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Quasi-plurisubharmonic envelopes 1: uniform estimates on Kähler manifolds

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Abstract. We develop a new approach to a priori L^∞-estimates for degenerate complex Monge–Ampère equations on complex manifolds. It only relies on compactness and envelopes properties of quasi-plurisubharmonic functions. Our method allows one to obtain new and efficient proofs of several fundamental results in Kähler geometry.

In a sequel we shall explain how this approach also applies to the hermitian setting, producing new relative a priori bounds, as well as existence results.

Keywords: Monge–Ampère equation, a priori estimates, envelopes.

Introduction

Complex Monge–Ampère equations have been one of the most powerful tools in Kähler geometry since Yau's solution to the Calabi conjecture [50]. A notable application is the construction of Kähler–Einstein metrics: given a compact Kähler manifold (X,ω) of complex dimension n and an appropriate volume form μ normalized by $\mu(X) = \int_X \omega^n$, one seeks for a solution $\varphi : X \to \mathbb{R}$ to

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi}\mu,$$

where $d = \partial + \overline{\partial}$, $d^c = i(\partial - \overline{\partial})$ and $\lambda \in \mathbb{R}$ is a constant whose sign depends on that of $c_1(X)$. The metric $\omega_\varphi := \omega + dd^c\varphi$ is then Kähler–Einstein as $\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi$.

When $\lambda \le 0$, Yau [50] (see also [2] for $\lambda < 0$) showed the existence of a unique (normalized) solution φ by establishing a priori estimates by a continuity method, the most delicate one being the uniform a priori estimate that he established by using Moser's iteration process.

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In recent years degenerate complex Monge–Ampère equations have been intensively studied by many authors. In relation to the Minimal Model Program, they led to the construction of singular Kähler–Einstein metrics (see [6, 32, 39] and the references therein). The main analytical input came here from pluripotential theory which allowed Kołodziej [41] to establish uniform a priori estimates when $\mu = f dV_X$ has density in L^p for some $p > 1$.

Using different methods (Gromov–Hausdorff techniques), the case $\lambda > 0$ (Yau–Tian–Donaldson conjecture) has been settled by Chen–Donaldson–Sun [16–18, 29]. Again establishing a uniform a priori estimate in this context turned out to be the most delicate issue, a key step being obtained by Donaldson–Sun [30] through a refinement of Hörmander's L^2-techniques. An alternative pluripotential variational approach has been developed by Berman–Boucksom–Jonsson [8], based on finite energy classes studied in [37] and variational tools obtained in [7]. This approach has been pushed one step further by Li–Tian–Wang who have settled the case of singular Fano varieties [42].

The main goal of this article is to provide yet another approach for establishing such uniform a priori estimates. While the pluripotential approach consists in measuring the Monge–Ampère capacity of sublevel sets $\{\varphi < -t\}$, we directly measure their volume, avoiding delicate integration by parts. Our approach thus extends with minor modifications to the hermitian (non-Kähler) setting, providing several new results discussed in companion papers [34,35]: the hermitian setting introduces several technicalities and new challenges that might affect the clarity of exposition.

In the whole article we let X denote a compact Kähler manifold of complex dimension n. We fix a closed semi-positive $(1, 1)$-form ω which is *big*, i.e.

$$V := \int_X \omega^n > 0.$$

We let $\mathrm{PSH}(X, \omega)$ denote the set of ω-plurisubharmonic functions; these are functions $u : X \to \mathbb{R} \cup \{-\infty\}$ which are locally given as the sum of a smooth and a plurisubharmonic function, and such that $\omega + dd^c u \geq 0$ is a positive current.

Our first main result is a brand new proof of the following a priori estimate.

Theorem A. *Let ω be semi-positive and big. Let μ be a probability measure such that* $\mathrm{PSH}(X, \omega) \subset L^m(\mu)$ *for some $m > n$. Any bounded solution $\varphi \in$* $\mathrm{PSH}(X, \omega)$ *to the equation $V^{-1}(\omega + dd^c\varphi)^n = \mu$ satisfies a uniform a priori bound*

$$\mathrm{Osc}_X(\varphi) \leq T_\mu$$

for some uniform constant $T_\mu = T(A_m(\mu))$ which depends on an upper bound on

$$A_m(\mu) := \sup \left\{ \int_X (-\psi)^m \, d\mu : \psi \in \mathrm{PSH}(X, \omega) \ \textit{with} \ \sup_X \psi = 0 \right\}.$$

The Hölder inequality shows that this result covers the case when $\mu = f dV_X$ is absolutely continuous with respect to Lebesgue measure, with density f belonging to L^p, $p > 1$, or to an appropriate Orlicz class, as we explain in Section 2.2.

A crucial particular case of this estimate is due to Kołodziej [41]. Other important special cases have been previously obtained in [22, 31, 32]. Our new method covers all these settings at once, and it also enables us to recover the main estimates of [10] (big cohomology classes) and [24] (collapsing families), as we explain in Sections 3.1 and 3.2. A slight refinement of our technique allows us to establish an important stability estimate (see Theorem 2.6).

There are several geometric situations when one cannot expect the Monge–Ampère potential φ to be globally bounded. We next consider the equation

$$V^{-1}(\omega + dd^c\varphi)^n = f dV_X,$$

where the density $f \in L^1(X)$ does not belong to any good Orlicz class. Since the measure $\mu = f dV_X$ is non-pluripolar, there exists a unique finite energy solution φ (see [28, 39]). It is crucial to understand its locally bounded locus.

As ω is a semi-positive and big $(1,1)$-form, we can find an ω-psh function ρ with analytic singularities such that $\omega + dd^c\rho \geq \delta\omega_X$ is a Kähler current (see [23, Theorem 0.5]). For ψ quasi-psh and $c > 0$, we set

$$E_c(\psi) := \{x \in X : \nu(\psi, x) \geq c\},$$

where $\nu(\psi, x)$ denotes the Lelong number of ψ at x. A celebrated theorem of Siu ensures that for any $c > 0$, the set $E_c(\psi)$ is a closed analytic subset of X.

Our second main result provides the following a priori estimate, which extends a result of Di Nezza–Lu [27]:

Theorem B. *Assume $f = ge^{-\psi}$, where $0 \leq g \in L^p(dV_X)$, $p > 1$, and ψ is a quasi-psh function. Then there exists a unique $\varphi \in \mathcal{E}(X, \omega)$ such that*

- $\alpha(\psi + \rho) - \beta \leq \varphi \leq 0$ *with* $\sup_X \varphi = 0$;
- φ *is locally bounded in the open set* $\Omega := X \setminus (\{\rho = -\infty\} \cup E_{1/q}(\psi))$;
- $V^{-1}(\omega + dd^c\varphi)^n = f dV_X$ *in* Ω,

where $\alpha, \beta > 0$ depend on an upper bound for $\|g\|_{L^p}$ and $1/p + 1/q = 1$.

Again the proof we provide is direct, and can be extended to the hermitian setting (see [35]). We finally show in Section 4 how the same arguments can be applied to efficiently solve the Dirichlet problem in pseudo-convex domains.

Comparison with other works. Yau's proof of his famous a priori L^∞-estimate [50] goes through a Moser iteration process. Although Yau could deal with some singularities, his method does not apply when the right hand side is too degenerate (see however [11, 49] for further applications of Yau's method).

An important generalization of Yau's estimate has been provided by Kołodziej [41] using pluripotential techniques. These have been further generalized in [10, 22, 31, 32] in order to deal with less positive or collapsing families of cohomology classes on Kähler manifolds. As this approach relies on delicate integration by parts, it is difficult to extend it to the hermitian setting.

Błocki [9] has provided a different approach based on the Alexandrov–Bakelman–Pucci maximum principle and a local stability estimate due to Cheng–Yau (L^2-case) and Kołodziej (L^p-case). This has been pushed further by Székelehydi [47]. It requires the reference form ω to be strictly positive.

A PDE proof of the L^∞-estimate has been recently provided by Guo–Phong–Tong [40] using an auxiliary Monge–Ampère equation, inspired by the recent breakthrough results by Chen–Cheng on constant scalar curvature metrics [14, 15].

Our approach consists in showing that the sublevel set $\{\varphi < -t\}$ becomes the empty set in finite time by directly measuring its μ-size. We only use weak compactness of normalized ω-plurisubharmonic functions and basic properties of quasi-psh envelopes, allowing us to deal with semi-positive forms.

