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On the regularity of the De Gregorio model for the 3D Euler equations

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Abstract. We study the regularity of the De Gregorio (DG) model $\omega_t + u\omega_x = u_x\omega$ on S^1 for initial data ω_0 with period π and in class X: ω_0 is odd and $\omega_0 \le 0$ (or $\omega_0 \ge 0$) on $[0, \pi/2]$. These sign and symmetry properties are the same as those of the smooth initial data that lead to singularity formation of the De Gregorio model on $\mathbb{R}$ or the generalized Constantin–Lax–Majda (gCLM) model on $\mathbb{R}$ or S^1 with a positive parameter. Thus, to establish global regularity of the DG model for general smooth initial data, which is a conjecture on the DG model, an important step is to rule out potential finite time blowup from smooth initial data in X. We accomplish this by establishing a one-point blowup criterion and proving global well-posedness for initial data $\omega_0 \in H^1 \cap X$ with $\omega_0(x)x^{-1} \in L^\infty$. On the other hand, for any $\alpha \in (0,1)$, we construct a finite time blowup solution from a class of initial data with $\omega_0 \in C^\alpha \cap C^\infty(S^1 \setminus \{0\}) \cap X$. Our results imply that singularities developed in the DG model and the gCLM model on S^1 can be prevented by stronger advection.

Keywords. 3D Euler equations, De Gregorio model, singularity, regularity

1. Introduction

To model the effect of vortex stretching in the three-dimensional (3D) incompressible Euler equations, Constantin, Lax, and Majda [12] proposed a one-dimensional model (CLM)

$$\omega_t = u_x\omega, \quad u_x = H\omega, \tag{1.1}$$

where H is the Hilbert transform. Singularity formation of (1.1) was established and studied in detail in [12]. The effect of advection in the 3D Euler equations is not modeled in (1.1).

De Gregorio [16,17] considered both effects by adding an advection term $u\omega_x$ to (1.1):

$$\omega_t + u\omega_x = u_x\omega, \quad u_x = H\omega, \tag{1.2}$$

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and provided some evidence that (1.2) admits no blowup. To understand the effect of advection in (1.2), we can neglect the vortex stretching term $u_x\omega$ in (1.2). The resulting model can also be seen as (1.3) below with infinite weight $a = \infty$ in the advection term. One can obtain the global well-posedness of this model using the conservation of $\|\omega\|_{L^\infty}$ (see, e.g., [46]). Numerical simulations performed in [38, 46] and the report in [30] suggest that a solution of (1.2) from smooth initial data exists globally. These lead to the conjecture that the De Gregorio (DG) model is globally well-posed for smooth initial data, which was made in [20, 38, 46]. Note that the question of regularity for the DG model is listed as one of the open problems in [23]. In contrast to the CLM model, there is a strong competition in the DG model between the nonlocal stabilizing effect due to advection and a destabilizing effect due to vortex stretching. These two effects are comparable, making it very challenging to analyze (1.2). We remark that the stabilizing effect of advection has been studied in [27, 28] for the 3D Navier–Stokes equations.

Regarding the global regularity of (1.2), the first result seems to be established only recently by Jia–Stewart–Šverák [30], who proved the nonlinear stability of a steady state $A\sin(2x)$ of (1.2) with period π using spectral theories. In [35], Lei–Liu–Ren discovered a novel equation (see (2.1)) and a conserved quantity for initial data ω_0 with a fixed sign and established the global regularity of (1.2) for such initial data. We note that for strictly positive or negative initial data ω_0, the CLM model (1.1) does not blow up. On the other hand, in recent joint work [9] with Hou and Huang, we established finite time blowup of (1.2) on $\mathbb{R}$ with initial data $\omega_0 \in C_c^\infty$ by proving the nonlinear stability of an approximate blowup profile. Thus the above conjecture on the regularity of the DG model is not valid for all smooth initial data in the case of $\mathbb{R}$.

In this paper, we study the regularity of the De Gregorio model (1.2) on S^1 with period π. We focus on odd initial data ω_0 in class X (see (1.4)): $\omega_0(x) \geq 0$ or $\omega_0(x) \leq 0$ for all $x \in [0, \pi/2]$. These properties are preserved dynamically. The class of initial data in X seems to provide the most promising scenario for a potential blowup solution of (1.2) on S^1 up to now for the following reasons. Firstly, the initial data considered in [9] that lead to finite time blowup of (1.2) on $\mathbb{R}$ have the same sign and symmetry properties as those in X. Secondly, for the generalized Constantin–Lax–Majda (gCLM) model [46]

$$\omega_t + au\omega_x = u_x\omega, \quad u_x = H\omega \tag{1.3}$$

with $a > 0$, which is closely related to (1.2), singularity formation [5, 6, 9, 19, 20] all develops from initial data with the same sign and symmetry properties as those in X. In particular, in [6], we established that the gCLM model on S^1 with a slightly less than 1, which can be seen as a slight perturbation to (1.2), develops finite time singularity from some smooth initial data in X. Thirdly, this scenario can be seen as a 1D analog of the hyperbolic blowup scenario for the 3D Euler equations reported by Hou–Luo [36, 37]. See also [8, 32, 33]. In fact, the restriction of the (angular) vorticity in [8, 32, 33, 36, 37] to the boundary has the same sign and symmetry properties as those in X. Thus, to establish global regularity of (1.2) for general smooth initial data, we need to address the important question of whether there is a finite time blowup in this class. We note that the initial data considered in [30] is close to the steady state $A\sin(2x)$ of (1.2). Thus it belongs to X or is close to one in X.

Note that the CLM model can only blow up in finite time at the zeros of ω [12]. Since vortex stretching is the driving force for a potential blowup of (1.2), it is likely that a potential singularity of (1.2) with general data is also located at the zeros of ω. For a zero x_0 of ω across which ω changes sign, the leading order term of ω near x_0 is $\partial_x^k \omega(x_0)(x-x_0)^k$ for some odd $k \in \mathbb{Z}_{>0}$. It has the same sign and symmetry properties as those in X. Thus, our analysis of (1.2) with $\omega \in X$ can provide valuable insights on the local analysis of these potential singularities. For a zero x_0 of ω across which ω does not change sign, the local analysis could benefit from [35].

There are other 1D models for the 3D Euler equations and SQG equation: see, e.g., [11, 13]. We refer to [11, 20] for excellent surveys and [10, 11, 20] for discussions on the connections.

1.1. *Main results*

Throughout this paper, we consider initial data ω_0 in the following class:

$$X := \{f : f \text{ is odd, } \pi\text{-periodic and } f(x) \le 0 \text{ for } x \in [0, \pi/2]\}, \tag{1.4}$$

unless we specify otherwise. We assume $\omega_0 \le 0$ on $[0, \pi/2]$ without loss of generality. For the case of $\omega_0 \ge 0$ on $[0, \pi/2]$, we can consider a new variable $\omega_{\text{new}}(x) := \omega(x + \pi/2)$ and then reduce it to the previous case. It is not difficult to show that the solution $\omega(t)$ remains in X.

Our first main result is a one-point blowup criterion. A similar blowup criterion has been obtained in our previous work [5] for the DG model and the gCLM model with dissipation.

Theorem 1. *Suppose that $\omega_0 \in X \cap H^1$ and*

$$A(\omega_0) := \int_0^{\pi/2} \left| \frac{\omega_{0,x}^2}{\omega_0} \sin(2x) \right| dx < \infty.$$

The unique local in time solution of (1.2) cannot be extended beyond $T > 0$ if and only if

$$\int_0^T u_x(0, t)\, dt = \infty. \tag{1.5}$$

For $\omega \in X \cap H^1$, we have $u_x(0,t) \ge 0$. Suppose that ω vanishes to the order $|x|^\beta$, $\beta > 0$, near $x = 0$. Then $\frac{\omega_x^2}{\omega}\sin(2x)$ is of order $|x|^{2(\beta-1)-\beta+1} = |x|^{\beta-1}$ near $x = 0$, which is locally integrable. A similar conclusion holds for the local integrability near $x = \pi/2$. For $\omega \in C^{1,\alpha} \cap X$, the sign condition in X implies that ω degenerates at its zeros in $S^1 \setminus \{0, \pi/2\}$ with an order $\beta > 1$, if the zeros exist, and thus $\frac{\omega_x^2}{\omega}\sin(2x)$ is still locally integrable. In particular, for $\omega_0 \in C^\infty \cap X$ with a finite number of zeros and a finite order of degeneracy, the assumption $A(\omega_0) < \infty$ holds automatically. Based on Theorem 1, we obtain the following global well-posedness result.

Theorem 2. *Suppose that $\omega_0 \in X \cap H^1$, $\omega_0(x)x^{-1} \in L^\infty$, and $A(\omega_0) < \infty$. There exists a global solution ω of (1.2) with initial data ω_0. In particular,*

(a) *for $\omega_0 \in X \cap C^{1,\alpha}$ with $\alpha \in (0,1)$ and $A(\omega_0) < \infty$, there exists a global solution from ω_0;*

(b) *for $\omega_0 \in X \cap C^1$ with $A(\omega_0) < \infty$, the unique local solution $\omega \in \bigcap_{\alpha<1} C^\alpha$ from ω_0 exists globally. If the initial data further satisfies $\omega_0 \in C^{1,\alpha}$ with $\alpha \in (0,1)$ and $\omega_{0,x}(0) = 0$, we have*

$$\|\omega(t)\|_{L^1} + |u_x(0,t)| \le K(\omega_0)e^{CQ(2)t},$$
$$\|\omega(t)\|_{L^\infty} \le K(\omega_0)\exp\Big(2\exp\big(K(\omega_0)\exp(CQ(2)t)\big)\Big),$$

where $Q(2) = \int_0^{\pi/2} |\omega_0| \cot^2 y \, dy$ and $K(\omega_0)$ is some constant depending on $H\omega_0(0)$, $H\omega_0(\pi/2)$, $\|\omega_0\|_{L^1}$, $Q(2)$, $A(\omega_0)$.

In the general case, the a priori estimates are much weaker. See Lemma 5.4 and Remark 5.5 for more discussion. Since $H^s \hookrightarrow C^{1,\alpha}$ for $s > \alpha + 3/2$, Theorem 2 implies the global well-posedness (GWP) in $H^s \cap X$ with $s > 3/2$. The condition $\omega_0(x)x^{-1} \in L^\infty$ in Theorem 2 is necessary since we can obtain a finite time blowup for ω_0 that is less regular near $x = 0$.

Theorem 3. *For any $0 < \alpha < 1$ and $s < 3/2$, there exists $\omega_0 \in X \cap C^\alpha \cap H^s \cap C^\infty(S^1 \setminus \{0\})$ with $A(\omega_0) < \infty$ such that the solution of (1.2) with initial data ω_0 develops a singularity in finite time. In particular, $\int_0^T u_x(0,t)\,dt = \infty$.*

One can establish the local well-posedness of (1.2) in $C^{k,\alpha}$ with any $k \in \mathbb{Z}_+ \cup \{0\}$ and $\alpha \in (0,1)$ using the particle trajectory method [39]. From the ill-posedness result for the incompressible Euler equations in [2], it is conceivable that (1.2) is ill-posed in C^1. For C^1 initial data, there is a unique local solution in $\bigcap_{\alpha<1} C^\alpha$. Thus, in view of the above theorems, in the class $\omega \in X$, the blowup criterion in Theorem 1 and the regularity results in Theorems 2 and 3 are sharp.

Theorem 2 verifies the conjecture on the GWP of (1.2) on S^1 and rules out potential blowup of (1.2) from initial data in $C^\infty \cap X$. It also addresses the conjecture made in [20] in the case of S^1 that the strong solution to (1.2) is global for C^1 initial data in X. Note that the smooth initial data that lead to singularity formation of the gCLM model (1.3) on S^1 [5, 6, 9] or the CLM model [12] can be chosen in the class of Theorem 2. Thus, Theorem 2 implies that advection in (1.2) can prevent singularity formation in the CLM model or the gCLM model for such initial data. The global regularity results in Theorem 2 can be generalized to the DG model (1.2) with an external force $f\omega$ linear in ω, where $f \in C^\infty$ is a given even function. Theorem 3 resolves the conjecture made in [20, 49] that (1.2) develops a finite time singularity from initial data $\omega_0 \in C^\alpha$ or $\omega_0 \in H^s$ for any $\alpha \in (0,1)$ and $s < 3/2$ in the case of S^1. The case of $\mathbb{R}$ has been resolved in [9] with $\omega_0 \in C_c^\infty$.

In [20], Elgindi–Jeong made an important observation that advection can be substantially weakened by choosing C^α data with sufficiently small α, and constructed a C^α self-similar blowup solution of (1.2) on $\mathbb{R}$ with small α. For (1.2) on S^1, a finite time blowup from C_c^α data with small α was obtained in [9]. In Theorem 3, the Hölder exponent α can be arbitrarily close to 1. As we will see in the proof, it suffices to weaken advection slightly. Theorem 3 is inspired by our previous work [6], where we constructed a finite time blowup solution for the gCLM model (1.3) with a slightly less than 1 and smooth initial data.

1.2. Connection with the CLM model

The CLM model (1.1) can be solved explicitly [12]:

$$\begin{aligned}\omega(x,t) &= \frac{4\omega_0(x)}{(2-tH\omega_0(x))^2+t^2\omega_0^2(x)},\\ H\omega(x,t) &= \frac{2H\omega_0(x)(2-tH\omega_0(x))-2t\omega_0^2(x)}{(2-tH\omega_0(x))^2+t^2\omega_0^2(x)}.\end{aligned} \tag{1.6}$$

We consider the solution of (1.1) with period π . From (1.6), the solution can blow up at x in finite time if and only if $\omega_0(x)=0$ and $H\omega_0(x)>0$. Consider odd ω_0 with $\omega_0<0$ on $(0,\pi/2)$. Since $H\omega_0(0)>0$ and $H\omega_0(\pi/2)<0$, the only point x with $\omega_0(x)=0$ and $H\omega_0(x)>0$ is $x=0$. Within this class of initial data, from Theorem 1, $u_x(0,t)$ controls the blowup in both the CLM model and the De Gregorio model. On the other hand, the CLM model blows up in finite time for smooth initial data, while from Theorems 2 and 3, the advection term in the De Gregorio model can prevent singularity formation if the initial data is smooth enough.

1.3. Competition between advection and vortex stretching

The competition between advection and vortex stretching and its relation to the vanishing order of $\omega\in X$ near $x=0$ can be illustrated by a simple Taylor expansion. Suppose that near $x=0$, $\omega=-x^a+\text{l.o.t.}$ for $a>0$ and $u=cx+\text{l.o.t.}$ for some $c>0$, where l.o.t. denotes lower order terms. We impose the assumption on u since $u=-(-\partial_{xx})^{-1/2}\omega$ is odd and at least C^1 with $u_x(0)>0$ for nontrivial $\omega\in X$. The leading order terms of $u\omega_x$ and $u_x\omega$ near $x=0$ are given by

$$u\omega_x=-acx^a+\text{l.o.t.},\quad u_x\omega=-cx^a+\text{l.o.t.}$$

This simple calculation suggests that $a-1$ characterizes the relative strength between the advection $|u\omega_x|$ and the vortex stretching $|u_x\omega|$ near $x=0$. Advection is weaker than, comparable to, and stronger than the vortex stretching if $a<1$, $a=1$, and $a>1$, respectively. Considering the stabilizing effect of advection [6,27,46] and the destabilizing effect of vortex stretching [12], one would expect that there exists singularity formation in the case of $a<1$ and global well-posedness in the case of $a\geq 1$. Theorems 2 and 3 confirm this formal analysis. In the case of $a=1$, e.g. $\omega_0\in C^{1,\alpha}$ with $\omega_{0,x}(0)\neq 0$ in Theorem 2, the effects of the two terms balance, making it very challenging to establish the GWP result in Theorem 2. To prove these results, we need to quantitatively characterize the competition in three different cases and precisely control the effects of advection and vortex stretching. See more discussion in Section 2.

1.4. Connections with incompressible fluids

1.4.1. The effect of advection. Theorem 2 provides some valuable insights on potential singularity formation in incompressible fluids. We consider the 2D Boussinesq equations

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \theta_x, \quad \theta_t + \mathbf{u} \cdot \nabla \theta = 0, \tag{1.7}$$

where ω is the vorticity, θ is the density, and $\mathbf{u}$ is the velocity field determined by $\nabla^{\perp}(-\Delta)^{-1}\omega$.

In the whole space, a promising potential blowup scenario is the hyperbolic-flow scenario with θ_x, ω being odd in both x, y, and positive θ_x, ω in the first quadrant. The induced flow is clockwise in the first quadrant near the origin. A similar scenario has been used in [26, 50]. In this scenario, the flow in the y-direction in the first quadrant moves away from the origin. To understand the effect of y-advection, we derive a model on θ_x, which is the driving force for the growth in (1.7). Taking the x-derivative of (1.7) and using the incompressibility condition $u_{2,y} = -u_{1,x}$ yields

$$\partial_t \theta_x + \mathbf{u} \cdot \nabla \theta_x = -u_{1,x}\theta_x - u_{2,x}\theta_y = u_{2,y}\theta_x - u_{2,x}\theta_y. \tag{1.8}$$

Dropping the θ_y term and the advection in the x direction and simplifying $\omega = \theta_x$, we further derive

$$\partial_t \theta_x + u_2 \partial_y \theta_x = u_{2,y}\theta_x, \tag{1.9}$$

$$\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\theta_x, \quad u_{2,y} = \partial_{xy}(-\Delta)^{-1}\theta_x. \tag{1.10}$$

See more motivations for these simplifications in Appendix A.2. Note that the θ-equation in (1.7) with (1.10) reduces to the incompressible porous media equation [14, 15]. Equation (1.9) captures the competition between the vortex stretching $u_{2,y}\theta_x$ and the y-advection $u_2\partial_y\theta_x$ in (1.8). This model relates to (1.2) via $\theta_x \to -\omega$, $\partial_{xy}(-\Delta)^{-1} \to -H$. Moreover, the solutions of the two models enjoy similar sign and symmetry properties. See more discussion in Appendix A.2. The connection between $\partial_{xy}(-\Delta)^{-1}$ and H can be justified under some assumptions [10, 11, 29], though it may not be consistent with the current setting.

Valuable insight from Theorem 2 and the connection between the above model and (1.2) is that if $\theta_x(x, y)$ vanishes near $y = 0$ to order $|y|^a$ with $a \geq 1$, advection may be strong enough to destroy potential singularity formation. In the hyperbolic flow scenario, due to the odd symmetry in y, a typical θ near the origin is of the form $\theta(x, y) \approx c_1 x^{1+\alpha} y + \text{l.o.t.}$ for $\theta \in C^{1,\alpha}$ and $\theta(x, y) \approx c_1 x^2 y + \text{l.o.t.}$ for $\theta \in C^{\infty}$. In both cases, θ_x vanishes linearly in y, and thus the effect of y-advection can be an obstacle to singularity formation. Such an effect can be overcome by imposing a solid boundary on $y = 0$, and singularity formation with $C^{1,\alpha}$ velocity has been established in [8]. For smooth data, the importance of boundary has been studied in [36, 37]. In the absence of a boundary, new mechanisms to overcome advection or a new scenario may be required to obtain singularity formation of (1.7) in $\mathbb{R}^2$.

1.4.2. Connections with the SQG equation. In [3], Castro–Córdoba observed that a solution $\omega(y,t)$ of the De Gregorio model (1.2) can be extended to a solution of the SQG equation

$$\theta_t + \mathbf{u} \cdot \nabla \theta = 0, \quad \mathbf{u} = \nabla^{\perp}(-\Delta)^{-1/2}\theta \tag{1.11}$$

with infinite energy via $\theta(x,y,t) = x\omega(y,t)$. We can perform derivations for (1.11) similar to those in (1.7)–(1.10). Under this connection, the terms dropped in the derivations are *exactly* 0, and the SQG equation in the hyperbolic-flow scenario [26] reduces *exactly* to the DG model (1.2) with a solution in class X. Hence, our analysis of (1.2) provides valuable insight into the effect of advection in (1.11) in such a scenario. Moreover, from Theorem 2, we obtain a new class of globally smooth nontrivial solutions to (1.11) with infinite energy. Note that a globally smooth solution to (1.11) with finite energy has been constructed in [4] (see also [24]). Singularity formation of (1.11) from smooth initial data with infinite energy follows from [9].

Under the radial homogeneity ansatz $\theta(t,r,\beta) = r^{2-2\alpha}g(t,\beta)$, Elgindi–Jeong [21] established a connection between a solution θ to the generalized SQG equation and a solution $g(t,\beta)$ to the gCLM model (1.3) with $a > 1$ up to some lower order term in the velocity operator. Our analysis of the global regularity of (1.2) sheds useful light on the analysis of (1.3) with $a > 1$ and constructing globally nontrivial solutions to the generalized SQG equation using the connection in [21]. In particular, our argument to analyze $u_x(0)$ and a singular integral, which is defined in (2.4) and characterizes the competition between advection and vortex stretching in (1.2), can be generalized to the gCLM model with $a > 1$. See more discussion in Section 7.

Organization of the paper. In Section 2, we discuss the main ideas in the proofs of the main theorems. In Section 3, we establish the one-point blowup criterion. In Section 4, we discuss the stabilizing effect of advection in (1.2) and study the positive-definiteness of several quadratic forms, which are the building blocks for the GWP results in Theorem 2. In Section 5, we prove Theorem 2. In Section 6, we construct finite time blowup of (1.2) with $C^\alpha \cap H^s$ data. We make some concluding remarks on the potential generalization of the results in Section 7. Some technical lemmas and derivations are deferred to the Appendix.

2. Main ideas and the outline of the proofs

In this section, we discuss the main ideas and outline the proofs of the main theorems.

2.1. Difference between the De Gregorio on $\mathbb{R}$ and on S^1

Note that the initial condition considered in [9] that leads to finite time blowup of (1.2) on $\mathbb{R}$ has the same sign and symmetry properties as those in X. To establish the well-posedness results in Theorems 1 and 2, we need to understand the mechanism on S^1 that prevents singularity formation similar to [9].

For (1.2) on S^1 with $\omega \in X$, we have two special points $x = 0, x = \pi/2$, which correspond to $x = 0, x = \infty$ in the case of $\mathbb{R}$. One of the key differences between the two cases is captured by the evolution of $\|\omega\|_{L^1}$:

$$\frac{d}{dt}\left(-\int_0^{\pi/2} \omega(x)\,dx\right) = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)\cot(x+y)\,dx\,dy,$$

which is derived in (3.10)–(3.11). Since $\omega \le 0$ on $[0, \pi/2]$, $-\int_0^{\pi/2} \omega(x)\,dx$ equals $\|\omega\|_{L^1}$.

For $x + y \le \pi/2$, the interaction on the right hand side has a positive sign due to $\cot(x+y) \ge 0$, which leads to the growth of $\|\omega\|_{L^1}$. On the other hand, for $x + y \ge \pi/2$, the interaction has a negative sign, which contributes to the decrease of $\|\omega\|_{L^1}$. The former and the latter interaction can be seen as the interaction near 0 and $\pi/2$, respectively. The latter plays a crucial role in our proof as a damping term. For comparison, a similar ODE can be derived for (1.2) on $\mathbb{R}$ with $\cot(x+y)$ replaced by $\frac{1}{x+y}$. The interaction is always positive and can contribute to the unbounded growth of the singular solution in [9] in the far field. Yet, for (1.2) on S^1, similar growth near $x = \pi/2$ is prevented due to the above damping term.

Moreover, for (1.2) on S^1 with $\omega \in X$, we have $-u \in X$ and thus $u_x(0) > 0$ and $u_x(\pi/2) < 0$ for nontrivial ω. The sign of $u_x(\pi/2)$ suggests that near $x = \pi/2$, the vortex stretching term $u_x\omega$ in (1.2) depletes the growth of the solution. Using these observations, we show that the nonlinear terms near $x = \pi/2$ are harmless. Thus, the main difficulty is the analysis of (1.2) near $x = 0$.

2.2. *The one-point blowup criterion*

In [35], an important equation was discovered:

$$\tfrac{1}{2}\partial_t((\sqrt{\omega})')^2 = -\tfrac{1}{2}u(((\sqrt{\omega})')^2)' - \tfrac{1}{2}H\omega((\sqrt{\omega})')^2 + \tfrac{1}{4}(H\omega)'\omega', \tag{2.1}$$

which implies

$$\frac{1}{2}\partial_t\frac{\omega_x^2}{\omega} = -\frac{1}{2}\left(u\frac{\omega_x^2}{\omega}\right)_x + \omega_x H\omega_x. \tag{2.2}$$

Identity (2.2) can also be obtained from the equation for ω_x and ω^{-1} using (1.2).

To prove Theorem 1, one of the key steps is the estimate of a new quantity $\int_0^{\pi/2} \frac{\omega_x^2}{\omega}\sin(2x)\,dx$. The vanishing property of $\sin(2x)$ near $x = 0, \pi/2$ cancels the singularity caused by $1/\omega$ for $\omega \in X$. Since $\omega(t)$ remains in X (see (1.4)) and $\omega \le 0$ on $[0, \pi/2]$, $\frac{\omega_x^2}{\omega}\sin(2x)$ has a fixed sign. To control the nonlinear terms in the energy estimate, we will exploit the conservation form $\left(u\frac{\omega_x^2}{\omega}\right)_x$, use an important cancellation for a quadratic form of ω_x and a crucial extrapolation inequality for u. Using some estimates in [5, 9], we derive a priori estimates on $u_x(0)$, $\|\omega\|_{L^1}$, $\int_0^{\pi/2} \frac{\omega_x^2}{\omega}\sin(2x)\,dx$, which controls $\omega(x)$ away from $x = \pi/2$ by interpolation. By exploiting the damping mechanisms near $x = \pi/2$ discussed in Section 2.1, we further show that $u_x(\pi/2, t)$ cannot blow up before the blowup of $u_x(0,t)$. With these estimates, we obtain an a priori estimate on $\|\omega\|_{L^\infty}$ in terms of $\int_0^t u_x(0,s)\,ds$, and establish the one-point blowup criterion by applying the Beale–Kato–Majda type blowup criterion [1, 30]. See also [46].

