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Solutions of the Ginzburg–Landau equations with vorticity concentrating near a nondegenerate geodesic

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Abstract. It is well-known that under suitable hypotheses, for a sequence of solutions of the (simplified) Ginzburg–Landau equations $-\Delta u_\varepsilon + \varepsilon^{-2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0$, the energy and vorticity concentrate as $\varepsilon \to 0$ around a codimension 2 stationary varifold – a (measure-theoretic) minimal surface. Much less is known about the question of whether, given a codimension 2 minimal surface, there exists a sequence of solutions for which the given minimal surface is the limiting concentration set. The corresponding question is very well-understood for minimal hypersurfaces and the scalar Allen–Cahn equation, and for the Ginzburg–Landau equations when the minimal surface is locally area-minimizing, but otherwise quite open.

We consider this question on a 3-dimensional closed Riemannian manifold (M, g), and we prove that any embedded nondegenerate closed geodesic can be realized as the asymptotic energy/vorticity concentration set of a sequence of solutions of the Ginzburg–Landau equations.

Keywords. 3d Ginzburg–Landau, geodesics, unstable critical points

1. Introduction

In this paper we construct certain geometrically meaningful solutions of the Ginzburg–Landau equations

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \tag{1.1}$$

for $u_\varepsilon : M \to \mathbb{C}$, where (M, g) is a closed n-dimensional Riemannian manifold, with $n = 3$ in our main results. Such solutions are critical points of the Ginzburg–Landau functional

$$E_\varepsilon(u) := \frac{1}{\pi|\log\varepsilon|}\int_M e_\varepsilon(u)\,\mathrm{vol}_g, \quad e_\varepsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - 1)^2.$$

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If M is simply connected, then given a sequence (u_ε) of solutions of (1.1) satisfying the energy bound

$$E_\varepsilon(u_\varepsilon) \le C, \tag{1.2}$$

the rescaled energy density $|\log \varepsilon|^{-1} e_\varepsilon(u_\varepsilon)$ is known to concentrate as $\varepsilon \to 0$, after possibly passing to a subsequence, around an $(n-2)$-dimensional stationary varifold – a weak, measure-theoretic minimal surface. This is proved in an appendix in [33], following earlier results in simply connected Euclidean domains, such as those in [5, 20, 22]. Similar but more complicated results hold when M is not simply connected; in this case, the limiting energy measure may have a diffuse part, but any concentrated part must again be an $(n-2)$-dimensional stationary varifold.

In this paper we address a sort of converse question:

When can a given codimension 2 minimal surface be realized as the energy concentration set of a sequence of solutions of (1.1)*?*

A first answer is provided by Gamma-convergence results [1, 15] that relate the Ginzburg–Landau functional and, roughly speaking, the $(n-2)$-dimensional area (with multiplicity) of a limiting vorticity concentration set, where the *vorticity* associated to a wave function u, denoted Ju, is the 2-form defined by

$$Ju := du^1 \wedge du^2, \quad \text{where } u = u^1 + iu^2 \text{ and } u^1, u^2 \text{ are real-valued.} \tag{1.3}$$

(We will also sometimes refer to Ju as the *Jacobian* of u.) These results imply as a general principle that one should be able to find solutions u_ε of (1.1) whose energy and vorticity concentrate around a *locally area-minimizing* minimal surface of codimension 2. In the Euclidean setting, specific instances of this general principle, for particular compatible choices of boundary conditions on the minimal surface and the solutions u_ε of (1.1), have been established in [1, 27, 31]. However, arguments based on Gamma-convergence are of limited use for capturing the behaviour of nonminimizing critical points.

The corresponding question is also very well-understood for minimal *hypersurfaces* and the Allen–Cahn equation, i.e. the scalar counterpart of (1.1); see for example [9, 18, 19, 28] among many others. Many of these results are based on gluing techniques and elliptic PDE arguments, which can be used to construct a great variety of solutions and establish detailed descriptions of them. These techniques seem to be hard to implement for the Ginzburg–Landau equations in three or more dimensions.

A particularly basic case in which our question remains open concerns the Ginzburg–Landau equations (1.1) on a smooth bounded domain $\Omega \subset \mathbb{R}^3$ containing an unstable geodesic with respect to natural boundary conditions, i.e. a line segment in Ω meeting $\partial\Omega$ orthogonally at both ends, admitting perturbations that decrease the arclength quadratically, and satisfying a natural nondegeneracy condition.

In this situation one would like to prove the existence of a sequence (u_ε) of solutions of the Ginzburg–Landau equations, also with natural (Neumann) boundary conditions, whose energy and vorticity concentrate around the given line segment. Such solutions would satisfy

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = L =: \text{the length of the geodesic.} \tag{1.4}$$

Partial progress toward this goal was achieved in [16], which develops a general framework for using Gamma-convergence to study convergence, not of *critical points*, but of *critical values*, then uses this framework to prove the existence of solutions of (1.1), in the situation described above, that satisfy (1.4), but without control over the limiting concentration set. An example in the same paper (Remark 4.5) shows that the general framework is too weak to characterize asymptotic behaviour of critical points – in this context, to determine where the energy and vorticity concentrate. For this, more detailed information about the sequence of solutions is needed.

The results of [16] were extended to the Riemannian setting in the Ph.D. thesis of Jeffrey Mesaric [25] which, starting with a nondegenerate unstable closed geodesic on a closed, oriented 3-dimensional Riemannian manifold (M, g), uses machinery from [16] to construct solutions to (1.1) satisfying (1.4). Again, this result does not establish whether the energy of the solutions concentrates along the geodesic.

In the main result of this paper, we fill in this gap in the Riemannian case. Our main result is the following theorem.

Theorem 1.1. *Let (M, g) be a closed oriented 3-dimensional Riemannian manifold, and let γ be a closed, embedded, nondegenerate geodesic of length L. Assume in addition that $\gamma = \partial S$ for some 2-dimensional orientable submanifold S of M.*

Then there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, there is a solution u_ε of the Ginzburg–Landau equations (1.1) such that

$$\frac{1}{\pi}\int_M \varphi \wedge Ju_\varepsilon \to \int_\gamma \varphi \quad \textit{for every smooth 1-form } \varphi \textit{ on } M$$

and

$$\frac{1}{\pi|\log\varepsilon|}\int_M \phi\, e_\varepsilon(u_\varepsilon) \to \int_\gamma \phi\, d\mathcal{H}^1 \quad \textit{for every } \phi \in C^\infty(M)$$

as $\varepsilon \to 0$, where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure.

In fact, we will prove a slightly stronger result; see Theorem 5.1 for the full statement.

We briefly sketch the main ideas, *not* in the order in which they appear in the body of the paper. Terminology such as ‘nondegenerate’ and ‘stationary varifold’ is defined in Section 2 below.

- In Section 4 we show that for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and any $\tau > 0$, one can find a solution u_ε of the Ginzburg–Landau heat flow whose vorticity is initially concentrated near the geodesic $\Gamma := \gamma([0, L))$, and such that

$$L - \delta \le E_\varepsilon(u_\varepsilon(\cdot, t)) \le L + \delta \quad \text{for all } t \in [0, \tau].$$

  See Proposition 4.1. This relies heavily on tools developed in the earlier papers [16,25].

The main point of the proof of Theorem 1.1 is to strengthen this by showing that for such solutions, if ε and δ are small enough, the vorticity $\frac{1}{\pi} J u_\varepsilon(\cdot, t)$ does not stray very far from Γ for any $t \in [0, \tau]$.

- We carry this out in Section 5, using an argument by contradiction and passing to limits to obtain a stationary 1-varifold that is close, but not equal, to the varifold associated to Γ. This argument requires, among other ingredients, an extension to the Riemannian setting of an important theorem of Bethuel, Orlandi, and Smets [6]. The extension we need is stated in Theorem 2.3 and is proved in a companion paper [8]. The stationary varifold satisfies additional good properties, notably including lower density bounds.
- To obtain a contradiction, we prove that this stationary varifold cannot exist. This is the content of Proposition 3.1, which is a measure-theoretic strengthening of the classical fact that a nondegenerate closed geodesic is isolated; it is the only closed geodesic in a tubular neighborhood of itself. The proof relies, among other ingredients, on results from [3] about the structure of stationary 1-varifolds on Riemannian manifolds.

We believe that something like Theorem 1.1 should be valid in much greater generality, including on higher-dimensional manifolds and on smooth, bounded subsets of $\mathbb{R}^n$, $n \geq 3$, with natural boundary conditions both for the geodesic Γ (or codimension 2 minimal surface, for $n \geq 4$) and the Ginzburg–Landau equation. Our proof does not adapt in a straightforward way to either of these settings.

- Our strategy requires a sufficiently good version of Theorem 2.3. On a bounded set $\Omega \subset \mathbb{R}^n$, even for $n = 3$, such a result is not known. If Ω is convex, a result of this type for the scalar parabolic Allen–Cahn equation was proved several years ago in [26]. A similar strategy could probably be pursued for the Ginzburg–Landau heat flow, but convexity is not a natural assumption for any analog of Theorem 1.1.
- Our reliance on results from [3] about stationary 1-varifolds would seriously complicate any effort to adapt our argument to dimensions $n \geq 4$, where one would confront stationary varifolds of dimension $n - 2 \geq 2$.

Remark added December 2022. This paper was posted on the arXiv in January 2021 and submitted in July 2021, after the completion of the companion paper [8]. In May 2022, we received a beautiful preprint of De Philippis and Pigati [11] that establishes a generalization of Theorem 1.1 to manifolds of arbitrary dimension and that also treats the Allen–Cahn equation and the Abelian Higgs model. The basic strategy follows that developed here and in [16, 25], with new ingredients that include an extension of Proposition 3.1 valid in arbitrary dimension and codimension, and requiring significant new ideas. The problem of establishing a counterpart of Theorem 1.1 on bounded subsets of $\mathbb{R}^n$, $n \geq 3$, or more generally on Riemannian manifolds with boundary, remains open, to the best of our knowledge.

2. Background and notation

2.1. Geometric notions regarding a nondegenerate geodesic

Throughout this document we use M or (M, g) to denote a closed oriented 3-dimensional Riemannian manifold where 'closed' means compact and without boundary. We let TM be the bundle over M whose fibre T_pM at $p \in M$ is the tangent space to M at p. We use the notation $(\cdot,\cdot)_g$ to denote the inner product on TM given by g. We also use $|\cdot|_g$ to denote the corresponding norm; we will omit mention of g when no confusion can arise. We write vol_g to denote the Riemannian volume form associated to the metric g.

Throughout this paper, r_0 denotes a positive number, to be fixed in Proposition 3.1, which among other properties will be required to satisfy

$$r_0 < \tfrac{1}{2}(\text{injectivity radius of } M). \tag{2.1}$$

Throughout, a central role will be played by a geodesic γ that we take to be parametrized by arclength. That is, we will assume the existence of an injective map $\gamma : \mathbb{R}/L\mathbb{Z} \to M$ whose range consists of a simple closed curve $\Gamma := \{\gamma(t) : t \in \mathbb{R}/L\mathbb{Z}\}$ of length L such that

$$|\gamma'| = 1, \quad \nabla_{\gamma'}\gamma' = 0 \quad \text{everywhere in } \mathbb{R}/L\mathbb{Z}. \tag{2.2}$$

We will insist that this curve Γ bounds an orientable smooth surface $S_\Gamma \subset M$, i.e.

$$\Gamma = \partial S_\Gamma. \tag{2.3}$$

We introduce here the notation

$$d_\Gamma(x) := \mathrm{dist}(x,\Gamma) := \inf\left\{\int_0^1 |\lambda'(t)|\,dt : \lambda \in \mathrm{Lip}([0,1]; M),\ \lambda(0) = x,\ \lambda(1) \in \Gamma\right\}$$

as well as

$$K_r := \{x \in M : d_\Gamma(x) < r\} \tag{2.4}$$

for a neighbourhood of Γ.

For $t \in \mathbb{R}/L\mathbb{Z}$, we then let

$$N_{\gamma(t)}\Gamma := \{u \in T_{\gamma(t)}M : (u, \gamma'(t))_g = 0\}.$$

A *normal vector field along* γ is a map $\xi : \mathbb{R}/L\mathbb{Z} \to TM$ such that $\xi(t) \in N_{\gamma(t)}\Gamma$ for every t. We also introduce the coordinates $\psi : B_r(0) \times (\mathbb{R}/L\mathbb{Z}) \to K_r$ defined by

$$\psi(y,t) := \exp_{\gamma(t)}\Big(\sum_{i=1}^{2} y^i\,\Xi_i(t)\Big), \tag{2.5}$$

where Ξ_1, Ξ_2 are fixed normal vector fields which are orthogonal for each $t \in \mathbb{R}/L\mathbb{Z}$. We note that for $r < r_0$, this map is smoothly invertible [12, Chapter 7]. For future use, we will use the notation $\psi^{-1}(x) = (y(x), \tau(x)) \in B_r(0) \times (\mathbb{R}/L\mathbb{Z})$, so that for $x \in K_r$,

$$\psi(y,t) = x \iff y(x) = y \text{ and } \tau(x) = t. \tag{2.6}$$

We observe that the mapping τ simply assigns to an $x \in K_r$ the parameter value t corresponding to the closest point on Γ to x.