1. Quasi-plurisubharmonic envelopes

In the whole article we let X denote a compact Kähler manifold of complex dimension $n \geq 1$. We fix a smooth closed real $(1,1)$-form ω on X.

1.1. Monge–Ampère operators

1.1.1. Quasi-plurisubharmonic functions. A function is *quasi-plurisubharmonic* if it is locally given as the sum of a smooth and a psh function. Quasi-psh functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ satisfying $\omega_\varphi := \omega + dd^c\varphi \geq 0$ in the weak sense of currents are called *ω-plurisubharmonic* (ω-psh for short).

Definition 1.1. We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-plurisubharmonic functions which are not identically $-\infty$.

Constant functions are ω-psh if (and only if) ω is semi-positive. A $\mathcal{C}^2$-smooth function u has bounded Hessian, hence εu is ω-psh if $\varepsilon > 0$ is small enough and ω is positive. It is useful to consider as well the case when ω is not necessarily positive, in order to study *big* cohomology classes (see Section 3.1).

Definition 1.2. A semi-positive closed $(1,1)$-form ω is *big* if $V_\omega := \int_X \omega^n > 0$.

The set $\mathrm{PSH}(X, \omega)$ is a closed subset of $L^1(X)$ for the L^1-topology. Subsets of ω-psh functions enjoy strong compactness and integrability properties; we mention notably the following: for any fixed $r \geq 1$,

- $\mathrm{PSH}(X, \omega) \subset L^r(X)$; the induced L^r-topologies are equivalent;
- $\mathrm{PSH}_A(X, \omega) := \{u \in \mathrm{PSH}(X, \omega) : -A \leq \sup_X u \leq 0\}$ is compact in L^r.

We refer the reader to [20, 39] for further basic properties of ω-psh functions.

1.1.2. Monge–Ampère measure. The complex Monge–Ampère measure

$$(\omega + dd^c u)^n = \omega_u^n$$

is well-defined for any ω-psh function u which is *bounded*, as follows from Bedford-Taylor theory (see [4] for the local theory, and [39] for the compact Kähler context). It also makes sense in the ample locus of a big cohomology class [10], as we shall briefly discuss in Section 3.1.

The mixed Monge–Ampère measures $(\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$ are also well defined for any $0 \leq j \leq n$, and any bounded ω-psh functions u, v. We note for later use the following classical inequality.

Lemma 1.3. *Let φ, ψ be bounded ω-psh functions such that $\varphi \leq \psi$. Then*

$$1_{\{\psi=\varphi\}}(\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j} \leq 1_{\{\psi=\varphi\}}(\omega + dd^c\psi)^n$$

for all $1 \leq j \leq n$.

Proof. To simplify notations we just treat the case $j = n$. It follows from Bedford–Taylor theory [4] that for any bounded ω-psh functions φ, ψ,

$$1_{\{\psi\leq\varphi\}}\omega_\varphi^n + 1_{\{\psi>\varphi\}}\omega_\psi^n \leq (\omega + dd^c \max(\varphi, \psi))^n.$$

When $\varphi \leq \psi$ we infer $1_{\{\psi=\varphi\}}\omega_\varphi^n \leq 1_{\{\psi=\varphi\}}\omega_\psi^n$. ∎

We shall also need the following (see [39, Proposition 10.11]).

Proposition 1.4 (Domination principle). *If u, v are bounded ω-psh functions such that $u \geq v$ a.e. with respect to ω_u^n, then $u \geq v$.*

1.2. Envelopes

Upper envelopes of (pluri)subharmonic functions are classical objects in potential theory. They were considered by Bedford and Taylor to solve the Dirichlet problem for the complex Monge–Ampère equation in strictly pseudo-convex domains [3]. We consider here envelopes of ω-psh functions.

1.2.1. Basic properties.

Definition 1.5. Given a Lebesgue measurable function $h : X \to \mathbb{R}$, we define the *ω-psh envelope* of h by

$$P_\omega(h) := (\sup\{u \in \mathrm{PSH}(X, \omega) : u \leq h \text{ in } X\})^*,$$

where the star means that we take the upper semi-continuous regularization.

The following is a combination of [36, Propositions 2.2 and 2.5, Lemma 2.3].

Proposition 1.6. *If h is bounded from below and quasi-continuous, then*

- *$P_\omega(h)$ is a bounded ω-plurisubharmonic function;*
- *$P_\omega(h) \leq h$ in $X \setminus P$, where P is pluripolar;*
- *$(\omega + dd^c P_\omega(h))^n$ is concentrated on the contact set $\{P_\omega(h) = h\}$.*

Recall that a function h is *quasi-continuous* if for any $\varepsilon > 0$, there exists an open set G of capacity smaller than ε such that h is continuous in $X \setminus G$. Quasi-psh functions are quasi-continuous (see [4]), as also are their differences: we shall use this fact during the proof of Theorem 3.3.

When h is $\mathcal{C}^{1,1}$-smooth, so is $P_\omega(h)$ on the ample locus of $[\omega]$, as shown in [19, Theorem 1.3], and further

$$(\omega + dd^c P_\omega(h))^n = 1_{\{P_\omega(h)=h\}}(\omega + dd^c h)^n. \tag{1.1}$$

1.2.2. A key lemma. The following is a key technical tool for our new approach:

Lemma 1.7. *Fix a concave increasing function* $\chi : \mathbb{R}^- \to \mathbb{R}^-$ *such that* $\chi'(0) \geq 1$. *Let* φ, ϕ *be bounded* ω*-psh functions with* $\varphi \leq \phi$. *If* $\psi = \phi + \chi \circ (\varphi - \phi)$ *then*

$$(\omega + dd^c P_\omega(\psi))^n \leq 1_{\{P_\omega(\psi)=\psi\}}(\chi' \circ (\varphi - \phi))^n(\omega + dd^c\varphi)^n.$$

Proof. Using $\chi'' \leq 0$ and $\chi' \geq 1$, we observe that

$$\begin{aligned}\omega + dd^c\psi &= \omega_\phi + \chi' \circ (\varphi - \phi)(\omega_\varphi - \omega_\phi) + \chi'' \circ (\varphi - \phi)d(\varphi - \phi) \wedge d^c(\varphi - \phi)\\ &\leq \chi' \circ (\varphi - \phi)\omega_\varphi + [1 - \chi' \circ (\varphi - \phi)]\omega_\phi \leq \chi' \circ (\varphi - \phi)\omega_\varphi.\end{aligned}$$

When φ, ϕ and χ are $\mathcal{C}^{1,1}$-smooth, we can invoke (1.1) to conclude that

$$(\omega + dd^c P_\omega(\psi))^n = 1_{\{P_\omega(\psi)=\psi\}}\omega_\psi^n \leq 1_{\{P_\omega(\psi)=\psi\}}(\chi' \circ (\varphi - \phi))^n\omega_\varphi^n.$$

The last inequality follows from $\omega + dd^c\psi \leq \chi' \circ (\varphi - \phi)\omega_\varphi$ and the fact that ψ is ω-psh on $\{P_\omega(\psi) = \psi\}$, where these inequalities can be interpreted pointwise.

When these functions are less regular, we take a different route. We set $\tau = \chi^{-1} : \mathbb{R}^- \to \mathbb{R}^-$. This is a convex increasing function such that $\tau' = (\chi' \circ \tau)^{-1} \leq 1$. Set $\rho = P_\omega(\psi) - \phi$. The function $v = \phi + \tau \circ (P_\omega(\psi) - \phi)$ is ω-psh with

$$\begin{aligned}\omega + dd^c v &= \omega_\phi + \tau'' \circ \rho\, d\rho \wedge d^c\rho + \tau' \circ \rho\, dd^c(P_\omega(\psi) - \phi)\\ &\geq [1 - \tau' \circ \rho]\omega_\phi + \tau' \circ \rho\,(\omega + dd^c P_\omega(\psi))\\ &\geq \tau' \circ \rho\,(\omega + dd^c P_\omega(\psi)).\end{aligned}$$

Thus $\omega^n_{P_\omega(\psi)} \leq 1_{\{P_\omega(\psi)=\psi\}}(\tau' \circ (P_\omega(\psi) - \phi))^{-n}\omega_v^n$. On $\{P_\omega(\psi) = \psi\}$ we get

$$\tau' \circ (P_\omega(\psi) - \phi) = \tau' \circ (\psi - \phi) = [\chi' \circ (\varphi - \phi)]^{-1}.$$

Now $v \leq \phi + \tau \circ (\psi - \phi) = \varphi$ on X, with equality on the contact set $\{P_\omega(\psi) = \psi\}$. It follows therefore from Lemma 1.3 that $\omega_v^n \leq \omega_\varphi^n$ on $\{P_\omega(\psi) = \psi\}$. ∎

2. Global L^∞-bounds

In this section we prove Theorem A, as well as a stability estimate.