2.3. Global well-posedness

To prove Theorem 2 using Theorem 1, we need to further control $u_x(0)$. In the special case of $\omega_0 \in C^{1,\alpha}$ with $\omega_{0,x}(0) = 0$, the key step is to establish

$$\frac{d}{dt}\int_0^{\pi/2} \omega \cot^2 x\,dx = \int_0^{\pi/2} (u_x\omega - u\omega_x)\cot^2 x\,dx \geq 0. \tag{2.3}$$

The quantity $\int_0^{\pi/2} \omega\cot^2 x\,dx$ is well-defined for $\omega \in C^{1,\alpha}$ with $\omega_x(0) = 0$ and $\alpha > 0$. The above inequality quantifies the fact that the stabilizing effect of advection is stronger than the effect of vortex stretching in some sense for ω in this case. We will exploit the convolution structure in the quadratic form in (2.3) and use an idea from Bochner's theorem for a positive-definite function to establish (2.3). We remark that an inequality similar to (2.3) has been established in the arXiv version of [9], where a more singular function $\cot^\beta x$ with $\beta \geq 2.2$ is used. The inequality (2.3) is stronger than that in [9] since $\int_0^{\pi/2} \omega(\cot x)^\beta\,dx$ is not well-defined for $\omega \in C^{1,\alpha}$ with $\alpha \in (0, \beta - 2)$ and $\omega_x(0) = 0$. Since $\omega \leq 0$ on $[0, \pi/2]$, (2.3) implies an a priori estimate of $\int_0^{\pi/2} |\omega\cot^2 x|\,dx$, based on which we can further control $\|\omega\|_{L^1}, u_x(0)$ and establish global well-posedness.

In the general case, ω_0 can vanish only linearly near $x = 0$. The proof is much more challenging since $\int_0^{\pi/2} |\omega\cot^2 x|\,dx$ is not well-defined, and there is no similar coercive conserved quantity. Note that in this case, for ω_0 close to $A\sin 2x$ in the C^2 norm, the solution $\omega(x,t)$ converges to $A\sin 2x$ as $t \to \infty$ [30]. As pointed out in [30], this imposes strong constraints on possible conserved quantities. Thus, it is not expected that there is any good conserved quantity similar to some weighted norm of ω.

To illustrate our main ideas, we consider $\omega_0 \in C^{1,\alpha} \cap X$ with $\omega_{0,x} \neq 0$. In this case, the only conserved quantities seem to be $\omega_x(x,t) \equiv \omega_{0,x}(x)$ for $x = 0, \pi/2$. Surprisingly, the one-point conservation law $\omega_x(0,t) \equiv \omega_{0,x}(0)$ allows us to control $Q(\beta,t)$ defined below for $\beta < 2$. We remark that we do not have monotonicity of $Q(\beta,t)$ in t similar to (2.3) when $\beta < 2$. A crucial observation is the following leading order structure:

$$Q(\beta,t) := \int_0^{\pi/2} -\omega(y,t)(\cot y)^\beta\,dy = \frac{-\omega_x(0)}{2-\beta} + \mathcal{R}(\beta,t), \quad |\mathcal{R}(\beta,t)| \lesssim_\alpha \|\omega\|_{C^{1,\alpha}}, \tag{2.4}$$

for any $\beta < 2$. As long as $\omega(t)$ remains in $C^{1,\alpha}$, we can choose β sufficiently close to 2, such that $(2-\beta)Q(\beta,t)$ is comparable to $-\omega_x(0)$, which is time-independent. Using this observation, an ODE for $Q(\beta,t)$ similar to (2.3) but with a nonlinear forcing term, and an additional extrapolation-type estimate, we can control $Q(\beta(t),t)$ with $\beta(t)$ sufficiently close to 2. In the case of less regular initial data $\omega_0 \in X \cap H^1$ with $\omega_0 x^{-1} \in L^\infty$, we will establish an estimate similar to (2.4). This enables us to further control $u_x(0)$ and establish global well-posedness.

2.4. Finite time blowup

To prove Theorem 3, we follow the method of Chen–Hou–Huang [9]. We also adopt an idea developed in our previous work [6] that a singular solution of the gCLM model (1.3) can be constructed by perturbing the equilibrium $\sin(2x)$ of (1.2). We first construct a C^α approximate self-similar profile of (1.2), $\omega_\alpha = C \cdot \mathrm{sgn}(x)|\sin 2x|^\alpha$ with $\alpha < 1$ sufficiently close to 1. Our key observation is that for $\alpha < 1$, the advection $u\omega_x$ is slightly weaker than the vortex stretching $u_x\omega$. See the discussion in the paragraph before Section 1.2 and in Section 1.3. Then we establish the nonlinear stability of the profile ω_α in the dynamic rescaling formulation of (1.2) based on the coercivity estimates of a linearized operator established in [35] and several weighted estimates. Using the nonlinear stability results and the argument in [6,9], we further establish finite time blowup.

The finite time singularity of (1.2) on $\mathbb{R}$ from C_c^∞ initial data established in [9] has expanding support, and the vorticity blows up at ∞. The singularities of the gCLM model (1.3) with weak advection constructed in [5,9,19,20] are focusing, and the blowups occur at the origin. Due to the relatively strong advection and the compactness of the circle, the C^α singular solution of (1.2) on S^1 we construct is neither expanding nor focusing, which is similar to the solution in [6]. Moreover, the solution blows up in most places at the blowup time. Compared to the analysis of the gCLM model in [6], the blowup analysis of (1.2) with C^α data is more complicated due to the less regular profile and its estimates in the nonlinear stability analysis with singular weights.

3. One-point blowup criterion

In this section, we establish the one-point blowup criterion in Theorem 1.

Recall the class X defined in (1.4) and the Hilbert transform on the circle with period π:

$$
\begin{aligned}
u_x &= H\omega = \frac{1}{\pi}\,\mathrm{P.V.}\int_{-\pi/2}^{\pi/2} \omega(y)\cot(x-y)\,dy, \\
u &= -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \omega(y)\log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy.
\end{aligned} \tag{3.1}
$$

For (1.2) with initial data $\omega_0 \in X$, it is not difficult to find that $\omega(\cdot,t), -u(\cdot,t)$ remain in X.

3.1. Energy estimate

To perform an energy estimate using (2.2), we multiply both sides of (2.2) by $-\sin(2x) \in X$ so that $-\frac{\omega_x^2}{\omega}\sin(2x) \geq 0$. Integrating over S^1, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{S^1} -\frac{\omega_x^2}{\omega}\sin(2x)\,dx &= \frac{1}{2}\int_{S^1}\left(u\frac{\omega_x^2}{\omega}\right)_x \sin(2x)\,dx - \int_{S^1}\omega_x H\omega_x \sin(2x)\,dx \\
&=: I + II.
\end{aligned} \tag{3.2}
$$

We introduce the following functionals:

$$
\begin{aligned}
A(\omega) &:= \int_{S^1} -\frac{\omega_x^2}{\omega}\sin(2x)\,dx, \qquad E(\omega) := A(\omega) + u_x(0) + \|\omega\|_{L^1}, \\
U(t) &:= \int_0^t u_x(0,s)\,ds.
\end{aligned}
\tag{3.3}
$$

We choose the special function $\sin 2x$ due to the crucial cancellation in Lemma A.3,

$$
II = \int_{S^1} \omega_x H\omega_x \sin(2x)\,dx = 0. \tag{3.4}
$$

For I, using integration by parts, we obtain

$$
\begin{aligned}
I &= -\frac{1}{2}\int_{S^1} u\frac{\omega_x^2}{\omega}(\sin(2x))_x\,dx = -\int_{S^1}\frac{u\cos(2x)}{\sin(2x)}\frac{\omega_x^2}{\omega}\sin(2x)\,dx \\
&= -2\int_0^{\pi/2}\frac{u\cos(2x)}{\sin(2x)}\frac{\omega_x^2}{\omega}\sin(2x)\,dx.
\end{aligned}
$$

A crucial observation is that by taking advantage of the conservation form $\big(u\frac{\omega_x^2}{\omega}\big)_x$ and estimating on (2.2) with an explicit function, the coefficient $\frac{u\cos(2x)}{\sin(2x)}$ in the nonlinear term I for x away from $x = 0, \pi/2$ is of lower order than u_x, ω. We further estimate I from above. Since $\omega, -u \in X$, we derive $-\frac{\omega_x^2}{\omega}\sin(2x) \geq 0$, $\frac{u}{\sin(2x)} \geq 0$, and $\cos(2x) \leq 0$ on $[\pi/4, \pi/2]$. It follows that

$$
I \leq -2\int_0^{\pi/4}\frac{u\cos(2x)}{\sin(2x)}\frac{\omega_x^2}{\omega}\sin(2x)\,dx \lesssim \left\|\frac{u}{\sin x}\right\|_{L^\infty[0,\pi/4]} A(\omega), \tag{3.5}
$$

where $A(\omega)$ is defined in (3.3). The fact that the nonlinear term in $[\pi/4, \pi/2]$ is harmless is related to the discussion in Section 2.1. To control $\frac{u}{\sin x}$, we use the following extrapolation.

Lemma 3.1. *Suppse that $\omega \in X$ satisfies $A(\omega) < \infty$, $u_x(0) < \infty$ and $\omega \in L^1$. Then*

$$
\left\|\frac{u}{\sin x}\right\|_{L^\infty[0,\pi/4]} \lesssim (u_x(0) + \|\omega\|_{L^1} + 1)\log(\|\omega\|_{L^\infty[0,\pi/3]} + 2), \tag{3.6}
$$

$$
\begin{aligned}
&\big\||\cos x|^{1/2}\omega\big\|_{L^\infty} \lesssim (A(\omega)(u_x(0) + \|\omega\|_{L^1}))^{1/2}, \\
&\|\sin x \cdot \omega\|_{L^\infty} \lesssim (A(\omega)|u_x(\pi/2)|)^{1/2}.
\end{aligned}
\tag{3.7}
$$

We remark that $\|\omega\|_{L^\infty[0,\pi/3]}$ can be further bounded by $\big\||\cos x|^{1/2}\omega\big\|_{L^\infty}$.

Proof. Denote

$$
\begin{aligned}
K(x,y) &= \frac{\sin y}{\sin x}\log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| = \frac{\sin y}{\sin x}\log\left|\frac{\tan x + \tan y}{\tan x - \tan y}\right|, \\
f(x) &= x\log\left|\frac{x+1}{x-1}\right|.
\end{aligned}
$$

From (3.1), we get

$$\frac{u}{\sin x} = -\frac{1}{\pi}\int_0^{\pi/2} \omega(y)\frac{1}{\sin x}\log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy = -\frac{1}{\pi}\int_0^{\pi/2} \frac{\omega(y)}{\sin y} K(x,y)\,dy. \quad (3.8)$$

For $\varepsilon < \frac{1}{10}$ to be determined, we decompose (3.8) as follows:

$$\begin{aligned}
&\left|\frac{u}{\sin x}\right| \\
&\lesssim \int_0^{\pi/2} \mathbf{1}_{|y/x-1|>\varepsilon}\left|\frac{\omega(y)}{\sin y}K(x,y)\right| dy + \int_0^{\pi/2} \mathbf{1}_{|y/x-1|\le\varepsilon}\left|\frac{\omega(y)}{\sin x}\right| \log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy \\
&=: I + II.
\end{aligned}$$

Denote $z = \frac{\tan y}{\tan x}$. For $|y/x-1| > \varepsilon$, $x, y \in [0, \pi/2]$, we have

$$|z-1| = \left|\frac{\tan y - \tan x}{\tan x}\right| = \left|\frac{\sin(x-y)}{\cos x \cdot \cos y \cdot \tan x}\right| = \left|\frac{\sin(x-y)}{\cos y \cdot \sin x}\right| \gtrsim \frac{|x-y|}{x} \gtrsim \varepsilon.$$

For $x \in [0, \pi/4]$ and $y \in [0, \pi/2]$, using $\sin x \asymp \tan x$, $\sin y \le \tan y$ and the above estimate, we get

$$K(x,y) \lesssim \frac{\tan y}{\tan x}\log\left|\frac{\tan x + \tan y}{\tan x - \tan y}\right| = z\log\left|\frac{z+1}{z-1}\right| = f(z) \lesssim \log\varepsilon^{-1},$$

where we have used $f(z) \lesssim 1$ for $z > 2$ and $z < 1/2$ to obtain the last inequality. Hence

$$\begin{aligned}
I &\lesssim \log\varepsilon^{-1} \cdot \int_0^{\pi/2} \frac{|\omega(y)|}{\sin y}\,dy \lesssim \log\varepsilon^{-1} \cdot \int_0^{\pi/2} (-\omega(y))(\cot y + 1)\,dy \\
&\lesssim \log\varepsilon^{-1} \cdot (u_x(0) + \|\omega\|_1).
\end{aligned}$$

For II, since $|y/x-1| \le \varepsilon < \frac{1}{10}$ and $x \in [0, \pi/4]$, we get $y \in [0, \pi/3]$. Since $\sin z \asymp z$ on $[0, 3\pi/4]$, we get $\left|\frac{\sin(x+y)}{\sin(x-y)}\right| \lesssim \left|\frac{x+y}{x-y}\right|$. Using these estimates, we derive

$$\begin{aligned}
II &\lesssim \|\omega\|_{L^\infty[0,\pi/3]} \int_{|y/x-1|\le\varepsilon} \left(1 + \log\left|\frac{y+x}{y-x}\right|\right)\frac{1}{x}\,dy \\
&= \|\omega\|_{L^\infty[0,\pi/3]} \int_{1-\varepsilon}^{1+\varepsilon} \left(1 + \log\left|\frac{1+z}{1-z}\right|\right) dz.
\end{aligned}$$

Using the change of variable $s = z - 1 \in [-\varepsilon, \varepsilon]$, we further obtain

$$II \lesssim \|\omega\|_{L^\infty[0,\pi/3]} \int_{|s|\le\varepsilon} \log|s|^{-1} ds \lesssim \varepsilon\log\varepsilon^{-1} \cdot \|\omega\|_{L^\infty[0,\pi/3]}.$$

Choosing $\varepsilon = (\|\omega\|_{L^\infty[0,\pi/3]} + 10)^{-1} < \frac{1}{10}$, we prove

$$\begin{aligned}
\|u(\sin x)^{-1}\|_{L^\infty[0,\pi/4]} &\lesssim (u_x(0) + \|\omega\|_1 + 1)\log\varepsilon^{-1} \\
&\lesssim (u_x(0) + \|\omega\|_1 + 1)\log(\|\omega\|_{L^\infty[0,\pi/3]} + 2),
\end{aligned}$$

which is exactly (3.6).

For $x \in [0, \pi/2]$, using the Cauchy–Schwarz inequality, we prove

$$\begin{aligned}
|\omega(x)(\cos x)^{1/2}| &\le (\cos x)^{1/2} \int_0^x |\omega_x(y)|\, dy \\
&\le \int_0^x |\omega_x(y)|(\cos y)^{1/2}\, dy \\
&\lesssim \Big(\int_0^{\pi/2} \frac{\omega_x^2}{|\omega|} \sin(2x)\, dx \int_0^{\pi/2} |\omega(x)|(\cot x + 1)\, dx \Big)^{1/2} \\
&\lesssim \big(A(\omega)(u_x(0) + \|\omega\|_{L^1})\big)^{1/2},
\end{aligned}$$

which is the first inequality in (3.7). The proof of the second one is similar. ■

3.1.1. Estimates of $\|\omega\|_{L^1}, u_x(0)$. To close the energy estimate using Lemma 3.1, we further estimate $\|\omega\|_{L^1}, u_x(0)$ in terms of $U(t)$. Similar estimates have been established in [5] and in the arXiv version of [9]. Integrating (1.2) over $[0, \pi/2]$ and using integration by parts yields

$$\frac{d}{dt} \int_0^{\pi/2} -\omega\, dx = \int_0^{\pi/2} (-u_x\omega + u\omega_x)\, dx = -2 \int_0^{\pi/2} u_x \omega\, dx =: III. \tag{3.9}$$

Since ω is odd, symmetrizing the kernel in (3.1) we obtain

$$\begin{aligned}
III &= -\frac{2}{\pi} \int_0^{\pi/2} \omega(x) \int_0^{\pi/2} \omega(y)\big(\cot(x-y) - \cot(x+y)\big)\, dy\, dx \\
&= \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) \cot(x+y)\, dx\, dy \\
&= \frac{4}{\pi} \int_0^{\pi/2} (-\omega(x)) \Big(-\int_0^x \omega(y) \cot(x+y)\, dy \Big).
\end{aligned} \tag{3.10}$$

Since $-\omega(x) \ge 0$ on $[0, \pi/2]$ and $\cot z$ is decreasing on $[0, \pi]$, we get

$$\begin{aligned}
-\int_0^x \omega(y) \cot(x+y)\, dy &\le -\int_0^x \omega(y) \cot y\, dy \\
&\le -\int_0^{\pi/2} \omega(y) \cot y\, dy \lesssim u_x(0).
\end{aligned} \tag{3.11}$$

It follows that

$$III \lesssim u_x(0) \int_0^{\pi/2} (-\omega(y))\, dy, \quad \frac{d}{dt} \int_0^{\pi/2} -\omega(y)\, dy = III \lesssim u_x(0) \int_0^{\pi/2} -\omega(y)\, dy.$$

Using Gronwall's inequality, we establish

$$\|\omega(t)\|_{L^1} \le \|\omega_0\|_{L^1} \exp\Big(C \int_0^t u_x(0, s)\, ds \Big) \lesssim \|\omega_0\|_{L^1} \exp(CU(t)).$$

Taking the Hilbert transform on both sides of (1.2) and applying Lemma A.1, we derive

$$\begin{aligned}\frac{d}{dt}u_x(0) &= H(u_x\omega - u\omega_x)(0) = 2H(u_x\omega)(0) - H(\partial_x(u\omega))(0)\\ &= u_x^2(0) - \omega^2(0) + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cot y \cdot (u\omega)_x(y)\,dy\\ &= u_x^2(0) + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{\sin^2 y} u\omega\,dy. \end{aligned} \tag{3.12}$$

Note that $u\omega \le 0$ for all x and $u_x(0) \ge 0$ for $\omega \in X$. It follows that

$$\frac{d}{dt}u_x(0) \le u_x^2(0).$$

Using Gronwall's inequality, we obtain

$$0 \le u_x(0,t) \le u_x(0,0)\exp(U(t)) = H\omega_0(0)\exp(U(t)).$$

Plugging the above estimates, (3.4), (3.5) and Lemma 3.1 in (3.2), we obtain

$$\begin{aligned}\frac{d}{dt}A(\omega) &\lesssim C(\|\omega_0\|_{L^1}, H\omega_0(0))\exp(CU(t))\\ &\quad \cdot A(\omega)\log\Big(\big(A(\omega)(u_x(0) + \|\omega\|_{L^1})\big)^{1/2} + 2\Big),\end{aligned}$$

where $C(\|\omega_0\|_{L^1}, H\omega_0(0))$ is some constant only depending on $\|\omega_0\|_{L^1}, H\omega_0(0)$. Recall the energy $E(\omega)$ in (3.3). Combining the above estimates, we establish

$$\frac{d}{dt}E(\omega) \lesssim C(\|\omega_0\|_{L^1}, H\omega_0(0))\exp(CU(t)) \cdot E\log(E+2).$$

Solving the differential inequality, we prove

$$E(\omega) \le (E(\omega_0)+2)\exp\bigg(\exp\bigg(C(\|\omega_0\|_{L^1}, H\omega_0(0))\int_0^t \exp(CU(s))\,ds\bigg)\bigg). \tag{3.13}$$

3.2. Estimate near $x = \pi/2$

In view of Lemma 3.1, we have control of $\|\omega\|_{L^\infty[0,a]}$ using $A(\omega), u_x(0)$ and $\|\omega\|_{L^1}$ only away from $x = \pi/2$, i.e. $a < \pi/2$, due to the vanishing weight $(\cos x)^{1/2}$. We further estimate $u_x(\pi/2, t)$ so that we can apply Lemma 3.1 to control $\|\omega\|_\infty$. This will enable us to apply the BKM type blowup criterion for (1.2) to establish Theorem 1.

Using a derivation similar to that in (3.12), we obtain

$$\frac{d}{dt}u_x\Big(\frac{\pi}{2}\Big) = u_x^2\Big(\frac{\pi}{2}\Big) + \frac{1}{\pi}\int_0^\pi \frac{1}{\cos^2 y}u\omega\,dy =: I + II. \tag{3.14}$$

A crucial observation is that for $\omega \in X$, $u_x(\pi/2) = \frac{1}{\pi}\int_0^\pi \omega(y)\tan y\,dy$ is negative. Thus the vortex stretching term $u_x^2(\pi/2)$ depletes the growth of $u_x(\pi/2)$, which is the main mechanism ensuring that $u_x(\pi/2)$ does not blow up as long as $U(t)$ is bounded. See also Section 2.1. On the other hand, since $u\omega \le 0$, the advection term $\frac{1}{\pi}\int_0^\pi \frac{1}{\cos^2 y} u\omega\, dy$ is negative and contributes to the growth of $u_x(\pi/2)$. Our goal is to show that the growing effect is weaker. The main difficulty is the singular functions $(\cos y)^{-2}$, $\tan y$ near $y = \pi/2$ in I and II since we can control ω away from $y = \pi/2$.

For II, we decompose it as follows:

$$II = \frac{1}{\pi}\int_0^\pi \tan^2 y \cdot u\omega\, dy + \frac{1}{\pi}\int_0^\pi u\omega\, dy =: II_1 + II_2.$$

Since II_2 does not involve a singular function, its estimate is simple. Using (3.1), we get

$$\begin{aligned} |u(x)| &\lesssim \int_0^\pi |\omega(y)|\, |\cos y|^{1/2} |\cos y|^{-1/2} \big|\log|\sin(x-y)|\big|\, dy \\ &\lesssim \big\| |\cos x|^{1/2}\omega \big\|_\infty \big\| |\cos x|^{-1/2} \big\|_{L^{4/3}} \|\log x\|_{L^4} \lesssim \big\| |\cos x|^{1/2}\omega \big\|_\infty . \end{aligned}$$

It follows that

$$|II_2| \le \|u\|_{L^\infty} \|\omega\|_{L^1} \lesssim \big\| |\cos x|^{1/2}\omega \big\|_\infty \|\omega\|_{L^1}. \tag{3.15}$$

For I and II_1, our goal is to establish

$$I + II_1 \ge \tfrac{1}{4} u_x^2(\pi/2) - C|u_x(\pi/2)| \cdot \|\omega\|_{L^\infty}. \tag{3.16}$$

We will further use Lemma 3.1 and the ε-Young inequality to estimate $|u_x(\pi/2)| \cdot \|\omega\|_{L^\infty}$ and close the estimate of $u_x(\pi/2)$ in (3.14). Note that near $y = \pi/2$, we have

$$(\cos y)^{-1}, \tan y = \frac{1}{\pi/2 - y} + O(|\pi/2 - y|).$$

For simplicity, we consider the coordinate near $\pi/2$ and introduce

$$f(x) = \omega(x + \pi/2), \quad g(x) = u(x + \pi/2), \quad s(x, y) = \frac{\tan y}{\tan x}. \tag{3.17}$$

Remark 3.2. Since $\tan z = z + O(z^3)$, $\sin z = z + O(z^3)$ near $z = 0$, in the following derivations, we essentially treat $\tan z$, $\sin z$ similarly to z.