Given two normal vector fields along γ, denoted by $\xi, \tilde{\xi}$, we can define their L^2 inner product in the natural way:

$$(\xi, \tilde{\xi})_{L^2} := \int_{\mathbb{R}/L\mathbb{Z}} (\xi(t), \tilde{\xi}(t))_g \, dt.$$

We will write $L^2(N\Gamma)$ to denote the space of square integrable normal vector fields, a Hilbert space with the above inner product.

For $\xi \in L^2(N\Gamma)$, we will use the notation

$$\gamma_\xi(t) := \exp_{\gamma(t)} \xi(t), \tag{2.7}$$

where exp denotes the exponential map.

We next recall the *Jacobi operator* L_J which acts on smooth normal vector fields ξ along γ, and is defined by

$$L_J \xi := -\xi'' + R(\xi, \gamma')\gamma', \tag{2.8}$$

where R denotes the curvature tensor. We say that a geodesic is *nondegenerate* if 0 is not an eigenvalue of L_J.

With this notion in hand, we add another crucial hypothesis on the geodesic by assuming henceforth that

$$\gamma : \mathbb{R}/L\mathbb{Z} \to M \text{ is a simple, closed, nondegenerate geodesic with } |\gamma'| \equiv 1. \tag{2.9}$$

One says that γ has *finite index* if the total number (algebraic multiplicity) of negative eigenvalues of L_J is finite. Since M is closed, this is always true, as a consequence of standard Sturm–Liouville theory. Our standing assumption (2.9) that γ is nondegenerate then implies there exists some $\ell \geq 0$ and a nondecreasing sequence of eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_\ell < 0 < \lambda_{\ell+1} \leq \cdots \tag{2.10}$$

of L_J, together with an associated orthonormal basis of $L^2(N\Gamma)$ consisting of (smooth) eigensections $\{\xi_j\}_{j=1}^\infty$. We will always assume that $\ell > 0$, since otherwise the results presented here admit much simpler proofs. We define

$$H_- := \operatorname{span}\{\xi_1, \ldots, \xi_\ell\}, \quad H_+ := H_-^\perp. \tag{2.11}$$

We will say that ξ is *Lipschitz*, and we will write $\xi \in \mathrm{Lip}$, if γ_ξ is Lipschitz continuous. It is clear that

$$H_-(r_0) := \{\xi \in H_- : \|\xi\|_{L^\infty} \leq r_0\} \subset \mathrm{Lip}$$

for r_0 and H_- from (2.1) and (2.11) respectively.

The standard fact that the Jacobi operator (2.8) is the second variation of arclength, together with the definition (2.11) of H_-, implies that there exist $c_0, r_0 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}/L\mathbb{Z}} |\gamma_\xi'(t)|\, dt &\leq L - c_0\|\xi\|_{L_2}^2 \quad \text{if } \xi \in H_-(r_0), \\ \int_{\mathbb{R}/L\mathbb{Z}} |\gamma_\xi'(t)|\, dt &\geq L + c_0\|\xi\|_{L_2}^2 \quad \text{if } \xi \in H_+ \text{ and } \|\xi\|_{W^{1,\infty}} \leq r_0. \end{aligned} \tag{2.12}$$

2.2. *Forms and currents*

We denote, for $k \in \mathbb{N} \cup \{0\}$, the space of smooth k-forms on M by

$$\mathcal{D}^k(M) := \{\phi \in C^\infty(M; \textstyle\bigwedge^k M)\}$$

where $\bigwedge^k M$ is an abbreviated notation for $\bigwedge^k T^*M$. We denote the dual space of $\mathcal{D}^k(M)$, for $k \in \mathbb{N} \cup \{0\}$, by

$$\mathcal{D}_k(M) := \{k\text{-currents on } M\}.$$

We refer to the elements of $\mathcal{D}_k(M)$ as *k-currents*. For a k-current T, we define the *mass* of T to be

$$\mathbf{M}(T) := \sup\{T(\phi) : \|\phi\|_\infty \leq 1\} \in [0, \infty].$$

We will be mostly interested in 1-currents. A basic class of examples consists of 1-currents we shall write as T_λ whose action on $\phi \in \mathcal{D}^1(M)$ takes the form

$$T_\lambda(\phi) := \int_\lambda \phi, \quad \text{where } \lambda : (a, b) \to M \text{ is a Lipschitz curve.} \tag{2.13}$$

We will say a 1-current is *integer multiplicity rectifiable* if it can be written as a countable sum of currents $\{T_{\lambda_i}\}_{i=1}^\infty$ of the form (2.13) such that

$$\sum_{i=1}^\infty \mathbf{M}(T_{\lambda_i}) < \infty.$$

We will write

$$\mathcal{R}_1(M) := \{T \in \mathcal{D}_1(M) : \mathbf{M}(T) < \infty,\ T \text{ is integer multiplicity rectifiable}\}.$$

For a 1-current J, we write $\|J\|$ to denote the associated total variation measure, defined through its action on continuous, nonnegative functions $f : M \to \mathbb{R}$ via

$$\int f\, d\|J\| := \sup\{J(\phi) : \phi \in \mathcal{D}^1(M),\ |\phi|_g \leq f\}. \tag{2.14}$$

For a k-current S, the *boundary of S* is the $(k-1)$-current ∂S defined by

$$\partial S(\phi) := S(d\phi) \quad \text{for all } \phi \in \mathcal{D}^{k-1}(M).$$

We define

$$\mathcal{F}_1'(M) := \{T \in \mathcal{D}_1(M) : T = \partial S \text{ for some } S \in \mathcal{D}_2(M),\ \mathbf{M}(S) < \infty\}$$

and for $T \in \mathcal{F}_1'(M)$, we will write

$$\|T\|_{\mathcal{F}} := \inf\{\mathbf{M}(S) : T = \partial S\}.$$

We also define

$$\mathcal{R}_1'(M) := \mathcal{R}_1(M) \cap \mathcal{F}_1'(M).$$

We note that the 1-current T_γ associated with the geodesic γ via (2.13), in particular, bounds a finite mass 2-current; that is,

$$T_\gamma \in \mathcal{R}_1'(M), \tag{2.15}$$

in light of the assumption (2.3).

Lastly, we will at times wish to identify the Jacobian (i.e. vorticity) of a map $u \in H^1(M;\mathbb{C})$ with an element of $\mathcal{D}_1(M)$, which we denote $\star J(u)$, and which is defined through its action on 1-forms ϕ by

$$\star J(u)(\phi) = \int \phi \wedge J(u), \tag{2.16}$$

where $J(u) = du^{(1)} \wedge du^{(2)}$ for $u = u^{(1)} + iu^{(2)}$ where $u^{(1)}, u^{(2)}$ are real-valued.

2.3. *Gamma-limit of the Ginzburg–Landau functional*

Below we state the version we will need of standard Gamma-convergence results for the Ginzburg–Landau functional.

We first fix the notation $V = \mathcal{F}_1'(M)$, with the flat norm $\|v\|_V := \|v\|_{\mathcal{F}}$. We also define the functional

$$E_V(T) := \begin{cases} \mathbf{M}(T) & \text{if } T \in \mathcal{R}_1'(M), \\ +\infty & \text{if not.} \end{cases} \tag{2.17}$$

Thus E_V is an extension to V of the 'arclength functional' in the sense that if $\lambda : (a,b) \to M$ is an injective Lipschitz continuous curve and T_λ is the corresponding current, then $E_V(T_\lambda) =$ arclength of $\text{im}(\lambda)$.

The following result is deduced in [25, Theorem 5.1] from corresponding Euclidean results [1, 15].

Theorem 2.1. *Let (M, g) be a closed 3-dimensional Riemannian manifold.*

(1) *Let $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a sequence in $H^1(M;\mathbb{C})$. If there exists $C > 0$ such that $E_\varepsilon(u_\varepsilon) \le C$ for all $\varepsilon \in (0, \varepsilon_0)$, then $(\frac{1}{\pi} \star Ju_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is precompact in V, and any limit as $\varepsilon \to 0$ belongs to $\mathcal{R}_1'(M)$.*

(2) *Let* $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *be a sequence in* $H^1(M;\mathbb{C})$*. If* $T \in V$ *and* $\|\frac{1}{\pi} \star Ju_\varepsilon - T\|_{\mathcal{F}} \to 0$ *as* $\varepsilon \to 0$*, then* $\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E_V(T)$.

The above theorem does not include a Gamma-limit upper bound, that is, the construction, for an arbitrary $T \in V$, of a sequence $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ in $H^1(M;\mathbb{C})$ such that $\|\frac{1}{\pi} \star Ju_\varepsilon - T\|_{\mathcal{F}} \to 0$ and $\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \leq E_V(T)$. Although this is not established in full generality in [25], it is proved there for particular currents T needed for our arguments, as we discuss in Lemma 4.3 and Appendix A below.

The geodesic γ is a saddle point of the arclength with respect to smooth perturbations, as reflected in (2.12). For use in combination with Theorem 2.1, one needs to identify a sense in which the corresponding current T_γ is a saddle point of E_V. We defer a discussion of this and related issues to Section 4.

2.4. Varifolds

We briefly recall the definition of a rectifiable 1-varifold and introduce some notation that will be used later. After doing this we will introduce the definition of a general 1-varifold. We note that the general definition will only be used in the proof of Proposition 3.1. For general varifolds we will follow [3] with some terminology from [32].

For any 1-dimensional rectifiable set Σ, basic theory (see for example [32, Lemma 11.1]) shows that there exists a countable family $(\Lambda_j)_{j\in\mathbb{N}}$ of C^1 curves in M such that

$$\Sigma \subset N_0 \cup \bigcup_{j\in N} \Lambda_j \quad \text{and} \quad \mathcal{H}^1(N_0) = 0,$$

and every point in $\Sigma \setminus N_0$ is contained in exactly one Λ_j. We then define, for $x \in \Sigma \setminus N_0$,

$$\mathrm{ap}T_x\Sigma = T_x\Lambda_j \quad \text{for the unique } j \text{ such that } x \in \Lambda_j.$$

We will write $\tau_\Sigma(x)$ to denote a unit vector in $\mathrm{ap}T_x\Sigma$.

First, we recall that if $\mathcal{S}$ is a countably 1-rectifiable, $\mathcal{H}^1$-measurable subset of M and $\Theta : \mathcal{S} \to (0,\infty)$ is a locally $\mathcal{H}^1$-integrable function on $\mathcal{S}$ then we can use the pair $(\mathcal{S},\Theta)$ to form the measure $\mathcal{H}^1 \llcorner \Theta$, where we have extended Θ to be zero outside of $\mathcal{S}$. We refer to such a measure as a *rectifiable* 1-*varifold*. We also refer to the function Θ as the *multiplicity function* of this rectifiable 1-varifold and, at times, we will write $\Theta_\mathcal{S}$ to emphasize the association. We will also sometimes use the alternative notation $\Theta\mathcal{H}^1 \llcorner \mathcal{S}$ for $\mathcal{H}^1 \llcorner \Theta_\mathcal{S}$. If Θ happens to be integer-valued $\mathcal{H}^1$-almost everywhere then we will say this rectifiable varifold is of *integer multiplicity*. Finally, if there is a $\lambda > 0$ such that $\Theta \geq \lambda$ at $\mathcal{H}^1$-almost every point then we say that the rectifiable varifold has *density bounded below*. A particular example of an integer multiplicity rectifiable 1-varifold that we will be interested in will be integration over a countable collection of geodesics.

Next, for a smooth Riemannian manifold, M, we let PM be the bundle whose fibre P_aM at $a \in M$ consists of the lines through the origin in T_aM. If $x \in M$ and ξ is a unit vector in T_xM, we will sometimes abuse notation slightly and write (x,ξ) to denote the element of PM

$$(x,\xi) \sim \{s\xi : s \in \mathbb{R}\} \subset T_x M. \tag{2.18}$$

Thus (x,ξ) and $(x,-\xi)$ correspond to the same element of PM. Suppose that η is a smooth function on M. When representing points in PM as described above, a mapping such as $(x,\xi) \in PM \mapsto |\nabla_\xi \eta(x)|^2$ is well-defined as a function $PM \to \mathbb{R}$, since it is independent of the choice of sign for the unit vector ξ.

We let $\pi : PM \to M$ be the bundle projection. We refer to a measure $\mathcal{V} \in \mathcal{M}(PM)$ as a 1-*varifold*. Observe that to a rectifiable 1-varifold $V = \mathcal{H}^1 \llcorner \Theta_{\mathcal{S}}$, we may associate a 1-varifold $\mathcal{V}$ defined by

$$\mathcal{V}(A) := V(\{a \in M : \mathrm{ap}T_a\mathcal{S} \in A\}) = \int_{\{a\in\mathcal{S} : \mathrm{ap}T_a\mathcal{S}\in A\}} \Theta_{\mathcal{S}}(a)\, d\mathcal{H}^1. \tag{2.19}$$

Roughly speaking, the difference between a rectifiable 1-varifold and the associated general 1-varifold is that the latter explicitly records information about the approximate tangent spaces to the set $\mathcal{S}$ on which the former lives.

