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Finiteness of reductions of Hecke orbits

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Abstract. We prove two finiteness results for reductions of Hecke orbits of abelian varieties over local fields: one in the case of supersingular reduction and one in the case of reductive monodromy. As an application, we show that only finitely many abelian varieties on a fixed isogeny leaf admit CM lifts, which in particular implies that in each fixed dimension g only finitely many supersingular abelian varieties admit CM lifts. Combining this with the Kuga–Satake construction, we also show that only finitely many supersingular K3 surfaces admit CM lifts. Our tools include p-adic Hodge theory and group-theoretic techniques.

Keywords. Hecke orbits, CM lifts

1. Introduction

CM lifts

Let $\bar{A}$ be an abelian variety over $\overline{\mathbb{F}}_p$. When $\bar{A}$ is *ordinary*, then $\bar{A}$ admits a canonical (CM) lift, and every isogeny from $\bar{A}$ lifts to an isogeny in characteristic zero with source any fixed lift of $\bar{A}$. The aim of this paper is to show that the situation is radically different for *supersingular* abelian varieties. In fact, we prove a general theorem for all Newton strata that interpolates between the ordinary case and the supersingluar case.

The first example of an abelian variety over $\overline{\mathbb{F}}_p$ without a CM lift was given by Oort [16]. Further examples of such abelian varieties, including supersingular abelian varieties, were then constructed by Conrad, Chai and Oort [3]. We prove that such examples are in fact quite abundant.

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Theorem 1.1. *Only finitely many supersingular abelian varieties of a given dimension admit CM lifts.*

A K3 surface over the complex numbers is said to have *complex multiplication* (or to be CM) if its Mumford–Tate group is commutative. We define a K3 surface X over $\overline{\mathbb{F}}_p$ to have a *CM lift* if there exists a finite extension K of $W(\overline{\mathbb{F}}_p)[1/p]$ and a K3 surface $\mathscr{X}/\mathcal{O}_K$, with special fiber isomorphic to X, such that $\mathscr{X}_K \times_K \mathbb{C}$ is a complex CM K3 surface for some complex embedding of K. By a result of Piatetski-Shapiro and Shafarevich [19, Theorem 4] every complex CM K3 surface is defined over a number field. It therefore follows that $\mathscr{X}_K \times_K \mathbb{C}$ is CM for some complex embedding of K if and only if it is CM for every complex embedding of K. Ito–Ito–Koshikawa [7] (unconditionally) and Yang [25] (for primes $p \geq 5$ and with a mild condition on the degree of polarization) have independently proved that every non-supersingular K3 surface over a finite field admits a CM lift. For more about CM K3 surfaces, see [7, Section 9.1].

We combine Theorem 1.1 with a refined analysis of the Kuga–Satake construction to answer a question of Ito–Ito–Koshikawa [7, Remark 1.3] pertaining to CM lifts of K3 surfaces:

Theorem 1.2. *If $p \geq 5$, then only finitely many supersingular K3 surfaces over $\overline{\mathbb{F}}_p$ admit CM lifts.*

We remark that the theorem above makes no mention of polarizations, so does not simply follow from the Kuga–Satake map for integral models of Shimura varieties. It instead requires an analysis of a Kuga–Satake construction at the level of p-divisible groups due to Yang [26].

Our results for general Newton strata use the notion of central leaves introduced by Oort [17, Theorem 5.3]. The central leaf through $\overline{A}$ is a closed subvariety inside its Newton stratum, and essentially consists of all abelian varieties whose p-divisible group is geometrically isomorphic to that of $\overline{A}$. In [18, Section 5], Oort computes the dimensions of central leaves and shows that they are 0-dimensional in the supersingular case, and equal the entire Newton stratum in the ordinary case.

Theorem 1.3. *Let W be a Newton stratum in the moduli space of principally polarized abelian varieties in characteristic p. The set of points of W admitting CM lifts is contained in a finite union of central leaves.*

In [17, Section 4], an *isogeny leaf* through $\overline{A}$ is defined to be a maximal irreducible subvariety of the Newton stratum, which parameterizes abelian varieties which are isogenous to $\overline{A}$ through a local-local isogeny. For example, any irreducible component of the supersingular locus is an isogeny leaf, while the isogeny leaf through an ordinary point is just that point. Oort further proves (in [17, Theorem 5.3]) that central leaves and isogeny leaves are "tranverse" to each other in a suitable sense, and in particular that a central leaf and an isogeny leaf meet only in finitely many points. Consequently, Theorem 1.3 implies that only finitely many abelian varieties in a fixed isogeny leaf admit CM lifts.

Hecke orbits

Let A be an abelian variety over a field K, with algebraic closure $\bar{K}$. By the *Hecke orbit* of an abelian variety A, we mean the set of isomorphism classes of abelian varieties over $\bar{K}$ which are isogenous to $A_{\bar{K}}$. If A is defined over a local field, and has good *reduction* $\bar{A}$, then the image of the Hecke orbit of A in the Hecke orbit of $\bar{A}$ is called the *reduction* of the former. We prove the following result, which is in stark contrast to the ordinary case. This is the key input for proving the CM lifting theorems stated earlier.

Theorem 1.4. *Let A be an abelian variety over a characteristic zero local field, and suppose A has good supersingular reduction. Then the reduction of its Hecke orbit is finite.*

This theorem answers a question posed by Poonen in an unpublished preprint, and also makes progress towards understanding the p-adic distribution of Hecke orbits. The proof of Theorem 1.4 is entirely local, and we prove an analogous theorem in the setting of p-divisible groups first.

Now suppose that K is a characteristic zero local field with ring of integers $\mathcal{O}_K$ and residue field k. For a p-divisible group $\mathscr{G}$ over $\mathcal{O}_K$, we define the Hecke orbit of $\mathscr{G}$ and its reduction in the same way as above. In particular, the reduction is a collection of isomorphism classes of p-divisible groups over an algebraic closure of k. We also establish a finiteness theorem under a semisimplicity hypothesis on the p-adic Galois representation.

Theorem 1.5 (Corollary 2.13). *Let $\mathscr{G}$ denote a p-divisible group over $\mathcal{O}_K$ such that the p-adic Galois representation associated to $\mathscr{G}$ is semisimple. Then the reduction of the Hecke orbit of $\mathscr{G}$ is finite.*

In light of Theorems 1.4 and 1.5, we make the following conjecture.

Conjecture 1.6. For any p-divisible group $\mathscr{G}$ over $\mathcal{O}_K$, the reduction of its Hecke orbit is finite.

By Theorems 1.1 and 1.5, we know that Conjecture 1.6 holds in the case of supersingular reduction (without any semisimplicity conditions on the Galois representation), and in the case that the Galois representation is semisimple (without any condition on the Newton polygon of $\mathscr{G}$). We remark that unlike the situation in characteristic zero, the Hecke orbit of $\bar{A}$ can contain positive-dimensional families.[1] This was first observed by Moret-Bailly [12], who constructed a complete family of supersingular abelian surfaces over $\mathbb{P}^1_{\mathbb{F}_p}$ such that all fibers are p-isogenous.

Outline of the paper. In Section 2, we prove our results on finiteness of reductions of Hecke orbits. This is done using a Galois-theoretic result which relies on work of Sen and

[1]This always happens unless the p-rank of the g-dimensional abelian variety is equal to g or $g-1$.

Serre. We apply this in Section 3 to show the results on CM lifts of p-divisible groups and abelian varieties. Here we make crucial use of Oort's results on central leaves. Finally, in Section 4, we prove the finiteness result for CM lifts of supersingular K3 surfaces, by comparing the deformation theory of K3's with that of GSpin p-divisible groups.

2. Finiteness for reductions of Hecke orbits

2.1. Let K be a field equipped with a rank-1 valuation. Throughout the paper, we will denote by $\mathcal{O}_K$ the ring of integers of K. Fix an algebraic closure $\bar{K}$ of K. We denote by $G_K = \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group of K, and by $I_K \subset G_K$ the inertia subgroup.

In this section we suppose that K is a finite extension of $\mathbb{Q}_p$. We write $\overline{\mathbb{F}}_p$ for the residue field of $\bar{K}$. We denote by $\breve{\mathbb{Q}}_p$ the maximal unramified extension of $\mathbb{Q}_p$, and by $\breve{K}$ the compositum of K and $\breve{\mathbb{Q}}_p$.