Clearly, $g_x = Hf$, g and f are odd and $f \ge 0$, $g \le 0$ on $(0, \pi/2)$. Using (3.1), (3.17), and $(\tan(x + \pi/2))^2 = (\tan x)^{-2}$, and symmetrizing the integrals in I, II_1, we get

$$\begin{aligned} I &= (H\omega(\pi/2))^2 = (Hf(0))^2 = \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} f(x)f(y)\cot x \cot y\, dx\, dy \\ &= \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{f(x)f(y)}{\tan x \cdot \tan y}\, dx\, dy, \\ II_1 &= \frac{1}{\pi}\int_0^\pi \frac{fg}{\tan^2 x}\, dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{fg}{\tan^2 x}\, dx \\ &= -\frac{2}{\pi^2}\int_0^{\pi/2} \frac{f(x)}{\tan^2 x}\int_0^{\pi/2} f(y) \log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy \\ &= -\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} f(x)f(y)\left(\frac{1}{\tan^2 x} + \frac{1}{\tan^2 y}\right) \log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy. \end{aligned}$$

Recall s from (3.17). Note that

$$\left|\frac{\sin(x+y)}{\sin(x-y)}\right| = \left|\frac{\tan x + \tan y}{\tan x - \tan y}\right| = \left|\frac{s+1}{1-s}\right|, \quad \frac{1}{\tan x} = s\frac{1}{\tan y}. \tag{3.18}$$

We further obtain

$$I + II_1 = \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{f(x)f(y)}{\tan^2 y}\left(4s - (1+s^2)\log\left|\frac{s+1}{1-s}\right|\right) dx\, dy.$$

Note that $f(x)f(y) \geq 0$ for $x, y \in [0, \pi/2]$. The competition between I, II_1 is characterized by the interaction kernel $K(s) = 4s - (1+s^2)\log|\frac{s+1}{1-s}|, s \in [0, \infty)$. An important observation is that for large s or small s, $K(s) \approx 2s$. In particular, it is easy to obtain

$$\begin{aligned} K(s) = s^2 K(s^{-1}), \quad K(s) &\geq s - (1+s^2)\log\left|\frac{1+s}{1-s}\right| \cdot \mathbf{1}_{a\leq s\leq a^{-1}} \\ &\geq s - C\log\left|\frac{1+s}{1-s}\right| \cdot \mathbf{1}_{a\leq s\leq a^{-1}} \end{aligned}$$

for some absolute constants $0 < a < 1$ and $C > 0$. It follows that

$$I + II_1 \geq \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\frac{f(x)f(y)}{\tan^2 y}\left(s - C\log\left|\frac{s+1}{1-s}\right| \cdot \mathbf{1}_{a\leq s\leq a^{-1}}\right) dx\, dy.$$

Repeating the above derivations, we get

$$I + II_1 \geq \frac{1}{4}(Hf(0))^2 - C\int_0^{\pi/2}\frac{f(y)}{\tan^2 y}\int_0^{\pi/2}\log\left|\frac{s+1}{1-s}\right| \cdot \mathbf{1}_{a\leq s\leq a^{-1}} f(x)\, dx\, dy. \tag{3.19}$$

Next, we show that

$$|J(y)| \lesssim \|f\|_{L^\infty}, \quad J(y) := \frac{1}{\tan y}\int_0^{\pi/2}\log\left|\frac{s+1}{1-s}\right| \cdot \mathbf{1}_{a\leq s\leq a^{-1}} f(x)\, dx.$$

We consider a change of variable $z = \tan x$. The restriction $s \in [a, a^{-1}]$ implies $z \in [a\tan y, a^{-1}\tan y]$. Using $dx = \frac{1}{1+z^2}dz$ and (3.18) yields

$$\begin{aligned} J(y) &\lesssim \frac{\|f\|_\infty}{\tan y}\int_{a\tan y}^{a^{-1}\tan y}\log\left|\frac{z+\tan y}{z-\tan y}\right|\frac{1}{1+z^2}dz \\ &\lesssim \frac{\|f\|_\infty}{\tan y}\int_{a\tan y}^{a^{-1}\tan y}\log\left|\frac{z+\tan y}{z-\tan y}\right| dz \\ &\lesssim \|f\|_{L^\infty}\int_a^{a^{-1}}\log\left|\frac{\tau+1}{\tau-1}\right| d\tau \lesssim \|f\|_{L^\infty}, \end{aligned}$$

where we have used another change of variable, $z = \tau\tan y$, to obtain the third estimate.

Recall $f(x) = \omega(x+\pi/2)$ from (3.17). Plugging the above estimates in (3.19), we establish

$$I + II_1 \geq \tfrac{1}{4}(Hf(0))^2 - C\int_0^{\pi/2}\frac{|f(y)|}{\tan y}dy\,\|f\|_{L^\infty} = \tfrac{1}{4}(Hf(0))^2 - C|Hf(0)| \cdot \|f\|_{L^\infty},$$

where we have used the facts that f is odd and that it has a fixed sign on $[0, \pi/2]$ to obtain the equality. We have thus proved (3.16).

3.2.1. Estimate of $u_x(\pi/2)$. Combining the estimates (3.14)–(3.16), we obtain

$$\frac{d}{dt}u_x(\pi/2) \geq \tfrac{1}{4}u_x^2(\pi/2) - C|u_x(\pi/2)| \cdot \|\omega\|_{L^\infty} - \big\||\cos x|^{1/2}\omega\big\|_\infty \|\omega\|_{L^1} =: J.$$

Recall the energies in (3.3). Using Lemma 3.1, we derive

$$\|\omega\|_{L^\infty} \lesssim |u_x(\pi/2)|^{1/2}(E(\omega))^{1/2} + E(\omega), \quad \big\||\cos x|^{1/2}\omega\big\|_\infty \|\omega\|_{L^1} \lesssim E^2(\omega).$$

Using the ε-Young inequality, we

$$\begin{aligned} J &\geq \tfrac{1}{4}u_x^2(\pi/2) - C|u_x(\pi/2)|\big(|u_x(\pi/2)|^{1/2}(E(\omega))^{1/2} + E(\omega)\big) - CE^2(\omega) \\ &\geq \tfrac{1}{8}u_x^2(\pi/2) - CE^2(\omega). \end{aligned}$$

Since $u_x(\pi/2) \leq 0$, we derive

$$\frac{d}{dt}|u_x(\pi/2)| \leq -\tfrac{1}{8}u_x^2(\pi/2) + CE^2(\omega).$$

Using the estimate (3.13), we get

$$\begin{aligned} |u_x(t,\pi/2)| &\leq |H\omega_0(\pi/2)| + C\int_0^t E^2(\omega(s))\,ds \\ &\leq |H\omega_0(\pi/2)| + C(E(\omega_0)+2)^2 \\ &\quad \cdot \exp\bigg(2\exp\bigg(C(\|\omega_0\|_{L^1}, H\omega_0)\int_0^t \exp(CU(s))\,ds\bigg)\bigg). \end{aligned} \tag{3.20}$$

3.2.2. The blowup criterion. Using (3.13), (3.20) and Lemma 3.1, we prove

$$\|\omega\|_{L^\infty} \leq K_1(\omega_0)\exp\bigg(2\exp\bigg(K_1(\omega_0)\int_0^t \exp(CU(s))\,ds\bigg)\bigg), \tag{3.21}$$

where C is some absolute constant, and the constant $K_1(\omega_0)$ depends on $H\omega_0(0)$, $H\omega_0(\pi/2)$, $\|\omega_0\|_{L^1}$ and $A(\omega_0)$. Applying the BKM-type blowup criterion, we conclude the proof of Theorem 1.

4. Stabilizing effect of advection and several quadratic forms

In order to apply Theorem 1 to establish the well-posedness result, we need to control $u_x(0)$. Yet, $u_x(0)$ itself does not enjoy a good estimate. Recall the ODE for $u_x(0)$ from (3.12).

$$\frac{d}{dt}u_x(0) = u_x^2(0) + \frac{2}{\pi}\int_0^{\pi/2}\frac{u\omega}{\sin^2 y}\,dy.$$

Since $u_x(0) \geq 0$ for $\omega \in X$, the quadratic nonlinearity $u_x^2(0)$ makes it very difficult to obtain a long time estimate on $u_x(0)$. Since $u_x(0) = \frac{-2}{\pi} \int_0^{\pi/2} \omega(y) \cot y \, dy$ can be viewed as a weighted integral of ω with a singular weight near 0, it motivates us to estimate another weighted integral that controls $u_x(0)$. For $\beta \in (1, 3)$, we introduce

$$Q(\beta, t) := -\int_0^{\pi/2} \omega(y, t)(\cot y)^\beta \, dy, \quad B(\beta, t) \triangleq \int_0^{\pi/2} (u_x \omega - u \omega_x)(\cot x)^\beta \, dx, \tag{4.1}$$

For $\omega \in X \cap H^1$, $Q(\beta, t)$ and $B(\beta, t)$ are well-defined if ω vanishes near $x = 0$ at order $|x|^\gamma$ with $\gamma > \beta - 1$. For $\omega \in X$, since $\omega \leq 0$ on $[0, \pi/2]$, we have $Q(\beta, t) \geq 0$. The boundedness of $Q(\beta, t)$ implies that ω cannot be too large near 0, and it allows us to control the weighted integral of ω near 0. In Section 5, we will combine it and $\|\omega\|_{L^1}$ to further control $u_x(0)$.

Remark 4.1. The special singular function $(\cot y)^\beta$ and the functional $Q(\beta, t)$ are motivated by the homogeneous function $|y|^{-\beta}$ and $\int_{\mathbb{R}^+} \omega / y^\beta \, dy$, which were used to analyze the gCLM model on the real line in the arXiv version of [9].

Using (1.2), we obtain the ODE for $Q(\beta, t)$:

$$\frac{d}{dt} Q(\beta, t) = -B(\beta, t). \tag{4.2}$$

We should further estimate $B(\beta, t)$. The key lemma to prove Theorem 2 is the following. To simplify the notation, we will drop "t" in some places.

Lemma 4.2. *Suppose that $\omega \in C^\alpha$ is odd with $\alpha \in (0, 1)$ and $\omega(x)x^{-1} \in L^\infty$. There exists some absolute constant $\beta_0 \in (1, 2)$ such that for $\beta \in [\beta_0, 2)$, we have*

$$B(\beta) \geq -(2 - \beta)\Big(u_x(0) Q(\beta) + \frac{1}{\pi} \iint_{[0,\pi/2]^2} \omega(x)\omega(y)(\cot y)^{\beta - 1} \frac{s(s^{\beta-1} - 1)}{s^2 - 1} \, dx \, dy \Big), \tag{4.3}$$

where $s(x, y) = \frac{\cot x}{\cot y}$. If in addition $\omega \in C^{1,\alpha}$ with $\alpha \in (0, 1)$ and $\omega_x(0) = 0$, then for $\beta = 2$ we have

$$B(2) \geq 0.$$

Note that in Lemma 4.2, we do not impose the sign condition $\omega \leq 0$ (or ≥ 0) on $[0, \pi/2]$. Thus, it is likely that Lemma 4.2 can be generalized to study (1.2) with a larger class of data.

Lemma 4.2 quantifies the stabilizing effect of advection, and reflects the fact that advection is stronger or almost stronger than vortex stretching for ω vanishing at least linearly near $x = 0$, which has been discussed heuristically in Section 1.3. In fact, if $\omega \in C^{1,\alpha}$ with $\omega_x(0) = 0$, using (4.2) and Lemma 4.2 we find that $Q(2, t)$ is bounded uniformly in t and thus ω cannot be too large near 0. In the general case, ω can vanish only linearly near $x = 0$. Then $Q(2, t)$ is not well-defined since $\omega(\cot y)^2$ is not integrable. In this case, we apply (4.3). Though $Q(\beta,t)$ may not be bounded uniformly in t, the critical small factor $2-\beta$ indicates that $Q(\beta,t)$ cannot grow too fast.

4.1. Symmetrization and derivation of the kernel

To prove Lemma 4.2, we first symmetrize the quadratic form $B(\beta)$ and derive its associated interaction kernel. The symmetrization idea has been used in [11] to analyze some quadratic forms in the Hou–Luo model. Denote

$$s = \frac{\tan y}{\tan x} = \frac{\cot x}{\cot y}. \tag{4.4}$$

Since ω is odd, applying (3.1) and following the symmetrization argument in the arXiv version of [9], we derive the following in Appendix A.3 if ω vanishes near $x=0$ at order $|x|^\gamma$ with $\gamma > \beta - 1$:

$$B(\beta) = \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)P_\beta(x,y)\,dx\,dy, \tag{4.5}$$

where

$$\begin{aligned} P_\beta(x,y) &= (\cot y)^{\beta-1}\Big(\frac{\beta}{2}(s^{\beta-1}+1)\log\Big|\frac{s+1}{s-1}\Big| - (s^{\beta-1}-1)\frac{2s}{s^2-1}\Big) \\ &\quad + (\cot y)^{\beta+1}\Big(\frac{\beta}{2}(s^{\beta+1}+1)\log\Big|\frac{s+1}{s-1}\Big| - (s^{\beta+1}-1)\frac{2s}{s^2-1}\Big) \\ &=: (\cot y)^{\beta-1}P_{2,\beta}(s) + (\cot y)^{\beta+1}P_{1,\beta}(s). \end{aligned} \tag{4.6}$$

Similar derivations and kernels were obtained in the arXiv version of [9]. The logarithmic terms come from the advection term $u\omega_x$ and are positive. The other terms $-(s^\tau-1)\frac{2s}{s^2-1}$, $\tau = \beta-1, \beta+1$, come from the vortex stretching term $u_x\omega$ and are negative. Thus, the kernel P_β captures the competition between the two terms. The main term in P_β is $(\cot y)^{\beta+1}P_{1,\beta}(s)$ since $(\cot y)^{\beta+1}$ is more singular. For s near 1, $P_\beta(s)$ is positive due to the singularity in $\log|\frac{1+s}{s-1}|$. It is not difficult to see that

$$\lim_{s\to\infty} P_{1,\beta}(s) = (\beta-2)s^\beta, \quad \lim_{s\to\infty} P_{2,\beta}(s) = (\beta-2)s^{\beta-2}. \tag{4.7}$$

Formally, as β increases, the kernel $P_\beta(x,y)$ becomes more positive-definite. Recall the ODE for $Q(\beta)$ from (4.1), (4.2). The higher the vanishing order of ω near 0, the larger β we can choose with $Q(\beta)$ being well-defined, and it is more likely that $Q(\beta,t)$ is decreasing and bounded uniformly in t. Therefore, the higher vanishing order of ω near 0 reflects the stronger effect of advection, which potentially depletes the growing effect of vortex stretching. The asymptotics (4.7) suggests that to obtain the positive-definiteness of P_β, β should be at least 2. Indeed, such a result is proved in the arXiv version of [9] for $\beta = 2.2$ under the sign condition $\omega \in X$ (see (1.4)) by showing that $P_{i,\beta}(s) \geq 0$ pointwise.

However, the method in [9] cannot be applied to the critical case $\beta = 2$ since a numerical result shows that $P_{1,2}(s) < 0$ for $s \le 0.5$ or $s \ge 2$.

For $\beta < 2$, it is not expected that P_β is positive-definite and the gap is of order $2-\beta$ quantified in Lemma 4.2. We study the modified kernel and its associated quadratic form:

$$\begin{aligned} K_{1,\beta}(s) &= P_{1,\beta}(s) + (2-\beta)(s+s^\beta), \\ K_{2,\beta}(s) &= P_{2,\beta}(s) + (2-\beta)\frac{(s^{\beta-1}-1)s}{s^2-1}, \\ K_\beta &= (\cot y)^{\beta+1} K_{1,\beta} + (\cot y)^{\beta-1} K_{2,\beta}, \\ \tilde{B}(\beta) &= \int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y) K_\beta(x,y)\, dx\, dy, \end{aligned} \tag{4.8}$$

where P_i, s are defined in (4.6), (4.4). Using (4.5), (4.6), (4.8), and the identities

$$\begin{aligned} &(s+s^\beta)(\cot y)^{\beta+1} = \cot x \cdot (\cot y)^\beta + (\cot x)^\beta \cot y, \\ &\frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)(s+s^\beta)(\cot y)^{\beta+1}\, dx\, dy \\ &\qquad = \frac{2}{\pi}\int_0^{\pi/2} \omega \cot y\, dy \int_0^{\pi/2} \omega (\cot y)^\beta\, dy = u_x(0) Q(\beta), \end{aligned} \tag{4.9}$$

we derive

$$\begin{aligned} \frac{\tilde{B}(\beta)}{\pi} &= \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y) \\ &\quad \cdot \left\{ P_\beta(x,y) + (2-\beta)\left((s+s^\beta)(\cot y)^{\beta+1} + \frac{(s^{\beta-1}-1)s}{s^2-1}(\cot y)^{\beta-1} \right) \right\} dx\, dy \\ &= B(\beta) \\ &\quad + (2-\beta)\left(u_x(0)Q(\beta) + \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)\frac{(s^{\beta-1}-1)s}{s^2-1}(\cot y)^{\beta-1}\, dx\, dy \right). \end{aligned} \tag{4.10}$$

Hence, Lemma 4.2 is equivalent to $\tilde{B}(\beta) \ge 0$, or the positive-definiteness of K_β for $\beta \in [\beta_0, 2]$.

Our key observation is that $s(x,y) = \frac{\cot x}{\cot y}$ can be written as $p(u-v)$ for some function p and variables u, v, and K_β can be written as a convolution kernel after a change of variable. This allows us to follow the idea in Bochner's theorem for a positive-definite function to leverage the positive part of $K_\beta(s)$ and establish that K_β is positive-definite.

In the following derivation, we restrict β to $\beta \in [1.9, 2]$. The reader can think of the special case $\beta = 2$, since we will choose β to be sufficiently close to 2.

4.1.1. Reformulation of $K_{1,\beta}$. We introduce

$$
\begin{aligned}
F_1(x) &:= \omega(x)(\cot x)^{\frac{\beta+1}{2}},\\
\tilde{K}_{1,\beta}(s) &:= s^{-\frac{\beta+1}{2}} K_{1,\beta}\\
&= \frac{\beta}{2}(s^{\frac{\beta+1}{2}} + s^{-\frac{\beta+1}{2}}) \log\left|\frac{s+1}{s-1}\right| - \frac{s^{\frac{\beta+1}{2}} - s^{-\frac{\beta+1}{2}}}{s^2-1} 2s\\
&\quad + (2-\beta)(s^{\frac{\beta-1}{2}} + s^{\frac{1-\beta}{2}}).
\end{aligned}
\tag{4.11}
$$

Recall $s \cot y = \cot x$ from (4.4). Using $s^{\frac{\beta+1}{2}}(\cot y)^{\beta+1} = (\cot y \cot x)^{\frac{\beta+1}{2}}$, we derive

$$
(\cot y)^{\beta+1} K_{1,\beta}(s) = (\cot y)^{\beta+1} s^{\frac{\beta+1}{2}} s^{-\frac{\beta+1}{2}} K_{1,\beta}(s) = (\cot y \cot x)^{\frac{\beta+1}{2}} \tilde{K}_{1,\beta}(s).
$$

Hence, we can rewrite the quadratic form associated with $K_{1,\beta}$ in $\tilde{B}(\beta)$ (see (4.8)) as follows:

$$
\begin{aligned}
B_1(\beta) &:= \int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)(\cot y)^{\beta+1} K_{1,\beta}(s)\, dx\, dy\\
&= \int_0^{\pi/2}\int_0^{\pi/2} F_1(x)F_1(y)\tilde{K}_{1,\beta}(s) dx\, dy.
\end{aligned}
\tag{4.12}
$$

For $x, y \in [0, \pi/2]$, we consider the change of variable

$$
x = \arctan e^r, \quad y = \arctan e^t, \quad F_2(z) = \frac{e^z F_1(\arctan e^z)}{1+e^{2z}}, \quad W_{1,\beta}(z) = \tilde{K}_{1,\beta}(e^z).
\tag{4.13}
$$

The variable $r = \log \tan x$ maps $(0, \pi/2)$ to $\mathbb{R}$. Using $\frac{dx}{dr} = \frac{e^r}{1+e^{2r}}$ and $s = \frac{\tan y}{\tan x} = e^{t-r}$, we obtain

$$
\begin{aligned}
B_1 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F_1(\arctan e^r)F_1(\arctan e^t)}{(1+e^{2r})(1+e^{2t})} \tilde{K}_{1,\beta}(e^{t-r}) e^r e^t\, dt\, dr\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} F_2(r)F_2(t)W_{1,\beta}(t-r)\, dt\, dr.
\end{aligned}
$$

Recall F_1 from (4.11). Since $\cot(\arctan e^r) = e^{-r}$, we can rewrite F_2 in terms of ω:

$$
F_2(r) = \frac{e^r \omega(\arctan e^r)(\cot(\arctan e^r))^{\frac{\beta+1}{2}}}{1+e^{2r}} = \frac{e^{-\frac{\beta-1}{2}r}\omega(\arctan e^r)}{1+e^{2r}}.
$$

Next, we discuss the integrability of $W_{1,\beta}$ and F_2. Since $\omega(x)x^{-1} \in L^\infty$ and $\arctan x \lesssim \min(x, 1)$, and since $\beta \in [1.9, 2]$, we get

$$
|F_2(r)| \lesssim e^{-\frac{\beta-1}{2}r} \min(1, e^r) \lesssim \min(e^{r/4}, e^{-r/4}).
$$

Recall the definition of $\tilde{K}_{1,\beta}$ in (4.11). Clearly, $|\tilde{K}_{1,\beta}(s)|^p, |W_{1,\beta}(z)|^p$ are locally integrable for any $p > 0$. Using (4.7) and $\left|\log|\frac{s+1}{s-1}| - \frac{2}{s}\right| \lesssim s^{-3}$ for $s > 2$, and a direct estimate, we obtain

$$
\tilde{K}_{1,\beta}(s) = \tilde{K}_{1,\beta}(s^{-1}), \quad |\tilde{K}_{1,\beta}(s)| \lesssim s^{-\frac{\beta-1}{2}} \lesssim s^{-1/4} \quad \text{for } s > 2.
$$

Note that for large s, the leading exponents $s^{\frac{\beta-1}{2}}$ appearing in each term of $\tilde{K}_{1,\beta}$ are canceled. As a result, we obtain

$$W_{1,\beta}(z) = \tilde{K}_{1,\beta}(e^z) = \tilde{K}_{1,\beta}(e^{-z}) = W_{1,\beta}(-z), \quad |W_{1,\beta}(z)| \lesssim e^{-|z|/4} \quad \text{for } |z| > 1. \tag{4.14}$$

Denote by $\hat{f}(\xi) = \int_{\mathbb{R}} \exp(-ix\xi) f(x)\, dx$ the Fourier transform of f. Using the Plancherel theorem, for some absolute constant $C_1 > 0$, we get

$$B_1(\beta) = C_1 \int_{\mathbb{R}} |\hat{F}_2(\xi)|^2 \widehat{W}_{1,\beta}(\xi)\, d\xi. \tag{4.15}$$

4.1.2. Reformulation of $K_{2,\beta}$. Similarly, we reformulate the kernel $K_{2,\beta}$ and its associated quadratic form in $\tilde{B}(\beta)$ in (4.8) as follows:

$$\begin{aligned} B_2(\beta) &:= \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y)(\cot y)^{\beta-1} K_{2,\beta}(s)\, dx\, dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} F_4(r) F_4(s) W_{2,\beta}(t-r)\, dt\, dr = C_1 \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{F}_4(\xi)|^2 \widehat{W}_{2,\beta}(\xi)\, d\xi \end{aligned} \tag{4.16}$$

for some absolute constant $C_1 > 0$, where

$$\begin{aligned} F_4(r) &= \frac{e^{\frac{3-\beta}{2} r} \omega(\arctan e^r)}{1 + e^{2r}}, \quad W_{2,\beta}(z) = \tilde{K}_{2,\beta}(e^z), \\ \tilde{K}_{2,\beta}(s) &= \frac{\beta}{2} \left((s^{\frac{\beta-1}{2}} + s^{-\frac{\beta-1}{2}}) \log \left| \frac{s+1}{s-1} \right| - (s^{\frac{\beta-1}{2}} - s^{-\frac{\beta-1}{2}}) \frac{2s}{s^2-1} \right). \end{aligned} \tag{4.17}$$

The variable F_4 corresponds to $\omega(x)(\cot x)^{\frac{\beta-1}{2}}$ after a change of variable. For $\tilde{K}_{2,\beta}$, $W_{2,\beta}$, F_4 with $s > 2$, $|z| > 1$, we have

$$\begin{aligned} &|F_4(r)| \lesssim \min(e^{r/4}, e^{-r/4}), \quad W_{2,\beta}(z) = W_{2,\beta}(-z), \quad \tilde{K}_{2,\beta}(s) = \tilde{K}_{2,\beta}(s^{-1}), \\ &|W_{2,\beta}(z)| \lesssim e^{-|z|/4}, \qquad |\tilde{K}_{2,\beta}(s)| \lesssim s^{-1/4}. \end{aligned}$$

4.2. *Positivity of $W_{j,\beta}$*

Recall formulas (4.15), (4.16) for $B_j(\beta)$. To show that $B_j(\beta) \geq 0$, it suffices to prove $\widehat{W}_{j,\beta}(\xi) \geq 0$ for any ξ. Since $W_{j,\beta}$ is even, it is equivalent to show that

$$\begin{aligned} G_{j,\beta}(\xi) &:= \frac{1}{2} \hat{W}_{j,\beta}(\xi) = \frac{1}{2} \int_{\mathbb{R}} W_{j,\beta}(x) e^{-ix\xi}\, dx = \int_{\mathbb{R}_+} W_{j,\beta}(x) \cos(x\xi)\, dx \\ &\geq 0 \end{aligned} \tag{4.18}$$

for any ξ. Since $G_{j,\beta}(\xi)$ and $\widehat{W}_{j,\beta}(\xi)$ are even, we can further restrict to $\xi \geq 0$. We first study the positivity of $G_{1,\beta}$, which is much more difficult than that of $G_{2,\beta}$.

4.2.1. Positivity of $G_{1,\beta}$. Since we are interested in the case where β is close to 2, using continuity, we can essentially reduce proving $G_{1,\beta} \geq 0$ to the special case $\beta = 2$.

Lemma 4.3. *Let $W = W_{1,2}$ and $G = G_{1,2}$. Suppose that there exist $x_0, M > 0$ such that*

$$G(\xi) > 0, \quad \xi \in [0, M], \tag{4.19}$$

$$W''(x) > 0, \quad x \in [0, x_0], \tag{4.20}$$

$$-W'(x_0) - \frac{1}{M}\left(|W''(x_0)| + \int_{x_0}^{\infty} |W'''(x)|\, dx\right) > 0. \tag{4.21}$$

Then there exists $\beta_0 \in (1, 2)$ such that for any $\beta \in [\beta_0, 2]$ and ξ, we have $G_{1,\beta}(\xi) \geq 0$.

Using continuity of $W_{1,\beta}$ in β and the smallness of $2 - \beta$, we will show that (4.19)–(4.21) hold for $W_{1,\beta}, G_{1,\beta}$. The proof of this part is standard and is deferred to Appendix A.4.

Next, we prove that (4.20)–(4.21) implies $G_{1,2}(\xi) \geq 0$ on $[M, \infty]$, which along with (4.19) proves $G_{1,2}(\xi) \geq 0$. The same argument applies to $G_{1,\beta}$. We simplify $W_{1,2}, G_{1,2}$ defined in (4.11), (4.13), (4.18) as W, G.