2.5. *Definitions: First variation, stationarity, Brakke flow*

For a rectifiable 1-varifold ν given by $\nu = \mathcal{H}^1 \llcorner \Theta_\Sigma$, where Σ is a 1-rectifiable set, the *first variation* of ν is a distribution, denoted $\delta\nu$, whose action on smooth vector fields X is defined by

$$\delta\nu(X) := \int_\Sigma (\tau_\Sigma(x), \nabla_{\tau_\Sigma(x)} X(x))_g\ \Theta(x)\, d\mathcal{H}^1. \tag{2.20}$$

(Note that since τ_Σ appears quadratically, the choice of unit vector in $\mathrm{ap}T_x\Sigma$ does not matter.) A 1-varifold ν of the given form is *stationary* if

$$\delta\nu = 0. \tag{2.21}$$

We remark that in light of (2.2), of course it follows from an integration by parts that one can associate a multiplicity-one stationary varifold with the geodesic γ. Properties of stationary varifolds will be recalled later as needed.

For simplicity, we discuss Brakke flows and related notions from geometric measure theory only in the case of 1-varifolds in the 3-dimensional manifold (M,g).

Let

$$\nu_*^t = \Theta_*(x,t)\mathcal{H}^1 \llcorner \Sigma_\nu^t, \quad t \geq 0, \tag{2.22}$$

be a family of rectifiable 1-varifolds in M. To say that $(\nu_*^t)_{t>0}$ is a *Brakke flow* means that for a.e. $t > 0$ there exists a ν_*^t-integrable vector field $H(\cdot,t)$ along Σ_ν^t (that is, $H(x,t) \in T_xM$ for ν_*^t-almost every $x \in \Sigma_\nu^t$) such that the following hold. First,

$$\delta\nu_*^t(X) = \int_M (X,H)_g\, d\nu_*^t = \int_{\Sigma_\nu^t} (X,H)_g\ \Theta_*(x,t)\, d\mathcal{H}^1$$

for all C^1 vector fields X. Second, for every $t > 0$ and every nonnegative $\chi \in C^2(M)$,

$$\limsup_{s\to t} \frac{\nu_*^t(\chi) - \nu_*^s(\chi)}{t-s} \leq \mathcal{B}(\nu^t,\chi), \tag{2.23}$$

where, if $\chi|H|^2 \in L^1(\nu_*^t)$, we have set

$$\mathcal{B}(\nu^t,\chi) = -\int_M \chi|H|^2\,d\nu_*^t + \int_M (\nabla\chi, P^\perp(H))_g\,d\nu_*^t,$$

where at a point $x \in \Sigma_\nu^t$ at which $T_x\Sigma_\nu^t$ exists, we write $P^\perp(\cdot)$ to denote orthogonal projection onto $(T_x\Sigma_\nu^t)^\perp \subset T_xM$, and we have set $\mathcal{B}(\nu^t,\chi) = -\infty$ otherwise.

For $(\nu_*^t)_{t>0}$ a Brakke flow in M of the form (2.22), it is an immediate consequence of (2.23) that

$$t \mapsto \nu_*^t(M) \text{ is nonincreasing.} \tag{2.24}$$

Another simple fact we will need is the following.

Lemma 2.2. *If there exist numbers* $0 \le a < b$ *such that*

$$t \mapsto \nu_*^t(M) \textit{ is constant for } a < t < b \tag{2.25}$$

then

$$\exists \text{ a stationary varifold } V_* \text{ in } M \text{ such that } \nu_*^t = V_* \text{ for all } a < t < b. \tag{2.26}$$

Proof. Clearly, if (2.25) holds, then by taking $\chi = 1$ in (2.23), we find that $H = 0$ a.e. in Σ_ν^t for every $t \in (a,b)$. It follows that ν_*^t is stationary for such t. It is also easy to see that $t \mapsto \nu_*^t$ is constant for $t \in (a,b)$. Indeed, given any nonnegative $\chi_1 \in C^2(M)$, choose $\chi_2 \in C^2(M)$ such that $\chi_1 + \chi_2$ is constant on M. Then it follows from (2.25) that

$$\int_M (\chi_1+\chi_2)\,d\nu_*^t = \int_M \chi_1\,d\nu_*^t + \int_M \chi_2\,d\nu_*^t = c\nu_*^t(M) \text{ is constant for } t \in (a,b).$$

On the other hand, since $H = 0$, it follows from (2.23) that

$$\int_M \chi_j\,d\nu_*^t \text{ is nonincreasing for } j = 1,2, \text{ for } t \in (a,b).$$

These together imply that $t \mapsto \int_M \chi_j\,d\nu_*^t$ is constant for $j = 1,2$. Since this holds for all nonnegative $\chi_1 \in C^2(M)$, it easily follows that ν_*^t does not depend on $t \in (a,b)$, proving (2.26). ∎

We will make heavy use of results from a paper of Allard and Almgren [3] on stationary 1-varifolds with positive density in a Riemannian manifold. Among other results, they prove that a stationary 1-varifold with density bounded away from 0 is supported on a finite or countable union of geodesic segments terminating at singular points. From these singular points multiple segments emanate, with a balance condition on the weighted sum, at each singular point, of the tangent vectors generating the geodesics that meet there. Other results from [3] will be cited as the need arises.

2.6. *Asymptotic analysis of the Ginzburg–Landau heat flow*

As a last preliminary, we state a recently established extension to the Riemannian setting of a theorem of Bethuel, Orlandi, and Smets [6], who built on prior work of a number of authors, including [4, 5, 14, 21].

The theorem quoted below is proved in [8].

Theorem 2.3. *Assume that (N, h) is a closed Riemannian manifold of dimension $n \geq 3$. Suppose we consider a sequence $(u_\varepsilon)_{\varepsilon \in (0,1]}$ where $u_\varepsilon : N \times [0, \infty) \to \mathbb{C}$ solves the Ginzburg–Landau heat flow*

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \quad \text{on } N \times (0, \infty)$$

with initial data $u_\varepsilon(x, 0) = u^0_\varepsilon(x)$. Assume that there exists $M_0 > 0$ such that

$$E_\varepsilon(u^0_\varepsilon) \leq M_0.$$

For every $t \geq 0$, let μ^t_ε be the measure on N defined by

$$\mu^t_\varepsilon(A) = \int_A \frac{e_\varepsilon(u_\varepsilon(\cdot, t))}{|\log \varepsilon|} \operatorname{vol} \quad \text{for every Borel } A \subset N.$$

Then after passing to a subsequence (still denoted simply by ε), there exist measures $\mu^t_* \in \mathcal{M}(N)$ for every $t > 0$ such that

$$\mu^t_\varepsilon \rightharpoonup \mu^t_* \quad \text{weakly as measures for every } t > 0.$$

Moreover, there exists a smooth function $\phi_* : N \times (0, \infty) \to \mathbb{R}$ solving the heat equation, a harmonic 1-form γ_* independent of t, and a family $(\nu^t_*)_{t>0}$ of measures on N, such that for every $t > 0$,

$$\mu^t_* = \tfrac{1}{2}|d\phi_*(x, t) + \gamma_*|^2 \operatorname{vol} + \nu^t_*$$

with ν^t_* taking the form

$$\nu^t_* = \Theta_*(x, t)\mathcal{H}^{n-2} \llcorner \Sigma^t_\nu \quad \text{for a.e. } t > 0 \tag{2.27}$$

where Σ^t_ν is an $(n-2)$-dimensional rectifiable subset of N and Θ_* is a bounded measurable function. In addition, there exists a function $\eta : (0, \infty) \to (0, \infty)$ such that

$$\Theta_*(x, t) = \lim_{r \to 0} \frac{\nu^t_*(B_r(x))}{\omega_n r^{n-2}} \geq \eta(t) \tag{2.28}$$

for $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma^t_\nu$, for a.e. $t > 0$. Finally, the family $(\nu^t_*)_{t>0}$ is a Brakke flow.

3. A nonexistence result for stationary 1-varifolds near a nondegenerate geodesic

The proof of our main result hinges crucially on showing there is no stationary varifold sitting over a 1-current that is near T_γ, the 1-current associated with the nondegenerate geodesic γ. While the nondegeneracy assumption (2.9) easily precludes the existence of another nearby smooth geodesic, it is the need to rule out proximity in the weaker sense of (3.2), (3.3) below and within the larger class of varifolds that makes the result below much more challenging to establish.

Proposition 3.1. *Let T_γ be the 1-current in M corresponding to integration over the nondegenerate geodesic γ, and let $\eta > 0$ be given. Then there exists $r_0 > 0$ depending on M, γ, and η such that for $0 < r < r_0$ there is no stationary rectifiable 1-varifold V_* and 1-current $J_1 \in \mathcal{R}_1(M) \cap \mathcal{F}_1'(M)$ satisfying the conditions*

$$V_* = \Theta_*(x)\mathcal{H}^1 \llcorner \Sigma_* \tag{3.1}$$

for Σ_* 1-rectifiable and $\Theta_* \geq \eta > 0$ $\mathcal{H}^1$-a.e. in Σ_*,

$$\|J_1 - T_\gamma\|_{\mathcal{F}} = r, \tag{3.2}$$

and

$$V_* \geq \|J_1\|, \quad V_*(M) \leq L. \tag{3.3}$$

Remark. If we knew that $\Theta_*(x) \geq 1$ for $\mathcal{H}^1$-a.e. $x \in \Sigma_*$, the proof of the proposition could be simplified significantly. Indeed, the main point of the proof is to show that Σ_* is smooth. If $\Theta_*(x) \geq 1$ a.e. in Σ_*, the Regularity Theorem of Allard [2, Section 8], adapted to the Riemannian setting and specialized to the 1-dimensional case, implies that there exist $R, \eta_1 > 0$ such that if $x \cap \Sigma_*$ and

$$V_*(B_R(x)) \leq (1 + \eta_1)2R \tag{3.4}$$

then $B_{R/2}(x) \cap \Sigma_*$ is a $C^{1,\alpha}$ graph and hence (since V_* is stationary) smooth. Using Lemma 3.2 below and for example an argument by contradiction, one can verify that (3.4) is satisfied at every $x \in \Sigma_*$ when r is sufficiently small. The same conclusion can be justified, with some more work, without invoking Lemma 3.2, allowing an extension to higher dimensions.

In our later arguments, we would be able to assume that $\Theta_* \geq 1$ a.e. if we knew that the constant $\eta(t)$ from Theorem 2.3 satisfies $\eta(t) = \pi$. This is expected to hold, and the corresponding estimate in the elliptic case has been established very recently in [30].

The starting point of the proof of Proposition 3.1 is provided by the following lemma, established by Mesaric [25].

Lemma 3.2. *For T_γ as above, let $J_1 \in \mathcal{R}_1(M) \cap \mathcal{F}_1'(M)$ be a current satisfying (3.2), and such that*

$$\partial J_1 = 0 \quad \text{and} \quad \mathbf{M}(J_1) \leq L.$$

Then provided r is taken sufficiently small, there is a 1-current $J_1^* \in \mathcal{R}_1(M)$ such that the support of J_1^*, denoted by Γ^*, consists of a single Lipschitz curve with no boundary satisfying

$$\Gamma^* \subset K_{2\sqrt[3]{r}} \cap \operatorname{spt}(J_1), \tag{3.5}$$

and

$$\Gamma^* \cap \tau^{-1}(t) \neq \varnothing \quad \text{for all } t \in \mathbb{R}/L\mathbb{Z} \tag{3.6}$$

(see (2.6) for the definition of τ).

In addition,

$$\mathbf{M}(J_1 - J_1^*) = \mathbf{M}(J_1) - \mathbf{M}(J_1^*), \tag{3.7}$$

there exists a constant $C_1 > 0$ *such that*

$$\mathbf{M}(J_1^*) \geq L - C_1 \sqrt[3]{r}, \tag{3.8}$$

and

$$\begin{aligned} &J_1^* - T_\gamma = \partial S^* \textit{ for some 2-current } S^* \textit{ with} \\ &\operatorname{spt}(S^*) \subset \overline{K_{5\sqrt[3]{r}}} \textit{ and } \mathbf{M}(S^*) < \infty. \end{aligned} \tag{3.9}$$

This is demonstrated in [25, Lemma 4.4 and comments following Lemma 4.6]. The proof is an adaptation to the Riemannian setting of arguments from [16, Lemma 5.5]. The idea is to use (3.2) and the definition of the flat norm to find a tubular neighbourhood K_s of $\Gamma = \mathrm{im}(\gamma)$ of radius $\sqrt[3]{r} < s < 2\sqrt[3]{r}$ and a 2-current S_1 which satisfies $\partial S_1 = J_1 - T_\gamma$ and for which the slice $\langle S_1, d_\Gamma, s\rangle$ is small in mass, where d_Γ denotes the distance function to Γ. Thus this slice is a 1-current supported in $K_s = d_\Gamma^{-1}(s)$. It is then shown that the 1-current $\widetilde{J}_1 := J_1 \llcorner K_s + \langle S_1, d_\Gamma, s\rangle$ has mass at most slightly larger than L, is supported in the tubular neighbourhood $\overline{K}_s$, and satisfies $\|\widetilde{J}_1 - T_\gamma\| \leq r$. The advantage in replacing J_1 by $\widetilde{J}_1$ is that it is now possible to slice normal to γ.