2.2. Let $\mathscr{G}/\mathcal{O}_K$ denote a p-divisible group. Denote by $T_p\mathscr{G}$ the p-adic Tate module of $\mathscr{G}$, and let

$$\rho = \rho_{\mathscr{G}} : G_K \to \mathrm{GL}(T_p\mathscr{G})$$

be the Galois representation associated to $\mathscr{G}$. We denote by G (resp. H) the Zariski closure of $\rho(G_K)$ (resp. $\rho(I_K)$) in $\mathrm{GL}(T_p\mathscr{G}[1/p])$. Note that, since $\rho(I_K)$ is normal in $\rho(G_K)$, H is normal in G.

Lemma 2.3. *We have* $[G : H] < \infty$ *if and only if* $[\rho(G_K) : \rho(I_K)] < \infty$.

Proof. If $\rho(G_K) = \bigcup_{g\in S} g\rho(I_K)$ for some finite set S, then $\bigcup_{g\in S} gH$ is a closed set containing $\rho(G_K)$ and hence equals G. This shows that $[G : H]$ is finite if $[\rho(G_K) : \rho(I_K)]$ is.

For the other direction, suppose that $[G : H] < \infty$. By a theorem of Sen [21, Theorem 2] and Serre [22, Theorem 1], $\rho(I_K)$ is open in $H(\mathbb{Q}_p)$, and hence in $G(\mathbb{Q}_p)$, as $H(\mathbb{Q}_p)$ has finite index in $G(\mathbb{Q}_p)$. In particular, this implies that $\rho(I_K)$ is open in $\rho(G_K) \subset G(\mathbb{Q}_p)$. Since $\rho(G_K)$ is compact, this implies $\rho(G_K)/\rho(I_K)$ is discrete and compact, hence finite. ∎

Definition 2.4. We say that two p-divisible groups over a finite field $\mathbb{F}_q$ are *equivalent* if they become isomorphic over $\overline{\mathbb{F}}_p$.

Lemma 2.5. *For any* $h \geq 1$*, the set of equivalence classes of* p*-divisible groups over* $\mathbb{F}_q$ *of height* h *is finite.*

Proof. By a result of Oort [17, Corollary 1.7], there is an integer $n = n(h)$ such that any two p-divisible groups over $\overline{\mathbb{F}}_p$ of height h are isomorphic if and only if their p^n-torsion subgroups are isomorphic. In particular, the equivalence class of a p-divisible group $\mathscr{H}$ over $\mathbb{F}_q$ of height h is determined by its p^n-torsion subgroup $\mathscr{H}[p^n]$. Since $\mathscr{H}[p^n]$ is a finite flat group scheme over $\mathbb{F}_q$ of order p^{nh}, there are only finitely many possibilities for $\mathscr{H}[p^n]$, and the lemma follows. ∎

2.6. Let $J(\mathscr{G})$ denote the set of isomorphism classes of p-divisible groups over $\mathcal{O}_{\bar{K}}$ which are isogenous to $\mathscr{G} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$. We define $J(\mathscr{G}, \overline{\mathbb{F}}_p)$ as

$$J(\mathscr{G}, \overline{\mathbb{F}}_p) = \{\mathscr{G}' \otimes_{\mathcal{O}_{K'}} \overline{\mathbb{F}}_p \mid \mathscr{G}' \in J(\mathscr{G})\},$$

the set of isomorphism classes of reductions of elements of $J(\mathscr{G})$.

Analogously, for an abelian variety $A/\mathcal{O}_K$, we denote by $I(A)$ the Hecke orbit of A, that is, the set of isomorphism classes of abelian varieties over $\mathcal{O}_{\bar{K}}$ which are isogenous to $A \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$. We then define $I(A, \overline{\mathbb{F}}_p)$ as

$$I(A, \overline{\mathbb{F}}_p) = \{A' \otimes_{\mathcal{O}_{K'}} \overline{\mathbb{F}}_p \mid A' \in I(A)\},$$

the reduction of the Hecke orbit of A.

We now give a Galois-theoretic criterion for the finiteness of $J(\mathscr{G}, \overline{\mathbb{F}}_p)$

Proposition 2.7. *If $[G : H] < \infty$, then $J(\mathscr{G}, \overline{\mathbb{F}}_p)$ is a finite set.*

Proof. By Lemma 2.3, our hypothesis implies that $\rho(I_K)$ has finite index in $\rho(G_K)$. Therefore, after replacing K by a finite extension if necessary, we may assume that $\rho(I_K) = \rho(G_K)$. Let K_ρ be the splitting field of ρ, that is, the Galois extension defined by the subgroup $\mathrm{Ker}(\rho)$. Since $\rho(I_K) = \rho(G_K)$, K_ρ/K is a totally ramified extension.

Note that any isogeny of p-divisible groups with source $\mathscr{G}$ can be defined over K_ρ. Hence if $\mathscr{G}'$ is in $J(\mathscr{G})$, then $\mathscr{G}'$ has a representative (again denoted $\mathscr{G}'$) which is defined over K_ρ. Since K_ρ/K is totally ramified, the reduction $\mathscr{G}' \otimes_{\mathcal{O}_{\bar{K}}} \overline{\mathbb{F}}_p$ is defined over the residue field of K, that is, over $\mathbb{F}_q$. The finiteness of $J(\mathscr{G}, \overline{\mathbb{F}}_p)$ now follows from Lemma 2.5. ■

2.8. We will apply Proposition 2.7 in two cases. To explain the first of these, recall that for a p-divisible group $\mathscr{H}$ over $\mathbb{F}_q$, its Dieudonné module $\mathbb{D}(\mathscr{H})$ is a finite free $W(\mathbb{F}_q)$-module equipped with a semilinear Frobenius φ. If $q = p^r$, then φ^r acts linearly on $\mathbb{D}(\mathscr{H})$, and we call this action the *q-Frobenius* Frob_q on $\mathbb{D}(\mathscr{H})$. Following Rapoport–Zink [20], we say that $\mathscr{H}$ is *decent* if the action of Frob_q on $\mathbb{D}(\mathscr{H})$ is semisimple with eigenvalues which are all rational powers of q. The latter condition means that each eigenvalue α satisfies $\alpha^m = q^n$ for some integers m, n.

The semisimplicity condition is always satisfied if $\mathscr{H} = B[p^\infty]$ is the p-divisible group arising from an abelian variety B over $\mathbb{F}_q$. Examples of decent p-divisible groups include those of the form $\mathscr{H} = B[p^\infty]$ with B a supersingular abelian variety over $\mathbb{F}_q$. Indeed, the Weil q-number associated to B is just $q^{1/2}$ times a root of unity.

Proposition 2.9. *If $\mathscr{H} = \mathscr{G} \otimes_{\mathcal{O}_K} \mathbb{F}_q$ is decent then $[G : H]$ is finite.*

Proof. Let Rep_G denote the $\mathbb{Q}_p$-linear Tannakian category of algebraic representations of G, and $\mathrm{Isoc}_{\mathbb{Q}_p}$ the Tannakian category of isocrystals over K_0, the maximal absolutely unramified subfield of K. Recall that $\mathrm{Frob}_q = \varphi^r$ acts linearly on $\mathbb{D}(\mathscr{H})$. Using Fontaine's functor D_{cris}, the representation $\rho : G_K \to G(\mathbb{Q}_p)$ gives rise to a functor

$$D_{\mathrm{cris}} : \mathrm{Rep}_G \to \mathrm{Isoc}_{\mathbb{Q}_p}, \quad w \mapsto D_{\mathrm{cris}}(w \circ \rho).$$

This functor sends $V = (T_p\mathscr{G}[1/p])^*$ to $\mathbb{D}[1/p]$, where $\mathbb{D} = \mathbb{D}(\mathscr{H})$.

Now let W be a faithful representation of G/H, viewed as a representation of G, and write $w : G \to \mathrm{GL}(W)$. Then $\rho \circ w$ is an unramified representation, so that the eigenvalues of Frob_q acting on $D_{\mathrm{cris}}(W)$ are p-adic units – indeed, all the Hodge–Tate weights of any unramified representation are 0, and the Newton–Hodge inequality then implies that the Frobenius eigenvalues must have p-adic valuation 0. Since W is in the Tannakian category generated by V, it follows that $D_{\mathrm{cris}}(W)$ is in the Tannakian category generated by $\mathbb{D}[1/p]$. Hence Frob_q acting on $D_{\mathrm{cris}}(W)$ is semisimple and each of its eigenvalues has the form $\prod_{i=1}^n \alpha_i^{d_i}$, where $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of Frob_q on $D_{\mathrm{cris}}(V)$. In particular, any eigenvalue α of Frob_q acting on $D_{\mathrm{cris}}(W)$ is a rational power of q. Since α is also a p-adic unit, it follows that α is a root of unity.

