{aligned}\int_{\gamma_+\backslash\gamma_+^\varepsilon}\int_{t-t_b(y,v)}^{t}&|[v_y\partial_Y-\alpha v_y\partial_{V_x}]f(Y(s;t,y,v),V(s;t,y,v))|\,|v_y|\,ds\,dv\\ &\le\int_{\mathbb{R}^3}\int_{-h}^{h}|[u_y\partial_{\tilde{y}}-\alpha u_y\partial_{u_x}]f(\tilde{y},u))|\,d\tilde{y}\,du.\end{aligned} \tag{3.9}$$

Consequently, the desired estimate (3.1) in the case of outgoing boundary follows from (3.8), (3.9) and (3.6).

We now turn to proving (3.2). For $f\in L^1([T_1,T]\times[-h,h]\times\mathbb{R}^3)$, we first show that

$$\begin{aligned}\int_{T_1}^{T}\int_{u\cdot n(y_f)>0}\int_{\max\{-t_b(y_f,u),T_1-\tilde{t}\}}^{0}&|f(\tilde{t}+s,Y(\tilde{t}+s;\tilde{t},y_f,u),V(\tilde{t}+s;\tilde{t},y_f,u))|\,|u_y|\\ &\times ds\,du\,d\tilde{t}\\ &\le\int_{T_1}^{T}\int_{-h}^{h}\int_{\mathbb{R}^3}|f(t,y,v)|\,dy\,dv\,dt,\end{aligned} \tag{3.10}$$

where $y_f=\pm h$, $T\ge T_1\ge 0$ and

$$Y(\tilde{t}+s;\tilde{t},y_f,u)=y_f+su_y,\quad V(\tilde{t}+s;\tilde{t},y_f,u)=(u_x-\alpha su_y,u_y,u_z)$$

with

$$Y(\tilde{t};\tilde{t},y_f,u)=y_f,\quad V(\tilde{t};\tilde{t},y_f,u)=u=(u_x,u_y,u_z).$$

Actually, given $(t,y,u)\in[T_1,T]\times[-h,h]\times\mathbb{R}^3$, let us define $y_f=y+t_{\mathbf{b}}(y,-u)u_y=\pm h$, and denote

$$y=Y(t;t-s,y_f,u)=y_f+su_y,\quad v=V(t;t-s,y_f,u)=(u_x-\alpha su_y,u_y,u_z),$$

for $u\cdot n(y_f)>0$. It is easy to see that $0\ge s\ge -t_{\mathbf{b}}(y_f,u)$, and it is natural to require that

$t-s \le T$. By the change of variable $(y,v)\to(s,u)$ and using (3.7), one has

$$\int_{T_1}^{T}\int_{u\cdot n(y_f)>0}\int_{\max\{-t_b(y_f,u),-(T-t)\}}^{0} |f(t,Y(t;t-s,y_f,u),V(t;t-s,y_f,u))|\,|u_y| \times ds\,du\,dt \le \int_{T_1}^{T}\int_{-h}^{h}\int_{\mathbb{R}^3}|f(t,y,v)|\,dy\,dv\,dt. \tag{3.11}$$

On the other hand, if we denote $\tilde{t}=t-s$, then it follows that $s\ge T_1-\tilde{t}$ due to $t\ge T_1$. In summary, one has

$$s\ge \max\{-t_{\mathbf{b}}(y_f,u),T_1-\tilde{t}\},\quad T_1\le\tilde{t}\le T.$$

Therefore, by the change of variable $t\to\tilde{t}$ we have

$$\begin{aligned}&\int_{T_1}^{T}\int_{u\cdot n(y_f)>0}\int_{\max\{-t_{\mathbf{b}}(y_f,u),-(T-t)\}}^{0} |f(t,Y(t;t-s,y_f,u),V(t;t-s,y_f,u))|\,|u_y| \times ds\,du\,dt\\ &=\int_{T_1}^{T}\int_{u\cdot n(y_f)>0}\int_{\max\{-t_{\mathbf{b}}(y_f,u),T_1-\tilde{t}\}}^{0} |f(\tilde{t}+s,Y(\tilde{t}+s;\tilde{t},y_f,u),V(\tilde{t}+s;\tilde{t},y_f,u))|u_y| \times ds\,du\,d\tilde{t}.\end{aligned} \tag{3.12}$$

Consequently, (3.11) and (3.12) imply (3.10). In addition, it follows that

$$\begin{aligned}f(t,y_f,u)&=f(t+s,Y(t+s;t,y_f,u),V(t+s;t,y_f,u))\\&\quad+\int_s^0\frac{d}{d\tau}f(t+\tau,Y(t+\tau;t,y_f,u),V(t+\tau;t,y_f,u))\,d\tau\\&=f(t+s,Y(t+s;t,y_f,u),V(t+s;t,y_f,u))\\&\quad+\int_s^0[\partial_t+u_y\partial_y-\alpha u_y\partial_{u_x}]f(t+\tau,Y(t+\tau;t,y_f,u),V(t+\tau;t,y_f,u))\,d\tau.\end{aligned} \tag{3.13}$$

For any $(t,y_f,u)\in[\varepsilon_1,T]\times\gamma_+\backslash\gamma_+^{\varepsilon}$ with $\varepsilon_1>0$ to be determined later and for $0\ge s\ge \max\{-t_{\mathbf{b}}(y_f,u),\varepsilon_1-t\}$, we then find from (3.13) and (3.10) that

$$\begin{aligned}&\min\{h\varepsilon,\varepsilon_1\}\int_{\varepsilon_1}^{T}\int_{u\cdot n(y_f)>0}|f(t,y_f,u)|\,|u_y|\,du\,dt\\&\le\int_{\varepsilon_1}^{T}\int_{u\cdot n(y_f)>0}\int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^{0}|f(t+s,Y(t+s;t,y_f,u),V(t+s;t,y_f,u))|\,|u_y| \times dt\,ds\,du\\&\quad+\int_{\varepsilon_1}^{T}\int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^{0}\int_{u\cdot n(y_f)>0}\int_s^0|[\partial_t+u_y\partial_y-\alpha u_y\partial_{u_x}] f(t+\tau,Y(t+\tau),V(t+\tau))|\,|u_y|\,d\tau\,du\,dt\end{aligned}$$

$$\leq \int_0^T \int_{u\cdot n(y_f)>0} \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 |f(t+s, Y(t+s;t,y_f,u), V(t+s;t,y_f,u))|\,|u_y| \times dt\,ds\,du$$

$$+ \int_0^T \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 \int_{u\cdot n(y_f)>0} \int_s^0 |[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t+\tau, Y(t+\tau), V(t+\tau))|\,|u_y|\,d\tau\,du\,dt$$

$$\leq \int_0^T \int_{-h}^h \int_{\mathbb{R}^3} |f(t,y,u))|\,dt\,dy\,du + \int_0^T \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 \int_{u\cdot n(y_f)>0} \int_s^0 |[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t+\tau, Y(t+\tau), V(t+\tau))|\,|u_y|\,d\tau\,du\,dt, \tag{3.14}$$

where we have used the fact that

$$h\varepsilon \leq t_{\mathbf{b}}(y_f,u) \leq h/\varepsilon$$

due to $(y_f,u) \in \gamma_+ \backslash \gamma_+^{\varepsilon}$.

Next, applying Fubini's theorem and using (3.10) again, one also has

$$\int_0^T \int_{u\cdot n(y_f)>0} \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 \int_s^0 |[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t+\tau, Y(t+\tau), V(t+\tau))|\,|u_y|\,d\tau\,du\,dt\,ds$$

$$= \int_0^T dt \int_{u\cdot n(y_f)>0} \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^{\tau} ds \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 d\tau \times |[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t+\tau)|\,|u_y|$$

$$\leq \int_0^T dt \int_{u\cdot n(y_f)>0} \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 d\tau \times |\max\{-t_{\mathbf{b}}(y_f,u),-t\}|\,|[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t+\tau)|\,|u_y|$$

$$\leq \max\{h\varepsilon, \varepsilon_1\} \int_0^T dt \int_{u\cdot n(y_f)>0} \int_{\max\{-t_{\mathbf{b}}(y_f,u),-t\}}^0 d\tau \times |[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t+\tau)|\,|u_y|$$

$$\leq \max\{h\varepsilon, \varepsilon_1\} \int_0^T dt \int_{-h}^h dy \int_{\mathbb{R}^3} du\,|[\partial_t + u_y\partial_y - \alpha u_y \partial_{u_x}] f(t,y,u)|. \tag{3.15}$$

Once (3.14) and (3.15) are obtained, it remains to compute

$$\int_0^{\varepsilon_1} \int_{u\cdot n(y_f)>0} |f(t,y_f,u)|\,|u_y|\,du\,dt.$$

In fact, if we choose ε_1 to be small enough so that $\varepsilon_1 \leq h\varepsilon$, at this stage, the backward trajectory hits the initial plane first. Therefore, for $(t, y_f, u) \in [0, \varepsilon_1] \times \gamma_+ \backslash \gamma_+^{\varepsilon}$, by directly using (3.7) and applying (3.10) once again, it follows that

$$\begin{aligned}
&\int_0^{\varepsilon_1}\int_{u\cdot n(y_f)>0} |f(t,y_f,u)|\,|u_y|\,du\,dt\\
&\qquad\le \int_0^{\varepsilon_1}\int_{u\cdot n(y_f)>0} |f(0,Y(0;t,y_f,u),V(0;t,y_f,u))|\,|u_y|\,du\,dt\\
&\qquad\quad+\int_0^{\varepsilon_1}\int_{u\cdot n(y_f)>0}\int_{-t}^{0} |[\partial_t+u_y\partial_y-\alpha u_y\partial_{u_x}]f(t+\tau)|\,|u_y|\,d\tau\,du\,dt\\
&\qquad\le C\int_{-h}^{h}\int_{\mathbb{R}^3}|f(0,y,u)|\,dy\,du\\
&\qquad\quad+C\int_0^{\varepsilon_1}\int_{-h}^{h}\int_{\mathbb{R}^3}|[\partial_t+u_y\partial_y-\alpha u_y\partial_{u_x}]f(t)|\,dy\,du\,dt.
\end{aligned}$$

The proof of Lemma 3.1 is thus complete. ∎

4. Steady problem: the first order correction

In this and the next sections, we are going to show Theorem 1.1 giving the existence of solutions to the steady problem (1.8). Recall (1.9) and (1.10). For this purpose, we will first study the first order correction term G_1 determined by the boundary value problem (1.11), (1.16). Notice that (1.15) and (1.12) are satisfied. Existence of the remainder G_R for the boundary value problem (1.13)–(1.14) will be considered in the next section. Indeed, we have the following proposition.

Proposition 4.1. *The boundary value problem* (1.11), (1.16) *admits a unique solution* $G_1 = G_1(y, v)$ *satisfying*

$$G_1(-v_x) = -G_1(v_x),\qquad \int_{-1}^{1}\int_{\mathbb{R}^3} G_1(y,v)\,dv\,dy = 0, \tag{4.1}$$

and

$$\|w_q\partial_{v_x}^m G_1\|_{L^\infty} \le \tilde{C}_1, \tag{4.2}$$

for any integers $m \ge 0$ *and* $q \ge 0$, *where* $\tilde{C}_1 > 0$ *is a constant depending only on* m *and* q.

To prove this proposition, let $0 < \epsilon < 1$ and $0 \le \sigma \le 1$. Then we consider the following general approximation problem:

$$\epsilon G_1 + v_y\partial_y G_1 + \nu_0 G_1 = \sigma K G_1 + \mathfrak{F}, \tag{4.3}$$

$$G_1(\pm 1, v)|_{v_y \lessgtr 0} = 0, \tag{4.4}$$

where the source term $\mathfrak{F} = \mathfrak{F}(y, v)$ is given and satisfies $\mathfrak{F}(-v_x) = -\mathfrak{F}(v_x)$. Recall that ν_0 and K are defined by (2.3). The above boundary value problem can be formally reduced to

$$v_y \partial_y G_1 + LG_1 = \mathfrak{F},$$
$$G_1(\pm 1, v)|_{v_y \lessgtr 0} = 0,$$

as $\sigma \to 1^-$ and $\epsilon \to 0^+$. To prove this rigorously, we deduce the following a priori estimate.

Lemma 4.1 (A priori estimate). *The solution to the boundary value problem* (4.3)–(4.4) *satisfies the following estimate uniform in both σ and ϵ:*

$$\sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} \le \mathfrak{C}_0 \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty}, \tag{4.5}$$

where N_0 is an arbitrary nonnegative integer and the constant $\mathfrak{C}_0 > 0$ is independent of ϵ and σ.

Proof. The proof of (4.5) is divided into two steps.

L^∞ estimates. Let $\mathfrak{G}_m = w_q \partial_{v_x}^m G_1$ for $m \ge 0$ and $q \ge 0$. Then $\mathfrak{G}_m$ solves the problem

$$\begin{aligned}\epsilon\mathfrak{G}_m &+ v_y\partial_y\mathfrak{G}_m + \nu_0\mathfrak{G}_m \\ &= \sigma w_q K\partial_{v_x}^m G_1 + \sigma \mathbf{1}_{m>0}\sum_{m'<m} C_m^{m'} w_q(\partial_{v_x}^{m-m'}K)(\partial_{v_x}^{m'}G_1) - w_q\partial_{v_x}^m\mathfrak{F},\end{aligned} \tag{4.6}$$

$$\mathfrak{G}_m(\pm 1, v)|_{v_y\lessgtr 0} = 0. \tag{4.7}$$

We write the solution of (4.6)–(4.7) in the following mild form:

$$\begin{aligned}\mathfrak{G}_m(y,v) = \sigma &\int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}\frac{w_q}{v_y}K(w_{-q}\mathfrak{G}_m)(y')\,dy' \\ &+ \sigma\mathbf{1}_{m>0}\sum_{m'<m} C_m^{m'}\int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}\frac{w_q}{v_y}(\partial_{v_x}^{m-m'}K)(\partial_{v_x}^{m'}G_1)(y')\,dy' \\ &- \int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}\frac{w_q}{v_y}\partial_{v_x}^m\mathfrak{F}\,dy' =: \sum_{i=1}^{3}\mathfrak{I}_i \quad \text{for } v_y > 0,\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\mathfrak{G}_m(y,v) = -\sigma &\int_{y}^{1} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}\frac{w_q}{v_y}K(w_{-q}\mathfrak{G}_m)(y')\,dy' \\ &- \sigma\mathbf{1}_{m>0}\sum_{m'<m} C_m^{m'}\int_{y}^{1} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}\frac{w_q}{v_y}(\partial_{v_x}^{m-m'}K)(\partial_{v_x}^{m'}G_1)(y')\,dy' \\ &+ \int_{y}^{1} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}\frac{w_q}{v_y}\partial_{v_x}^m\mathfrak{F}\,dy' =: \sum_{i=4}^{6}\mathfrak{I}_i \quad \text{for } v_y < 0.\end{aligned}$$

We next compute $\mathfrak{I}_i$ $(1 \le i \le 6)$ term by term. Since

$$\begin{cases} \mathbf{1}_{v_y>0}\displaystyle\int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1}\,dy' \le \frac{1}{\nu_0+\epsilon}\big(1-e^{-\frac{2(\nu_0+\epsilon)}{|v_y|}}\big) < \frac{1}{\nu_0+\epsilon}, \\ \mathbf{1}_{v_y<0}\Big|\displaystyle\int_{y}^{1} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1}\,dy'\Big| < \frac{1}{\nu_0+\epsilon}, \end{cases} \tag{4.9}$$

we see that

$$|\mathfrak{I}_3|, |\mathfrak{I}_6| \le C\|w_q\partial_{v_x}^m \mathfrak{F}\|_{L^\infty}.$$

In view of definition (2.3) and Lemma 2.2,

$$|\mathfrak{I}_2|, |\mathfrak{I}_5| \le C\mathbf{1}_{m>0}\sum_{m'<m}\|w_q(\partial_{v_x}^{m-m'}K)(\partial_{v_x}^{m'}G_1)\|_{L^\infty} \le C\mathbf{1}_{m>0}\sum_{m'<m}\|w_q\partial_{v_x}^{m'}G_1\|_{L^\infty}.$$

Consequently, we have

$$\begin{aligned} |\mathfrak{G}_m(y,v)| \le{}& \mathbf{1}_{v_y>0}\,\sigma\int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v')|\mathfrak{G}_m(v',y')|\,dv'\,dy' \\ &+ \mathbf{1}_{v_y<0}\,\sigma\int_{y}^{1} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')}|v_y|^{-1}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v')|\mathfrak{G}_m(v',y')|\,dv'\,dy' \\ &+ C\mathbf{1}_{m>0}\sum_{m'<m}\|w_q\partial_{v_x}^{m'}G_1\|_{L^\infty} + C\|w_q\partial_{v_x}^m\mathfrak{F}\|_{L^\infty}, \end{aligned} \tag{4.10}$$

where $\mathbf{k}_w$ is given in Lemma 2.1. Then we iterate (4.10) once more to obtain

$$|\mathfrak{G}_m(y,v)| \le \sum_{i=1}^{6}\mathfrak{I}_{1,i} \tag{4.11}$$

with

$$\begin{aligned} \mathfrak{I}_{1,1} ={}& \mathbf{1}_{v_y>0}\,\sigma^2\int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v')\mathbf{1}_{v'_y>0}\int_{-1}^{y'} e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} {v'_y}^{-1} \\ &\times\int_{\mathbb{R}^3}\mathbf{k}_w(v',v'')|\mathfrak{G}_m(v'',y'')|\,dv''\,dy''\,dv'\,dy', \\ \mathfrak{I}_{1,2} ={}& \mathbf{1}_{v_y>0}\,\sigma^2\int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v')\mathbf{1}_{v'_y<0}\int_{y'}^{1} e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} |v'_y|^{-1} \\ &\times\int_{\mathbb{R}^3}\mathbf{k}_w(v',v'')|\mathfrak{G}_m(v'',y'')|\,dv''\,dy''\,dv'\,dy', \\ \mathfrak{I}_{1,3} ={}& \mathbf{1}_{v_y<0}\,\sigma^2\int_{y}^{1} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} |v_y|^{-1}\int_{\mathbb{R}^3}\mathbf{k}_w(v,v')\mathbf{1}_{v'_y>0}\int_{-1}^{y'} e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} {v'_y}^{-1} \\ &\times\int_{\mathbb{R}^3}\mathbf{k}_w(v',v'')|\mathfrak{G}_m(v'',y'')|\,dv''\,dy''\,dv'\,dy', \end{aligned}$$

$$\mathfrak{I}_{1,4} = \mathbf{1}_{v_y<0}\,\sigma^2 \int_y^1 e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} |v_y|^{-1} \int_{\mathbb{R}^3} \mathbf{k}_w(v,v')\mathbf{1}_{v'_y<0} \int_{y'}^1 e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} |v'_y|^{-1}$$
$$\times \int_{\mathbb{R}^3} \mathbf{k}_w(v',v'')|\mathfrak{G}_m(v'',y'')|\, dv''\, dy''\, dv'\, dy',$$
$$\mathfrak{I}_{1,5} = \mathbf{1}_{v_y>0}\,\sigma \int_{-1}^y e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1} \int_{\mathbb{R}^3} \mathbf{k}_w(v,v')$$
$$\times \Big(C\mathbf{1}_{m>0} \sum_{m'<m} \|w_q \partial_{v_x}^{m'} G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty} \Big)\, dv'\, dy',$$
$$\mathfrak{I}_{1,6} = \mathbf{1}_{v_y<0}\,\sigma \int_y^1 e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} |v_y|^{-1} \int_{\mathbb{R}^3} \mathbf{k}_w(v,v')$$
$$\times \Big(C\mathbf{1}_{m>0} \sum_{m'<m} \|w_q \partial_{v_x}^{m'} G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty} \Big)\, dv'\, dy'.$$

By using (4.9) and Lemma 2.1, we see that the last two terms can be bounded as

$$|\mathfrak{I}_{1,5}|, |\mathfrak{I}_{1,6}| \le C\mathbf{1}_{m>0} \sum_{m'<m} \|w_q \partial_{v_x}^{m'} G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty}.$$

For the other four terms, we only compute $\mathfrak{I}_{1,2}$ because the other three terms can be treated similarly. The estimates are divided into three cases. First of all, we take $M > 0$ large enough.

Case 1: $|v| > M$. In this case, Lemma 2.1 and (4.9) directly give

$$\mathfrak{I}_{1,2} \le \frac{C}{1+M} \|\mathfrak{G}_m\|_{L^\infty}.$$

Case 2: $|v| \le M$ and $|v'| > 2M$, or $|v'| \le 2M$ and $|v''| > 3M$. In this case, we have either $|v-v'| > M$ or $|v'-v''| > M$ so that one of the following two estimates holds respectively:

$$\mathbf{k}_w(v,v') \le Ce^{-\varepsilon M^2/16}\mathbf{k}_w(v,v')e^{\varepsilon|v-v'|^2/16},$$
$$\mathbf{k}_w(v',v'') \le Ce^{-\varepsilon M^2/16}\mathbf{k}_w(v',v'')e^{\varepsilon|v'-v''|^2/16}.$$

This together with Lemma 2.1 and (4.9) gives

$$\mathfrak{I}_{1,2} \le Ce^{-\varepsilon M^2/16} \|\mathfrak{G}_m\|_{L^\infty}.$$

Case 3: $|v| \le M$, $|v'| \le 2M$ and $|v''| \le 3M$. In this situation, we make use of the boundedness of the operator K on the complement of a singular set. For any large $N > 0$, we choose a number $M(N)$ to define

$$\begin{aligned} \mathbf{k}_{w,M}(v,v') &\equiv \mathbf{1}_{|v-v'|\ge 1/M,\, |v'|\le 2M}\mathbf{k}_w(v,v'), \\ \mathbf{k}_{w,M}(v',v'') &\equiv \mathbf{1}_{|v'-v''|\ge 1/M,\, |v''|\le 3M}\mathbf{k}_w(v',v''), \end{aligned} \tag{4.12}$$

such that

$$\sup_v \int_{\mathbb{R}^3} |\mathbf{k}_{w,M}(v,v') - \mathbf{k}_w(v,v')|\, dv' \le \frac{1}{N},$$
$$\sup_{v'} \int_{\mathbb{R}^3} |\mathbf{k}_{w,M}(v',v'') - \mathbf{k}_w(v',v'')|\, dv'' \le \frac{1}{N}.$$

Moreover, note that $\mathbf{k}_{w,M}(v,v'), \mathbf{k}_{w,M}(v',v'') \le C_M$. We further rewrite

$$\begin{aligned}\mathbf{k}_w(v,v')\mathbf{k}_w(v',v'') = {} & [\mathbf{k}_w(v,v') - \mathbf{k}_{w,M}(v,v')]\mathbf{k}_w(v',v'') \\ & + \mathbf{k}_{w,M}(v,v')[\mathbf{k}_w(v',v'') - \mathbf{k}_{w,M}(v',v'')] \\ & + \mathbf{k}_{w,M}(v',v'')\mathbf{k}_{w,M}(v,v').\end{aligned}$$

The first two difference terms lead to the small contribution to $\Im_{1,2}$ bounded by

$$\frac{C}{N}\|\mathfrak{G}_m\|_{L^\infty}.$$

For the last term, we use the decomposition

$$\begin{aligned}&\mathbf{1}_{v_y>0}\,\sigma^2 \int_{-1}^{y} e^{-\frac{\nu_0+\epsilon}{v_y}(y-y')} v_y^{-1} \int_{|v'|\le 2M,\ |v''|\le 3M} \mathbf{k}_{w,M}(v,v')\mathbf{k}_{w,M}(v',v'') \\ &\quad \times \mathbf{1}_{v'_y<0}\left[\int_{y'+\eta_0}^{1} + \int_{y'}^{y'+\eta_0}\right] e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} |v'_y|^{-1} |\mathfrak{G}_m(v'',y'')|\, dv''\, dy''\, dv'\, dy' \\ &\quad := \Im^{I}_{1,2} + \Im^{II}_{1,2},\end{aligned}$$

where $\eta_0 > 0$ is suitably small. For $\Im^I_{1,2}$, since $y'' - y' \ge \eta_0$, it follows that

$$\mathbf{1}_{v'_y<0}\, e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} |v'_y|^{-1} \le \frac{C}{\eta_0},$$

which together with Lemma 2.1 as well as (4.9) implies

$$\Im^I_{1,2} \le \frac{C_M}{\eta_0}\left\{\int_{|v''|\le 3M}\int_{-1}^{1} |\partial^m_{v_x} G_1(v'',y'')|^2\, dv''\, dy''\right\}^{1/2}.$$

As for $\Im^{II}_{1,2}$, since $y'' - y' \le \eta_0$, we find that for $\beta \in (0,1)$,

$$\begin{aligned}&\int_{|v'|\le 2M} \mathbf{1}_{v'_y<0} \int_{y'}^{y'+\eta_0} e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} |v'_y|^{-1}\, dy''\, dv' \\ &\quad = \int_{|v'|\le 2M} \mathbf{1}_{v'_y<0} \int_{y'}^{y'+\eta_0} e^{-\frac{\nu_0+\epsilon}{v'_y}(y'-y'')} \left|\frac{y'-y''}{v'_y}\right|^{\beta} |y'-y''|^{-\beta} |v'_y|^{-1+\beta}\, dy''\, dv' \\ &\quad \le C\int_{|v'|\le 2M} |v'_y|^{-1+\beta}\, dv' \int_{y'}^{y'+\eta_0} |y'-y''|^{-\beta}\, dy'' \le C_M \eta_0^{1-\beta}, \qquad (4.13)\end{aligned}$$

where we have used the fact that

$$e^{-\frac{\nu_0+\epsilon}{|v_y'|}|y'-y''|}\left|\frac{y'-y''}{v_y'}\right|^{\beta}<\infty.$$

Plugging (4.13) into $\mathfrak{I}_{1,2}^{II}$, we get

$$\mathfrak{I}_{1,2}^{II}\le C_M\eta_0^{1-\beta}\|\mathfrak{G}_m\|_{L^\infty}.$$

As a consequence,

$$\mathfrak{I}_{1,2}\le\left\{\frac{C}{N}+C_M\eta_0^{1-\beta}+Ce^{-\varepsilon M^2/16}\right\}\|\mathfrak{G}_m\|_{L^\infty}+C_M\|\partial_{v_x}^m G_1\|.$$

Substituting the above estimates into (4.11), we conclude

$$\|\mathfrak{G}_m\|_{L^\infty}\le C\mathbf{1}_{m>0}\sum_{m'<m}\|w_q\partial_{v_x}^{m'}G_1\|_{L^\infty}+C\|\partial_{v_x}^m G_1\|+C\|w_q\partial_{v_x}^m\mathfrak{F}\|_{L^\infty}. \quad (4.14)$$

A linear combination of (4.14) from $m=0$ to $m=N_0$ gives the a priori estimate

$$\sum_{0\le m\le N_0}\|\mathfrak{G}_m\|_{L^\infty}\le C\sum_{0\le m\le N_0}\|\partial_{v_x}^m G_1\|+C\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathfrak{F}\|_{L^\infty}, \quad (4.15)$$

where $C>0$ depends on N_0 and q. This concludes the L^∞ estimate.