Next, it is shown by a slicing argument that most normal slices of $\widetilde{J}_1$ can only intersect $\widetilde{J}_1$ once and the intersection occurs within the smaller tubular neighbourhood $K_{\sqrt[3]{r}/4}$. Finally, appealing to Federer's decomposition of integral 1-currents [13, Theorem 4.2.25], we are able to find the desired current J_1^* and to prove that it is confined within $K_{2\sqrt[3]{r}}$. The idea of this last point is that its failure would entail a long excursion from $K_{\sqrt[3]{r}/4}$, which one can show would violate the mass bound $\mathbf{M}(J_1) \leq L$.

We remark that the lemma is not valid as stated for higher-dimensional currents.

Proof of Proposition 3.1

Step 1: First we show that the rectifiable varifold V_* does not have any mass outside of $K_{\sqrt[4]{r}}$. The idea is that if this fails, then the monotonicity formula and (3.8) would contradict the assumption $\mathbf{M}(V_*) \leq L$. This argument relies crucially on the uniform lower density bound for V_*.

We recall that the Hessian Comparison Theorem [17, Theorem 6.6.1] gives us that if $\mu > 0$ is an upper bound on the absolute value of the sectional curvature over M and $r > 0$ is chosen so that

$$r < \frac{1}{2} \min\left\{r_0, \frac{\pi}{2\sqrt{\mu}}\right\},$$

then for each $p \in M$ and all $x \in B_r(p)$ and $v \in T_x M$ we have

$$\sqrt{\mu}\,\rho(x)\cot(\sqrt{\mu}\,\rho(x))|v|^2 \leq \mathrm{Hess}_g\left(\frac{\rho^2(x)}{2}\right)(v,v) \leq \sqrt{\mu}\,\rho(x)\coth(\sqrt{\mu}\,\rho(x))|v|^2 \tag{3.10}$$

where $\rho(x) = d(x,p)$. In view of the lower density bound $\Theta_* \geq \eta$, V_*-almost everywhere, it follows from a Riemannian version of the Monotonicity Formula (established with different notation in [3, Theorem, (5), p. 87]) that there exists $r_{\mathrm{con}}(M) > 0$ such that for every $0 < s < r_{\mathrm{con}}$,

$$\eta \le \frac{1}{2s} \int_{B_s(p)} \mathrm{Hess}_g \left(\frac{\rho^2(x)}{2} \right)(v, v)\, dV_* \tag{3.11}$$

for V_*-almost every $p \in M$, where v is a unit vector in $\mathrm{ap}T_x\Sigma_*$. (Clearly the value of $\mathrm{Hess}_g(\rho^2(x)/2)(v, v)$ does not depend on which unit tangent is chosen.)

It follows from (3.10) that

$$2s\eta \le (1 + \mu s^2) V_*(B_s(p))$$

for all $0 < s < \frac{1}{2} \min\{r_0, r_{\mathrm{con}}, \frac{\pi}{2\sqrt{\mu}}\}$ and V_*-almost every $p \in M$. We conclude that for all $0 < s < \frac{1}{2} \min\{r_0, r_{\mathrm{con}}, \frac{\pi}{2\sqrt{\mu}}, 1\}$ we have

$$V_*(B_s(p)) \ge \frac{2s\eta}{1+\mu} \quad \text{for } V_*\text{-a.e. } p \in M. \tag{3.12}$$

We now use this to prove that if r is chosen sufficiently small, then

$$V_*(M \setminus K_{\sqrt[4]{r}}) = 0. \tag{3.13}$$

To verify (3.13), suppose to the contrary that $V_*(M \setminus K_{\sqrt[4]{r}}) > 0$ and so there is a point p of $\mathrm{spt}(V_*)$ in $M \setminus K_{\sqrt[4]{r}}$ for which (3.12) holds. By (3.12) we have

$$V_*(B_{\frac{1}{2}\sqrt[4]{r}}(p)) \ge \frac{\eta}{1+\mu} \sqrt[4]{r} \quad \text{if } r < \min\left\{ r_{\mathrm{con}}, r_0, \frac{\pi}{2\sqrt{\mu}}, 1 \right\}^4.$$

By shrinking r_0 if necessary, we may assume that $B_{\frac{1}{2}\sqrt[4]{r}}(p) \cap K_{2\sqrt[3]{r}} = \varnothing$. Hence, appealing to (3.8), we find that

$$L \ge V_*(M) \ge V_*(K_{2\sqrt[3]{r}}) + V_*(B_{\frac{1}{2}\sqrt[4]{r}}(p)) > L - C_1\sqrt[3]{r} + \frac{\eta}{1+\mu}\sqrt[4]{r}.$$

Choosing r smaller if necessary, depending on C_1, η, μ, yields a contradiction. We conclude that (3.13) holds. Since $\|J_1\| \le V_*$, we remark that J_1 is also supported in $K_{\sqrt[4]{r}}$.

Step 2: Next we demonstrate that

$$|\nabla\tau(x)| = 1 + O(\sqrt[4]{r}), \quad [\mathrm{Hess}_g(\tau)(x)](v, v) = O(\sqrt[4]{r}), \tag{3.14}$$

where $v \in T_x M$ is a unit vector, $x \in K_{\sqrt[4]{r}}$, and τ is the mapping defined in (2.6).

We prove only the statement about the Hessian, as the gradient estimate follows by similar arguments. In coordinates introduced by $\psi : B_{\sqrt[4]{r}}(0) \times (0, L) \to K_{\sqrt[4]{r}}$ as defined in (2.5), we can write τ as

$$\tilde{\tau}(y, t) = t$$

where $\tilde{\tau} = \tau \circ \psi$. These are what are called Fermi coordinates, and a basic fact, proved for example in [23, Section V], is that the vectors Ξ_1, Ξ_2 in (2.5) can be chosen so that all Christoffel symbols vanish along the central geodesic, that is, when $y = 0$,

$$\Gamma_{ij}^k(0,t) = 0 \quad \text{for } i,j,k = 1,2,3,$$

where $x^k = y^k$ for $k = 1,2$, and $x^3 = t$. In general, the expression for the Hessian in coordinates is

$$\mathrm{Hess}_g(\tau)(\psi(0,t)) = \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial^2\tilde{\tau}}{\partial x^i \partial x^j}(0,t) - \sum_{k=1}^{3}\Gamma_{ij}^k(0,t)\frac{\partial\tilde{\tau}}{\partial x^k}(0,t)\right)\mathrm{d}x^i \otimes \mathrm{d}x^j$$

(see for example [17, Definition 4.3.5]). By combining these, we readily deduce that

$$\mathrm{Hess}_g(\tau)(\psi(0,t)) = 0$$

and thus $\mathrm{Hess}_g(\tau)(\psi(y,t))_{ij} = O(|y|)$ for $1 \leq i,j \leq 3$. The Hessian estimate in (3.14) follows directly.

Step 3: Let $\mathcal{V}_*$ be the 1-varifold associated as in (2.19) to the rectifiable 1-varifold V_*. We next demonstrate that for each $\delta > 0$ there is $r_1 > 0$ such that if $0 < r < r_1$ in (3.2) and hence in (3.13), then

$$\mathcal{V}_*\big(\{(x,\xi) \in PM : |\nabla_\xi \tau(x)|^2 \leq (1-\delta)^2\}\big) < \delta, \tag{3.15}$$

where we recall our convention that a generic element of PM – that is, a line in T_xM for some $x \in M$ – is represented by a pair (x,ξ), where ξ is a unit vector in T_xM spanning the given line (see (2.18)). This will establish that most tangent vectors to the support of V_* are, according to the measure $\mathcal{V}_*$, nearly parallel to $\nabla\tau$.

We suppose toward a contradiction that there is a $\delta > 0$, a sequence $(r_k)_{k\in\mathbb{N}}$ tending to 0 from the right, and a sequence $(V_k)_{k\in\mathbb{N}}$ of stationary rectifiable varifolds on M satisfying the hypotheses of Proposition 3.1 with r replaced by r_k in (3.2), and such that the associated 1-varifolds $\mathcal{V}_k$ satisfy

$$\mathcal{V}_k\big(\{(x,\xi) \in PM : |\nabla_\xi \tau(x)|^2 \leq (1-\delta)^2\}\big) \geq \delta \tag{3.16}$$

for all $k \in \mathbb{N}$. In particular, we have

$$V_k(M) \leq L, \quad \mathrm{spt}(V_k) \subset K_{\sqrt[4]{r_k}}, \quad \Theta_{V_k}(x) \geq \eta \quad \text{for } x \in \mathrm{spt}(V_k). \tag{3.17}$$

Since $(\mathcal{V}_k)_{k\in\mathbb{N}}$ is a sequence of stationary rectifiable varifolds, we may combine (3.17) with the compactness result of [32, Theorem 42.7, p. 247] to conclude that there is a subsequence $(\mathcal{V}_{k_j})_{j\in\mathbb{N}}$ and a rectifiable varifold V with associated multiplicity Θ_V such that

(1) $\mathcal{V}_{k_j} \rightharpoonup \mathcal{V}$ weakly as measures,

(2) $\Theta_V(x) \geq \eta$ on $\mathrm{spt}(V)$,

(3) $\|\delta V\|(W) \leq \liminf_{j\to\infty}\|\delta V_{k_j}\|(W)$ for all $W \subset\subset M$.

It follows from (3) and the fact that each $\mathcal{V}_{k_j}$ is stationary that $\mathcal{V}$ is also stationary. Then from (1) and the fact that $\mathrm{spt}(V_{k_j}) \subset K_{\sqrt[4]{r_{k_j}}}$ we conclude that $\mathrm{spt}(V) \subset \Gamma$. Next, we observe that, due to (3.6), each V_{k_j} has support that contains a closed curve that meets every level set of τ. Hence, $\mathrm{spt}(V) = \Gamma$ as a result of (1). Observe that since $\mathcal{V}$ is a stationary varifold with density bounded below and $\mathrm{spt}(V) = \Gamma$, by [3, Theorem, p. 89] we know that V is simply a constant multiplicity multiple of the stationary varifold V_Γ associated with the geodesic γ. Applying the weak convergence (1) to (3.16), however, we see that

$$\mathcal{V}\big(\{(x,\xi) \in PM : |\nabla_\xi \tau(x)|^2 \le (1-\delta)^2\}\big) \ge \delta,$$

an impossibility given that all tangent vectors ξ along Γ coincide with $\pm\nabla\tau$.

Step 4: Next we introduce three sets corresponding to slices normal to the central geodesic that are in some sense bad. We will argue that two of them correspond to sets of t-values of measure zero while the third is of small measure.

We introduce the first such set, $\mathcal{B}_1$, through the function $h : (0, L) \to \mathbb{R}$ given by

$$h(t) = \int_{\pi^{-1}(\{0\le\tau\le t\}\cap K_{\sqrt[4]{r}})} |\nabla_\xi \tau(x)|^2 \, d\mathcal{V}_*(x,\xi),$$

with $\mathcal{B}_1$ defined by

$$\mathcal{B}_1 := \{a \in (0, L) : h \text{ is not differentiable at } a\}. \tag{3.18}$$

Since h is nondecreasing, it is differentiable $\mathcal{L}^1$-almost everywhere and consequently $\mathcal{L}^1(\mathcal{B}_1) = 0$.

Now we recall that the singular set S_{V_*}, as defined in [3], is the set of points of M near which Θ_{V_*}, restricted to $\mathrm{spt}(V_*)$, is not constant. Then we introduce the set $\mathcal{B}_2$ as the set of slices meeting the singular set:

$$\mathcal{B}_2 := \big\{t \in (0, L) : \{\tau = t\} \cap S_{V_*} \ne \varnothing\big\} = \{\tau(x) : x \in S_{V_*} \cap K_{\sqrt[4]{r}}\}.$$

We claim that $\mathcal{L}^1(\mathcal{B}_2) = 0$ as well.

To see this, we note that in [3, Remark, p. 89], it is stated that

$$V_*(S_{V_*}) = 0, \tag{3.19}$$

and so by (2.28) and (3.19) we have

$$0 = \int_{S_{V_*}} \Theta(x) \, d\mathcal{H}^1(x) \ge \eta \mathcal{H}^1(S_{V_*}).$$

We conclude that

$$\mathcal{H}^1(S_{V_*}) = 0. \tag{3.20}$$

Since $\mathcal{B}_2$ is the image of a subset of S_{V_*} by the Lipschitz map $\tau : K_{\sqrt[4]{r}} \to (0, L)$, it follows that $\mathcal{L}^1(\mathcal{B}_2) = 0$ as claimed.