It follows that Frob_q acting on $D_{\mathrm{cris}}(W)$ has finite order. This implies that for some power q' of q, the representation W viewed as a representation of G_K/I_K can be identified with $(D_{\mathrm{cris}}(W) \otimes W(\mathbb{F}_{q'}))^{\varphi=1}$, with G_K/I_K acting via its action on $W(\mathbb{F}_{q'})$. In particular, we see that G_K/I_K acts on W through a finite quotient. Since $\rho(G_K)$ is dense in G, it is dense in the image of G in $\mathrm{GL}(W)$, and hence this image is finite. As W was a faithful representation of G/H, this proves the proposition. ∎

Corollary 2.10. *If $\mathscr{H} = \mathscr{G} \otimes_{\mathcal{O}_K} \mathbb{F}_q$ is decent, then $J(\mathscr{G}, \overline{\mathbb{F}}_p)$ is a finite set. If moreover $\mathscr{G} = A[p^\infty]$ for an abelian scheme A over $\mathcal{O}_K$ with supersingular reduction, then $I(\mathscr{G}, \overline{\mathbb{F}}_p)$ is a finite set.*

Proof. The first statement follows immediately from Propositions 2.9 and 2.7.

For the second statement, as the special fiber of A is supersingular, we may replace K by a finite extension so that the Galois action on the prime-to-p torsion of A is through scalars (with Frobenius mapping to the scalar $q^{1/2}$ where q is the size of the residue field of K). Further, we observe that $A[p^\infty] \otimes_{\mathcal{O}_K} \mathbb{F}_q$ is decent as A has supersingular reduction, and hence the hypothesis of Proposition 2.7 holds. Let K_ρ be as in the proof of Proposition 2.7. We know that every isogeny from A is defined over K_ρ and hence all A' isogenous to A have reductions defined over $\mathbb{F}_q$, the residue field of K. By Zarkhin's trick [27, Theorem 4.1] there are only finitely many isomorphism classes of abelian varieties over $\mathbb{F}_q$ of any fixed dimension, and hence $I(A, \overline{\mathbb{F}}_p)$ is a finite set. ∎

2.11. Our second application of Proposition 2.7 is more indirect, and proceeds by showing that even if $\mathscr{G}$ does not satisfy the hypothesis of Proposition 2.7 one can sometimes construct an auxiliary p-divisible group which does.

Lemma 2.12. *Suppose that the connected component of the identity in G is reductive. Then after replacing K by a finite extension, there exists a p-divisible group $\mathscr{G}'/\mathcal{O}_K$ such that:*

(1) *$\mathscr{G}$ and $\mathscr{G}'$ are isomorphic over $\mathcal{O}_{\breve{K}}$.*

(2) *If G' (resp. H') denotes the Zariski closure of the image of G_K (resp. I_K) in the group $\mathrm{GL}(T_p\mathscr{G}'[1/p])$, then $G' = H'$.*

Proof. After replacing K by a finite extension, we may assume that G is reductive, and we set $T = G/H$. As T is abelian, it is a torus. Let Z_G denote the center of G, and G^{der} its derived subgroup. The map $Z_G \times G^{\mathrm{der}} \to G$ is surjective with finite kernel, so we obtain a surjective map $Z_G \to T$. This implies that there is a subtorus $T_{\mathrm{sub}} \subset Z_G$ such that the map $T_{\mathrm{sub}} \to T$ is an isogeny.

Let $\chi : G_K \to T(\mathbb{Q}_p)$ be the map induced by ρ, and let $\sigma \in G_K$ be a lift of the q-Frobenius. For some positive integer m, the element $\tau = \chi(\sigma^m)$ lifts to an element of $T_{\mathrm{sub}}(\mathbb{Q}_p)$. Thus after replacing K by a finite extension, we may assume that τ lifts to an element $\tau' \in T_{\mathrm{sub}}(\mathbb{Q}_p)$. Since $T_{\mathrm{sub}} \to T$ is an isogeny, the subgroup generated by τ' is bounded, so there is an unramified Galois character of the form

$$\psi : G_K \to G_K/I_K \to T_{\mathrm{sub}}(\mathbb{Q}_p), \quad \sigma \mapsto \tau'.$$

Now let $C \subset Z_G(\mathbb{Q}_p)$ denote the subgroup preserving $T_p\mathscr{G}$. After replacing K by a finite extension, and so τ' by a power, we may assume that $\tau' \in C$.

The action of C on $T_p\mathscr{G}$ commutes with the action of G_K. Hence by Tate's theorem, it induces a map $C \to \operatorname{Aut}\mathscr{G}$. Since $\mathscr{G}$ is defined over $\mathcal{O}_K$, for any $\gamma \in G_K/I_K$ we have a canonical isomorphism $\gamma^*\mathscr{G} \simeq \mathscr{G}$ over $\mathcal{O}_{\breve{K}}$. Denote by c_γ the composite of this isomorphism and the automorphism $\psi(\gamma)^{-1} \in C$ viewed as an automorphism of $\mathscr{G}$ (here we view ψ as a character on G_K/I_K). Then c_γ defines a descent datum on $\mathscr{G}[p^n]|_{\mathcal{O}_{\breve{K}}}$, for each n. By étale descent, c_γ arises from a unique p-divisible group $\mathscr{G}'$ over $\mathcal{O}_K$.

By construction, the underlying $\mathbb{Z}_p$-modules of $T_p\mathscr{G}$ and $T_p\mathscr{G}'$ are canonically identified, and the action of G_K on $\mathscr{G}'$ is obtained by multiplying its action on $T_p\mathscr{G}$ by $\psi(\sigma)^{-1}$. Hence we have $H = H' \subset G' \subset G$, and the map $G' \to T$ is trivial. It follows that $G' = H'$. ∎

Corollary 2.13. *If the p-adic Galois representation associated to $\mathscr{G}$ is semisimple, then the set $J(\mathscr{G}, \overline{\mathbb{F}}_p)$ is finite.*

Proof. By Lemma 2.12 we can find $\mathscr{G}'$ such that $H' = G'$. By Proposition 2.7, $J(\mathscr{G}', \overline{\mathbb{F}}_p)$ is a finite set. On the other hand, the two p-divisible groups $\mathscr{G}$ and $\mathscr{G}'$ are isomorphic over $\breve{K}$ and therefore $J(\mathscr{G}, \overline{\mathbb{F}}_p) = J(\mathscr{G}', \overline{\mathbb{F}}_p)$. ∎

3. Finiteness of p-divisible groups admitting a CM lift

3.1. In this section we assume that K is a finite extension of $K_0 = W(\overline{\mathbb{F}}_p)[1/p]$.

A p-divisible group $\mathscr{G}$ of (constant) height h, over any base, is said to *have CM* by a commutative semisimple $\mathbb{Q}_p$-algebra F if there is an injective homomorphism

$$F \hookrightarrow \operatorname{End}(\mathscr{G}) \otimes \mathbb{Q}$$

such that $\dim_{\mathbb{Q}_p} F = \operatorname{height}(\mathscr{G})$. We say that $\mathscr{G}$ is CM, or has CM, if $\mathscr{G}$ has CM by some F as above.

If $\mathscr{G}$ is a p-divisible group over $\mathcal{O}_K$, we can form its formal group $\widehat{\mathscr{G}}$. If $\mathscr{G}$ has CM by F, then $\operatorname{Lie}\widehat{\mathscr{G}} \otimes_{\mathcal{O}_K} \bar{K} \simeq \bigoplus_\sigma V_\sigma$ where σ runs over $\mathbb{Q}_p$-algebra maps $F \to \bar{K}$, and for $a \in F$, we have $aV_\sigma = \sigma(a)V_\sigma$. For each σ, the summand V_σ is either trivial, or one-dimensional over $\bar{K}$ [3, Lemma 3.7.1.3]. We denote by Φ the set of σ for which V_σ is one-dimensional, and we call Φ the *CM type* of $\mathscr{G}$.

Lemma 3.2. *Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$ with CM by F. Then there exists a finite extension $K'/\mathbb{Q}_p$ contained in K, and a p-divisible group $\mathscr{G}'$ over $\mathcal{O}_{K'}$ with CM by F, such that $\mathscr{G}' \otimes_{\mathcal{O}_{K'}} \mathcal{O}_K$ is F-linearly isomorphic to $\mathscr{G}$.*

Proof. This is well known. Let D be the weakly admissible module $D_{\mathrm{cris}}(T_p\mathscr{G}[1/p])$, so that D is an $F \otimes_{\mathbb{Q}_p} K_0$-module equipped with an injective, semilinear Frobenius and a one-step filtration $\mathrm{Fil}^1 D_K \subset D_K = D \otimes_{K_0} K$ by an $F \otimes_{\mathbb{Q}_p} K$-submodule.