L^2 estimates. To close the L^∞ estimate (4.15), we need to derive an L^2 estimate for G_1. For this, we first consider the zeroth order L^2 estimate on G_1. Notice that $G_1=\mathbf{P}_0G_1+\mathbf{P}_1G_1$ and $\mathbf{P}_0G_1=[a_1+\mathbf{b}_1\cdot v+c_1(|v|^2-3)]\sqrt{\mu}$ with $\mathbf{b}_1=[b_{1,1},b_{1,2},b_{1,3}]$. Moreover,

$$a_1=\langle G_1,\sqrt{\mu}\rangle,\quad \mathbf{b}_1=\langle G_1,v\sqrt{\mu}\rangle,\quad c_1=\tfrac{1}{6}\langle G_1,|v|^2\sqrt{\mu}\rangle.$$

On the other hand, from (4.8) with $m=0$, one has $G_1(y,-v_x,v_y,v_z)=-G_1(y,v_x,v_y,v_z)$, i.e. G_1 is odd in v_x. This implies

$$a_1=b_{1,2}=b_{1,3}=c_1=0. \quad (4.16)$$

To obtain an L^2 estimate of the macroscopic component, it remains now to deduce an L^2 estimate of $b_{1,1}$. Actually, one can show that

$$\|b_{1,1}\|^2\le C\|\mathbf{P}_1G_1\|^2+C\int_{v_y\gtrless 0}|v_y|G_1^2(\pm1)\,dv+C\|w_q\mathfrak{F}\|_{L^\infty}^2, \quad (4.17)$$

where $C>0$ is a constant independent of ϵ and σ. For this, we define

$$\Psi=\Psi_{b_{1,1}}=v_yv_x\frac{d}{dy}\phi_{b_{1,1}}(y)\sqrt{\mu},$$

where

$$-\phi_{b_{1,1}}''=b_{1,1},\quad \phi_{b_{1,1}}(\pm1)=0.$$

For the above boundary value problem on $b_{1,1}$, one has

$$\|\phi_{b_{1,1}}\|_{H^2} \le C\|b_{1,1}\|, \quad |\phi'_{b_{1,1}}(\pm 1)| \le C\|b_{1,1}\|. \tag{4.18}$$

Taking the inner product of (4.3) and $\Psi_{b_{1,1}}$ over $(-1,1)\times\mathbb{R}^3$, we get

$$\begin{aligned}\epsilon(G_1,\Psi_{b_{1,1}}) - (v_y G_1, \partial_y \Psi_{b_{1,1}}) + \langle v_y G_1(1), \Psi_{b_{1,1}}(1)\rangle - \langle v_y G_1(-1), \Psi_{b_{1,1}}(-1)\rangle \\ + (1-\sigma)\nu_0(G_1,\Psi_{b_{1,1}}) + \sigma(LG_1,\Psi_{b_{1,1}}) = (\mathfrak{F},\Psi_{b_{1,1}}).\end{aligned} \tag{4.19}$$

We now compute the terms in (4.19) one by one. By Cauchy–Schwarz's inequality and (4.18), one has

$$\begin{aligned}&[\epsilon + (1-\sigma)\nu_0]|(G_1,\Psi_{b_{1,1}})| \\ &\quad \le [\epsilon + (1-\sigma)\nu_0]|(\mathbf{P}_0 G_1,\Psi_{b_{1,1}})| + [\epsilon + (1-\sigma)\nu_0]|(\mathbf{P}_1 G_1,\Psi_{b_{1,1}})| \\ &\quad \le \eta[\epsilon + (1-\sigma)\nu_0]\|b_{1,1}\|^2 + C_\eta[\epsilon + (1-\sigma)\nu_0]\|\mathbf{P}_1 G_1\|^2, \\ -(v_y G_1, \partial_y \Psi_{b_{1,i}}) &= -(v_y \mathbf{P}_0 G_1, \partial_y \Psi_{b_{1,i}}) - (v_y \mathbf{P}_1 G_1, \partial_y \Psi_{b_{1,i}}) \\ &\ge \|b_{1,1}\|^2 - \eta\|b_{1,1}\|^2 - C_\eta\|\mathbf{P}_1 G_1\|^2, \\ |(\mathfrak{F},\Psi_{b_{1,1}})| &\le \eta\|b_{1,1}\|^2 + C_\eta\|w_q\mathfrak{F}\|_{L^\infty}^2.\end{aligned}$$

And by Lemma 2.2,

$$\sigma|(LG_1,\Psi_{b_{1,1}})| \le \eta\|b_{1,1}\|^2 + C_\eta\|\mathbf{P}_1 G_1\|^2.$$

For the boundary term, one deduces from (4.4) and (4.18) that

$$\begin{aligned}&\langle v_y G_1(1), \Psi_{b_{1,1}}(1)\rangle - \langle v_y G_1(-1), \Psi_{b_{1,1}}(-1)\rangle \\ &\quad = \int_{v_y>0} v_y G_1(1)\Psi_{b_{1,1}}(1)\,dv - \int_{v_y<0} v_y G_1(-1)\Psi_{b_{1,1}}(-1)\,dv \\ &\quad \le \eta\|b_{1,1}\|^2 + C_\eta \int_{v_y \gtrless 0} |v_y| G_1^2(\pm 1)\,dv.\end{aligned}$$

Combining the above estimates for the terms in (4.19), we get (4.17).

We now deduce an L^2 estimate of the microscopic component $\mathbf{P}_1 G_1$. A direct energy estimate for (4.3) gives

$$\begin{aligned}[\epsilon + (1-\sigma)\nu_0]\|G_1\|^2 + \delta_0\sigma\|\mathbf{P}_1 G_1\|^2 + \frac{1}{2}\int_{v_y \gtrless 0} |v_y| G_1^2(\pm 1)\,dv \\ \le \eta\|G_1\|^2 + C_\eta\|w_q\mathfrak{F}\|_{L^\infty}^2.\end{aligned} \tag{4.20}$$

Thus, (4.17) and (4.20) as well as (4.16) yield

$$\|G_1\|^2 + \int_{v_y \gtrless 0} |v_y| G_1^2(\pm 1)\,dv \le C\|w_q\mathfrak{F}\|_{L^\infty}^2. \tag{4.21}$$

Furthermore, to get higher order L^2 estimates on G_1, we see from $(\partial_{v_x}^m G_1, \partial_{v_x}^m(4.3))$ with $m \geq 1$ that

$$[\epsilon + (1-\sigma)\nu_0]\|\partial_{v_x}^m G_1\|^2 + \delta_0\sigma\|\partial_{v_x}^m G_1\|^2 + \frac{1}{2}\int_{v_y \gtrless 0} |v_y| \partial_{v_x}^m G_1^2(\pm 1)\, dv \leq C\|G_1\|^2 + C\|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty}^2, \tag{4.22}$$

where Lemma 2.3 has been used for $\sigma(\partial_{v_x}^m LG_1, \partial_{v_x}^m G_1)$.

Finally, the a priori estimate (4.5) follows from (4.15), (4.21) and (4.22). This completes the proof of Lemma 4.1. ∎

With the a priori estimate (4.5), we now prove the following existence result for general linear equations (4.3) and (4.4). Before doing this, we first define the function space

$$\mathfrak{X}_{N_0} = \left\{ g = g(y, v) \;\middle|\; \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m g\|_{L^\infty} < \infty,\ g(-v_x) = -g(v_x) \right\}$$

endowed with the norm

$$\|g\|_{\mathfrak{X}_{N_0}} = \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m g\|_{L^\infty}.$$

And for convenience, we also define a linear operator $\mathfrak{L}_\sigma$ by

$$\mathfrak{L}_\sigma g = [\epsilon + v_y \partial_y + \nu_0 - \sigma K] g.$$

Lemma 4.2. *Assume $\mathfrak{F} = \mathfrak{F}(y, v)$ satisfies*

$$\mathfrak{F}(-v_x) = -\mathfrak{F}(v_x), \qquad \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty} < \infty. \tag{4.23}$$

Then there exists a unique solution $G_1 = G_1(y, v)$ to (4.3)–(4.4) with $\sigma = 1$ satisfying

$$G_1(-v_x) = -G_1(v_x),$$

and

$$\sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} \leq \mathfrak{C}_0 \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty}, \tag{4.24}$$

where $\mathfrak{C}_0 > 0$ is a constant depending only on N_0 and q.

Proof. The proof is based on a bootstrap argument in the following three steps.

Step 1: Existence for $\sigma = 0$. If $\sigma = 0$, then (4.3)–(4.4) is reduced to the problem

$$\epsilon G_1 + v_y \partial_y G_1 + \nu_0 G_1 = \mathfrak{F},$$
$$G_1(\pm 1, v)|_{v_y \lessgtr 0} = 0,$$

which has a unique explicit solution

$$G_1(y,v) = \mathbf{1}_{v_y>0}\int_{-1}^{y} e^{-\frac{(\nu_0+\epsilon)(y-y')}{v_y}} v_y^{-1}\mathfrak{F}(y',v)\,dy' + \mathbf{1}_{v_y<0}\int_{y}^{1} e^{-\frac{(\nu_0+\epsilon)(y-y')}{v_y}} v_y^{-1}\mathfrak{F}(y',v)\,dy'. \tag{4.25}$$

Moreover, one sees that $G_1(-v_x) = -G_1(v_x)$ according to (4.23), and a direct calculation implies

$$\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m G_1\|_{L^\infty} \le \mathcal{C}_0 \sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m \mathfrak{F}\|_{L^\infty}.$$

Step 2: Existence for any $\sigma\in[0,\sigma_*]$ with some $\sigma_*>0$. Suppose $\sigma\in(0,1]$, and consider a more general problem

$$\epsilon G_1 + v_y\partial_y G_1 + \nu_0 G_1 = \sigma K G_1 + \mathfrak{F}, \tag{4.26}$$

$$G_1(\pm1,v)|_{v_y\lessgtr 0} = 0. \tag{4.27}$$

To solve this boundary value problem, we design the following approximation problem:

$$\begin{aligned}&\epsilon G_1^{n+1} + v_y\partial_y G_1^{n+1} + \nu_0 G_1^{n+1} = \sigma K G_1^n + \mathfrak{F},\\ &G_1^{n+1}(\pm1,v)|_{v_y\lessgtr 0} = 0,\end{aligned}$$

starting from $G_1^0=0$. Once G_1^n is given, G_1^{n+1} is well-defined by Step 1 and satisfies the estimate

$$\begin{aligned}\sum_{0\le m\le N_1}\|w_q\partial_{v_x}^m G_1^{n+1}\|_{L^\infty} &\le \mathcal{C}_0\sigma\sum_{0\le m\le N_1}\|w_q\partial_{v_x}^m K G_1^n\|_{L^\infty} + \mathcal{C}_0\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathfrak{F}\|_{L^\infty}\\ &\le \mathcal{C}_0\mathcal{C}_1\sigma\sum_{0\le m\le N_1}\|w_q\partial_{v_x}^m G_1^n\|_{L^\infty} + \mathcal{C}_0\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathfrak{F}\|_{L^\infty},\end{aligned} \tag{4.28}$$

where $\mathcal{C}_1>0$ depends only on K. If we choose $\sigma_*>0$ such that $\mathcal{C}_0\mathcal{C}_1\sigma_*\le\frac{1}{2}$, then (4.28) implies

$$\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m G_1^n\|_{L^\infty} \le 2\mathcal{C}_0\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathfrak{F}\|_{L^\infty} \tag{4.29}$$

for any $n\ge0$. Furthermore, one can also show that for $\sigma\in[0,\sigma_*]$,

$$\|[G_1^{n+1}-G_1^n]\|_{\mathfrak{X}_{N_0}} \le \mathcal{C}_0\mathcal{C}_1\sigma\|[G_1^n-G_1^{n-1}]\|_{\mathfrak{X}_{N_0}} \le \tfrac{1}{2}\|[G_1^n-G_1^{n-1}]\|_{\mathfrak{X}_{N_0}}, \tag{4.30}$$

which implies that $G_1^n\to G_1$ strongly in $\mathfrak{X}_{N_0}$. In addition, it is easy to see that $G_1^{n+1}(-v_x) = -G_1^{n+1}(v_x)$ if $G_1^n(-v_x) = -G_1^n(v_x)$. Thus, for $\sigma\in[0,\sigma_*]$, there exists a unique solution $G_1 \in \mathfrak{X}_{N_0}$ to the problem (4.26)–(4.27). Actually, the a priori estimate (4.5) implies that we still have the bound

$$\sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} \le \mathfrak{C}_0 \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m \mathfrak{F}\|_{L^\infty}.$$

In other words, it follows that

$$\|\mathfrak{L}_{\sigma_*}^{-1}\mathfrak{F}\|_{\mathfrak{X}_{N_0}} \le \mathfrak{C}_0\|\mathfrak{F}\|_{\mathfrak{X}_{N_0}}. \tag{4.31}$$

Step 3: Existence for $\sigma \in [0, 2\sigma_*]$. By using (4.31) and performing similar calculations to those resulting in (4.29) and (4.30), one can see that there exists a unique solution $G_1 \in \mathfrak{X}_{N_0}$ to the lifted equation

$$\epsilon G_1 + v_y\partial_y G_1 + \nu_0 G_1 - \sigma_* K G_1 = \sigma K G_1 + \mathfrak{F}, \quad G_1(\pm 1, v)|_{v_y \lessgtr 0} = 0,$$

with $\sigma \in [0, \sigma_*]$. Therefore, the solution mapping $\mathfrak{L}_{2\sigma_*}^{-1}$ is also well-defined on $\mathfrak{X}_{N_0}$ and the estimate (4.24) holds for $\sigma = 2\sigma_*$.

Finally, repeating the above procedure step by step, one can reach $\sigma = 1$ so that $\mathfrak{L}_1^{-1}$ exists and (4.24) also follows simultaneously. This completes the proof of Lemma 4.2. ■

Proof of Proposition 4.1. By setting $\mathfrak{F} = -v_x v_y\sqrt{\mu}$ in Lemma 4.2, we see that for any $\epsilon > 0$ there exists a unique solution $G_1^\epsilon \in \mathfrak{X}_{N_0}$ to the boundary value problem

$$\epsilon G_1^\epsilon + v_y\partial_y G_1^\epsilon + L G_1^\epsilon = -v_x v_y\sqrt{\mu}, \quad G_1^\epsilon(\pm 1, v)|_{v_y \lessgtr 0} = 0.$$

Notice that G_1^ϵ satisfies (4.1) and the estimate

$$\|G_1^\epsilon\|_{\mathfrak{X}_{N_0}} \le \tilde{C}_1,$$

where $\tilde{C}_1 > 0$ is independent of ϵ. Furthermore, we consider a positive sequence $\{\epsilon_n\}_{n=1}^\infty$ such that $|\epsilon_{n+1} - \epsilon_n| \le 2^{-n}$, and $\epsilon_n \to 0^+$ as $n \to \infty$. We consider the following approximation problems:

$$\begin{aligned}&\epsilon_n G_1^{\epsilon_n} + v_y\partial_y G_1^{\epsilon_n} + L G_1^{\epsilon_n} = -v_x v_y\sqrt{\mu},\\ &G_1^{\epsilon_n}(\pm 1, v)|_{v_y \lessgtr 0} = 0.\end{aligned}$$

Then letting $\bar{\mathfrak{G}}_{n+1} = G_1^{\epsilon_{n+1}} - G_1^{\epsilon_n}$, one sees that $\bar{\mathfrak{G}}_{n+1}$ satisfies

$$\begin{aligned}&\epsilon^{n+1}\bar{\mathfrak{G}}_{n+1} + v_y\partial_y\bar{\mathfrak{G}}_{n+1} + L\bar{\mathfrak{G}}_{n+1} = -(\epsilon^{n+1} - \epsilon^n)G_1^{\epsilon_n},\\ &\bar{\mathfrak{G}}_{n+1}|_{v_y \lessgtr 0} = 0.\end{aligned}$$

Thanks to Lemma 4.2, it follows that

$$\|\bar{\mathfrak{G}}_{n+1}\|_{\mathfrak{X}_{N_0}} \le \mathfrak{C}_0|\epsilon^{n+1} - \epsilon^n|\,\|G_1^{\epsilon_n}\|_{\mathfrak{X}_{N_0}} \le \mathfrak{C}\tilde{C}_1|\epsilon^{n+1} - \epsilon^n|.$$

This means that $\{G_1^{\epsilon_n}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathfrak{X}_{N_0}$. Thus, letting $n \to \infty$, the limit function denoted by G_1 is the unique solution of (1.11), (1.16). Moreover, G_1 satisfies (4.1) and the bound (4.2). The proof of Proposition 4.1 is thus complete. ■

5. Steady problem: remainder

Based on Proposition 4.1, one can further show the following existence result for the remainder G_R. Recall the steady problem (1.8) as well as (1.9) and (1.10).

Proposition 5.1. *The boundary value problem* (1.13)–(1.14) *admits a unique solution* $G_R = G_R(y, v)$ *with* $\widetilde{G}_R = \sqrt{\mu}\, G_R$ *satisfying*

$$\int_{-1}^{1}\int_{\mathbb{R}^3} \widetilde{G}_R(y, v)\, dv\, dy = 0.$$

And there is an integer $q_0 > 0$ *such that for any integer* $q \geq q_0$*, there is* $\alpha_0 = \alpha_0(q) > 0$ *such that for any* $\alpha \in (0, \alpha_0)$ *and any integer* $m \geq 0$*,*

$$\|w_q \partial_{v_x}^m \widetilde{G}_R\|_{L^\infty} \leq \tilde{C}_R,$$

where $\tilde{C}_R > 0$ *is a constant depending only on* m *and* q *but independent of* α*.*

5.1. Caflisch's decomposition

To prove Proposition 5.1, we follow the strategy of the proof in [15] for treating the shear force term in the framework of perturbation. In fact, notice that there is a growth term $\frac{1}{2}\alpha v_x v_y G_R$ in equation (1.13). To treat this growth in velocity, the key point is to use Caflisch's decomposition [10] and an algebraic weighted estimate introduced originally by Arkeryd–Esposito–Pulvirenti [2]. For this purpose, we first decompose the remainder G_R as

$$\sqrt{\mu}\, G_R = G_{R,1} + \sqrt{\mu}\, G_{R,2}, \tag{5.1}$$

where $G_{R,1}$ and $G_{R,2}$ satisfy the following two boundary value problems, respectively:

$$\begin{aligned} &v_y \partial_y G_{R,1} - \alpha v_y \partial_{v_x} G_{R,1} + \nu_0 G_{R,1} \\ &\qquad = \chi_M \mathcal{K} G_{R,1} - \tfrac{1}{2}\alpha \sqrt{\mu}\, v_x v_y G_{R,2} - \tfrac{1}{2}\sqrt{\mu}\, v_x v_y G_1 + \sqrt{\mu}\, v_y \partial_{v_x} G_1 \\ &\qquad\quad + Q(\sqrt{\mu}\, G_1, \sqrt{\mu}\, G_1) \\ &\qquad\quad + \alpha\{Q(\sqrt{\mu}\, G_R, \sqrt{\mu}\, G_1) + Q(\sqrt{\mu}\, G_1, \sqrt{\mu}\, G_R)\} \\ &\qquad\quad + \alpha^2 Q(\sqrt{\mu}\, G_R, \sqrt{\mu}\, G_R), \quad y \in (-1, 1),\ v \in \mathbb{R}^3, \end{aligned} \tag{5.2}$$

$$G_{R,1}(\pm 1, v)|_{v_y \lessgtr 0} = 0, \quad v \in \mathbb{R}^3, \tag{5.3}$$

and

$$\begin{aligned} &v_y \partial_y G_{R,2} - \alpha v_y \partial_{v_x} G_{R,2} + L G_{R,2} \\ &\qquad = (1 - \chi_M)\mu^{-1/2} \mathcal{K} G_{R,1}, \quad y \in (-1, 1),\ v \in \mathbb{R}^3, \end{aligned} \tag{5.4}$$

$$G_{R,2}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu}\, G_R(\pm 1, v)|v_y|\, dv, \quad v \in \mathbb{R}^3. \tag{5.5}$$

Here $\chi_M(v)$ is a nonnegative smooth cutoff function such that

$$\chi_M(v) = \begin{cases} 1, & |v| \geq M+1, \\ 0, & |v| \leq M, \end{cases}$$

and $\mathcal{K}$ is defined by (2.4). Existence of solutions for (5.2)–(5.3) and (5.4)–(5.5) can be proved via the approximation sequence by iteratively solving the systems

$$\begin{aligned} &\epsilon G^{n+1}_{R,1} + v_y \partial_y G^{n+1}_{R,1} - \alpha v_y \partial_{v_x} G^{n+1}_{R,1} + \nu_0 G^{n+1}_{R,1} \\ &\qquad = \chi_M \mathcal{K} G^{n+1}_{R,1} - \tfrac{1}{2}\alpha\sqrt{\mu}\, v_x v_y G^{n+1}_{R,2} - \tfrac{1}{2}\sqrt{\mu}\, v_x v_y G_1 + \sqrt{\mu}\, v_y \partial_{v_x} G_1 \\ &\qquad\quad + Q(\sqrt{\mu}\, G_1, \sqrt{\mu}\, G_1) \\ &\qquad\quad + \alpha\{Q(\sqrt{\mu}\, G^n_R, \sqrt{\mu}\, G_1) + Q(\sqrt{\mu}\, G_1, \sqrt{\mu}\, G^n_R)\} \\ &\qquad\quad + \alpha^2 Q(\sqrt{\mu}\, G^n_R, \sqrt{\mu}\, G^n_R), \quad y \in (-1,1),\ v \in \mathbb{R}^3, \end{aligned} \tag{5.6}$$

$$G^{n+1}_{R,1}(\pm 1, v)|_{v_y \lessgtr 0} = 0, \quad v \in \mathbb{R}^3, \tag{5.7}$$

and

$$\begin{aligned} &\epsilon G^{n+1}_{R,2} + v_y \partial_y G^{n+1}_{R,2} - \alpha v_y \partial_{v_x} G^{n+1}_{R,2} + L G^{n+1}_{R,2} \\ &\qquad = (1-\chi_M)\mu^{-1/2} \mathcal{K} G^{n+1}_{R,1}, \quad y \in (-1,1),\ v \in \mathbb{R}^3, \end{aligned} \tag{5.8}$$

$$G^{n+1}_{R,2}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu}\, G^{n+1}_R(\pm 1, v)|v_y|\, dv, \quad v \in \mathbb{R}^3, \tag{5.9}$$

for a small parameter $\epsilon > 0$, where we have set $[G^0_{R,1}, G^0_{R,2}] = [0,0]$ for $n = 0$.

The proof of Proposition 5.1 is in three steps. First, similarly to treating the existence of G_1, we introduce a modified coupled boundary value problem with two parameters $\epsilon > 0$ and $0 \leq \sigma \leq 1$. This boundary value problem is directly solvable via the method of characteristics in the case of $\sigma = 0$ corresponding to the homogeneous inflow data, and we then lift the value of σ from $\sigma = 0$ for the zero inflow data to $\sigma = 1$ for the full diffuse reflection boundary condition by a bootstrap argument. Second, we establish the limit $n \to \infty$ for any fixed $\epsilon > 0$. Third, we let $\epsilon \to 0^+$ to obtain the desired solution which satisfies (5.2)–(5.3) and (5.4)–(5.5). As a result, with the help of (5.1), we get the solution to the original boundary value problem (1.13)–(1.14).

5.2. *A priori estimates with parameters ϵ and σ*

Let us first show that $[G^{n+1}_{R,1}, G^{n+1}_{R,2}]$ is well-defined once $[G^n_{R,1}, G^n_{R,2}]$ is given. To do this, we apply the contraction method. We define the linear vector operator parameterized by $\sigma \in [0,1]$ as follows:

$$\mathcal{L}_\sigma[\mathcal{G}_1, \mathcal{G}_2] = [\mathcal{L}^1_\sigma, \mathcal{L}^2_\sigma][\mathcal{G}_1, \mathcal{G}_2],$$

where

$$
\begin{aligned}
&\mathcal{L}_\sigma^1[\mathcal{G}_1,\mathcal{G}_2]\\
&=\begin{cases}\epsilon\mathcal{G}_1+v_y\partial_y\mathcal{G}_1-\alpha v_y\partial_{v_x}\mathcal{G}_1+\nu_0\mathcal{G}_1-\sigma\chi_M\mathcal{K}G_1+\alpha\frac{v_xv_y}{2}\sqrt{\mu}\,\mathcal{G}_2, & y\in(-1,1),\\ \mathcal{G}_1(\pm1,v)\mathbf{1}_{\{v_y\lessgtr 0\}}, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{L}_\sigma^2[\mathcal{G}_1,\mathcal{G}_2]\\
&=\begin{cases}\epsilon\mathcal{G}_2+v_y\partial_y\mathcal{G}_2-\alpha v_y\partial_{v_x}\mathcal{G}_2+\nu_0\mathcal{G}_2-\sigma K\mathcal{G}_2-\sigma(1-\chi_M)\mu^{-1/2}\mathcal{K}\mathcal{G}_1, & y\in(-1,1),\\ \mathcal{G}_2(\pm1,v)\mathbf{1}_{\{v_y\lessgtr 0\}}-\sigma\sqrt{2\pi\mu}\displaystyle\int_{v_y\gtrless 0}(\mathcal{G}_1+\sqrt{\mu}\,\mathcal{G}_2)(\pm1,v)|v_y|\,dv. \end{cases}
\end{aligned}
$$

We then consider the solvability of the following coupled linear system:

$$
\begin{cases}\mathcal{L}_\sigma^1[\mathcal{G}_1,\mathcal{G}_2]=\mathcal{F}_1, \quad \mathcal{L}_\sigma^2[\mathcal{G}_1,\mathcal{G}_2]=\mathcal{F}_2, \quad y\in(-1,1),\\ \mathcal{L}_\sigma^1[\mathcal{G}_1,\mathcal{G}_2]=0, \quad \mathcal{L}_\sigma^2[\mathcal{G}_1,\mathcal{G}_2]=\mathcal{F}_{2,b}, \quad y=\pm1,\end{cases} \tag{5.10}
$$

where $\mathcal{F}_1$, $\mathcal{F}_1$ and $\mathcal{F}_{2,b}$ are given, and $\langle\mathcal{F}_1,1\rangle+\langle\mathcal{F}_2,\sqrt{\mu}\rangle=0$. In the rest of the proof, for brevity, we denote

$$
\tilde{\mathcal{F}}_1=\begin{cases}\mathcal{F}_1, & y\in(-1,1),\\ 0, & y=\pm1,\end{cases}\qquad \tilde{\mathcal{F}}_2=\begin{cases}\mathcal{F}_2, & y\in(-1,1),\\ \mathcal{F}_{2,b}(\pm1,v), & y=\pm1.\end{cases}
$$

In what follows, we look for solutions to the system (5.10) in the Banach space

$$
\mathbf{X}_{\alpha,N_0}=\left\{[\mathcal{G}_1,\mathcal{G}_2]\;\Big|\;\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m[\mathcal{G}_1,\mathcal{G}_2]\|_{L^\infty}<\infty\right\} \tag{5.11}
$$

endowed with the norm

$$
\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\mathbf{X}_{\alpha,N_0}}=\sum_{0\le m\le N_0}\{\|w_q\partial_{v_x}^m\mathcal{G}_1\|_{L^\infty}+\|w_q\partial_{v_x}^m\mathcal{G}_2\|_{L^\infty}\}.
$$

Let us now deduce an a priori estimate for the parameterized linear system (5.10).

Lemma 5.1 (A priori estimate). *Let $[\mathcal{G}_1,\mathcal{G}_2]\in\mathbf{X}_{\alpha,N_0}$ with $\alpha>0$ and $N_0\ge0$ be a solution to (5.10) with $\epsilon>0$ suitably small and $\sigma\in[0,1]$, and let $[\tilde{\mathcal{F}}_1,\tilde{\mathcal{F}}_2]\in\mathbf{X}_{\alpha,N_0}$ with $\langle\mathcal{F}_1,1\rangle+\langle\mathcal{F}_2,\sqrt{\mu}\rangle=0$. There is $q_0>0$ such that for any $q\ge q_0$ arbitrarily large, there are $\alpha_0=\alpha_0(q)>0$ and large $M=M(q)>0$ such that for any $0<\alpha<\alpha_0$, the solution $[\mathcal{G}_1,\mathcal{G}_2]$ satisfies the estimate*

$$
\begin{aligned}
\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\mathbf{X}_{\alpha,N_0}}&=\|\mathcal{L}_\sigma^{-1}[\tilde{\mathcal{F}}_1,\tilde{\mathcal{F}}_2]\|_{\mathbf{X}_{\alpha,N_0}}\\
&\le C_{\mathcal{L}}\sum_{0\le m\le N_0}\{\|w_q\partial_{v_x}^m\mathcal{F}_1\|_{L^\infty}+\|w_q\partial_{v_x}^m\mathcal{F}_2\|_{L^\infty}+\|w_q\mathcal{F}_{2,b}\|_{L^\infty}\},
\end{aligned} \tag{5.12}
$$

where the constant $C_{\mathcal{L}}>0$ may depend on ϵ but not on σ or α.

Proof. The proof is divided into two steps.

