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Fractal dimensions of fractal transformations and quantization dimensions for bi-Lipschitz mappings

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Abstract. In this paper, we study the fractal dimension of the graph of a fractal transformation and also determine the quantization dimension of a probability measure supported on the graph of the fractal transformation. Moreover, we estimate the quantization dimension of the invariant measure corresponding to a weighted iterated function system consisting of bi-Lipschitz mappings under the strong open set condition.

1. Introduction

In fractal geometry, the iterated function systems (IFSs) play an important role. They are commonly used to generate fractals. In fact, most of the fractals are the attractors of some IFS. Plenty of literature is available on the study of an IFS, its attractor and the fractal dimensions of the attractor (see, for instance, [2, 7, 10, 12, 13, 18]). In 2009, Barnsley [3] introduced the idea of fractal transformation. Basically, fractal transformation is a map between the attractor of one IFS to the attractor of another IFS. After that, Barnsley et al. [4] discussed many applications of fractal transformation. In 2014, Barnsely and his collaborators [5] determined some conditions under which a fractal transformation is measure preserving. In 2016, Bandt et al. [1] proved that under some conditions a fractal transformation becomes a homeomorphism. In 2018, Vince [19] showed that we can extend a fractal transformation from the non-empty attractor to the whole space and described some conditions under which a fractal transformation is the attractor of some IFS, which is constructed from the given IFSs.

The quantization dimension is one of the most important thing in the quantization theory. In 1964, Zador [20] was the first, who introduced the term quantization dimension and also discussed some properties of this dimension. In 2002, Graf and Luschgy [9] gave a formula of the quantization dimension of the self-similar

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measures under the open set condition (OSC). After that, Lindsay et al. [11] generalized the result of Graf and Luschgy [8] for the self-conformal measures and described some connection between quantization theory and fractal geometry. The quantization dimension is also related to some other known fractal dimensions like the Hausdorff dimension and the box-counting dimension, see, for more details [8]. In 2010, Roychowdhury [15] obtained the quantization dimension of the Moran measures. After that, Roychowdhury [16] also determined the quantization dimension of a Borel probability measure supported on the attractor of the bi-Lipschitz IFS by taking some conditions on the bi-Lipschitz constants and assuming that the IFS satisfies the strong open set condition (SOSC). In 2021, Roychowdhury and Selmi [17] estimated bounds of the quantization dimension of the invariant measure generated by the hyperbolic recurrent IFS under the strong separation condition. In this paper, we determine bounds of the quantization dimension of the invariant Borel probability measures supported on the attractor of a general class of bi-Lipschitz IFSs under the SOSC. This result also generalizes the result of Graf and Luschgy [8]. We also estimate the quantization dimension of the invariant Borel probability measure supported on the graph of a fractal transformation. Here, we discuss some dimensional results for the graph of a fractal transformation.

Let (Y, ρ) be a metric space. A map $\theta : (Y, \rho) \to (Y, \rho)$ is called a contraction if there exists a constant $c < 1$ such that

$$\rho\big(\theta(a), \theta(b)\big) \leq c\, \rho(a, b) \text{ for all } a, b \in Y.$$

Definition 1.1. The system $\mathcal{I} = \{(Y, \rho); \theta_1, \theta_2, \dots, \theta_N\}$ is called an iterated function system (IFS) if each θ_i is a contraction self-map on Y for $i \in \{1, 2, \dots, N\}$.

We know that depending on the type of the maps θ_i, we name the IFS. For example, an IFS is said to be a bi-Lipschitz IFS, an affine IFS and a conformal IFS, respectively, if the maps θ_i's are bi-Lipschitz transformations, affine maps and conformal maps, respectively. Note that a bi-Lipschitz IFS has the capability to handle all the well-known classes of fractal sets such as self-similar, self-affine and self-conformal sets as listed below.

(1) Let $\mathcal{I} = \{(Y, \rho); \theta_1, \theta_2, \dots, \theta_N\}$ be a bi-Lipschitz IFS, that is,

$$c_i \rho(a, b) \leq \rho\big(\theta(a), \theta(b)\big) \leq C_i \rho(a, b) \text{ for all } a, b \in Y.$$

Then, by choosing $c_i = C_i$, we get an IFS consisting of similarity transformations, and a self-similar set as the associated fractal set.

(2) An affine transformation on $\mathbb{R}^d$ is a map $\theta : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\theta(x) = Ax + b$, where A is a linear transformation on $\mathbb{R}^d$ and $b \in \mathbb{R}^d$. If the map A

is injective, then

$$\frac{1}{\|A^{-1}\|}\|x-y\|_2 \le \|Ax-Ay\|_2 = \|\theta(x)-\theta(y)\|_2$$
$$= \|Ax-Ay\|_2 \le \|A\|\|x-y\|_2.$$

(3) Let $\mathcal{I} = \{(X,\rho); \theta_1, \theta_2, \ldots, \theta_N\}$ be a conformal IFS, where $X \subset \mathbb{R}^d$ is a compact set such that $X = \overline{\text{int}(X)}$. Then, by [14, Lemma 2.2], there exists an open set V such that $X \subset V$ and a constant $C \ge 1$ such that

$$C^{-1}\|\theta_\sigma'\|\|x-y\| \le \|\theta_\sigma(x)-\theta_\sigma(y)\| \le C\|\theta_\sigma'\|\|x-y\|$$

for all $x, y \in V$ and $\sigma \in \cup_{n=1}\{1,2,\ldots,N\}^n$.