The final 'bad' set of slices is $\mathcal{B}_3$ defined by

$$\mathcal{B}_3 := \{t \in (0, L) : \exists (x, \xi) \in \operatorname{spt}(\mathcal{V}_*) \text{ s.t. } |\nabla_\xi \tau(x)|^2 < (1-\delta)^2,\ \tau(x) = t\}.$$

Replacing the role of the equality (3.19) by the inequality (3.15) in the argument above, the same line of reasoning shows that there is a constant $C_2 > 0$ such that

$$\mathcal{L}^1(\mathcal{B}_3) < C_2 \delta/\eta,$$

where r is chosen sufficiently small.

Step 5: We now use the results obtained in Steps 1 and 2 to show that, for r and δ chosen sufficiently small, and $a, b \in (0, L) \setminus \mathcal{B}_1$, we have

$$h'(b) = h'(a) + O(\sqrt[4]{r}),$$

where h is as defined in Step 4.

For s small and positive, we define the function

$$H_{a,b;s}(t) = \begin{cases} 1 & \text{if } a + s < t < b - s, \\ 0 & \text{if } 0 < t < a \text{ or } b < t < L, \\ \frac{t-a}{s} & \text{if } a \le t \le a + s, \\ \frac{b-t}{s} & \text{if } b - s \le t \le b, \end{cases}$$

and we let X be a smooth vector field on M such that $X(x) = H_{a,b;s}(\tau(x))\nabla\tau(x)$ for $x \in K_{r_0/2}$. The fact that $\mathcal{V}_*$ is stationary implies that $\delta V_*(X) = 0$ (see (2.20)), and this means that

$$\int_{\pi^{-1}(K_{\sqrt[4]{r}})} H'_{a,b;s}(\tau)|\nabla_\xi \tau|^2 \, d\mathcal{V}_*(x,\xi) + \int_{\pi^{-1}(K_{\sqrt[4]{r}})} H_{a,b;s}(\tau) \operatorname{Hess}_g(\tau)(\xi,\xi) \, d\mathcal{V}_*(x,\xi) = 0. \tag{3.21}$$

We have used Step 1 to find that V_* is concentrated on $K_{\sqrt[4]{r}}$. Next, we use Step 2, $V_*(M) \le L$, and the fact that $\|H_{a,b;s}\|_{L^\infty(\mathbb{R})} = 1$ to conclude that

$$\int_{\pi^{-1}(K_{\sqrt[4]{r}})} H_{a,b;s}(\tau) \operatorname{Hess}(\tau)(\xi,\xi) \, d\mathcal{V}_*(x,\xi) = O(\sqrt[4]{r}). \tag{3.22}$$

Then we observe that, by the definition of $H_{a,b;s}$, we have

$$\int_{\pi^{-1}(K_{\sqrt[4]{r}})} H'_{a,b;s}|\nabla_\xi \tau|^2 \, d\mathcal{V}_*(x,\xi) = \frac{1}{s}\int_{\pi^{-1}(\{a \le \tau \le a+s\} \cap K_{\sqrt[4]{r}})} |\nabla_\xi \tau|^2 \, d\mathcal{V}_*(x,\xi) - \frac{1}{s}\int_{\pi^{-1}(\{b-s \le \tau \le b\} \cap K_{\sqrt[4]{r}})} |\nabla_\xi \tau|^2 \, d\mathcal{V}_*(x,\xi).$$

Combining this with (3.21) and (3.22) yields

$$\frac{h(b)-h(b-s)}{s}=\frac{h(a+s)-h(a)}{s}+O(\sqrt[4]{r}).$$

Letting s tend to zero, we find that

$$h'(b)=h'(a)+O(\sqrt[4]{r}). \tag{3.23}$$

Step 6: We next introduce a set of 'good' slices via

$$\mathcal{G}:=\big\{a\in(0,L):\mathcal{H}^0\big(\{\tau=a\}\cap\operatorname{spt}(V_*)\big)=1\big\},$$

and in this step we will demonstrate that

$$(0,L)\setminus(\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3)\subset\mathcal{G}. \tag{3.24}$$

From this and Step 4 it will follow that

$$\mathcal{L}^1(\mathcal{G})\geq L-C_2\delta/\eta \tag{3.25}$$

provided that r and δ are chosen sufficiently small.

Suppose by way of contradiction that there exists a value a such that

$$a\in(0,L)\setminus(\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{G}). \tag{3.26}$$

Then in light of (3.6) we have

$$\mathcal{H}^0(\{\tau=a\}\cap\operatorname{spt}(V_*))\geq 2$$

for some $a\in(0,L)\setminus(\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3)$. We first argue that for such an a and all $c,\delta\in(0,1)$ we necessarily have

$$h'(a)\geq(1-c)(1-\delta)^2(1+\eta) \tag{3.27}$$

provided r is chosen sufficiently small, depending on c and δ. Fix $0<c,\delta<1$. Note that since $a\notin\mathcal{B}_2$, by [3, Theorem, (3), p. 89] we see that $\{\tau=a\}\cap\operatorname{spt}(V_*)$ consists only of interior points of the constituent geodesics (or 'intervals' as they are referred to in [3]) making up $\mathcal{V}_*$. Moreover, the endpoints of these geodesics cannot accumulate at a. We conclude by compactness of $\operatorname{spt}(V_*)$ that $\{\tau=a\}\cap\operatorname{spt}(V_*)$ can intersect only finitely many of these constituents of $\operatorname{spt}(V_*)$. Also since $a\notin\mathcal{B}_3$, it follows that this slice must intersect $\operatorname{spt}(V_*)$ transversally, so that $\{\tau=a\}\cap\operatorname{spt}(V_*)$ consists of finitely many points, say $x_1,\dots,x_K$, where $K\geq 2$ by the choice of a.

It follows from the gradient estimate $|\nabla\tau|=1+O(\sqrt[4]{r})$ in $K_{\sqrt[4]{r}}$, established in (3.14), that for $0<t_1<t_2<L$, the geodesic distance between the sets $\{\tau=t_1\}\cap K_{\sqrt[4]{r}}$ and $\{\tau=t_2\}\cap K_{\sqrt[4]{r}}$ is at least $(t_2-t_1)/(1+O(\sqrt[4]{r}))$ if $t_2-t_1<L/2$. Hence if r is small enough, then

$$B_{(1-c)s}(x_i)\cap K_{\sqrt[4]{r}}\subset\{x\in K_{\sqrt[4]{r}}:a-s<\tau(x)<a+s\}.$$

(Here and below, we tacitly assume that $0<a-s<a+s<L$ and $s<L/4$.)

Next we again use the fact that $a \notin \mathcal{B}_3$ to choose $s_0 > 0$ small enough so that

$$|\nabla_\xi \tau(y)|^2 \geq (1 - \alpha\delta)^2 \tag{3.28}$$

if $y \in \bigcup_{i=1}^K B_{s_0}(x_i)$ and $(y, \xi) \in \operatorname{spt}(\mathcal{V}_*)$. Combining these facts, for each $0 < s \leq s_0$ we estimate

$$\frac{h(a+s) - h(a-s)}{2s} \geq (1 - \alpha\delta)^2 \sum_{i=1}^K \frac{V_*(B_{(1-c)s}(x_i))}{2s}.$$

We now fix $1 < \alpha < 1/\delta$, apply [3, Theorem 1 (5), p. 87] and let $s \to 0^+$, using the differentiability of h at a guaranteed by the assumption $a \notin \mathcal{B}_1$, to find

$$h'(a) \geq (1-c)(1-\alpha\delta)^2 \sum_{i=1}^K \Theta_*(x_i) \geq (1-c)(1-\alpha\delta)^2(1+\eta).$$

Here we have used Lemma 3.2 to assert that Γ^* intersects each level set of τ with $\Theta_*(x) \geq 1$ for $x \in \Gamma^*$ by (3.3), and that $\Theta_* \geq \eta$ in general by (5.12). Since $\alpha > 1$ was arbitrary we may let $\alpha \to 1^+$ to obtain (3.27).

In light of (3.23), it then follows from (3.27) that for any $b \notin \mathcal{B}_1$ we obtain

$$h'(b) \geq (1-c)(1-\delta)^2(1+\eta) + O(\sqrt[4]{r}).$$

Thus, choosing c, δ, and then r sufficiently small and recalling that $\mathcal{L}^1(\mathcal{B}_1) = 0$, we deduce that

$$h'(b) \geq 1 + \eta/2 \quad \text{for a.e. } b \in (0, L). \tag{3.29}$$

Thus, if there were a value $a \in (0, L)$ satisfying (3.26), then

$$\begin{aligned}\left[1 + C_3\sqrt[4]{r}\right]L &\geq \left[1 + C_3\sqrt[4]{r}\right]V_*(\pi^{-1}\{0 \leq \tau \leq L\}) \\ &\geq h(L) - h(0) = \int_0^L h'(s)\,ds \geq (1 + \eta/2)L.\end{aligned}$$

Here we use (3.14) in the second inequality, and the constant C_3 depends only on M and Γ. If we choose r sufficiently small, the contradiction is reached, establishing (3.24) and (3.25).

Step 7: In this step we show that

$$S_{V_*} = \varnothing. \tag{3.30}$$

According to [3], this asserts that V_* can be identified with a disjoint union of closed geodesics, each with constant weight, and hence that the Lipschitz curve Γ^* guaranteed by Lemma 3.2 is in fact a closed geodesic.

Crucial use in this step will be made of the following general property of stationary 1-varifolds (cf. [3, p. 88]):

Every point $p \in M$ is contained in an open set U_p such that if V is any stationary varifold on M with support in U_p, and if the support of δV consists of exactly two points, then V is the varifold corresponding to a constant multiple of the geodesic joining these two points.

This result is proved in [3] for possibly noncompact manifolds. Since M is compact, we may invoke the Lebesgue Number Lemma to conclude that there exists $\lambda > 0$ such that for any $p \in M$, the geodesic ball $B_\lambda(p)$ has the stated property.

For any $A \subset (0, L)$, we will write

$$K_{\sqrt[4]{r}}(A) := \{x \in K_{\sqrt[4]{r}} : \tau(x) \in A\} = \{\psi(y,t) : |y| < \sqrt[4]{r},\ t \in A\}.$$

By extending ψ to be periodic with respect to the t variable in the natural way, we can define $K_{\sqrt[4]{r}}(A)$ for any $A \subset \mathbb{R}$.

By shrinking r and δ, we may arrange that if I is any interval of length at most $2C_2\delta/\eta$, where C_2 is the constant appearing in (3.25), then $K_{\sqrt[4]{r}}(I)$ is contained in a ball of radius λ.

We will prove the claim by showing that

$$\text{for any } t \in [0, L), \quad K_{\sqrt[4]{r}}(\{t\}) \cap S_{V_*} = \varnothing.$$

Indeed, for any t we can appeal to (3.25) to find some $s_1, s_2 \in \mathcal{G}$ such that

$$s_1 < t < s_2, \quad s_2 - s_1 < 2C_2\delta/\eta.$$

We now apply the result stated at the outset of this step to the varifold

$$\tilde{V} = V_* \llcorner K_{\sqrt[4]{r}}([s_1, s_1])$$

in an open ball $B_\lambda(p)$ that contains $K_{\sqrt[4]{r}}([s_1, s_1])$. The definition of $\mathcal{G}$ implies that V_* intersects $\{\tau = s_j\}$ in exactly one point, say x_j, and that $\delta\tilde{V}$ is supported in $\{x_1, x_2\}$. Hence, this restriction of V_* consists of a multiple of the geodesic joining these two points. This immediately implies (3.30).

As noted above, it follows that Γ^* is a closed geodesic. By perhaps shrinking r one more time and applying the Morse–Palais Lemma [29, p. 307] we may conclude that the central geodesic Γ, being a nondegenerate critical point of length, is isolated and so necessarily $\Gamma^* = \Gamma$. Then the 1-current J_1^* from Lemma 3.2 satisfies $\mathbf{M}(J_1^*) \geq \mathcal{H}^1(\Gamma^*) = \mathcal{H}^1(\Gamma) = L$. Then (3.3) and (3.7) imply that $J_1 = J_1^* = T_\gamma$. However, this contradicts (5.13) since $r > 0$, and the proof of Proposition 3.1 complete. ∎

4. Finding good trajectories

The critical points of Ginzburg–Landau that we seek will be obtained as limits of certain carefully chosen trajectories of the Ginzburg–Landau heat flow. In this section we identify these trajectories.