Step 1: L^∞ estimates. Let $0 \le m \le N_0$ and $q > 0$, and denote

$$[H_{1,m}, H_{2,m}] = [w_q \partial_{v_x}^m \mathcal{G}_1, w_q \partial_{v_x}^m \mathcal{G}_2].$$

Then we see that $[H_{1,m}, H_{2,m}]$ satisfies

$$\begin{aligned}\epsilon H_{1,m} &+ v_y \partial_y H_{1,m} - \alpha v_y \partial_{v_x} H_{1,m} + 2q\alpha \frac{v_y v_x}{1+|v|^2} H_{1,m} + \nu_0 H_{1,m} \\ &- \sigma \chi_M w_q \mathcal{K}\left(\frac{H_{1,m}}{w_q}\right) - \sigma \mathbf{1}_{m>0} \sum_{1\le m'\le m} C_m^{m'} w_q \partial_{v_x}^{m'} (\chi_M \mathcal{K}) \partial_{v_x}^{m-m'} \mathcal{G}_1 \\ &+ \alpha \sum_{0\le m'\le m} C_m^{m'} w_q \partial_{v_x}^{m'} \left(\frac{v_x v_y}{2}\sqrt{\mu}\right) \partial_{v_x}^{m-m'} \mathcal{G}_2 = w_q \partial_{v_x}^m \mathcal{F}_1, \end{aligned} \tag{5.13}$$

$$H_{1,m}(\pm 1, v)|_{\{v_y \lessgtr 0\}} = 0, \tag{5.14}$$

and

$$\begin{aligned}\epsilon H_{2,m} &+ v_y \partial_y H_{2,m} - \alpha v_y \partial_{v_x} H_{2,m} + 2q\alpha \frac{v_y v_x}{1+|v|^2} H_{1,m} + \nu_0 H_{2,m} \\ &- \sigma w_q K\left(\frac{H_{2,m}}{w_q}\right) - \sigma \mathbf{1}_{m>0} \sum_{1\le m'\le m} C_m^{m'} w_q \partial_{v_x}^{m'} K \partial_{v_x}^{m-m'} \mathcal{G}_2 \\ &- \sigma \sum_{0\le m'\le m} C_m^{m'} w_q \partial_{v_x}^{m'} ((1-\chi_M)\mu^{-1/2}) \partial_{v_x}^{m-m'} \mathcal{K} \mathcal{G}_1 = w_q \partial_{v_x}^m \mathcal{F}_2, \end{aligned} \tag{5.15}$$

$$\begin{aligned} H_{2,m}(\pm 1, v)\mathbf{1}_{\{v_y \lessgtr 0\}} - \sigma w_q \partial_{v_x}^m (\sqrt{2\pi\mu}) \int_{v_y \gtrless 0} (\mathcal{G}_1 + \sqrt{\mu}\, \mathcal{G}_2)(\pm 1, v) |v_y|\, dv \\ = w_q \partial_{v_x}^m \mathcal{F}_{2,b}(\pm 1). \end{aligned} \tag{5.16}$$

Recall the trajectory $[Y(s;t,y,v), V(s;t,y,v)]$ defined in (3.4). In addition, for $(y, v) \in [-1, 1] \times \mathbb{R}^3$, we define the backward exit time $t_{\mathbf{b}}(y, v)$ as

$$t_{\mathbf{b}}(y, v) = \inf\{s : y - s v_y \notin (-1, 1),\ s > 0\}, \tag{5.17}$$

which is the first time that the backward characteristic line $[Y(s;t,y,v), V(s;t,y,v)]$ exits $(-1, 1)$. Note that at the boundary $y = \pm 1$, $t_{\mathbf{b}}(y, v)$ is well-defined if $\pm v_y > 0$. For any (y, v), we use $t_{\mathbf{b}}(y, v)$ when it is well-defined. Furthermore, we denote

$$y_{\mathbf{b}}(y, v) = y - t_{\mathbf{b}}(y, v) v_y \in \{-1, 1\},$$

and for a random variable v_k, we define the backward time cycle

$$\begin{aligned} &(t_0, y_0, v_0) = (t, y, v), \\ &(t_{k+1}, y_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(y_k, v_k), y_{\mathbf{b}}(y_k, v_k), v_{k+1}), \quad k \ge 0. \end{aligned} \tag{5.18}$$

We also set

$$Y_{\mathbf{cl}}^{l}(s;t,y,v)=\mathbf{1}_{[t_{l+1},t_l)}(s)\{y_l+(s-t_l)v_{ly}\},$$
$$V_{\mathbf{cl}}^{l}(s;t,y,v)=\mathbf{1}_{[t_{l+1},t_l)}(s)(v_{lx}+\alpha(t_l-s)v_{ly},v_{ly},v_{lz}).$$

Note that $[Y_{\mathbf{cl}}^{0}(s),V_{\mathbf{cl}}^{0}(s)]=[Y(s),V(s)]$ and $y_l=\pm 1$ for $l\geq 1$. Moreover, t_l can be negative.

Define $\mathcal{V}_j=\{v_j\in\mathbb{R}^3\mid v_j\cdot n(y_j)>0\}$, where $n(y_j)=(0,1,0)$ if $y_j=1$ and $n(y_j)=(0,-1,0)$ if $y_j=-1$. Let the iterated integral for $k\geq 2$ be defined as

$$\int_{\prod_{l=1}^{k-1}\mathcal{V}_l}\prod_{l=1}^{k-1}d\sigma_l:=\int_{\mathcal{V}_1}\cdots\left\{\int_{\mathcal{V}_{k-1}}d\sigma_{k-1}\right\}d\sigma_1, \tag{5.19}$$

where $d\sigma_l=\sqrt{2\pi}\,\mu(v_l)|v_{ly}|\,dv_l$ is a probability measure.

Without loss of generality, we assume $\lim_{|t|\to\infty}[H_{1,m},H_{2,m}](t)=0$. Along the characteristic line (3.3), for $(y,v)\in[-1,1]\times\mathbb{R}^3\setminus(\gamma_-\cup\gamma_0)$, we write the solution of the system (5.13)–(5.14) in the mild form as follows:

$$\begin{aligned}
&H_{1,m}(t)=H_{1,m}(y(t),v(t))\\
&=\sigma\int_{t_1}^{t}e^{-\int_s^t\mathcal{A}^{\epsilon}(\tau,V(\tau))\,d\tau}\left\{\chi_M w_q\mathcal{K}\left(\frac{H_{1,m}}{w_q}\right)\right\}(Y(s),V(s))\,ds\\
&\quad+\sigma\mathbf{1}_{m>0}\int_{t_1}^{t}e^{-\int_s^t\mathcal{A}^{\epsilon}(\tau,V(\tau))\,d\tau}\sum_{1\leq m'\leq m}C_m^{m'}\{w_q\partial_{v_x}^{m'}(\chi_M\mathcal{K})\partial_{v_x}^{m-m'}\mathcal{G}_1\}(Y(s),V(s))\,ds\\
&\quad-\alpha\int_{t_1}^{t}e^{-\int_s^t\mathcal{A}^{\epsilon}(\tau,V(\tau))\,d\tau}\sum_{0\leq m'\leq m}C_m^{m'}\left\{w_q\partial_{v_x}^{m'}\left(\frac{v_xv_y}{2}\sqrt{\mu}\right)\partial_{v_x}^{m-m'}\mathcal{G}_2\right\}(Y(s),V(s))\,ds\\
&\quad+\int_{t_1}^{t}e^{-\int_s^t\mathcal{A}^{\epsilon}(\tau,V(\tau))\,d\tau}(w_q\partial_{v_x}^{m}\mathcal{F}_1)(Y(s),V(s))\,ds,
\end{aligned}$$

where

$$\mathcal{A}^{\epsilon}(\tau,V(\tau))=\nu_0+\epsilon+2q\alpha\frac{V_y(\tau)V_x(\tau)}{1+|V(\tau)|^2}\geq\nu_0/2, \tag{5.20}$$

provided that $\epsilon>0$ and $q\alpha>0$ are suitably small. By Lemma 2.4, it is straightforward to see that

$$\begin{aligned}
&\sup_{-\infty<t<\infty}\|H_{1,m}(t)\|_{L^\infty}\\
&\qquad\leq\frac{C}{q}\sum_{m'\leq m}\sup_{-\infty<t<\infty}\|H_{1,m'}(t)\|_{L^\infty}+C\alpha\sum_{m'\leq m}\sup_{-\infty<t<\infty}\|H_{2,m'}(t)\|_{L^\infty}\\
&\qquad\quad+\sup_{-\infty<t<\infty}\|w_q\partial_{v_x}^{m}\mathcal{F}_1(t)\|_{L^\infty}.
\end{aligned}\tag{5.21}$$

By taking q sufficiently large, (5.21) further gives

$$\sum_{0\le m\le N_0}\|H_{1,m}\|_{L^\infty}\le C\alpha\sum_{0\le m\le N_0}\|H_{2,m}\|_{L^\infty}+\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathcal{F}_1\|_{L^\infty}. \tag{5.22}$$

Similarly, one can also write the solution of (5.15)–(5.16) in the mild form of

$$\begin{aligned}
H_{2,m}(t) &= \underbrace{\sigma\int_{t_1}^t e^{-\int_s^t \mathcal{A}^\epsilon(\tau,V(\tau))\,d\tau}\Big\{w_q K\Big(\frac{H_{2,m}}{w_q}\Big)\Big\}(Y(s),V(s))\,ds}_{I_1}\\
&\underbrace{+\sigma\mathbf{1}_{m>0}\int_{t_1}^t e^{-\int_s^t \mathcal{A}^\epsilon(\tau,V(\tau))\,d\tau}\sum_{1\le m'\le m}C_m^{m'}\{w_q(\partial_{v_x}^{m'}K)\partial_{v_x}^{m-m'}\mathcal{G}_2\}(Y(s),V(s))\,ds}_{I_2}\\
&\underbrace{+\sigma\int_{t_1}^t e^{-\int_s^t \mathcal{A}^\epsilon(\tau,V(\tau))\,d\tau}\sum_{0\le m'\le m}C_m^{m'}\{w_q\partial_{v_x}^{m'}((1-\chi_M)\mu^{-1/2})\partial_{v_x}^{m-m'}\mathcal{K}\mathcal{G}_1\}(Y(s),V(s))\,ds}_{I_3}\\
&\underbrace{+\int_{t_1}^t e^{-\int_s^t \mathcal{A}^\epsilon(\tau,V(\tau))\,d\tau}(w_q\partial_{v_x}^m\mathcal{F}_2)(Y(s),V(s))\,ds+e^{-\int_{t_1}^t \mathcal{A}^\epsilon(\tau,V(\tau))\,d\tau}(w_q\partial_{v_x}^m\mathcal{F}_{2,b})(t_1,y_1,V(t_1))}_{I_4}\\
&+\sum_{n=5}^{10}I_n,
\end{aligned}\tag{5.23}$$

where $(y,v)\in[-1,1]\times\mathbb{R}^3\setminus(\gamma_-\cup\gamma_0)$, and for $k\ge 2$,

$$\begin{aligned}
I_5 &= \sigma^{k-1}\underbrace{\sqrt{2\pi}\,e^{-\int_{t_1}^t \mathcal{A}^\epsilon(\tau,V(\tau))\,d\tau}[w_q\partial_{v_x}^m(\sqrt{\mu})](V(t_1))}_{\mathcal{W}}\\
&\quad\times\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}(w_q\mathcal{G}_2)(t_k,y_k,V_{\mathbf{cl}}^{k-1}(t_k))\,d\Sigma_{k-1}(t_k),\\
I_6 &= \sum_{l=2}^{k-1}\sigma^{l-1}\mathcal{W}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}(w_q\mathcal{F}_{2,b})(t_l,y_l,V_{\mathbf{cl}}^{l-1}(t_l))\,d\Sigma_l(t_l),\\
I_7 &= \sigma^l\sum_{l=1}^{k-1}\mathcal{W}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}w_q\mathcal{F}_2(Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\Sigma_l(s)\,ds,\\
I_8 &= \sigma^l\sum_{l=1}^{k-1}\mathcal{W}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}\Big\{w_q K\Big(\frac{H_{2,0}}{w_q}\Big)\Big\}(Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\Sigma_l(s)\,ds,\\
I_9 &= \sigma^l\sum_{l=1}^{k-1}\mathcal{W}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}\Big\{(1-\chi_M)w_q\mu^{-1/2}\mathcal{K}\Big(\frac{H_{1,0}}{w_q}\Big)\Big\}(Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\Sigma_l(s)\,ds,\\
I_{10} &= \sigma^l\mathcal{W}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\Big(\frac{w_q}{\sqrt{\mu}}\mathcal{G}_1\Big)(t_l,y_l,V_{\mathbf{cl}}^{l-1}(t_l))d\Sigma_l(t_l).
\end{aligned}$$

In the above expressions we have used the notations

$$\Sigma_l(s) = \prod_{j=l+1}^{k-1} d\sigma_j \, e^{-\int_s^{t_l} \mathcal{A}^\epsilon(\tau, V_{\mathbf{cl}}^l(\tau))\, d\tau} \tilde{w}_2(v_l)\, d\sigma_l \times \prod_{j=1}^{l-1} \frac{\tilde{w}_2(v_j)}{\tilde{w}_2(V_{\mathbf{cl}}^j(t_{j+1}))} e^{-\int_{t_{j+1}}^{t_j} \mathcal{A}^\epsilon(\tau, V_{\mathbf{cl}}^l(\tau))\, d\tau} d\sigma_j, \tag{5.24}$$

and

$$\tilde{w}_2(v) = (\sqrt{2\pi} w_q \sqrt{\mu})^{-1}. \tag{5.25}$$

L^∞ estimates for $H_{2,m}$ are more complicated because K has no smallness property. To overcome this, we have to iterate (5.23) twice. Let us first compute I_n $(1 \le n \le 10)$ term by term. Recalling the definition (2.5) of $\mathbf{k}_w$, one directly has, by (5.20),

$$|I_1| \le \int_{t_1}^t e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s), v') |H_{2,m}(s, Y(s), v')| \, dv' \, ds.$$

By Lemma 2.2, it follows that

$$\begin{aligned} |I_2| &\le C \mathbf{1}_{m>0} \sum_{m' \le m-1} \|w_q \partial_{v_x}^{m'} \mathcal{G}_2\|_{L^\infty} \int_{t_1}^t e^{-\frac{\nu_0}{2}(t-s)} \, ds \\ &\le C \mathbf{1}_{m>0} \sum_{m' \le m-1} \|w_q \partial_{v_x}^{m'} \mathcal{G}_2\|_{L^\infty}, \end{aligned}$$

and similarly

$$|I_3| \le C \sum_{m' \le m} \|w_q \partial_{v_x}^{m'} \mathcal{G}_1\|_{L^\infty}.$$

It is straightforward to see that

$$|I_4| \le C \|w_q \partial_{v_x}^m \mathcal{F}_2\|_{L^\infty} + C \|w_q \partial_{v_x}^m \mathcal{F}_{2,b}\|_{L^\infty}.$$

Next, notice that

$$|\mathcal{W}| \le C m! 4^q q! e^{-\nu_0 (t-t_1)/2}.$$

In the sequel, for simplicity, we denote by $\mathcal{C}_{m,q}$ the constant $m!4^q q!$. By Lemma 8.1,

$$\begin{aligned} |I_5| &\le C \mathcal{C}_{m,q} 2^{-C_2 T_0^{5/4}} e^{-\frac{\nu_0}{2}(t-t_1)} \|H_{2,0}\|_{L^\infty}, \\ |I_6| + |I_7| &\le C k \mathcal{C}_{m,q} e^{-\frac{\nu_0}{2}(t-t_1)} \{\|w_q \mathcal{F}_2\|_{L^\infty} + \|w_q \mathcal{F}_{2,b}\|_{L^\infty}\}, \\ |I_9|, \ |I_{10}| &\le C \mathcal{C}_{m,q} k e^{-\frac{\nu_0}{2}(t-t_1)} \|H_{1,0}\|_{L^\infty}, \end{aligned}$$

$$|I_8| \le C\mathscr{C}_{m,q}e^{-\frac{\nu_0}{2}(t-t_1)}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}\int_{\mathbb{R}^3}\mathbf{k}_w(V_{\mathbf{cl}}^l(s),v')|H_{2,0}(s,Y_{\mathbf{cl}}^l(s),v')| \\ \times dv'\,d\Sigma_l(s)\,ds,$$

where we have taken $T_0 = t - t_k$ with $k = C_1T_0^{5/4}$, and both $C_1 > 0$ and $C_2 > 0$ are given in Lemma 8.1.

Putting all the estimates for I_n $(1 \le n \le 10)$ above together and adjusting the constants, we have

$$\begin{aligned}
&|H_{2,m}(t)| \\
&\le C\mathscr{C}_{m,q}e^{-\frac{\nu_0}{2}(t-t_1)}\int_{t_1}^{t}e^{-\frac{\nu_0}{2}(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_w(V(s),v')|H_{2,m}(s,Y(s;t,y,v),v')|\,dv'\,ds \\
&\quad + C\mathscr{C}_{m,q}e^{-\frac{\nu_0}{2}(t-t_1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}\int_{\mathbb{R}^3}\mathbf{k}_w(V_{\mathbf{cl}}^l(s),v') \\
&\qquad\qquad \times|H_{2,0}(s,Y_{\mathbf{cl}}^l(s;t,y,v),v')|\,dv'\,d\Sigma_l(s)\,ds \\
&\quad + \mathcal{Q}(t),
\end{aligned}\tag{5.26}$$

where

$$\begin{aligned}
\mathcal{Q}(t) = {}& C\mathbf{1}_{m>0}\sum_{m'\le m-1}\sup_{-\infty<s\le t}\|H_{2,m'}(s)\|_{L^\infty} + C\sum_{m'\le m}\sup_{-\infty<s\le t}\|H_{1,m'}(s)\|_{L^\infty} \\
&+ C\sup_{-\infty<s\le t}\|w_q\partial_{v_x}^m\mathcal{F}_2(s)\|_{L^\infty} + C\sup_{-\infty<s\le t}\|w_q\partial_{v_x}^m\mathcal{F}_{2,b}(s)\|_{L^\infty} \\
&+ C\mathscr{C}_{m,q}2^{-C_2T_0^{5/4}}\sup_{-\infty<s\le t}\|H_{2,0}(s)\|_{L^\infty} + C\mathscr{C}_{m,q}k\sup_{-\infty<s\le t}\|H_{1,0}(s)\|_{L^\infty} \\
&+ C\mathscr{C}_{m,q}k\sup_{-\infty<s\le t}\|w_q\mathcal{F}_2(s)\|_{L^\infty} + C\mathscr{C}_{m,q}k\sup_{-\infty<s\le t}\|w_q\mathcal{F}_{2,b}(s)\|_{L^\infty}.
\end{aligned}$$

Then let us define a new backward time cycle as

$$(t'_{\ell+1}, y'_{\ell+1}, v'_{\ell+1}) = (t'_\ell - t_{\mathbf{b}}(y'_\ell, v'_\ell), y_{\mathbf{b}}(y'_\ell, v'_\ell), v'_{\ell+1}),$$

and the starting point

$$(t'_0, y'_0, v'_0) = (s, y', v') := (s, Y(s), v') \text{ or } (s, Y_{\mathbf{cl}}^l(s), v'),$$

for some $s \in \mathbb{R}$ and $l \in \mathbb{Z}^+$. Furthermore, for $\ell \in \mathbb{Z}^+$, we also denote

$$\begin{aligned}
\bar{Y}_{\mathbf{cl}}^\ell(s'; s, y', v') &= \mathbf{1}_{[t'_{\ell+1}, t'_\ell)}(s')\{y'_\ell + (s' - t'_\ell)v'_{\ell y}\}, \\
\bar{V}_{\mathbf{cl}}^\ell(s'; s, y', v') &= \mathbf{1}_{[t'_{\ell+1}, t'_\ell)}(s')(v'_{\ell x} + \alpha(t'_\ell - s')v_{\ell y}, v'_{\ell y}, v'_{\ell z}).
\end{aligned}$$

To be consistent, we set $[\bar{Y}_{\mathbf{cl}}^0(s'), \bar{V}_{\mathbf{cl}}^0(s')] := [\bar{Y}(s'), \bar{V}(s')]$.

Iterating (5.26) again, one has

$$
\begin{aligned}
&|H_{2,m}(t)| \\
&\le C\mathcal{C}_{m,q}^2 \int_{t_1}^{t} e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v') \int_{t_1'}^{s} e^{-\frac{\nu_0}{2}(s-s')} \int_{\mathbb{R}^3} \mathbf{k}_w(\bar{V}(s';Y(s),v'),v'') \\
&\qquad\qquad \times |H_{2,m}(s',\bar{Y}(s';Y(s),v'),v'')|\, dv''\, ds'\, dv'\, ds \\
&\quad + C\mathcal{C}_{m,q}^2 \int_{t_1}^{t} e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v') e^{-\frac{\nu_0}{2}(s-t_1')} \\
&\qquad\qquad \times \sum_{\ell=1}^{\imath-1} \int_{\prod_{j=1}^{\imath-1}\mathcal{V}_j} \int_{t_{\ell+1}'}^{t_\ell'} \int_{\mathbb{R}^3} \mathbf{k}_w(\bar{V}_{\mathbf{cl}}^{\ell}(s';Y(s),v'),v'') \\
&\qquad\qquad \times |H_{2,0}(s',\bar{Y}_{\mathbf{cl}}^{\ell}(s';Y(s),v'),v'')|\, dv''\, d\Sigma_\ell(s')\, ds'\, dv'\, ds \\
&\quad + C\mathcal{C}_{m,q}\mathcal{C}_{0,q} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{t_{l+1}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^{l}(s),v') \\
&\qquad\qquad \times \int_{t_1'}^{s} e^{-\frac{\nu_0}{2}(s-s')} \int_{\mathbb{R}^3} \mathbf{k}_w(\bar{V}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'') \\
&\qquad\qquad \times |H_{2,0}(s',\bar{Y}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|\, dv''\, ds'\, dv'\, d\Sigma_l(s)\, ds \\
&\quad + C\mathcal{C}_{m,q}\mathcal{C}_{0,q} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{t_{l+1}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^{l}(s;v),v') e^{-\frac{\nu_0}{2}(s-t_1')} \\
&\quad \times \sum_{\ell=1}^{\imath-1} \int_{\prod_{j=1}^{\imath-1}\mathcal{V}_j} \int_{t_{\ell+1}'}^{t_\ell'} \int_{\mathbb{R}^3} \mathbf{k}_w(\bar{V}_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|H_{2,0}(s',\bar{Y}_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')| \\
&\qquad\qquad \times dv''\, d\Sigma_\ell(s')\, ds'\, dv'\, d\Sigma_l(s)\, ds \\
&\quad + C\mathcal{C}_{m,q}\mathcal{C}_{0,q} \int_{t_1}^{t} e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v')\mathcal{Q}(s)\, dv'\, ds \\
&\quad + C\mathcal{C}_{m,q}^2 \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{t_{l+1}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^{l}(s),v')\mathcal{Q}(s)\, dv'\, d\Sigma_l(s)\, ds,
\end{aligned}
\tag{5.27}
$$

where according to Lemma 8.1 and Remark 8.1, we choose $\imath \in \mathbb{Z}^+$ such that $\imath \sim (\tilde{T}_0)^{5/4}$ with $\tilde{T}_0 = s - t_\imath'$ being suitably large. We claim that

$$
\begin{aligned}
\|H_{2,m}\|_{L^\infty} \le \eta\{\|H_{2,m}\|_{L^\infty} + \|H_{2,0}\|_{L^\infty}\} + C(\tilde{T}_0)\{\|\partial_{v_x}^m \mathcal{G}_2\| + \|\mathcal{G}_2\|\} \\
+ C \sup_{s\le t} \mathcal{Q}(s),
\end{aligned}
\tag{5.28}
$$

where $\eta > 0$ is suitably small. To prove (5.28), we only estimate the fourth term on the right hand side of (5.27), because the other terms can be estimated similarly. For any sufficiently small $\eta_0 > 0$, we first divide $[t_{\ell+1}', t_\ell']$ as $[t_{\ell+1}', t_\ell' - \eta_0] \cup (t_\ell' - \eta_0, t_\ell']$, then rewrite the fourth term on the right hand side of (5.27) as

$$
\begin{aligned}
\mathcal{J} := C\mathscr{C}_{m,q}\mathscr{C}_{0,q} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{t_{l+1}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^l(s;v), v') e^{-\frac{\nu_0}{2}(s-t_1')} \\
\times \sum_{\ell=1}^{\iota-1} \int_{\prod_{j=1}^{\iota-1}\mathcal{V}_j} \left(\int_{t_{\ell+1}'}^{t_\ell'-\eta_0} + \int_{t_\ell'-\eta_0}^{t_\ell'} \right) \int_{\mathbb{R}^3} \mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v'), v'') \\
\times |H_{2,0}(s', \bar{Y}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v'), v'')| \, dv'' \, d\Sigma_\ell(s') \, ds' \, dv' \, d\Sigma_l(s) \, ds \\
=: \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}
$$

By Lemma 8.1, it is easy to see that

$$
\mathcal{J}_2 \le C\mathscr{C}_{m,q}\mathscr{C}_{0,q}\eta_0 \iota \|H_{2,0}\|_{L^\infty}.
$$

For $\mathcal{J}_1$, the computation is divided into the following three cases.

Case 1: $|V_{\mathbf{cl}}^l(s;v)| > M$ *or* $|\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v')| > M$. In this case, by Lemma 2.1,

$$
\int_{\mathbb{R}^3} |\mathbf{k}_w(V_{\mathbf{cl}}^l(s;v), v')| \, dv' \le \frac{C(q)}{1+M} \text{ or } \int_{\mathbb{R}^3} |\mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v'), v'')| \, dv'' \le \frac{C(q)}{1+M},
$$

where $C(q) > 0$ and depends on $q!$. Therefore, by using Lemma 8.1 again one has

$$
|\mathcal{J}_1| \le \frac{C\mathscr{C}_{m,q}\mathscr{C}_{0,q}C^2(q)}{1+M} \|H_{2,0}\|_{L^\infty}.
$$

Note that here and in the sequel, l and ℓ run over $[1, k-1]$ and $[1, \iota-1]$, respectively.

Case 2: $|V_{\mathbf{cl}}^l(s;v)| \le M$ *and* $|v'| > 2M$, *or* $|\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v')| \le M$ *and* $|v''| > 2M$. In this regime, we have either $|V_{\mathbf{cl}}^l(s;v) - v'| > M$ or $|\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v') - v''| > M$. Then the following two estimates hold respectively:

$$
\begin{aligned}
\mathbf{k}_w(V_{\mathbf{cl}}^l(s;v), v') &\le Ce^{-\varepsilon M^2/16} \mathbf{k}_w(V_{\mathbf{cl}}^l(s;v), v') e^{\varepsilon |V_{\mathbf{cl}}^l - v'|^2/16}, \\
\mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v'), v'') &\le Ce^{-\varepsilon M^2/16} \mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v'), v'') e^{\varepsilon |\bar{V}_{\mathbf{cl}}^\ell - v''|^2/16}.
\end{aligned}
$$

This together with Lemma 2.1 implies

$$
|\mathcal{J}_1| \le C\mathscr{C}_{m,q}\mathscr{C}_{0,q}C^2(q) e^{-\varepsilon M^2/16} \|H_{2,0}\|_{L^\infty}.
$$

Case 3: $|V_{\mathbf{cl}}^l(s;v)| \le M$, $|v'| \le 2M$, $|\bar{V}_{\mathbf{cl}}^\ell(s'; Y_{\mathbf{cl}}^l(s), v')| \le M$ *and* $|v''| \le 2M$. The key point here is to convert the L^1 integral with respect to the double v variables into the L^2 norm with respect to the variables y and v. To do so, for any large $N > 0$, we choose a number $M(N)$ to define $\mathbf{k}_{w,M}(u, v')$ as in (4.12), then decompose

$$
\begin{aligned}
\mathbf{k}_w(V_{\mathbf{cl}}^l, v') \mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell, v'') = \{\mathbf{k}_w(V_{\mathbf{cl}}^l, v') - \mathbf{k}_{w,M}(V_{\mathbf{cl}}^l, v')\} \mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell, v'') \\
+ \{\mathbf{k}_w(\bar{V}_{\mathbf{cl}}^\ell, v'') - \mathbf{k}_{w,M}(\bar{V}_{\mathbf{cl}}^\ell, v'')\} \mathbf{k}_{w,M}(V_{\mathbf{cl}}^l, v') \\
+ \mathbf{k}_{w,M}(V_{\mathbf{cl}}^l, v') \mathbf{k}_{w,M}(\bar{V}_{\mathbf{cl}}^\ell, v'').
\end{aligned}
$$

From Lemma 8.1, the first two difference terms lead to a small contribution to $\mathcal{J}_1$ bounded by

$$\frac{C\mathscr{C}_{m,q}\mathscr{C}_{\mathbf{0},q}C^2(q)}{N}\|H_{2,\mathbf{0}}\|_{L^\infty}.$$

For the remaining main contribution of the bounded product $\mathbf{k}_{w,M}(V_{\mathbf{cl}}^{l},v')\mathbf{k}_{w,M}(\bar{V}_{\mathbf{cl}}^{\ell},v'')$, we denote

$$\tilde{y}=\bar{Y}_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v')=y'_{\ell}-(t'_{\ell}-s')v'_{\ell y}$$

and apply the change of variable $v'_{ly}\to\tilde{y}$. Then one has

$$\left|\frac{\partial\tilde{y}}{\partial v'_{ly}}\right|=\left|\frac{\partial(y'_{\ell}-(t'_{\ell}-s')v'_{\ell y})}{\partial v'_{ly}}\right|=|t'_{\ell}-s'|\geq\eta_0.$$

We now estimate this part as follows:

$$\begin{aligned}
&C\mathscr{C}_{m,q}\mathscr{C}_{\mathbf{0},q}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}\int_{|v'|\leq 2M}\mathbf{k}_{w,M}(V_{\mathbf{cl}}^{l}(s;v),v')e^{-\frac{\nu_0}{2}(s-t'_1)}\\
&\qquad\times\sum_{\ell=1}^{\imath-1}\int_{\prod_{\jmath=1}^{\imath-1}\mathcal{V}_\jmath}\int_{t'_{\ell+1}}^{t'_{\ell}-\eta_0}\int_{|v''|\leq 2M}\mathbf{k}_{w,M}(\bar{V}_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')\\
&\qquad\times|H_{2,\mathbf{0}}(s',\bar{Y}_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|\,dv''\,d\Sigma_\ell(s')\,ds'\,dv'\,d\Sigma_l(s)\,ds\\
&\leq C(M,m,q)\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\int_{t_{l+1}}^{t_l}e^{-\frac{\nu_0}{2}(s-t'_1)}\\
&\qquad\times\sum_{\ell=1}^{\imath-1}\int_{\prod_{\jmath=1}^{\imath-1}\mathcal{V}_\jmath}\int_{t'_{\ell+1}}^{t'_{\ell}-\eta_0}\int_{|v'|\leq 2M,\,|v''|\leq 2M}|\mathscr{G}_2(s',\bar{Y}_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|\\
&\qquad\times dv''\,dv'\,d\Sigma_\ell(s')\,ds'\,d\Sigma_l(s)\,ds\\
&\leq\frac{C(M,m,q)}{\sqrt{\eta_0}}\sup_{s\leq t}\|\mathscr{G}_2(s)\|\sup_{v,v'}\left\{\int_{t_k}^{t_1}e^{-\frac{\nu_0(t_1-s)}{2}}e^{-\frac{\nu_0}{2}(s-t'_1)}\int_{t'_\imath}^{t'_1}e^{-\frac{\nu_0(t'_1-s')}{2}}\,ds'\,ds\right\}\\
&\leq\frac{C(M,m,q)}{\sqrt{\eta_0}}\sup_{s\leq t}\|\mathscr{G}_2(s)\|.
\end{aligned}$$