2.1. *Measures which integrate quasi-plurisubharmonic functions*

Theorem 2.1. *Let ω be semi-positive and big. Let μ be a probability measure such that* $\mathrm{PSH}(X,\omega) \subset L^m(\mu)$ *for some $m > n$. Any solution $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ to $V^{-1}(\omega + dd^c\varphi)^n = \mu$ satisfies*

$$\mathrm{Osc}_X(\varphi) \le T_\mu$$

for some uniform constant $T_\mu = T(A_m(\mu))$ which depends on an upper bound on

$$A_m(\mu) := \sup\left\{ \left(\int_X (-\psi)^m \, d\mu \right)^{1/m} : \psi \in \mathrm{PSH}(X,\omega) \text{ with } \sup_X \psi = 0 \right\}.$$

Let us stress that this result is not new: it can be derived from the celebrated a priori estimate of Kołodziej [41], together with its extensions [22, 31, 32]. We provide here an elementary proof that does not use the theory of Monge–Ampère capacities, and merely relies on the compactness properties of sup-normalized ω-psh functions and Lemma 1.7.

Proof of Theorem 2.1. Shifting by an additive constant, we normalize φ by $\sup_X \varphi = 0$. Set

$$T_{\max} := \sup\{t > 0 : \mu(\varphi < -t) > 0\}.$$

Our goal is to establish a precise bound on $T_{\max}$. By definition, $-T_{\max} \le \varphi$ almost everywhere with respect to μ, hence everywhere by the domination principle (Proposition 1.4), providing the desired a priori bound $\mathrm{Osc}_X(\varphi) \le T_{\max}$.

We let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ denote a *concave* increasing function such that $\chi(0) = 0$ and $\chi'(0) = 1$. We set $\psi = \chi \circ \varphi$, $u = P_\omega(\psi)$ and observe that

$$\omega + dd^c\psi = \chi' \circ \varphi \, \omega_\varphi + [1 - \chi' \circ \varphi]\, \omega + \chi'' \circ \varphi \, d\varphi \wedge d^c\varphi \le \chi' \circ \varphi \, \omega_\varphi.$$

It follows from Lemma 1.7 that

$$\mathrm{MA}(u) := \frac{1}{V}(\omega + dd^c u)^n \le 1_{\{u=\psi\}} (\chi' \circ \varphi)^n \mu.$$

Controlling the norms $\|u\|_{L^m}$. We fix $\varepsilon > 0$ so that $n < n + 3\varepsilon = m$. The concavity of χ and the normalization $\chi(0) = 0$ yield $|\chi(t)| \le |t|\chi'(t)$. Since $u = \chi \circ \varphi$ on the contact set $\{P_\omega(\psi) = \psi\}$, the Hölder inequality yields

$$\begin{aligned}
\int_X (-u)^\varepsilon \, \mathrm{MA}(u) &\le \int_X (-\chi \circ \varphi)^\varepsilon (\chi' \circ \varphi)^n \, d\mu \le \int_X (-\varphi)^\varepsilon (\chi' \circ \varphi)^{n+\varepsilon} \, d\mu \\
&\le \left(\int_X (-\varphi)^{n+2\varepsilon} \, d\mu \right)^{\frac{\varepsilon}{n+2\varepsilon}} \left(\int_X (\chi' \circ \varphi)^{n+2\varepsilon} \, d\mu \right)^{\frac{n+\varepsilon}{n+2\varepsilon}} \\
&\le A_m(\mu)^\varepsilon \left(\int_X (\chi' \circ \varphi)^{n+2\varepsilon} \, d\mu \right)^{\frac{n+\varepsilon}{n+2\varepsilon}}
\end{aligned}$$

by using the fact that φ belongs to the set of ω-psh functions v normalized by $\sup_X v = 0$, which is compact in $L^{n+2\varepsilon}(\mu)$, and observing that $A_{n+2\varepsilon}(\mu) \le A_m(\mu)$.

We are going to choose the weight χ in such a way that $\int_X (\chi' \circ \varphi)^{n+2\varepsilon} \, d\mu = B \leq 2$ is a finite constant under control. This provides a uniform lower bound on $\sup_X u$: indeed,

$$0 \leq \left(-\sup_X u\right)^{\varepsilon} = \left(-\sup_X u\right)^{\varepsilon} \int_X \mathrm{MA}(u) \leq \int_X (-u)^{\varepsilon} \, \mathrm{MA}(u) \leq 2A_m(\mu)^{\varepsilon}$$

yields $-2^{1/\varepsilon} A_m(\mu) \leq \sup_X u \leq 0$. We infer that u belongs to a compact set of ω-psh functions, hence its norm $\|u\|_{L^m(\mu)}$ is under control with

$$\|u\|_{L^m(\mu)} \leq A_m(\mu) + 2^{1/\varepsilon} A_m(\mu) \leq [1 + 2^{1/\varepsilon}] A_m(\mu).$$

Since $u \leq \chi \circ \varphi \leq 0$ we infer $\|\chi \circ \varphi\|_{L^m} \leq \|u\|_{L^m}$. The Chebyshev inequality thus yields

$$\mu\{\varphi < -t\} \leq \frac{\tilde{A}}{|\chi(-t)|^m}, \quad \text{where} \quad \tilde{A} = [1 + 2^{1/\varepsilon}] A_m(\mu). \tag{2.1}$$

Choice of χ. Lebesgue's formula ensures that if $g : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $g(0) = 1$, then

$$\int_X g \circ (-\varphi) \, d\mu = \mu(X) + \int_0^{T_{\max}} g'(t) \mu\{\varphi < -t\} \, dt.$$

Fix $0 < T_0 < T_{\max}$. Setting $g(t) = \chi'(-t)^{n+2\varepsilon}$ we define χ by imposing $\chi(0) = 0$, $\chi'(0) = 1$, and

$$g'(t) = \begin{cases} \dfrac{1}{(1+t)^2 \mu\{\varphi < -t\}} & \text{if } t \leq T_0, \\ \dfrac{1}{(1+t)^2} & \text{if } t > T_0. \end{cases}.$$

This choice guarantees that $\chi : \mathbb{R}^- \to \mathbb{R}^-$ is concave increasing with $\chi' \geq 1$, and

$$\int_X (\chi' \circ \varphi)^{n+2\varepsilon} \, d\mu \leq \mu(X) + \int_0^{+\infty} \frac{dt}{(1+t)^2} = 2.$$

Conclusion. We set $h(t) = -\chi(-t)$ and work with the positive counterpart of χ. Note that $h(0) = 0$ and $h'(t) = g(t)^{1/(n+2\varepsilon)}$ is positive increasing, hence h is convex. Observe also that $g(t) \geq g(0) = 1$, hence $h'(t) = g(t)^{1/(n+2\varepsilon)} \geq 1$ yields

$$h(1) = \int_0^1 h'(s) \, ds \geq 1. \tag{2.2}$$

Together with (2.1) our choice of χ yields, for all $t \in [0, T_0]$,

$$\frac{1}{(1+t)^2 g'(t)} = \mu\{\varphi < -t\} \leq \frac{\tilde{A}}{h(t)^m}.$$

For $t \in [0, T_0]$, this reads

$$h(t)^m \leq \tilde{A}(1+t)^2 g'(t) = (n + 2\varepsilon) \tilde{A} (1+t)^2 h''(t) h'(t)^{n+2\varepsilon-1}.$$

Multiplying by h' and integrating between 0 and t, we infer that for all $t \in [0, T_0]$,

$$\begin{aligned}\frac{h(t)^{m+1}}{m+1} &\leq (n+2\varepsilon)\tilde{A}\int_0^t (1+s)^2 h''(s)h'(s)^{n+2\varepsilon} \\ &\leq \frac{(n+2\varepsilon)\tilde{A}(t+1)^2}{n+2\varepsilon+1}(h'(t)^{n+2\varepsilon+1}-1) \\ &\leq \tilde{A}(1+t)^2 h'(t)^{n+2\varepsilon+1}.\end{aligned}$$

Recall that $m = n + 3\varepsilon$ so that $\alpha := m + 1 > \beta := n + 2\varepsilon + 1 > 2$. The previous inequality then reads

$$(1+t)^{-2/\beta} \leq C h'(t) h(t)^{-\alpha/\beta}$$

for some uniform constant C depending on $n, m, \tilde{A}$. Since $\alpha > \beta > 2$ and $h(1) \geq 1$, integrating the above inequality between 1 and T_0 we obtain $T_0 \leq C'$ for some uniform constant C' depending on C, α, β. Since T_0 was chosen arbitrarily in $(0, T_{\max})$, the result follows. ■

2.2. *Absolutely continuous measures*

Assume $\mu = f dV_X$ is absolutely continuous with respect to a volume form dV_X, with density $0 \leq f \in L^p(dV_X)$ for some $p > 1$. Since $\mathrm{PSH}(X, \omega) \subset L^r(dV_X)$ for any $1 \leq r < +\infty$, we obtain

$$\int_X |u|^m \, d\mu \leq \|f\|_{L^p(dV_X)} \cdot \left(\int_X |u|^{qm} \, dV_X\right)^{1/q}$$

for all $u \in \mathrm{PSH}(X, \omega)$, where $1/p + 1/q = 1$, so that $\mathrm{PSH}(X, \omega) \subset L^m(d\mu)$ for all $m \geq 1$. Thus Theorem 2.1 applies to this type of measures, providing a new proof of the celebrated a priori estimate of Kołodziej [41] (see also [32]).