From the above points, it is clear that using the dimensional results for a bi-Lipschitz IFS, one can obtain some estimates for the dimensions of self-similar, self-affine, and self-conformal IFSs. However, it should be noted the exact dimensional value of self-affine and self-conformal sets may not be obtained by bi-Lipschitz IFS, see, for instance, [7, 11, 13, 14]. The novelty of our result on the quantization dimension of a bi-Lipschitz IFS is evident in view of Remark 4.13 and Remark 4.14 when comparing to [16].

The paper is organized as follows. In the upcoming Section 2, we discuss some preliminary results and the required definitions for the forthcoming section. In Section 3, we give some results on the fractal transformation and the product of two IFSs. Firstly, we determine some results related to the product of two IFSs. Next, we obtain the bounds on the Hausdorff dimension of the graph of the fractal transformation under some conditions. After that, we determine a relation between the invariant measures of two IFSs and their product IFS. We also give a relation between the quantization dimensions of these invariant measures. Finally, in Section 4, we provide the bounds on the quantization dimension of the invariant probability measure corresponding to a bi-Lipschitz IFS under the SOSC and also give bounds on the quantization dimension of the invariant measure supported on the graph of the fractal transformation.

2. Preliminaries

Definition 2.1. Let F be a subset of a metric space (Y,ρ). The Hausdorff dimension of F is defined as follows: $\dim_H F = \inf\{\beta > 0 : \text{for every } \epsilon > 0, \text{there is a countable cover } \{V_i\} \text{ of } F \text{ with } \sum |V_i|^\beta < \epsilon\}$, where $|V_i|$ denotes the diameter of V_i.

Definition 2.2. The box dimension of a non-empty bounded subset F of a metric space (Y, ρ) is defined as

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ is the minimum number of sets of diameter $\delta > 0$ that can cover F, provided the limit exists. If this limit does not exist, then by taking the limsup and the liminf, we define the upper and the lower box dimensions, and are denoted by $\overline{\dim}_B(F)$ and $\underline{\dim}_B(F)$, respectively.

Let (Y, ρ) be a complete metric space, and we denote the family of all non-empty compact subsets of Y by $H(Y)$. For any $A, B \in H(Y)$, we define the Hausdorff metric by

$$h(A, B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where A_δ and B_δ denote the δ-neighbourhoods of sets A and B, respectively. It is well known that $(H(Y), h)$ is a complete metric space.

Definition 2.3. The system $\{(Y, \rho); \theta_1, \theta_2, \dots, \theta_N; p_1, p_2, \dots, p_N\}$ is called a weighted iterated function system (WIFS) if $\{(Y, \rho) : \theta_1, \theta_2, \dots, \theta_N\}$ is an IFS with probability vector $(p_1, p_2, \dots, p_N)$.

Remark 2.4. Notice that $(p_1, p_2, \dots, p_N)$ is a probability vector if and only if $\sum_{i=1}^N p_i = 1$ and $p_i > 0$ for all $i \in \{1, 2, \dots, N\}$.

Note 2.5. Let $\mathcal{I} = \{(Y, \rho); \theta_1, \theta_2, \dots, \theta_N\}$ be an IFS. We define the Hutchinson mapping S from $H(Y)$ into $H(Y)$ given by

$$S(A) = \bigcup_{i=1}^N \theta_i(A).$$

The map S is a contraction map under the Hausdorff metric h. If (Y, ρ) is a complete metric space, then, by Banach contraction principle, there exists a unique $E \in H(Y)$ such that

$$E = \bigcup_{i=1}^N \theta_i(E).$$

The set E is called the attractor of the IFS. Furthermore, if $(p_1, p_2, \dots, p_N)$ is a probability vector corresponding to the IFS $\mathcal{I}$, then there exists a unique Borel probability measure μ supported on the attractor E such that

$$\mu = \sum_{i=1}^N p_i \mu \circ \theta_i^{-1}.$$

We call μ the invariant measure corresponding to the WIFS $\{(Y,\rho);\theta_1,\theta_2,\ldots,\theta_N;p_1, p_2,\ldots,p_N\}$. We refer the reader to see [2,7] for details.

Definition 2.6. We say that an IFS $\mathcal{I}=\{(Y,\rho);\theta_1,\theta_2,\ldots,\theta_N\}$ satisfies the open set condition (OSC) if there is a non-empty open set O with $\theta_i(O)\subset O$ for all $i\in\{1,2,\ldots,N\}$ and $\theta_i(O)\cap\theta_j(O)=\emptyset$ for $i\neq j$. Moreover, if $O\cap E\neq\emptyset$, where E is the attractor of the IFS $\mathcal{I}$, then we say that $\mathcal{I}$ satisfies the strong open set condition (SOSC). If $\theta_i(E)\cap\theta_j(E)=\emptyset$ for $i\neq j$, then we say that the IFS $\mathcal{I}$ satisfies the strong separation condition (SSC).