Putting all the estimates for $\mathcal{J}_1$ and $\mathcal{J}_2$ together, we now obtain

$$\begin{aligned}
\mathcal{J}\leq{}&C\mathscr{C}_{m,q}\mathscr{C}_{\mathbf{0},q}\eta_0\imath\|H_{2,\mathbf{0}}\|_{L^\infty}+\frac{C\mathscr{C}_{m,q}\mathscr{C}_{\mathbf{0},q}C^2(q)}{1+M}\|H_{2,\mathbf{0}}\|_{L^\infty}\\
&+C\mathscr{C}_{m,q}\mathscr{C}_{\mathbf{0},q}C^2(q)e^{-\frac{\varepsilon M^2}{16}}\|H_{2,\mathbf{0}}\|_{L^\infty}\\
&+\frac{C\mathscr{C}_{m,q}\mathscr{C}_{\mathbf{0},q}C(q)}{N}\|H_{2,\mathbf{0}}\|_{L^\infty}+\frac{C(M,m,q)}{\sqrt{\eta_0}}\sup_{s\leq t}\|\mathscr{G}_2(s)\|.
\end{aligned}$$

As mentioned before, by performing similar calculations for the other terms on the right hand side of (5.27), one gets

$$\begin{aligned}\|H_{2,m}\|_{L^\infty} \le{}& \frac{C\mathscr{C}_{m,q}\mathscr{C}_{0,q}C^2(q)}{1+M}\|H_{2,m}\|_{L^\infty} + C\mathscr{C}_{m,q}\mathscr{C}_{0,q}C^2(q)e^{-\varepsilon M^2/16}\|H_{2,m}\|_{L^\infty}\\ &+ C\mathscr{C}_{m,q}\mathscr{C}_{0,q}\eta_0\iota\|H_{2,0}\|_{L^\infty} + \frac{C\mathscr{C}_{m,q}\mathscr{C}_{0,q}(4^q q!)^2}{1+M}\|H_{2,0}\|_{L^\infty}\\ &+ C\mathscr{C}_{m,q}\mathscr{C}_{0,q}C^2(q)e^{-\varepsilon M^2/16}\|H_{2,0}\|_{L^\infty} + \frac{C\mathscr{C}_{m,q}\mathscr{C}_{0,q}C(q)}{N}\|H_{2,0}\|_{L^\infty}\\ &+ \frac{C(M,m,q)}{\sqrt{\eta_0}}\sup_{s\le t}\|\mathscr{G}_2(s)\| + C(M,m,q)\sup_{s\le t}\|\partial_{v_x}^m\mathscr{G}_2(s)\| + C\sup_{s\le t}\mathcal{Q}(s). \end{aligned} \quad (5.29)$$

Since $\iota \sim (\tilde{T}_0)^{5/4}$, by taking M and N large enough and $\eta_0 = (\tilde{T}_0)^{-5/2}$ small enough, (5.29) further yields (5.28). Finally, taking a linear combination of (5.28) with $m = 0, 1, \dots, N_0$, we conclude that

$$\begin{aligned}\sum_{0\le m\le N_0}&\|H_{2,m}\|_{L^\infty}\\ &\le C(N_0,q,\tilde{T}_0)\sum_{0\le m\le N_0}\|\partial_{v_x}^m\mathscr{G}_2\| + C(N_0,q,T_0)\sum_{0\le m\le N_0}\|H_{1,m}\|_{L^\infty}\\ &\quad + C(N_0,q,T_0)\sum_{0\le m\le N_0}\{\|w_q\partial_{v_x}^m\mathscr{F}_2\|_{L^\infty} + \|w_q\partial_{v_x}^m\mathscr{F}_{2,b}\|_{L^\infty}\}. \end{aligned} \quad (5.30)$$

Remark 5.1. We point out that the estimates (5.22) and (5.30) obtained above are independent of ϵ. Moreover, both $\tilde{T}_0$ and T_0 are independent of t, because starting from any $t \in (-\infty, \infty)$ we can trace back k times to some t_k which can be negative.

Step 2: L^2 estimate. To close the final estimate, we turn to the L^2 estimate of $\partial_{v_x}^m\mathscr{G}_2$ with $0 \le m \le N_0$. The goal is to prove that for a given $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\begin{aligned}\sum_{0\le m\le N_0}&\|\partial_{v_x}^m\mathscr{G}_2\|^2 + \sum_{0\le m\le N_0}|\partial_{v_x}^m\mathscr{G}_2|_{2,+}^2\\ &\le C(\epsilon)\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathscr{G}_1\|_{L^\infty}^2 + C(\epsilon)\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathscr{F}_2\|_{L^\infty}^2\\ &\quad + C(\epsilon)\sum_{0\le m\le N_0}\|w_q\partial_{v_x}^m\mathscr{F}_{2,b}\|_{L^\infty}^2. \end{aligned} \quad (5.31)$$

For this, we begin with the following equations for $\mathscr{G}_2$:

$$\begin{cases}\epsilon\mathscr{G}_2 + v_y\partial_y\mathscr{G}_2 - \alpha v_y\partial_{v_x}\mathscr{G}_2 + \nu_0\mathscr{G}_2 - \sigma K\mathscr{G}_2 - \sigma(1-\chi_M)\mu^{-1/2}\mathcal{K}\mathscr{G}_1 = \mathscr{F}_2,\\ \qquad\qquad y\in(-1,1),\\ \mathscr{G}_2(\pm1,v)\mathbf{1}_{\{v_y\lessgtr0\}} - \sigma\sqrt{2\pi\mu}\int_{v_y\gtrless0}\sqrt{\mu}\,\mathscr{G}(\pm1,v)|v_y|\,dv = \mathscr{F}_{2,b}.\end{cases} \quad (5.32)$$

Taking the inner product of (5.32)$_1$ and $\mathscr{G}_2$ over $(y,v)\in(-1,1)\times\mathbb{R}^3$ we have, for $\eta>0$,

$$\begin{aligned}(\epsilon+(1-\sigma)\nu_0)\|\mathscr{G}_2\|^2&+\sigma\delta_0\|\mathbf{P}_1\mathscr{G}_2\|^2+\tfrac{1}{2}|\{I-P_\gamma\}\mathscr{G}_2|^2_{2,+}+\tfrac{1}{2}(1-\sigma)|P_\gamma\mathscr{G}_2|^2_{2,+}\\&\le|(\mathscr{G}_2,\mathscr{F}_2)|+|((1-\chi_M)\mu^{-1/2}\mathcal{K}\mathscr{G}_1,\mathscr{G}_2)|\\&\quad+\eta|P_\gamma\mathscr{G}_2|^2_{2,+}+C_\eta\|w_q\mathscr{G}_1\|^2_{L^\infty}+C_\eta\|w_q\mathscr{F}_{2,b}\|^2_{L^\infty},\end{aligned}\tag{5.33}$$

where the following estimate on the boundary term has been used:

$$\begin{aligned}&\int_{\mathbb{R}^3}v_y\mathscr{G}_2^2(1)\,dv-\int_{\mathbb{R}^3}v_y\mathscr{G}_2^2(-1)\,dv\\&=\int_{v_y>0}v_y\mathscr{G}_2^2(1)\,dv-\int_{v_y<0}v_y(\sigma P_\gamma\mathscr{G}_2+\sigma\bar{P}_\gamma\mathscr{G}_1+\mathscr{F}_{2,b})^2(1)\,dv\\&\quad-\int_{v_y<0}v_y\mathscr{G}_2^2(-1)\,dv-\int_{v_y>0}v_y(\sigma P_\gamma\mathscr{G}_2+\sigma\bar{P}_\gamma\mathscr{G}_1+\mathscr{F}_{2,b})^2(-1)\,dv\\&\ge(1-\sigma^2)\int_{v_y>0}v_y(P_\gamma\mathscr{G}_2)^2(1)\,dv+\int_{v_y>0}v_y(\{I-P_\gamma\}\mathscr{G}_2)^2(1)\,dv\\&\quad+(1-\sigma^2)\int_{v_y<0}|v_y|(P_\gamma\mathscr{G}_2)^2(-1)\,dv+\int_{v_y<0}|v_y|(\{I-P_\gamma\}\mathscr{G}_2)^2(-1)\,dv\\&\quad-\eta\int_{v_y<0}v_y(P_\gamma\mathscr{G}_2)^2(1)\,dv-C_\eta\|w_q\mathscr{F}_{2,b}(1)\|^2_{L^\infty}-\eta\int_{v_y>0}|v_y|(P_\gamma\mathscr{G}_2)^2(-1)\,dv\\&\quad-C_\eta\|w_q\mathscr{F}_{2,b}(-1)\|^2_{L^\infty}-C_\eta\int_{v_y<0}|v_y|\,|\bar{P}_\gamma\mathscr{G}_1(1)|^2\,dv\\&\quad-C_\eta\int_{v_y>0}|v_y|\,|\bar{P}_\gamma\mathscr{G}_1(-1)|^2\,dv\\&\ge|\{I-P_\gamma\}\mathscr{G}_2|^2_{2,+}+(1-\sigma)|P_\gamma\mathscr{G}_2|^2_{2,+}-\eta|P_\gamma\mathscr{G}_2|^2_{2,+}\\&\quad-C_\eta\|w_q\mathscr{F}_{2,b}(\pm1)\|^2_{L^\infty}-C_\eta\|w_q\mathscr{G}_1\|^2_{L^\infty}.\end{aligned}$$

Here we have used the notation

$$\bar{P}_\gamma\mathscr{G}_1(\pm1)=\sqrt{2\pi\mu}\int_{v_y\gtrless0}\mathscr{G}_1(\pm1)|v_y|\,dv$$

and the estimate

$$\left|\int_{v_y\gtrless0}\mathscr{G}_1(\pm1)|v_y|\,dv\right|\le C\|w_q\mathscr{G}_1\|_{L^\infty}\quad\text{for } q>5/2.$$

Next, since

$$\begin{aligned}|P_\gamma\mathscr{G}_2(\pm1)|^2_{2,\pm}&=\int_{v_y\gtrless0}\left[\int_{v_y\gtrless0}\mathscr{G}_2(\pm1)\sqrt{\mu}\,|v_y|\,dv\right]^2 2\pi\mu(v)|v_y|\,dv\\&=\left[\int_{v_y\gtrless0}\mathscr{G}_2(\pm1)\sqrt{\mu}\,|v_y|\,dv\right]^2,\end{aligned}$$

by dividing the integration domain as

$$\{v \in \mathbb{R}^3 : v_y > 0\} = \underbrace{\{v \in \mathbb{R}^3 : 0 < v_y < \varepsilon \text{ or } v_y > 1/\varepsilon\}}_{V^\varepsilon} \cup \{v \in \mathbb{R}^3 : \varepsilon \le v_y \le 1/\varepsilon\},$$

one sees that the grazing part of $|P_\gamma \mathcal{G}_2(1)|_{2,+}^2$ is bounded by the Hölder inequality as

$$\left(\int_{V^\varepsilon} \mu(v)|v_y|\,dv\right)\int_{v_y>0} |\mathcal{G}_2(1)|^2 v_y\,dv \lesssim \varepsilon \int_{v_y>0} |\mathcal{G}_2(1)|^2 v_y\,dv. \tag{5.34}$$

For the nongrazing region, we see by using the trace Lemma 3.1 that

$$\begin{aligned}\int_{\{v\in\mathbb{R}^3 : v_y>0\}\setminus V^\varepsilon} &|\mathcal{G}_2(1)|^2 v_y\,dv \le C\|\mathcal{G}_2\|^2 + C\|v_y\partial_y \mathcal{G}_2^2 - \alpha v_y \partial_{v_x}\mathcal{G}_2^2\|_{L^1}\\ &\le C\|\mathcal{G}_2\|^2 + C|(L\mathcal{G}_2,\mathcal{G}_2)| + C|((1-\chi_M)\mu^{-1/2}\mathcal{K}\mathcal{G}_1,,\mathcal{G}_2)| + |(\mathcal{F}_2,\mathcal{G}_2)|\\ &\le C\|\mathcal{G}_2\|^2 + C\|w_q\mathcal{G}_1\|_{L^\infty}^2 + C\|\mathcal{F}_2\|^2.\end{aligned} \tag{5.35}$$

Putting (5.34) and (5.35) together, one has

$$|P_\gamma\mathcal{G}_2|_{2,+}^2 \le \varepsilon|\{I-P_\gamma\}\mathcal{G}_2|_{2,+}^2 + C\|\mathcal{G}_2\|^2 + C\|w_q\mathcal{G}_1\|_{L^\infty}^2 + C\|\mathcal{F}_2\|^2. \tag{5.36}$$

Consequently, (5.33) and (5.36) give

$$\|\mathcal{G}_2\|^2 + |\mathcal{G}_2|_{2,+}^2 \le C(\epsilon)\{\|w_q\mathcal{F}_2\|_{L^\infty}^2 + \|w_q\mathcal{F}_{2,b}\|_{L^\infty}^2 + \|w_q\mathcal{G}_1\|_{L^\infty}^2\}. \tag{5.37}$$

Remark 5.2. Note that the constant $C(\epsilon)$ in (5.37) is independent of the parameter σ.

It remains to deduce an L^2 estimate of higher order velocity derivatives. For this, applying $\partial_{v_x}^m$ $(m \ge 1)$ to (5.32) we have

$$\begin{cases}\epsilon\partial_{v_x}^m\mathcal{G}_2 + v_y\partial_y\partial_{v_x}^m\mathcal{G}_2 - \alpha v_y\partial_{v_x}^{m+1}\mathcal{G}_2 + \nu_0\partial_{v_x}^m\mathcal{G}_2 - \sigma\partial_{v_x}^m K\mathcal{G}_2\\ \qquad -\sigma\partial_{v_x}^m[(1-\chi_M)\mu^{-1/2}\mathcal{K}\mathcal{G}_1] = \partial_{v_x}^m\mathcal{F}_2, \quad y\in(-1,1),\\ \partial_{v_x}^m\mathcal{G}_2(\pm1,v)\mathbf{1}_{\{v_y\lessgtr 0\}} - \sigma\sqrt{2\pi}\,\partial_{v_x}^m(\mu^{1/2})\int_{v_y\gtrless 0}\sqrt{\mu}\,\mathcal{G}(\pm1,v)|v_y|\,dv = \partial_{v_x}^m\mathcal{F}_{2,b}.\end{cases} \tag{5.38}$$

Taking the inner product of $(5.38)_1$ and $\partial_{v_x}^m\mathcal{G}_2$, we deduce

$$\begin{aligned}(\epsilon + (1-\sigma)\nu_0)\|\partial_{v_x}^m\mathcal{G}_2\|^2 + \sigma\delta_0\|\partial_{v_x}^m\mathcal{G}_2\|^2 + \tfrac{1}{2}|\partial_{v_x}^m\mathcal{G}_2|_{2,+}^2&\\ \le C\|\mathcal{G}_2\|^2 + C\sum_{m'\le m}\|w_q\partial_{v_x}^{m'}\mathcal{G}_1\|_{L^\infty}^2 + C(m)|P_\gamma\mathcal{G}_2|_{2,+}^2&\\ + C\|\partial_{v_x}^m\mathcal{F}_2\|^2 + C\|w_q\partial_{v_x}^m\mathcal{F}_{2,b}\|_{L^\infty}^2,&\end{aligned} \tag{5.39}$$

where we have used the following estimates for the incoming boundary term in $(5.38)_2$:

$$\begin{aligned}\int_{v_y\lessgtr 0} &|v_y|\,|\partial_{v_x}^m\mathcal{G}_2(\pm1,v)\mathbf{1}_{\{v_y\lessgtr 0\}}|^2\,dv\\ &\le C(m)|P_\gamma\mathcal{G}_2|_{2,+}^2 + C\|w_q\mathcal{G}_1\|_{L^\infty}^2 + C\|w_q\partial_{v_x}^m\mathcal{F}_{2,b}\|_{L^\infty}^2.\end{aligned}$$

Then (5.39) and (5.37) give (5.31). With (5.31), (5.12) follows from (5.22) and (5.30). This completes the proof of the lemma. ∎

5.3. Existence for the linear problem with $\sigma = 1$ and $\epsilon > 0$

With Lemma 5.1, we now turn to the existence of solution to (5.10) for a fixed parameter $\epsilon > 0$ in the L^∞ framework by the contraction mapping argument.

Lemma 5.2. *Under the same assumption of Lemma 5.1, there exists a unique solution $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha,N_0}$ to (5.10) with $\sigma = 1$ satisfying*

$$\sum_{0\le m\le N_0} \{\|w_q \partial_{v_x}^m \mathcal{G}_1\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{G}_2\|_{L^\infty}\}$$
$$\le C \sum_{0\le m\le N_0} \{\|w_q \partial_{v_x}^m \mathcal{F}_1\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_2\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_{2,b}(s)\|_{L^\infty}\}. \tag{5.40}$$

Proof. The proof is based on the a priori estimate (5.12) established in Lemma 5.1 and a bootstrap argument. Just as for Lemma 4.2, the proof is divided into three steps.

Step 1: Existence for $\sigma = 0$. If $\sigma = 0$, then (5.10) is reduced to

$$\epsilon\mathcal{G}_1 + v_y\partial_y\mathcal{G}_1 - \alpha v_y\partial_{v_x}\mathcal{G}_1 + \nu_0\mathcal{G}_1 + \alpha\frac{v_xv_y}{2}\sqrt{\mu}\,\mathcal{G}_2 = \mathcal{F}_1, \quad y \in (-1,1),$$
$$\mathcal{G}_1(\pm1, v)\mathbf{1}_{\{v_y\lessgtr 0\}} = 0,$$

and

$$\epsilon\mathcal{G}_2 + v_y\partial_y\mathcal{G}_2 - \alpha v_y\partial_{v_x}\mathcal{G}_2 + \nu_0\mathcal{G}_2 = \mathcal{F}_2, \quad y \in (-1,1),$$
$$\mathcal{G}_2(\pm1, v)\mathbf{1}_{\{v_y\lessgtr 0\}} = \mathcal{F}_{2,b},$$

respectively. Then, in this simple case, the existence of L^∞ solutions can be directly proved by the method of characteristics so that

$$\|\mathcal{L}_0^{-1}[\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2]\|_{\mathbf{X}_{\alpha,N_0}} \le C_{\mathcal{L}}\|[\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2]\|_{\mathbf{X}_{\alpha,N_0}}. \tag{5.41}$$

Step 2: Existence for $\sigma \in [0, \sigma_*]$ for some $\sigma_* > 0$. Letting $\sigma \in (0,1]$, we now consider

$$\epsilon\mathcal{G}_1 + v_y\partial_y\mathcal{G}_1 - \alpha v_y\partial_{v_x}\mathcal{G}_1 + \nu_0\mathcal{G}_1 + \alpha\frac{v_xv_y}{2}\sqrt{\mu}\,\mathcal{G}_2$$
$$= \sigma\chi_M\mathcal{K}\mathcal{G}_1 + \mathcal{F}_1, \quad y \in (-1,1), \tag{5.42}$$
$$\mathcal{G}_1(\pm1, v)\mathbf{1}_{\{v_y\lessgtr 0\}} = 0, \tag{5.43}$$

and

$$\epsilon\mathcal{G}_2 + v_y\partial_y\mathcal{G}_2 - \alpha v_y\partial_{v_x}\mathcal{G}_2 + \nu_0\mathcal{G}_2$$
$$= \sigma K\mathcal{G}_2 + \sigma(1-\chi_M)\mu^{-1/2}\mathcal{K}\mathcal{G}_1 + \mathcal{F}_2, \quad y \in (-1,1), \tag{5.44}$$
$$\mathcal{G}_2(\pm1, v)\mathbf{1}_{\{v_y\lessgtr 0\}} = \sigma\sqrt{2\pi\mu}\int_{v_y\gtrless 0}(\mathcal{G}_1 + \sqrt{\mu}\,\mathcal{G}_2)(\pm1, v)|v_y|\,dv + \mathcal{F}_{2,b}. \tag{5.45}$$

For the above system, we design the following approximation scheme:

$$\epsilon \mathcal{G}_1^{n+1} + v_y \partial_y \mathcal{G}_1^{n+1} - \alpha v_y \partial_{v_x} \mathcal{G}_1^{n+1} + \nu_0 \mathcal{G}_1^{n+1} + \alpha \frac{v_x v_y}{2} \sqrt{\mu}\, \mathcal{G}_2^{n+1} = \sigma \chi_M \mathcal{K} \mathcal{G}_1^n + \mathcal{F}_1 =: \mathcal{F}_1^{(1)}, \tag{5.46}$$

$$\mathcal{G}_1^{n+1}(\pm 1, v) \mathbf{1}_{\{v_y \lessgtr 0\}} = 0, \tag{5.47}$$

and

$$\epsilon \mathcal{G}_2^{n+1} + v_y \partial_y \mathcal{G}_2^{n+1} - \alpha v_y \partial_{v_x} \mathcal{G}_2^{n+1} + \nu_0 \mathcal{G}_2^{n+1} = \sigma K \mathcal{G}_2^n + \sigma(1-\chi_M)\mu^{-1/2} \mathcal{K} \mathcal{G}_1^n + \mathcal{F}_2 =: \mathcal{F}_2^{(1)}, \tag{5.48}$$

$$\mathcal{G}_2^{n+1}(\pm 1, v) \mathbf{1}_{\{v_y \lessgtr 0\}} = \sigma \sqrt{2\pi\mu} \int_{v_y \gtrless 0} (\mathcal{G}_1^n + \sqrt{\mu}\, \mathcal{G}_2^n)(\pm 1, v) |v_y| \, dv + \mathcal{F}_{2,b} =: \mathcal{F}_{2,b}^{(1)}, \tag{5.49}$$

with $[\mathcal{G}_1^0, \mathcal{G}_2^0] = [0, 0]$. The goal in the following proof is twofold: (i) $[\mathcal{G}_1^n, \mathcal{G}_2^n]_{n=0}^{\infty}$ is uniformly bounded in $\mathbf{X}_{\alpha, N_0}$, and (ii) $[\mathcal{G}_1^n, \mathcal{G}_2^n]_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbf{X}_{\alpha, N_0}$. By (5.41), it follows that

$$\begin{aligned} \|[\mathcal{G}_1^{n+1}, \mathcal{G}_2^{n+1}]\|_{\mathbf{X}_{\alpha,N_0}} &\le C_{\mathcal{L}} \Big\{ \|[\mathcal{F}_1^{(1)}, \mathcal{F}_2^{(1)}]\|_{\mathbf{X}_{\alpha,N_0}} + \sum_{0 \le m \le N_0} \|w_q \partial_{v_x}^m \mathcal{F}_{2,b}\|_{L^\infty} \Big\} \\ &\le C_{\mathcal{L}} \sigma \bar{C}_1 \|[\mathcal{G}_1^n, \mathcal{G}_2^n]\|_{\mathbf{X}_{\alpha,N_0}} \\ &\quad + \underbrace{C_{\mathcal{L}} \sum_{0 \le m \le N_0} \{ \|w_q \partial_{v_x}^m \mathcal{F}_1\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_2\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_{2,b}\|_{L^\infty} \}}_{\mathcal{M}_0}, \end{aligned} \tag{5.50}$$

where $\bar{C}_1 > 0$ is independent of σ and n. Choosing $0 < \sigma_* < 1$ suitably small such that

$$C_{\mathcal{L}} \sigma_* \bar{C}_1 \le 1/2, \tag{5.51}$$

(5.50) implies that

$$\|[\mathcal{G}_1^n, \mathcal{G}_2^n]\|_{\mathbf{X}_{\alpha,N_0}} \le 2\mathcal{M}_0 \tag{5.52}$$

for all $n \ge 0$. Moreover, by (5.46)–(5.49) and applying (5.41), one has

$$\begin{aligned} \|[\mathcal{G}_1^{n+1}, \mathcal{G}_2^{n+1}] - [\mathcal{G}_1^n, \mathcal{G}_2^n]\|_{\mathbf{X}_{\alpha,N_0}} &\le C_{\mathcal{L}} \sigma \bar{C}_1 \|[\mathcal{G}_1^n, \mathcal{G}_2^n] - [\mathcal{G}_1^{n-1}, \mathcal{G}_2^{n-1}]\|_{\mathbf{X}_{\alpha,N_0}} \\ &\le \tfrac{1}{2} \|[\mathcal{G}_1^n, \mathcal{G}_2^n] - [\mathcal{G}_1^{n-1}, \mathcal{G}_2^{n-1}]\|_{\mathbf{X}_{\alpha,N_0}} \end{aligned} \tag{5.53}$$

with the condition (5.51). Consequently, (5.53) and (5.52) imply that the systems (5.42)–(5.43) and (5.44)–(5.45) have a unique solution $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha, N_0}$ for any $\sigma \in [0, \sigma_*]$. Moreover, by Lemma 5.1, we have the uniform estimate

$$\|[\mathcal{G}_1, \mathcal{G}_2]\|_{\mathbf{X}_{\alpha,N_0}} \le C_{\mathcal{L}} \sum_{0 \le m \le N_0} \{ \|w_q \partial_{v_x}^m \mathcal{F}_1\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_2\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_{2,b}\|_{L^\infty} \},$$

which is equivalent to

$$\|\mathcal{L}_{\sigma_*}^{-1}[\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2]\|_{\mathbf{X}_{\alpha,N_0}} \leq C_{\mathcal{L}} \|[\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2]\|_{\mathbf{X}_{\alpha,N_0}}. \tag{5.54}$$

Step 3: Existence for $\sigma \in [0, 2\sigma_*]$ for some $\sigma_* > 0$. By using (5.54) and performing similar calculations to those leading to (5.52) and (5.53), for $\sigma \in [0, \sigma_*]$, one can see that there exists a unique solution $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha,N_0}$ to the lifted system

$$\begin{aligned}
&\epsilon\mathcal{G}_1 + v_y\partial_y\mathcal{G}_1 - \alpha v_y\partial_{v_x}\mathcal{G}_1 + \nu_0\mathcal{G}_1 + \alpha\frac{v_xv_y}{2}\sqrt{\mu}\,\mathcal{G}_2 - \sigma_*\chi_M\mathcal{K}\mathcal{G}_1\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \sigma\chi_M\mathcal{K}\mathcal{G}_1 + \mathcal{F}_1, \quad y \in (-1,1),\\
&\mathcal{G}_1(\pm 1, v)\mathbf{1}_{\{v_y \lessgtr 0\}} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\epsilon\mathcal{G}_2 + v_y\partial_y\mathcal{G}_2 - \alpha v_y\partial_{v_x}\mathcal{G}_2 + \nu_0\mathcal{G}_2 - \sigma_* K\mathcal{G}_2 - \sigma_*(1-\chi_M)\mu^{-1/2}\mathcal{K}\mathcal{G}_1\\
&\qquad\qquad\qquad = \sigma K\mathcal{G}_2 + \sigma(1-\chi_M)\mu^{-1/2}\mathcal{K}\mathcal{G}_1 + \mathcal{F}_2, \quad y \in (-1,1),\\
&\mathcal{G}_2(\pm 1, v)\mathbf{1}_{\{v_y \lessgtr 0\}} - \sigma_*\sqrt{2\pi\mu}\int_{v_y \gtrless 0}(\mathcal{G}_1 + \sqrt{\mu}\,\mathcal{G}_2)(\pm 1, v)|v_y|\,dv\\
&\qquad\qquad\qquad = \sigma\sqrt{2\pi\mu}\int_{v_y \gtrless 0}(\mathcal{G}_1 + \sqrt{\mu}\,\mathcal{G}_2)(\pm 1, v)|v_y|\,dv + \mathcal{F}_{2,b}.
\end{aligned}$$

In other words, we have shown the existence of $\mathcal{L}_{2\sigma_*}^{-1}$ on $\mathbf{X}_{\alpha,N_0}$ and (5.12) holds true for $\sigma = 2\sigma_*$.

Therefore, by repeating this procedure finitely many times, one can see that $\mathcal{L}_1^{-1}$ exists when $\sigma = 1$ and (5.40) follows correspondingly. This completes the proof of Lemma 5.2. ∎

5.4. *Estimates on the remainder*

We are ready to complete the proof of Proposition 5.1.

Proof of Proposition 5.1. We now prove existence for the coupled system (5.2) and (5.4) under the diffuse boundary conditions (5.3) and (5.5), respectively.