As in [41] our technique also covers the case of more general densities as we briefly indicate. Let $w : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing weight. A measurable function f belongs to the *Orlicz class* $L^w(dV_X)$ if there exists $\alpha > 0$ such that

$$\int_X w(\alpha|f|) \, dV_X < +\infty.$$

The *Luxemburg norm* of f is defined as

$$\|f\|_w := \inf\left\{r > 0 : \int_X w(|f|/r) \, dV_X \leq 1\right\};$$

it turns $L^w(dV_X)$ into a Banach space.

If w^* denotes the conjugate convex weight of w (its Legendre transform), the Hölder–Young inequality ensures that for all measurable functions f, g,

$$\int_X |fg| \, dV_X \leq 2\|f\|_w \|g\|_{w^*}.$$

We refer the reader to [45] for more information on Orlicz classes.

Theorem 2.1 thus allows us to re-prove [41, Theorem 2.5.2].

Corollary 2.2. *Let $\mu = f dV_X$ be a probability measure. Let $w : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing weight that grows at infinity at least like $t(\log t)^m$ with $m > n$. If f belongs to the Orlicz class L^w then any solution $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ to $V^{-1}(\omega + dd^c\varphi)^n = \mu$ satisfies*

$$\mathrm{Osc}_X(\varphi) \le T_\mu$$

for some uniform constant $T_\mu \in \mathbb{R}^+$.

Proof. While this was not required for the case of L^p-densities, we need here to invoke Skoda's uniform integrability result (see [39, Theorem 8.11]): there exist $\alpha > 0$ and $C = C(\alpha, M) > 0$ such that

$$\sup \left\{ \int_X e^{2\alpha|u|}\, dV_X : u \in \mathrm{PSH}(X,\omega) \text{ and } -M \le \sup_X u \le 0 \right\} \le C.$$

The reader will check that, as $s \to +\infty$, the conjugate weight $w^*(s)$ grows like

$$w^*(s) \sim s^{1-1/m} \exp(s^{1/m}) \le \exp(2s^{1/m}).$$

It follows therefore from the Young inequality that any ω-psh function u satisfies

$$\alpha^m \int_X |u|^m\, d\mu \le \int_X w \circ f\, dV_X + \int_X \exp(2\alpha|u|)\, dV_X < +\infty.$$

Thus $\mathrm{PSH}(X,\omega) \subset L^m(\mu)$ and the conclusion follows from Theorem 2.1. ∎

One can slightly improve the assumption on the density as in [41, Theorem 2.5.2]; we leave the technical details to the interested reader.

Remark 2.3. The Chern–Levine–Nirenberg inequality implies that if $\mu = (\omega + dd^c\varphi)^n$ is the Monge–Ampère measure of a bounded ω-psh function, then $\mathrm{PSH}(X,\omega) \subset L^1(\mu)$. If $n = 1$ this condition is equivalent to μ having bounded potential (see [25, Lemma 3.2]). Note however that when $n \ge 2$,

- the condition $\mathrm{PSH}(X,\omega) \subset L^n(\mu)$, $\mu = (\omega + dd^c\varphi)^n$, is not sufficient to guarantee that the ω-psh function φ is bounded;
- one cannot improve the C-L-N inequality: there are examples of Monge–Ampère measures with bounded potential and $\mathrm{PSH}(X,\omega) \not\subset L^{1+\varepsilon}(\mu)$,

as the following examples show.

Example 2.4. We consider the function v from [25, Section 1.3.1]. It is a smooth function in $X \setminus \{p\}$ that is given locally near p by

$$v = \chi \circ L, \quad L = \log \|z\|, \quad \chi(t) = -\log(\log(-t)).$$

A direct computation shows that

$$\mu := (\omega_X + dd^c v)^n \simeq \frac{dV(z)}{\|z\|^{2n} \big|\log \|z\|\big|^{n+1} \big|\log\big|\log \|z\|\big|\big|^n}.$$

One can easily check that $\mathrm{PSH}(X, \omega_X) \in L^n(\mu)$ if $n \geq 2$ (it suffices to check this for $\log \|z\|$), but the local Monge–Ampère potential is not bounded near 0.

Example 2.5. Consider v as above with $n = 1$ and $\chi(t) = \frac{1}{\log(-t)}$. Then

$$\omega_X + dd^c v \simeq \frac{dV(z)}{\|z\|^2 \left|\log \|z\|\right|^2 \left|\log\left|\log \|z\|\right|\right|^2}$$

does not satisfy $\mathrm{PSH}(X, \omega_X) \subset L^{1+\varepsilon}(\omega_X + dd^c v)$, for any $\varepsilon > 0$.

2.3. Stability estimate

We now establish the following stability estimate, which can be seen as a refinement of [38, Proposition 5.2].

Theorem 2.6. *Let ω, μ be as in Theorem 2.1. Let $\varphi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ be such that $\sup_X \varphi = 0$ and $V^{-1}(\omega + dd^c\varphi)^n = \mu$. Then*

$$\sup_X (\phi - \varphi)_+ \leq T\left(\int_X (\phi - \varphi)_+ \, d\mu\right)^\tau$$

for any $\phi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$, where $\tau = \tau(n, m) > 0$ and

$$T = T(\mu, \|\phi\|_{L^\infty})$$

is a uniform constant which depends on an upper bound on $\|\phi\|_{L^\infty}$ and on

$$A_m(\mu) := \sup\left\{\left(\int_X (-\psi)^m \, d\mu\right)^{1/m} : \psi \in \mathrm{PSH}(X, \omega) \text{ with } \sup_X \psi = 0\right\}.$$

Proof. Replacing ϕ by $\max(\varphi, \phi)$, we can assume that $\varphi \leq \phi$. Define

$$T_{\max} := \sup\{t > 0 : \mu\{\varphi < \phi - t\} > 0\}.$$

It follows from Theorem 2.1 that $T_{\max}$ is uniformly controlled by μ and $\|\phi\|_{L^\infty}$.

We let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ denote a *concave* increasing function such that $\chi(0) = 0$ and $\chi'(0) = 1$. We set $\psi = \phi + \chi \circ (\varphi - \phi)$, $u = P(\psi)$ and observe that

$$\begin{aligned}\omega + dd^c\psi &= \omega_\phi + \chi' \circ (\varphi - \phi)(\omega_\varphi - \omega_\phi) + \chi'' \circ (\varphi - \phi) d(\varphi - \phi) \wedge d^c(\varphi - \phi)\\ &\leq \chi' \circ (\varphi - \phi)\omega_\varphi.\end{aligned}$$

It follows from Lemma 1.7 that

$$\mathrm{MA}(u) := \frac{1}{V}(\omega + dd^c u)^n \leq 1_{\{u=\psi\}}(\chi' \circ (\varphi - \phi))^n \mu.$$