Large ξ. We will choose M to be relatively large. This allows us to exploit the oscillation in the integral $G(\xi)$ (see (4.18)) for $\xi \geq M$. From the definition of $W(x)$ in (4.11) and (4.13), we know that $W(x)$ is smooth away from $x = 0$ and $W(x)$ is singular of order $\log |x|$ near $x = 0$. Using integration by parts twice, we obtain

$$\begin{aligned} G(\xi) &= \xi^{-1} \int_{\mathbb{R}_+} W(x)\partial_x \sin(x\xi)\, dx = -\xi^{-1} \int_{\mathbb{R}_+} W'(x) \sin(x\xi)\, dx \\ &= -\xi^{-2} \int_{\mathbb{R}_+} W'(x)\partial_x(1 - \cos(x\xi))\, dx = \xi^{-2} \int_{\mathbb{R}_+} W''(x)(1 - \cos(x\xi))\, dx, \end{aligned} \tag{4.22}$$

where the boundary term vanishes since we have $W(x)\sin(x\xi) = O(x \log x)$ and $W'(x)(1 - \cos x\xi) = O(\frac{1}{x}x^2) = O(x)$ and the fast decay (4.14). The advantage of the above formula is that we obtain a nonnegative coefficient $1 - \cos(x\xi)$. For some $x_0 > 0$, we define

$$G_1(\xi) := \int_0^{x_0} W''(x)(1 - \cos(x\xi))\, dx, \quad G_2(\xi) := \int_{x_0}^{\infty} W''(x)(1 - \cos(x\xi))\, dx, \tag{4.23}$$

It suffices to verify $G_1(\xi) \geq 0$ and $G_2(\xi) \geq 0$. Thanks to (4.20) and $1 - \cos(\xi x) \geq 0$, we obtain $G_1(\xi) \geq 0$. For $G_2(\xi)$, the main term is associated with 1 since $\cos(x\xi)$ oscillates. In fact, using integration by parts again, we obtain

$$\begin{aligned} G_2(\xi) &= -W'(x_0) - \int_{x_0}^{\infty} W''(x)\cos(x\xi)\, dx \\ &= -W'(x_0) - \xi^{-1} \int_{x_0}^{\infty} W''(x)\partial_x \sin(x\xi)\, dx \\ &= -W'(x_0) + W''(x_0)\frac{\sin(x_0\xi)}{\xi} + \int_{x_0}^{\infty} W'''(x)\frac{\sin(x\xi)}{\xi}\, dx \\ &\geq -W'(x_0) - \frac{1}{M}\left(|W''(x_0)| + \int_{x_0}^{\infty} |W'''(x)|\, dx\right), \end{aligned}$$

where we have used $\xi \geq M$ in the last inequality. We choose $x_0 > 0$ and decompose the integral into two domains $x \leq x_0$ and $x > x_0$ in (4.23) since W''' in the above derivation is not integrable near $x = 0$. Using the assumption (4.21), we obtain $G_2(\xi) \geq 0$.

4.2.2. Verification of the conditions in Lemma 4.3. We now discuss how to verify conditions (4.19)–(4.21).

Firstly, $G(\xi)$ is smooth in ξ and the Lipschitz constant satisfies

$$|\partial_\xi G| \leq \int_{\mathbb{R}_+} |W(x)| x \, dx =: b_1. \tag{4.24}$$

The constant b_1 will be estimated rigorously. For small $\xi \in [0, M]$, we compute a lower bound of the integral $G(\xi)$ rigorously for the discrete points $\xi = ih$, $i = 0, 1, \dots, n$, $M = nh$, and verify $G(ih) > 0$. For $\xi \in [ih, (i+1)h]$, we use

$$G(\xi) \geq \min(G(ih), G((i+1)h)) - \frac{h}{2} b_1 > 0 \tag{4.25}$$

and verify the second inequality to obtain $G(\xi) > 0$. This enables us to establish (4.19).

For (4.20) and (4.21), let us first motivate why they hold for some x_0 and M. Using (4.11) and (4.13), we obtain the asymptotic behavior of $W(x)$ for x near 0:

$$W(x) \approx -C \log|e^x - 1| \approx -C \log x, \quad W'(x) \approx -\frac{C}{x} < 0, \quad W''(x) \approx \frac{C}{x^2} > 0,$$

for some constant $C > 0$. See also (A.3) for a detailed derivation. Note that W''' is integrable away from 0. Thus, (4.20) and (4.21) hold for small x_0 and large M.

In practice, we choose $x_0 = \log \frac{5}{3}$ and $M = 20$ in Lemma 4.3. Note that $W_{1,2}$ is an explicit function. We prove (4.20) for $x_0 = \log \frac{5}{3}$ in Appendix A.4. We discuss how to compute the integrals in (4.25) and (4.21) and verify these conditions, which are independent of ξ, rigorously in Appendix A.6. This allows us to establish the conditions in Lemma 4.3. The rigorous lower bound of $G(\xi)$ for $\xi = ih \in [0, M]$ is plotted in Figure 1, and $G(\xi)$ is strictly positive.

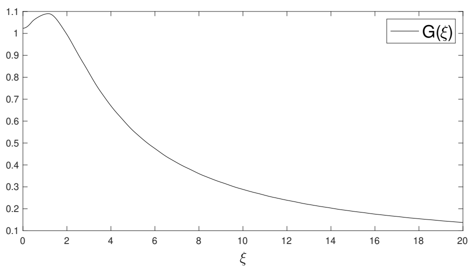

Fig. 1. Rigorous lower bound of $G(\xi)$ for $\xi = ih, h = 0.05, 0 \leq i \leq 400$, $G(ih) > 0$.

4.2.3. Positivity of $G_{2,\beta}$. Recall $W_{2,\beta}, G_{2,\beta}$ defined in (4.17) and (4.18). For $G_{2,\beta}$, it is easier to establish positivity than for $G_{1,\beta}$. From the argument in Section 4.2.1 and (4.22), a sufficient condition for $G_{2,\beta}(\xi) \geq 0$ is the convexity of $W_{2,\beta}$. We have the following result.

Lemma 4.4. *For any $\beta \in (1, 2]$, we have $W_{2,\beta}''(x) \geq 0$ for $x \geq 0$. As a result, $G_{2,\beta}(\xi) \geq 0$ for any ξ and $\beta \in (1, 2]$.*

The proof is based on estimating $W_{2,\beta}''$ directly using its explicit formula and elementary inequalities, which is not difficult and deferred to Appendix A.4.

4.2.4. Proof of Lemma 4.2. Combining Lemmas 4.3 and 4.4, we establish that there exists $\beta_0 \in (1, 2)$ such that for $\beta \in [\beta_0, 2]$ and any ξ, $\widehat{W}_{j,\beta}(\xi) = 2G_{1,\beta}(\xi) \geq 0$ for $j = 1, 2$. From (4.15), (4.16), we prove $B_j(\beta) \geq 0$. Recall the definitions of $\tilde{B}(\beta), B_1(\beta), B_2(\beta)$ from (4.8), (4.12), and (4.16). We obtain $\tilde{B}(\beta) = B_1(\beta) + B_2(\beta) \geq 0$.

Note that to obtain the equivalence between the forms of $B(\beta)$ in (4.1) and (4.5), we require that ω vanishes near $x = 0$ at order $|x|^\gamma$ with $\gamma > \beta - 1$. Using the relation (4.10) between $\tilde{B}(\beta)$ and $B(\beta)$, we prove (4.3) of Lemma 4.2 for $\beta \in [\beta_0, 2)$ and odd $\omega \in C^\alpha$ with $\omega x^{-1} \in L^\infty$. If in addition $\omega \in C^{1,\alpha}$ and $\omega_x(0) = 0$, we find that the vanishing order of ω near $x = 0$ is larger than 1 and choose $\beta = 2$ to establish $B(2) = \tilde{B}(2) \geq 0$. This concludes the proof of Lemma 4.2.

5. Global well-posedness

In this section, we use the crucial Lemma 4.2 to control $u_x(0, t)$ and then establish the global well-posedness result in Theorem 2 using the one-point blowup criterion of Theorem 1. We impose the assumptions $\omega_0 \in H^1 \cap X$, $\omega_0(x)x^{-1} \in L^\infty$, and $A(\omega_0) < \infty$ stated in Theorem 2.

Recall $Q(\beta)$ defined in (4.1). To apply Theorem 1, from Hölder's inequality

$$|u_x(0)| \lesssim \int_0^{\pi/2} |\omega(y)| \cot y \, dy \lesssim Q(\beta)^{1/\beta} \|\omega\|_{L^1}^{1-1/\beta}, \tag{5.1}$$

we only need to control $\|\omega\|_{L^1}$ and $Q(\beta)$. In (3.10) and (3.11), we derive the evolution of $\|\omega\|_{L^1}$:

$$\frac{d}{dt}\left(-\int_0^{\pi/2} \omega(x) \, dx\right) = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) \cot(x + y) \, dx \, dy. \tag{5.2}$$

Recall the discussion of the interaction on the right hand side in Section 2.1. For $x + y \geq \pi/2$, the interaction has a negative sign and it will play a crucial role as a damping term.

5.1. Special case: $\omega_0 \in C^{1,\alpha}$, $\omega_{0,x}(0) = 0$

For initial data ω_0 with $\omega_{0,x}(0) = 0$, $\omega_x(0,t) = 0$ is preserved and the value $Q(2,t) = -\int_0^{\pi/2} \omega(y) \cot^2 y \, dy$ is well-defined. Using (4.2) and Lemma 4.2, we obtain

$$\frac{d}{dt} Q(2,t) = -B(2,t) \le 0.$$

Since $\omega \le 0$ on $(0, \pi/2)$, we derive $Q(2,t) \ge 0$ and

$$\int_0^{\pi/2} |\omega| \cot^2 y \, dy = Q(2,t) \le Q(2,0) = \int_0^{\pi/2} |\omega_0| \cot^2 y \, dy < \infty.$$

Next, we estimate $\|\omega\|_{L^1}$. We first establish an estimate similar to (3.11):

$$-\int_0^x \omega(y) \cot(x+y) \, dy \le -\int_0^{\pi/2} \omega(y) \cot^2 y \, dy = Q(2,t) \tag{5.3}$$

for $x \in [0, \pi/2]$. Since $\cot z \le 0$ for $z \ge \pi/2$, $\cot y \le 1$ on $[0, \pi/4]$, and $\cot y$ is decreasing on $[0, \pi]$, for $0 \le y \le x \le \pi/2$ we get

$$\mathbf{1}_{y \le x} \cot(x+y) \le \mathbf{1}_{y \le \pi/4} \mathbf{1}_{y \le x} \cot(x+y) \le \mathbf{1}_{y \le \pi/4} \mathbf{1}_{y \le x} \cot y \le \cot y^2,$$

where we have used $x + y \ge \pi/2$, $\cot(x+y) \le 0$ if $\pi/4 \le y \le x$ in the first inequality. Since $\omega \le 0$ on $[0, \pi/2]$, we get (5.3). Plugging (5.3) in the estimates (3.9)–(3.10), we derive

$$\frac{d}{dt} \int_0^{\pi/2} -\omega \, dx \lesssim Q(2,t) \int_0^{\pi/2} -\omega \, dx \le Q(2,0) \int_0^{\pi/2} -\omega \, dx.$$

Using the above estimate and the interpolation (5.1) with $\beta = 2$, we obtain

$$\begin{aligned} &\|\omega\|_{L^1} \le \|\omega_0\|_{L^1} e^{CQ(2,0)t}, \\ &|u_x(0)| \lesssim (Q(2,t)\|\omega\|_{L^1})^{1/2} \lesssim (Q(2,0)\|\omega_0\|_{L^1})^{1/2} e^{CQ(2,0)t}, \end{aligned}$$

for some constant $C > 0$. Applying the same argument as in Sections 3.1 and 3.2 with $U(t)$ replaced by $CQ(2,0)t$, we establish

$$\|\omega\|_{L^\infty} \le K(\omega_0) \exp\Big(2 \exp\big(K(\omega_0) \exp(CQ(2,0)t)\big)\Big),$$

where we have used $\int_0^t \exp(CQ(2,0)s) \, ds \lesssim K(\omega_0) e^{CQ(2,0)t}$ where $K(\omega_0)$ is some constant depending on $H\omega_0(0)$, $H\omega_0(\pi/2)$, $\|\omega_0\|_{L^1}$, $Q(2,0)$ and $A(\omega_0)$. We have proved the result in Theorem 2 for the case of $\omega_0 \in C^{1,\alpha}$ with $\omega_{0,x}(0) = 0$.

We remark that the above a priori estimates can be generalized to initial data ω_0 with lower regularity, e.g. $\omega_0/|x|^{1+\alpha} \in L^\infty$ for some $\alpha > 0$ and $\omega_0 \in X \cap H^1$.

5.2. General case

Recall from Section 2.3 the difficulties and ideas in the general case where ω_0 can vanish only linearly near $x = 0$. In this case, the monotone quantity $Q(2,t)$ of the previous case is not well-defined and not applicable. We will exploit a relation similar to the conservation law $\omega_x(0,t) = \omega_{0,x}(0)$ and control $Q(\beta,t)$ for β sufficiently close to 2.

5.2.1. Estimate of ωx^{-1} . For the less regular initial data $\omega_0 \in H^1$ with $\omega_0 x^{-1} \in L^\infty$, $\omega_x(0,t)$ is not well-defined. Instead of using the conservation law $\omega_x(0,t) = \omega_{0,x}(0)$, we show that $\omega(x,t)x^{-1}$ cannot grow too fast for x near 0. Consider the flow map

$$\frac{d}{dt}\Phi(x,t) = u(\Phi(x,t),t), \quad \Phi(x,0) = x. \tag{5.4}$$

We focus on $x \in [0,\pi/2]$. Since $u(x,t) \geq 0$, $u(0,t) = 0$, and $u(\pi/2,t) = 0$, we get

$$\frac{d}{dt}\Phi(x,t) \geq 0, \quad 0 \leq \Phi(x,t_1) \leq \Phi(x,t_2), \tag{5.5}$$

for $t_1 \leq t_2$. Using (1.2), we derive the equation for ω/x:

$$\partial_t \frac{\omega}{x} + u\partial_x\left(\frac{\omega}{x}\right) = \left(u_x - \frac{u}{x}\right)\frac{\omega}{x}.$$

Fix $\gamma \in (0,1/2)$. Using the embedding $H^1 \hookrightarrow C^\gamma$, we have $\omega, u_x \in C^\gamma$. Since $u_x(x) - u(x)/x = 0$ at $x = 0$ and $\omega \leq 0$ on $[0,\pi/2]$, for $x \in [0,\pi/2]$ we get

$$\begin{aligned}\frac{d}{dt}\left(-\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\right) &= \left(u_x(\Phi(x,t),t) - \frac{u(\Phi(x,t),t)}{\Phi(x,t)}\right)\left(-\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\right)\\ &\lesssim |\Phi(x,t)|^\gamma \|\omega\|_{H^1}\left|\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\right|.\end{aligned}$$

Denote

$$m := \|\omega_0 x^{-1}\|_{L^\infty}.$$

Using Gronwall's inequality and (5.5), we derive

$$\begin{aligned}\left|\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\right| &\leq \exp\left(C\int_0^t |\Phi(x,s)|^\gamma \|\omega(s)\|_{H^1}\, ds\right)\left\|\frac{\omega_0}{x}\right\|_{L^\infty}\\ &\leq m\exp\left(C|\Phi(x,t)|^\gamma \int_0^t \|\omega(s)\|_{H^1}\, ds\right).\end{aligned}$$

Since $\Phi(\cdot,t)$ is a bijection from $[0,\pi/2]$ to $[0,\pi/2]$ and x is arbitrary, we get

$$\begin{aligned}\left|\frac{\omega(x,t)}{x}\right| &\leq m\exp\left(C|x|^\gamma \int_0^t \|\omega(s)\|_{H^1}\, ds\right)\\ &\leq m\left(1 + C|x|^\gamma \exp\left(C\int_0^t \|\omega(s)\|_{H^1}\, ds\right)\right),\end{aligned} \tag{5.6}$$

where we have used $|x| \leq \pi/2$ and

$$e^{Ax} \leq 1 + Ax\cdot e^{Ax} \leq 1 + Cxe^{CA}$$

for some absolute constant C in the last inequality. The above estimate shows that $\limsup_{x\to 0} |\omega(x,t)/x|$ is bounded uniformly in t, which is an analog of $\omega_x(0,t) = \omega_{0,x}(0)$. Moreover, we find that $\omega(x,t)x^{-1} \in L^\infty$.

5.2.2. Weighted L^1 estimates. From the local well-posedness result and (5.6), we have $\omega(t) \in X \cap H^1$ and $\omega(x,t)x^{-1} \in L^\infty$, and $\omega(t)$ satisfies the assumptions in Lemma 4.2. A key step to control $Q(\beta,t)$ is establishing the following weighted L^1 estimates.

Lemma 5.1. *Let β_0 be the parameter in Lemma 4.2. For $\beta \in [\beta_0, 2)$, we have*

$$\begin{aligned} \frac{d}{dt} Q(\beta,t) &\le C(2-\beta)Q^2(\beta,t) + C(2-\beta)D(t), \\ \frac{d}{dt}\|\omega\|_{L^1} &\le CQ^2(\beta,t) - C_2 D(t), \end{aligned} \tag{5.7}$$

for some absolute constants $C, C_2 > 0$, where $D(t) \ge 0$ is a damping term given by

$$D(t) = -\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)\cot(x+y)\mathbf{1}_{x+y>\pi/2}\, dx\, dy. \tag{5.8}$$

As a result, for some absolute constant $\lambda > 0$, we have

$$\frac{d}{dt}(Q(\beta,t) + \lambda(2-\beta)\|\omega\|_{L^1}) \lesssim (2-\beta)Q^2(\beta,t). \tag{5.9}$$

At first glance, the estimate (5.9) looks terrible due to the quadratic nonlinearity $Q^2(\beta,t)$. Yet, we have a crucial small factor $2-\beta$, which can compensate the nonlinearity. The boundedness of ωx^{-1} for x near 0 (see (5.6)) implies the following leading order structure of $Q(\beta,t)$:

$$\begin{aligned} Q(\beta,t) &= -\int_0^{\pi/2} \omega(x,t)(\cot x)^\beta\, dx \le m\int_0^1 x\cdot x^{-\beta}\, dx + \mathcal{R}(\beta,t) \\ &\le \frac{m}{2-\beta} + \mathcal{R}(\beta,t), \end{aligned}$$

where the remainder $\mathcal{R}(\beta,t)$ is of order lower than $(2-\beta)^{-1}$. For β sufficiently close to 2, we get $(2-\beta)Q(\beta,t) \lesssim m$, which is time-independent. Formally, the nonlinearity in (5.9) becomes linear. In Section 5.2.3, we will apply (5.9) and this key observation to prove Theorem 2.

The first estimate in (5.7) is highly nontrivial since the forcing term $u_x(0)Q(\beta)$ (see (5.13)) cannot be controlled by $Q^2(\beta)$. The idea behind Lemma 5.1 is that for the forcing terms $B(\beta,t)$ in (4.2) and (4.3) and that in (5.2), we use the more singular integral $Q(\beta,t)$ to control them near $x=0$, and the magic damping term $D(t)$ from (5.2) to control them near $x=\pi/2$. To prove Lemma 5.1, we need several inequalities, whose proofs are deferred to Appendix A.5.

Lemma 5.2. *Denote $a \wedge b = \min(a,b)$. For $x, y \in [0,\pi/2]$ and $\beta \in [3/2, 2]$, we have*

$$\cot(x+y) \le \mathbf{1}_{x+y\ge\pi/2}\cot(x+y) + (\cot x\cot y)^\beta, \tag{5.10}$$

$$\cot y\cdot(\cot x)^{\beta-2} \wedge \cot x(\cot y)^{\beta-2} \lesssim (\cot x\cot y)^\beta + \mathbf{1}_{x+y\ge\pi/2}\cot(\pi-x-y), \tag{5.11}$$

$$\cot y\cdot\mathbf{1}_{y\ge\pi/3} \lesssim (\cot x\cot y)^\beta + \mathbf{1}_{x+y\ge\pi/2}\cot(\pi-x-y). \tag{5.12}$$

Proof of Lemma 5.1. Using $\omega(x)\omega(y) \geq 0$ for $x, y \in [0, \pi/2]^2$ and (5.10), we obtain

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\pi/2} &\omega(x)\omega(y)\cot(x+y)\,dx\,dy\\ &\leq -D(t) + \int_0^{\pi/2}\int_0^{\pi/2}\omega(x)\omega(y)(\cot x\cot y)^\beta\,dx\,dy\\ &\leq -D(t) + Q^2(\beta,t),\end{aligned}$$

where $D(t)$ is defined in (5.8). Using the above estimate and (5.2), we prove the second estimate in (5.7). Recall the ODE (4.2) for $Q(\beta,t)$. Applying Lemma 4.2, for $\beta \in [\beta_0, 2)$ we get

$$\begin{aligned}&\frac{d}{dt}Q(\beta,t)\\ &\leq (2-\beta)\Big(u_x(0)Q(\beta,t) + \frac{1}{\pi}\iint_{[0,\pi/2]^2}\omega(x)\omega(y)(\cot y)^{\beta-1}\frac{s(s^{\beta-1}-1)}{s^2-1}\,dx\,dy\Big)\\ &=: (2-\beta)(I_1+I_2),\end{aligned} \tag{5.13}$$

where $s = \frac{\cot x}{\cot y}$. Next, we estimate $f(s) = \frac{s(s^{\beta-1}-1)}{s^2-1}$. Note that $\beta \in (3/2, 2)$. For $s \geq 0$, the following estimate is straightforward:

$$0 \leq f(s) \lesssim \mathbf{1}_{s<1/2}s + \mathbf{1}_{1/2\leq s\leq 2} + \mathbf{1}_{s\geq 2}s^{\beta-2} \lesssim s \wedge s^{\beta-2}.$$

Since $s = \frac{\cot x}{\cot y}$, using the above estimate and (5.11) yields

$$\begin{aligned}f(s)(\cot y)^{\beta-1} &\lesssim (s\wedge s^{\beta-2})\cdot(\cot y)^{\beta-1} = \cot y\cdot(\cot x)^{\beta-2}\wedge\cot x\cdot(\cot y)^{\beta-2}\\ &\lesssim (\cot x\cot y)^\beta + \mathbf{1}_{x+y\geq\pi/2}\cot(\pi-x-y) = (\cot x\cot y)^\beta - \mathbf{1}_{x+y\geq\pi/2}\cot(x+y).\end{aligned}$$

Using $\omega(x)\omega(y) \geq 0$ for $x, y \in [0, \pi/2]$, the above estimate and (5.8), we derive

$$\begin{aligned}0 \leq I_2 &\lesssim \int_0^{\pi/2}\int_0^{\pi/2}\omega(x)\omega(y)\big((\cot x\cot y)^\beta - \mathbf{1}_{x+y\geq\pi/2}\cot(x+y)\big)\,dx\,dy\\ &= Q^2(\beta,t) + D(t).\end{aligned}$$

For I_1, we cannot establish the desired estimate by comparing the kernel similar to the above since

$$\cot y\cdot(\cot x)^\beta \lesssim (\cot x\cot y)^\beta - \mathbf{1}_{x+y\geq\pi/2}\cot(x+y)$$

does not hold for x close to 0 and y close to $\pi/2$. In fact, for $\pi/2 - y = t^\beta, x = t$, with t sufficiently small, the left hand side is $O(1)$, while the right hand side is $o(t)$. The main difficulty lies in $(\cot y)^\beta$ being too weak to control $\cot y$ for y close to $\pi/2$.

A key observation is that we can further impose the restriction $Q(\beta,t) \leq u_x(0) \lesssim \|\omega\|_{L^1}$. In fact, if $u_x(0) \leq Q(\beta,t)$, we obtain the trivial estimate

$$I_1 = u_x(0)Q(\beta,t) \leq Q^2(\beta,t).$$

In the other case $Q(\beta,t) \le u_x(0)$, thanks to the interpolation (5.1), we derive

$$u_x(0) \lesssim Q(\beta,t)^{1/\beta} \|\omega\|_{L^1}^{1-1/\beta} \le (u_x(0))^{1/\beta} \|\omega\|_{L^1}^{1-1/\beta},$$

which implies $u_x(0) \lesssim \|\omega\|_{L^1}$. Now, we decompose $I_1 = u_x(0) Q(\beta,t)$ as follows:

$$\begin{aligned} I_1 &\lesssim \int_0^{\pi/2} |\omega(y)| \cot y \, dy \, Q(\beta,t) \\ &= \int_0^{\pi/3} |\omega(y)| \cot y \, dy \, Q(\beta,t) + \int_{\pi/3}^{\pi/2} |\omega(y)| \cot y \, dy \, Q(\beta,t) =: J_1 + J_2. \end{aligned}$$

For J_1, since $\cot y \lesssim (\cot y)^\beta$ for $y \le \pi/3$, we get $J_1 \lesssim Q^2(\beta,t)$. For J_2, using $Q(\beta,t) \le u_x(0) \lesssim \|\omega\|_{L^1}$, we obtain

$$J_2 \lesssim \int_{\pi/3}^{\pi/2} |\omega(y)| \cot y \, dy \, \|\omega\|_{L^1} \lesssim \int_{\pi/3}^{\pi/2} \omega(y) \cot y \, dy \int_0^{\pi/2} \omega(x) \, dx,$$

where we have used $\omega(x) \le 0$ on $[0, \pi/2]$ to obtain the last inequality. Applying (5.12) and $\cot(\pi - x - y) = -\cot(x+y)$, we obtain

$$\begin{aligned} J_2 &\lesssim \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) \big((\cot x \cot y)^\beta - \mathbf{1}_{x+y \ge \pi/2} \cot(x+y) \big) \, dx \, dy \\ &= Q^2(\beta,t) + D(t). \end{aligned}$$

Combining the above estimates on J_1, J_2, in the other case $Q(\beta,t) \le u_x(0)$, we prove

$$I_1 \lesssim J_1 + J_2 \lesssim Q^2(\beta,t) + D(t).$$

Combining the above estimates on I_1, I_2, we establish the first inequality in (5.7). Estimate (5.9) follows directly from (5.7) by choosing $\lambda > 0$ with $C_2 \lambda \ge 2C$, e.g. $\lambda = 2C/C_2$. ∎

Remark 5.3. We cannot apply (5.1) to estimate $u_x(0)$ in I_1 directly, since such an estimate only offers

$$\frac{d}{dt}(Q(\beta,t) + \mu\|\omega\|_{L^1}) \lesssim (2-\beta)^\gamma (Q(\beta,t) + \mu\|\omega\|_{L^1})^2$$

with power $\gamma < 1$ for any well chosen μ, which is not sufficient for our purpose. Compared to (5.9), the above estimate loses a small factor $(2-\beta)^{1-\gamma}$, which is due to the fact that we do not have a good estimate on $\|\omega\|_{L^1}$, while for $Q(\beta,t)$ we have the crucial small factor $2-\beta$. We only add a minimal amount of $\|\omega\|_{L^1}$ in the energy in (5.9) for a similar reason.