The filtration on D_K is induced by a cocharacter $\mu \in X_*(\mathrm{Res}_{F/\mathbb{Q}_p} \mathbb{G}_{\mathrm{m}})$. Choose $K'/\mathbb{Q}_p$ finite and contained in K such that μ is defined over K', and let K'_0 denote the maximal unramified subfield of K'. Then there exists a free $F \otimes_{\mathbb{Q}_p} K'_0$-module D', an $F \otimes_{\mathbb{Q}_p} K'$-submodule $\mathrm{Fil}^1 D'_{K'} \subset D'_{K'} = D' \otimes K'$, and an F-linear isomorphism $\iota : D' \otimes_{K'_0} K_0 \simeq D$ respecting filtrations.

Now identify D' with $F \otimes_{\mathbb{Q}_p} K'_0$. Then we can identify D with $F \otimes_{\mathbb{Q}_p} K_0$ via ι, and the Frobenius on D is given by $\delta\sigma$, where $\delta \in (F \otimes_{\mathbb{Q}_p} K_0)^\times$ and σ denotes the Frobenius on K_0. After possibly replacing K' by a larger field, there exists $\delta' \in (F \otimes_{\mathbb{Q}_p} K'_0)^\times$ such that $\delta'\delta^{-1} \in (\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{K'_0})^\times$. We equip D' with the Frobenius $\delta'\sigma$.

As $\delta'\delta^{-1} \in (\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_{K'_0})^\times$ there exists $c \in (F \otimes_{\mathbb{Q}_p} K_0)^\times$ such that $\delta' = c^{-1}\delta\sigma(c)$. Then $c \cdot \iota$ respects Frobenius, and also respects filtrations as ι does. D' along with the Frobenius $\delta'\sigma$ and filtration $\mathrm{Fil}^1 D'_{K'} \subset D'_{K'}$ is a weakly admissible module as D is. This weakly admissible module equals $D_{\mathrm{cris}}(T_p\mathscr{G}'[1/p])$ where $\mathscr{G}'/\mathcal{O}_{K'}$ is a p-divisible group, for example by [8]. The isomorphism $c \cdot \iota : D'_{K_0} \simeq D$ induces a quasi-isogeny between the Tate modules of $\mathscr{G}'$ and $\mathscr{G}$ as $\mathrm{Gal}(\bar{K}/K)$-representations, and hence (after multiplication by a power of p) an isogeny $\mathscr{G}'_{\mathcal{O}_K} \to \mathscr{G}$. After replacing K' by a finite extension, we may assume that the kernel of this isogeny is defined over $\mathcal{O}_{K'}$ and the theorem follows. ∎

3.3. Let $\mathscr{H}$ be a p-divisible group over $\overline{\mathbb{F}}_p$. We say that $\mathscr{H}$ *admits a CM lift* if there exists a finite extension $K/W(\overline{\mathbb{F}}_p)[1/p]$, and a CM p-divisible group $\mathscr{G}$ over $\mathcal{O}_K$, such that $\mathscr{G} \otimes_{\mathcal{O}_K} \overline{\mathbb{F}}_p$ is isomorphic to $\mathscr{H}$.

We remark that there is essentially no extra generality gained by considering CM lifts to more general base rings. More precisely, if R is an integral, normal, flat $W(\overline{\mathbb{F}}_p)$-algebra, and $\mathscr{G}$ is a CM deformation of $\mathscr{H}$ to R, then there is a finite extension $K/W(\overline{\mathbb{F}}_p)[1/p]$, and an inclusion $\mathcal{O}_K \to R$, such that $\mathscr{G}$ arises from a CM deformation of $\mathscr{H}$ over $\mathcal{O}_K$. This can be deduced from the fact that the rigid analytic period morphism in [20, Section 5] is étale, together with the fact that any $F \otimes_{W(\overline{\mathbb{F}}_p)} R$-direct summand of the free, rank-1 $F \otimes_{W(\overline{\mathbb{F}}_p)} R$-module $\mathbb{D}(\mathscr{G})(R)$ is defined over $F \otimes_{W(\overline{\mathbb{F}}_p)} \mathcal{O}_K$, for some $\mathcal{O}_K \subset R$, as above.

Theorem 3.4. *Let $\mathscr{H}/\overline{\mathbb{F}}_p$ be a p-divisible group. Then the isogeny class of $\mathscr{H}$ contains only finitely many isomorphism classes of p-divisible groups which admit a CM lift.*

Proof. Since the algebra $\mathrm{End}(\mathscr{H}) \otimes \mathbb{Q}$ has finite dimension over $\mathbb{Q}_p$, there are only finitely many choices for the CM algebra F. Given F, there are only finitely many choices for the CM type Φ. Thus, we may fix F and Φ, and consider only those p-divisible groups in the isogeny class of $\mathscr{H}$ which admit a CM lift having CM by F and CM type Φ.

Let $\mathscr{G}, \mathscr{G}_1$ be such lifts, defined over some finite extension $K/W(\overline{\mathbb{F}}_p)[1/p]$. By Lemma 3.2, there exists a p-divisible group $\mathscr{G}'$ with CM by F, defined over a finite extension $K'/\mathbb{Q}_p$, such that $\mathscr{G}' \otimes_{\mathcal{O}_{K'}} \mathcal{O}_K \simeq \mathscr{G}$. Since $\mathscr{G}_1$ and $\mathscr{G}$ have the same CM type, there is an F-linear isogeny $\mathscr{G} \to \mathscr{G}_1$, by [3, Proposition 3.7.4]. Thus $\mathscr{G}_1 \in J(\mathscr{G}')$.

Now the Zariski closure of the image of $G_{K'}$ acting on $T_p\mathscr{G}'$ is a closed subgroup of the torus $\mathrm{Res}_{F/\mathbb{Q}_p} \mathbb{G}_{\mathrm{m}}$, hence is reductive. Hence $J(\mathscr{G}', \overline{\mathbb{F}}_p)$ is finite by Corollary 2.13, and the theorem follows. ∎

3.5. Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g. For a given Newton polygon ν we denote by $W_\nu \subset \mathcal{A}_{g,\overline{\mathbb{F}}_p}$ the corresponding Newton stratum. It is a locally closed subscheme.

For any $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$, the associated p-divisible group $\mathscr{G}_x$ carries a principal polarization, an isomorphism ψ_x from $\mathscr{G}_x$ to its Cartier dual. The *polarized central leaf* through a point $x \in \mathcal{A}_g$ is the locus of points where the associated polarized p-divisible group is geometrically *isomorphic* to the polarized p-divisible group parameterized by x. Oort [17, Theorem 5.3] has shown that this locus is a closed subvariety of the Newton stratum through x.

Theorem 3.6. *Let S be the set of points in $\mathcal{W}_\nu$ which admit CM lifts. Then S is contained in a finite number of central leaves.*

Proof. Let n denote some large enough integer such that $\mathcal{A}_g[n]$, the moduli space of principally polarized g-dimensional abelian varieties with full symplectic level n structure, is a fine moduli space. The notion of a polarized central leaf in $\mathcal{A}_g[n]$ through any point is defined exactly as in the case of $\mathcal{A}_g$, and without any reference to level structure. Then the result for $\mathcal{A}_g$ follows directly from the result for $\mathcal{A}_g[n]$.

Let $x \in \mathcal{W}_\nu(\overline{\mathbb{F}}_p) \subset \mathcal{A}_g[n](\overline{\mathbb{F}}_p)$, and let $\mathscr{H}_x$ and $\mathcal{A}_x$ be the associated principally polarized p-divisible group and abelian scheme, respectively. By [17, Theorem 2.2], the set of points $y \in \mathcal{A}_g[n](\overline{\mathbb{F}}_p)$ with $\mathscr{H}_y$ isomorphic to $\mathscr{H}_x$ (as unpolarized p-divisible groups over $\overline{\mathbb{F}}_p$) is a closed subvariety of $\mathcal{W}_\nu$. By [17, Theorem 3.3], this closed subvariety is a union of finitely many central leaves[2] in $\mathcal{W}_\nu$. The result now follows immediately from Theorem 3.4. ∎

Corollary 3.7. *There are only finitely many supersingular principally polarized abelian varieties of dimension g which admit a CM lift.*

Proof. This follows from Theorem 3.6, and the fact that central leaves in the supersingular stratum are zero-dimensional [18, Section 5]. ∎

[2]Both of Oort's results are stated in the setting of families of p-divisible groups, which is the only reason we need to work with $\mathcal{A}_g[n]$ instead of $\mathcal{A}_g$.