2.1. Code space

For this part, we refer the reader to [7]. Let (X,ρ) be a complete metric space. Let $\mathcal{F}:=\{X;f_1,f_2,\ldots,f_N\}$ be an IFS. Next, let Ω be the set of all infinite sequences $\Omega=\{\{\sigma_k\}_{k=1}^{\infty};\sigma_k\in\{1,2,\ldots,N\}\}$. We write $\sigma=\sigma_1\sigma_2\sigma_3\cdots\in\Omega$ to denote a typical element of Ω, and we write σ_k to denote the kth element of $\sigma\in\Omega$. Then (Ω,d_Ω) is a compact metric space, where the metric d_Ω is defined by $d_\Omega(\sigma,\omega)=0$ when $\sigma=\omega$ and $d_\Omega(\sigma,\omega)=2^{-k}$ when k is the least index for which $\sigma_k\neq\omega_k$. We call Ω the code space associated with the IFS $\mathcal{F}$.

Let $\sigma\in\Omega$ and $x\in X$. Then, using the contractivity of $\mathcal{F}$, it is not difficult to prove that

$$\phi_{\mathcal{F}}(\sigma):=\lim_{k\to\infty}f_{\sigma_1}\circ f_{\sigma_2}\circ\cdots\circ f_{\sigma_k}(x)$$

exists, is independent of x, and depends continuously on σ. Furthermore, let $A_{\mathcal{F}}=\{\phi_{\mathcal{F}}(\sigma):\sigma\in\Omega\}$. Then, it is easy to show that $A_{\mathcal{F}}\subset X$ is the attractor of $\mathcal{F}$. The continuous function

$$\phi_{\mathcal{F}}:\Omega\to A_{\mathcal{F}}$$

is called the address function of $\mathcal{F}$. We call $\phi_{\mathcal{F}}^{-1}(\{x\})=\{\sigma\in\Omega:\phi_{\mathcal{F}}(\sigma)=x\}$ the set of addresses of the point $x\in A_{\mathcal{F}}$. We order the elements of Ω according to

$$\sigma\prec\omega \text{ if and only if } \sigma_k<\omega_k,$$

where k is the least index for which $\sigma_k\neq\omega_k$. We observe that all elements of Ω are less than or equal to $\overline{N}=NNN\ldots$ and greater than or equal to $\overline{1}=111\ldots$ Note that $\phi_{\mathcal{F}}^{-1}(\{x\})$ contains a unique largest element. Let $\mathcal{F}$ be an IFS with attractor $A_{\mathcal{F}}$ and address function $\phi_{\mathcal{F}}:\Omega\to A_{\mathcal{F}}$. Let

$$\tau_{\mathcal{F}}(x)=\max\big\{\sigma\in\Omega:\phi_{\mathcal{F}}(\sigma)=x\big\}$$

for all $x\in A_{\mathcal{F}}$. Then

$$\Omega_{\mathcal{F}}:=\big\{\tau_{\mathcal{F}}(x):x\in A_{\mathcal{F}}\big\}$$

is called the tops code space and

$$\tau_{\mathcal{F}} : A_{\mathcal{F}} \to \Omega_{\mathcal{F}}$$

is called the tops function corresponding to the IFS $\mathcal{F}$. It can be seen that the tops function $\tau_{\mathcal{F}} : A_{\mathcal{F}} \to \Omega_{\mathcal{F}}$ is one-one and onto.

The address structure of the IFS $\mathcal{F}$ is defined to be the set of sets

$$\mathcal{C}_{\mathcal{F}} = \left\{\phi_{\mathcal{F}}^{-1}(x) \cap \overline{\Omega}_{\mathcal{F}} : x \in A_{\mathcal{F}}\right\}.$$

Let $\mathcal{C}_{\mathcal{G}}$ be the address structure of another IFS $\mathcal{G} = \{Y; g_1, g_2, \ldots, g_N\}$. We write $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$, if for each $P \in \mathcal{C}_{\mathcal{F}}$, there is a $Q \in \mathcal{C}_{\mathcal{G}}$ such that $P \subset Q$.

Definition 2.7. Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{Y; g_1, g_2, \ldots, g_N\}$ be two IFSs. Suppose $A_{\mathcal{F}}$ and $A_{\mathcal{G}}$ are the attractors of $\mathcal{F}$ and $\mathcal{G}$, respectively. The associated fractal transformation $T_{\mathcal{F}\mathcal{G}} : A_{\mathcal{F}} \to A_{\mathcal{G}}$ is defined by

$$T_{\mathcal{F}\mathcal{G}} = \phi_{\mathcal{G}} \circ \tau_{\mathcal{F}},$$

where $\phi_{\mathcal{G}}$ is the code map corresponding to $\mathcal{G}$ and $\tau_{\mathcal{F}}$ is the tops function corresponding to $\mathcal{F}$.

Remark 2.8 ([3, Theorem 1]). Let $\mathcal{F}$ and $\mathcal{G}$ be two IFSs, $\mathcal{F} = \{X; f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{Y; g_1, g_2, \ldots, g_N\}$, such that $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$. Then, the fractal transformation map $T_{\mathcal{F}\mathcal{G}}$ is continuous.