5.2.3. A bootstrap estimate. Now, we can establish the global well-posedness result of Theorem 2 in the general case. It follows from a bootstrap lemma.

Lemma 5.4. *Suppose that ω_0 satisfies the assumptions in Theorem 2. Set $m = \|\omega_0 x^{-1}\|_{L^\infty}$. There exists an absolute constant c such that for $\delta = c/m$, if $\int_0^T u_x(0,s)\,ds < \infty$, we have $\int_0^{T+\delta} u_x(0,s)\,ds < \infty$.*

Proof. Without loss of generality, we assume $m > 0$. Recall $Q(\beta, t)$ from (4.1). Denote

$$H(\beta,t) = Q(\beta,t) + \lambda(2-\beta)\|\omega\|_{L^1}.$$

In view of Theorem 1 and (5.1), for $\omega_0 \in H^1 \cap X$, the solution $\omega(x,t)$ remains in H^1 if $H(\beta,t) < \infty$ for some $\beta < 2$. Thus, it suffices to control H. Using Lemma 5.1, we have

$$\frac{d}{dt} H(\beta,t) \le \mu(2-\beta) H^2(\beta,t) \tag{5.14}$$

for some absolute constant $\mu > 0$ and any $\beta \in [\beta_0, 2)$. Since $\int_0^T u_x(0,s)\,ds < 0$, using Theorem 1 we obtain $\sup_{t\le T} \|\omega(t)\|_{H^1} < \infty$ and $\|\omega(T)\|_{L^1} < \infty$. Using (5.6), we get

$$\begin{aligned} Q(\beta,T) &= \int_0^{\pi/2} |\omega(y)|(\cot y)^\beta\,dy \le \int_0^1 |\omega(y)| y^\beta\,dy + C\int_0^{\pi/2} |\omega(y)|\,dy \\ &\le m\int_0^1 \Big(y^{1-\beta} + C y^{\gamma+1-\beta} \exp\Big(CT \sup_{t\le T}\|\omega(t)\|_{H^1}\Big)\Big)\,dy + C\|\omega(T)\|_{L^1} \\ &\le \frac{m}{2-\beta} + Cm\exp\Big(CT\sup_{t\le T}\|\omega(t)\|_{H^1}\Big) + C\|\omega(T)\|_{L^1}, \end{aligned} \tag{5.15}$$

where C is some absolute constant and we have used

$$|(\cot x)^\beta - x^{-\beta}| \lesssim |\cot x - x^{-1}| x^{-\beta+1} \lesssim x^{-\beta+2} \lesssim 1$$

in the first inequality. Thus, there exists β_1 slightly less than 2 such that

$$\begin{aligned} H(\beta_1,T) &= Q(\beta_1,T) + \lambda(2-\beta_1)\|\omega(T)\|_{L^1} \\ &\le \frac{m}{2-\beta_1} + Cm\exp\Big(CT\sup_{t\le T}\|\omega(t)\|_{H^1}\Big) + C\|\omega(T)\|_{L^1} \le \frac{2m}{2-\beta_1}. \end{aligned}$$

Solving the ODE (5.14) with $\beta = \beta_1$ on the interval $t \ge T$ yields

$$\frac{d}{dt} H^{-1}(\beta_1,t) \ge -\mu(2-\beta_1),$$

which along with the estimate on $H(\beta_1,T)$ implies

$$H^{-1}(\beta_1, T+\tau) \ge H^{-1}(\beta_1,T) - \mu(2-\beta_1)\tau \ge \frac{2-\beta_1}{2m} - \mu(2-\beta_1)\tau.$$

Note that μ is absolute. We choose $\delta = \frac{1}{4m\mu}$. Then, for $t \in [T, T+\delta]$, we get

$$H^{-1}(\beta_1,t) \ge \frac{2-\beta_1}{2m} - \frac{2-\beta_1}{4m} = \frac{2-\beta_1}{4m}, \quad H(\beta_1,t) \le \frac{4m}{2-\beta_1}. \tag{5.16}$$

Applying (5.1), we obtain $u_x(0,t) \lesssim \frac{m}{(2-\beta_1)^2}$ on $[T, T+\delta]$, which concludes the proof. ∎

Remark 5.5. Denote $V(t) = \int_0^t (u_x(0,s)+1)\,ds$. We can obtain an a priori estimate for $V(t)$ by tracking the bounds in the above proof. Using standard energy estimates and (3.21), we obtain

$$
\begin{aligned}
&Cm\exp\Big(Ct\sup_{s\le t}\|\omega(s)\|_{H^1}\Big) + C\|\omega(t)\|_{L^1} \le g(V(t),C_1),\\
&g(x,c) := c\cdot\exp(c\cdot\exp(c\cdot\exp(c\cdot\exp(c\cdot\exp(c\cdot\exp(cx)))))),
\end{aligned}
$$

for some constant $C_1 > 1$ depending only on the initial data. Note that the estimate (3.21) of $\|\omega\|_{L^\infty}$ gives triple exponential growth, and then the estimate of $\|\omega\|_{H^1}$ gives quintuple one due to extrapolation in bounding $\|u_x\|_{L^\infty}$. These estimates further lead to the above sextuple exponential growth. For any $T \ge 0$, choosing β_1 with $2-\beta_1 = c\cdot\frac{m}{g(V(T),C_1)}$ for some absolute constant c and using (5.1) and (5.16) yields

$$
V(T+\delta) \le g(V(T),C_2)
$$

for some constant $C_2 > 0$ depending only on ω_0. Since δ and C_2 are independent of T, iterating the above estimate yields an a priori estimate for $V(t)$ with any $t \ge 0$.

Remark 5.6. The above estimate is consistent with the heuristic in the paragraph below (5.9) that the nonlinearity $(2-\beta)Q^2$ in (5.9) or $(2-\beta)H^2$ is essentially linear. In fact, for $t\in[T,T+\delta]$, (5.16) implies $(2-\beta_1)Q(\beta_1,t) \le (2-\beta_1)H(\beta_1,t) \le 4m$. Formally, $Q(\beta,t)$ grows exponentially in t for β close to 2, which we can barely control, while in the previous case, $Q(2,t)$ is bounded uniformly. This argument is similar in spirit to extrapolation, e.g. the BKM blowup criterion [1].

6. Finite time blowup for $C^\alpha \cap H^s$ data

In this section, we prove Theorem 3 on finite time blowup for (1.2) with $C^\alpha\cap H^s$ data for any $\alpha\in(0,1)$ and $s\in(1/2,3/2)$. We will use the ideas outlined in Section 2.

Since we will adopt several estimates established in [6, 35], for consistency, throughout this section, we assume that the solution ω is 2π-periodic. This modification also simplifies our notations. Theorem 3 can be established by applying the same argument to $\omega_\pi(x) := \omega_{2\pi}(2x)$. As a result, the Hilbert transform and the set X of (1.4) become

$$
\begin{aligned}
&Hf(x) := \frac{1}{2\pi}\,\mathrm{P.V.}\int_{-\pi}^{\pi}\cot\frac{x-y}{2}\,f(y)\,dy,\\
&X := \{f : f \text{ is odd, } 2\pi\text{-periodic and } f(x)\le 0 \text{ for } x\in[0,\pi]\}.
\end{aligned}
$$

6.1. Slightly weakening the effect of advection

Recall the discussion on the competition between advection and vortex stretching in Section 1.3. To show that advection is relatively weak for $\omega\in C^\alpha\cap X$ with $\omega\approx -Cx^\alpha$ near $x=0$, we study (1.2) using the dynamic rescaling formulation

$$
\omega_t + u\omega_x = (c_\omega + u_x)\omega, \quad u_x = H\omega \tag{6.1}
$$

derived in (6.3)–(6.5) with the normalization condition

$$c_\omega(t) = (\alpha - 1)u_x(0,t), \tag{6.2}$$

where c_ω is a rescaling factor. If $u_x(0,t)$ is bounded away from 0, $u_x(0,t) \geq C > 0$ for all t, the competition between advection and vortex stretching is encoded in the sign of c_ω since $\mathrm{sign}(c_\omega) = \mathrm{sign}(\alpha - 1)$, which can determine the long time behavior of the solution. See the discussion below (6.5). We remark that the idea and condition (6.2) are similar to those in [6], which play a crucial role in establishing singularity formation for the gCLM model.

6.2. *Dynamic rescaling formulation*

We follow the method of [6,9] to construct a finite time blowup solution using the dynamic rescaling formulation of (1.2). Let $\omega(x,t), u(x,t)$ be the solutions of equation (1.2). It is easy to show that

$$\tilde\omega(x,\tau) = C_\omega(\tau)\omega(x,t(\tau)), \quad \tilde u(x,\tau) = C_\omega(\tau)u(x,t(\tau)) \tag{6.3}$$

are the solutions to the dynamic rescaling equations

$$\tilde\omega_\tau + \tilde u\tilde\omega_x = c_\omega\tilde\omega + \tilde u_x\tilde\omega, \quad \tilde u_x = H\tilde\omega, \tag{6.4}$$

where

$$C_\omega(\tau) = \exp\left(\int_0^\tau c_\omega(s)\,ds\right), \quad t(\tau) = \int_0^\tau C_\omega(s)\,ds. \tag{6.5}$$

We will impose some normalization condition on the time-dependent scaling parameter $c_\omega(\tau)$, and establish that $-C_1 \leq c_\omega(\tau) \leq -C < 0$ for all $\tau > 0$ and some $C_1, C > 0$. Then the solution of (6.4) is equivalent to that of the original equation (1.2) via the transformations in (6.3)–(6.5). Moreover, we will establish that the solution $\tilde\omega(\cdot,\tau)$ is nontrivial, e.g. $\|\tilde\omega(\cdot,\tau)\|_{L^\infty} \geq c > 0$ for all $\tau > 0$. Then the rescaling relationship (6.3)–(6.5) implies that

$$C_\omega(\tau) \leq e^{-C\tau}, \quad t(\infty) \leq \int_0^\infty e^{-C\tau}\,d\tau = C^{-1} < \infty$$

and that the solution

$$|\omega(x,t(\tau))| = C_\omega(\tau)^{-1}|\tilde\omega(x,\tau)| \geq e^{C\tau}|\tilde\omega(x,\tau)|$$

blows up at finite time $T = t(\infty)$.

Note that a similar dynamic rescaling formulation was employed in [34, 41] to study the nonlinear Schrödinger (and related) equation. This formulation is closely related to the modulation technique, which has been developed by Merle, Raphaël, Martel, Zaag and others; see, e.g., [31, 40, 42–44]. It has been a very effective tool to study singularity formation for many problems like the nonlinear Schrödinger equation [31, 42], the nonlinear wave equation [44], the nonlinear heat equation [43], and the generalized KdV equation [40]. Recently, it has been used to establish finite time blowup from smooth initial data in model problems for the 3D Euler equations, including the DG model [9], the gCLM model [5, 6, 9, 19] and the Hou–Luo model [10].

To simplify our presentation, we still use t to denote the rescaled time in the rest of this section, unless specified, and drop $\tilde{\cdot}$ in (6.4). Then (6.4) reduces to (6.1).

6.3. *Construction of an C^α approximate steady state*

Based on the discussion in Sections 1.3 and 6.1, we first construct an approximate steady state $(\omega_\alpha, c_{\omega,\alpha})$ of (6.1) with $\omega_\alpha \in C^\alpha$ and $\omega_\alpha \approx -Cx^\alpha$ near $x = 0$. Following the idea in [6], we perform the construction by perturbing the equilibrium $\sin x$ of (1.2). A natural choice of ω_α is

$$\omega_\alpha = -\mathrm{sgn}(x)|\sin x|^\alpha c_\alpha, \quad c_\alpha = \left(\frac{1}{\pi} \int_0^\pi (\sin x)^\alpha \cot \frac{x}{2} \, dx \right)^{-1}. \tag{6.6}$$

We choose the above c_α to normalize $H\omega_\alpha(0) = 1$. Let u_α be the associated velocity with $u_{\alpha,x} = H\omega_\alpha$. We choose $c_{\omega,\alpha}$ according to (6.2),

$$c_{\omega,\alpha} = (\alpha - 1)u_{\alpha,x}(0) = \alpha - 1. \tag{6.7}$$

Denote

$$\omega_1 = -\sin x, \quad u_1 = \sin x, \quad \eta_\alpha = \omega_\alpha - \omega_1. \tag{6.8}$$

For α close to 1, we expect that $(\omega_\alpha, u_\alpha)$ is close to (ω_1, u_1).

Lemma 6.1. *Let $\kappa_1 = \frac{3}{4}$, $\kappa_2 = \frac{7}{8}$. For $\kappa_2 < \frac{9}{10} < \alpha < 1$ and $x \in [-\pi, \pi]$, we have*

$$|\partial_x^i \eta_\alpha| \lesssim (1-\alpha)|\sin x|^{\kappa_2 - i}, \quad i = 1, 2, 3, \tag{6.9}$$

$$|H\eta_\alpha| \lesssim (1-\alpha)|x|^{\kappa_1}, \quad |\partial_x H\eta_\alpha| \lesssim (1-\alpha)|\sin x|^{\kappa_1 - 1}, \tag{6.10}$$

$$|(\alpha-1)\omega_\alpha - \sin x \cdot (\omega_{\alpha,xx} - \omega_{1,xx})| \lesssim ((1-\alpha) \wedge |x|^2)|\sin x|^{\alpha-1}. \tag{6.11}$$

For x near 0, the above estimates on ω_α are similar to those for $\omega_\alpha = -x^\alpha$ and $\omega_1 = -x$. The reader can think of κ_1, κ_2 as close to 1, and that α is even closer to 1.

Proof of Lemma 6.1. By symmetry, it suffices to consider $x \geq 0$.

Firstly, using Lemma A.4 and $1 \lesssim \alpha - \kappa_2$, we obtain

$$|(\sin x)^\alpha - \sin x| = (\sin x)^{\kappa_2}(\sin x)^{\alpha-\kappa_2}(1 - (\sin x)^{1-\alpha}) \lesssim (1-\alpha)(\sin x)^{\kappa_2}. \tag{6.12}$$

Recall c_α defined in (6.6). Using the above estimate, we obtain

$$\frac{1}{\pi}\int_0^\pi |(\sin x)^\alpha - \sin x| \cot \frac{x}{2} \, dx \lesssim 1-\alpha, \quad |c_\alpha - 1| \lesssim 1 - \alpha. \tag{6.13}$$

Next, we establish the estimate of ω_α defined in (6.6). A direct calculation yields

$$\begin{aligned} \omega_{\alpha,x} &= -c_\alpha \alpha (\sin x)^{\alpha-1} \cos x, \\ \omega_{\alpha,xx} &= -c_\alpha \alpha(\alpha-1)(\sin x)^{\alpha-2}\cos^2 x + \alpha c_\alpha (\sin x)^\alpha. \end{aligned} \tag{6.14}$$

We consider a typical case $i = 3$ in (6.9), and the case $i = 1$ or 2 can be proved similarly. Recall $\omega_1, u_1, \eta_\alpha$ from (6.8). Using (6.12), (6.13) and $\kappa_2 < \alpha$, we get

$$\begin{aligned}|\eta_{\alpha,xx}| = |\omega_{\alpha,xx} - \sin x| &\lesssim |\alpha c_\alpha (\sin x)^\alpha - \sin x| + (1-\alpha)(\sin x)^{\alpha-2} \\ &\lesssim |(\sin x)^\alpha - \sin x| + (1-\alpha)(\sin x)^{\alpha-2} \lesssim (1-\alpha)(\sin x)^{\kappa_2-2}.\end{aligned}$$

For (6.11), the first bound $(1-\alpha)|\sin x|^{\alpha-1}$ follows directly from (6.9). Using (6.14), $|\omega_{1,xx}| = \sin x$ and a direct calculation, we obtain

$$\begin{aligned}&|(\alpha-1)\omega_\alpha - \sin x \cdot (\omega_{\alpha,xx} - \omega_{1,xx})| \\ &\quad \leq |c_\alpha(\alpha-1)\alpha(\sin x)^{\alpha-1}(\cos x - \cos^2 x)| + C(\sin x)^{\alpha+1} + \sin x \cdot |\omega_{1,xx}| \\ &\quad \lesssim (\sin x)^{a-1}|x|^2 + (\sin x)^{a+1} \lesssim (\sin x)^{a-1}|x|^2,\end{aligned}$$

where we have used $|1 - \cos x| \lesssim x^2$.

Next, we prove (6.10). Denote $D_x = \sin x \cdot \partial_x$. Using (6.9) and $\kappa_2 = \frac{7}{8}$ close to 1, we have

$$\begin{aligned}&\|\partial_x \eta_\alpha\|_{L^4} \lesssim (1-\alpha)\big\| |\sin x|^{\kappa_2-1}\big\|_{L^4} \lesssim 1-\alpha, \\ &\|\partial_x(D_x\eta_\alpha)\|_{L^4} \lesssim \big\| |\partial_x\eta_\alpha| + |\sin x \cdot \partial_x^2\eta_\alpha|\big\|_{L^4} \lesssim (1-\alpha)\||\sin x|^{\kappa_2-1}\|_{L^4} \lesssim 1-\alpha.\end{aligned} \tag{6.15}$$

Recall from (6.6) that $u_{\alpha,x}(0) = H\omega_\alpha(0) = 1 = u_{1,x}(0)$. This implies $H\eta_\alpha(0) = 0$. Since the Hilbert transform is L^4-bounded, using Hölder's inequality and (6.15) yields

$$\begin{aligned}|H\eta_\alpha(x)| &= \left|\int_0^x \partial_x H\eta_\alpha(y)\,dy\right| \leq \|\partial_x H\eta_\alpha\|_{L^4}\left(\int_0^x 1\,dy\right)^{3/4} \lesssim \|\partial_x H\eta_\alpha\|_{L^4} x^{3/4} \\ &= x^{3/4}\|H\partial_x\eta_\alpha\|_{L^4} \lesssim x^{3/4}\|\partial_x\eta_\alpha\|_{L^4} \lesssim (1-\alpha)x^{3/4}.\end{aligned}$$

Since $D_x H\eta_\alpha$ vanishes at $x = 0, \pi$, using an estimate similar to the above yields

$$\begin{aligned}|D_x H\eta_\alpha(x)| &\lesssim \|\partial_x(D_x H\eta_\alpha(x))\|_{L^4}(|x|^{3/4} \wedge |\pi - x|^{3/4}) \\ &\lesssim \|\partial_x(D_x H\eta_\alpha(x))\|_{L^4}|\sin x|^{3/4}.\end{aligned}$$

Applying Lemma A.2 ($n = 2$) yields

$$\partial_x(D_x H\eta_\alpha) = \partial_x\big(H(D_x\eta_\alpha) - H(D_x\eta_\alpha)(0)\big) = \partial_x(H(D_x\eta_\alpha) = H(\partial_x D_x\eta_\alpha).$$

Applying (6.15) and the fact that H is L^4-bounded, we establish

$$\begin{aligned}|D_x H\eta_\alpha(x)| &\lesssim \|H(\partial_x D_x\eta_\alpha)\|_{L^4}|\sin x|^{3/4} \\ &\lesssim \|\partial_x D_x\eta_\alpha\|_{L^4}|\sin x|^{3/4} \lesssim (1-\alpha)|\sin x|^{3/4},\end{aligned}$$

which implies the second inequality in (6.10). ∎

The above L^4 estimate on $H\eta_\alpha$ can be replaced by L^p estimates with larger p, which yields a higher vanishing order of $H\eta_\alpha$ near $x = 0$. Here, the power $|x|^{3/4}$ is sufficient for our later weighted energy estimates.

6.4. Nonlinear stability of the approximate steady state

In this section, we follow [6, 9] in performing stability analysis around $(\omega_\alpha, c_{\omega,\alpha})$ constructed in (6.6), (6.7), and we establish the finite time blowup results. We first introduce some weighted norms and spaces.

Definition 6.2. Define the singular weight $\rho = (\sin\frac{x}{2})^{-2}$, the standard inner product $\langle\cdot,\cdot\rangle$ on S^1, the weighted norms $\|\cdot\|_{\mathcal{H}}$ and the Hilbert space $\mathcal{H}$ as follows:

$$\langle f, g\rangle = \int_0^{2\pi} fg\,dx, \quad \|f\|_{\mathcal{H}}^2 := \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{|f_x|^2}{\sin^2\frac{x}{2}}\,dx, \quad \mathcal{H} := \{f : f(0)=0, \|f\|_{\mathcal{H}} < \infty\} \tag{6.16}$$

with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ induced by the $\mathcal{H}$ norm.

The $\mathcal{H}$ norm was introduced in [35] for the stability analysis of the De Gregorio model. By definition, we have

$$\langle f, g\rangle_{\mathcal{H}} = (4\pi)^{-1}\langle f_x, g_x\rho\rangle. \tag{6.17}$$

6.4.1. Linearized equation. Linearizing (6.1) around $\omega_\alpha, c_{\omega,\alpha}$, we obtain the equation for the perturbation ω, c_ω ($(\omega+\omega_\alpha, c_\omega + c_{\omega,\alpha})$ is the solution of (6.1)):

$$\begin{aligned}\omega_t &= -u_\alpha\omega_x + u_{\alpha,x}\omega + u_x\omega_\alpha - u\omega_{\alpha,x} + c_{\omega,\alpha}\omega + c_\omega\omega_\alpha + N(\omega) + F(\omega_\alpha)\\ &=: \mathcal{L}_\alpha\omega + N(\omega) + F(\omega_\alpha),\end{aligned} \tag{6.18}$$

where the nonlinear term $N(\omega)$ and the error term $F(\omega_\alpha)$ are given by

$$N(\omega) = (c_\omega + u_x)\omega - u\omega_x, \quad F(\omega_\alpha) = (c_{\omega,\alpha} + u_{\alpha,x})\omega_\alpha - u_\alpha\omega_{\alpha,x}. \tag{6.19}$$

We choose the normalization condition on c_ω according to (6.2),

$$c_\omega = (\alpha-1)u_x(0). \tag{6.20}$$

Under conditions (6.2) and (6.20), it is easy to see that the slope of ω/x^α is fixed, i.e.

$$\lim_{x\to 0}\frac{\omega(x,t)+\omega_\alpha(x)}{x^\alpha} = \lim_{x\to 0}\frac{\omega(x,0)+\omega_\alpha(x)}{x^\alpha}, \quad \lim_{x\to 0}\frac{\omega(x,t)}{x^\alpha} = \lim_{x\to 0}\frac{\omega(x,0)}{x^\alpha}.$$

In particular, if the initial perturbation $\omega_0(x)$ vanishes near $x=0$ with order higher than x^α, e.g. $x^{2\alpha}$, the perturbation $\omega(x,t)$ will also vanish near $x=0$ with higher order. This allows us to perform energy estimates on ω with a singular weight near $x=0$.

We treat the linearized operator $\mathcal{L}_\alpha$ as a perturbation to $\mathcal{L}_1$, where

$$\mathcal{L}_1\omega = -u_1\omega_x + u_{1,x}\omega + u_x\omega_1 - u\omega_{1,x} = -\sin x\cdot\omega_x + \cos x\cdot\omega - u_x\sin x + u\cos x,$$

where we have used the explicit formulas (6.8), and perform the following decomposition:

$$\begin{aligned}\mathcal{L}_\alpha\omega &= \mathcal{L}_1\omega - (u_\alpha - u_1)\omega_x + (u_{\alpha,x} - u_{1,x})\omega + u_x(\omega_\alpha - \omega_1)\\ &\quad - u(\omega_{\alpha,x} - \omega_{1,x}) + c_{\omega,\alpha}\omega + c_\omega\omega_\alpha\\ &= \mathcal{L}_1\omega - u(\eta_\alpha)\omega_x + H\eta_\alpha\cdot\omega + u_x\eta_\alpha - u\eta_{\alpha,x} + c_{\omega,\alpha}\omega + c_\omega\omega_\alpha\\ &=: \mathcal{L}_1\omega + \mathcal{R}_\alpha\omega,\end{aligned} \tag{6.21}$$

where $u(\eta_\alpha)$ denotes the odd velocity u with $u_x = H\eta_\alpha$. In fact, we have $u(\eta_\alpha) = -(-\partial_{xx})^{-1/2}\eta_\alpha$.

The operator $\mathcal{L}_1$ enjoys an important coercive estimate established in [35]. The following slight modification of the result in [35] is taken from [6].

Lemma 6.3. *Suppose that $f, g \in \mathcal{H}$ and $\int_{S^1} f\, dx = 0$. Denote $e_0(x) = \cos x - 1$ and*

$$f_e = \langle f, e_0\rangle_{\mathcal{H}}, \quad \langle f, g\rangle_Y := \langle f - f_e e_0, g - g_e e_0\rangle_{\mathcal{H}}.$$

We have:

(a) Equivalence of norms: $(\mathcal{H}/\mathbb{R}\cdot e_0, \langle\cdot,\cdot\rangle_Y)$ *is a Hilbert space and the induced norm* $\|\cdot\|_Y$ *satisfies* $\frac{1}{2}\|f\|_{\mathcal{H}} \le \|f\|_Y \le \|f\|_{\mathcal{H}}$.