4. CM lifts of supersingular K3 surfaces

In this section we deduce Theorem 1.2 for CM lifts of K3 surfaces when $p \geq 5$. The weaker statement for polarized K3 surfaces is an immediate consequence of Theorem 3.6 and the theory of integral models of Shimura varieties [11]. Here we prove the stronger result for K3 surfaces with no reference to polarizations. The main inputs will be the Kuga–Satake construction for K3 crystals (without any reference to polarizations) due to Yang, and the crystalline Torelli theorem, due to Ogus.

4.1. Let R be a p-adically complete and separated, p-torsion free ring. Suppose that R is equipped with a lift σ of Frobenius. For $i \geq 0$, we denote by $R(i)$ the R-module R equipped with a Frobenius $\varphi = p^i \cdot \sigma$ and a filtration given by $\mathrm{Fil}^i R(i) = R(i)$ and $\mathrm{Fil}^{i+1} R(i) = 0$.

A *K*3 *crystal* over R (cf. [14, Definition 3.1]) is a free R-module $\mathbb{L}$ of rank 22, endowed with a Frobenius-linear endomorphism $\varphi : \mathbb{L} \to \mathbb{L}$ and a φ-compatible, perfect, symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathbb{L} \times \mathbb{L} \to R(2)$ satisfying:

(a) $p^2\mathbb{L} \subset \varphi(\mathbb{L})$.

(b) The image of $\varphi \otimes_R R/pR$ is projective of rank 1.

When $R = W(k)$ for a perfect field k, we also call this a *K*3 *crystal over* k.

A *filtered* K3 crystal over R is a K3 crystal $\mathbb{L}$ over R equipped with a filtration $0 = \mathrm{Fil}^3 \mathbb{L} \subset \mathrm{Fil}^2 \mathbb{L} \subset \mathrm{Fil}^1 \mathbb{L} \subset \mathrm{Fil}^0 \mathbb{L} = \mathbb{L}$ such that $\mathrm{gr}^\bullet \mathbb{L}$ is a projective R-module and the following conditions hold:

(c) $\langle\cdot,\cdot\rangle : \mathbb{L} \otimes \mathbb{L} \to R(2)$ is strictly compatible with filtrations.

(d) $\mathrm{Fil}^1 \mathbb{L} \otimes_R R/pR$ is the kernel of φ on $\mathbb{L} \otimes_R R/pR$.

Note that these conditions imply that $\mathrm{Fil}^1 \mathbb{L} = (\mathrm{Fil}^2 \mathbb{L})^\perp$, that $\varphi(\mathrm{Fil}^2 \mathbb{L}) \subset p^2\mathbb{L}$, and that for $i = 0, 1, 2$, $\mathrm{gr}^i \mathbb{L}$ has rank $1, 20, 1$ respectively.

4.2. Now suppose that $R = W(k)$ with k a perfect field, which will be either $\overline{\mathbb{F}}_p$ or a finite field in applications.

A K3 crystal $\mathbb{L}$ over k is said to be *supersingular* if the slopes of φ are all 1. If $\mathbb{L}$ is supersingular and k is algebraically closed, then $\mathbb{L}^{\varphi=p}$ is a free $\mathbb{Z}_p$-module of rank 22 [14, Theorem 3.3], which also admits a bilinear form (which will no longer be perfect).

If $k = \mathbb{F}_q$ is a finite field, then we say a K3 crystal $\mathbb{L}$ over k is *decent* if the q-Frobenius on $\mathbb{L}$ has eigenvalues which are rational powers of q. We note that this implies the $\mathbb{Z}_p$-module $\mathbb{L}^{\varphi=p} \subset \mathbb{L}$ has rank 22. Note that every K3 crystal over $\overline{\mathbb{F}}_p$ admits a decent model over $\mathbb{F}_q$ for some q [10, Section 4.3]

Lemma 4.3. *Let* $\mathbb{L}$ *be a filtered K*3 *crystal over* R*, as above. Then the filtration on* $\mathbb{L}$ *is induced by a cocharacter* $\mu : \mathbb{G}_\mathrm{m} \to \mathrm{GO}(\mathbb{L}, \langle\cdot,\cdot\rangle)$*. In particular, the subgroup* $P \subset \mathrm{GO}(\mathbb{L}, \langle\cdot,\cdot\rangle)$ *preserving the filtration is parabolic.*

Proof. Let $\mathbb{L}^2 = \mathrm{Fil}^2 \mathbb{L}$, and choose a submodule $\mathbb{L}^1 \subset \mathrm{Fil}^1 \mathbb{L}$ so that $\mathrm{Fil}^1 \mathbb{L} = \mathbb{L}^2 \oplus \mathbb{L}^1$. Then $(\mathbb{L}^1)^\perp$ is free of rank 2 and surjects on $\mathrm{gr}^0 \mathbb{L}$, as $\langle\cdot,\cdot\rangle$ is perfect and strict for filtrations. Thus, we can choose a rank-1 direct summand $\mathbb{L}^0 \subset (\mathbb{L}^1)^\perp$ which maps isomorphically to $\mathrm{gr}^0\,\mathbb{L}$. Then $\langle\cdot,\cdot\rangle$ induces a perfect pairing between the rank-1 subspaces $\mathbb{L}^0, \mathbb{L}^2 \subset (\mathbb{L}^1)^\perp$. Thus, modifying $\mathbb{L}^0 \subset (\mathbb{L}^1)^\perp$ we may assume that $\mathbb{L}^0$ is isotropic for $\langle\cdot,\cdot\rangle$.

Now $\mathbb{L} = \mathbb{L}^2 \oplus \mathbb{L}^1 \oplus \mathbb{L}^0$, and we define μ by requiring that $\mu(z)$ by z^i on $\mathbb{L}^i$. Since $\mathbb{L}^i$ and $\mathbb{L}^j$ are orthogonal for $i + j \neq 2$, we see that $\mu(z)$ acts on $\langle\cdot,\cdot\rangle$ by z^2. In particular, μ factors through $\mathrm{GO}(\mathbb{L}, \langle\cdot,\cdot\rangle)$. ∎

4.4. Now let k be a perfect field of characteristic p which is either algebraically closed or an algebraic extension of $\mathbb{F}_p$. Let X/k denote a K3 surface. Then $H^2_{\mathrm{cris}}(X)$ is a K3 crystal over $W(k)$. It is well known that the deformation functor of X is smooth and pro-representable and formally smooth of dimension 20 over $W = W(k)$. Let $\mathrm{Spf}\,\widehat{R}_X$ denote the universal deformation space of X, and let $\mathbb{X}^{\mathrm{u}} \xrightarrow{\pi} \mathrm{Spf}\,\widehat{R}_X$ denote the universal deformation of X. Choose a set $\{p, x_1, \ldots, x_{20}\}$ of elements that generate the maximal ideal of $\widehat{R}_X$; define σ to be the lift of the Frobenius endomorphism of $\widehat{R}_X$ mod p such that σ is the usual Frobenius on $W(\mathbb{F}_q)$, and $\sigma(x_i) = (x_i)^p$.

Then $\mathbb{L}_{\mathrm{u}} := H^2_{\mathrm{dR}}(\mathbb{X}^{\mathrm{u}}/\widehat{R}_X)$ is equipped with the structure of a filtered K3 crystal over $\widehat{R}_X$. To give a more explicit description of this filtered K3 crystal, we need the following.

Lemma 4.5. *Let $\mathbb{L}$ be the K3 crystal attached to X. There exists a free $\mathbb{Z}_p$-module with quadratic form $(T, \langle\cdot,\cdot\rangle')$ and an isomorphism $\iota : (T, \langle\cdot,\cdot\rangle') \otimes_{\mathbb{Z}_p} W \to (\mathbb{L}, \langle\cdot,\cdot\rangle)$.*

Proof. The bilinear form on $\mathbb{L}$ is self-dual. It follows from the theory of non-degenerate bilinear forms over finite fields that after replacing k by at most a quadratic extension, $(\mathbb{L}, \langle\cdot,\cdot\rangle)$ has determinant a square and also admits a rank-11 isotropic subspace. Indeed, there is a unique quadratic form on $\mathbb{L}$ (up to isomorphism) that satisfies these conditions, whence it follows that $(T, \langle\cdot,\cdot\rangle')_{\mathbb{Z}_p} \otimes W$ is isomorphic to $(\mathbb{L}, \langle\cdot,\cdot\rangle)$ where T is a rank-22 free $\mathbb{Z}_p$-module, and $\langle\cdot,\cdot\rangle'$ is the unique self-dual quadratic form on T that has square determinant and admits a rank-11 isotropic subspace. This weaker statement (namely, after replacing k with a quadratic extension) will actually be enough for our applications, so we only sketch the proof of the full statement of the lemma.