Proposition 4.1. *Given $\delta > 0$, there exists $\varepsilon_0(\delta) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $\tau > 0$, there is a solution $u_\varepsilon = u_\varepsilon(x,t;\delta,\tau)$ of the Ginzburg–Landau heat flow*

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon \quad \textit{in } M \times (0,\infty), \tag{4.1}$$
$$u_\varepsilon(x,0) = u_\varepsilon^0(x) \quad \textit{for } x \in M,$$

such that $|u_\varepsilon(x,t)| \le 1$ *for every* $(x,t) \in M \times [0,\infty)$, *and*

$$\left\| \frac{1}{\pi} \star J u_\varepsilon^0 - T_\gamma \right\|_{\mathcal{F}} < \delta, \quad E_\varepsilon(u_\varepsilon^0) \le L + \delta, \quad E_\varepsilon(u_\varepsilon(\cdot,t)) \ge L - \delta \text{ for all } t \in [0,\tau]. \tag{4.2}$$

The proposition follows from small modifications of the asymptotic minmax theory developed in [16, 25]. Indeed, in the end the proof amounts to this:

Short proof of Proposition 4.1. This follows from arguments in [16, 25]. ∎

In the remainder of this section we expand on this, aiming to provide enough detail to convey the main ideas, to explain where we depart from [16, 25], and to make it possible, in principle, to check the terse proof given above.

The main difference between [16, 25] and our present treatment is that those earlier works use a pseudo-gradient flow for the energy E_ε, whereas we employ a small modification of the Ginzburg–Landau heat flow (4.1) for similar purposes. The use of (4.1) is necessary for our approach, due to our reliance in Section 5 below on Theorem 2.3.

4.1. Saddle point property of E_V

Our assumptions about γ imply, roughly speaking, that the 'arclength functional' has a local minmax geometry near γ, as reflected in (2.12), with respect to smooth perturbations. In particular, there is an ℓ-parameter family of arclength-decreasing perturbations of γ, and arclength increases for sufficiently transverse smooth perturbations. Here and below, ℓ is the index of γ (see (2.10)).

The result below states that the 'generalized arclength functional' E_V defined in (2.17) has a saddle point, in a suitable weak sense, at the current $T_\gamma \in V$ corresponding to γ, where $V = \mathcal{F}_1'(M)$ is the space of integer multiplicity rectifiable currents with finite mass and which are the boundary of another current of finite mass. The relevant notion of saddle point was first introduced in [16].

Lemma 4.2 (cf. [25, Theorem 4.1]). *For the geodesic* γ *satisfying* (2.9) *and* (2.15), *the associated current* T_γ *is a saddle point of* E_V *in the sense that there exist* $R, \delta_0 > 0$ *and continuous functions*

$$P_{WV} : V \to \mathbb{R}^\ell, \quad Q_{VW} : W \to V$$

for

$$W = B_R^\ell = \{w \in \mathbb{R}^\ell : |w| < R\} \tag{4.3}$$

such that $P_{WV}(T_\gamma) = 0$, *and the following conditions are satisfied:*

$$E_V(T_\gamma) < E_V(T) \quad \text{for } T \in V \text{ with } 0 < \|T - T_\gamma\|_{\mathcal{F}} \le \delta_0,\ P_{WV}(T) = 0, \tag{4.4}$$

$$Q_{VW}(0) = T_\gamma, \tag{4.5}$$

$$P_{WV} \circ Q_{VW}(w) = w \quad \text{for all } w \in W, \tag{4.6}$$

$$\sup_{\{w \in W : |w| \ge r\}} E_V(Q_{VW}(w)) < E_V(T_\gamma) \quad \text{for every } r > 0. \tag{4.7}$$

We sketch the proof from [25], although we note that this will not play any role in what follows, except that the notation γ_w for the curve defined in (4.9) and T_{γ_w} for the associated 1-current (see (2.13)) will be used below.

To start, for $w \in W$ we define

$$\xi(w) = w_1\xi_1 + \cdots + w_\ell\xi_\ell, \tag{4.8}$$

where ξ_j denote eigenfunctions of the Jacobi operator (see in particular (2.10)). We then define the curve γ_w via

$$\gamma_w(t) := \exp_{\gamma(t)} \xi(w)(t) \tag{4.9}$$

(see (2.7)). We will always assume that R is small enough that

$$\|\xi(w)\|_{W^{1,\infty}} \leq r_0, \quad \text{and hence } \xi(w) \in H_-(r_0), \quad \text{for } w \in W.$$

With this in hand, we define $Q_{VW} : B^\ell_R \to \mathcal{R}'_1$ via

$$Q_{VW}(w) := T_{\gamma_w} \quad \text{for } w \in W.$$

Then (4.5) is immediate, and (4.7) follows directly from (2.12).

The construction of P_{WV} is carried out in [25, Lemma 4.3] by designing an $\mathbb{R}^\ell$-valued 1-form Φ such that

$$T_{\gamma_\xi}(\Phi) = ((\xi,\xi_1)_{L^2}, \ldots, (\xi,\xi_\ell)_{L^2}) \tag{4.10}$$

for $\xi \in L^2(N\Gamma) \cap \text{Lip}$ as long as $\|\xi\|_{L^\infty} \leq r_0$, a condition that can be guaranteed by a suitable choice of R. Here we recall that γ_ξ is the curve given by (2.7) and T_{γ_ξ} is its associated 1-current. We then simply define $P_{WV}(T) = T(\Phi)$. With this choice, (4.6) follows directly from (4.8)–(4.10).

The hard part of the proof of Lemma 4.2 is the verification of (4.4). This is carried out in [25, Proposition 4.1] via a 'selection principle' argument inspired by a 1994 paper [34] of White. One might hope to show show that T_γ minimizes

$$T \mapsto E^*_V(T) := E_V(T) + C|P_{WV}(T)|^2,$$

for a suitable C, among all currents T such that $\|T - T_\gamma\|_{\mathcal{F}} \leq \delta_0$. This is formally clear from (2.12) and (4.10) when $T = T_{\gamma_\xi}$ for some ξ with small C^1 norm. To reduce to this situation, Mesaric first uses a result [25, Lemma 4.4] we have stated above as Lemma 3.2. This enables him to minimize E^*_V among currents T that are homologous to T_γ and with support constrained to lie in a small tubular neighbourhood of Γ, rather than among currents close to T_γ in the flat norm. One then uses the fact that the constrained minimizer is an almost-minimizer of mass, to which standard regularity theory applies, to obtain the needed $C^{1,\gamma}$ estimates.

Many points in the proof, including the construction of Φ with the property (4.10), are similar to elements in the proof of [34, Theorem 5]. Many more sophisticated versions of arguments in the same spirit have been developed in recent years by various authors; see for example [7, 10].

Note that we may shrink at will the parameter R in the definition (4.3) of W, and the conclusions of the lemma remain valid.

4.2. An ℓ-parameter family of solutions of (4.1)

To prove Proposition 4.1, we will define an ℓ-parameter family of solutions of (4.1) for every sufficiently small $\varepsilon > 0$. In the final step, given ε and τ (where ultimately we will take $\varepsilon < \varepsilon_0(\delta)$), we will choose from this family one solution u_ε such that $E_\varepsilon(u_\varepsilon(\cdot,t)) \geq L - o(1)$ for all times $t \in [0, \tau]$.

The initial data for this family of solutions is provided by the following result.

Lemma 4.3. *There exist $R, \varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ and $w \in B^\ell_R$, there exists a function $U^{\varepsilon,0}_w \in H^2(M)$ satisfying the conditions:*

(1) *$w \mapsto U^{\varepsilon,0}_w$ is Lipschitz continuous from B^ℓ_R into $H^2(M)$,*

(2) *$\|U^{\varepsilon,0}_w\|_{L^\infty} \leq 1$ and $\|U^{\varepsilon,0}_w\|_{H^2} \leq C_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_1)$ and $w \in B^\ell_R$,*

(3) *$E_\varepsilon(U^{\varepsilon,0}_w) \leq L - c_0|w|^2 + o(1)$ as $\varepsilon \to 0$,*

(4) $\|\frac{1}{\pi} \star J U^{\varepsilon,0}_w - T_{\gamma_w}\|_V \to 0$

uniformly for $w \in B^\ell_R$.

For fixed w, in view of (2.12) and the construction of γ_w, conclusions (3) and (4) hold if $U^{\varepsilon,0}_w$ is a recovery sequence for the current T_{γ_w}, the Gamma-limit in Theorem 2.1. Such constructions are rather standard. It is easy to arrange that $\|U^{\varepsilon,0}_w\|_{L^\infty} \leq 1$. The only points requiring attention are that the construction has to be carried out so that it depends continuously on w, in the H^1 norm, and with some control over the H^2 norm. The former point is carried out in [25], and the latter can be achieved by a small modification of the construction of [25]. We defer a more detailed discussion to Appendix A.

The H^2 estimate facilitates the proof of Lemma 4.4 below, whose need arises because we require the Ginzburg–Landau heat flow rather than the pseudo-gradient flows employed in [16,25].

Having constructed appropriate initial data for the Ginzburg–Landau flow, we are now ready to define the flow that we will use in our arguments below.

Lemma 4.4. *For $\varepsilon \in (0, \varepsilon_1)$ and $w \in W$, let $U^{\varepsilon,1}_w(x, t)$ solve the Ginzburg–Landau heat flow with initial data $U^{\varepsilon,0}_w$. Then*

$$[0, \infty) \times W \ni (t, w) \mapsto U^{\varepsilon,1}_w(\cdot, t) \in H^1(M; \mathbb{C}) \quad \textit{is continuous}.$$

See Appendix A for the proof, which involves rather standard parabolic estimates.

Finally, we define $U_\varepsilon : [0, \infty) \times W \to H^1(M; \mathbb{C})$ by

$$U_\varepsilon(t, w) := U^{\varepsilon,1}_w(\cdot, \chi(w)t) \tag{4.11}$$

for a smooth, compactly supported $\chi : B^\ell_R \to [0, 1]$ such that $\chi = 1$ in $B^\ell_{R/2}$. We point out that

$$\text{if } |w| = R, \quad \text{then} \quad U_\varepsilon(t, w) = U^{\varepsilon,0}_w \quad \text{for all } t \geq 0. \tag{4.12}$$

For c_ε as in Lemma 4.5 below we can guarantee that $E^\varepsilon_U(U_\varepsilon(0, w)) < c_\varepsilon - \zeta$ whenever $|w| > R/2$ and ε is small enough, for suitable $\zeta > 0$.

4.3. Choosing a good trajectory

We finally make use of the asymptotic saddle point geometry of E_ε, inherited from E_V via the Gamma-convergence Theorem 2.1, to complete the proof of Proposition 4.1.

We will use the notation

$$P_{WU}(u) = P_{WV}\left(\frac{1}{\pi} \star Ju\right) \quad \text{for } u \in H^1(M;\mathbb{C}).$$

Lemma 4.5. *If $\delta_0 > 0$ is as in Lemma 4.2 then there exists $R_0 > 0$ such that for every $R \in (0, R_0)$, there is some $\zeta = \zeta(R) > 0$ such that if we define*

$$\begin{aligned} a_\varepsilon &:= \max\{E_\varepsilon(U_w^{\varepsilon,0}) : |w| = R\}, \\ c_\varepsilon &:= \min\left\{E_\varepsilon(u) : P_{WU}(u) = 0, \left\|\frac{1}{\pi} \star Ju - T_\gamma\right\|_V \le \delta_0\right\}, \\ d_\varepsilon &:= \max\{E_\varepsilon(U_w^{\varepsilon,0}) : |w| \le R\}, \end{aligned}$$

then

$$a_\varepsilon \to L - 2\zeta, \quad c_\varepsilon \to L, \quad d_\varepsilon \to L \tag{4.13}$$

as $\varepsilon \to 0$, where L is the length of the geodesic γ.

The assertions about a_ε and d_ε follow directly from (2.12) and Lemma 4.3, and Step 3 of the proof of [16, Theorem 4.4] shows exactly that $c_\varepsilon \to L$. The proof uses only ingredients that we have collected in Theorem 2.1 and Lemma 4.2.

Below we will not refer explicitly to the assertion about $\lim_{\varepsilon\to 0} a_\varepsilon$, but it plays a role in the proof of Lemma 4.6, and together with the lower bound for $\liminf c_\varepsilon$, it reflects the asymptotic minmax geometry of E_ε.

Proposition 4.1 will essentially follow from the next fact.

Lemma 4.6. *For each $r > 0$ there exist $\varepsilon_0 > 0$ and $R > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and every $\tau > 0$, there exists $w = w(\varepsilon, \tau)$ such that*

$$P_{WU}(U_\varepsilon(\tau, w)) = 0, \qquad \left\|\frac{1}{\pi} \star JU_\varepsilon(\tau, w) - J_\gamma\right\|_V \le r. \tag{4.14}$$

As a result, $w = w(\varepsilon, \tau)$ satisfies

$$d_\varepsilon \ge E_\varepsilon(U_\varepsilon(t, w)) \ge E_\varepsilon(U_\varepsilon(\tau, w)) \ge c_\varepsilon \quad \text{for all } t \in [0, \tau] \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{4.15}$$

Finally, $w(\varepsilon, \tau) \to 0$ as $\varepsilon \to 0$.