Let (X, ρ) be a complete metric space. Given a Borel probability measure μ on X, a number $r \in (0, +\infty)$ and $n \in \mathbb{N}$, the nth quantization error of order r for μ is defined by

$$V_{n,r}(\mu) := \inf\left\{\int \rho(x, A)^r \, d\mu(x) : A \subset X, \, \mathrm{Card}(A) \leq n\right\},$$

where $\rho(x, A)$ represents the distance of the point x from the set A. Next, we define $e_{n,r}(\mu) := V_{n,r}^{\frac{1}{r}}(\mu)$. Finally, we define the *quantization dimension* of order r of μ by

$$D_r = D_r(\mu) := \lim_{n \to \infty} \frac{\log n}{-\log\left(e_{n,r}(\mu)\right)},$$

if the limit exists. If the limit does not exist, then we define the lower and upper quantization dimensions by taking the limit inferior and the limit superior of the sequence and are denoted by $\underline{D}_r(\mu)$ and $\overline{D}_r(\mu)$, respectively.

Remark 2.9. Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^r d\mu(x) < \infty$. Then for every $n \in \mathbb{N}$ there exists a finite set $A_n \subset \mathbb{R}^d$ such that

$$V_{n,r}(\mu) = \int \min_{a \in A_n} \|x - a\|^r \, d\mu(x).$$

This A_n is called an n-optimal set for measure μ of order r.

Remark 2.10. Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. Then

$$\int \|x\|^r d\mu(x) < \infty,$$

for any $r \in (0, +\infty)$.

Lemma 2.11 ([8, Lemma 6.1]). *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^r d\mu(x) < \infty$. Then, for any $r \in (0, \infty)$*

$$V_{n,r}(\mu) \to 0, \text{ as } n \to \infty.$$

Definition 2.12. Let A be a subset of $\mathbb{R}^d$. The Voronoi region of $a \in A$ is defined by

$$W(a|A) = \left\{x \in \mathbb{R}^d : \ \|x - a\| = \min_{b \in A} \|x - b\|\right\}$$

and the set $\{W(a|A); \ a \in A\}$ is called the Voronoi diagram of A.

Lemma 2.13 ([8, Lemma 13.8]). *Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support A_* and $r \in (0, \infty)$. Let A_n be an n-optimal set for measure μ of order r. Define*

$$\|A_n\|_\infty = \max_{a \in A_n} \max_{x \in W(a|A_n) \cap A_*} \|x - a\|.$$

Then

$$\left(\frac{\|A_n\|_\infty}{2}\right)^r \min_{x \in A_*} \mu\left(B\left(x, \frac{\|A_n\|_\infty}{2}\right)\right) \leq V_{n,r}(\mu).$$

Proposition 2.14 ([8, Proposition 11.3]). *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^r d\mu(x) < \infty$. Then, we have the following:*

(1) *If $0 \leq t_1 < \overline{D}_r(\mu) < t_2$, then*

$$\limsup_{n \to \infty} n \cdot e_{n,r}^{t_1}(\mu) = \infty \quad \text{and} \quad \lim_{n \to \infty} n \cdot e_{n,r}^{t_2}(\mu) = 0.$$

(2) *If $0 \leq t_1 < \underline{D}_r(\mu) < t_2$, then*

$$\liminf_{n \to \infty} n \cdot e_{n,r}^{t_2}(\mu) = 0 \quad \text{and} \quad \lim_{n \to \infty} n \cdot e_{n,r}^{t_1}(\mu) = \infty.$$

Let $\{(X, \rho); f_1, f_2, \ldots, f_N\}$ be an IFS with probability vector $(p_1, p_2, \ldots, p_N)$. We denote the set of all finite sequences of symbols belonging to the set $\{1, 2, \ldots, N\}$ by $\{1, 2, \ldots, N\}^*$ and $|\sigma|$ denotes the length of $\sigma \in \{1, 2, \ldots, N\}^*$. We denote the set of all finite sequences of length n over the symbols belonging to the set $\{1, 2, \ldots, N\}$ by $\{1, 2, \ldots, N\}^n$. Let $\sigma \in \{1, 2, \ldots, N\}^*$ and $m \leq |\sigma|$, we define $\sigma|_m$ as follows

$$\sigma|_m = \begin{cases} \emptyset, & m = 0 \\ \sigma_1\sigma_2 \cdots \sigma_m, & m \neq 0. \end{cases}$$

We define a natural order on $\{1,2,\dots,N\}^*$ by

$$\sigma \leq \tau \quad \text{iff} \quad |\sigma| \leq |\tau|, \tau_{|\sigma|} = \sigma,$$

where $\sigma, \tau \in \{1,2,\dots,N\}^*$. We use the notations $p_\sigma = p_{\sigma_1} \cdot p_{\sigma_2} \cdot \dots \cdot p_{\sigma_m}$ and $p_{\sigma^-} = p_{\sigma_1} \cdot p_{\sigma_2} \cdot \dots \cdot p_{\sigma_{m-1}}$ for $\sigma \in \{1,2,\dots,N\}^*$, $|\sigma| = m$. Let $\sigma, \tau \in \{1,2,\dots,N\}^*$. We say that σ and τ are incomparable if neither $\sigma \leq \tau$ nor $\tau \leq \sigma$.

A finite set $\Gamma \subset \{1,2,\dots,N\}^*$ is called a finite antichain if and only if any two elements of Γ are incomparable. A finite antichain Γ is called maximal if and only if for every finite antichain $\Gamma' \subset \{1,2,\dots,N\}^*$ with $\Gamma \subseteq \Gamma'$, we have $\Gamma' = \Gamma$.