Choose a lift $\tilde{X}$ of X to a smooth formal scheme over W. We will apply the theory of prismatic cohomology [2] to $\tilde{X}$, using the map $\mathfrak{S} = W[\![z]\!] \xrightarrow{z \mapsto 0} W$, where $\mathfrak{S}$ is equipped with the Frobenius which sends z to z^p; this is an example of a prism. We obtain a finite free $\mathfrak{S}$-module $\mathfrak{M}$, equipped with a semilinear Frobenius and a bilinear pairing $\langle\cdot,\cdot\rangle_{\mathfrak{M}}$ such that:

(1) $(\mathfrak{M}, \langle\cdot,\cdot\rangle_{\mathfrak{M}}) \otimes_{\mathfrak{S}} W \simeq (\mathbb{L}, \langle\cdot,\cdot\rangle)$.

(2) There is a faithfully flat $\mathfrak{S}$-algebra A and an isomorphism

$$(\mathfrak{M}, \langle\cdot,\cdot\rangle_{\mathfrak{M}}) \otimes_{\mathfrak{S}} A \simeq (H^2(\tilde{X}_{\bar{K}}, \mathbb{Z}_p), \langle\cdot,\cdot\rangle) \otimes_{\mathbb{Z}_p} A$$

where $\bar{K}$ is an algebraic closure of $W[1/p]$, and $H^2(\tilde{X}_{\bar{K}}, \mathbb{Z}_p)$ is equipped with a perfect symmetric bilinear form, using Poincaré duality.

Then by the Key Lemma of [9, Proposition 1.3.4], one finds that there exists an isomorphism

$$(\mathbb{L}, \langle \cdot, \cdot \rangle) \simeq (H^2(\tilde{X}_{\bar{K}}, \mathbb{Z}_p), \langle \cdot, \cdot \rangle) \otimes W. \qquad \blacksquare$$

4.6. Let $\mathfrak{n} \in \operatorname{Spec} \widehat{R}_X$ be the kernel of $\widehat{R}_X \to W$ sending the x_i to 0, and let $\mathbb{L}_0 = \mathbb{L}_{\mathrm{u}} \otimes_{\widehat{R}_X} W$ be the corresponding filtered K3 crystal over W. The underlying K3 crystal of $\mathbb{L}_0$ is canonically identified with $\mathbb{L}$.

Fix an isomorphism as in Lemma 4.5. This allows us to regard $\mathrm{GO} = \mathrm{GO}(\mathbb{L}, \langle \cdot, \cdot \rangle)$, as a group over $\mathbb{Z}_p$. Choose an isomorphism $j : (\mathbb{L}_{\mathrm{u}}, \langle \cdot, \cdot \rangle) \simeq (\mathbb{L}, \langle \cdot, \cdot \rangle) \otimes \widehat{R}_X$ inducing the identity mod $\mathfrak{n}$. This gives rise to an isomorphism $\mathrm{GO}(\mathbb{L}_{\mathrm{u}}, \langle \cdot, \cdot \rangle) \simeq \mathrm{GO}(\mathbb{L}, \langle \cdot, \cdot \rangle) \otimes \widehat{R}_X$, and the Frobenius maps on $\mathbb{L}_{\mathrm{u}}$ and $\mathbb{L}_0$ are then given by $b_u\sigma$ and $b\sigma$ for elements $b_u \in \mathrm{GO}(\widehat{R}_X[1/p])$ and $b \in \mathrm{GO}(W[1/p])$ such that b_u specializes to b.

Proposition 4.7. *The isomorphism* $(\mathbb{L}_{\mathrm{u}}, \langle \cdot, \cdot \rangle) \simeq (\mathbb{L}, \langle \cdot, \cdot \rangle) \otimes \widehat{R}_X$ *can be chosen so that:*

(1) j *respects filtrations.*[3]

(2) $b_u = u \cdot b$ *where* $u \in U^{\mathrm{opp}}(\widehat{R}_X)$*, and* U^{opp} *is the opposite unipotent of the parabolic* $P \subset \mathrm{GO}(\mathbb{L}_{\mathrm{u}}, \langle \cdot, \cdot \rangle)$ *corresponding to the filtration on* $\mathbb{L}_{\mathrm{u}}$.

Moreover, if these conditions are satisfied, the tautological map $\operatorname{Spf} \widehat{R}_X \xrightarrow{u} \widehat{U}^{\mathrm{opp}}$ *is an isomorphism.*

Proof. To show (1), we have to show that j can be chosen so that the parabolics $P \subset \mathrm{GO}(\mathbb{L}_{\mathrm{u}}, \langle \cdot, \cdot \rangle)$ and $P_0 \subset \mathrm{GO}(\mathbb{L}_0, \langle \cdot, \cdot \rangle)$ corresponding to the filtrations on $\mathbb{L}_{\mathrm{u}}$ and $\mathbb{L}_0$ are identified. This follows from the fact that any deformation of P_0 to a parabolic in $\mathrm{GO}(\mathbb{L}_{\mathrm{u}}, \langle \cdot, \cdot \rangle)$ is conjugate to the constant deformation [5]. The choice of such a j is unique up to conjugation by elements of $P(\widehat{R}_X)$.

Next let $\mu : \mathbb{G}_{\mathrm{m}} \to \mathrm{GO}$ be a cocharacter corresponding to P. We claim that

$$b_u \in \mathrm{SO}(\widehat{R}_X)\sigma(\mu)(p).$$

Let

$$\mathbb{L}'_{\mathrm{u}} = (b_u\sigma)\mu(p^{-1})\mathbb{L}_{\mathrm{u}} = b_u\sigma(\mu)(p^{-1})\mathbb{L}_{\mathrm{u}}.$$

The conditions on the filtration in a filtered K3 crystal imply that $\mathbb{L}'_{\mathrm{u}} \subset \mathbb{L}_{\mathrm{u}}$, and that $\mu(p^{-1})$ acts by p^{-2} on $\langle \cdot, \cdot \rangle$. Thus for $x, y \in \mathbb{L}_{\mathrm{u}}$ we have

$$\langle (b_u\sigma)\mu(p^{-1})x, (b_u\sigma)\mu(p^{-1})y \rangle = p^2\sigma(\langle \mu(p^{-1})x, \mu(p^{-1})y \rangle) = \sigma(\langle x, y \rangle).$$

It follows that $\langle \cdot, \cdot \rangle$ is perfect on $\mathbb{L}'_{\mathrm{u}}$, and hence $\mathbb{L}'_{\mathrm{u}} = \mathbb{L}_{\mathrm{u}}$. This proves the claim.

Consequently, $w = b_u b^{-1} \in \mathrm{SO}(\widehat{R}_X)$. Since the map $U^{\mathrm{opp}} \to P_0 \backslash \mathrm{GO}(\mathbb{L}_0, \langle \cdot, \cdot \rangle)$ is an open immersion, and w is the identity mod $\mathfrak{n}$, we can write $w = \lambda \cdot u$ with $u \in U^{\mathrm{opp}}(\widehat{R}_X)$ and $\lambda \in P(\widehat{R}_X)$ both reducing to 1 mod $\mathfrak{n}$. Conjugating j by an element of $P(\widehat{R}_X)$ has

[3] See [9, Sections 1.4–1.5] for analogous descriptions of deformation spaces of p-divisible groups. We also note that the filtration is not parallel for the Gauss–Manin connection.

the effect of replacing $b_u = \lambda ub$ by its σ-conjugate by the same element. Now let $m \geq 1$, and suppose that j can be chosen so that $b_u = b_u(m) \equiv ub \bmod \mathfrak{m}^m$, so that $\lambda = \lambda(m) \equiv 1 \bmod \mathfrak{m}^m$. Then

$$\lambda^{-1} b_u \sigma(\lambda) = ub\sigma(\lambda) = u(b\sigma(\lambda)b^{-1})b.$$

Since $\lambda \equiv 1 \bmod \mathfrak{m}^n$, we have $\sigma(\lambda) \equiv 1 \bmod \mathfrak{m}^{pm}$, and hence $b_u(m+1) = \lambda^{-1} b_u(m)\sigma(\lambda) \equiv ub \bmod \mathfrak{m}^{pm}$. This shows that ub is the σ-conjugate of $b_u(1)$ by the convergent product $\ldots\lambda(2)\lambda(1)$, so j can be chosen with $b_u = ub$.