Proof. One may prove (4.14) by simply repeating the arguments from Steps 5–8 of the proof of [16, Theorem 4.4], for $r > 0$ such that $0 < r < \delta_0$, where δ_0 is the constant from (4.4) of Lemma 4.2. Some comments are in order:

First, the argument in [16] is stated for a pseudo-gradient flow (see [16, Lemma 4.8]) with certain properties that our flow $(t, w) \mapsto U_\varepsilon(t, w)$ does not possess. These are in fact not needed for the proof of (4.14), and some of them may appear in [16] only because there Lemma 4.8 is quoted directly from a standard text, which provides more than is actually needed. These are the only properties of the flow that are required for the proof of (4.14):

- $t \mapsto E_\varepsilon(U_\varepsilon(t,w))$ is nonincreasing,
- $t \mapsto U_\varepsilon(t,w)$ is constant for $w \in \partial W$ (see (4.12)),
- continuity properties of the flow, as summarized in Lemma 4.4.

All of these are available here.

Without going into detail, we remark that the basic strategy of the proof is to apply degree theory arguments to the maps $w \mapsto P_{WU}(U_\varepsilon(t,w)) : W \to \mathbb{R}^\ell$ as t varies from 0 to τ (where $\tau = 1$ in [16], a harmless normalization).

Next, (4.15) follows directly from (4.14) and Lemma 4.5. Finally, we deduce from (4.15) and Lemma 4.5 that

$$E_\varepsilon(U_\varepsilon(0,w)) = E_\varepsilon(U_w^{\varepsilon,0}) \to L \quad \text{as } \varepsilon \to 0.$$

Then conclusion (3) of Lemma 4.3 implies that $w = w(\varepsilon,\tau) \to 0$ as $\varepsilon \to 0$. ∎

We are now in a position to present:

Proof of Proposition 4.1. Let $0 < \delta < \delta_0$ be given where δ_0 is as in (4.4) of Lemma 4.2. We take $\varepsilon_0(\delta)$ and $R(\delta)$ as defined in Lemma 4.6 and set $u_\varepsilon(x,t;\delta,\tau) = U_\varepsilon(t, w(\varepsilon,\tau))$. By shrinking ε_0 if necessary, we may ensure that $|w(\varepsilon,\tau)| < R(\delta)/2$, as shown in Lemma 4.6. Thus, $u_\varepsilon(x,t;\delta,\tau)$ solves (4.1) since

$$U_\varepsilon(t, w(\varepsilon,\tau)) = U_w^{\varepsilon,1}(x,t)$$

for such $\varepsilon > 0$.

Since $\|U_w^{\varepsilon,0}\|_{L^\infty} \le 1$, it is clear that $|u_\varepsilon(x,t)| \le 1$ everywhere, and all other conclusions of the proposition follow directly from Lemmas 4.5 and 4.6. ∎

5. Proof of the main result

The main result of this paper, stated more informally in the introduction as Theorem 1.1, can now be phrased precisely as

Theorem 5.1. *Let (M,g) be a closed oriented 3-dimensional Riemannian manifold, and let γ be a closed, embedded, nondegenerate geodesic of length L. Assume in addition that $\gamma = \partial S$ for some 2-dimensional orientable submanifold S of M.*

Then for every $r > 0$, there exists $\varepsilon_2(r) > 0$ such that if $0 < \varepsilon < \varepsilon_2(r)$, then there is a solution u_ε of the Ginzburg–Landau equation

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \quad \text{on } M \tag{5.1}$$

such that

$$\left\| \frac{1}{\pi} \star Ju_\varepsilon - T_\gamma \right\|_{\mathcal{F}} \le r, \quad |E_\varepsilon(u_\varepsilon) - L| < r.$$

As a result, there exists a sequence $(u_\varepsilon)_{\varepsilon>0} \subset H^1(M;\mathbb{C})$ *of solutions of the Ginzburg–Landau equations such that*

$$\left\| \frac{1}{\pi} \star Ju_\varepsilon - T_\gamma \right\|_{\mathcal{F}} \to 0, \quad E_\varepsilon(u_\varepsilon) \to L \quad \textit{as } \varepsilon \to 0. \tag{5.2}$$

We remark that standard Gamma-convergence results (see Theorem 2.1) imply that the sequence of solutions in (5.2) satisfies

$$\frac{e_\varepsilon(u_\varepsilon)}{\pi|\log\varepsilon|} \rightharpoonup \|T_\gamma\| = \mathcal{H}^1 \llcorner \Gamma \quad \text{weakly as measures,} \tag{5.3}$$

which is the last conclusion of Theorem 1.1. Indeed, since $E_\varepsilon(u_\varepsilon)$ is uniformly bounded, there exists some measure μ such that

$$\frac{e_\varepsilon(u_\varepsilon)}{\pi|\log\varepsilon|} \rightharpoonup \mu \quad \text{weakly as measures,}$$

after perhaps passing to a subsequence, and $\mu(M) = \lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = L$. Standard Gamma-convergence results and (5.2) imply

$$\mu \ge \|T_\gamma\|,$$

and since $\|T_\gamma\|(M) = L = \mu(M)$, it follows that $\mu = \|T_\gamma\|$, proving (5.3).

Proof of Theorem 5.1. The proof relies on an improvement on the properties of the flow defined in the previous section. The assertion is that the trajectory solving the Ginzburg–Landau flow identified in Proposition 4.1 remains close to T_γ in the flat norm. More precisely, we will show:

Claim. *For every* $r > 0$*, there exist positive constants* $\delta_1(r)$ *and* $\varepsilon_2(r) < \varepsilon_0(\delta_1(r))$*, depending on r, such that for every* $\varepsilon \in (0, \varepsilon_2(r))$ *and* $\delta \in (0, \delta_1(r))$*, and for every* $\tau > 0$*, if* $u_\varepsilon = u_\varepsilon(x, t; \delta, \tau)$ *is the solution of* (4.1) *found in Proposition* 4.1*, then*

$$\left\| \frac{1}{\pi} \star Ju_\varepsilon(\cdot,t) - T_\gamma \right\|_{\mathcal{F}} < r \quad \textit{for all } t \in [0, \tau]. \tag{5.4}$$

To establish this, we assume toward a contradiction that there exists $r > 0$ and sequences $\delta_k, \varepsilon_k \to 0$ and $0 < t_k \le \tau_k$ such that $u_k(x,t) := u_{\varepsilon_k}(x, t; \delta_k, \tau_k)$ satisfies

$$\left\| \frac{1}{\pi} \star Ju_k(\cdot, t_k) - T_\gamma \right\|_{\mathcal{F}} = r. \tag{5.5}$$

We will assume that $r < r_0$, where r_0 is the constant found in Proposition 3.1. This clearly does not entail any loss of generality.

Step 1: Under the assumption (5.5), we will argue that necessarily $t_k \to \infty$ as $k \to \infty$. (In fact, below we only need to know that t_k is bounded away from 0.)

Assume toward a contradiction that $\liminf_k t_k < K$ for some $K > 0$. By passing to subsequences, relabelling, and invoking standard compactness, continuity, and Gamma-convergence results (i.e. Theorem 2.1), we may assume that the following hold.

First, $t_k \le K$ for all k.

Second, there exists a 1-current $J_1 \in \mathcal{R}_1 \cap \mathcal{F}_1'(M)$ such that

$$\left\| \frac{1}{\pi} \star J u_k(\cdot, t_k) - J_1 \right\|_{\mathcal{F}} \to 0.$$

Hence, by (5.5),

$$\|J_1 - T_\gamma\|_{\mathcal{F}} = r. \tag{5.6}$$

Third, there exists a Radon measure μ_1 such that

$$\frac{e_{\varepsilon_k}(u_k(\cdot, t_k))}{\pi |\log \varepsilon_k|} \rightharpoonup \mu_1 \quad \text{weakly as measures,}$$

and in addition

$$\mu_1 \ge \|J_1\| \tag{5.7}$$

(see (2.14)).

Finally, since $\delta_k \to 0$ and M is compact, it follows from Proposition 4.1 that

$$\mu_1(M) = L = \|T_\gamma\|(M). \tag{5.8}$$

We next claim that

$$\frac{e_{\varepsilon_k}(u_k^0(\cdot))}{\pi |\log \varepsilon_k|} \rightharpoonup \mu_0 = \|T_\gamma\| \quad \text{weakly as measures,}$$

where we have set $u_k^0(\cdot) := u_k(\cdot, 0)$. Indeed, we may assume that $\frac{e_{\varepsilon_k}(u_k^0(\cdot))}{\pi |\log \varepsilon_k|}$ converges weakly as measures to a limit μ_0 as $k \to \infty$. Then recalling that $\|\frac{1}{\pi} \star J u_k^0 - T_\gamma\|_{\mathcal{F}} \le \delta_k \to 0$, standard Gamma-convergence results as in (5.7) imply that $\mu_0 \ge \|T_\gamma\|$. On the other hand, as in (5.8),

$$\mu_0(M) = L = \text{length}(\gamma) = \|T_\gamma\|(M),$$

so it follows that in fact $\mu_0 = \|T_\gamma\|$ as claimed. It then follows from (5.6) and (5.7) that $\mu_1 \ne \mu_0 = \|T_\gamma\|$.

We will obtain a contradiction, completing Step 1, by showing that under our assumptions μ_0 and μ_1 must be equal. Indeed, after taking the inner product of (4.1) with $\partial_t u_k$, standard computations show that

$$\partial_t e_{\varepsilon_k}(u_k) = -|\partial_t u_k|^2 + \text{div}(\partial_t u_k \cdot \nabla u_k).$$

Multiplying by a function $\phi \in C^1(M)$ and integrating by parts on the right-hand side, we get

$$\int_M \phi e_{\varepsilon_k}(u_k)\,\mathrm{vol}\Big|_0^{t_k} = -\int_0^{t_k}\int_M [\phi|\partial_t u_k|^2 + (\nabla\phi, \partial_t u_k \cdot \nabla u_k)_g]\,\mathrm{vol}\,dt. \tag{5.9}$$

Clearly

$$\frac{1}{\pi|\log\varepsilon_k|}\int_M \phi e_{\varepsilon_k}(u_k)\,\mathrm{vol}\Big|_0^{t_k} \to \int_M \phi(\mu_1 - \mu_0).$$

On the other hand, it is not hard to see that after dividing by $\pi|\log\varepsilon_k|$, the right-hand side of (5.9) tends to 0 as $k \to \infty$. First, taking $\phi = 1$ in (5.9), we see that

$$\frac{1}{\pi|\log\varepsilon_k|}\int_0^{t_k}\int_M |\partial_t u_k|^2\,\mathrm{vol}\,dt = E_{\varepsilon_k}(u_k^0) - E_{\varepsilon_k}(u_k(\cdot, t_k)) \le 2\delta_k \to 0$$

as $k \to \infty$. It immediately follows that

$$\left|\int_0^{t_k}\int_M \phi\frac{|\partial_t u_k|^2}{\pi|\log\varepsilon_k|}\,\mathrm{vol}\,dt\right| \to 0 \quad \text{as } k \to \infty.$$

Similarly, from Cauchy–Schwarz, the fact that

$$\int_M \frac{|\nabla u_k(\cdot,t)|^2}{\pi|\log\varepsilon_k|}\,\mathrm{vol}\,dt \le E_{\varepsilon_k}(u_k(\cdot,t)) \le L + \delta_k \quad \text{for all } t \ge 0$$

and the assumption that $t_k \le K$, we easily see that

$$\frac{1}{\pi|\log\varepsilon_k|}\left|\int_0^{t_k}\int_M (\nabla\phi, \partial_t u_k \cdot \nabla u_k)_g\ \mathrm{vol}\,dt\right| \to 0 \quad \text{as } k \to \infty.$$

Combining these, we conclude that $\int_M \phi(\mu_1 - \mu_0) = 0$ for all $\phi \in C^1(M)$, and hence that $\mu_0 = \mu_1$. This contradiction yields the conclusion that if (5.5) holds, then $t_k \to \infty$, completing Step 1.

Step 2: We will now reach a contradiction to (5.5), and so obtain the Claim.

To this end, we apply Theorem 2.3 to the sequence of functions

$$\tilde{u}_k(x,t) = u_k(x, t + t_k - 1),$$

which, in particular, satisfies (4.1) on $M \times [0,\infty)$ with $\varepsilon = \varepsilon_k \to 0$. This yields a function $\phi_*: M \times (0,1] \to \mathbb{R}$ that solves the heat equation, a harmonic 1-form γ_* that is not a function of t, as well as measures $(\mu_*^t)_{0<t\le 1}$ and $(\nu_*^t)_{0<t\le 1}$ such that

$$\frac{e_{\varepsilon_k}(\tilde{u}_k(\cdot,t))}{\pi|\log\varepsilon_k|}\,\mathrm{vol} \rightharpoonup \mu_*^t = \tfrac{1}{2}|\mathrm{d}\phi_*(\cdot,t) + \gamma_*|^2\,\mathrm{vol} + \nu_*^t$$

weakly as measures and $(\nu_*^t)_{0<t\le 2}$ is a 1-dimensional Brakke flow satisfying (2.27) and (2.28) (with $n = 2$).