Let us first go back to the approximation system (5.6)–(5.9). By applying Lemma 5.2, for fixed $\epsilon > 0$, we see that $[G_{R,1}^{n+1}, G_{R,2}^{n+1}]$ is well defined when $[G_{R,1}^n, G_{R,2}^n]$ is given and the solution belongs to $\mathbf{X}_{\alpha,N_0}$ defined in (5.11) for $N_0 \geq 0$.

We now show that $\{[G_{R,1}^n, G_{R,2}^n]\}_{n=0}^\infty$ is a Cauchy sequence in $\mathbf{X}_{\alpha,N_0}$, which implies that its limit denoted by $[G_{R,1}^\epsilon, G_{R,2}^\epsilon]$ is the unique solution of the system

$$\begin{aligned}
&\epsilon G_{R,1}^\epsilon + v_y\partial_y G_{R,1}^\epsilon - \alpha v_y\partial_{v_x}G_{R,1}^\epsilon + \nu_0 G_{R,1}^\epsilon - \chi_M\mathcal{K}G_{R,1}^\epsilon\\
&\quad = -\tfrac{1}{2}\alpha\sqrt{\mu}\,v_xv_yG_{R,2}^\epsilon - \tfrac{1}{2}\sqrt{\mu}\,v_xv_yG_1 + \sqrt{\mu}\,v_y\partial_{v_x}G_1 + Q(\sqrt{\mu}\,G_1, \sqrt{\mu}\,G_1)\\
&\qquad + \alpha\{Q(\sqrt{\mu}\,G_R^\epsilon, \sqrt{\mu}\,G_1) + Q(\sqrt{\mu}\,G_1, \sqrt{\mu}\,G_R^\epsilon)\} + \alpha^2 Q(\sqrt{\mu}\,G_R^\epsilon, \sqrt{\mu}\,G_R^\epsilon)\\
&\quad =: \mathcal{N}_\epsilon,\ \ y \in (-1,1), \quad v \in \mathbb{R}^3,
\end{aligned} \tag{5.55}$$

$$G_{R,1}^{\epsilon}(\pm 1, v)|_{v_y \lessgtr 0} = 0, \quad v \in \mathbb{R}^3, \tag{5.56}$$

and

$$\begin{aligned} &\epsilon G_{R,1}^{\epsilon} + v_y \partial_y G_{R,2}^{\epsilon} - \alpha v_y \partial_{v_x} G_{R,2}^{\epsilon} + L G_{R,2}^{\epsilon} \\ &\qquad = (1-\chi_M)\mu^{-1/2}\mathcal{K} G_{R,1}^{\epsilon}, \quad y \in (-1,1), \quad v \in \mathbb{R}^3, \end{aligned} \tag{5.57}$$

$$G_{R,2}^{\epsilon}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu}\, G_R^{\epsilon}(\pm 1, v)|v_y|\, dv, \quad v \in \mathbb{R}^3. \tag{5.58}$$

Furthermore, we will show that the convergence of the sequence $\{[G_{R,1}^n, G_{R,2}^n]\}_{n=0}^{\infty}$ is independent of ϵ. For this, we first prove that

$$\|[G_{R,1}^n, G_{R,2}^n]\|_{\mathbf{X}_{\alpha,N_0}} \le 2\mathcal{C}_0, \tag{5.59}$$

where $\mathcal{C}_0 > 0$ is independent of ϵ and n for all $n \ge 0$. We apply induction on n. Notice $[G_{R,1}^0, G_{R,2}^0] = [0,0]$. If $n = 1$, then the system (5.6)–(5.9) reads

$$\begin{aligned} &\epsilon G_{R,1}^1 + v_y \partial_y G_{R,1}^1 - \alpha v_y \partial_{v_x} G_{R,1}^1 + \nu_0 G_{R,1}^1 - \chi_M \mathcal{K} G_{R,1}^1 + \tfrac{1}{2}\alpha\sqrt{\mu}\, v_x v_y G_{R,2}^1 \\ &\quad = -\tfrac{1}{2}\sqrt{\mu}\, v_x v_y G_1 + \sqrt{\mu}\, v_y \partial_{v_x} G_1 + Q(G_1, G_1) := \mathcal{S}^0, \ y \in (-1,1), \quad v \in \mathbb{R}^3, \end{aligned} \tag{5.60}$$

$$G_{R,1}^1(\pm 1, v)|_{v_y \lessgtr 0} = 0, \quad v \in \mathbb{R}^3, \tag{5.61}$$

and

$$\begin{aligned} &\epsilon G_{R,2}^1 + v_y \partial_y G_{R,2}^1 - \alpha v_y \partial_{v_x} G_{R,2}^1 + L G_{R,2}^1 \\ &\qquad = (1-\chi_M)\mu^{-1/2}\mathcal{K} G_{R,1}^1, \quad y \in (-1,1), \ v \in \mathbb{R}^3, \end{aligned} \tag{5.62}$$

$$G_{R,2}^1(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu}\, G_R^1(\pm 1, v)|v_y|\, dv, \quad v \in \mathbb{R}^3. \tag{5.63}$$

Performing similar calculations to those in deriving (5.22) and (5.30), one has

$$\begin{aligned} \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_{R,1}^1\|_{L^\infty} &\le C\alpha \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_{R,2}^1\|_{L^\infty} + C \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m \mathcal{S}^0\|_{L^\infty} \\ &\le C\alpha \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_{R,2}^1\|_{L^\infty} + C, \end{aligned} \tag{5.64}$$

and

$$\sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_{R,2}^1\|_{L^\infty} \le C \sum_{0\le m\le N_0} \|\partial_{v_x}^m G_{R,2}^1\| + C \sum_{0\le m\le N_0} \|w_q \partial_{v_x}^m G_{R,1}^{k+1}\|_{L^\infty}, \tag{5.65}$$

where the constant $C > 0$ is independent of ϵ; see also Remark 5.1.

Since the mass of $G^1_{R,2}$ is not conserved, in order to estimate the macroscopic component of $G^1_{R,2}$ we instead turn to the L^2 estimate of G^1_R. Recall $\sqrt{\mu}\, G^1_R = G^1_{R,1} + \sqrt{\mu}\, G^1_{R,2}$. By (5.60)–(5.63), it is easy to see that G^1_R satisfies

$$\epsilon G^1_R + v_y \partial_y G^1_R - \alpha v_y \partial_{v_x} G^1_R + \tfrac{1}{2}\alpha v_x v_y G^1_R + L G^1_R = \mu^{-1/2} \mathcal{S}^0, \tag{5.66}$$

and

$$G^1_R(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu}\, G^1_R(\pm 1, v)|v_y|\, dv.$$

Next, for $n \geq 1$, denote

$$\begin{aligned} \mathbf{P}_0 G^n_R &= (a^n + \mathbf{b}^n \cdot v + c^n(|v|^2 - 3))\sqrt{\mu}, \\ \mathbf{P}_0 G^n_{R,2} &= (a^n_2 + \mathbf{b}^n_2 \cdot v + c^n_2(|v|^2 - 3))\sqrt{\mu}, \end{aligned} \tag{5.67}$$

and define the projection $\bar{\mathbf{P}}_0$ from L^2 to $\ker \mathcal{L}$ as

$$\bar{\mathbf{P}}_0 G^n_{R,1} = (a^n_1 + \mathbf{b}^n_1 \cdot v + c^n_1(|v|^2 - 3))\mu. \tag{5.68}$$

We will also use the notations

$$\mathbf{b}^n_i = [b^n_{i,1}, b^n_{i,2}, b^n_{i,3}], \quad i = 1, 2, \quad \mathbf{b}^n = [b^n_1, b^n_2, b^n_3].$$

Note that

$$a^n = a^n_1 + a^n_2, \quad \mathbf{b}^n = \mathbf{b}^n_1 + \mathbf{b}^n_2, \quad c^n = c^n_1 + c^n_2, \quad \int_{-1}^{1} a^n(y)\, dy = 0. \tag{5.69}$$

Since

$$\|[a^n_1, \mathbf{b}^n_1, c^n_1]\| \leq C \|\bar{\mathbf{P}}_0 G^n_{R,1}\| \leq C \|w_q G^n_{R,1}\|_{L^\infty} \tag{5.70}$$

for $q > 5/2$, to obtain the estimate of $\|[a^1_2, \mathbf{b}^1_2, c^1_2]\|$ it suffices to derive L^2 estimates of $[a^1, \mathbf{b}^1, c^1]$. In what follows, we will show that the L^2 norm of the macroscopic part of G^1_R can indeed be dominated by its microscopic component and other known terms. We estimate $[a^1, \mathbf{b}^1, c^1]$ by the dual argument. First of all, we let $\Psi(y, v) \in C^\infty([-1, 1] \times \mathbb{R}^3)$, and take the inner product of (5.66) and Ψ over $(-1, 1) \times \mathbb{R}^3$ to obtain

$$\begin{aligned} &\epsilon(G^1_R, \Psi) - (v_y G^1_R, \partial_y \Psi) + \langle v_y G^1_R(1), \Psi(1)\rangle - \langle v_y G^1_R(-1), \Psi(-1)\rangle + \alpha(v_y G^1_R, \partial_{v_x} \Psi) \\ &\quad + \tfrac{1}{2}\alpha(v_x v_y G^1_R, \Psi) + (L G^1_R, \Psi) = (\mu^{-1/2} \mathcal{S}^0, \Psi). \end{aligned} \tag{5.71}$$

Estimate of a^1. Let

$$\Psi = \Psi_{a^1} = v_y \frac{d}{dy} \phi_{a^1}(y)(|v|^2 - 10)\sqrt{\mu},$$

where

$$\phi''_{a^1} = a^1, \quad \phi'_{a^1}(\pm 1) = 0. \tag{5.72}$$

Thus

$$\|\phi_{a^1}\|_{H^2} \le C\|a^1\|. \tag{5.73}$$

Plugging $\Psi = \Psi_{a^1}$ into (5.71), we now compute the equation term by term. First of all, by the Cauchy–Schwarz inequality with $\eta > 0$ and using (5.73), one has

$$\begin{aligned}
\epsilon|(G_R^1, \Psi_{a^1})| &\le \epsilon|(\mathbf{P}_0 G_R^1, \Psi)| + \epsilon|(\mathbf{P}_1 G_R^1, \Psi)| \\
&\le C\epsilon\|a^1\|^2 + C\epsilon\{\|\mathbf{P}_1 G_{R,2}^1\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2\} + C\epsilon\|b_2^1\|^2, \\
-(v_y G_R^1, \partial_y \Psi_{a^1}) &= -(v_y \mathbf{P}_0 G_R^1, \partial_y \Psi_{a^1}) - (v_y \mathbf{P}_1 G_R^1, \partial_y \Psi_{a^1}) \\
&\ge 5\|a^1\|^2 - \eta\|a^1\|^2 - C_\eta\{\|\mathbf{P}_1 G_{R,2}^1\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2\},
\end{aligned}$$

and

$$\alpha|(v_y G_R^1, \partial_{v_x}\Psi_{a^1})| + \tfrac{1}{2}\alpha|(v_x v_y G_R^1, \Psi_{a^1})| \le C\alpha(\|b_1^1\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2 + \|G_{R,2}^1\|^2).$$

Then by Lemmas 4.1 and 2.2, it follows that

$$\begin{aligned}
(\mu^{-1/2}\mathcal{S}^0, \Psi_{a^1}) &= |(v_y \partial_{v_x} G_1 - \tfrac{1}{2} v_x v_y G_1 + \Gamma(G_1, G_1), \Psi_{a^1})| \\
&\le \eta\|a^1\|^2 + C_\eta\|G_1\|^2 + C_\eta \int_{\mathbb{R}^3}\int_{-1}^{1} |\Gamma(G_1, G_1)|^2 \, dv\, dy \\
&\le \eta\|a^1\|^2 + C_\eta\|w_q G_1\|_{L^\infty}^2 + C_\eta \int_{-1}^{1} \|G_1\|^4 dy \\
&\le \eta\|a^1\|^2 + C_\eta\|w_q G_1\|_{L^\infty}^2 + C_\eta\|w_q G_1\|_{L^\infty}^2\|G_1\|^2 \le \eta\|a^1\|^2 + C_\eta,
\end{aligned} \tag{5.74}$$

provided that $q > 3/2$.

Next, noting that $LG_R^1 = -\{\Gamma(G_R^1, \sqrt{\mu}) + \Gamma(\sqrt{\mu}, G_R^1)\}$, by a similar argument one has

$$\begin{aligned}
|(LG_R^1, \Psi_{a^1})| &\le |(L(G_{R,1}^1 \mu^{-1/2}), \Psi_{a^1})| + |(LG_{R,2}^1, \Psi_{a^1})| \\
&\le \eta\|a^1\|^2 + C_\eta(\|w_q G_{R,1}^1\|_{L^\infty}^2 + \|\mathbf{P}_1 G_{R,2}^1\|^2).
\end{aligned} \tag{5.75}$$

The last boundary term $\langle v_y G_R^1(1), \Psi_{a^1}(1)\rangle - \langle v_y G_R^1(-1), \Psi_{a^1}(-1)\rangle$ vanishes because of (5.72).

Putting all the estimates above together, we have

$$\begin{aligned}
\|a^1\|^2 &\le C\|\mathbf{P}_1 G_{R,2}^1\|^2 + C\|w_q G_{R,1}^1\|_{L^\infty}^2 + C\alpha\{\|G_{R,2}^1\|^2 + \|b_1^1\|^2\} \\
&\quad + C\|b_2^1\|^2 + C.
\end{aligned} \tag{5.76}$$

Estimate of $\mathbf{b}^1$. Let

$$\Psi = \Psi_{b_i^1} = \begin{cases} v_y v_x \frac{d}{dy}\phi_{b_1^1}(y)\sqrt{\mu}, & i = 1, \\ v_y v_z \frac{d}{dy}\phi_{b_3^1}(y)\sqrt{\mu}, & i = 3, \\ v_y^2(|v|^2 - 5)\frac{d}{dy}\phi_{b_2^1}(y)\sqrt{\mu}, & i = 2, \end{cases}$$

where

$$-\phi''_{b_i^1} = b_i^1, \quad \phi_{b_i}(\pm 1) = 0.$$

Then

$$\|\phi_{b_i^1}\|_{H^2} \le C\|b_i^1\|, \quad |\phi'_{b_i^1}(\pm 1)| \le C\|b_i^1\|. \tag{5.77}$$

We now compute each term in (5.71) with $\Psi = \Psi_{b_i^1}$. By the Cauchy–Schwarz inequality and (5.77), one has

$$\begin{aligned}
\epsilon|(G_R^1, \Psi_{b_i^1})| &\le \epsilon|(\mathbf{P}_0 G_R^1, \Psi_{b_i^1})| + \epsilon|(\mathbf{P}_1 G_R^1, \Psi_{b_i^1})| \\
&\le C\epsilon\|b_i^1\|^2 + C\epsilon\{\|\mathbf{P}_1 G_{R,2}^1\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2\} + C\epsilon \mathbf{1}_{i=2}\|c^1\|^2, \\
-(v_y G_R^1, \partial_y \Psi_{b_i^1}) &= -(v_y \mathbf{P}_0 G_R^1, \partial_y \Psi_{b_i^1}) - (v_y \mathbf{P}_1 G_R^1, \partial_y \Psi_{b_i^1}) \\
&\ge \begin{cases} \|b_i^1\|^2 - \eta\|b_i^1\|^2 - C_\eta\{\|\mathbf{P}_1 G_{R,2}^1\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2\}, & i = 1, 3, \\ 6\|b_i^1\|^2 - \eta\|b_i^1\|^2 - C_\eta\{\|\mathbf{P}_1 G_{R,2}^1\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2\}, & i = 2, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\alpha|(v_y G_R^1, \partial_{v_x} \Psi_{b_i^1})| + \tfrac{1}{2}\alpha|(v_x v_y G_R^1, \Psi_{b_i^1})| \\
&\qquad \le C\alpha(\|[a^1, c^1]\|^2 + \|w_q G_{R,1}^1\|_{L^\infty}^2 + \|G_{R,2}^1\|^2).
\end{aligned}$$

Similar to (5.74) and (5.75), it follows that

$$(\mu^{-1/2}\mathcal{S}^0, \Psi_{b_i^1}) \le \eta\|b_i^1\|^2 + C,$$

and

$$\begin{aligned}
|(LG_R^1, \Psi_{b_i})| &\le |(L(G_{R,1}^1 \mu^{-1/2}), \Psi_{b_i^1})| + |(LG_{R,2}, \Psi_{b_i^1})| \\
&\le \eta\|b_i\|^2 + C_\eta(\|w_q G_{R,1}\|_{L^\infty}^2 + \|\mathbf{P}_1 G_{R,2}\|^2).
\end{aligned}$$

For the boundary term, noting that

$$G_R^1(\pm 1)|_{v_y \neq 0} = P_\gamma G_R^1(\pm 1) + \{I - P_\gamma\}G_R^1(\pm 1)|_{v_y \gtrless 0}, \tag{5.78}$$

we have

$$\begin{aligned}
&\langle v_y G_R^1(1), \Psi_{b_i}(1)\rangle - \langle v_y \mathcal{G}(-1), \Psi_{b_i}(-1)\rangle \\
&\quad = \langle v_y P_\gamma G_R^1(1), \Psi_{b_i}(1)\rangle + \langle v_y\{I - P_\gamma\}G_R^1(1)|_{v_y>0}, \Psi_{b_i}(1)\rangle \\
&\qquad - \langle v_y P_\gamma G_R^1(-1), \Psi_{b_i}(-1)\rangle - \langle v_y\{I - P_\gamma\}G_R^1(-1)|_{v_y<0}, \Psi_{b_i}(-1)\rangle \\
&\quad = \langle v_y\{I - P_\gamma\}G_R^1(1)|_{v_y>0}, \Psi_{b_i}(1)\rangle - \langle v_y\{I - P_\gamma\}G_R^1(-1)|_{v_y<0}, \Psi_{b_i}(-1)\rangle \\
&\quad \le \eta\|b_i^1\|^2 + C_\eta|\{I - P_\gamma\}G_{R,2}^1|_{2,+}^2 + C_\eta\|w_q G_{R,1}^1\|_{L^\infty}^2,
\end{aligned}$$

where the fact that $\langle v_y P_\gamma G_R^1(\pm 1), \Psi_{b_i}(\pm 1)\rangle = 0$ has been used.

We now conclude from the above estimates for b_i^1 with $1 \le i \le 3$ that

$$\|\mathbf{b}^1\|^2 \le C\|\mathbf{P}_1 G^1_{R,2}\|^2 + C\|w_q G^1_{R,1}\|^2_{L^\infty} + C\alpha\{\|G^1_{R,2}\|^2 + \|[a^1, c^1]\|^2\} + C|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} + C\|c^1\|^2 + C. \tag{5.79}$$

Estimate of c^1. Let

$$\Psi = \Psi_{c^1} = v_y(|v|^2 - 5)\frac{d}{dy}\phi_{c^1}(y)\sqrt{\mu},$$

where

$$-\phi''_{c^1} = c^1, \quad \phi_{c^1}(\pm 1) = 0.$$

One has

$$\|\phi_{c^1}\|_{H^2} \le C\|c^1\|, \quad |\phi'_{c^1}(\pm 1)| \le C\|c^1\|. \tag{5.80}$$

By the Cauchy–Schwarz inequality and (5.80), it follows that

$$\begin{aligned}\epsilon|(G^1_R, \Psi_{c^1})| &\le \epsilon|(\mathbf{P}_0 G^1_R, \Psi_{c^1})| + \epsilon|(\mathbf{P}_1 G^1_R, \Psi_{c^1})| \\ &\le C\epsilon\|c\|^2 + C\epsilon\{\|\mathbf{P}_1 G^1_{R,2}\|^2 + \|w_q G^1_{R,1}\|^2_{L^\infty}\}, \\ -(v_y G^1_R, \partial_y \Psi_{c^1}) &= -(v_y \mathbf{P}_0 G^1_R, \partial_y \Psi_{c^1}) - (v_y \mathbf{P}_1 G^1_R, \partial_y \Psi_{c^1}) \\ &\ge 30\|c^1\|^2 - \eta\|c^1\|^2 - C_\eta\{\|\mathbf{P}_1 G^1_{R,2}\|^2 + \|w_q G^1_{R,1}\|^2_{L^\infty}\},\end{aligned}$$

and

$$\alpha|(v_y G^1_R, \partial_{v_x}\Psi_{c^1})| + \tfrac{1}{2}\alpha|(v_x v_y G^1_R, \Psi_{c^1})| \le C\alpha(\|b^1_1\|^2 + \|w_q G^1_{R,1}\|^2_{L^\infty} + \|G^1_{R,2}\|^2).$$

Also similar to (5.74) and (5.75), one has

$$(\mu^{-1/2}\mathcal{S}^0, \Psi_{c^1}) \le \eta\|c^1\|^2 + C,$$

and

$$\begin{aligned}|(LG^1_R, \Psi_{c^1})| &\le |(L(G^1_{R,1}\mu^{-1/2}), \Psi_{c^1})| + |(LG_{R,2}, \Psi_{c^1})| \\ &\le \eta\|c^1\|^2 + C_\eta(\|w_q G^1_{R,1}\|^2_{L^\infty} + \|\mathbf{P}_1 G^1_{R,2}\|^2).\end{aligned}$$

For the boundary term, by applying (5.78) and using

$$\langle v_y P_\gamma G^1_R(\pm 1), \Psi_{c^1}(\pm 1)\rangle = 0,$$

we have

$$\begin{aligned}&\langle v_y G^1_R(1), \Psi_{c^1}(1)\rangle - \langle v_y G^1_R(-1), \Psi_{c^1}(-1)\rangle \\ &\qquad \le \eta\|c^1\|^2 + C_\eta|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} + C_\eta\|w_q G^1_{R,1}\|^2_{L^\infty}.\end{aligned}$$

Combining the above estimates of c^1 gives

$$\|c^1\|^2 \le C\|\mathbf{P}_1 G^1_{R,2}\|^2 + C\|w_q G^1_{R,1}\|^2_{L^\infty} + C\alpha\{\|G^1_{R,2}\|^2 + \|b^1_1\|^2\} + C\|w_q G^1_{R,1}\|^2_{L^\infty} + C|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} + C\alpha^2. \tag{5.81}$$

Finally, a linear combination of (5.76), (5.79) and (5.81) gives

$$\|[a^1, \mathbf{b}^1, c^1]\|^2 \leq C\|\mathbf{P}_1 G^1_{R,2}\|^2 + C\|w_q G^1_{R,1}\|^2_{L^\infty} + C\alpha\|G^1_{R,2}\|^2 + C|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} + C. \tag{5.82}$$

This together with (5.69) and (5.70) implies that

$$\|[a^1_2, \mathbf{b}^1_2, c^1_2]\|^2 \leq C\|\mathbf{P}_1 G^1_{R,2}\|^2 + C\|w_q G^1_{R,1}\|^2_{L^\infty} + C|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} + C. \tag{5.83}$$

In order to obtain estimates for $\|\mathbf{P}_1 G^1_{R,2}\|$, we have to further consider the BVP for $G^1_{R,2}$ as follows:

$$\begin{cases} \epsilon G^1_{R,2} + v_y \partial_y G^1_{R,2} - \alpha v_y \partial_{v_x} G^1_{R,2} + L G^1_{R,2} - (1-\chi_M)\mu^{-1/2}\mathcal{K} G^1_{R,1} = 0, \\ G^1_{R,2}(\pm 1, v)\mathbf{1}_{\{v_y \lessgtr 0\}} = \sqrt{2\pi\mu}\displaystyle\int_{v_y \gtrless 0} \sqrt{\mu}\, G^1_R(\pm 1, v)|v_y|\, dv. \end{cases}$$

Applying the estimates (5.33), (5.36) and (5.39) with $\sigma = 1$, $\mathcal{F}_2 = 0$ and $\mathcal{F}_{2,b} = 0$, one has

$$\epsilon\|G^1_{R,2}\|^2 + \delta_0\|\mathbf{P}_1 G^1_{R,2}\|^2 + \tfrac{1}{2}|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} \leq \eta|P_\gamma G^1_{R,2}|^2_{2,+} + \eta\|G^1_{R,2}\|^2 + C_\eta\|w_q G^1_{R,1}\|^2_{L^\infty}, \tag{5.84}$$

$$|P_\gamma G^1_{R,2}|^2_{2,+} \leq \varepsilon|\{I - P_\gamma\}G^1_{R,2}|^2_{2,+} + C\|G^1_{R,2}\|^2 + C\|w_q G^1_{R,1}\|^2_{L^\infty}, \tag{5.85}$$

and

$$\epsilon\|\partial^m_{v_x} G^1_{R,2}\|^2 + \delta_0\|\partial^m_{v_x} G^1_{R,2}\|^2 + \tfrac{1}{2}|\partial^m_{v_x} G^1_{R,2}|^2_{2,+} \leq C\|G^1_{R,2}\|^2 + C\sum_{m' \leq m}\|w_q \partial^{m'}_{v_x} G^1_{R,1}\|^2_{L^\infty} + C(m)|P_\gamma G^1_{R,2}|^2_{2,+}, \tag{5.86}$$

where all the constants on the right hand side are independent of ϵ. Then (5.82) and (5.84)–(5.86) give

$$\sum_{0\leq m\leq N_0}\|\partial^m_{v_x} G^1_{R,2}\|^2 + \sum_{0\leq m\leq N_0}|\partial^m_{v_x} G^1_{R,2}|^2_{2,+} \leq C\sum_{0\leq m\leq N_0}\|w_q\partial^m_{v_x} G^1_{R,1}\|^2_{L^\infty} + C. \tag{5.87}$$

Consequently, a linear combination of (5.64), (5.65) and (5.87) gives

$$\sum_{0\leq m\leq N_0}\{\|w_q\partial^m_{v_x} G^1_{R,1}\|_{L^\infty} + \|w_q\partial^m_{v_x} G^1_{R,1}\|_{L^\infty}\} \leq \mathcal{C}_0$$

for some suitably large $\mathcal{C}_0 > 0$. Therefore (5.59) holds for $n = 1$.

We now assume that (5.59) is valid for $n = k \geq 1$ and we will prove it for $n = k + 1$. In fact, applying the estimates (5.22) and (5.30) to the systems (5.6)–(5.7) and (5.8)–(5.9) with $n = k$, one has

$$\sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m G_{R,1}^{k+1}\|_{L^\infty} \leq C\alpha \sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m G_{R,2}^{k+1}\|_{L^\infty} + C \sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m \mathcal{S}^k\|_{L^\infty}, \tag{5.88}$$

and

$$\sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m G_{R,2}^{k+1}\|_{L^\infty} \leq C \sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m G_{R,2}^{k+1}\| + C \sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m G_{R,1}^{k+1}\|_{L^\infty}, \tag{5.89}$$

where

$$\mathcal{S}^k = -\tfrac{1}{2}\sqrt{\mu}\, v_x v_y G_1 + \sqrt{\mu}\, v_y \partial_{v_x} G_1 + Q(\sqrt{\mu}\, G_1, \sqrt{\mu}\, G_1) + \alpha\{Q(\sqrt{\mu}\, G_R^k, \sqrt{\mu}\, G_1) + Q(\sqrt{\mu}\, G_1, \sqrt{\mu}\, G_R^k)\} + \alpha^2 Q(\sqrt{\mu}\, G_R^k, \sqrt{\mu}\, G_R^k).$$

By Lemmas 4.1 and 2.2 and the induction assumption, we have

$$\sum_{0\leq m\leq N_0} \|w_q \partial_{v_x}^m \mathcal{S}^k\|_{L^\infty} \leq C + C\alpha \sum_{0\leq m\leq N_0} \{\|w_q \partial_{v_x}^m G_{R,1}^k\|_{L^\infty} + \|w_q \partial_{v_x}^m G_{R,1}^k\|_{L^\infty}\} + C\alpha^2 \sum_{0\leq m\leq N_0} \{\|w_q \partial_{v_x}^m G_{R,1}^k\|_{L^\infty}^2 + \|w_q \partial_{v_x}^m G_{R,1}^k\|_{L^\infty}^2\}. \tag{5.90}$$

For the L^2 estimate, by performing a parallel calculation to (5.83), one has

$$\|[a_1^{k+1}, \mathbf{b}_1^{k+1}, c_1^{k+1}]\|^2 \leq C\|\mathbf{P}_1 G_{R,2}^{k+1}\|^2 + C\|w_q G_{R,1}^{k+1}\|_{L^\infty}^2 + C|\{I - P_\gamma\}G_{R,2}^{k+1}|_{2,+}^2 + C\sum_{j=1}^{3} |(\mu^{-1/2}\mathcal{S}^k, \Psi_j)|. \tag{5.91}$$

Here Ψ_j $(1 \leq j \leq 3)$ are chosen as $\Psi_{a^{k+1}}$, $\Psi_{b_i^{k+1}}$ and $\Psi_{c^{k+1}}$ in the same way as for Ψ_{a^1}, $\Psi_{b_i^1}$ and Ψ_{c^1}, respectively. Hence, (5.91) also gives

$$\|[a_2^{k+1}, \mathbf{b}_2^{k+1}, c_2^{k+1}]\|^2 \leq C\|\mathbf{P}_1 G_{R,2}^{k+1}\|^2 + C\|w_q G_{R,1}^{k+1}\|_{L^\infty}^2 + C|\{I - P_\gamma\}G_{R,2}^{k+1}|_{2,+}^2 + C + C\alpha^2\{\|w_q G_{R,1}^k\|_{L^\infty}^2 + \|w_q G_{R,2}^k\|_{L^\infty}^2\} + C\alpha^4\{\|w_q G_{R,1}^k\|_{L^\infty}^4 + \|w_q G_{R,2}^k\|_{L^\infty}^4\}, \tag{5.92}$$

by applying Lemma 2.2 and (5.69).