(b) Orthogonality: $\|e_0\|_{\mathcal{H}} = 1$ *and*

$$\langle f - f_e e_0, e_0\rangle_{\mathcal{H}} = 0, \quad \|f\|_{\mathcal{H}}^2 = f_e^2 + \|f\|_Y^2.$$

(c) Coercivity: $\langle \mathcal{L}_1 f, f\rangle_Y \le -\frac{3}{8}\|f\|_Y^2$.

Using (6.17) and the above result (b), we can represent $\langle\cdot,\cdot\rangle_Y$ as follows:

$$\langle f, g\rangle_Y = \langle f - f_e e_0, g\rangle_{\mathcal{H}} = (4\pi)^{-1}\langle f_x + f_e \sin x, g_x \rho\rangle, \tag{6.22}$$

where we have used $\partial_x e_0 = -\sin x$.

6.4.2. Weighted H^1 estimates. We consider an odd perturbation ω which satisfies $\int_{S^1}\omega\, dx = 0$. Recall the linearized equation (6.18) and the decomposition (6.21). Performing an energy estimate on $\langle\omega,\omega\rangle_Y$ yields

$$\frac{1}{2}\frac{d}{dt}\langle\omega,\omega\rangle_Y = \langle\mathcal{L}_1\omega,\omega\rangle_Y + \langle\mathcal{R}_\alpha\omega,\omega\rangle_Y + \langle N(\omega),\omega\rangle_Y + \langle F(\omega_\alpha),\omega\rangle_Y. \tag{6.23}$$

The estimate of the first term $\langle\mathcal{L}_1\omega,\omega\rangle_Y$ follows from Lemma 6.3:

$$\langle\mathcal{L}_1\omega,\omega\rangle_Y \le -\tfrac{3}{8}\|\omega\|_Y^2. \tag{6.24}$$

For the remainder $\mathcal{R}_\alpha$ in (6.21), a direct calculation yields

$$\begin{aligned}\partial_x\mathcal{R}_\alpha\omega &= -u(\eta_\alpha)\omega_{xx} + \partial_x H\eta_\alpha\cdot\omega + u_{xx}\eta_\alpha - u\eta_{\alpha,xx} + c_{\omega,\alpha}\omega_x + c_\omega\omega_{a,x}\\ &=: -u(\eta_\alpha)\omega_{xx} + \mathcal{R}_{\alpha,2}\omega.\end{aligned}$$

Applying (6.22), we derive

$$\begin{aligned}\langle\mathcal{R}_\alpha\omega,\omega\rangle_Y &= (4\pi)^{-1}\langle\partial_x\mathcal{R}_\alpha\omega, (\omega_x + \omega_e\sin x)\rho\rangle\\ &= (4\pi)^{-1}\langle -u(\eta_\alpha)\omega_{xx}, (\omega_x + \omega_e\sin x)\rho\rangle\\ &\quad + (4\pi)^{-1}\langle\mathcal{R}_{\alpha,2}, (\omega_x + \omega_e\sin x)\rho\rangle =: I + II.\end{aligned} \tag{6.25}$$

Recall $\rho = (\sin\frac{x}{2})^{-2}$. Since $\sin\frac{x}{2} \asymp x$, we can essentially treat ρ as x^{-2}. For II, it suffices to estimate $\|\mathcal{R}_{\alpha,2}\omega\rho^{1/2}\|_2$. Since $c_\omega = (\alpha-1)u_x(0)$ by (6.20), we decompose $\mathcal{R}_{\alpha,2}$

as follows:

$$\mathcal{R}_{\alpha,2} = \partial_x H\eta_\alpha \cdot \omega + u_{xx}\eta_\alpha - (u - u_x(0)\sin x)\eta_{\alpha,xx} + u_x(0)((\alpha-1)\omega_{\alpha,x} - \sin x \cdot \eta_{\alpha,xx}) + c_{\omega,\alpha}\omega_x. \tag{6.26}$$

Next, we estimate the $L^2(\rho)$ norm of each term. The main difficulty is the estimate of the nonlocal term, e.g. $\|u_{xx}\eta_\alpha\rho^{1/2}\|_2$, due to the singular weight ρ near $x = 0$ and that the profiles $\omega_\alpha, \eta_\alpha$ are not smooth near $x = 0, \pi$. Since $\eta_\alpha\rho^{1/2} \notin L^\infty$ (see (6.6) and (6.8)), we need to perform a weighted estimate on u_{xx}. It is based on the lemma below, which shows that the Hilbert transform commutes with $1/x$ up to some lower order terms.

Lemma 6.4. *Suppose that $f/x \in L^2([-\pi,\pi])$. Then*

$$\left|\frac{Hf - Hf(0)}{x} - H\left(\frac{f}{x}\right)\right| \lesssim \int_{-\pi}^{\pi}\left|\frac{f(y)}{y}\right| dy.$$

The proof is deferred to Appendix A.1. Since u, ω are odd, we get $u_{xx}(0) = 0$. Applying the lemma with $f = u_x$ and using the fact that H is L^2-bounded, we get

$$\left\|\frac{u_{xx}}{x}\right\|_{L^2} = \left\|\frac{H\omega_x}{x}\right\|_{L^2} \lesssim \left\|H\left(\frac{\omega_x}{x}\right)\right\|_{L^2} + \left\|\frac{\omega_x}{x}\right\|_{L^1} \lesssim \left\|\frac{\omega_x}{x}\right\|_{L^2} \lesssim \|\omega\|_{\mathcal{H}}. \tag{6.27}$$

Applying (6.9), we obtain

$$\|u_{xx}\eta_\alpha\rho^{1/2}\|_{L^2} \lesssim \|u_{xx}x^{-1}\|_{L^2}\|\eta_\alpha\|_{L^\infty} \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}.$$

Denote $\tilde{u} = u - u_x(0)\sin x$. Next we estimate $\|\tilde{u}\eta_{\alpha,xx}\rho^{1/2}\|_2$. From (6.6) and (6.9), $\eta_{\alpha,xx}$ is similar to $|\sin x|^{\alpha-2}$, which is singular at both $x = 0, \pi$. To overcome the singularities from $\eta_{\alpha,xx}$ and $\rho^{1/2}$, we estimate $\tilde{u}(\sin x)^{-1}x^{-1}$. For $|x| \geq \pi/2$, since $\tilde{u}(\pi) = 0$ and $|\sin x| \lesssim \big|\pi - |x|\big|^{-1}$, we get

$$|\tilde{u}(\sin x)^{-1}x^{-1}| \lesssim \left|\tilde{u}\cdot\big|\pi - |x|\big|^{-1}\right| \lesssim \|\partial_x\tilde{u}\|_\infty \lesssim \|u_{xx}\|_2 \lesssim \|\omega\|_{\mathcal{H}}.$$

For $|x| \leq \pi/2$, since $\tilde{u}(0) = \partial_x\tilde{u}(0) = 0$, using integration by parts, we obtain

$$\begin{aligned}|\tilde{u}(\sin x)^{-1}x^{-1}| &\lesssim \frac{|\tilde{u}|}{x^2} = \frac{1}{x^2}\left|\int_0^x \partial_{yy}\tilde{u}(y)\cdot(x-y)\,dy\right| \\ &\lesssim \frac{1}{x^2}\|\partial_{yy}\tilde{u}\cdot y^{-1}\|_2\left(\int_0^x y^2(x-y)^2\,dy\right)^{1/2}.\end{aligned}$$

Since $\partial_{yy}\tilde{u}(y) = \partial_{yy}u + u_x(0)\sin y$, using (6.27) we derive

$$|\tilde{u}(\sin x)^{-1}x^{-1}| \lesssim x^{-2}(\|u_{xx}x^{-1}\|_2 + |u_x(0)|)x^{5/2} \lesssim \|\omega\|_{\mathcal{H}}.$$

Since $\rho^{1/2} = (\sin\frac{x}{2})^{-1} \asymp x^{-1}$, applying the above estimate and (6.9) we obtain

$$\begin{aligned}\|(u - u_x(0)\sin x)\eta_{\alpha,xx}\rho^{1/2}\|_2 &\lesssim \|\tilde{u}(\sin x)^{-1}\rho^{1/2}\|_\infty\|\eta_{\alpha,xx}\sin x\|_2 \\ &\lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}\left\||\sin x|^{\kappa_2-1}\right\|_2 \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}.\end{aligned}$$

The estimates of other terms in (6.26) and I in (6.25) are relatively simple. Since ω vanishes at $x = 0, \pi$, using the Hardy-type inequality of Lemma A.5 we get

$$\begin{aligned}\|\omega(\sin x)^{-1}\rho^{1/2}\|_2 &\lesssim \|\omega x^{-2}\|_2 + \big\|\omega|\pi-|x||^{-1}\big\|_2 \lesssim \|\omega_x x^{-1}\|_2 + \|\omega_x\|_2\\ &\lesssim \|\omega_x x^{-1}\|_2 \lesssim \|\omega\|_{\mathcal{H}}.\end{aligned}$$

Applying the above estimates and (6.10) in Lemma 6.1, we obtain

$$\|\partial_x H\eta_\alpha \cdot \omega\rho^{1/2}\|_2 \lesssim \|\omega(\sin x)^{-1}\rho^{1/2}\|_2\|\sin x\cdot\partial_x H\eta_\alpha\|_{L^\infty} \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}.$$

Applying (6.11) in Lemma 6.1 and $(1-\alpha)\wedge x^2 \lesssim (1-\alpha)^{1/2}|x|$ yields

$$\begin{aligned}&\|u_x(0)((\alpha-1)\omega_{\alpha,x} - \sin x\cdot\eta_{\alpha,xx})\rho^{1/2}\|_2\\ &\qquad\lesssim \|u_x\|_{L^\infty}\big\|(1-\alpha)\wedge|x|^2)|\sin x|^{\alpha-1}x^{-1}\big\|_2\\ &\qquad\lesssim \|u_{xx}\|_2(1-\alpha)^{1/2}\big\||\sin x|^{\alpha-1}\big\|_2\\ &\qquad\lesssim (1-\alpha)^{1/2}\|\omega_x\|_2 \lesssim (1-\alpha)^{1/2}\|\omega\|_{\mathcal{H}}.\end{aligned}$$

Recall $c_{\omega,\alpha} = (\alpha-1)$ from (6.7). The estimate of the last term in (6.26) is trivial,

$$\|c_{\omega,\alpha}\omega_x\rho^{1/2}\|_2 \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}.$$

Combining the above $L^2(\rho)$ estimates of each term in (6.26), we establish

$$\begin{aligned}|II| &\lesssim \|\mathcal{R}_{\alpha,2}\rho^{1/2}\|_2\|(\omega_x+\omega_e\sin x)\rho^{1/2}\|_2 \lesssim (1-\alpha)^{1/2}\|\omega\|_{\mathcal{H}}(\|\omega\|_{\mathcal{H}}+|\omega_e|)\\ &\lesssim (1-\alpha)^{1/2}\|\omega\|_{\mathcal{H}}^2,\end{aligned}\tag{6.28}$$

where we have applied $|\omega_e| \lesssim \|\omega\|_{\mathcal{H}}$ from Lemma 6.3 in the last inequality.

Next, we estimate the term I from (6.25). Applying integration by parts, we get

$$\begin{aligned}I_1 &:= \langle -u(\eta_\alpha)\omega_{xx}, \omega_x\rho\rangle = \big\langle -u(\eta_\alpha)\rho, \tfrac{1}{2}\partial_x(\omega_x)^2\big\rangle = \tfrac{1}{2}\langle\partial_x(u(\eta_\alpha)\rho)\rho^{-1}, \omega_x^2\rho\rangle,\\ I_2 &:= \langle -u(\eta_\alpha)\omega_{xx}, \omega_e\sin x\cdot\rho\rangle = \omega_e\langle\partial_x(u(\eta_\alpha)\rho\cdot\sin x), \omega_x\rangle.\end{aligned}$$

Since $\rho = (\sin\frac{x}{2})^{-2}$, $|\partial_x\rho| \lesssim \rho|x|^{-1}$, and $\partial_x u(\eta_\alpha) = H\eta_\alpha$, applying (6.10) we derive

$$\begin{aligned}|\partial_x(u(\eta_\alpha)\rho)| &\lesssim \left(|\partial_x u(\eta_\alpha)| + \left|\frac{u(\eta_\alpha)}{x}\right|\right)\rho \lesssim \|\partial_x u(\eta_\alpha)\|_\infty\rho \lesssim (1-\alpha)\rho,\\ |\partial_x(u(\eta_\alpha)\rho\cdot\sin x)| &\lesssim |u(\eta_\alpha)\rho| + |x\partial_x(u(\eta_\alpha)\rho)|\\ &\lesssim \|\partial_x u(\eta_\alpha)\|_\infty|x\rho| + (1-\alpha)|x|\rho \lesssim (1-\alpha)|x|\rho.\end{aligned}$$

Using the above estimate and Lemma 6.3 (b), we establish

$$\begin{aligned}|I_1| &\lesssim \|\partial_x(u(\eta_\alpha)\rho)\rho^{-1}\|_\infty\|\omega_x\rho^{1/2}\|_2^2 \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}^2,\\ |I_2| &\lesssim (1-\alpha)|\omega_e|\cdot\|\omega_x x\rho\|_{L^1} \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}\|\omega_x\rho^{1/2}\|_{L^1} \lesssim (1-\alpha)\|\omega\|_{\mathcal{H}}^2.\end{aligned}\tag{6.29}$$

Plugging the estimates (6.28) and (6.29) in (6.25) and then applying Lemma 6.3, we obtain

$$|\langle\mathcal{R}_\alpha\omega,\omega\rangle_Y| \lesssim (1-\alpha)^{1/2}\|\omega\|_{\mathcal{H}}^2 \lesssim (1-\alpha)^{1/2}\|\omega\|_Y^2.\tag{6.30}$$

6.4.3. Estimates of nonlinear and error terms. Recall the nonlinear term $N(\omega)$ and the error term $F(\omega_\alpha)$ from (6.19). Since $N(\omega)$ is similar to that in [6,35] and the perturbation ω lies in the same space $\mathcal{H}$, the estimate of $N(\omega)$ is almost identical to that in [6, 35]. In particular, we get

$$|\langle N(\omega), \omega\rangle_Y| \lesssim \|\omega\|_{\mathcal{H}}^3 \lesssim \|\omega\|_Y^3 \tag{6.31}$$

and refer for the detailed estimates to [6, 35].

In the following derivation, we use the implicit notation $O(f)$ to denote some term g that satisfies $|g| \lesssim f$. It can vary from line to line. By symmetry, we focus on $x \in [0, \pi]$.

For the error term $F(\omega_\alpha)$, we first compute

$$\partial_x F(\omega_\alpha) = u_{\alpha,xx}\omega_\alpha - u_\alpha \omega_{\alpha,xx} + c_{\omega,\alpha}\omega_{\alpha,x}. \tag{6.32}$$

Recall $u_1 = \sin x$, $\omega_1 = -\sin x$, $\eta_\alpha = \omega_\alpha - \omega_1$ from (6.8), and $u_{\alpha,x} - u_{1,x} = H\eta_\alpha$. Applying Lemma 6.1 and $|\omega_\alpha| \lesssim |\sin x|^\alpha$ (see (6.6)) yields

$$\begin{aligned} u_{\alpha,xx}\omega_\alpha &= (u_{1,xx} + \partial_x H\eta_{\alpha,x})\omega_\alpha = u_{1,xx}\omega_\alpha + O((1-\alpha)|\sin x|^{\kappa_1 - 1+\alpha}) \\ &= u_{1,xx}\omega_1 - \sin x \cdot \eta_\alpha + O((1-\alpha)|\sin x|^{\kappa_1-1+\alpha}) \\ &= (\sin x)^2 + O((1-\alpha)|\sin x|^{\kappa_1-1+\alpha}). \end{aligned} \tag{6.33}$$

We decompose the second term on the RHS in (6.32) as follows:

$$\begin{aligned} u_\alpha \omega_{\alpha,xx} &= u_\alpha \eta_{\alpha,xx} + u_\alpha \omega_{1,xx} = (u_\alpha - \sin x)\eta_{\alpha,xx} + \sin x \cdot \eta_{\alpha,xx} + u_\alpha \omega_{1,xx} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{6.34}$$

Using (6.10) yields

$$\begin{aligned} &|u_{\alpha,xx}| \lesssim |u_{1,xx}| + |\partial_x H\eta_\alpha| \lesssim |\sin x|^{\kappa_1 - 1}, \\ &|u_\alpha - \sin x| \lesssim (\|u_x(\eta_\alpha)\|_\infty + 1)|\sin x| \lesssim |\sin x|. \end{aligned}$$

Recall $u_{\alpha,x}(0) = 1$ from (6.6). For $0 \le x \le \pi/2$, the above estimate implies

$$\begin{aligned} |u_\alpha - \sin x| &\le |u_\alpha - x| + C|x|^3 = \left| \int_0^x (u_{\alpha,x}(x) - u_{\alpha,x}(0))\, dx \right| + C|x|^3 \\ &= \left| \int_0^x u_{\alpha,xx}(y) \cdot (x-y)\, dy \right| + C|x|^3 \lesssim \int_0^x y^{\kappa_1 - 1}(x-y)\, dy + C|x|^3 \\ &\lesssim |x|^{\kappa_1+1}. \end{aligned}$$

Therefore,

$$|u_\alpha - \sin x| \lesssim \mathbf{1}_{x \le \pi/2}|x|^{\kappa_1+1} + \mathbf{1}_{x > \pi/2}|\sin x| \lesssim |\sin x| \cdot |x|^{\kappa_1},$$

which along with (6.9) implies the estimate of I_1 in (6.34),

$$|I_1| \lesssim (1-\alpha)|\sin x|^{\kappa_2 - 1}|x|^{\kappa_1}.$$

For I_3 in (6.34), applying (6.10) and $u_1 = \sin x, \omega_1 = -\sin x$, we get

$$\begin{aligned} I_3 &= u_1\omega_{1,xx} + (u_\alpha - u_1)\omega_{1,xx} = (\sin x)^2 + O(|\sin x|^2\|u_{\alpha,x}\|_\infty) \\ &= (\sin x)^2 + O((1-\alpha)|\sin x|^2). \end{aligned}$$

Recall $c_{\omega,\alpha} = \alpha - 1$ from (6.7). We combine I_2 of (6.34) and $c_{\omega,\alpha}\omega_{\alpha,x}$ of (6.32) and then apply (6.11) to obtain

$$\begin{aligned} |c_{\omega,\alpha}\omega_{\alpha,x} - I_2| &= |(\alpha-1)\omega_{\alpha,x} - \sin x \cdot \eta_{\alpha,xx}| \lesssim ((1-\alpha)\wedge |x|^2)|\sin x|^{\alpha-1} \\ &\lesssim (1-\alpha)^{1/2}|x| \cdot |\sin x|^{\alpha-1}. \end{aligned}$$

Plugging the above estimates on I_i and $c_{\omega,\alpha}$ in (6.34), we establish

$$\begin{aligned} u_\alpha\omega_{\alpha,xx} - c_{\omega,\alpha}\omega_{\alpha,x} &= I_1 + I_3 + (I_2 - c_{\omega,\alpha}\omega_{\alpha,x}) \\ &= (\sin x)^2 + O((1-\alpha)^{1/2}|x|^{\kappa_1}|\sin x|^{\kappa_2-1}), \end{aligned} \tag{6.35}$$

where we have used $|\sin x| \le |\sin x|^{\kappa_2-1}$, $|\sin x| \lesssim |x| \lesssim 1$ and $\kappa_2 < \alpha$ to combine the estimates of I_i in the last estimate.

Recall $\kappa_1 = \frac{3}{4}, \kappa_2 = \frac{7}{8}$ from Lemma 6.1. Combining (6.32), (6.33) and (6.35), we establish

$$\begin{aligned} \partial_x F(\omega_\alpha) &= (\sin x)^2 \cdot (1-1) + O((1-\alpha)|\sin x|^{\kappa_1 - 1 + \alpha}) \\ &\quad + O(1-\alpha)^{1/2}|x|^{\kappa_1}|\sin x|^{\kappa_2 - 1} \\ &= (1-\alpha)^{1/2}|\sin x|^{\kappa_2-1}|x|^{\kappa_1}, \end{aligned}$$

where we have used $|\sin x|^{\kappa_1+\alpha-\kappa_2} \lesssim |\sin x|^{\kappa_1} \lesssim |x|^{\kappa_1}$ to obtain the last estimate. Using the above estimate and Lemma 6.3, we prove

$$\begin{aligned} |\langle F(\omega_\alpha), \omega\rangle_Y| &\lesssim \|F(\omega_\alpha)\|_Y\|\omega\|_Y \lesssim \|\partial_x F(\omega_\alpha)\rho^{1/2}\|_2\|\omega\|_Y \\ &\lesssim (1-\alpha)^{1/2}\left\| |\sin x|^{\kappa_2-1}|x|^{\kappa_1-1}\right\|_2\|\omega\|_Y \lesssim (1-\alpha)^{1/2}\|\omega\|_Y, \end{aligned} \tag{6.36}$$

where the integral is bounded since $2\kappa_2 - 2 = -\frac{1}{4} > -1, 2\kappa_2 + 2\kappa_1 - 4 = -\frac{3}{4} > -1$.

6.4.4. Nonlinear stability and finite time blowup. Combining (6.24), (6.30), (6.31) and (6.36), we establish the following nonlinear estimate for some absolute constant $C > 0$:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_Y^2 \le -\left(\frac{3}{8} - C|1-\alpha|^{1/2}\right)\|\omega\|_Y^2 + C|1-\alpha|^{1/2}\|\omega\|_Y + C\|\omega\|_Y^3.$$

Therefore, there exist absolute constants $\alpha_0 < 1$ sufficiently close to 1 and $\mu > 0$ such that for any $\alpha \in (\alpha_0, 1)$, if the initial perturbation satisfies $\|\omega_0\|_Y < \mu|1-\alpha|^{1/2}$, then

$$\begin{aligned} &\|\omega(t)\|_Y < \mu|1-\alpha|^{1/2}, \\ &c_{\omega,\alpha} + c_\omega(t) = (\alpha-1)(1+u_x(0)) \le (\alpha-1)(1-C|\alpha-1|^{1/2}) \le \tfrac{1}{2}(\alpha-1) \end{aligned}$$

for all $t > 0$. Since $\rho = O(1)$ near $x = \pi$ and $(\partial_x\omega_\alpha)^2\rho$ is integrable near $x = \pi$, we can choose the initial perturbation ω_0 such that $\|\omega_0\|_Y < \mu|1-\alpha|^{1/2}, \omega_0 \in C^2((-\pi/3, \pi/3))$

and $\omega_0 + \omega_\alpha \in C^\alpha \cap C^\infty(S^1 \setminus \{0\})$. For example, ω_0 can be $-\omega_\alpha$ near $x = \pi$, zero near $x = 0$ and smooth in the intermediate region. A simple Lemma A.6 shows that $\omega_0 + \omega_\alpha \in H^s$ for any $s < \alpha + 1/2$, and a direct calculation gives $\int_0^\pi |\sin x \cdot f_x^2 / f| \, dx < \infty$ where $f = \omega_0 + \omega_\alpha$. Using the rescaling argument in Section 6.2, we establish finite time blowup of (1.2) from $\omega_0 + \omega_\alpha$.

The condition $\int_0^T u_{\mathrm{phy},x}(0,t)\,dt = \infty$ in Theorem 3, where u_{phy} is the velocity in (1.2), follows from Theorem 1 or a calculation using the above a priori estimates on the perturbation and the rescaling relations (6.3)–(6.5). Due to the inclusions $C^\alpha \subset C^{\alpha_1}$ and $H^s \subset H^{s_1}$ for $0 < \alpha_1 < \alpha$ and $s_1 < s$, we conclude the proof of Theorem 3.

7. Concluding remarks

We have constructed a finite time blowup solution of the De Gregorio model (1.2) from C^α initial data for any $0 < \alpha < 1$, and established the global well-posedness (GWP) from initial data $\omega_0 \in H^1 \cap X$ with $\omega_0(x)x^{-1} \in L^\infty$, based on a one-point blowup criterion. These results verified the conjecture on global regularity of the DG model on S^1 for smooth data in X, and showed that advection can prevent singularity formation if the initial data is smooth enough.

Our analysis provides valuable insights on the global well-posedness of (1.2) with more general data, and it is likely that some results are generalizable. A potential direction is to generalize the one-point blowup criterion to a finitely-many-points version. For simplicity, we assume that the number of zeros of $\omega(x,t)$ is finite, and the zeros are $x_i(t)$, $i = 1, \dots, n$, with $\partial_x \omega(x_i(t), t) \neq 0$. It is shown in [30] that the number n and $\partial_x \omega(x_i(t), t)$, $i = 1, \dots, n$, are conserved. Denote $N_\pm(t) := \{x : \omega(x,t) = 0, \operatorname{sgn}(\omega_x(x,t)) = \pm 1\}$. A natural generalization of Theorem 1 is that the solution of (1.2) cannot be extended beyond T if and only if

$$\int_0^T \sum_{x \in N_-(t)} |u_x(x,t)| \, dt = \infty. \tag{7.1}$$

A weaker version is that $\sum_{i=1}^n |u_x(x_i(t), t)|$ controls the breakdown of the solution. These blowup criteria are consistent with that of the CLM model. See the discussion in Section 1.2. We believe that these criteria are important for the GWP from general smooth initial data.

Passing from (7.1) to the GWP, a possible approach is to estimate functionals and quadratic forms similar to those in Section 4 in suitable moving frames. We remark that our proof of Lemma 4.2 does not require the assumption on the sign of ω. Thus, it is conceivable that the argument can be adapted to study other scenarios.