We fix $0 < a < b < c < 2c < \varepsilon$ so small that

$$q := \frac{(\varepsilon - a)(n + b)}{b - a} < m = n + \varepsilon.$$

The concavity of χ and the normalization $\chi(0)=0$ yield $|\chi(t)| \leq |t|\chi'(t)$. Since $u = \phi + \chi \circ (\varphi - \phi)$ on the support of $(\omega + dd^c u)^n$ and $\mathrm{PSH}(X,\omega) \subset L^{n+2c}(\mu)$, the Hölder inequality yields

$$\begin{aligned}
0 &\leq \int_X (-u+\phi)^c \,\mathrm{MA}(u) \leq \int_X (-\chi \circ (\varphi-\phi))^c (\chi' \circ (\varphi - \phi))^n \, d\mu \\
&\leq \int_X (-\varphi + \phi)^c (\chi' \circ (\varphi-\phi))^{n+c} \, d\mu \\
&\leq \left(\int_X (-\varphi+\phi)^{n+2c} \, d\mu \right)^{\frac{c}{n+2c}} \left(\int_X (\chi' \circ (\varphi - \phi))^{n+2c} \, d\mu \right)^{\frac{n+c}{n+2c}} \\
&\leq A_m(\mu)^c \left(\int_X (\chi' \circ (\varphi-\phi))^{n+2c} \, d\mu \right)^{\frac{n+c}{n+2c}}.
\end{aligned}$$

Controlling the norms $\|u\|_{L^m}$. Below we will choose χ such that $\int_X (\chi' \circ (\varphi - \phi))^{n+2c} \, d\mu \leq B$ is under control. This provides a uniform lower bound on $\sup_X u$. Indeed, our normalizations yield $\chi(t) \leq t$, hence $u \leq \phi + \chi(\varphi - \phi) \leq \varphi \leq 0$, while

$$0 \leq \left(-\sup_X (u - \phi) \right)^c \leq \int_X (-u+\phi)^c \,\mathrm{MA}(u) \leq A_m(\mu)^c B^{\frac{n+c}{n+2c}}$$

yields a lower bound on $\sup_X (u - \phi)$. Now $u = u - \phi + \phi \geq u - \phi + \inf_X \phi$, so $\sup_X u \geq \sup_X (u-\phi) + \inf_X \phi \geq -A_m(\mu) B^{\frac{n+c}{c(n+2c)}} + \inf_X \phi$.

Thus u belongs to a compact set of ω-psh functions: its norm $\|u\|_{L^q(\mu)}$ is under control for any $q \leq m$. Since $u - \phi \leq \chi \circ (\varphi - \phi) \leq 0$, the Hölder inequality yields

$$\begin{aligned}
\int_X |\chi \circ (\varphi - \phi)|^m \, d\mu &\leq \int_X |\chi \circ (\varphi-\phi)|^{n+a} (\phi - u)^{\varepsilon - a} \, d\mu \\
&\leq \left(\int_X |\chi \circ (\varphi - \phi)|^{n+b} \, d\mu \right)^{\frac{n+a}{n+b}} \left(\int_X (\phi - u)^q \, d\mu \right)^{\frac{b-a}{n+b}} \\
&\leq C'_\mu \left(\int_X |(\phi - \varphi) \chi' \circ (\varphi - \phi)|^{n+b} \, d\mu \right)^{\frac{n+a}{n+b}} \\
&\leq C'_\mu \left(\int_X (\phi - \varphi)^{\frac{(n+c)(n+b)}{c-b}} \, d\mu \right)^{\frac{(c-b)(n+a)}{(n+c)(n+b)}} \left(\int_X |\chi' \circ (\varphi - \phi)|^{n+c} \, d\mu \right)^{\frac{n+a}{n+c}} \\
&\leq C_1 B^{\frac{n+a}{n+c}} \left(\int_X (\phi - \varphi) \, d\mu \right)^{\gamma} =: \tilde{A}, \qquad (2.3)
\end{aligned}$$

where

$$\gamma = \frac{(c-b)(n+a)}{(n+c)(n+b)},$$

and C_1 depends on C_μ, $\|\varphi\|_{L^\infty}$ and $\|\phi\|_{L^\infty}$.

It follows therefore from the Chebyshev inequality that

$$\mu\{\varphi < \phi - t\} \leq \frac{\tilde{A}}{|\chi(-t)|^m}. \qquad (2.4)$$

Choice of χ. Fix $T_0 \in (0, T_{\max})$. We set $g(t) = \chi'(-t)^{n+2c}$ and define χ by imposing $\chi(0) = 0$, $\chi'(0) = 1$, and

$$g'(t) = \begin{cases} \dfrac{1}{\mu\{\varphi < \phi - t\}} & \text{if } t \le T_0, \\ 1 & \text{if } t > T_0. \end{cases}$$

This choice guarantees that

$$\int_X (\chi' \circ (\varphi - \phi))^{n+2c} \, d\mu \le \mu(X) + \int_0^{T_{\max}} dt = 1 + T_{\max}.$$

It follows from Theorem 2.1 that $T_{\max} \le T_\mu$ is uniformly bounded from above, hence $B := 1 + T_\mu$ is under control. Together with (2.3) and (2.4) we thus obtain

$$\mu\{\varphi < \phi - t\} \le \frac{C_2 \delta}{|\chi(-t)|^m}, \tag{2.5}$$

where $\delta := (\int_X (\phi - \varphi) \, d\mu)^\gamma$.

Conclusion. Set $h(t) = -\chi(-t)$. It follows from (2.5) that for all $t \in [0, T_0]$,

$$\frac{1}{g'(t)} = \mu\{\varphi < \phi - t\} \le \frac{C_2 \delta}{h(t)^m},$$

hence

$$h(t)^m \le C_2 \delta g'(t) = (n + 2c) C_2 \delta h''(t) h'(t)^{n+2c-1}.$$

Multiplying by h' and integrating between 0 and t, we infer that for all $t \in [0, T_0]$,

$$\begin{aligned} h(t)^{m+1} &\le (m+1)(n+2c) C_2 \delta \int_0^t h''(s) h'(s)^{n+2c} \, ds \\ &\le C_3 \delta (h'(t)^{n+2c+1} - 1), \end{aligned}$$

which yields

$$1 \le \frac{C_3 \delta h'(t)^{n+2c+1}}{h(t)^{m+1} + C_3 \delta}. \tag{2.6}$$

Recall that we have set $m = n + \varepsilon$ so that

$$\alpha := m + 1 = n + \varepsilon + 1 > \beta := n + 2c + 1.$$

Raising both sides of (2.6) to the power $1/\beta$ we obtain

$$1 \le \frac{C_4 \delta^{1/\beta} h'(t)}{(h(t)^\alpha + C_3 \delta)^{1/\beta}}.$$

We integrate between 0 and T_0 and make the change of variables $x = h(t) \delta^{-1/\alpha}$ to conclude that $T_0 \le C_5 \delta^{1/\alpha} \le C_5 (\int_X (\phi - \varphi)_+ \, d\mu)^\tau$ with $\tau = \gamma / \alpha$. Letting $T_0 \to T_{\max}$ we obtain the desired estimate. ∎

3. Refinements and extensions

We now explain how minor modifications of the proof of Theorem 2.1 provide other important uniform estimates in various contexts of Kähler geometry.

3.1. Big cohomology classes

Let θ be a smooth closed $(1,1)$-form that represents a big cohomology class α. We set

$$V_\theta(x) := \sup\{v(x) : v \in \mathrm{PSH}(X,\theta) \text{ with } v \leq 0\}.$$

This is a θ-psh function with minimal singularities, i.e. any other θ-psh function φ satisfies $\varphi \leq V_\theta + C$ for some constant C. It is locally bounded in the ample locus $\mathrm{Amp}(\alpha)$, a Zariski open subset of X where the cohomology class α behaves like a Kähler class.

The Monge–Ampère measure $(\theta + dd^c\varphi)^n$ of a θ-psh function φ with minimal singularities is well-defined in $\mathrm{Amp}(\alpha)$, and one can show that it has finite mass independent of φ and equal to

$$V_\alpha = \mathrm{Vol}(\alpha) = \int_{\mathrm{Amp}(\alpha)} (\theta + dd^c V_\theta)^n > 0,$$

the volume of the class α.

We refer the reader to [10] for more details on these notions and focus here on slightly extending [10, Theorem B] by our new approach:

Theorem 3.1. *Let μ be a probability measure on X. If $\mathrm{PSH}(X,\theta) \subset L^m(\mu)$ for some $m > n$, then there exists a unique $\varphi \in \mathrm{PSH}(X,\theta)$ with minimal singularities such that $V_\alpha^{-1}(\theta + dd^c\varphi)^n = \mu$ and $\sup_X \varphi = 0$. Moreover,*

$$\|\varphi - V_\theta\|_{L^\infty(X)} \leq T_\mu$$

for some uniform constant T_μ.