On the other hand, similar to the estimates (5.84)–(5.86), it also follows that

$$\epsilon\|G_{R,2}^{k+1}\|^2+\delta_0\|\mathbf{P}_1G_{R,2}^{k+1}\|^2+\frac{1}{2}|\{I-P_\gamma\}G_{R,2}^{k+1}|_{2,+}^2 \\ \leq \eta|P_\gamma G_{R,2}^{k+1}|_{2,+}^2+\eta\|G_{R,2}^{k+1}\|^2+C_\eta\|w_qG_{R,1}^{k+1}\|_{L^\infty}^2, \tag{5.93}$$

$$|P_\gamma G_{R,2}^{k+1}|_{2,+}^2\leq \varepsilon|\{I-P_\gamma\}G_{R,2}^{k+1}|_{2,+}^2+C\|G_{R,2}^{k+1}\|^2+C\|w_qG_{R,1}^{k+1}\|_{L^\infty}^2, \tag{5.94}$$

and

$$\epsilon\|\partial_{v_x}^mG_{R,2}^{k+1}\|^2+\delta_0\|\partial_{v_x}^mG_{R,2}^{k+1}\|^2+\frac{1}{2}|\partial_{v_x}^mG_{R,2}^{k+1}|_{2,+}^2 \\ \leq C\|G_{R,2}^{k+1}\|^2+C\sum_{m'\leq m}\|w_q\partial_{v_x}^{m'}G_{R,1}^{k+1}\|_{L^\infty}^2+C|P_\gamma G_{R,2}^{k+1}|_{2,+}^2. \tag{5.95}$$

As a consequence, combining the estimates (5.92)–(5.95) gives

$$\sum_{0\leq m\leq N_0}\|\partial_{v_x}^mG_{R,2}^{k+1}\|^2+\sum_{0\leq m\leq N_0}|\partial_{v_x}^mG_{R,2}^{k+1}|_{2,+}^2 \\ \leq C\sum_{0\leq m\leq N_0}\|w_q\partial_{v_x}^mG_{R,1}^{k+1}\|_{L^\infty}^2+C\alpha^2\{\|w_qG_{R,1}^k\|_{L^\infty}^2+\|w_qG_{R,1}^k\|_{L^\infty}^2\} \\ +C\alpha^4\{\|w_qG_{R,1}^k\|_{L^\infty}^4+\|w_qG_{R,2}^k\|_{L^\infty}^4\}. \tag{5.96}$$

Finally, by taking $C_1>0$ suitably large, we infer from (5.88)–(5.90) and (5.96) that

$$\|[G_{R,1}^{k+1},G_{R,2}^{k+1}]\|_{\mathbf{X}_{\alpha,N_0}}\leq \mathscr{C}_0+C_1\alpha\|[G_{R,1}^k,G_{R,2}^k]\|_{\mathbf{X}_{\alpha,N_0}}+C_1\alpha^2\|[G_{R,1}^k,G_{R,2}^k]\|_{\mathbf{X}_{\alpha,N_0}}^2 \\ \leq \mathscr{C}_0\{1+2\mathscr{C}_0C_1\alpha+4C_1\mathscr{C}_0^2\alpha^2\}\leq \frac{5}{4}\mathscr{C}_0$$

provided that α is chosen to be sufficiently small. Thus (5.59) holds for $n=k+1$. Therefore, it holds for all $n\geq 0$.

We now prove that $\{[G_{R,1}^n,G_{R,1}^n]\}_{n=0}^\infty$ is a Cauchy sequence in $\mathbf{X}_{\alpha,N_0}$. For this, denote

$$\mu^{1/2}\tilde{G}_R^{n+1}=\tilde{G}_{R,1}^{n+1}+\mu^{1/2}\tilde{G}_{R,2}^{n+1}$$

with

$$[\tilde{G}_{R,1}^{n+1},\tilde{G}_{R,2}^{n+1}]=[G_{R,1}^{n+1}-G_{R,1}^n,G_{R,2}^{n+1}-G_{R,2}^n].$$

Then $[\tilde{G}_{R,1}^{n+1},\tilde{G}_{R,2}^{n+1}]$ satisfies

$$\epsilon\tilde{G}_{R,1}^{n+1}+v_y\partial_y\tilde{G}_{R,1}^{n+1}-\alpha v_y\partial_{v_x}\tilde{G}_{R,1}^{n+1}+\nu_0\tilde{G}_{R,1}^{n+1}-\chi_M\mathcal{K}\tilde{G}_{R,1}^{n+1}+\frac{1}{2}\alpha\sqrt{\mu}\,v_xv_y\tilde{G}_{R,2}^{n+1} \\ =\alpha\{Q(\sqrt{\mu}\tilde{G}_R^n,\sqrt{\mu}\,G_1)+Q(\sqrt{\mu}\,G_1,\sqrt{\mu}\tilde{G}_R^n)\} \\ +\alpha^2\{Q(\sqrt{\mu}\tilde{G}_R^n,\sqrt{\mu}\tilde{G}_R^n)+Q(\sqrt{\mu}\tilde{G}_R^n,\sqrt{\mu}\,G_R^{n-1})+Q(\sqrt{\mu}\,G_R^{n-1},\sqrt{\mu}\tilde{G}_R^n)\} \\ =:\mathcal{N},\quad y\in(-1,1),\quad v\in\mathbb{R}^3, \\ \tilde{G}_{R,1}^{n+1}(\pm 1,v)|_{v_y\lessgtr 0}=0,\quad v\in\mathbb{R}^3,$$

and

$$\epsilon\tilde{G}^{n+1}_{R,2} + v_y\partial_y\tilde{G}^{n+1}_{R,2} - \alpha v_y\partial_{v_x}\tilde{G}^{n+1}_{R,2} + L\tilde{G}^{n+1}_{R,2} - (1-\chi_M)\mu^{-1/2}\mathcal{K}\tilde{G}^{n+1}_{R,1} = 0,$$
$$y\in(-1,1),\quad v\in\mathbb{R}^3,$$
$$\tilde{G}^{n+1}_{R,2}(\pm1,v)|_{v_y\lessgtr0} = \sqrt{2\pi\mu}\int_{v_y\gtrless0}\sqrt{\mu}\,\tilde{G}^{n+1}_{R}(\pm1,v)|v_y|\,dv,\quad v\in\mathbb{R}^3.$$

We claim that

$$\|[\tilde{G}^{n+1}_{R,1},\tilde{G}^{n+1}_{R,2}]\|_{\mathbf{X}_{\alpha,N_0}} \le C_m\alpha\|[\tilde{G}^{n}_{R,1},\tilde{G}^{n}_{R,2}]\|_{\mathbf{X}_{\alpha,N_0}} \tag{5.97}$$

under the condition (5.59). In fact, on the one hand, by performing a similar calculation to those leading to (5.88), (5.89) and (5.96), one has

$$\|[\tilde{G}^{n+1}_{R,1},\tilde{G}^{n+1}_{R,2}]\|_{\mathbf{X}_{\alpha,N_0}} \le C\sum_{0\le m\le N_0}\|w_q\partial^m_{v_x}\mathcal{N}(\tilde{G}^n_R,\tilde{G}^n_R)\|_{L^\infty}.$$

On the other hand, from Lemma 2.2 we have

$$\sum_{0\le m\le N_0}\|w_q\partial^m_{v_x}\mathcal{N}(\tilde{G}^n_R,\tilde{G}^n_R)\|_{L^\infty} \le C\alpha\sum_{0\le m\le N_0}\{\|w_q\partial^m_{v_x}\tilde{G}^n_{R,1}\|_{L^\infty}+\|w_q\partial^m_{v_x}\tilde{G}^n_{R,2}\|_{L^\infty}\}$$
$$+ C\alpha^2\sum_{0\le m\le N_0}\|w_q\partial^m_{v_x}\tilde{G}^n_R\|_{L^\infty}\sum_{0\le m\le N_0}\|w_q\partial^m_{v_x}G^n_R\|_{L^\infty},$$

which is further bounded by

$$C\alpha\sum_{0\le m\le N_0}\{\|w_q\partial^m_{v_x}\tilde{G}^n_{R,1}\|_{L^\infty}+\|w_q\partial^m_{v_x}\tilde{G}^n_{R,2}\|_{L^\infty}\},$$

according to (5.59). Thus, the claim (5.97) holds. In other words, if $\alpha>0$ is suitably small, then $\{[G^n_{R,1},G^n_{R,2}]\}_{n=0}^\infty$ is a Cauchy sequence in $\mathbf{X}_{\alpha,N_0}$. Hence,

$$[G^n_{R,1},G^n_{R,2}]\to[G^\epsilon_{R,1},G^\epsilon_{R,2}]$$

strongly in $\mathbf{X}_{\alpha,N_0}$ as $n\to\infty$. Moreover, the convergence is uniform with respect to ϵ, and the limit $[G^\epsilon_{R,1},G^\epsilon_{R,2}]$ is a unique solution to (5.55)–(5.56) and (5.57)–(5.58). In addition, $[G^\epsilon_{R,1},G^\epsilon_{R,2}]$ also satisfies

$$\|[G^\epsilon_{R,1},G^\epsilon_{R,2}]\|_{\mathbf{X}_{\alpha,N_0}}\le C, \tag{5.98}$$

where $C>0$ is independent of ϵ.

Furthermore, by taking the limit $\epsilon\to0$, we can repeat the same procedure as when letting $n\to\infty$ so that the limit function $[G_{R,1},G_{R,2}]\in\mathbf{X}_{\alpha,N_0}$ is the unique solution of (5.2)–(5.3) and (5.4)–(5.5) with the same bound as in (5.98). Thus the proof of Proposition 5.1 is complete. ∎

Finally, Theorem 1.1 is an immediate consequence of Propositions 4.1 and 5.1, except for the nonnegativity of the solution $F_{\text{st}}(y,v)$ that will be proved from the dynamical stability of $F_{\text{st}}(y,v)$ in Theorem 1.2. ∎

6. Unsteady problem: local existence

We now turn to the time-dependent situation. To solve the initial boundary value problem (1.21), we set the perturbation as

$$F(t,y,v) = F_{\rm st}(y,v) + \sqrt{\mu}\, f(t,y,v). \tag{6.1}$$

Then $f = f(t,y,v)$ satisfies

$$\begin{cases} \partial_t f + v_y\partial_y f - \alpha v_y \partial_{v_x} f + \frac{1}{2}\alpha v_x v_y f + Lf \\ \qquad = \Gamma(f,f) + \alpha\{\Gamma(G_1+\alpha G_R, f) + \Gamma(f, G_1+\alpha G_R)\}, \\ \qquad\qquad\qquad\qquad t>0,\ y\in(-1,1),\ v=(v_x,v_y,v_z)\in\mathbb{R}^3, \\ \sqrt{\mu}\, f(0,y,v) =: f_0(y,v) = F(0,y,v) - F_{\rm st}(y,v), \quad y\in(-1,1),\ v\in\mathbb{R}^3, \\ f(t,\pm1,v)|_{v_y\lessgtr 0} = \sqrt{2\pi\mu}\displaystyle\int_{v_y\gtrless 0} f(t,\pm1,v)\sqrt{\mu}\,|v_y|\,dv, \quad t\geq 0,\ v\in\mathbb{R}^3. \end{cases} \tag{6.2}$$

The goal of this section is to construct a local-in-time solution to the initial boundary value problem (6.2). The proof of the global existence of solutions as well as the large time behavior will be left to the next section. To resolve the difficulty caused by the growth term $\frac{1}{2}\alpha v_x v_y f$, it is still necessary to introduce the decomposition

$$\sqrt{\mu}\, f = f_1 + \sqrt{\mu}\, f_2, \tag{6.3}$$

where f_1 and f_2 satisfy the initial boundary value problems

$$\begin{aligned} &\partial_t f_1 + v_y\partial_y f_1 - \alpha v_y\partial_{v_x} f_1 + \nu_0 f_1 \\ &\qquad = \chi_M \mathcal{K} f_1 - \tfrac{1}{2}\alpha\sqrt{\mu}\, v_x v_y f_2 + \alpha\{Q(\sqrt{\mu}\, f, \sqrt{\mu}\{G_1+\alpha G_R\}) \\ &\qquad\quad + Q(\sqrt{\mu}\{G_1+\alpha G_R\}, \sqrt{\mu}\, f)\} + Q(\sqrt{\mu}\, f, \sqrt{\mu}\, f), \end{aligned} \tag{6.4}$$

$$\begin{aligned} &f_1(0,y,v) = f_0(y,v) = F_0 - F_{\rm st}, \\ &f_1(\pm1,v)|_{v_y\lessgtr 0} = \sqrt{2\pi}\,\mu\int_{v_y\gtrless 0} f_1(\pm1,v)|v_y|\,dv, \end{aligned} \tag{6.5}$$

and

$$\partial_t f_2 + v_y\partial_y f_2 - \alpha v_y\partial_{v_x} f_2 + Lf_2 = (1-\chi_M)\mu^{-1/2}\mathcal{K} f_1, \tag{6.6}$$

$$f_2(0,y,v) = 0, \quad f_2(\pm1,v)|_{v_y\lessgtr 0} = \sqrt{2\pi\mu}\int_{v_y\gtrless 0}\sqrt{\mu}\, f_2(\pm1,v)|v_y|\,dv, \tag{6.7}$$

respectively. Note that the initial data for f_2 is set to be zero.

We will look for solutions to (6.4)–(6.5) and (6.6)–(6.7) in the function space

$$\mathbf{Y}_{\alpha,T} = \left\{[\mathcal{G}_1,\mathcal{G}_2]\ \middle|\ \sup_{0\leq t\leq T}\{\|w_q\mathcal{G}_1(t)\|_{L^\infty} + \|w_q\mathcal{G}_2(t)\|_{L^\infty}\} < \infty\right\}$$

endowed with the norm

$$\|[\mathcal{G}_1,\mathcal{G}_2]\|_{\mathbf{Y}_{\alpha,T}} = \sup_{0\leq t\leq T}\{\|w_q\mathcal{G}_1(t)\|_{L^\infty} + \|w_q\mathcal{G}_2(t)\|_{L^\infty}\}.$$

Theorem 6.1 (Local existence). *Under the assumptions of Theorem 1.2, there exists $T_* > 0$ depending on α such that the coupled systems (6.4)–(6.5) and (6.6)–(6.7) admit a unique local-in-time solution $[f_1(t,y,v), f_2(t,y,v)]$ satisfying*

$$\|[f_1, f_2]\|_{\mathbf{Y}_{\alpha,T_*}} \leq C_0\varepsilon_0$$

for some $C_0 > 0$.

Proof. We first consider the following systems for approximation solutions:

$$\partial_t f_1^{n+1} + v_y\partial_y f_1^{n+1} - \alpha v_y \partial_{v_x} f_1^{n+1} + \nu_0 f_1^{n+1} = \chi_M \mathcal{K} f_1^n - \tfrac{1}{2}\alpha\sqrt{\mu}\, v_x v_y f_2^n + H(f_1^n, f_2^n), \tag{6.8}$$

$$\begin{aligned} &f_1^{n+1}(0,y,v) = f_0(y,v) = F_0 - F_{\text{st}},\\ &f_1^{n+1}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi}\,\mu \int_{v_y \gtrless 0} f_1^n(\pm 1, v)|v_y|\,dv, \end{aligned} \tag{6.9}$$

and

$$\partial_t f_2^{n+1} + v_y\partial_y f_2^{n+1} - \alpha v_y \partial_{v_x} f_2^{n+1} + \nu_0 f_2^{n+1} = K f_2^n + (1-\chi_M)\mu^{-1/2}\mathcal{K} f_1^n, \tag{6.10}$$

$$\begin{aligned} &f_2^{n+1}(0,y,v) = 0,\\ &f_2^{n+1}(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu} \int_{v_y \gtrless 0} \sqrt{\mu}\, f_2^n(\pm 1, v)|v_y|\,dv, \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} H(f_1^n, f_2^n) = \alpha\{&Q(\sqrt{\mu}\, f^n, \sqrt{\mu}\{G_1 + \alpha G_R\}) + Q(\sqrt{\mu}\{G_1 + \alpha G_R\}, \sqrt{\mu}\, f^n)\}\\ &+ Q(\sqrt{\mu}\, f^n, \sqrt{\mu}\, f^n), \end{aligned}$$

and $\sqrt{\mu}\, f^n = f_1^n + \sqrt{\mu}\, f_2^n$. Set $[f_1^0, f_2^0] = [f_0, 0]$.

Next, one can show inductively that there exists a finite $T_* > 0$ such that

$$\sup_{0\leq t\leq T_*} \|w_q[f_1^m, f_2^m](t)\|_{L^\infty} \leq C_0\varepsilon_0 \tag{6.12}$$

for any $m \geq 0$ provided that

$$\|w_q[f_1^0, f_2^0]\|_{L^\infty} = \|w_q[F_0(y,v) - F_{\text{st}}(y,v)]\|_{L^\infty} \leq \varepsilon_0.$$

This also implies that $[f_1^{n+1}, f_2^{n+1}]$ is well-defined by (6.8)–(6.9) and (6.10)–(6.11) if $[f_1^n, f_2^n]$ is bounded as in (6.12). Denote

$$[\mathfrak{G}_1^n, \mathfrak{G}_2^n] = w_q[f_1^n, f_2^n], \quad \sqrt{\mu}\,\mathfrak{G}^n = \mathfrak{G}_1^n + \sqrt{\mu}\,\mathfrak{G}_2^n.$$

Then $\mathfrak{G}_1^n$ and $\mathfrak{G}_2^n$ satisfy

$$\begin{aligned}\partial_t \mathfrak{G}_1^{n+1} + v_y \partial_y \mathfrak{G}_1^{n+1} - \alpha v_y \partial_{v_x} \mathfrak{G}_1^{n+1} + 2q\alpha \frac{v_x v_y}{1+|v|^2} \mathfrak{G}_1^{n+1} + \nu_0 \mathfrak{G}_1^{n+1} \\ = \chi_M w_q \mathcal{K}\left(\frac{\mathfrak{G}_1^n}{w_q}\right) - \frac{1}{2}\alpha\sqrt{\mu}\, v_x v_y \mathfrak{G}_2^n + w_q H(f_1^n, f_2^n),\end{aligned} \tag{6.13}$$

$$\begin{aligned}&\mathfrak{G}_1^{n+1}(0,y,v) = w_q f_0(y,v),\\ &\mathfrak{G}_1^{n+1}(\pm 1, v)|_{v_y \lessgtr 0} = \tilde{w}_1^{-1} \int_{v_y \gtrless 0} \tilde{w}_1 \sqrt{2\pi}\, \mu \mathfrak{G}_1^n(\pm 1, v)|v_y|\, dv,\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}\partial_t \mathfrak{G}_2^{n+1} + v_y \partial_y \mathfrak{G}_2^{n+1} - \alpha v_y \partial_{v_x} \mathfrak{G}_2^{n+1} + 2q\alpha \frac{v_x v_y}{1+|v|^2} \mathfrak{G}_2^{n+1} + \nu_0 \mathfrak{G}_2^{n+1} \\ = w_q K\left(\frac{\mathfrak{G}_2^n}{w_q}\right) + (1-\chi_M) w_q \mu^{-1/2} \mathcal{K} f_1^n,\end{aligned} \tag{6.15}$$

$$\begin{aligned}&\mathfrak{G}_2^{n+1}(0,y,v) = 0,\\ &\mathfrak{G}_2^{n+1}(\pm 1, v)|_{v_y \lessgtr 0} = \tilde{w}_2^{-1} \int_{v_y \gtrless 0} \tilde{w}_2 \sqrt{2\pi}\, \mu \mathfrak{G}_2^n(\pm 1, v)|v_y|\, dv,\end{aligned} \tag{6.16}$$

with $[\mathfrak{G}_1^0, \mathfrak{G}_2^0] = w_q[f_1^0, f_2^0] = w_q[f_0, 0]$. Here

$$\tilde{w}_1 = \tilde{w}_1(v) = (\sqrt{2\pi}\, w_q \mu)^{-1},$$

and $\tilde{w}_2$ is given by (5.25).

Along the same characteristic line (3.3), by noting that s is no longer a parameter and it is nonnegative, (6.13)–(6.14) and (6.15)–(6.16) are equivalent to

$$\begin{aligned}&\mathfrak{G}_1^{n+1}(t,y,v)\\ &= \mathbf{1}_{t_1 \le 0}\, e^{-\int_0^t \mathcal{A}(\tau, V(\tau))\, d\tau} (w_q f_0)(Y(0), V(0))\\ &\quad + \underbrace{\mathbf{1}_{t_1>0}\, e^{-\int_{t_1}^t \mathcal{A}(\tau,V(\tau))\,d\tau} \tilde{w}_1^{-1}(V(t_1)) \int_{n(y_1)\cdot v_1>0} \tilde{w}_1 \sqrt{2\pi}\, \mu \mathfrak{G}_1^n(t_1, y_1, v_1)|v_{1y}|\, dv_1}_{\mathcal{I}_b^{(1)}}\\ &\quad + \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,d\tau} \left\{\chi_M w_q \mathcal{K}\left(\frac{\mathfrak{G}_1^n}{w_q}\right)\right\}(V(s))\, ds\\ &\quad - \alpha \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,d\tau} \frac{V_x(s)V_y(s)}{2} \sqrt{\mu}(V(s)) \mathfrak{G}_2^n(V(s))\, ds\\ &\quad + \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,d\tau} (w_q H(f_1^n, f_2^n))(V(s))\, ds,\end{aligned} \tag{6.17}$$

and

$$
\begin{aligned}
&\mathfrak{G}_2^{n+1}(t,y,v)\\
&\quad = \underbrace{\mathbf{1}_{t_1>0}\, e^{-\int_{t_1}^t \mathcal{A}(\tau,V(\tau))\,d\tau}\tilde{w}_2^{-1}(V(t_1))\int_{n(y_1)\cdot v_1>0}\tilde{w}_2\sqrt{2\pi}\,\mu\mathfrak{G}_2^n(t_1,y_1,v_1)|v_{1y}|\,dv_1}_{\mathcal{I}_b^{(2)}}\\
&\qquad + \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,d\tau}\left\{(1-\chi_M)\mu^{-1/2}w_q\mathcal{K}\left(\frac{\mathfrak{G}_1^n}{w_q}\right)\right\}(V(s))\,ds\\
&\qquad + \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}(\tau,V(\tau))\,d\tau}\left[w_q K\left(\frac{\mathfrak{G}_2^n}{w_q}\right)\right](V(s))\,ds,
\end{aligned}
\tag{6.18}
$$

where

$$
\mathcal{A}(\tau,V(\tau)) = \nu_0 + 2q\alpha\frac{V_y(\tau)V_x(\tau)}{1+|V(\tau)|^2} \geq \nu_0/2
$$

provided that $q\alpha$ is suitably small.

For the boundary terms $\mathcal{I}_b^{(1)}$ and $\mathcal{I}_b^{(2)}$, we use equations (6.17) and (6.18) recursively to obtain

$$
\mathcal{I}_b^{(1)} = \sum_{j=1}^5 \mathcal{I}_j^{(1)},\quad \mathcal{I}_b^{(2)} = \sum_{j=1}^3 \mathcal{I}_j^{(2)}
$$

with

$$
\begin{aligned}
\mathcal{I}_1^{(1)} &= \underbrace{\mathbf{1}_{t_1>0}\, e^{-\int_{t_1}^t \mathcal{A}(\tau,V(\tau))\,d\tau}\tilde{w}_1^{-1}(V(t_1))}_{\mathcal{W}_0^{(1)}}\\
&\qquad\qquad \times \int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{t_k>0}\mathfrak{G}_1^{n+1-k}(t_k,y_k,V_{\mathbf{cl}}^{k-1}(t_k))\,d\bar{\Sigma}_{k-1}^{(1)}(t_k),\\
\mathcal{I}_2^{(1)} &= \mathcal{W}_0^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}(w_q f_0)(Y_{\mathbf{cl}}^l(0),V_{\mathbf{cl}}^l(0))\,d\bar{\Sigma}_l^{(1)}(0),\\
\mathcal{I}_3^{(1)} &= \mathcal{W}_0^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\int_0^{t_l} + \mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\right\}\left\{\chi_M w_q\mathcal{K}\left(\frac{\mathfrak{G}_1^{n-l}}{w_q}\right)\right\}\\
&\qquad\qquad \times (Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\bar{\Sigma}_l^{(1)}(s)\,ds,\\
\mathcal{I}_4^{(1)} &= -\alpha\mathcal{W}_0^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\int_0^{t_l} + \mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\right\}\left\{\frac{v_x v_y}{2}\sqrt{\mu}\,\mathfrak{G}_2^{n-l}\right\}\\
&\qquad\qquad \times (Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\bar{\Sigma}_l^{(1)}(s)\,ds,\\
\mathcal{I}_5^{(1)} &= \mathcal{W}_0^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\mathbf{1}_{\{t_{l+1}\leq 0<t_l\}}\int_0^{t_l} + \mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\right\}(w_q H(f_1^{n-l},f_2^{n-l}))\\
&\qquad\qquad \times (Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\bar{\Sigma}_l^{(1)}(s)\,ds,
\end{aligned}
$$

$$\mathcal{J}_1^{(2)} = \underbrace{\mathbf{1}_{t_1>0}\, e^{-\int_{t_1}^{t} \mathcal{A}(\tau,V(\tau))\,d\tau} \tilde{w}_2^{-1}(V(t_1))}_{\mathcal{W}_0^{(2)}} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{t_k>0}\, \mathfrak{G}_2^{n+1-k}(t_k, y_k, V_{\mathbf{cl}}^{k-1}(t_k)) \times d\bar{\Sigma}_{k-1}^{(2)}(t_k),$$

$$\mathcal{J}_2^{(2)} = \mathcal{W}_0^{(2)} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \left\{\mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \int_0^{t_l} + \mathbf{1}_{\{t_{l+1}>0\}} \int_{t_{l+1}}^{t_l}\right\} \left\{w_q K\left(\frac{\mathfrak{G}_2^{n-l}}{w_q}\right)\right\} \times (Y_{\mathbf{cl}}^l, V_{\mathbf{cl}}^l)(s)\, d\bar{\Sigma}_l^{(2)}(s)\, ds,$$

$$\mathcal{J}_3^{(2)} = \mathcal{W}_0^{(2)} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \left\{\mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \int_0^{t_l} + \mathbf{1}_{\{t_{l+1}>0\}} \int_{t_{l+1}}^{t_l}\right\} \left\{(1-\chi_M) w_q \mu^{-1/2} \mathcal{K}\left(\frac{\mathfrak{G}_1^{n-l}}{w_q}\right)\right\} (Y_{\mathbf{cl}}^l, V_{\mathbf{cl}}^l)(s)\, d\bar{\Sigma}_l^{(2)}(s)\, ds,$$

where $k \ge 2$. Here, similar to (5.24), $\bar{\Sigma}_l^{(i)}(s)$ $(i = 1, 2)$ is given by

$$\bar{\Sigma}_l^{(i)}(s) = \prod_{j=l+1}^{k-1} d\sigma_j\, e^{-\int_s^{t_l} \mathcal{A}(\tau, V_{\mathbf{cl}}^l(\tau))\,d\tau} \tilde{w}_i(v_l)\, d\sigma_l \times \prod_{j=1}^{l-1} \frac{\tilde{w}_i(v_j)}{\tilde{w}_i(V_{\mathbf{cl}}^j(t_{j+1}))} e^{-\int_{t_{j+1}}^{t_j} \mathcal{A}(\tau, V_{\mathbf{cl}}^l(\tau))\,d\tau}\, d\sigma_j.$$

To obtain (6.12), one can first prove that for fixed finite $k > 0$ and any $t \ge 0$,

$$\sup_{0\le l\le k}\ \sup_{0\le s\le t} \|[\mathfrak{G}_1^l, \mathfrak{G}_2^l](s)\|_{L^\infty} \le C(k)\|w_q f_0\|_{L^\infty} \le \tfrac{1}{2} C_0 \varepsilon_0, \tag{6.19}$$

by choosing $C_0 > 0$ suitably large. Note that (6.19) can be easily obtained by using (6.17) and (6.18) recursively because k is finite.