Our analysis has benefited from the property that the zeros of ω with $\omega \in X$ (see (1.4)) are essentially fixed. For more general data, controlling the locations of the zeros of ω can be a challenging problem.

For the gCLM model on a circle with a parameter $a > 1$ and $\omega_0 \in C^\infty \cap X$, monotonicity of $\int_0^{\pi/2} |\omega(y)| (\cot y)^\beta \, dy$ with $\beta = \beta(a) < 2$ and a priori estimates of

$\|\omega(t)\|_{L^1}, u_x(0,t)$ can be studied by the argument in Sections 4 and 5. These a priori estimates shed some helpful light on the regularity of the gCLM model with $\omega_0 \in C^\infty \cap X$. Note that for $a > a_0$ with $a_0 \approx 1.05$, these estimates have been established in the arXiv version of [9].

Appendix A. Some technical lemmas and derivations

A.1. Properties of the Hilbert transform and functional inequalities

The following Cotlar identity for the Hilbert transform is well known; see, e.g., [9,18,20]).

Lemma A.1. *For $f \in C^\infty(S^1)$, we have*

$$H(fHf) = \tfrac{1}{2}((Hf)^2 - f^2).$$

We have the following commutator identity from [6, Lemma 2.6].

Lemma A.2. *For $f \in H^1(S^1)$ with period $n\pi$, we have*

$$H\left(\sin\left(\frac{2x}{n}\right) f_x\right) - \sin\left(\frac{2x}{n}\right) Hf_x = -\frac{2}{n^2\pi}\int f \sin(2y)\, dy = H\left(\sin\left(\frac{2x}{n}\right) f_x\right)(0).$$

The case $n = 2$ is proved in [6]. The general case follows by a rescaling argument.

We use the following important lemma to establish the energy estimate in Section 3.

Lemma A.3. *Suppose that $\omega \in H^1$ is π-periodic and odd. Then*

$$\int_{S^1} \omega_x H\omega_x \cdot \sin(2x)\, dx = 0.$$

Proof. We prove the identity for $\omega \in C^\infty$, and the general case $\omega \in H^1$ can be obtained by approximation. Applying Lemma A.2 with $f = \omega$ and $n = 1$ yields

$$\begin{aligned} S &:= \int_{S^1} \omega_x H\omega_x \cdot \sin(2x)\, dx \\ &= \int_{S^1} \omega_x \big(H(\sin(2x)\omega_x) - H(\sin(2x)\omega_x)(0)\big)\, dx \\ &= \int_{S^1} \omega_x H(\sin(2x)\omega_x)\, dx. \end{aligned}$$

Denote $f = \sin(2x)\omega_x$. Using $\frac{1}{\sin(2x)} = \frac{1}{2}(\tan x + \cot x) = \frac{1}{2}(\cot(\pi/2 - x) + \cot(x))$, (3.1) and Lemma A.1, we obtain

$$\begin{aligned} S &= \frac{1}{2}\int_{S^1}\left(\cot\left(\frac{\pi}{2} - x\right) + \cot(x)\right) f \cdot Hf\, dx \\ &= \frac{\pi}{2}\left(H(fHf)\left(\frac{\pi}{2}\right) - H(fHf)(0)\right) \\ &= \frac{\pi}{4}\left((Hf)^2\left(\frac{\pi}{2}\right) - f^2\left(\frac{\pi}{2}\right) - (Hf)^2(0) - f^2(0)\right). \end{aligned}$$

Since $\omega \in C^\infty$ and it is odd, we get $f(0) = f(\pi/2) = 0$. Note that

$$Hf\left(\frac{\pi}{2}\right) - Hf(0) = \frac{1}{\pi}\int_{S^1}\left(\cot\left(\frac{\pi}{2} - x\right) + \cot x\right)\sin(2x)\omega_x\,dx$$
$$= \frac{1}{\pi}\int_{S^1}\frac{2}{\sin(2x)}\sin(2x)\omega_x\,dx = 0.$$

We obtain $S = 0$, as desired. ■

We use the following simple lemma from [8] to estimate the profile in Section 6.

Lemma A.4. *For $x \in [0,1]$, $\alpha, \lambda > 0$, we have*

$$(1 - x^\alpha)x^\lambda \le \alpha/\lambda.$$

Proof. For the sake of completeness, we present the proof. Using Young's inequality, we get

$$(1 - x^\alpha)x^\lambda = \frac{\alpha}{\lambda}\cdot\left(\frac{\lambda}{\alpha}(1 - x^\alpha)\right)(x^\alpha)^{\lambda/\alpha}$$
$$\le \frac{\alpha}{\lambda}\left(\frac{\frac{\lambda}{\alpha}(1 - x^\alpha) + \frac{\lambda}{\alpha}x^\alpha}{1 + \lambda/\alpha}\right)^{\lambda/\alpha+1} = \frac{\alpha}{\lambda}\left(\frac{\lambda/\alpha}{1 + \lambda/\alpha}\right)^{\lambda/\alpha+1} \le \frac{\alpha}{\lambda}. \quad ■$$

We have the following Hardy-type inequality [25] in a bounded domain.

Lemma A.5. *For $p > 1$ and $L > 0$, suppose that $f x^{-p/2}, f_x x^{-p/2+1} \in L^2([0, L])$. Then*

$$\int_0^L \frac{f^2}{x^p}\,dx \lesssim_p \int_0^L \frac{f_x^2}{x^{p-2}}\,dx.$$

It can be proved by applying an integration by parts argument. A proof can be found in [10, Supplementary material].

Next, we prove the commutator-type Lemma 6.4.

Proof of Lemma 6.4. A direct calculation yields

$$S := \frac{1}{x}(Hf - Hf(0)) - H\left(\frac{f}{x}\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{x}\cot\frac{x-y}{2} + \frac{1}{x}\cot\frac{y}{2} - \frac{1}{y}\cot\frac{x-y}{2}\right)f(y)\,dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{x}\left(y\cot\frac{y}{2} - (x-y)\cot\frac{x-y}{2}\right)\frac{f(y)}{y}\,dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{x}(g(y) - g(y-x))\frac{f(y)}{y}\,dy,$$

where $g(z) = z\cot\frac{z}{2}$ satisfies $g(z) = g(-z)$. Since g is Lipschitz on $[-3\pi/2, 3\pi/2]$,

$$|g'(z)| = \left|\cot\frac{z}{2} - \frac{z}{2\left(\sin\frac{z}{2}\right)^2}\right| = \frac{|\sin z - z|}{2\left(\sin\frac{z}{2}\right)^2} \lesssim \frac{z^3}{z^2} \lesssim 1,$$

and applying $|g(y) - g(y-x)| \lesssim |x|$, we get the desired result. ■

A.2. Derivation of a model for 2D Boussinesq equations

We derive the model (1.9)–(1.10), and discuss its connections with (1.2). Recall the Boussinesq equations (1.7) and (1.8):

$$\partial_t \theta_x + \mathbf{u} \cdot \nabla \theta_x = -u_{1,x}\theta_x - u_{2,x}\theta_y = u_{2,y}\theta_x - u_{2,x}\theta_y.$$

Inspired by the anisotropic property of θ in [8], i.e. $|\theta_y| \ll |\theta_x|$ near the origin, we drop the θ_y term. To study y-advection, we further drop the x-advection term. Then we obtain (1.9),

$$\partial_t \theta_x + u_2 \partial_y \theta_x = u_{2,y}\theta_x.$$

Since θ_x is the forcing term in the ω-equation of (1.7), it leads to a strong alignment between θ_x and ω. Thus, we simplify the ω-equation in (1.7) by $\omega = \theta_x$, which leads to the following Biot–Savart law in (1.10):

$$\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\theta_x, \quad u_{2,y} = \partial_{xy}(-\Delta)^{-1}\theta_x.$$

This model relates to (1.2) via $\theta_x \to -\omega$, $\partial_{xy}(-\Delta)^{-1} \to -H$. The velocities of the two models u_2 and u are related via $u_{2,y} = \partial_{xy}(-\Delta)^{-1}\theta_x \approx -H(-\omega) = H\omega = u_x$. Moreover, the solutions of the two models enjoy similar sign and symmetry properties. Suppose that θ_x satisfies the sign and symmetry properties in the hyperbolic-flow scenario. The induced flow $u_2(x, y)$ is odd in y with $u_2(x, y) > 0$ in the first quadrant near $(0, 0)$. The odd symmetries of θ_x, u_2 in y are the same as those of ω, u in (1.2) for the class X of (1.4). Moreover, for fixed $x > 0$, $-\theta_x(x, \cdot)$ and ω satisfy similar sign conditions, and $u_2(x, \cdot)$ and u satisfy similar sign conditions near the origin.

A.3. Derivation of (4.5)–(4.6)

Recall the formulas for u_x, u in (3.1) and the quadratic form in (4.1). Using integration by parts, we obtain

$$\begin{aligned} B(\beta) &= \int_0^{\pi/2} (2u_x\omega - (u\omega)_x) \cot^{\beta} x \, dx \\ &= 2\int_0^{\pi/2} u_x \omega \cot^{\beta} x \, dx - \beta \int_0^{\pi/2} u\omega \cot^{\beta-1} x \cdot \frac{1}{\sin^2 x} \, dx =: I + II. \end{aligned}$$

The boundary terms $u\omega \cot^{\beta} x\big|_0^{\pi/2}$ in the integration by parts vanish since $u(\pi/2) = 0$ and $u(x) = O(x)$, $\omega(x) = O(x^{\gamma})$ with $\gamma > \beta - 1$ near $x = 0$ by the assumption in Lemma 4.2.

Since ω is odd, using (3.1) and symmetrizing the kernel we get

$$\begin{aligned} I &= \frac{2}{\pi} \int_0^{\pi/2} \omega(x) \cot^{\beta} x \int_0^{\pi/2} \omega(y)(\cot(x-y) - \cot(x+y)) \, dy \\ &= \frac{1}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) P_1(x, y) \, dx \, dy, \end{aligned}$$

where

$$P_1(x,y) = \cot^\beta x \cdot (\cot(x-y) - \cot(x+y)) + \cot^\beta y \cdot (\cot(y-x) - \cot(x+y)).$$

Recall $s = \frac{\cot x}{\cot y}$ from (4.4), so $\cot x = s \cot y$. We expand $\cot(x-y), \cot(x+y)$ as follows:

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x} = \frac{s\cot^2 y + 1}{\cot y \cdot (1-s)},$$
$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x} = \frac{s\cot^2 y - 1}{\cot y \cdot (1+s)}.$$

Thus, we obtain

$$\begin{aligned}
\cot(x-y) - \cot(x+y) &= \cot y \cdot \left(\frac{s}{1-s} - \frac{s}{1+s}\right) + \frac{1}{\cot y}\left(\frac{1}{1-s} + \frac{1}{1+s}\right) \\
&= \cot y \cdot \frac{2s^2}{1-s^2} + \frac{1}{\cot y}\frac{2}{1-s^2}, \\
\cot(y-x) - \cot(x+y) &= \cot y \cdot \left(-\frac{s}{1-s} - \frac{s}{1+s}\right) + \frac{1}{\cot y}\left(-\frac{1}{1-s} + \frac{1}{1+s}\right) \\
&= -\cot y \cdot \frac{2s}{1-s^2} - \frac{1}{\cot y}\frac{2s}{1-s^2}.
\end{aligned}$$

Using the above formulas and $\cot^\beta x = s^\beta \cot^\beta y$, we get

$$\begin{aligned}
P_1 &= \cot^\beta y \cdot s^\beta \left(\cot y \cdot \frac{2s^2}{1-s^2} + \frac{1}{\cot y}\frac{2}{1-s^2}\right) \\
&\quad + \cot^\beta y \cdot \left(-\cot y \cdot \frac{2s}{1-s^2} - \frac{1}{\cot y}\frac{2s}{1-s^2}\right) \\
&= \cot^{\beta+1} y \cdot (s^{\beta+1} - 1)\frac{2s}{1-s^2} + \cot^{\beta-1} y \cdot (s^{\beta-1} - 1)\frac{2s}{1-s^2}.
\end{aligned}$$

We remark that $P_1 \le 0$ since $\frac{1-s^\tau}{1-s^2} \ge 0$ for $s, \tau > 0$.

For II, using (3.1), we get

$$\begin{aligned}
II &= \frac{\beta}{\pi}\int_0^{\pi/2} \frac{\omega(x)}{\sin^2 x}\cot^{\beta-1} x \int_0^{\pi/2} \omega(y) \log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy \\
&= \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} P_2(x,y)\omega(x)\omega(y)\,dx\,dy,
\end{aligned}$$

where

$$P_2 = \frac{\beta}{2}\left(\frac{\cot^{\beta-1} x}{\sin^2 x} + \frac{\cot^{\beta-1} y}{\sin^2 y}\right)\log\left|\frac{\sin(x+y)}{\sin(x-y)}\right|.$$

Note that

$$\frac{\cot^{\beta-1} z}{\sin^2 z} = \cot^{\beta-1} z + \cot^{\beta+1} z, \qquad \left|\frac{\sin(x+y)}{\sin(x-y)}\right| = \left|\frac{\cot x + \cot y}{\cot x - \cot y}\right| = \left|\frac{1+s}{1-s}\right|.$$

We derive

$$P_2(x,y) = \frac{\beta}{2}\Big(\cot^{\beta+1} y \cdot (1+s^{\beta+1}) \log\Big|\frac{1+s}{1-s}\Big| + \cot^{\beta-1} y \cdot (1+s^{\beta-1}) \log\Big|\frac{1+s}{1-s}\Big|\Big).$$

We remark that P_2 is positive. Combining the formulas for P_1, P_2, we derive (4.5)–(4.6).

A.4. *Positive-definiteness of the kernel*

In this subsection, we prove Lemmas 4.3 and 4.4, which are related to the positive-definiteness of the kernel $K_{i,\beta}$. We establish (4.20) for $x_0 = \log \frac{5}{3}$ in Appendix A.4.1.

Proof of Lemma 4.3. We show that there exists $\beta_0 \in (1,2)$ such that conditions (4.19)–(4.21) hold for $W = W_{1,\beta}, G = G_{1,\beta}$ with $\beta \in [\beta_0, 2]$. Then using the same argument as in Section 4.2.1, we obtain $G_{1,\beta}(\xi) \geq 0$ for all ξ and $\beta \in [\beta_0, 2]$.

Firstly, we impose $\beta \in [1.9, 2]$. Recall $G_{j,\beta}$ defined in (4.18),

$$G_{j,\beta}(\xi) = \int_0^\infty W_{j,\beta}(x)\cos(x\xi)\,dx, \tag{A.1}$$

and $W_{1,\beta}$ from (4.11), (4.13). Clearly, $W_{1,\beta}(x)$ converges to $W_{1,2}(x)$ as $\beta \to 2$ almost everywhere. Moreover, from the formula for $W_{1,\beta}$ and the decay estimate (4.14), we have

$$|W_{1,\beta}(z)| \lesssim \mathbf{1}_{|z|>1} e^{-|z|/4} + \mathbf{1}_{|z|\leq 1}\big(1 + \big|\log|z|\big|\big), \tag{A.2}$$

where the term $\log|z|$ is due to the logarithm singularity $\log|s-1| = \log|e^z - 1|$ in (4.11). Thus, using the dominated convergence theorem yields

$$\lim_{\beta\to 2^-} G_{1,\beta}(\xi) = G_{1,2}(\xi).$$

Using (A.1) and (A.2), we find that $G_{1,\beta}(\xi)$ is equicontinuous:

$$|\partial_\xi G_{j,\beta}(\xi)| \leq \int_0^\infty |W_{j,\beta}(x)|\,|x|\,dx \lesssim 1.$$

Thus, $G_{1,\beta}(\xi)$ converges to $G_{1,2}(\xi)$ uniformly for $\xi \in [0, M]$, where M is the parameter in Lemma 4.3.

For x near 0, from (4.11) and (4.13) we have

$$W_{1,\beta}(x) = -\frac{\beta}{2}(e^{\frac{\beta+1}{2}x} + e^{-\frac{\beta+1}{2}x}) \log|e^x - 1| + S_\beta(x),$$

where $S_\beta(x)$ is smooth near $x = 0$. Thus a direct calculation yields

$$\begin{aligned}\partial_{xx} W_{1,\beta}(x) &\geq -\frac{\beta}{2}(e^{\frac{\beta+1}{2}x} + e^{-\frac{\beta+1}{2}x})\partial_{xx}\log|e^x-1| - \frac{C}{|x|}\\ &\geq \frac{\beta}{2}(e^{\frac{\beta+1}{2}x} + e^{-\frac{\beta+1}{2}x})\frac{e^x}{(e^x-1)^2} - \frac{C}{|x|} \geq \frac{\beta}{x^2} - \frac{C}{|x|}\end{aligned} \tag{A.3}$$

for some absolute constant $C > 0$ and $|x| < 1/2$. Therefore, there exists $\delta > 0$ such that

$$\partial_{xx} W_{1,\beta}(x) > 0, \quad x \in [0, \delta]. \tag{A.4}$$

Note that $W_{1,\beta}(x) = \tilde{K}_{1,\beta}(e^x)$ (see (4.11)) is smooth for $(\beta, x) \in [1.9, 2] \times [\delta, x_0]$, where x_0 is the parameter in Lemma 4.3. We find that $\partial_{xx} W_{1,\beta}(x)$ converges to $\partial_{xx} W_{1,2}(x)$ uniformly for $x \in [\delta, x_0]$ as $\beta \to 2$, and $\partial_x W_{1,\beta}(x_0) \to \partial_x W_{1,2}(x_0)$ as $\beta \to 2$.

Next, we consider the integral of W''' in (4.21). We need a decay estimate of $W'''_{1,\beta}$. For $r = e^{x_0} > 1$ and $s \geq r > 1$, performing Taylor expansion of $\log |\frac{s+1}{s-1}|$ and $\frac{1}{s^2-1}$, we find that the kernel $\tilde{K}_{1,\beta}$ enjoys the expansion

$$\tilde{K}_{1,\beta} = \sum_{i \geq 1} a_i(\beta) s^{-\alpha_i(\beta)}, \quad |a_i(\beta)| \lesssim 1, \quad \max\left(\frac{\beta - 1}{2}, \frac{i-2}{10}\right) \leq \alpha_i(\beta) \leq 10(i+1), \tag{A.5}$$

with $\alpha_i(\beta)$ increasing. Since the expansions for $\log |\frac{s+1}{s-1}|$ and $\frac{1}{s^2-1}$ converge uniformly for $s \geq r > 1$, the above expansion also converges uniformly. Thus, we can exchange the summation and derivatives when computing $\partial_x^k \tilde{K}_{1,\beta}$. We are interested in the leading order term in the above expansion. It decays at least as $s^{-(\beta-1)/2}$ since the other terms in $\tilde{K}_{1,\beta}$ that decay more slowly, such as $s^{(\beta-1)/2}$, are canceled. Using $W_{1,\beta}(x) = \tilde{K}_{1,\beta}(e^x)$ and (A.5), for $x \geq x_0 > 0$, we get

$$\begin{aligned}|\partial_x^3 W_{1,\beta}(x)| &= \left|\partial_x^3 \sum_{i \geq 1} a_i(\beta) e^{-\alpha_i(\beta) x}\right| = \left|\sum_{i \geq 1} a_i(\beta)(-\alpha_i(\beta))^3 e^{-\alpha_i(\beta) x}\right| \\ &\lesssim e^{-\frac{\beta-1}{2} x} \lesssim e^{-x/4},\end{aligned}$$

where the implicit constant can depend on x_0. Note that $\partial_x^3 W_{1,\beta}(x) \to \partial_x^3 W_{1,2}(x)$ for any $x \geq x_0 > 0$ as $\beta \to 2$. Using the dominated convergence theorem, we get

$$\lim_{\beta \to 2^-} \int_{x_0}^{\infty} |\partial_x^3 W_{1,\beta}(x)|\, dx = \int_{x_0}^{\infty} |\partial_x^3 W_{1,2}(x)|\, dx. \tag{A.6}$$

Note that conditions (4.19)–(4.21) hold with strict inequality for $W = W_{1,2}, G = G_{1,2}$. From the uniform convergences $G_{1,\beta}(\xi) \to G_{1,2}(\xi)$ on $[0, M]$, $\partial_x^2 W_{1,\beta}(x) \to \partial_x^2 W_{1,2}(x)$ on $[\delta, x_0]$, $\partial_x W_{1,\beta}(x_0) \to \partial_x W_{1,2}(x_0)$ as $\beta \to 2$, (A.4) and (A.6), we conclude that there exists $\beta_0 \in (1, 2)$ such that (4.19)–(4.21) hold for $W = W_{1,\beta}, G = G_{1,\beta}$ with $\beta \in [\beta_0, 2]$. ∎

A.4.1. Convexity of $W_{i,\beta}$. We first establish (4.20) for $x_0 = \log \frac{5}{3}$ and then prove Lemma 4.4.

Since $W_{i,\beta}$ is given explicitly in (4.11), (4.13) and (4.17), to simplify the derivations we have used Mathematica. All the symbolic derivations and simplification steps are given in Mathematica (version 12) [7]. We only provide the steps that require estimates.

Suppose that $W(x) = K(e^x)$ and denote $s = e^x$. Using the chain rule, we get

$$\begin{aligned}\partial_{xx} W_{i,\beta}(x) &= \partial_{xx} \tilde{K}_{i,\beta}(e^x) = e^{2x}(\partial^2 \tilde{K}_{i,\beta})(e^x) + e^x(\partial \tilde{K}_{i,\beta})(e^x) \\ &= s^2 \partial^2 \tilde{K}_{i,\beta}(s) + s\partial \tilde{K}_{i,\beta}(s) =: I_i(s, \beta).\end{aligned} \tag{A.7}$$

To establish (4.20), i.e., $\partial_{xx} W_{1,2}(x) > 0$ for $x \in [0, x_0]$, $x_0 = \log \frac{5}{3}$, it suffices to prove $I_1(s, 2) > 0$ for $s \in [1, \frac{5}{3}]$. For $i = 1, \beta = 2$, using symbolic calculation, we get

$$I_1(s,2) = \frac{P_1 + P_2}{4s^{3/2}(1+s)^3}, \quad P_2 = 9(1+s)^4(1-s+s^2)\log\left|\frac{s+1}{s-1}\right|.$$

We do not write down the expression of P_1 since it is an intermediate term and is not used directly. We provide its formula in Mathematica [7]. Using $\log(1+z) \le z$ for $z > -1$, we obtain

$$\log\left|\frac{1+s}{1-s}\right| = -\log\left|\frac{1-s}{1+s}\right| \ge -\left(-\frac{2}{1+s}\right) = \frac{2}{1+s}. \tag{A.8}$$

Using the above inequality and simplifying the expression yields

$$\begin{aligned} I_1(s,2) &\ge \frac{1}{4s^{3/2}(1+s)^3}\left(P_1 + 9(1+s)^4(1-s+s^2)\frac{2}{s+1}\right) \\ &= \frac{P_3}{4s^{3/2}(1+s)^3}, \\ P_3 &= -\frac{2(-9+9s+27s^2-18s^3-59s^4+9s^5+9s^6)}{(s-1)^2}. \end{aligned}$$

Since $s \in [1, \frac{5}{3}]$, using $s^i \le s^j$ for $i \le j$ and $9s + 9s^2 \le 15 + 25 < 41$ we obtain

$$\begin{aligned} &-9+9s+27s^2-18s^3-59s^4+9s^5+9s^6 \\ &\qquad < (9s+27s^2-18s^3-18s^4) + s^4(9s+9s^2-41) < 0, \end{aligned}$$

which implies $P_3 > 0$ on $[1, \frac{5}{3}]$. Hence $I_1(s, 2) > 0$ on $[1, \frac{5}{3}]$ and we get (4.20) with $x_0 = \log \frac{5}{3}$.

Proof of Lemma 4.4. Recall $W_{2,\beta}(x) = \tilde{K}_{2,\beta}(e^x)$ and formulas (4.17). Denote $s = e^x$. Using (A.7), it suffices to prove that $I_2(s, \beta) \ge 0$ for all $s = e^x \ge 1$. Using symbolic calculation, we have

$$I_2(s,\beta) = \frac{\beta}{2}s^{-a}\left(I_{2,1}(s,\beta) + a^2(1+s^{2a})\log\frac{1+s}{s-1}\right), \quad a = \frac{\beta-1}{2},$$

where $I_{2,1}(s, \beta)$ is an intermediate term and its formula is given in Mathematica [7]. Since $\beta > 0$, using (A.8) we get

$$I_2(s,\beta) \ge \frac{\beta}{2}s^{-a}\left(I_{2,1}(s,\beta) + a^2(1+s^{2a})\frac{2}{1+s}\right) =: \frac{\beta}{2}s^{-a}I_{2,2}(s,\beta).$$

Next, we show that $I_{2,2}(s, \beta) \ge 0$. Simplifying the expression, we obtain

$$\begin{aligned} I_{2,2}(s,\beta) &= \frac{P_1+P_2+P_3}{(s^2-1)^3}, \qquad P_1 = -2a^2(s^2-1)^2(1-2s+s^{2a}), \\ P_2 &= 8as(s^2-1)(s^2+s^{2a}), \quad P_3 = 4s(3s^2+s^4-s^{2a}-3s^{2+2a}). \end{aligned}$$

Since $a = \frac{\beta-1}{2} \in [0, \frac{1}{2}]$ and $s \ge 1$, we get $2s - 1 - s^{2a} \ge 2s - 1 - s = s - 1 \ge 0$. Thus, we obtain $P_1, P_2 \ge 0$. Using $s^{2a} \le s$ again, we derive

$$\begin{aligned} P_3 \ge 4s(3s^2 + s^4 - s - 3s^3) &= 4s^2(s^3 - 1 + 3s - 3s^2) \\ &= 4s^2(s-1)(s^2 + s + 1 - 3s) = 4s^2(s-1)^3 \ge 0. \end{aligned}$$

Combining the above estimates of P_i, we establish $I_2(s, \beta) \ge 0$ for $s \ge 1, \beta > 1$, which further implies $\partial_{xx} W_{2,\beta} \ge 0$ for $x \ge 0$. ∎

A.5. *Proof of other lemmas*

Proof of Lemma 5.2. Recall that $x, y \in [0, \pi/2]$ and $\beta \in [3/2, 2]$. In the following estimates, the reader can think of the special case $\beta = 2$.