Lemma 2.15 ([8], [16, Lemma 3.8]). *Let $\{\mathbb{R}^d; f_1, f_2, \dots, f_N; p_1, p_2, \dots, p_N\}$ be a WIFS. Let μ be the invariant measure corresponding to this WIFS. If Γ is a finite maximal antichain, then*

$$\sum_{\sigma \in \Gamma} p_\sigma = 1 \quad \text{and} \quad \mu = \sum_{\sigma \in \Gamma} p_\sigma \mu \circ f_\sigma^{-1}.$$

Lemma 2.16 ([8, p. 192]). *Let $(p_1, p_2, \dots, p_N)$ be a probability vector. If*

$$0 < \epsilon \leq \min\{p_1, p_2, \dots, p_N\},$$

then

$$\Gamma_\epsilon = \{\sigma \in \{1,2,\dots,N\}^*; \ \ p_{\sigma^-} \geq \epsilon > p_\sigma\}$$

is a finite maximal antichain.

Let $\{\mathbb{R}^d; f_1, f_2, \dots, f_N; p_1, p_2, \dots, p_N\}$ be a WIFS such that each f_i is a contractive similarity transformation such that

$$\|f_i(x_1) - f_i(x_2)\| = c_i \|x_1 - x_2\|,$$

where $0 < c_i < 1$. Then, there is a unique Borel probability measure μ supported on the attractor E of the IFS $\{\mathbb{R}^d; f_1, f_2, \dots, f_N\}$ such that

$$\mu = \sum_{i=1}^N p_i \mu \circ f_i^{-1}.$$

In this case, we call the measure μ an invariant self-similar measure. As it was proved by Graf and Luschgy [8, 9], the quantization dimension function l_r of the invariant self-similar measure μ exists, and satisfies the following equation:

$$\sum_{i=1}^N (p_i c_i^r)^{\frac{l_r}{r+l_r}} = 1,$$

provided that the given WIFS satisfies the OSC.

3. On the product IFS and the dimension of the graph of a fractal transformation

First, we define a metric on the product space as follows.

Let (X, ρ_1) and (Y, ρ_2) be two complete metric spaces. We define a metric $\mathcal{D}$ on $X \times Y$ by

$$\mathcal{D}\big((x, y), (x', y')\big) = \max\{\rho_1(x, x'), \rho_2(y, y')\},$$

where $x, x' \in X$ and $y, y' \in Y$. It is well known that $(X \times Y, \mathcal{D})$ is a complete metric space.

In the following theorem, we determine the bounds of the Hausdorff dimension of graph of fractal transformation without any separation condition.

Theorem 3.1. *Let $\mathcal{F} = \{(X, \rho_1); f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{(Y, \rho_2); g_1, g_2, \ldots, g_N\}$ be two IFSs. Consider the iterated function system $\mathcal{H} = \{X \times Y; h_1, h_2, \ldots, h_N\}$, where $h_i(x, y) = (f_i(x), g_i(y))$. Then*

$$\dim_H A_{\mathcal{F}} \leq \dim_H G(T_{\mathcal{F}\mathcal{G}}) \leq \dim_H A_{\mathcal{H}},$$

where $G(T_{\mathcal{F}\mathcal{G}})$ denotes the graph of the fractal transformation $T_{\mathcal{F}\mathcal{G}}$.

Proof. We define a mapping $\Psi : G(T_{\mathcal{F}\mathcal{G}}) \to A_{\mathcal{F}}$ by

$$\Psi\big(x, T_{\mathcal{F}\mathcal{G}}(x)\big) = x.$$

Let $(x, T_{\mathcal{F}\mathcal{G}}(x)), (x', T_{\mathcal{F}\mathcal{G}}(x')) \in G(T_{\mathcal{F}\mathcal{G}})$. Then, we have

$$\begin{aligned} \rho_1\big(\Psi(x, T_{\mathcal{F}\mathcal{G}}(x)), \Psi(x', T_{\mathcal{F}\mathcal{G}}(x'))\big) &= \rho_1(x, x') \\ &\leq \max\{\rho_1(x, x'), \rho_2\big(T_{\mathcal{F}\mathcal{G}}(x), T_{\mathcal{F}\mathcal{G}}(x')\big)\} \\ &= \mathcal{D}\big((x, T_{\mathcal{F}\mathcal{G}}(x)), (x', T_{\mathcal{F}\mathcal{G}}(x'))\big). \end{aligned}$$

Thus, Ψ is a Lipschitz map. Therefore, by the Lipschitz invariance property of the Hausdorff dimension, we get

$$\dim_H A_{\mathcal{F}} \leq \dim_H G(T_{\mathcal{F}\mathcal{G}}). \tag{3.1}$$

For the other inequality, let $(x, T_{\mathcal{F}\mathcal{G}}(x)) \in G(T_{\mathcal{F}\mathcal{G}})$. So, there is a $\sigma \in \Omega_{\mathcal{F}}$ such that $\phi_{\mathcal{F}}(\sigma) = x$. Now,

$$\big(x, T_{\mathcal{F}\mathcal{G}}(x)\big) = \big(\phi_{\mathcal{F}}(\sigma), \phi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ \phi_{\mathcal{F}}(\sigma)\big) = \big(\phi_{\mathcal{F}}(\sigma), \phi_{\mathcal{G}}(\sigma)\big) \in A_{\mathcal{H}}.$$

Hence, $G(T_{\mathcal{F}\mathcal{G}}) \subset A_{\mathcal{H}}$. Therefore, using the monotonic property of the Hausdorff dimension, we have

$$\dim_H G(T_{\mathcal{F}\mathcal{G}}) \leq \dim_H A_{\mathcal{H}}. \tag{3.2}$$

Combining inequalities 3.1 and 3.2, we get our required result. ■

In the next lemma, we show that the product IFS satisfies SSC, OSC and SOSC, respectively, provided one of the IFSs satisfies SSC, OSC and SOSC, respectively.