Proof. It follows from [10, Theorem A] that there exists a unique finite energy solution φ. The key point for us here is to establish the a priori estimate. Note that $\varphi \leq V_\theta$ since $\sup_X \varphi = 0$. Our goal is to show that $V_\theta - T_{\max} \leq \varphi$, obtaining a uniform upper bound on $T_{\max}$.

A difficulty lies in the fact that θ is not a positive form. We consider the positive current $\omega = \theta + dd^c V_\theta$ and set $\tilde{\varphi} = \varphi - V_\theta \leq 0$. Observe that

$$\theta_\varphi := \theta + dd^c\varphi = \omega + dd^c\tilde{\varphi} =: \omega_{\tilde{\varphi}} \geq 0.$$

Our plan is thus to show that the "ω-psh" function $\tilde{\varphi}$ is bounded.

As in the proof of Theorem 2.1, we let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ denote a concave increasing function such that $\chi(0) = 0$ and $\chi'(0) = 1$. We set $\psi = V_\theta + \chi \circ \tilde{\varphi}$ and consider

$$u = P_\theta(\psi) = P_\theta(V_\theta + \chi \circ (\varphi - V_\theta)).$$

Observe that

$$\begin{aligned}\theta + dd^c\psi &= \chi' \circ \tilde{\varphi}\theta_{\varphi} + [1 - \chi' \circ \tilde{\varphi}]\omega + \chi'' \circ \tilde{\varphi} d\tilde{\varphi} \wedge d^c\tilde{\varphi} \\ &\leq \chi' \circ (\varphi - V_{\theta})(\theta + dd^c\varphi).\end{aligned}$$

The envelopes in the context of big cohomology classes enjoy similar properties to those reviewed in Section 1.2. In particular, the complex Monge–Ampère measure $(\theta + dd^c P_{\theta}(\psi))^n$ is concentrated on the contact set $\{P_{\theta}(\psi) = \psi\}$ (see [36, Theorem 2.7]) and the big version of Lemma 1.7 holds, showing that

$$V_{\alpha}^{-1}(\theta + dd^c u)^n \leq 1_{\{P_{\theta}(\psi)=\psi\}}(\chi' \circ (\varphi - V_{\theta}))^n \mu.$$

The rest of the proof is identical to that of Theorem 2.1. ∎

3.2. Degenerating families

Families of Kähler–Einstein varieties have been intensively studied in the past decade, requiring one to analyze the associated family of complex Monge–Ampère equations. We refer the reader to [24,33,44,46,48,49] for detailed examples and geometrical motivations.

The most delicate situation is when the volume of the fiber collapses. Theorem 2.1 yields a uniform bound in this case, providing an alternative proof and an extension of the main results of [22, 31]:

Corollary 3.2. *Fix a non-empty open set $I \subset \mathbb{R}$. Let $(\omega_t)_{t\in I}$ be a family of semi-positive and big forms on X, and assume there is a fixed form Θ such that $0 \leq \omega_t \leq \Theta$. Let $V_t := \int_X \omega_t^n > 0$ denote the volume of (X, ω_t). Let μ be a probability measure. If $\mathrm{PSH}(X, \Theta) \subset L^m(\mu)$ for some $m > n$, then any solution $\varphi_t \in \mathrm{PSH}(X, \omega_t) \cap L^{\infty}(X)$ to*

$$\frac{1}{V_t}(\omega_t + dd^c\varphi_t)^n = \mu$$

satisfies $\mathrm{Osc}_X(\varphi_t) \leq T_{\mu}$ for some uniform constant T_{μ}.

The point is that the estimate is uniform in t, and independent of the behavior of V_t (in particular, V_t may degenerate to zero at any boundary point of I).

Proof. Theorem 2.1 provides a uniform bound $\mathrm{Osc}_X(\varphi_t) \leq T(A_m(\omega_t, \mu))$, where

$$A_m(\omega_t, \mu) := \sup\left\{\int_X (-\psi)^m\, d\mu : \psi \in \mathrm{PSH}(X, \omega_t) \text{ with } \sup_X \psi = 0\right\}.$$

Observe now that $\mathrm{PSH}(X, \omega_t) \subset \mathrm{PSH}(X, \Theta)$ and $\mathrm{PSH}(X, \Theta) \subset L^m(\mu)$, hence we obtain $A_m(\omega_t, \mu) \leq A_m(\Theta, \mu) < +\infty$. The uniform upper bound follows. ∎

This uniform estimate shows in particular that in many geometrical contexts, uniform control on the $L^{n+\varepsilon}$-norm of the Monge–Ampère potentials φ_t suffices to obtain L^{∞}-control of the latter.

One can obtain similarly uniform estimates when the underlying complex structure is also changing: Let $\mathcal{X}$ be an irreducible and reduced complex Kähler space, and let $\pi : \mathcal{X} \to \mathbb{D}$ denote a proper, surjective holomorphic map such that each fiber $X_t = \pi^{-1}(t)$ is an n-dimensional, reduced, irreducible, compact Kähler space, for any $t \in \mathbb{D}$. Given a Kähler form ω on $\mathcal{X}$ and $\omega_t := \omega_{|X_t}$, one can consider the complex Monge–Ampère equations

$$\frac{1}{V}(\omega_t + dd^c\varphi_t)^n = \mu_t,$$

where

- the volume $V = \int_{X_t} \omega_t^n$ turns out to be independent of t;
- μ_t is a family of probability measures on each fiber X_t (e.g. the normalized Calabi–Yau measures of a degenerating family of Calabi–Yau manifolds).

In many concrete geometrical situations (see e.g. [24, 33, 44]), one can check that $A_m(\omega_t, \mu_t) \leq A$ is uniformly bounded from above for some $m > n$ (often any $m > 1$). If one can further uniformly compare $\sup_{X_t} \varphi_t$ and $\int_{X_t} \varphi_t \frac{\omega_t^n}{V}$, then Theorem 2.1 applies and provides a uniform L^∞-estimate. It is thus sometimes not necessary to establish a uniform Skoda integrability theorem in families (compare with [24, 43]).

3.3. Relative a priori L^∞-bounds

Fix a semi-positive and big $(1,1)$-form ω, and an ω-psh function ρ with analytic singularities such that $\omega + dd^c\rho \geq \delta\omega_X$ is a Kähler current which is smooth in the ample locus $\mathrm{Amp}(\omega)$. We normalize ρ so that $\sup_X \rho = 0$ and set $V = \int_X \omega^n > 0$.

In this section we consider the degenerate complex Monge–Ampère equation

$$V^{-1}(\omega + dd^c\varphi)^n = \mu = f dV_X, \tag{3.1}$$

where μ is a probability measure whose density $f \in L^1(X)$ does not belong to any good Orlicz class (see Section 2.2). Since μ does not charge pluripolar sets, there exists a unique "finite energy solution" $\varphi \in \mathcal{E}(X,\omega)$ (see [39]), but one cannot expect any longer that φ is globally bounded on X.

Given a quasi-plurisubharmonic function ψ on X and $c > 0$, we set

$$E_c(\psi) := \{x \in X : \nu(\psi, x) \geq c\},$$

where $\nu(\psi, x)$ denotes the Lelong number of ψ at x. A celebrated theorem of Siu ensures that for any $c > 0$, the set $E_c(\psi)$ is a closed analytic subset of X.

Theorem 3.3. *Assume $f = ge^{-\psi}$, where $0 \leq g \in L^p(dV_X)$, $p > 1$, and ψ is a quasi-psh function. Then there exists a unique $\varphi \in \mathcal{E}(X,\omega)$ such that*

- $\alpha(\psi + \rho) - \beta \leq \varphi \leq 0$ *with* $\sup_X \varphi = 0$;
- φ *is locally bounded in the Zariski open set* $\Omega := \mathrm{Amp}(\omega) \setminus E_{1/q}(\psi)$;
- $V^{-1}(\omega + dd^c\varphi)^n = f dV_X$ *in* Ω,

where $\alpha, \beta > 0$ depend on an upper bound for $\|g\|_{L^p}$ and $1/p + 1/q = 1$.

When $f \leq e^{-\psi}$ for some quasi-psh function ψ, it has been shown by Di Nezza–Lu [27, Theorem 2] that the normalized solution φ to (3.1) is locally bounded in the complement of the set $\{\psi = -\infty\}$. The proof of Di Nezza–Lu is a generalization of the method of Kołodziej [41] that makes use of a theory of generalized Monge–Ampère capacities further developed in [26]. We slightly extend this result here and propose a brand new proof using envelopes and Theorem 2.1.