It remains to show that the map $u : \operatorname{Spf} \widehat{R}_X \to \widehat{U}^{\mathrm{opp}}$ is an isomorphism. Given that these are both smooth 20-dimensional formal schemes over W, it suffices to prove this modulo the ideal $(p, \mathfrak{m}^2)$. Now let $S = \widehat{R}_X/(p, \mathfrak{m}^2)$, and let $\mathbb{L}_S$ denote the filtered K3 crystal over S given by $\mathbb{L}_{\mathrm{u}}|_S$. After modifying our chosen isomorphism j by u, $\mathbb{L}_S$ may be identified with

$$(\mathbb{L} \otimes S, u^{-1}ub\sigma(u), u \cdot (\mathrm{Fil}^\bullet \mathbb{L})_S, \langle\cdot,\cdot\rangle) = (\mathbb{L} \otimes S, b, u \cdot (\mathrm{Fil}^\bullet \mathbb{L})_S, \langle\cdot,\cdot\rangle)$$

where we have used $\sigma(u) = 1$ in S.

Work of Nygaard–Ogus [13, Theorems 5.2, 5.3] implies that deformations of X to S correspond bijectively to *isotropic lifts* of $\mathrm{Fil}^2 \mathbb{L} \bmod \mathfrak{m}$. These lifts are in bijection with points of $U^{\mathrm{opp}}(S)$ which are 1 mod $\mathfrak{m}$, and hence with $\widehat{U}^{\mathrm{opp}}(S)$. This implies that u induces an isomorphism on S points, and hence is an isomorphism. ■

4.8. Let H denote the Clifford module associated to $(T, \langle\cdot,\cdot\rangle')$, and let $\{s_{\alpha,p}\}_\alpha \subset H^\otimes$ denote tensors whose pointwise stabilizer is the group GSpin (such tensors exist by [9, Proposition 1.3.2]) of spinor similitudes associated to $(T, \langle\cdot,\cdot\rangle')$.

Definition 4.9. A *GSpin-structure* on a p-divisible group $\mathscr{H}/k$ is the data of an isomorphism $\iota : H \otimes W(k) \to \mathbb{D}(\mathscr{H})$ such that $\iota(s_{\alpha,p}) \in \mathbb{D}(\mathscr{H})^\otimes$ are Frobenius-invariant. We say that ι_1 and ι_2 are *isomorphic GSpin-structures* if $\iota_1(s_{\alpha,p}) = \iota_2(s_{\alpha,p})$.

4.10. We now recall a Kuga–Satake construction for K3 crystals due to Yang [26, Appendix A], which associates to a K3 crystal a p-divisible group with GSpin-structure.

Denote by $\mathbb{H}$ the Clifford module associated to $(\mathbb{L}, \langle\cdot,\cdot\rangle)$. The isomorphism ι from Lemma 4.5 induces a canonical isomorphism $H \otimes W(k) \to \mathbb{H}$ of Clifford modules (where H is as in Section 4.8), which we shall also denote by ι.

Let μ be as in the proof of Proposition 4.7: a $\mathrm{GO}(\mathbb{L})$-valued cocharacter, defined over $W(k)$, that induces the mod p filtration on $\mathbb{L}$. Let μ_{SO} denote the cocharacter of SO given by $\mu_{\mathrm{SO}}(z) = z^{-1}\mu(z)$. The natural map

$$\mathrm{GSpin}(\mathbb{H}) \to \mathrm{SO}(\mathbb{L})$$

has kernel $\mathbb{G}_{\mathrm{m}}$, and there is a unique lift of μ_{SO} to a cocharacter $\tilde{\mu}$ of $\mathrm{GSpin}(\mathbb{H})$ such that $\tilde{\mu}$ acts with weights 0 and 1 on $\mathbb{H}$. Both weights then occur with equal multiplicity. We saw in the proof of Proposition 4.7 that $b \in \mathrm{GO}(W(k))\sigma(\mu)(p)$. Hence $p^{-1}b \in \mathrm{SO}(W(k))\sigma(\mu_{\mathrm{SO}})(p)$, and we may lift $p^{-1}b$ to an element $\tilde{b} \in \mathrm{GSpin}(\mathbb{H})(W(k))\sigma(\tilde{\mu})(p)$.

Proposition 4.11. *There exists a p-divisible group $\mathscr{H}$ over k whose Dieudonné module $\mathbb{D}(\mathscr{H})$ is given by $\mathbb{H}$, with Frobenius acting as $\tilde{b}\sigma$.*

Proof. As $\tilde{\mu}$, and therefore $\sigma(\tilde{\mu})$, acts on $\mathbb{H}$ with weights 0 and 1, it follows that $p\mathbb{H} \subset \sigma(\tilde{\mu})(p)\mathbb{H} \subset \mathbb{H}$. Therefore, we also have $p\mathbb{H} \subset \tilde{b}\mathbb{H} \subset \mathbb{H}$. It then follows by Dieudonné theory that there exists a p-divisible group over k whose Dieudonné module is given by $\mathbb{H}$ with Frobenius acting by $\tilde{b}\sigma$. ∎

4.12. By [6, Section 4.2.1], if X is supersingular, then $\mathscr{H}$ is a supersingular p-divisible group. Since $\tilde{b} \in \mathrm{GSpin}(W[1/p])$, the tensors $s_{\alpha,0} = \iota(s_{\alpha,p}) \in \mathbb{H}^{\otimes}$ are stable by Frobenius, and therefore $\mathscr{H}$ is equipped with a canonical GSpin-structure.

We will now use the description of the universal deformation space of X to prove that $\mathscr{H}$ admits a CM lift if X does. Write $\widehat{R} = \widehat{R}_X$. The opposite unipotent of $\tilde{\mu}$ in GSpin is canonically isomorphic to U^{opp}, the opposite unipotent of μ_{SO} in SO. Thus, we may regard $u \in U^{\mathrm{opp}}(\widehat{R})$ as an endomorphism of $\mathbb{H} \otimes \widehat{R}$.

Let $\widetilde{\mathrm{Fil}} \subset \mathbb{H}$ denote the filtration of $\mathbb{H}$ induced by $\tilde{\mu}$. By [9, Section 1.5], the data $(\mathbb{H} \otimes \widehat{R}, \widetilde{\mathrm{Fil}} \otimes \widehat{R}, u \cdot (\tilde{b}\sigma))$ arises from the Dieudonné module of a p-divisible group $\mathscr{H}_{\widehat{R}}$ over Spf $\widehat{R}$, which deforms $\mathscr{H}$. Note that the tensors $s_{\alpha,0} \in \mathbb{D}(\mathscr{H}_{\widehat{R}})^{\otimes}$ are Frobenius-invariant and in $\widetilde{\mathrm{Fil}}^0$.

Lemma 4.13. *Let $K/W(k)[1/p]$ be a finite extension, $y : \widehat{R} \to \mathcal{O}_K$ a map of $W(k)$-algebras, and X_y (resp. $\mathscr{H}_y$) the K3 surface (resp. p-divisible group) over $\mathcal{O}_K$ corresponding to y. Let $D(X_y)$ (resp. $D(\mathscr{H}_y)$) denote the weakly admissible module over K associated to X_y (resp. $\mathscr{H}_y$). Then there is a canonical inclusion*

$$D(X_y)(1) \subset D(\mathscr{H}_y)^{\otimes}$$

compatible with filtrations and Frobenius.

Proof. The weakly admissible module $D(X_y)$ is constructed using the crystalline and de Rham cohomology of X_y, and the isomorphism between them as in [1, Section 2], and similarly for $D(\mathscr{H}_y)$. Let us briefly recall the construction.

Let $\widehat{R}_y$ denote the PD-completion of $\widehat{R}$ with respect to $\mathrm{Ker}(y) + p\widehat{R}$. Then $\mathbb{L}_{\widehat{R}_y} = \mathbb{L}_{\mathrm{u}} \otimes_{\widehat{R}} \widehat{R}_y$ is equipped with a Frobenius and filtration. Moreover, there is a unique Frobenius equivariant map $\mathbb{L} = \mathbb{L}_0 \to \mathbb{L}_{\widehat{R}_y}[1/p]$ which lifts the identity over $\mathfrak{n}$. It may be constructed by choosing any lift of the identity s_0, and taking the limit $s = \lim_i \varphi^i(s_0) = \lim_i \varphi^i \circ s_0 \circ \varphi^{-i}$, which exists. This allows us to identify $\mathbb{L} \otimes_{W(k)} K$ with $\mathbb{L}_y[1/p]$, where $\mathbb{L}_y = \mathbb{L}_{\mathrm{u}} \otimes_{\widehat{R},y} \mathcal{O}_K$. If $K_0 \subset K$ denotes the maximal unramified subfield, this gives $D(X_y) \cong \mathbb{L} \otimes_{W(k)} K_0$ the structure of a weakly admissible module over K. Note that this identification is not in general given by the identity of $\mathbb{L}$. There is an analogous construction starting with $\mathbb{H}_{\widehat{R}}$ in place of $\mathbb{L}_{\mathrm{u}}$.