The proof of the following lemma is not difficult. But, we give the complete proof for the convenience of the reader.

Lemma 3.2. *Let $\mathcal{F} = \{X; f_1, f_2, \dots, f_N\}$ and $\mathcal{G} = \{Y; g_1, g_2, \dots, g_N\}$ be two IFSs. For the IFS $\mathcal{H} := \{X \times Y; h_1, h_2, \dots, h_N\}$, where $h_i(x, y) = (f_i(x), g_i(y))$, we have the following:*

(1) *If $\mathcal{F}$ satisfies SSC, then $\mathcal{H}$ also satisfies SSC.*

(2) *If $\mathcal{F}$ satisfies OSC, then $\mathcal{H}$ also satisfies OSC.*

(3) *If $\mathcal{F}$ satisfies SOSC, then $\mathcal{H}$ also satisfies SOSC.*

Proof. (1) Since $\mathcal{F}$ satisfies SSC, we have

$$f_i(A_{\mathcal{F}}) \cap f_j(A_{\mathcal{F}}) = \emptyset \text{ for all } i \neq j.$$

This further yields

$$\begin{aligned} h_i(A_{\mathcal{F}} \times A_{\mathcal{G}}) \cap h_j(A_{\mathcal{F}} \times A_{\mathcal{G}}) \\ = \big(f_i(A_{\mathcal{F}}) \times g_i(A_{\mathcal{G}})\big) \cap \big(f_j(A_{\mathcal{F}}) \times g_j(A_{\mathcal{G}})\big) = \emptyset \end{aligned}$$

for all $i \neq j$.
Since $A_{\mathcal{H}} \subset A_{\mathcal{F}} \times A_{\mathcal{G}}$, we obtain

$$h_i(A_{\mathcal{H}}) \cap h_j(A_{\mathcal{H}}) = \emptyset \text{ for all } i \neq j.$$

Therefore, the IFS $\mathcal{H}$ satisfies SSC.

(2) Since $\mathcal{F}$ satisfies OSC, we have an open set U such that

$$\bigcup_{i=1}^{N} f_i(U) \subset U \text{ and } f_i(U) \cap f_j(U) = \emptyset \text{ for all } i \neq j.$$

This further yields

$$h_i(U \times Y) \cap h_j(U \times Y) = \big(f_i(U) \times g_i(Y)\big) \cap \big(f_j(U) \times g_j(Y)\big) = \emptyset$$

for all $i \neq j$.
Now, define an open set $W = U \times Y$. Then

$$\bigcup_{i=1}^{N} h_i(W) \subset W \text{ and } h_i(W) \cap h_j(W) = \emptyset \text{ for all } i \neq j.$$

Therefore, the IFS $\mathcal{H}$ satisfies OSC.

(3) Since $\mathcal{F}$ satisfy SOSC, we have an open set U such that

$$\bigcup_{i=1}^{N} f_i(U) \subset U, \quad U \cap A_{\mathcal{F}} \neq \emptyset, f_i(U) \cap f_j(U) = \emptyset \text{ for all } i \neq j.$$

This further yields

$$h_i(U \times Y) = (f_i(U) \times g_i(Y)) \subset U \times Y, \quad (U \times Y) \cap (G(T_{\mathcal{F}\mathcal{G}})) \neq \emptyset,$$

$$h_i(U \times Y) \cap h_j(U \times Y) = (f_i(U) \times g_i(Y)) \cap (f_j(U) \times g_j(Y)) = \emptyset,$$

for all $i \neq j$.
Now, we define an open set $W = U \times Y$. Since $G(T_{\mathcal{F}\mathcal{G}}) \subset A_{\mathcal{H}}$, we have

$$\bigcup_{i=1}^{N} h_i(W) \subset W, \quad h_i(W) \cap h_j(W) = \emptyset, \text{ for all } i \neq j, \text{ and } W \cap (A_{\mathcal{H}}) \neq \emptyset.$$

Therefore, the IFS $\mathcal{H}$ satisfies SOSC. ■

Lemma 3.3. *Let $\mathcal{F} = \{(X, \rho_1); f_1, f_2, \dots, f_N\}$ and $\mathcal{G} = \{(Y, \rho_2); g_1, g_2, \dots, g_N\}$ be two IFSs. For the IFS $\mathcal{H} := \{X \times Y; h_1, h_2, \dots, h_N\}$, where $h_i(x, y) = (f_i(x), g_i(y))$, for all $i \in \{1, 2, \dots, N\}$, we have the following:*

(1) *If f_i and g_i are similarity transformations such that $\rho_1(f_i(x), f_i(x')) = c_i \rho_1(x, x')$ and $\rho_2(g_i(y), g_i(y')) = c_i \rho_2(y, y')$ for all $x, x' \in X$, and $y, y' \in Y$, then so is h_i.*

(2) *If f_i and g_i are bi-Lipschitz mappings, then so is h_i.*

Proof. Since the proof is easy, we skip it. ■

Lemma 3.4. *Let $\mathcal{F} = \{\mathbb{R}^d; f_1, f_2, \dots, f_N\}$ and $\mathcal{G} = \{\mathbb{R}^d; g_1, g_2, \dots, g_N\}$ be two IFSs. Consider the IFS*

$$\mathcal{H} := \{\mathbb{R}^d \times \mathbb{R}^d; h_1, h_2, \dots, h_N\},$$

where $h_i(x, y) = (f_i(x), g_i(y))$. If, for each i, f_i and g_i are affine transformations, then so is h_i.