In the following, we prove (6.12) for $m = n+1$ under the assumption that it holds for $m \le n$. By letting $t \le T_*$ with $T_* > 0$ suitably small and applying Lemma 8.1, we have

$$\begin{aligned}
&\sup_{0\le t\le T_*} \|\mathfrak{G}_1^{n+1}\|_{L^\infty} \\
&\quad\le (C/q + C\bar{\varepsilon}) \sup_{1\le l\le k}\ \sup_{0\le t\le T_*} \|\mathfrak{G}_1^{n+1-l}\|_{L^\infty} + C\alpha \sup_{1\le l\le k}\ \sup_{0\le t\le T_*} \|\mathfrak{G}_2^{n+1-l}\|_{L^\infty} \\
&\qquad + C \sup_{1\le l\le k}\ \sup_{0\le t\le T_*} (\|\mathfrak{G}_1^{n+1-l}\|_{L^\infty}^2 + \|\mathfrak{G}_2^{n+1-l}\|_{L^\infty}^2) + C\|w_q f_0\|_{L^\infty} \\
&\quad\le C\|w_q f_0\|_{L^\infty} + (C/q + C\alpha + C\bar{\varepsilon}) C_0 \varepsilon_0 + C C_0^2 \varepsilon_0^2,
\end{aligned} \tag{6.20}$$

and

$$\sup_{0\le t\le T_*} \|\mathfrak{G}_2^{n+1}\|_{L^\infty} \le C T_* \sup_{1\le l\le k}\ \sup_{0\le t\le T_*} \|\mathfrak{G}_1^{n+1-l}\|_{L^\infty} + C(T_* + \bar{\varepsilon}) \sup_{1\le l\le k}\ \sup_{0\le t\le T_*} \|\mathfrak{G}_2^{n+1-l}\|_{L^\infty}, \tag{6.21}$$

where Lemma 2.4 has been used to have the factor $1/q$ in (6.20), and the coefficient T_* on the right hand side of (6.21) comes from the last two terms in (6.18) as well as $\mathcal{J}_2^{(2)}$ and $\mathcal{J}_3^{(2)}$. Choosing T_* and $\bar{\varepsilon}$ suitably small so that $C(T_*+\bar{\varepsilon})\leq 1/8$, and using an induction argument, we deduce from (6.21) and (6.20) that

$$\begin{aligned}\|\mathfrak{G}_2^{n+1}\|_{L^\infty} &\leq \frac{1}{8^{[n/k]}}\sup_{0\leq l\leq k}\|\mathfrak{G}_2^l\|_{L^\infty}\\ &\quad+\frac{8kCT_*}{7}\{C\|w_q f_0\|_{L^\infty}+(C/q+C\alpha+C\bar{\varepsilon})C_0\varepsilon_0+CC_0^2\varepsilon_0^2\},\quad n\geq k,\end{aligned}$$

where $[n/k]$ stands for the largest integer no more than n/k. Therefore,

$$\begin{aligned}\|\mathfrak{G}_1^{n+1}\|_{L^\infty}+\|\mathfrak{G}_2^{n+1}\|_{L^\infty} &\leq \frac{1}{8^{[n/k]}}\sup_{0\leq l\leq k}\|\mathfrak{G}_2^l\|_{L^\infty}\\ &\quad+\frac{8kCT_*+7}{7}\{C\|w_q f_0\|_{L^\infty}+(C/q+C\alpha+C\bar{\varepsilon})C_0\varepsilon_0+CC_0^2\varepsilon_0^2\},\quad n\geq k.\end{aligned}$$

This together with (6.19) implies that (6.12) holds for $m=n+1$ because $q>0$ can be sufficiently large and $\varepsilon_0>0$ as well as $\alpha>0$ can be suitably small.

Let us now show that $\{[f_1^n,f_2^n]\}_{n=1}^\infty$ converges strongly in the space $\mathbf{Y}_{\alpha,T_*}$. We denote $[\tilde{\mathfrak{G}}_1^n,\tilde{\mathfrak{G}}_2^n]=[\mathfrak{G}_1^n-\mathfrak{G}_1^{n-1},\mathfrak{G}_2^n-\mathfrak{G}_2^{n-1}]$ for $n\geq 1$. Then $[\tilde{\mathfrak{G}}_1^n,\tilde{\mathfrak{G}}_2^n]$ satisfies

$$\begin{aligned}&\partial_t\tilde{\mathfrak{G}}_1^{n+1}+v_y\partial_y\tilde{\mathfrak{G}}_1^{n+1}-\alpha v_y\partial_{v_x}\tilde{\mathfrak{G}}_1^{n+1}+2q\alpha\frac{v_xv_y}{1+|v|^2}\tilde{\mathfrak{G}}_1^{n+1}+\nu_0\tilde{\mathfrak{G}}_1^{n+1}\\ &\quad=\chi_M w_q\mathcal{K}\left(\frac{\tilde{\mathfrak{G}}_1^n}{w_q}\right)-\tfrac{1}{2}\alpha\sqrt{\mu}\,v_xv_y\tilde{\mathfrak{G}}_2^n+w_q[H(f_1^n,f_2^n)-H(f_1^{n-1},f_2^{n-1})],\\ &\tilde{\mathfrak{G}}_1^{n+1}(0,y,v)=0,\quad \tilde{\mathfrak{G}}_1^{n+1}(\pm1,v)|_{v_y\lessgtr0}=\tilde{w}_1^{-1}\int_{v_y\gtrless0}\tilde{w}_1\sqrt{2\pi}\,\mu\tilde{\mathfrak{G}}_1^n(\pm1,v)|v_y|\,dv,\end{aligned}$$

and

$$\begin{aligned}&\partial_t\tilde{\mathfrak{G}}_2^{n+1}+v_y\partial_y\tilde{\mathfrak{G}}_2^{n+1}-\alpha v_y\partial_{v_x}\tilde{\mathfrak{G}}_2^{n+1}+2q\alpha\frac{v_xv_y}{1+|v|^2}\tilde{\mathfrak{G}}_2^{n+1}+\nu_0\tilde{\mathfrak{G}}_2^{n+1}\\ &\qquad\qquad=w_qK\left(\frac{\tilde{\mathfrak{G}}_2^n}{w_q}\right)+(1-\chi_M)w_q\mu^{-1/2}\mathcal{K}\tilde{f}_1^n,\\ &\tilde{\mathfrak{G}}_2^{n+1}(0,y,v)=0,\quad \tilde{\mathfrak{G}}_2^{n+1}(\pm1,v)|_{v_y\lessgtr0}=\tilde{w}_2^{-1}\int_{v_y\gtrless0}\tilde{w}_2\sqrt{2\pi}\,\mu\tilde{\mathfrak{G}}_2^n(\pm1,v)|v_y|\,dv,\end{aligned}$$

where $\tilde{f}_1^n=f_1^n-f_1^{n-1}$, and $\sqrt{\mu}\,\tilde{\mathfrak{G}}^n=\tilde{\mathfrak{G}}_1^n+\sqrt{\mu}\,\tilde{\mathfrak{G}}_2^n$. Then similar to (6.20) and (6.21), one has

$$\begin{aligned}&\sup_{0\leq t\leq T_*}\|\tilde{\mathfrak{G}}_1^{n+1}\|_{L^\infty}\\ &\qquad\leq (C/q+C\bar{\varepsilon})\sup_{1\leq l\leq k}\sup_{0\leq t\leq T_*}\|\tilde{\mathfrak{G}}_1^{n-l}\|_{L^\infty}+C\alpha\sup_{1\leq l\leq k}\sup_{0\leq t\leq T_*}\|\tilde{\mathfrak{G}}_2^{n-l}\|_{L^\infty}\\ &\qquad\quad+C\varepsilon_0\sup_{1\leq l\leq k}\sup_{0\leq t\leq T_*}(\|\tilde{\mathfrak{G}}_1^{n-l}\|_{L^\infty}+\|\tilde{\mathfrak{G}}_2^{n-l}\|_{L^\infty}),\end{aligned}\tag{6.22}$$

and

$$
\begin{aligned}
&\sup_{0\le t\le T_*} \|\tilde{\mathfrak{G}}_2^{n+1}\|_{L^\infty} \\
&\quad \le CT_* \sup_{1\le l\le k}\sup_{0\le t\le T_*} \|\tilde{\mathfrak{G}}_1^{n+1-l}\|_{L^\infty} + C(T_*+\bar{\varepsilon}) \sup_{1\le l\le k}\sup_{0\le t\le T_*} \|\tilde{\mathfrak{G}}_2^{n+1-l}\|_{L^\infty}.
\end{aligned} \tag{6.23}
$$

Plugging (6.22) into (6.23) gives

$$
\begin{aligned}
&\sup_{0\le t\le T_*} \|\tilde{\mathfrak{G}}_2^{n+1}\|_{L^\infty} \\
&\quad \le C(1/q+\alpha+\varepsilon_0+T_*+\bar{\varepsilon}) \sup_{1\le l\le k}\sup_{0\le t\le T_*} \|[\tilde{\mathfrak{G}}_1^{n+1-l}, \tilde{\mathfrak{G}}_2^{n+1-l}]\|_{L^\infty}.
\end{aligned} \tag{6.24}
$$

By taking $q>0$ sufficiently large and $\alpha>0$, $\varepsilon_0>$ as well $T_*>0$ suitably small, we see from (6.24) and (6.22) that

$$
\sup_{0\le t\le T_*} (\|\tilde{\mathfrak{G}}_1^{n+1}\|_{L^\infty} + \|\tilde{\mathfrak{G}}_2^{n+1}\|_{L^\infty}) \le \frac{1}{8^{[n/k]}} \sup_{0\le l\le k}\sup_{0\le t\le T_*} \|[\tilde{\mathfrak{G}}_1^{l}, \tilde{\mathfrak{G}}_2^{l}]\|_{L^\infty}, \quad n\ge k.
$$

On the other hand, $\sup_{0\le l\le k}\sup_{0\le t\le T_*}\|[\tilde{\mathfrak{G}}_1^{l}, \tilde{\mathfrak{G}}_2^{l}]\|_{L^\infty}$ is bounded due to (6.12). Hence $\{[f_1^n, f_2^n]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbf{Y}_{\alpha,T_*}$, and there is a unique $[f_1, f_2]\in\mathbf{Y}_{\alpha,T_*}$ such that $[f_1^n, f_2^n]$ converges strongly to $[f_1, f_2]$ as $n\to\infty$ and $[f_1, f_2]$ is the desired local-in-time solution to the coupled systems (6.4)–(6.5) and (6.6)–(6.7). This completes the proof of Theorem 6.1. ∎

7. Unsteady problem: asymptotic stability and positivity

This section is about the global existence and large time behavior of solution to the initial boundary value problem (6.2). Recall the decomposition (6.3) with f_1 and f_2 satisfying the coupled systems (6.4)–(6.5) and (6.6)–(6.7). Firstly, we focus on uniform $L^\infty\cap L^2$ estimates under the a priori assumption

$$
\sup_{s\ge 0}\{e^{\lambda_0 s}\|w_q f_1(s,y,v)\|_{L^\infty} + e^{\lambda_0 s}\|w_q f_2(s,y,v)\|_{L^\infty}\} \le \tilde{\varepsilon}, \tag{7.1}
$$

for a constant $\tilde{\varepsilon}>0$ suitably small, where $\lambda_0>0$ independent of α is to be determined later. And then we will give the proof of Theorem 1.2.

7.1. L^∞ estimates

As in the proof of Theorem 6.1, an L^∞ estimate of f follows from uniform L^∞ estimates of f_1 and f_2.

Lemma 7.1. *Let* $0 < \lambda_0 \le \nu_0/4$. *Then under the assumption* (7.1),

$$\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_1(s)\|_{L^\infty} \le C_q\|w_q f_0\|_{L^\infty} + C(\alpha+\tilde\varepsilon)\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_2(s)\|_{L^\infty}, \tag{7.2}$$

$$\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_2(s)\|_{L^\infty} \le C\|w_q f_0\|_{L^\infty} + C\sup_{0\le s\le t}\|e^{\lambda_0 s} f_2(s)\|, \tag{7.3}$$

for any $t \ge 0$.

Proof. For brevity, set

$$[g_1, g_2](t,y,v) = e^{\lambda_0 t} w_q(v)[f_1, f_2](t,y,v) \tag{7.4}$$

with $\lambda_0 > 0$ to be chosen. Then the IBVP for $[g_1, g_2]$ is as follows:

$$\begin{aligned}
&\partial_t g_1 + v_y\partial_y g_1 - \alpha v_y \partial_{v_x} g_1 + 2q\alpha\frac{v_x v_y}{1+|v|^2} g_1 + (\nu_0-\lambda_0)g_1 \\
&\qquad = \chi_M w_q \mathcal{K}\left(\frac{g_1}{w_q}\right) - \tfrac{1}{2}\alpha\sqrt{\mu}\, v_x v_y g_2 + e^{\lambda_0 t} w_q H(f_1,f_2),\\
&g_1(0,y,v) = w_q f_0(x,v),\\
&g_1(\pm1,v)|_{v_y\lessgtr 0} = \tilde w_1^{-1}\int_{v_y\gtrless 0}\tilde w_1\sqrt{2\pi}\,\mu g_1(\pm1,v)|v_y|\,dv,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t g_2 + v_y\partial_y g_2 - \alpha v_y \partial_{v_x} g_2 + 2q\alpha\frac{v_x v_y}{1+|v|^2} g_2 + (\nu_0-\lambda_0)g_2 \\
&\qquad = w_q K\left(\frac{g_2}{w_q}\right) + (1-\chi_M) w_q \mu^{-1/2}\mathcal{K}\left(\frac{g_1}{w_q}\right),\\
&g_2(0,y,v) = 0,\quad g_2(\pm1,v)|_{v_y\lessgtr 0} = \tilde w_2^{-1}\int_{v_y\gtrless 0}\tilde w_2\sqrt{2\pi}\,\mu g_2(\pm1,v)|v_y|\,dv.
\end{aligned}$$

Along the characteristic line (3.3) the solution to the above problem can be written in the mild form

$$\begin{aligned}
g_1(t,y,v) &= \mathbf{1}_{t_1\le 0}\, e^{-\int_0^t \mathcal{A}_1(\tau,V(\tau))\,d\tau}(w_q f_0)(Y(0),V(0))\\
&\quad + \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}_1(\tau,V(\tau))\,d\tau}\left\{\chi_M w_q\mathcal{K}\left(\frac{g_1}{w_q}\right)\right\}(V(s))\,ds\\
&\quad - \alpha\int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}_1(\tau,V(\tau))\,d\tau}\frac{V_x(s)V_y(s)}{2}\sqrt{\mu}(V(s))g_2(V(s))\,ds\\
&\quad + \int_{\max\{0,t_1\}}^t e^{-\int_s^t \mathcal{A}_1(\tau,V(\tau))\,d\tau} e^{\frac{\nu_0 s}{4}}(w_q H(f_1,f_2))(V(s))\,ds + \sum_{n=1}^5 \mathcal{J}_n^{(1)},
\end{aligned} \tag{7.5}$$

and

$$g_2(t,y,v)=\int_{\max\{0,t_1\}}^{t} e^{-\int_s^t \mathcal{A}_1(\tau,V(\tau))\,d\tau}\left\{(1-\chi_M)\mu^{-1/2}w_q\mathcal{K}\left(\frac{g_1}{w_q}\right)\right\}(V(s))\,ds$$
$$+\int_{\max\{0,t_1\}}^{t} e^{-\int_s^t \mathcal{A}_1(\tau,V(\tau))\,d\tau}\left[w_qK\left(\frac{g_2}{w_q}\right)\right](V(s))\,ds+\sum_{n=1}^{3}\mathcal{J}_n^{(2)}, \tag{7.6}$$

where

$$\mathcal{A}_1(\tau,V(\tau))=\nu_0-\lambda_0+2q\alpha\frac{V_y(\tau)V_x(\tau)}{1+|V(\tau)|^2}.$$

We will take $0<\lambda_0\le\frac{\nu_0}{4}$ and let $2q\alpha\le\frac{\nu_0}{4}$ such that $\mathcal{A}_1(\tau,V(\tau))\ge\frac{\nu_0}{2}$. Moreover, for an integer $k\ge 2$, the terms $\mathcal{J}_n^{(1)}$ $(1\le n\le 5)$ in (7.5) are given by

$$\mathcal{J}_1^{(1)}=\underbrace{\mathbf{1}_{t_1>0}\,e^{-\int_{t_1}^t\mathcal{A}_1(\tau,V(\tau))\,d\tau}\tilde{w}_1^{-1}(V(t_1))}_{\mathcal{W}_1^{(1)}}$$
$$\times\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{t_k>0}\,g_1^{n+1-k}(t_k,y_k,V_{\mathbf{cl}}^{k-1}(t_k))\,d\tilde{\Sigma}_{k-1}^{(1)}(t_k),$$

$$\mathcal{J}_2^{(1)}=\mathcal{W}_1^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}(w_qf_0)(Y_{\mathbf{cl}}^l(0),V_{\mathbf{cl}}^l(0))\,d\tilde{\Sigma}_l^{(1)}(0),$$

$$\mathcal{J}_3^{(1)}=\mathcal{W}_1^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}+\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\right\}\left\{\chi_Mw_q\mathcal{K}\left(\frac{g_1}{w_q}\right)\right\}$$
$$\times(Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\tilde{\Sigma}_l^{(1)}(s)\,ds,$$

$$\mathcal{J}_4^{(1)}=-\alpha\mathcal{W}_1^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}+\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\right\}\left\{\frac{v_xv_y}{2}\sqrt{\mu}\,g_2\right\}$$
$$\times(Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\tilde{\Sigma}_l^{(1)}(s)\,ds,$$

$$\mathcal{J}_5^{(1)}=\mathcal{W}_1^{(1)}\sum_{l=1}^{k-1}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\left\{\mathbf{1}_{\{t_{l+1}\le 0<t_l\}}\int_0^{t_l}+\mathbf{1}_{\{t_{l+1}>0\}}\int_{t_{l+1}}^{t_l}\right\}(w_qH(f_1,f_2))$$
$$\times(Y_{\mathbf{cl}}^l,V_{\mathbf{cl}}^l)(s)\,d\tilde{\Sigma}_l^{(1)}(s)\,ds.$$

And the terms $\mathcal{J}_n^{(2)}$ $(1\le n\le 3)$ in (7.6) are

$$\mathcal{J}_1^{(2)}=\underbrace{\mathbf{1}_{t_1>0}\,e^{-\int_{t_1}^t\mathcal{A}_1(\tau,V(\tau))\,d\tau}\tilde{w}_2^{-1}(V(t_1))}_{\mathcal{W}_1^{(2)}}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{t_k>0}\,g_2(t_k,y_k,V_{\mathbf{cl}}^{k-1}(t_k))$$
$$\times\,d\tilde{\Sigma}_{k-1}^{(2)}(t_k),$$

$$\mathcal{J}_2^{(2)} = \mathcal{W}_1^{(2)} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \left\{ \mathbf{1}_{\{t_{l+1} \le 0 < t_l\}} \int_0^{t_l} + \mathbf{1}_{\{t_{l+1} > 0\}} \int_{t_{l+1}}^{t_l} \right\} \left\{ w_q K \left(\frac{g_2}{w_q} \right) \right\} \times (Y_{\mathbf{cl}}^l, V_{\mathbf{cl}}^l)(s) \, d\tilde{\Sigma}_l^{(2)}(s) \, ds,$$

$$\mathcal{J}_3^{(2)} = \mathcal{W}_1^{(2)} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \left\{ \mathbf{1}_{\{t_{l+1} \le 0 < t_l\}} \int_0^{t_l} + \mathbf{1}_{\{t_{l+1} > 0\}} \int_{t_{l+1}}^{t_l} \right\} \left\{ (1 - \chi_M) w_q \mu^{-1/2} \mathcal{K} \left(\frac{g_1}{w_q} \right) \right\} (Y_{\mathbf{cl}}^l, V_{\mathbf{cl}}^l)(s) \, d\tilde{\Sigma}_l^{(2)}(s) \, ds,$$

where

$$\tilde{\Sigma}_l^{(i)}(s) = \prod_{j=l+1}^{k-1} d\sigma_j \, e^{-\int_s^{t_l} \mathcal{A}_1(\tau, V_{\mathbf{cl}}^l(\tau)) \, d\tau} \tilde{w}_i(v_l) \, d\sigma_l \times \prod_{j=1}^{l-1} \frac{\tilde{w}_i(v_j)}{\tilde{w}_i(V_{\mathbf{cl}}^j(t_{j+1}))} e^{-\int_{t_{j+1}}^{t_j} \mathcal{A}_1(\tau, V_{\mathbf{cl}}^l(\tau)) \, d\tau} \, d\sigma_j, \quad i = 1, 2.$$

Consequently, for any $t \ge 0$, by applying Lemmas 2.2, 2.4 and 8.1 as well as the a priori assumption (7.1), we deduce from (7.5) that

$$\sup_{0 \le s \le t} \|g_1(s, y, v)\|_{L^\infty} \le C_q \|w_q f_0\|_{L^\infty} + (C/q + C\bar{\varepsilon}) \sup_{0 \le s \le t} \|g_1(s, y, v)\|_{L^\infty} + C(\alpha + \tilde{\varepsilon}) \sup_{0 \le s \le t} \{\|g_1(s, y, v)\|_{L^\infty} + \|g_2(s, y, v)\|_{L^\infty}\},$$

which gives (7.2).

For g_2, similar to (5.26), one has

$$\begin{aligned} &|g_2(t, y, v)| \\ &\le C_q e^{-\frac{\nu_0}{2}(t - t_1)} \int_{\max\{t_1, 0\}}^t e^{-\frac{\nu_0}{2}(t - s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s), v') |g_2(s, Y(s; t, y, v), v')| \, dv' \, ds \\ &\quad + C_q e^{-\frac{\nu_0}{2}(t - t_1)} \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \int_{\max\{t_{l+1}, 0\}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^l(s), v') \\ &\qquad \times |g_2(s, Y_{\mathbf{cl}}^l(s; t, y, v), v')| \, dv' \, d\tilde{\Sigma}_l^{(2)}(s) \, ds \\ &\quad + \mathcal{P}(t), \end{aligned} \tag{7.7}$$

where

$$\mathcal{P}(t) = C_q \sup_{0 \le s \le t} \|g_1(s)\|_{L^\infty} + \tilde{\varepsilon} C_q \sup_{0 \le s \le t} \|g_2(s)\|_{L^\infty}. \tag{7.8}$$

We now have, by iterating (7.7),

$$
\begin{aligned}
&|g_2(t,y,v)| \\
&\leq C_q \int_{\max\{t_1,0\}}^{t} e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v') \int_{\max\{t_1',0\}}^{s} e^{-\frac{\nu_0}{2}(s-s')} \\
&\qquad \times \int_{\mathbb{R}^3} \mathbf{k}_w(\bar V(s';Y(s),v'),v'')|g_2(s',\bar Y(s';Y(s),v'),v'')|\,dv''\,ds'\,dv'\,ds \\
&\quad + C_q \int_{\max\{t_1,0\}}^{t} e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v')e^{-\frac{\nu_0}{2}(s-t_1')} \sum_{\ell=1}^{\imath-1} \int_{\prod_{\jmath=1}^{\imath-1}\mathcal{V}_\jmath} \int_{\max\{t_{\ell+1}',0\}}^{t_\ell'} \int_{\mathbb{R}^3} \\
&\qquad \mathbf{k}_w(\bar V_{\mathbf{cl}}^{\ell}(s';Y(s),v'),v'')|g_2(s',\bar Y_{\mathbf{cl}}^{\ell}(s';Y(s),v'),v'')|\,dv''\,d\tilde\Sigma_\ell^{(2)}(s')\,ds'\,dv'\,ds \\
&\quad + C_q \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{\max\{t_{l+1},0\}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^{l}(s),v') \int_{\max\{t_1',0\}}^{s} e^{-\frac{\nu_0}{2}(s-s')} \int_{\mathbb{R}^3} \\
&\qquad \mathbf{k}_w(\bar V(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|g_2(s',\bar Y(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|\,dv''\,ds'\,dv'\,d\tilde\Sigma_l^{(2)}(s)\,ds \\
&\quad + C_q \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{\max\{t_{l+1},0\}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^{l}(s;v),v')e^{-\frac{\nu_0}{2}(s-t_1')} \\
&\qquad \times \sum_{\ell=1}^{\imath-1} \int_{\prod_{\jmath=1}^{\imath-1}\mathcal{V}_\jmath} \int_{\max\{t_{\ell+1}',0\}}^{t_\ell'} \int_{\mathbb{R}^3} \mathbf{k}_w(\bar V_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'') \\
&\qquad \times |g_2(s',\bar Y_{\mathbf{cl}}^{\ell}(s';Y_{\mathbf{cl}}^{l}(s),v'),v'')|\,dv''\,d\tilde\Sigma_\ell^{(2)}(s')\,ds'\,dv'\,d\tilde\Sigma_l^{(2)}(s)\,ds \\
&\quad + C_q \int_{\max\{t_1,0\}}^{t} e^{-\frac{\nu_0}{2}(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v')\mathcal{P}(s)\,dv'\,ds \\
&\quad + C_q \sum_{l=1}^{k-1} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \int_{\max\{t_{l+1},0\}}^{t_l} \int_{\mathbb{R}^3} \mathbf{k}_w(V_{\mathbf{cl}}^{l}(s),v')\mathcal{P}(s)\,dv'\,d\tilde\Sigma_l^{(2)}(s)\,ds. \qquad (7.9)
\end{aligned}
$$

With (7.9), similar to (5.28), for sufficiently large $T_0 > 0$, we have

$$
\begin{aligned}
\sup_{0\leq s\leq T_0} \|g_2(s)\|_{L^\infty} \leq C\bar\varepsilon \sup_{0\leq s\leq T_0} \|g_2(s)\|_{L^\infty} + C(T_0) \sup_{0\leq s\leq T_0} \|f_2(s)\| \\
+ C \sup_{0\leq s\leq T_0} \mathcal{P}(s),
\end{aligned}
$$

which together with (7.8) gives

$$
\sup_{0\leq s\leq T_0} \|g_2(s)\|_{L^\infty} \leq C \sup_{0\leq s\leq T_0} \|g_1(s)\|_{L^\infty} + C(T_0) \sup_{0\leq s\leq T_0} \|f_2(s)\|. \qquad (7.10)
$$

Next, combining (7.2) at $t = T_0$ and (7.10), one gets

$$
\begin{aligned}
\sup_{0\leq s\leq T_0} \|[g_1,g_2](s)\|_{L^\infty} &\leq C\|w_q[f_1(0,y,v),f_2(0,y,v)]\|_{L^\infty} + C(T_0) \sup_{0\leq s\leq T_0} \|f_2(s)\| \\
&\leq C\|w_q f_0\|_{L^\infty} + C(T_0) \sup_{0\leq s\leq T_0} \|f_2(s)\|.
\end{aligned}
$$

Then it follows that for any $t \in [0, T_0]$,

$$\|w_q[f_1, f_2](t)\|_{L^\infty} \le Ce^{-\lambda_0 t}\|w_q f_0\|_{L^\infty} + C(T_0)e^{-\lambda_0 t} \sup_{0\le s\le T_0} \|f_2(s)\|. \tag{7.11}$$

In particular, we have

$$\begin{aligned}
&\|w_q[f_1, f_2](T_0)\|_{L^\infty} \\
&\quad \le Ce^{-\lambda_0 T_0}\|w_q[f_1(0, y, v), f_2(0, y, v)]\|_{L^\infty} + C(T_0)e^{-\lambda_0 T_0} \sup_{0\le s\le T_0} \|f_2(s)\| \\
&\quad \le Ce^{-\lambda_0 T_0}\|w_q f_0\|_{L^\infty} + C(T_0)e^{-\lambda_0 T_0} \sup_{0\le s\le T_0} \|f_2(s)\|.
\end{aligned} \tag{7.12}$$

Moreover, (7.11) can be extended to

$$\|w_q[f_1, f_2](t)\|_{L^\infty} \le Ce^{-\lambda_0(t-s)}\|w_q[f_1, f_2](s)\|_{L^\infty} + C(T_0)e^{-\lambda_0(t-s)} \sup_{s\le\tau\le t} \|f_2(\tau)\| \tag{7.13}$$

for any $t \in [s, s + T_0]$ with $s \ge 0$.