For $x + y \le \pi/2$, since $y \le \pi/2 - x$ and $\cot z$ is decreasing on $[0, \pi]$, we have

$$\cot x \cot y \ge \cot x \cot(\pi/2 - x) = 1. \tag{A.9}$$

Since $\min(x, y) \le \frac{1}{2}(x + y) \le \pi/4$, we obtain $\max(\cot x, \cot y) \ge 1$ and

$$(\cot x \cot y)^\beta \ge \cot x \cot y \ge \min(\cot x, \cot y) \ge \cot(x + y).$$

The case $x + y \ge \pi/2$ is trivial, and we get (5.10) in Lemma 5.2. Next, we consider (5.11):

$$\begin{aligned} I &:= \cot y \cdot (\cot x)^{\beta-2} \wedge \cot x \cdot (\cot y)^{\beta-2} \\ &\lesssim (\cot x \cot y)^\beta + \mathbf{1}_{x+y\ge\pi/2} \cot(\pi - x - y) =: J. \end{aligned}$$

Note that $\mathbf{1}_{x+y\ge\pi/2} \cot(\pi - x - y)$ is nonnegative. Without loss of generality, we assume $x \le y$. Since $\beta \le 2$ and $\cot x \ge \cot y$, we get

$$I = \cot y \cdot (\cot x)^{\beta-2}.$$

Case 1: $x + y \le \pi/2$. Since $x \le y$ and $x \le \frac{1}{2}(x + y) \le \pi/4$, using (A.9), $\cot x \ge 1$, $\cot x \ge \cot y$ and $\beta \in [1, 2]$ we get

$$J \ge (\cot x \cot y)^\beta \ge (\cot x \cot y)^{\beta-1} \ge (\cot y)^{\beta-1} \ge \cot y \cdot (\cot x)^{\beta-2} = I.$$

Case 2: $x + y > \pi/2$. In this case, J contains the term $\cot(\pi - x - y) \ge 0$.

Case 2.a: $x > \pi/3$. Since $y \ge x \ge \pi/3$, we know that $\cot y \le \cot x$, $\cot x \lesssim 1$ and $\cot(\pi - x - y) \ge \cot(\pi/3) \gtrsim 1$. It follows that

$$I \le \cot x \cdot (\cot x)^{\beta-2} = (\cot x)^{\beta-1} \lesssim 1 \lesssim \cot(\pi - x - y) \lesssim J.$$

Case 2.b: $x \le \pi/3$ and $\pi - x - y \le y$. Since $1 \lesssim \cot x$ and $\cot z$ is decreasing on $[0, \pi]$, we get

$$I \lesssim \cot y \le \cot(\pi - x - y).$$

Case 2.c: $x \leq \pi/3$ and $\pi - x - y \geq y$. Since $y \geq \frac{1}{2}(x+y) \geq \pi/4$ and $x \leq \pi/3$, we have

$$\cot x \gtrsim x^{-1}, \quad \cot y \gtrsim \cos y \gtrsim \pi/2 - y.$$

Note that $\pi - x - y \geq y$ implies $\pi/2 - y \geq x/2$. We find that

$$\cot x \cot y \gtrsim \frac{\pi/2 - y}{x} \gtrsim 1,$$

which along with $1 \lesssim \cot x$, $\cot y \leq \cot x$, $\beta \in [1,2]$ implies

$$I \leq \cot y \cdot (\cot y)^{\beta-2} = (\cot y)^{\beta-1} \lesssim (\cot x \cot y)^{\beta-1} \lesssim (\cot x \cot y)^{\beta} \lesssim J.$$

This concludes the proof of (5.11).

Next, we prove (5.12),

$$II := \cot y \cdot \mathbf{1}_{y \geq \pi/3} \lesssim (\cot x \cot y)^{\beta} + \mathbf{1}_{x+y \geq \pi/2} \cot(\pi - x - y) = J.$$

We focus on $y \geq \pi/3$. We consider three cases: (a) $x + y \leq \pi/2$, (b) $x + y > \pi/2$ and $\pi - x - y \leq y$, (c) $x + y > \pi/2, \pi - x - y \geq y$. In the first case, from (A.9), we have $J \geq 1 \gtrsim II$. In the second case, since $\cot z$ is decreasing, we get

$$J \geq \cot(\pi - x - y) \geq \cot y \geq II.$$

In the third case, since $x \leq \pi - 2y \leq \pi - 2\pi/3 \leq \pi/3$, $y \geq \pi/3$ and $\pi/2 - y \geq x/2$, using the same argument as in Case 2.c we get

$$\cot x \cot y \gtrsim 1, \quad J \geq (\cot x \cot y)^{\beta} \gtrsim 1 \gtrsim II.$$

This concludes the proof of (5.12) and of Lemma 5.2. ■

The initial data constructed in Section 6.4.4 enjoys the following regularity in Sobolev space.

Lemma A.6. *Suppose that $\omega_0 + \omega_\alpha \in C^\infty(S^1 \setminus \{0\})$ and $\omega_0 \in C^2(-\pi/3, \pi/3)$. Then $\omega_0 + \omega_\alpha \in H^s$ for any $s < \alpha + 1/2$.*

Proof. Let χ be a smooth even cutoff function on S^1 (2π-periodic) with $\chi(x) = 1$ for $|x| \leq \pi/8$ and $\chi(x) = 0$ for $|x| \geq \pi/4$. We decompose $\omega_0 + \omega_\alpha$ as follows:

$$\omega_0 + \omega_\alpha = \chi\omega_\alpha + \chi\omega_0 + (1-\chi)(\omega_0 + \omega_\alpha) =: I + II + III.$$

Clearly, $II, III \in C^2 \subset H^{s_1}$ for any $s_1 \leq 2$. Denote $f_\alpha = \chi\omega_\alpha$. Since f_α is odd, it enjoys an expansion $\omega_\alpha(x) = \sum_{k \geq 1} a_k \sin(kx)$. Next, we estimate a_k. Using integration by parts, we get

$$\begin{aligned} a_k &= C \int_0^\pi f_\alpha \sin(kx)\,dx = \frac{C}{k} \int_0^\pi f_\alpha' \cos kx\,dx \\ &= \frac{C}{k} \int_0^\pi (\mathbf{1}_{x \leq 1/k} + \mathbf{1}_{1/k \leq x \leq \pi/4}) f_\alpha' \cos kx\,dx =: J_1 + J_2, \end{aligned}$$

where the restriction $\mathbf{1}_{x \leq \pi/4}$ is due to the fact that χ is supported in $|x| \leq \pi/4$. Recall formula (6.6) for ω_α. A direct calculation yields

$$|J_1| \lesssim_\alpha k^{-1} \int_0^{1/k} |f_\alpha'| \, dx \lesssim \int_0^{1/k} |x|^{\alpha-1} \, dx \lesssim_\alpha k^{-1-\alpha}.$$

For J_2, using $\cos kx = \partial_x \frac{\sin kx}{k}$, $|\partial_x^i \omega_\alpha(x)| \lesssim |x|^{\alpha-i}$ and integration by parts again, we derive

$$\begin{aligned}
|J_2| &\lesssim_\alpha k^{-1} \left(\left| \frac{\sin(k \cdot k^{-1})}{k} f_\alpha'\left(\frac{1}{k}\right) \right| + \frac{1}{k} \int_{1/k}^{\pi/4} |f_\alpha'' \sin kx| \, dx \right) \\
&\lesssim_\alpha k^{-1} \left(\frac{1}{k} \left(\frac{1}{k}\right)^{\alpha-1} + \frac{1}{k} \int_{1/k}^{\pi/4} |x|^{\alpha-2} \, dx \right) \\
&\lesssim_\alpha k^{-1} (k^{-\alpha} + k^{-1} (k^{-1})^{\alpha-1}) \lesssim_\alpha k^{-\alpha-1}.
\end{aligned}$$

Therefore, for $s < \alpha + 1/2$, we establish

$$\sum_{k \geq 1} |a_k|^2 k^{2s} \leq \sum_{k \geq 1} k^{-2-2\alpha+2s} < \infty,$$

which implies $\omega_\alpha \chi = f_\alpha \in H^s$ and concludes the proof. ∎

A.6. Rigorous verification

To establish Lemma 4.2, we need to verify conditions (4.19) and (4.21) in Lemma 4.3. Note that condition (4.20) has been verified in Appendix A.4.1.

Since the kernel $W_{1,2}$ is explicit (see (4.11) and (4.13)), to simplify the derivations we have used Mathematica. All the symbolic derivations and simplification steps are given in Mathematica (version 12). We only provide the steps that require estimates. All the numerical computations and quantitative verifications are performed in MATLAB (version 2019a) in double-precision floating-point operations. The Mathematica and MATLAB codes can be found via [7]. We will also use interval arithmetic [45, 48] and refer for the discussion to Appendix A.6.4.

To obtain (4.19), using the approach in Section 4.2.2, we only need to verify (4.25). Conditions (4.25) and (4.21) involve a finite number of integrals and the Lipschitz constant b_1 in (4.24). Since these conditions are not tight, we use the following simple method to verify them.

To estimate the integral of f on $[A, \infty)$ with $A \geq 0$, we first choose B sufficiently large and partition $[A, B]$ into $A = y_0 < y_1 < \cdots < y_N = B$. We will estimate the decay of f in the far field in Appendix A.6.3, and treat the integral in $[B, \infty)$ as a small error. For each small interval $I = [y_i, y_{i+1}]$, we use a trivial first order method to estimate the integral

$$|I| \min_{x \in I} f(x) \leq \int_I f(x) \, dx \leq |I| \max_{x \in I} f(x), \quad |I| = y_{i+1} - y_i. \tag{A.10}$$

Denote by $f^u(I)$, $f^l(I)$ the upper and lower bounds for f in I. To use (A.10), we estimate $f^l(I)$, $f^u(I)$ for each interval $I = [y_i, y_{i+1}]$. For simplicity, we drop the dependence on I.

We simplify $W_{1,2}$ defined in (4.11), (4.13) as W. All the integrands involved in (4.25), (4.21), (4.24) are $W(x)\cos(x\xi)$ for $\xi = ih, i = 0, 1, \ldots, M/h$, and $|W(x)x|, |W'''(x)|$. To obtain the piecewise upper and lower bounds for these integrands, using basic interval arithmetic (see, e.g., [22])

$$
\begin{aligned}
&(fg)^u = \max(f^u g^u, f^l g^u, f^u g^l, f^l g^l),\\
&(fg)^l = \min(f^u g^u, f^l g^u, f^u g^l, f^l g^l),\\
&|f|^u = \max(|f^l|, |f^u|), \quad (f-g)^l = f^l - g^u, \quad (f-g)^u = f^u - g^l,
\end{aligned}
\tag{A.11}
$$

we only need to obtain the bounds for $\cos(x\xi)$, W, $|Wx|$, W'''. Those for x are trivial.

A.6.1. Upper and lower bounds for W, Wx, W'''. We simplify $\tilde{K}_{1,2}$ in (4.11) to $\tilde{K}$. Denote $s = e^x$. Using the chain rule and $W(x) = \tilde{K}(e^x) = \tilde{K}(s)$, we get

$$
\begin{aligned}
\partial_x^3 W(x) = \partial_x^3 \tilde{K}(e^x) &= e^{3x}(\partial^3\tilde{K})(e^x) + 3e^{2x}(\partial^2\tilde{K})(e^x) + e^x(\partial\tilde{K})(e^x)\\
&= s^3\partial^3\tilde{K}(s) + 3s^2\partial^2\tilde{K}(s) + s\partial\tilde{K}(s) =: D^3\tilde{K}(s).
\end{aligned}
$$

Since e^x is increasing, the bounds for W on $[x_l, x_u]$ and those for $\tilde{K}$ on $[e^{x_l}, e^{x_u}]$ enjoy

$$
\begin{aligned}
f^l = g^l(e^{x_l}, e^{x_u}), \quad f^u = g^u(e^{x_l}, e^{x_u}),\\
(f, g) = (W, \tilde{K}), (\partial_x^3 W, D^3\tilde{K}), (W(x)x, \tilde{K}(s)\log s).
\end{aligned}
\tag{A.12}
$$

Thus it suffices to get bounds for $\tilde{K}$, $\tilde{K}\log s$, $D^3\tilde{K}$. Recall $\tilde{K}$ from (4.11) with $\beta = 2$:

$$
\begin{aligned}
\tilde{K}(s) &= (s^{3/2} + s^{-3/2})\log\left|\frac{s+1}{s-1}\right| - \frac{s^{3/2} - s^{-3/2}}{s^2 - 1}2s\\
&= (s^{3/2} + s^{-3/2})\log\left|\frac{s+1}{s-1}\right| - 2s^{-1/2}\frac{s^2+s+1}{s+1}.
\end{aligned}
\tag{A.13}
$$

In the interval $s \in [s_l, s_u]$ with $1 \le s_l < s_u$, using monotonicity, e.g. $s^{3/2} \in [s_l^{3/2}, s_u^{3/2}]$, the fact that $\log|\frac{s+1}{s-1}|$ is decreasing and (A.11), we get the upper and lower bounds for $\tilde{K}$:

$$
\begin{aligned}
\tilde{K}^l(s_l, s_u) &= (s_l^{3/2} + s_u^{-3/2})\log\left|\frac{s_u+1}{s_u-1}\right| - 2s_l^{-1/2}\frac{s_u^2+s_u+1}{s_l+1},\\
\tilde{K}^u(s_l, s_u) &= (s_u^{3/2} + s_l^{-3/2})\log\left|\frac{s_l+1}{s_l-1}\right| - 2s_u^{-1/2}\frac{s_l^2+s_l+1}{s_u+1}.
\end{aligned}
\tag{A.14}
$$

Next, we consider $\tilde{K}\log s$. For $s \in [s_l, s_u]$ with $s_l \ge 1$, since $\log s \ge 0$ we get

$$
\tilde{K}(s)\log s \le \tilde{K}^u \log s \le \max(\tilde{K}^u\log s_l, \tilde{K}^u\log s_u).
$$

Similarly, we obtain the lower bound for $\tilde{K}\log s$. Yet, near $s=1$, the upper bound blows up due to $\log|s_l-1|$ in $\tilde{K}^u$. Note that $\log s \le s-1$. Using (A.8), for $s\ge 1$ we get

$$\partial_s\Big((s-1)\log\Big|\frac{s+1}{s-1}\Big|\Big) = \Big(\frac{1}{s+1}-\frac{1}{s-1}\Big)(s-1)+\log\frac{s+1}{s-1}$$
$$= -\frac{2}{s+1}+\log\frac{s+1}{s-1}\ge 0.$$

Thus, $\log|\frac{s+1}{s-1}|(s-1)$ is increasing on $[s_l,s_u]$ and

$$\log\Big|\frac{s+1}{s-1}\Big|\log s\le\log\Big|\frac{s+1}{s-1}\Big|\cdot(s-1)\le\log\Big|\frac{s_u+1}{s_u-1}\Big|\cdot(s_u-1).$$

We obtain the following improvement for the upper bound of $\tilde{K}(s)\log s$ on $[s_l,s_u]$:

$$\tilde{K}(s)\log(s)\le(s_u^{3/2}+s_l^{-3/2})\log\Big|\frac{s_u+1}{s_u-1}\Big|\cdot(s_u-1)-2s_u^{-1/2}\frac{s_l^2+s_l+1}{s_u+1}\cdot\log s_l. \tag{A.15}$$

For $D^3\tilde{K}(s)$, firstly, using symbolic computation, we get

$$D^3\tilde{K}(s)=\frac{P_{42}(s)-P_{41}(s)+P_5(s)}{P_6(s)},\quad P_{42}(s)=180s^3+180s^7,$$
$$P_{41}(s)=54s+54s^2+266s^4+124s^5+266s^6+54s^8+54s^9,$$
$$P_5(s)=27(s^2-1)^4(1+s+s^2)\log\Big|\frac{s+1}{s-1}\Big|,\quad P_6(s)=8(s-1)^3s^{3/2}(1+s)^4. \tag{A.16}$$

Since $1\le s_l<s_u$ and P_{41},P_{42},P_6 are increasing, we get $P_m^u=P_m(s_u)$, $P_m^l=P_m(s_l)$ for $m=41,42,6$. The bounds for P_5 are also trivial:

$$P_5^l=27(s_l^2-1)^4(1+s_l+s_l^2)\log\Big|\frac{s_u+1}{s_u-1}\Big|,$$
$$P_5^u=27(s_u^2-1)^4(1+s_u+s_u^2)\log\Big|\frac{s_l+1}{s_l-1}\Big|.$$

Using the bounds for P_{41},P_{42},P_5,P_6 and (A.11), we can further derive the bounds for $D^3\tilde{K}$.

A.6.2. Upper and lower bounds for $\cos(x\xi)$. For $f\in C^2([a,b])$ and $x\in[a,b]$, the basic linear interpolation implies $f(x)=\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)+\frac{1}{2}f''(x_1)(x-a)(x-b)$ for some $x_1\in[a,b]$ and

$$\min(f(a),f(b))-\frac{(b-a)^2}{8}\|f''\|_{L^\infty[a,b]}$$
$$\le f(x)\le\max(f(a),f(b))+\frac{(b-a)^2}{8}\|f''\|_{L^\infty[a,b]}.$$

Applying the above estimate to $f(x)=\cos(x\xi)$ and $|f''(x)|\le\xi^2$, we derive the upper and lower bounds for $\cos(x\xi)$ on $[a,b]$.

To verify (4.25), it suffices to get a lower bound for $G(\xi)$ with $\xi = jh$. Applying (A.12), (A.14), the above estimate for $\cos(x\xi)$ and (A.10), we get

$$\int_{y_i}^{y_{i+1}} \cos(x\xi) W(x)\, dx \geq (y_{i+1} - y_i) \cdot I^l, \quad I(x) := \cos(x\xi) W(x).$$

The term I^l can be obtained using (A.11). For y_i close to 0, we should avoid using (A.11) to derive I^l since it involves $W^u(x_l, x_u) = \tilde{K}^u(e^{x_l}, e^{x_u})$ (see (A.14)), which blows up near $x = 0$. For $x\xi \leq \pi/2$, since $\cos(x\xi) \geq 0$, we derive I^l using

$$\cos(x\xi) W(x) \geq \cos(x\xi) W^l \geq \min((\cos(\cdot\xi))^l W^l, (\cos(\cdot\xi))^u W^l).$$

For large ξ, the above estimate is not sharp due to large oscillation in $\cos(x\xi)$. Denote $m = \frac{W_l + W_u}{2}$, $h_0 = b - a$. We consider an improved estimate:

$$\begin{aligned}\int_a^b \cos(x\xi) W(x)\, dx &= \int_a^b \cos(x\xi)(W(x) - m)\, dx + m \int_a^b \cos(x\xi)\, dx \\ &\geq m \frac{\sin(x\xi)}{\xi}\Big|_a^b - h_0 |\cos(x\xi)|^u |W - m|^u \\ &\geq \frac{W_l + W_u}{2} \frac{\sin(b\xi) - \sin(a\xi)}{\xi} - h_0 |\cos(x\xi)|^u \frac{W_u - W_l}{2},\end{aligned}$$

where we have used $W - m \in [W_l - m, W_u - m] = [-\frac{W_u - W_l}{2}, \frac{W_u - W_l}{2}]$.

Using the above estimates, we obtain the lower bound of the integral in $G(\xi)$ of (4.18) in a finite domain. The integrals in (4.24) and (4.21) in a finite domain are estimated similarly.

A.6.3. Decay estimates of $W, \partial_x^3 W$. It remains to estimate the integrals in (4.25), (4.18), (4.24) and (4.21) in the far field. For $s > 1$, using Taylor expansion, we get

$$\begin{aligned}\log\left|\frac{s+1}{s-1}\right| &= \sum_{k \geq 1} \frac{2}{2k-1} s^{-(2k-1)}, \\ \left|\log\left|\frac{s+1}{s-1}\right| - \frac{2}{s}\right| &\leq \frac{2}{3} \sum_{k \geq 2} s^{-(2k-1)} = \frac{2}{3} \frac{s^{-3}}{1 - s^{-2}}.\end{aligned} \tag{A.17}$$

Using the above estimate and (A.13), we obtain

$$\begin{aligned}|\tilde{K}| &\leq \left| s^{3/2} \cdot \frac{2}{s} - 2s^{-1/2} \frac{s^2 + s + 1}{1 + s} \right| + s^{3/2} \cdot \frac{2}{3} \frac{s^{-3}}{1 - s^{-2}} + s^{-3/2} \log\left|\frac{s+1}{s-1}\right| \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Note that $I_1 = \frac{2s^{-1/2}}{s+1} \leq 2s^{-3/2}$. We derive

$$|\tilde{K}| \leq s^{-3/2} \left(2 + \frac{2}{3} \frac{1}{1 - s^{-2}} + \log\left|\frac{s+1}{s-1}\right| \right) =: s^{-3/2} \tilde{K}_{\text{tail}}(s). \tag{A.18}$$

Next, we estimate $D^3 \tilde{K}$ (see (A.16)). Using (A.17), we decompose P_5 in (A.16) as follows:

$$|P_5 - P_{5,M}| \le P_{5,\mathrm{err}}, \quad P_{5,M} = 27(s^2-1)^4(1+s+s^2)\frac{2}{s},$$
$$P_{5,\mathrm{err}} = 27(s^2-1)^4(1+s+s^2)\frac{2}{3}\frac{s^{-3}}{1-s^{-2}}.$$

Recall P_{41}, P_{42}, P_6 from (A.16). Denote $P_7 = P_{42} - P_{41} + P_{5,M}$. We estimate (A.16) as follows:

$$|D^3 \tilde{K}| \le \frac{|P_{42} - P_{41} + P_{5,M}| + P_{5,\mathrm{err}}}{P_6} \le \frac{|P_7|}{P_6} + \frac{P_{5,\mathrm{err}}}{P_6}. \tag{A.19}$$

By definition, P_7 is a sum of a polynomial in s and s^{-1}. Simplifying the expression of P_7 (see details in [7]) and using the triangle inequality, we find that

$$|P_7| \le P_8 = 54 + 54s^{-1} + 216s + 270s^2 + 288s^3 + 58s^4 + 16s^5 + 482s^6 + 18s^7$$
$$=: s^7 P_{8,\mathrm{tail}}(s),$$

where $P_{8,\mathrm{tail}} := P_8(s)s^{-7}$ is decreasing in s. For P_6 (see (A.16)) and the error term $P_{5,\mathrm{err}}$, we have

$$P_6 = 8(-1+s)^3 s^{3/2}(1+s)^4 \ge s^{7+3/2} \cdot 8(1-s^{-1})^3 =: s^{7+3/2} P_{6,\mathrm{tail}}(s),$$
$$\frac{P_{5,\mathrm{err}}}{P_6} = \frac{9(1+s+s^2)}{4s^{5/2}(1+s)} \le s^{-5/2}\frac{9(1+s)}{4} \le s^{-3/2}\frac{9}{4}(1+s^{-1}) =: s^{-3/2} E_{\mathrm{tail}}(s).$$

Plugging the above estimates in (A.16), (A.19), we obtain

$$|D^3 \tilde{K}(s)| \le \frac{|P_7|}{P_6} + \frac{P_{5,\mathrm{err}}}{P_6} \le \frac{P_8}{P_6} + \frac{P_{5,\mathrm{err}}}{P_6}$$
$$\le s^{-3/2}\left(\frac{P_{8,\mathrm{tail}}}{P_{6,\mathrm{tail}}} + E_{\mathrm{tail}}\right) =: s^{-3/2}\tilde{K}_{\mathrm{tail},2}. \tag{A.20}$$

Clearly, $\tilde{K}_{\mathrm{tail}}(s)$ is decreasing. Since $P_{8,\mathrm{tail}}, E_{\mathrm{tail}}$ are decreasing and $P_{6,\mathrm{tail}}$ is increasing, $\tilde{K}_{\mathrm{tail},2}$ is decreasing. Using $W(x) = \tilde{K}(e^x)$, we estimate the integrals in $G(\xi)$ (see (4.18) and (4.24)) in the far field as follows:

$$\left|\int_B^\infty W(x)\cos(x\xi)\,dx\right| \le \tilde{K}_{\mathrm{tail}}(e^B)\int_B^\infty e^{-3x/2}\,dx = \tilde{K}_{\mathrm{tail}}(e^B)\frac{2}{3}e^{-3B/2},$$
$$\int_B^\infty |W(x)x|\,dx \le \tilde{K}_{\mathrm{tail}}(e^B)\int_B^\infty e^{-3x/2}x\,dx \tag{A.21}$$
$$= \tilde{K}_{\mathrm{tail}}(e^B)\left(\frac{2B}{3} + \frac{4}{9}\right)e^{-3B/2},$$

and treat them as errors. Similarly, we estimate the integral in (4.21) in the far field.

This concludes the estimates of all the integrals in (4.25), (4.18), (4.24) and (4.21).

A.6.4. Interval arithmetic. To implement the above estimates and verify (4.25), (4.21) rigorously, we adopted the standard method of interval arithmetic [45, 48]. In particular, we used the MATLAB toolbox INTLAB (version 11 [47]) for the interval computations. Every single real number p involved in the above estimates is represented by an interval $[p_l, p_r]$ that contains p, where $[p_l, p_r]$ are some floating-point numbers. We refer to [9, 10, 22] for related discussion.

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