Proof. Since the proof is easy, we skip it. ■

Remark 3.5. We define three mappings $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x}{2}, \quad f_3(x) = \frac{x}{2} + \frac{1}{2}.$$

Let $\mathcal{I} = \{\mathbb{R}; f_1, f_2, f_3\}$ and $\mathcal{J} = \{\mathbb{R}; f_2, f_3\}$ be two IFSs. Then $\mathcal{J}$ is a sub IFS of $\mathcal{I}$. One can easily show that the sub IFS $\mathcal{J}$ satisfies SOSC and SOC but the IFS $\mathcal{I}$ does not.

In the following proposition, we give the exact value of the Hausdorff dimension of the graph of the fractal transformation under some separation condition.

Proposition 3.6. *Let* $\mathcal{F} = \{(X, \rho_1); f_1, f_2, \ldots, f_N\}$ *and* $\mathcal{G} = \{(Y, \rho_2); g_1, g_2, \ldots, g_N\}$ *be two IFSs such that*

$$\rho_1\big(f_i(x), f_i(x')\big) = c_i \rho_1(x, x'),$$

$$\rho_2\big(g_i(y), g_i(y')\big) \le r_i \rho_2(y, y'),$$

where $c_i, r_i \in (0,1)$. *If* $r_i \le c_i$ *for all* $i \in \{1, 2, \ldots, N\}$, $\mathcal{F}$ *satisfies SOSC, then* $\dim_H G(T_{\mathcal{F}\mathcal{G}}) = s_0$, *where* s_0 *is given by* $\sum_{i=1}^N {c_i}^{s_0} = 1$.

Proof. If we consider the IFS $\mathcal{H} = \{X \times Y; h_1, h_2, \ldots, h_N\}$, then

$$\begin{aligned}\mathcal{D}\big(h_i(x, y), h_i(x', y')\big) &= \mathcal{D}\big((f_i(x), g_i(y)), (f_i(x'), g_i(y'))\big)\\ &= \max\big(\rho_1(f_i(x), f_i(x')), \rho_2(g_i(y), g_i(y'))\big)\\ &\le \max\big(c_i \rho_1(x, x'), r_i \rho_2(y, y')\big)\\ &\le c_i \mathcal{D}\big((x, y), (x', y')\big).\end{aligned}$$

Therefore, by [7, Theorem 9.6], $\dim_H A_{\mathcal{H}} \le s_0$ where s_0 is given by $\sum_{i=1}^N {c_i}^{s_0} = 1$. Since $\mathcal{F}$ satisfies SOSC and each f_i is a similarity transformation, by [18, Theorem 2.6] $\dim_H A_{\mathcal{F}} = s_0$, where s_0 is uniquely determined by $\sum_{i=1}^N {c_i}^{s_0} = 1$. Theorem 3.1, yields that

$$s_0 = \dim_H A_{\mathcal{F}} \le \dim_H G(T_{\mathcal{F}\mathcal{G}}) \le \dim_H A_{\mathcal{H}} \le s_0.$$

Hence, $\dim_H G(T_{\mathcal{F}\mathcal{G}}) = s_0$. This completes the proof. ∎

Proposition 3.7. *Let* $\mathcal{F} = \{(X, \rho_1); f_1, f_2, \ldots, f_N\}$ *and* $\mathcal{G} = \{(Y, \rho_2); g_1, g_2, \ldots, g_N\}$ *be two IFSs such that*

$$\rho_1\big(f_i(x), f_i(x')\big) = c_i \rho_1(x, x'),$$

$$\rho_2\big(g_i(y), g_i(y')\big) \le r_i \rho_2(y, y'),$$

where $c_i, r_i \in (0,1)$. *If* $c_i \le r_i$ *for all* $i \in \{1, 2, \ldots, N\}$, $\mathcal{F}$ *satisfies SOSC. Then*

$$s_0 \le \dim_H G(T_{\mathcal{F}\mathcal{G}}) \le t_0,$$

where t_0 *and* s_0 *are given by* $\sum_{i=1}^N {r_i}^{t_0} = 1$ *and* $\sum_{i=1}^N {c_i}^{s_0} = 1$, *respectively.*

Proof. Using similar arguments of Proposition 3.6, one can easily prove this. ∎

In the upcoming theorem, we obtain bounds of the Hausdorff dimension of the product IFS provided some conditions hold.