Proof of Theorem 3.3. Reduction to analytic singularities. We let q denote the conjugate exponent of p, set $r = \frac{2p}{p+1}$, and note that $1 < r < p$. If the Lelong numbers of ψ are all less than $1/q$, it follows from the Hölder inequality that $f \in L^r(dV_X)$, since

$$\int_X f^r \, dV_X = \int_X g^r e^{-r\psi} \, dV_X \leq \left(\int_X g^p \, dV_X\right)^{\frac{r}{p}} \cdot \left(\int_X e^{-\frac{pr}{p-r}\psi} \, dV_X\right)^{\frac{p-r}{p}},$$

where the last integral is finite by Skoda's integrability theorem [39, Theorem 8.11] if $\frac{pr}{p-r}\nu(\psi, x) < 2$ for all $x \in X$, which is equivalent to $\nu(\psi, x) < 1/q$.

It is thus natural to expect that the solution φ will be locally bounded in the complement of the closed analytic set $E_{q^{-1}}(\psi)$. It follows from Demailly's equisingular approximation technique (see [21]) that there exists a sequence (ψ_m) of quasi-psh functions on X such that

- $\psi_m \geq \psi$ and $\psi_m \to \psi$ (pointwise and in L^1);
- ψ_m has analytic singularities concentrated along $E_{m^{-1}}(\psi)$;
- $dd^c \psi_m \geq -K\omega_X$ for some uniform constant $K > 0$;
- $\int_X e^{2m(\psi_m - \psi)} \, dV_X < +\infty$ for all m.

We choose $m = [q]$, set $g_m := g e^{\psi_m - \psi}$, and observe that

$$\begin{aligned}\int_X g_m^r &\leq \left(\int_X e^{2m(\psi_m-\psi)} \, dV_X\right)^{\frac{1}{2m}} \cdot \left(\int_X g_m^{\frac{2mr}{2m-r}} \, dV_X\right)^{\frac{2m-r}{2m}} \\ &\leq \left(\int_X e^{2m(\psi_m-\psi)} \, dV_X\right)^{\frac{1}{2m}} \cdot \left(\int_X g_m^p \, dV_X\right)^{\frac{2m-r}{2m}} < +\infty\end{aligned}$$

if we choose $r^{-1} = p^{-1} + (2m)^{-1} < 1$ so that $\frac{2mr}{2m-r} = p$. By replacing ψ by $\psi_{[q]} \geq \psi$ and g by $g_m \in L^r$, we can thus assume that

- ψ has analytic singularities and is smooth in $X \setminus E_{q^{-1}}(\psi)$;
- the function $\tilde{\psi} := a\psi + \rho$ with $a := \delta/K$ is ω-psh.

Uniform integrability of φ. It is a standard measure-theoretic fact that the density f belongs to an Orlicz class L^w for some convex increasing weight $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that $w(t)/t \to +\infty$ as $t \to +\infty$. Set $\chi_1(t) := -(w^*)^{-1}(-t)$, where w^* denotes the Legendre transform of w. Thus $\chi_1 : \mathbb{R}^- \to \mathbb{R}^-$ is a convex increasing weight such that $\chi_1(-\infty) = -\infty$ and

$$\int_X (-\chi_1 \circ \varphi)(\omega + dd^c\varphi)^n \leq \int_X w \circ f \, dV_X + \int_X (-\varphi) \, dV_X \leq C_0,$$

as follows from the additive version of the Hölder–Young inequality and the compactness of sup-normalized ω-psh functions.

It follows that φ belongs to a compact subset of the finite energy class $\mathcal{E}_{\chi_1}(X,\omega)$, hence for all $\lambda \in \mathbb{R}$,

$$\int_X \exp(-\lambda\varphi)\, dV_X \leq C_\lambda \tag{3.2}$$

for some C_λ independent of φ (see [37, 39] for more information).

The envelope construction. Let $u = P(2\varphi - \tilde{\psi})$ denote the greatest ω-psh function that lies below $2\varphi - \tilde{\psi}$. Since $h = 2\varphi - \tilde{\psi}$ is bounded from below and quasi-continuous, it follows from Proposition 1.6 that the measure $(\omega + dd^c u)^n$ is supported on the contact set $\mathcal{C} = \{u = 2\varphi - \tilde{\psi}\}$. Thus

$$(\omega + dd^c u)^n \leq 1_{\mathcal{C}}(\omega + dd^c(2\varphi - \tilde{\psi}))^n \leq 1_{\mathcal{C}}(2\omega + dd^c(2\varphi))^n.$$

Since $v \leq w$ on X, it follows from Lemma 1.3 that

$$1_{\{v=w\}}(2\omega + dd^c v)^n \leq 1_{\{v=w\}}(2\omega + dd^c w)^n, \tag{3.3}$$

where

- $v = u + \tilde{\psi}$ is 2ω-psh and $u + \tilde{\psi} \leq 2\varphi = w$ on X;
- $\{u + \tilde{\psi} = 2\varphi\}$ coincides with the contact set $\mathcal{C}$.

Therefore, it follows from (3.3) that

$$\begin{aligned} 1_{\mathcal{C}}(2\omega + dd^c(u + \tilde{\psi}))^n &\leq 1_{\mathcal{C}}(2\omega + dd^c(2\varphi))^n \leq 1_{\mathcal{C}} 2^n c g e^{-\psi}\, dV_X \\ &\leq 1_{\mathcal{C}} 2^n c g e^{u/a} e^{-2\varphi/a}\, dV_X \leq c_1 g e^{-2\varphi/a}\, dV_X, \end{aligned}$$

since $\sup_X u \leq c_2$ is uniformly bounded from above, as we explain below.

It follows from the Hölder inequality and (3.2) that the measure $ge^{-2\varphi/a} dV_X$ satisfies the assumption of Theorem 2.1. We infer that $u \geq -M$ is uniformly bounded below, hence

$$2\varphi = (2\varphi - \tilde{\psi}) + \tilde{\psi} \geq u + \tilde{\psi} \geq \frac{\delta}{K}\psi + \rho - M.$$

The desired a priori estimate follows with $\beta = M/2$ and $\alpha = \max(1, \delta/(2K))$.

Bounding $\sup_X u$ *from above.* We can assume without loss of generality that $\sup_X \tilde{\psi} = 0$. Consider $G = \{\tilde{\psi} > -1\}$; this is a non-empty plurifine open set. Observe that for all $x \in G$, $u(x) \leq (2\varphi - \tilde{\psi})(x) \leq 1$, hence

$$u(x) - 1 \leq V_{G,\omega}(x) := \sup\{w(x) : w \in \mathrm{PSH}(X,\omega) \text{ with } w \leq 0 \text{ on } G\}.$$

It follows from [39, Theorem 9.17.1] that $\sup_X V_{G,\omega} = C$ is finite since G is non-pluripolar, thus

$$\sup_X u \leq c_2 = 1 + \sup_X V_{G,\omega} = 1 + C.$$ ■

4. The local context

4.1. Cegrell classes

We fix a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$, i.e. there exists a continuous plurisubharmonic function $\rho : \Omega \to [-1, 0)$ whose sublevel sets $\{\rho < -c\} \Subset \Omega$ are relatively compact for all $c > 0$.

Let $\mathcal{T}(\Omega)$ denote the set of bounded plurisubharmonic functions u in Ω such that $\lim_{z \to \zeta} u(z) = 0$ for every $\zeta \in \partial\Omega$, and $\int_\Omega (dd^c u)^n < +\infty$. Cegrell [12, 13] has studied the complex Monge–Ampère operator $(dd^c \cdot)^n$ and introduced different classes of plurisubharmonic functions on which the latter is well-defined:

- $\mathrm{DMA}(\Omega)$ is the set of psh functions u such that for all $z_0 \in \Omega$, there exists a neighborhood V_{z_0} of z_0 and a decreasing sequence $u_j \in \mathcal{T}(\Omega)$ which converges to u in V_{z_0} and satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$.
- A function u belongs to $\mathcal{F}(\Omega)$ iff there exists a sequence $u_j \in \mathcal{T}(\Omega)$ decreasing to u in all of Ω, which satisfies $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$.
- A function u belongs to $\mathcal{E}^p(\Omega)$ if there exists a sequence $u_j \in \mathcal{T}(\Omega)$ decreasing to u in Ω with $\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty$.
- A function u belongs to $\mathcal{F}^p(\Omega)$ if there exists a sequence $u_j \in \mathcal{T}(\Omega)$ decreasing to u in Ω with $\sup_j \int_\Omega [1 + (-u_j)^p](dd^c u_j)^n < +\infty$.