As with (5.8), and using Step 1, it follows that

$$\mu_*^t(M) = L \quad \text{for all } t \in (0,2). \tag{5.10}$$

We recall the standard estimate

$$\frac{d}{dt}\int_M \frac{|\mathrm{d}\phi_*(\cdot,t)+\gamma_*|^2}{2}\,\mathrm{vol} = -\int_M |\partial_t\phi_*|^2\,\mathrm{vol} \le 0 \tag{5.11}$$

(the counterpart for the linear heat equation of (5.9)). Since $t \mapsto \nu_*^t(M)$ is also nonincreasing, as noted in (2.24), we conclude from (5.10) that both

$$\nu_*^t(M) \quad \text{and} \quad \int_M \frac{|\mathrm{d}\phi_*(\cdot,t)+\gamma_*|^2}{2}\,\mathrm{vol} \quad \text{are independent of } t \in (0,2).$$

It then follows from (5.11) that $\partial_t\phi_* = 0$ and hence $\phi_* = \phi_*(x)$ is independent of t and harmonic. Similarly, it follows from (2.25) and (2.26) that there exists a stationary 1-varifold V_* such that

$$\nu_*^t = V_* \quad \text{for all } t \in (0,2),$$

and so, in particular, at $t = 1$. Also, (2.27) and (2.28) imply that there exists a 1-rectifiable set $\Sigma_* \subset M$ and a function Θ_* such that

$$V_* = \Theta_*(x)\mathcal{H}^1 \llcorner \Sigma_*, \quad \Theta_* \ge \eta > 0 \quad \mathcal{H}^1\text{-a.e. in } \Sigma_*. \tag{5.12}$$

Moreover, as in the proof of Step 1, there exists a 1-current $J_1 \in \mathcal{R}_1(M) \cap \mathcal{F}_1'(M)$ such that

$$\left\| \frac{1}{\pi} \star J\tilde{u}(\cdot,1) - J_1 \right\|_{\mathcal{F}} = \left\| \frac{1}{\pi} \star Ju(\cdot,t_k) - J_1 \right\|_{\mathcal{F}} \to 0$$

and thus

$$\|J_1 - T_\gamma\|_{\mathcal{F}} = r. \tag{5.13}$$

We claim that in addition

$$V_* \ge \|J_1\|, \quad V_*(M) \le L. \tag{5.14}$$

The second assertion follows from (5.10), and the first assertion is a consequence of standard Gamma-convergence results, which imply that

$$\mu_*^1 = \tfrac{1}{2}|\mathrm{d}\phi_*(x)+\gamma_*|^2\,\mathrm{vol} + V_* \ge \|J_1\|.$$

Since $\frac{1}{2}|\mathrm{d}\phi_*(x)+\gamma_*|^2$ vol is absolutely continuous with respect to vol and $\|J_1\|$ is concentrated on a 1-rectifiable set, this implies that $V_* \ge \|J_1\|$, as claimed.

However, recalling that $0 < r < r_0$, we may appeal to Proposition 3.1 to find that no such varifold can exist. The Claim is established.

Step 3: Fix $r \in (0, r_0)$, where r_0 is fixed to ensure the previous steps hold, and let $\delta_1(r)$ and $\varepsilon_2(r)$ be as provided in the Claim. We may assume that $\delta_1(r) \le r$. For $\varepsilon \in (0, \varepsilon_2(r))$, let

$$u_k = u_\varepsilon\big(\cdot,\cdot;\tfrac{1}{2}\delta_1(r), 2^k\big).$$

It follows from Proposition 4.1 that

$$\delta_1(r) \geq E_U^\varepsilon(u_k(\cdot,0)) - E_U^\varepsilon(u_k(\cdot,2^k)) = \frac{1}{\pi|\log\varepsilon|}\int_0^{2^k}\int_M |\partial_t u_k|^2 \operatorname{vol}\, dt,$$

so there exists $\sigma_k \in (0, 2^k)$ such that $w_k := u_k(\cdot, \sigma_k)$ satisfies

$$\int_M \left|\Delta w_k - \frac{1}{\varepsilon^2}(|w_k|^2 - 1)w_k\right|^2 \operatorname{vol} = \int_M |\partial_t u_k|^2 \operatorname{vol}\Big|_{t=\sigma_k} \leq \delta_1(r)\pi|\log\varepsilon|2^{-k}.$$

Also, it follows from the Claim that

$$\left\|\frac{1}{\pi}\star Jw_k - T_\gamma\right\|_{\mathcal{F}} = \left\|\frac{1}{\pi}\star Ju_k(\cdot,\sigma_k) - T_\gamma\right\|_{\mathcal{F}} < r.$$

Since $|u_k| \leq 1$ everywhere, we have $\|\Delta w_k\|_{L^2(M)} \leq C_\varepsilon$, and hence by elliptic regularity, $\|w_k\|_{H^2} \leq C_\varepsilon$. One may thus extract a subsequence and a function $u_\varepsilon \in H^2(M;\mathbb{C})$ such that $w_k \rightharpoonup u_\varepsilon$ weakly in H^2, and it easily follows from the above that

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \quad \text{and} \quad \left\|\frac{1}{\pi}\star Ju_\varepsilon - T_\gamma\right\|_{\mathcal{F}} \leq r.$$

Finally, since we require $\delta < \delta_1(r) \leq r$, Proposition 4.1 yields $|E_\varepsilon(u_\varepsilon) - L| < r$. ■

Appendix A. Proofs of Lemmas 4.3 and 4.4

A.1. On the proof of Lemma 4.3

As remarked above, this lemma is essentially proved in [25]. We describe the proof given there and the extremely small modifications that we need.

The idea of the proof is first to construct $U_0^{\varepsilon,0}$, with its vorticity concentrating around the central geodesic Γ, then for $w \in W$, to define

$$U_w^{\varepsilon,0} := U_0^{\varepsilon,0} \circ O_w^{-1}, \tag{A.1}$$

where $O_w : M \to M$ is a suitable family of diffeomorphisms indexed by $w \in W$ such that $(w, x) \mapsto O_w(x)$ is smooth, described below.

Construction of $U_0^{\varepsilon,0}$. Recall that in (2.6) we defined a map $y : K_{r_0} \to \mathbb{R}^2$, smooth and nonvanishing away from Γ. Let $y^0 : M \setminus K_{r_0/2} \to S^1$ be any smooth function such that $y^0(x) = y(x)/|y(x)|$ in $K_{r_0} \setminus K_{r_0/2}$. The existence of such a function is a consequence of the topological assumption (2.3).

Then we set $\tilde{v}^\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by $\tilde{v}^\varepsilon(p) = f(\frac{|p|}{\varepsilon})\frac{p}{|p|}$ where $f : [0,\infty) \to [0,1]$ is a smooth nondecreasing function such that $f(s) = s$ for $s \in [0, 1/2]$ and $f(s) = 1$ for $s \geq 1$. Finally, we define

$$U_0^{\varepsilon,0}(x) := \begin{cases} \tilde{v}^\varepsilon(y(x)) & \text{for } x \in K_{r_0}, \\ y^0(x) & \text{for } x \in M \setminus K_{r_0}. \end{cases}$$

The only way in which this construction differs from that in [25] is that there, f is chosen to be $f(s) = \min(s, 1)$, which is Lipschitz continuous but not smooth. With this change, $U_0^{\varepsilon,0}$ is smooth.

Construction of O_w. We take O_w in (A.1) to be *exactly* the same map as in [25, p. 62]. The construction easily implies that $(w, x) \mapsto U_w^{\varepsilon,0}(x)$ is smooth and hence $\|U_w^{\varepsilon,0}\|_{H^2} \le C_\varepsilon$ for all $w \in W$. All other conclusions are proved in [25], and some are obvious anyway, such as $\|U_w^{\varepsilon,0}\|_{L^\infty} \le 1$. In particular, (3), which follows from a Gamma-limsup type estimate together with (2.12), is verified in [25, Lemma 5.5]. Finally, (4) follows from [25, Lemma 5.4].

A.2. *Proof of Lemma* 4.4

The maximum principle and standard energy estimates imply that for every $t > 0$,

$$\|U_w^{\varepsilon,1}(\cdot,t)\|_{L^\infty(M)} \le 1,$$
$$E_\varepsilon(U_w^{\varepsilon,1}(\cdot,t)) + \frac{1}{\pi|\log\varepsilon|}\int_0^t\int_M |\partial_t U_w^{\varepsilon,1}|^2 \operatorname{vol}\, dt' \le E_\varepsilon(U_w^{\varepsilon,1}(\cdot,0)) \le L+1$$

for all $|w| \le R$, provided ε and R are small enough. We next claim that for every $t > 0$, there exists $C = C_{\varepsilon,\tau}$ such that

$$\|U_w^{\varepsilon,1}(\cdot,t)\|_{H^2} \le C_{\varepsilon,\tau} \quad \text{for all } t \in [0,\tau] \text{ and } |w| \le R. \tag{A.2}$$

To specify the norm, we fix an open cover $\{U_j\}_{j\in J}$ of M, with local coordinates $\varphi_j : U_j \to V_j \subset \mathbb{R}^3$ on each patch, and a finite partition of unity $\{\eta_j\}$ subordinate to $\{U_j\}$. We then define

$$\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \||\nabla u|_g\|_{L^2}^2 + \sum_{j\in J}\sum_{k=1}^{3} \|\sqrt{\eta_j}\,|\nabla\partial_k(u\circ\varphi_j^{-1})|_g\|_{L^2(V_j)}^2,$$

where ∂_k denotes differentiation with respect to local coordinates on V_j. To prove (A.2), we write (4.1) in local coordinates on each patch, and apply ∂_k to derive an equation for $V_k := \partial_k U_w^{\varepsilon,1}$ of the form

$$\partial_t V_k - \Delta_g V_k = \text{terms involving } U_w^{\varepsilon,1}, \nabla U_w^{\varepsilon,1}.$$

Multiplying by $\eta_j\partial_t V_k$, using the fact from Lemma 4.3 (2) that $\|\nabla V_k(\cdot,0)\| \le C_\varepsilon$, integrating by parts, and carrying out rather standard estimates leads to (A.2).

It follows from the above estimates and the equation that $\|\partial_t U_w^{\varepsilon,1}(\cdot,t)\|_{L^2} \le C_{\varepsilon,\tau}$ for $0 < s \le \tau$. Thus, for $0 \le t_1 < t_2 \le \tau$ and any $w \in W$, we have

$$\begin{aligned}\|U_w^{\varepsilon,1}(\cdot,t_2) - U_w^{\varepsilon,1}(\cdot,t_2)\|_{L^2}^2 &\le (t_2-t_1)\int_{M\times[t_1,t_2]} |\partial_t U_w^{\varepsilon,1}(x,t)|^2 \operatorname{vol}\, dt \\ &\le C_{\varepsilon,\tau}(t_2-t_1).\end{aligned}$$

Then the interpolation estimate $\|u\|_{H^1} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}$ and (A.2) imply that

$$\|U_w^{\varepsilon,1}(\cdot,t_2) - U_w^{\varepsilon,1}(\cdot,t_1)\|_{H^1} \le C_{\varepsilon,\tau}\sqrt{t_2-t_1} \quad \text{for } w \in W,\ 0 \le t_1 < t_2 \le \tau.$$

Now consider $w_1, w_2 \in W$. Writing $f_\varepsilon(u) = \frac{1}{\varepsilon^2}(1-|u|^2)u$ and using the identity

$$f_\varepsilon(b) - f_\varepsilon(a) = \int_0^1 \frac{d}{d\sigma} f_\varepsilon(\sigma b + (1-\sigma)a)\,d\sigma = \int_0^1 f_\varepsilon'(\sigma b + (1-\sigma)a)\,d\sigma\,(b-a),$$

we find that $V := U_{w_2}^{\varepsilon,1} - U_{w_1}^{\varepsilon,1}$ satisfies the equation

$$\partial_t V - \Delta V = gV,$$

where $\|g(\cdot,t)\|_{L^\infty} \le C_\varepsilon$ for every t. In addition, it follows from Lemma 4.3 that $\|V(\cdot,0)\|_{H^1} \le C_\varepsilon|w_2 - w_1|$. Thus carrying out further standard parabolic estimates (multiplying by V or $\partial_t V$, integrating by parts etc.) leads to

$$\|U_{w_2}^{\varepsilon,1}(\cdot,t) - U_{w_1}^{\varepsilon,1}(\cdot,t)\|_{H^1} \le C_{\varepsilon,\tau}|w_2 - w_1| \quad \text{for } 0 \le t \le \tau.$$

We conclude that the map $[0,\tau] \times W \ni (t,w) \mapsto U_w^{\varepsilon,1}(\cdot,t) \in H^1(M;\mathbb{C})$ is continuous, since it is separately uniformly continuous in t and w. A more detailed reference for such parabolic estimates on manifolds can be found, e.g., in [24, Appendix A]. ■

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