Now let $\mathbb{L}_{\mathrm{u}}(1)$ denote the K3 crystal with underlying module $\mathbb{L}_{\mathrm{u}}$, equipped with the Frobenius given by $p^{-1}b_u\sigma$, and the filtration given by $\mathrm{Fil}^i\,\mathbb{L}_{\mathrm{u}}(1) = \mathrm{Fil}^{i+1}\,\mathbb{L}_{\mathrm{u}}$. By construction, there is an inclusion $\mathbb{L}_{\mathrm{u}}(1) \subset \mathbb{H}_{\widehat{R}}^{\otimes}$. Applying the above construction to both sides, we obtain $D(X_y)(1) \subset D(\mathscr{H}_y)^{\otimes}$, as required. ∎

Proposition 4.14. *Suppose that the K3 surface X admits a CM lift. Then so does the associated Kuga–Satake p-divisible group $\mathscr{H}$.*

Proof. Let K, K_0 and $y : \widehat{R} \to \mathcal{O}_K$ be as in the previous lemma, and denote by $G(D(X_y)(1))$ and $G(D(\mathscr{H}_y))$ the Tannakian groups of the weakly admissible modules $D(X_y)(1)$ and $D(\mathscr{H}_y)^{\otimes}$, associated to the fiber functor which takes a weakly admissible module to its underlying K_0-vector space. By the last assertion of Lemma 4.13, we have the commutative diagram

$$\begin{array}{ccc} G(D(\mathscr{H}_y)) & \longrightarrow & G(D(X_y)(1)) \\ \downarrow & & \downarrow \\ \mathrm{GSpin}(\mathbb{H}_{K_0}) & \longrightarrow & \mathrm{SO}(\mathbb{L}_{K_0}(1)) \end{array}$$

where the vertical maps are the natural inclusions. Since the bottom map is surjective, with kernel $\mathbb{G}_{\mathrm{m}}$, it follows that $G(D(\mathscr{H}_y))$ contains a finite-index abelian subgroup if $G(D(X_y)(1))$ contains a finite-index abelian subgroup.

Now suppose that X_y is CM. By what we just saw, it suffices to show that $G(D(X_y)(1))$, or equivalently $G(D(X_y))$, contains a finite-index abelian subgroup. If $\bar{K}$ denotes an algebraic closure of K, this is equivalent to asking that the $\mathrm{Gal}(\bar{K}/K)$ action on $H^2_{\text{ét}}(X_{y,\bar{K}}, \mathbb{Q}_p)$ is abelian. As X_y descends to a number field F [19, Theorem 4], it suffices to show that the image of the Galois representation of $\mathrm{Gal}(\bar{F}/F)$ on $H^2_{\text{ét}}(X_{y,\bar{F}}, \mathbb{Q}_p)$ contains a finite-index abelian subgroup. This follows from work of Deligne [4, Section 6.6.1], which relates this Galois representation to the Galois representation on the cohomology of the associated Kuga–Satake abelian variety A_y. As X_y is CM, so is A_y, and the image of the latter Galois representation contains a finite-index abelian subgroup. We could also have used the Mumford–Tate conjecture for K3 surfaces, proved by Tankeev [23, 24]. ∎

Theorem 4.15. *If $p \geq 5$, then only finitely many supersingular K3 surfaces over $\overline{\mathbb{F}}_p$ admit CM lifts.*

Proof. Let $\mathcal{S}$ denote the set of supersingular K3 surfaces over $\overline{\mathbb{F}}_p$ that admit CM lifts. By Proposition 4.14, the Kuga–Satake p-divisible group $\mathscr{H}(X)$ associated to every $X \in \mathcal{S}$ has a CM lift. Let $\mathcal{S}'$ denote the set $\{\mathscr{H}(X) \mid X \in \mathcal{S}\}$. Theorem 3.4 implies that $\mathcal{S}'$ contains only finitely[4] many $\overline{\mathbb{F}}_p$-isomorphism classes of p-divisible groups, and therefore there exists a finite field $\mathbb{F}_q$ such that every $\mathscr{H} \in \mathcal{S}'$ admits a decent model over $\mathbb{F}_q$ (which we will again denote by $\mathscr{H}/\mathbb{F}_q$). By replacing $\mathbb{F}_q$ by a finite extension if necessary, we may assume that Frob_q acts by the scalar $q^{1/2}$ (that is, we assume that the eigenvalues are equal, whereas without replacing $\mathbb{F}_q$ by a finite extension, we only know that the ratios of the eigenvalues are roots of unity). By Lemma 4.16 below, every GSpin-structure on $\mathscr{H}$ is defined over $\mathbb{F}_q$. Since the data of a p-divisible group $\mathscr{H}$ along with a GSpin-structure

[4]Theorem 1.2 does not follow immediately – it is a priori possible that infinitely many non-isomorphic K3 surfaces X might yield the same Kuga–Satake p-divisible group.

uniquely determines the K3 crystal, it follows that the K3 crystal $\mathbb{L}(X)$ admits a decent model over $\mathbb{F}_q$ for every $X \in \mathcal{S}$.

By Ogus' crystalline Torelli theorem (the main result of [15]), it suffices to prove that there are only finitely many isomorphism classes of decent K3 crystals defined over any fixed finite field $\mathbb{F}_q$. This is a direct consequence of the discussion in [14, Section 3, Definition 3.19, Theorem 3.20] pertaining to characteristic subspaces, and therefore the result follows. ■

Lemma 4.16. *Let $\mathscr{H}/\mathbb{F}_q$ denote a p-divisible group such that* Frob_q *acts by the scalar $q^{1/2}$. Then every* GSpin*-structure on $\mathscr{H}_{\overline{\mathbb{F}}_p}$ is defined over $\mathbb{F}_q$.*

Proof. Let $\mathbb{H} = \mathbb{D}(\mathscr{H})$, and $\iota : H \otimes_W W(\overline{\mathbb{F}}_p) \to \mathbb{H}_{W(\overline{\mathbb{F}}_p)}$ a GSpin-structure on $\mathscr{H}$. Let $s_{\alpha,0} = \iota(s_{\alpha,p}) \in \mathbb{H}^{\otimes}_{W(\overline{\mathbb{F}}_p)}$. Write φ for the Frobenius on $\mathbb{H}$. By definition, the $s_{\alpha,0}$ are Frobenius-invariant tensors. As $s_{\alpha,0}$ is $\varphi \otimes \sigma$-invariant, it lies in the slope-0 part of $\mathbb{H}^{\otimes}_{W(\overline{\mathbb{F}}_p)}$.

Our assumption is that Frob_q acts on $\mathbb{H}$ as the scalar $q^{1/2}$. Therefore if $q = p^r$, then φ^r acts on $\mathbb{D}(\mathscr{H}_{\overline{\mathbb{F}}_p}) = \mathbb{H} \otimes_{W(\mathbb{F}_q)} W(\overline{\mathbb{F}}_p)$ as the map $\mathrm{Frob}_q \otimes \sigma^r$. As φ^r acts trivially on $s_{\alpha,0}$, we see that σ^r acts trivially on $s_{\alpha,0}$. Hence, $s_{\alpha,0} \in \mathbb{H}^{\otimes}$.

Now consider the $W(\mathbb{F}_q)$-scheme that represents the functor which assigns to an $W(\mathbb{F}_q)$-algebra R the set of isomorphisms $H \otimes R \to \mathbb{H} \otimes R$ that send $s_{\alpha,p}$ to $s_{\alpha,0}$. This scheme is a GSpin_W-torsor as it has a point (given by the isomorphism ι) defined over $W(\overline{\mathbb{F}}_p)$. As GSpin is a connected reductive group, Lang's lemma implies that the torsor must be trivial over $W(\mathbb{F}_q)$, and hence there exists an isomorphism $\iota_q : H \otimes W(\mathbb{F}_q) \to \mathbb{H}$ that respects tensors. It follows that the GSpin-structure is indeed defined over $\mathbb{F}_q$, as claimed.[5] ■

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