Given $u \in \mathcal{E}^p(\Omega)$ we define the weighted energy of u by

$$E_p(u) := \int_\Omega (-u)^p (dd^c u)^n < +\infty.$$

The operator $(dd^c \cdot)^n$ is well-defined on these sets, and continuous under decreasing limits. If $u \in \mathcal{E}^p(\Omega)$ for some $p > 0$ then $(dd^c u)^n$ vanishes on all pluripolar sets [5, Theorem 2.1]. If $u \in \mathcal{E}^p(\Omega)$ and $\int_\Omega (dd^c u)^n < +\infty$ then $u \in \mathcal{F}^p(\Omega)$. Also, note that

$$\mathcal{T}(\Omega) \subset \mathcal{F}^p(\Omega) \subset \mathcal{F}(\Omega) \subset \mathrm{DMA}(\Omega) \quad \text{and} \quad \mathcal{T}(\Omega) \subset \mathcal{E}^p(\Omega) \subset \mathrm{DMA}(\Omega).$$

Cegrell has characterized the range of the complex Monge–Ampère operator acting on the classes $\mathcal{E}^p(\Omega)$:

Theorem 4.1 ([12, Theorem 5.1]). *Let μ be a probability measure in Ω. There exists a function $u \in \mathcal{F}^p(\Omega)$ such that $(dd^c u)^n = \mu$ if and only if $\mathcal{F}^p(\Omega) \subset L^p(\Omega)$.*

A simplified variational proof of this result has been provided in [1].

4.2. Dirichlet problem

We have the following local analogue of Theorem 2.1:

Theorem 4.2. *Assume μ is a probability measure in Ω and $\mathcal{F}(\Omega) \subset L^m(\mu)$ for some $m > n$. Then there exists a unique bounded function $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n = \mu$.*

The upper bound on $\sup_\Omega |u|$ *only depends on* $A_m(\mu), m, n$, *where*

$$A_m(\mu) := \sup\left\{\int_\Omega (-u)^m \, d\mu : u \in \mathcal{T}(\Omega) \text{ with } \int_\Omega (dd^c u)^n \le 1\right\}.$$

Proof. We first explain why the integrability condition $\mathcal{F}(\Omega) \subset L^m(\Omega, \mu)$ is equivalent to the finiteness of A_m. Indeed, if A_m is not finite then there exists a sequence (u_j) in $\mathcal{T}(\Omega)$ such that $\int_\Omega (dd^c u_j)^n \le 1$ but $\int_\Omega |u_j|^m \, d\mu \ge 4^{jm}$. Let $u := \sum_{j=1}^{+\infty} 2^{-j} u_j$. Then, by [13, Corollary 5.6], we have $u \in \mathcal{F}(\Omega)$, but

$$\int_\Omega (-u)^m \, d\mu \ge 2^{-jm} \int_\Omega (-u_j)^m \, d\mu \ge 2^{jm} \to +\infty.$$

It follows from Theorem 4.1 that there exists $\varphi \in \mathcal{F}(\Omega)$ such that $(dd^c\varphi)^n = \mu$. We assume for the moment that $u \in \mathcal{T}$ is bounded and we establish a uniform bound for φ. Set

$$T_{\max} := \sup\{t > 0 : \mu\{\varphi < -t\} > 0\}.$$

Our goal is to establish a precise bound on $T_{\max}$. By definition, $-T_{\max} \le \varphi$ almost everywhere with respect to μ, hence $(dd^c \max(\varphi, -T_{\max}))^n \ge (dd^c\varphi)^n$ and the domination principle, [39, Corollary 3.31] gives $\varphi \ge -T_{\max}$, providing the desired a priori bound $|\varphi| \le T_{\max}$.

We let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ denote a *concave* increasing function such that $\chi(0) = 0$ and $\chi'(0) = 1$. We set $\psi = \chi \circ \varphi$, $u = P(\psi) \in \mathcal{T}(\Omega)$ the largest psh function in Ω which lies below ψ, and observe that

$$dd^c\psi = \chi' \circ \varphi \omega_\varphi + \chi'' \circ \varphi d\varphi \wedge d^c\varphi \le \chi' \circ \varphi dd^c\varphi.$$

Since $\psi \ge \chi'(-T_{\max})\varphi$ and the latter is in $\mathcal{T}(\Omega)$, we deduce that $u \ge \chi'(-T_{\max})\varphi$ and $u \in \mathcal{T}(\Omega)$.

Although the function ψ is not psh, this provides a bound from above on the positivity of $dd^c\psi$ which allows us to control the Monge–Ampère measure of its envelope (see [25, Lemmas 4.1 and 4.2]):

$$(dd^c u)^n \le 1_{\{u=\psi\}}(dd^c\psi)^n \le (\chi' \circ \varphi)^n \mu.$$

The above inequalities hold for smooth functions and the general case of bounded psh functions can be obtained as in the proof of Lemma 1.7.

We thus get uniform control on the Monge–Ampère mass of u:

$$\int_\Omega (dd^c u)^n \le \int_\Omega (\chi' \circ \varphi)^n \, d\mu.$$

We are going to choose below the weight χ in such a way that $\int_\Omega (\chi' \circ \varphi)^n \, d\mu = B \le 2$ is a finite constant under control. This provides a uniform upper bound on $\|u\|_{L^m(\mu)}$. Using the Chebyshev inequality we thus obtain

$$\mu\{\varphi < -t\} \le \frac{\int_\Omega |\chi(\varphi)|^m \, d\mu}{|\chi(-t)|^m} \le \frac{\int_\Omega |u|^m \, d\mu}{|\chi(-t)|^m} \le \frac{A_m}{|\chi(-t)|^m}, \tag{4.1}$$

where $A_m \ge 1$ is an upper bound for $\int_\Omega |u|^m \, d\mu$.

Choice of χ. We use again Lebesgue's formula: if $g:\mathbb{R}^+\to\mathbb{R}^+$ is increasing and normalized by $g(0)=1$ then

$$\int_\Omega g\circ(-\varphi)\,d\mu=\mu(\Omega)+\int_0^{T_{\max}} g'(t)\mu\{\varphi<-t\}\,dt.$$

Setting $g(t)=\chi'(-t)^n$ we define χ by imposing $\chi(0)=0$, $\chi'(0)=1$, and

$$g'(t)=\begin{cases}\dfrac{1}{(1+t)^2\mu\{\varphi<-t\}} & \text{if } t\in[0,T_0],\\[2ex] \dfrac{1}{(1+t)^2} & \text{if } t>T_0.\end{cases}$$

This choice guarantees that

$$\int_\Omega (\chi'\circ\varphi)^n\,d\mu\le\mu(\Omega)+\int_0^{+\infty}\frac{dt}{(1+t)^2}=2.$$

Conclusion. We set $h(t)=-\chi(-t)$ and work with the positive counterpart of χ. Note that $h(0)=0$ and $h'(t)=g(t)^{1/n}$ is positive increasing, hence h is convex increasing (so χ is concave increasing and negative).

Together with (4.1) our choice of χ yields, for all $t\in[0,T_0]$,

$$\frac{1}{(1+t)^2g'(t)}=\mu\{\varphi<-t\}\le\frac{A_m}{h(t)^m}.$$

This reads

$$h(t)^m\le A_m(1+t)^2g'(t)=nA_m(1+t)^2h''(t)h'(t)^{n-1}.$$

We integrate this inequality as in the proof of Theorem 2.1 and obtain

$$T_0\le C'$$

for some uniform constant C' depending on n,m,A_m.

To finish the proof we write $\mu=f(dd^c\phi)^n$, where $0\le f\in L^1(\Omega,(dd^c\phi)^n)$ and $\phi\in\mathcal{T}(\Omega)$. This is known as Cegrell's decomposition theorem [12, Theorem 6.3]. We next solve $(dd^c\varphi_j)^n=\min(f,j)(dd^c\phi)^n$ with $\varphi_j\in\mathcal{T}(\Omega)$. Since $(dd^c\varphi_j)^n\le\mu$, our estimate above shows that $|\varphi_j|\le C$ for a uniform constant C. The comparison principle also implies that φ_j is decreasing and $\varphi\le\varphi_j$, thus $u:=\lim_j\varphi_j\in\mathcal{F}(\Omega)$ is bounded and $(dd^cu)^n=\mu$. It then follows from [13, Theorem 5.15] that $u=\varphi$, finishing the proof. ∎

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