Next, for any integer $m \ge 1$, we can repeat the estimate (7.12) finitely many times so that the functions $[f_1, f_2](lT_0 + s)$ for $l = m - 1, m - 2, \dots, 0$ satisfy

$$\begin{aligned}
&\|w_q[f_1, f_2](mT_0)\|_{L^\infty} \\
&\quad \le Ce^{-\lambda_0 T_0}\|w_q[f_1, f_2](\{m-1\}T_0)\|_{L^\infty} + C(T_0)e^{-\lambda_0 T_0} \sup_{\{m-1\}T_0\le s\le mT_0} \|f_2(s)\| \\
&\quad \le Ce^{-\lambda_0 T_0}\|w_q[f_1, f_2](\{m-1\}T_0)\|_{L^\infty} \\
&\qquad + C(T_0)e^{-\lambda_0 T_0}e^{-\lambda_0(m-1)T_0} \sup_{\{m-1\}T_0\le s\le mT_0} \|e^{\lambda_0 s} f_2(s)\| \\
&\quad \le Ce^{-\lambda_0 mT_0}\|w_q[f_1, f_2](0)\|_{L^\infty} \\
&\qquad + C(T_0)\sum_{l=0}^{m-1} e^{-m\lambda_0 T_0} \sup_{\{m-l-1\}T_0\le s\le (m-l)T_0} \|e^{\lambda_0 s} f_2(s)\| \\
&\quad \le Ce^{-\lambda_0 mT_0}\|w_q f_0\|_{L^\infty} + C(T_0)e^{-m\lambda_0 T_0} \sup_{0\le s\le mT_0} \|e^{\lambda_0 s} f_2(s)\|.
\end{aligned} \tag{7.14}$$

Furthermore, for any $t \ge T_0$, we can find an integer $m \ge 0$ such that $t = mT_0 + s$ with $0 \le s \le T_0$. Then, on the one hand, by (7.14) we have

$$\|[g_1, g_2](mT_0)\|_\infty \le C\|w_q f_0\|_{L^\infty} + C(T_0) \sup_{0\le s\le mT_0} \|e^{\lambda_0 s} f_2(s)\|. \tag{7.15}$$

On the other hand, (7.13) implies that

$$\begin{aligned}
\|w_q[f_1, f_2](t)\|_\infty &= \|w_q[f_1, f_2](mT_0 + s)\|_{L^\infty} \\
&\le Ce^{-\lambda_0 s}\|w_q[f_1, f_2](mT_0)\|_\infty + C(T_0)e^{-\lambda_0 s} \sup_{mT_0\le\tau\le mT_0+s} \|e^{\lambda_0 \tau} f_2(\tau)\|,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}\|w_q[g_1,g_2](t)\|_\infty &= \|w_q[g_1,g_2](mT_0+s)\|_{L^\infty}\\ &\le C\|w_q[g_1,g_2](mT_0)\|_\infty + C(T_0)\sup_{mT_0\le\tau\le mT_0+s}\|e^{\lambda_0\tau}f_2(\tau)\|. \end{aligned}\tag{7.16}$$

Consequently, applying (7.15) to (7.16) gives the second estimate (7.3). This together with (7.2) yields the L^∞ estimate of f_1 and f_2, and thus completes the proof of Lemma 7.1. ■

7.2. L^2 estimates

In order to close the L^∞ estimate in terms of (7.2) and (7.3), we need to deduce the L^2 estimate of $e^{\lambda_0 t}f_2(t,y,v)$. As pointed out in Section 4, the key is to obtain the dissipation estimate of the macroscopic component of f_2 as well as f_1 through the conservation of mass. Therefore, we need to resort to the original perturbation $\sqrt{\mu}\,g_\lambda := g_1+\sqrt{\mu}\,g_2$ with some abuse of notation,

$$[g_1,g_2](t,y,v) := e^{\lambda_0 t}[f_1,f_2](t,y,v), \tag{7.17}$$

compared to (7.4) in the previous subsection. Note that the velocity weight is no longer needed for the L^2 estimates. Indeed, the only time-weighted function g_λ satisfies the IBVP

$$\begin{cases}\partial_t g_\lambda + v_y\partial_y g_\lambda - \alpha v_y\partial_{v_x}g_\lambda + \frac{1}{2}\alpha v_x v_y g_\lambda + Lg_\lambda - \lambda_0 g_\lambda\\ \quad = \underbrace{e^{\lambda_0 t}\Gamma(f,f) + \alpha e^{\lambda_0 t}\{\Gamma(G_1+\alpha G_R, f) + \Gamma(f, G_1+\alpha G_R)\}}_{\mathcal{H}},\\ \qquad\qquad t>0,\ y\in(-1,1),\ v=(v_x,v_y,v_z)\in\mathbb{R}^3,\\ \sqrt{\mu}\,g_\lambda(0,y,v) = f_0(y,v) = F(0,y,v) - F_{\rm st}(y,v),\quad y\in(-1,1),\ v\in\mathbb{R}^3,\\ g_\lambda(t,\pm1,v)|_{v_y\lessgtr 0} = \sqrt{2\pi\mu}\int_{v_y\gtrless 0}\sqrt{\mu}\,g_\lambda(t,\pm1,v)|v_y|\,dv,\quad t\ge 0,\ v\in\mathbb{R}^3.\end{cases}\tag{7.18}$$

Note that since $\int_{-1}^1\int_{\mathbb{R}^3} f(t,y,v)\sqrt{\mu}\,dv\,dy = 0$ according to (1.22) and (6.2), it is direct to see that

$$\int_{-1}^1\int_{\mathbb{R}^3} g_\lambda(t,y,v)\sqrt{\mu}\,dv\,dy = 0,\quad \forall t\ge 0.$$

Next, as in (5.67) and (5.68), we define

$$\begin{aligned}\mathbf{P}_0 g_\lambda &= (a_\lambda + \mathbf{b}_\lambda\cdot v + c_\lambda(|v|^2-3))\sqrt{\mu},\\ \mathbf{P}_0 g_2 &= (a_{\lambda,2} + \mathbf{b}_{\lambda,2}\cdot v + c_{\lambda,2}(|v|^2-3))\sqrt{\mu},\\ \bar{\mathbf{P}}_0 g_1 &= (a_{\lambda,1} + \mathbf{b}_{\lambda,1}\cdot v + c_{\lambda,1}(|v|^2-3))\mu.\end{aligned}$$

We also use the notation $\mathbf{b}_\lambda = (b_\lambda^1, b_\lambda^2, b_\lambda^3)$. Obviously,

$$a_\lambda = a_{\lambda,1} + a_{\lambda,2}, \quad \mathbf{b}_\lambda = \mathbf{b}_{\lambda,1} + \mathbf{b}_{\lambda,2}, \quad c_\lambda = c_{\lambda,1} + c_{\lambda,2},$$
$$\int_{-1}^{1} a_\lambda(t,y)\,dy = 0, \quad \forall t \geq 0. \tag{7.19}$$

As in Section 4, we are able to prove the following result in order to capture the macroscopic dissipation of g_λ.

Lemma 7.2. *Under the assumption* (7.1)*, there exists an instant functional* $\mathcal{E}_{\text{int}}(t)$ *satisfying*

$$|\mathcal{E}_{\text{int}}(t)| \leq \|g_2\|^2 + \|w_q g_1\|_{L^\infty}^2 \tag{7.20}$$

such that for any $t \geq 0$,

$$\frac{d}{dt}\mathcal{E}_{\text{int}}(t) + \lambda \|[a_\lambda, \mathbf{b}_\lambda, c_\lambda]\|^2 \leq C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2$$
$$+ C(\alpha + \tilde{\varepsilon})\|g_2\|^2 + C|\{I - P_\gamma\}g_2|_{2,+}^2. \tag{7.21}$$

Proof. The proof of (7.21) is similar to that of (5.82) in Section 5. For brevity, we only show how to derive an L^2 estimate of a_λ. By letting $\Psi = \Psi(t,y,v) \in C^\infty([0,\infty) \times [-1,1] \times \mathbb{R}^3)$ be a test function and taking the inner product of (7.18) and Ψ, one has

$$\frac{d}{dt}(g_\lambda, \Psi) - (g_\lambda, \partial_t \Psi) - (v_y g_\lambda, \partial_y \Psi) + \langle v_y, (g_\lambda \Psi)(1)\rangle - \langle v_y, (g_\lambda \Psi)(-1)\rangle$$
$$+ \alpha(v_y g_\lambda, \partial_{v_x}\Psi) + \tfrac{1}{2}\alpha(v_x v_y g_\lambda, \Psi) + ((-\lambda_0 + L)g_\lambda, \Phi) = (\mathcal{H}, \Phi). \tag{7.22}$$

Choose

$$\Psi = \Psi_{a_\lambda} = v_y \partial_y \phi_{a_\lambda}(t,y)(|v|^2 - 10)\sqrt{\mu},$$

where

$$\partial_y^2 \phi_{a_\lambda} = a_\lambda, \quad \partial_y \phi_{a_\lambda}(\pm 1) = 0, \quad \int_{-1}^{1} a_\lambda(y)\,dy = 0. \tag{7.23}$$

It follows that

$$\|\phi_{a_\lambda}\|_{H^2} \leq C\|a_\lambda\|. \tag{7.24}$$

We now estimate the terms in (7.22) one by one. The Cauchy–Schwarz inequality and (7.24) directly give

$$|(g_\lambda, \Psi_{a_\lambda})| \leq C\|g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2,$$
$$\alpha|(v_y g_\lambda, \partial_{v_x}\Psi_{a_\lambda})| \leq C\alpha\|g_2\|^2 + C\alpha\|w_q g_1\|_{L^\infty}^2,$$
$$\tfrac{1}{2}\alpha|(v_x v_y g_\lambda, \Psi_{a_\lambda})| \leq C\alpha\|g_2\|^2 + C\alpha\|w_q g_1\|_{L^\infty}^2,$$
$$|(L g_\lambda, \Psi_{a_\lambda})| \leq \eta\|a_{\lambda,2}\|^2 + C_\eta\|g_2\|^2 + C_\eta\|w_q g_1\|_{L^\infty}^2.$$

And from Lemma 2.2 and the a priori assumption (7.1) we have

$$|(\mathcal{H}, \Psi_{a_\lambda})| \le C(\alpha+\tilde{\varepsilon}+\eta)\|a_\lambda\|^2 + C_\eta(\alpha+\tilde{\varepsilon})\{\|w_q g_2\|_{L^\infty}^2 + \|w_q g_1\|_{L^\infty}^2\},$$

where we have used

$$\begin{aligned}
&|(e^{\lambda_0 t}\Gamma(f,f), \Psi_{a_\lambda})| \\
&\le \eta\|a_\lambda\|^2 + C_\eta \int_{-1}^{1}\int_{\mathbb{R}^3} [e^{\lambda_0 t}\Gamma(f,f)|v_y|(|v|^2-10)^2\sqrt{\mu}]^2\,dv\,dy \\
&\le \eta\|a_\lambda\|^2 + C_\eta\|w_q Q(\mu^{1/2}g_\lambda, \mu^{1/2}g_\lambda)\|_{L^\infty}^2 \int_{-1}^{1}\left(\int_{\mathbb{R}^3} w_{-q}[|v_y|(|v|^2-10)^2]^2\,dv\right)^2 dy \\
&\le \eta\|a_\lambda\|^2 + C_\eta\|w_q\mu^{1/2}g_\lambda\|_{L^\infty}^4 \\
&\le \eta\|a_\lambda\|^2 + C_\eta\tilde{\varepsilon}^2\{\|w_q g_2\|_{L^\infty}^2 + \|w_q g_1\|_{L^\infty}^2\}.
\end{aligned}$$

For the second term on the left hand side of (7.22), from the inner product $\langle(7.31), \sqrt{\mu}\rangle$, we have

$$\partial_t a_\lambda + \partial_y b_\lambda^2 = 0$$

in the weak sense, which yields

$$|(g_\lambda, \partial_t\Psi_{a_\lambda})| \le C\|b_\lambda^2\|^2 + C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2.$$

In particular, the third term on the left hand side of (7.22) gives the following main contribution:

$$\begin{aligned}
-(v_y g_\lambda, \partial_y\Psi_{a_\lambda}) &= -(v_y\mathbf{P}_0 g_\lambda, \partial_y\Psi_{\lambda,2}) - (v_y\mathbf{P}_1 g_\lambda, \partial_y\Psi_\lambda) \\
&\ge 5\|a_\lambda\|^2 - \eta\|a_\lambda\|^2 - C_\eta\|\mathbf{P}_1 g_\lambda\|^2.
\end{aligned}$$

The boundary term $\langle v_y, (g_\lambda\Psi_{a_\lambda})(1)\rangle - \langle v_y, (g_\lambda\Psi_{a_\lambda})(-1)\rangle$ vanishes due to the boundary condition in (7.23). Putting all the above estimates for a_λ together, we have

$$\begin{aligned}
&\frac{d}{dt}(g_\lambda, \Psi_{a_\lambda}) + \kappa\|a_\lambda\|^2 \\
&\qquad \le C\|b_\lambda^y\|^2 + C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2 + C(\alpha+\tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2. \qquad (7.25)
\end{aligned}$$

Next, let

$$\Psi = \Psi_{b_\lambda^i} = \begin{cases} v_y v_x \frac{d}{dy}\phi_{b_{\lambda,1}}(y)\sqrt{\mu}, & i=1, \\ v_y v_z \frac{d}{dy}\phi_{b_{\lambda,3}}(y)\sqrt{\mu}, & i=3, \\ v_y^2(|v|^2-5)\frac{d}{dy}\phi_{b_{\lambda,2}}(y)\sqrt{\mu}, & i=2, \end{cases}$$

where

$$-\phi''_{b_\lambda^i} = b_\lambda^i, \quad \phi_{b_\lambda^i}(\pm 1) = 0,$$

and

$$\Psi = \Psi_{c_\lambda} = v_y(|v|^2-5)\frac{d}{dy}\phi_{c_\lambda}(y)\sqrt{\mu},$$

where

$$-\phi_{c_\lambda}'' = c_\lambda, \quad \phi_{c_\lambda}(\pm 1) = 0.$$

Similar to (5.79) and (5.81), one can show that

$$\frac{d}{dt}(g_\lambda, \Psi_{\mathbf{b}_\lambda}) + \kappa \|\mathbf{b}_\lambda\|^2 \le C\|c_\lambda\|^2 + C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2 + C(\alpha + \tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2 + C|\{I - P_\gamma\}g_2|_{2,+}^2, \tag{7.26}$$

and

$$\frac{d}{dt}(g_\lambda, \Psi_{c_\lambda}) + \kappa \|c_\lambda\|^2 \le C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2 + C(\alpha + \tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2 + C|\{I - P_\gamma\}g_2|_{2,+}^2, \tag{7.27}$$

respectively. Note that the decomposition $\sqrt{\mu}\, g_\lambda = g_1 + \sqrt{\mu}\, g_2$ has also been used to handle the terms involving $\langle v_y, (\{I - P_\gamma\}g_\lambda \Psi)(1)\rangle - \langle v_y, (\{I - P_\gamma\}g_\lambda \Psi)(-1)\rangle$.

Consequently, by choosing $0 < \kappa_1 \ll \kappa_2 \ll 1$, from $\kappa_1 \times$ (7.25) + $\kappa_2 \times$ (7.26) + (7.27) we get

$$\frac{d}{dt}\{\kappa_1(g_\lambda, \Psi_{\mathbf{b}_\lambda}) + \kappa_2(g_\lambda, \Psi_{\mathbf{b}_\lambda}) + (g_\lambda, \Psi_{c_\lambda})\} + \kappa\|[a_\lambda, \mathbf{b}_\lambda, c_\lambda\|^2 \le C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2 + C(\alpha + \tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2 + C|\{I - P_\gamma\}g_2|_{2,+}^2. \tag{7.28}$$

Finally, (7.21) follows from (7.28) by defining

$$\mathcal{E}_{\text{int}}(t) = \kappa_1(g_\lambda, \Psi_{\mathbf{b}_\lambda}) + \kappa_2(g_\lambda, \Psi_{\mathbf{b}_\lambda}) + (g_\lambda, \Psi_{c_\lambda}). \tag{7.29}$$

Note that (7.20) is satisfied. Thus the proof of Lemma 7.2 is complete. ∎

Now, with Lemmas 7.2 and 7.1, we are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. The global existence of solution to the problem (6.2) follows from the local existence established in Section 6 and the a priori estimates in the weighted L^∞ space by the continuity argument. Therefore, to prove Theorem 1.2, it remains to show the uniform estimate (1.23) under the a priori assumption (7.1). Indeed, by (7.19), we can rewrite (7.21) as

$$\frac{d}{dt}\mathcal{E}_{\text{int}}(t) + \lambda\|\mathbf{P}_0 g_2\|^2 \le C\|\mathbf{P}_1 g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2 + C(\alpha + \tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2 + C|\{I - P_\gamma\}g_2|_{2,+}^2, \tag{7.30}$$

where $[g_1, g_2]$ is defined in (7.17). For an L^2 estimate of $\mathbf{P}_1 g_2$, note that g_2 satisfies

$$\partial_t g_2 + v_y \partial_y g_2 - \alpha v_y \partial_{v_x} g_2 + (-\lambda_0 + L)g_2 = (1 - \chi_M)\mu^{-1/2}\mathcal{K} g_1, \tag{7.31}$$

and

$$g_2(0, y, v) = 0, \quad g_2(\pm 1, v)|_{v_y \lessgtr 0} = \sqrt{2\pi\mu}\int_{v_y \gtrless 0} \sqrt{\mu}\, g_2(\pm 1, v)|v_y|\, dv.$$

By taking the inner product of (7.31) and g_2 with respect to y and v over $(-1,1)\times\mathbb{R}^3$, one has

$$\frac{d}{dt}\|g_2\|^2+|\{I-P_\gamma\}g_2|_{2,+}^2+\delta_0\|\mathbf{P}_1 g_2\|^2 \le C_\eta\|w_q g_1\|_{L^\infty}^2+C(\eta+\lambda_0)\|w_q g_2\|^2. \tag{7.32}$$

Let $\tilde{C}>0$ be a constant sufficiently large. By taking the sum of $\tilde{C}\times$(7.32) and (7.30) we have

$$\frac{d}{dt}\{\tilde{C}\|g_2(t)\|^2+\mathcal{E}_{\text{int}}(t)\}+\lambda\|g_2\|^2+\lambda|\{I-P_\gamma\}g_2|_{2,+}^2 \le C\|w_q g_1\|_{L^\infty}^2+C(\alpha+\tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2. \tag{7.33}$$

Denote

$$\mathcal{E}(t)=\tilde{C}\|g_2(t)\|^2+\mathcal{E}_{\text{int}}(t).$$

For $\tilde{C}>0$ large enough, from (7.20) there exist constants $C_1, C_2>0$ such that for any $t\ge 0$,

$$2\tilde{C}\|g_2(t)\|^2+C_2\|w_q g_1(t)\|_{L^\infty}^2\ge\mathcal{E}(t)\ge\tfrac{1}{2}\tilde{C}\|g_2(t)\|^2-C_1\|w_q g_1(t)\|_{L^\infty}^2. \tag{7.34}$$

Then, from (7.33) and (7.34), it follows that

$$\frac{d}{dt}\mathcal{E}(t)+\frac{2\lambda}{\tilde{C}}\mathcal{E}(t)+\lambda|\{I-P_\gamma\}g_2|_{2,+}^2\le C\|w_q g_1\|_{L^\infty}^2+C(\alpha+\tilde{\varepsilon})\|w_q g_2\|_{L^\infty}^2.$$

Hence

$$\begin{aligned}
\mathcal{E}(t)&+\lambda\int_0^t e^{-\frac{2\lambda}{\tilde{C}}(t-s)}|\{I-P_\gamma\}g_2(s)|_{2,+}^2\,ds\\
&\le \mathcal{E}(0)e^{-\frac{2\lambda}{\tilde{C}}t}+C\int_0^t e^{-\frac{2\lambda}{\tilde{C}}(t-s)}\|w_q g_1(s)\|_{L^\infty}^2\,ds\\
&\quad+C(\alpha+\tilde{\varepsilon})\int_0^t e^{-\frac{2\lambda}{\tilde{C}}(t-s)}\|w_q g_2(s)\|_{L^\infty}^2\,ds\\
&\le \mathcal{E}(0)+C\sup_{0\le s\le t}\|w_q g_1(s)\|_{L^\infty}^2+C(\alpha+\tilde{\varepsilon})\sup_{0\le s\le t}\|w_q g_2(s)\|_{L^\infty}^2
\end{aligned} \tag{7.35}$$

for any $t\ge 0$. Therefore, by using (7.34) and (7.17), it follows from (7.35) that

$$\sup_{0\le s\le t} e^{\lambda_0 s}\|f_2(s)\|\le C\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_1(s)\|_{L^\infty} + C(\alpha+\tilde{\varepsilon})\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_2(s)\|_{L^\infty}.$$

By putting the above estimate back to (7.3) and using the smallness of α and $\tilde{\varepsilon}$, one has

$$\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_2(s)\|_{L^\infty}\le C\|w_q f_0\|_{L^\infty}+C\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q f_1(s)\|_{L^\infty}. \tag{7.36}$$

Moreover, by plugging (7.36) into (7.2) and using the smallness of α and $\tilde{\varepsilon}$ as well as (7.36), one obtains

$$\sup_{0\le s\le t} e^{\lambda_0 s}\|w_q[f_1,f_2](s)\|_{L^\infty} \le C\|w_q f_0\|_{L^\infty},$$

which gives (1.23). Since $\|w_q f_0\|_{L^\infty}$ is sufficiently small, the a priori assumption (7.1) is closed.

Finally, the nonnegativity of the global solution constructed above can be proved similarly to [15], so that the proof of Theorem 1.2 is complete. ∎

8. Appendix

Recall the backward time cycle starting at $(t_0, y_0, v_0) = (t, y, v)$ in (5.18), the boundary probability measure $d\sigma_l$ on $\mathcal{V}_l$ in (5.19) and the product measure $d\Sigma_l(s)$ over $\prod_{j=1}^{k-1}\mathcal{V}_j$ in (5.24). The following lemma gives an estimate on the measure of the phase space $\prod_{j=1}^{k-1}\mathcal{V}_j$ when there are k bounces.

Lemma 8.1. *For any $\bar{\varepsilon}>0$ and any $T_0>0$, there exists an integer $k_0=k_0(\bar{\varepsilon},T_0)$ such that for any integer $k\ge k_0$ and any $1\gg\eta_0\ge 0$ and for all $(t,y,v)\in[0,T_0]\times[-1,1]\times\mathbb{R}^3$,*

$$\int_{\prod_{l=1}^{k-1}\mathcal{V}_l} \mathbf{1}_{\{t_k(t,y,v,v_1,\dots,v_{k-1})>0\}} \prod_{l=1}^{k-1} e^{\frac{\eta_0}{2}|v_{lx}|^2}\, d\sigma_l \le \bar{\varepsilon}. \tag{8.1}$$

In particular, for $T_0>0$ large enough, there exist constants C_1 and $C_2>0$ independent of T_0 such that for $k=C_1T_0^{5/4}$ with a suitable choice of C_1 such that k is an integer and for all $(t,y,v)\in[0,\infty)\times[-1,1]\times\mathbb{R}^3$, one has

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k(t,y,v,v_1,\dots,v_{k-1})>0\}} \prod_{l=1}^{k-1} e^{\frac{\eta_0}{2}|v_{lx}|^2}\, d\sigma_l \le \left\{\frac{1}{2}\right\}^{C_2T_0^{5/4}}. \tag{8.2}$$

Furthermore, for any $q>0$ in the weight function $w_q(v)$, there exist constants $C_3, C_4>0$ independent of k and T_0 such that

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}} \int_0^{t_l} d\Sigma_l(s)\, ds \le C_3, \tag{8.3}$$

$$\int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1}>0\}} \int_{t_{l+1}}^{t_l} d\Sigma_l(s)\, ds \le C_4. \tag{8.4}$$

Proof. We only give the proof for (8.1) and (8.3), since (8.2) and (8.4) can be proved similarly by using Lemma 23 in [25, p. 781]. By definition (5.19), we rewrite

$$\begin{aligned} e^{\eta_0|v_{lx}|^2}\, d\sigma_l &= \frac{1}{2\pi} e^{\frac{\eta_0}{2}|v_{lx}|^2} e^{-\frac{|v_l|^2}{2}} |v_{ly}|\, dv_l \\ &= \frac{1}{2\pi} e^{\frac{1}{2}(\eta_0-1)|v_{lx}|^2}\, dv_{lx}\, e^{-\frac{|v_{ly}|^2+|v_{lz}|^2}{2}} |v_{ly}|\, dv_{ly}\, dv_{lz}. \end{aligned}$$

On the other hand, if we choose η_0 so that $0 \le \eta_0 \le \frac{1}{k-1}$, then we have

$$\prod_{l=1}^{k-1} (\sqrt{2\pi})^{-1} \int_{\mathbb{R}} e^{\frac{1}{2}(\eta_0-1)|v_{lx}|^2}\, dv_{lx} = (1-\eta_0)^{-(k-1)/2} \le e^{1/2}.$$

Thus for $0 \le \eta_0 \le \frac{1}{k-1}$, one gets

$$\begin{aligned}\int_{\prod_{l=1}^{k-1} \mathcal{V}_l} & \mathbf{1}_{\{t_k(t,y,v,v_1,\ldots,v_{k-1})>0\}} \prod_{l=1}^{k-1} e^{\frac{\eta_0}{2}|v_{lx}|^2}\, d\sigma_l \\ & \le e^{1/2} \int_{\prod_{l=1}^{k-1} \bar{\mathcal{V}}_l} \mathbf{1}_{\{t_k(t,y,v,v_1,\ldots,v_{k-1})>0\}} d\bar{\sigma}_l, \end{aligned} \tag{8.5}$$

where

$$\bar{\mathcal{V}}_j = \{(v_{jy}, v_{jz}) \in \mathbb{R}^2 \mid v_j \cdot n(y_j) > 0\}, \quad \bar{\sigma}_l = e^{-\frac{|v_{ly}|^2+|v_{lz}|^2}{2}} |v_{ly}|\, dv_{ly}\, dv_{lz}.$$

Next performing similar calculations to the proof of (194) in [25, Lemma 23, p. 781], we can obtain

$$\int_{\prod_{l=1}^{k-1} \bar{\mathcal{V}}_l} \mathbf{1}_{\{t_k(t,y,v,v_1,\ldots,v_{k-1})>0\}}\, d\bar{\sigma}_l \le e^{-1/2}\bar{\varepsilon},$$

which together with (8.5) implies that (8.1) is valid.

We now turn to the proof of (8.3). Recall the definition (5.24). We have

$$\begin{aligned} & \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \le 0 < t_l\}} \int_0^{t_l} d\Sigma_l(s)\, ds \\ & = \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \le 0 < t_l\}} \int_0^{t_l} \prod_{j=l+1}^{k-1} d\sigma_j\, e^{-\int_s^{t_l} \mathcal{A}^\epsilon(\tau, V_{\mathbf{cl}}^l(\tau))\, d\tau} \tilde{w}(v_l)\, d\sigma_l \\ & \qquad \times \prod_{j=1}^{l-1} \frac{\tilde{w}(v_j)}{\tilde{w}(V_{\mathbf{cl}}^j(t_{j+1}))} e^{-\int_{t_{j+1}}^{t_j} \mathcal{A}^\epsilon(\tau, V_{\mathbf{cl}}^l(\tau))\, d\tau}\, d\sigma_j\, ds, \end{aligned}$$

which, using direct calculations, can be bounded by

$$\begin{aligned} & \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{t_k \le 0} \int_0^{t_l} \prod_{j=l+1}^{k-1} d\sigma_j e^{-\frac{\nu_0}{2}(t_l - s)} \tilde{w}(v_l)\, d\sigma_l \\ & \qquad \times \prod_{j=1}^{l-1} \frac{\tilde{w}(v_j)}{\tilde{w}(V_{\mathbf{cl}}^j(t_{j+1}))} e^{-\frac{\nu_0}{2}(t_j - t_{j+1})}\, d\sigma_j ds \\ & \qquad \le C \int_{\prod_{j=1}^{l} \mathcal{V}_j} \int_0^{t_l} e^{-\frac{\nu_0}{2}(t_1 - s)} \tilde{w}(v_l)\, d\sigma_l \prod_{j=1}^{l-1} d\sigma_j\, ds \le C. \end{aligned}$$

Here we have used, for $0 \le \eta_0 \ll 1$,

$$\int_{\mathcal{V}_l} \tilde{w}_2(v_l)\, d\sigma_l < \infty, \qquad \int_{\mathcal{V}_j} e^{\eta_0 |v_{jx}|^2/4}\, d\sigma_j < \infty,$$

and

$$\begin{aligned}\frac{\tilde{w}(v_j)}{\tilde{w}(V_{\mathbf{cl}}^j(t_{j+1}))} &= \frac{w_q(V_{\mathbf{cl}}^j(t_{j+1}))\mu^{1/2}(V_{\mathbf{cl}}^j(t_{j+1}))}{w_q(v_j)\mu^{1/2}(v_j)} \\ &= \frac{(1+|V_{\mathbf{cl}}^j(t_{j+1})|^2)^q}{(1+|v_j|^2)^q} \cdot e^{\frac{|v_j|^2-|V_{\mathbf{cl}}^j(t_{j+1})|^2}{4}} \\ &\le 2^q(1+|V_{\mathbf{cl}}^j(t_{j+1})-v_j|^2)^q e^{\frac{C_{\eta_0}\alpha^2(t_b(v_j)v_{jy})^2+\eta_0|v_{jx}|^2}{4}} \\ &\le 2^q(1+4\alpha^2)^q e^{C_{\eta_0}\alpha^2} e^{\frac{\eta_0|v_{jx}|^2}{4}},\end{aligned}$$

by Peetre's inequality and the fact that $\alpha|t_b(v_j)v_{jy}| \le 2\alpha$. Thus the proof of the lemma is complete. ∎

Remark 8.1. The time interval $[0, T_0]$ in Lemma 8.1 can be replaced by any interval $[s, t]$ of length $t - s = T_0$. In addition, since

$$\int_{\mathcal{V}_l} \tilde{w}_1(v_l)\, d\sigma_l < \infty, \quad q > 3/2,$$

and $\mathcal{A}^\epsilon$, $\mathcal{A}$ and $\mathcal{A}_1$ have the same lower bound $\nu_0/2$, the statement in Lemma 8.1 is also valid if $\Sigma_l(s)$ is replaced by either $\bar{\Sigma}_l(s)$ or $\tilde{\Sigma}_l^{(i)}(s)$ $(i = 1, 2)$.

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