Theorem 3.8. *Let the iterated function system* $\mathcal{H} := \{X \times Y; h_1, h_2, \dots, h_N\}$ *satisfy SOSC, where* $h_i(x, y) = (f_i(x), g_i(y))$*, and assume that*

$$c_i \mathcal{D}\big((x, y), (x', y')\big) \leq \mathcal{D}\big(h_i(x, y), h_i(x', y')\big) \leq C_i \mathcal{D}\big((x, y), (x', y')\big),$$

where $(x, y), (x', y') \in X \times Y$ *and* $0 < c_i \leq C_i < 1$ *for all* $i \in \{1, 2, \dots, N\}$*. Then* $r \leq \dim_H(A_{\mathcal{H}}) \leq R$*, where* r *and* R *are given by* $\sum_{i=1}^{N} c_i^r = 1$ *and* $\sum_{i=1}^{N} C_i^R = 1$*, respectively.*

Proof. For the upper bound of $\dim_H(A_{\mathcal{H}})$, follow Proposition 9.6 in [7]. For the lower bound of $\dim_H(A_{\mathcal{H}})$, we proceed as follows.

Since the IFS $\mathcal{H}$ satisfies SOSC, there exists an open set V of $X \times Y$ such that

$$\bigcup_{i=1}^{N} h_i(V) \subset V, \;\; V \cap A_{\mathcal{H}} \neq \emptyset, h_i(V) \cap h_j(V) = \emptyset \text{ for all } i \neq j, 1 \leq i, j \leq N.$$

Since $V \cap A_{\mathcal{H}} \neq \emptyset$, therefore there exists an index $\omega \in \{1, 2, \dots, N\}^*$ such that $h_\omega(A_{\mathcal{H}}) \subset V$. We denote $h_\omega(A_{\mathcal{H}})$ by $(A_{\mathcal{H}})_\omega$ for any $\omega \in \{1, 2, \dots, N\}^*$. Now, by using the condition $h_i(V) \cap h_j(V) = \emptyset \;\forall\, i \neq j, \; 1 \leq i, j \leq N$ and $h_\omega(A_{\mathcal{H}}) \subset V$, it is clear that for each $n \in \mathbb{N}$, the sets $\{(A_{\mathcal{H}})_{i\omega} : i \in \{1, 2, \dots, N\}^n\}$ are pairwise disjoint. We define an IFS $\mathcal{L}_n = \{h_{i\omega} : i \in \{1, 2, \dots, N\}^n\}$. Let A_n^* be the attractor of IFS $\mathcal{L}_n$. By analysing the code space of IFS $\mathcal{L}_n$ and $\mathcal{H}$, we deduce that $A_n^* \subset A_{\mathcal{H}}$. This further yields that the IFS $\mathcal{L}_n = \{h_{i\omega} : i \in \{1, 2, \dots, N\}^n\}$ satisfies SSC. Thus, the IFS $\mathcal{L}_n$ fulfills all the assumptions of Proposition 9.7 in [7]. Hence, by Proposition 9.7 in [7], we obtain that $r_n \leq \dim_H(A_n^*)$, where r_n is given by $\sum_{i \in \{1,2,\dots,N\}^n} c_{i\omega}^{r_n} = 1$. Then, $r_n \leq \dim_H(A_n^*) \leq \dim_H(A_{\mathcal{H}})$ because $A_n^* \subset A_{\mathcal{H}}$. Suppose that $\dim_H(A_{\mathcal{H}}) < r$. This implies that $r_n < r$. Let $c_{\max} = \max\{c_1, c_2, \dots, c_N\}$. Then, we have

$$c_\omega^{-r_n} = \sum_{i \in \{1,2,\dots,N\}^n} c_i^{r_n} \geq \sum_{i \in \{1,2,\dots,N\}^n} c_i^r c_i^{\dim_H(A_{\mathcal{H}}) - r} \geq \sum_{i \in \{1,2,\dots,N\}^n} c_i^r c_{\max}^{n(\dim_H(A_{\mathcal{H}}) - r)}$$

This implies that

$$c_\omega^{-r} \geq c_{\max}^{n(\dim_H(A_{\mathcal{H}}) - r)}.$$

We have a contradiction for large values of $n \in \mathbb{N}$. Therefore, we get $\dim_H(A_{\mathcal{H}}) \geq r$, proving the assertion. ∎

Remark 3.9. We may compare the above result with the work of Edgar and Golds [6], wherein the authors determined the upper and the lower bounds of the Hausdorff dimension of the graph directed bi-Lipschitz IFS using the Perron–Frobenius theory. We have obtained the upper and the lower bounds of the Hausdorff dimension for the bi-Lipschitz IFS using Moran–Hutchinson technique. We note that our result can be obtained as a special case of Edgar and Golds [6]. However, our technique is different and the proof is simpler. For determining the Hausdorff dimension using the Perron–Frobenius theory in more general setting, one can see [13].

In the next result, we estimate bounds of the Hausdorff dimension of the graph of the fractal transformation by using the previous theorem.

Proposition 3.10. *Let $\mathcal{F}=\{(X,\rho_1); f_1, f_2, \dots, f_N\}$ and $\mathcal{G}=\{(Y,\rho_2); g_1, g_2, \dots, g_N\}$ be two IFSs such that f_i and g_i are bi-Lipschitz mappings as follow!*

$$c_i\rho_1(x,x') \le \rho_1\big(f_i(x), f_i(x')\big) \le c_i'\rho_1(x,x'),$$

$$r_i\rho_2(y,y') \le \rho_2\big(g_i(y), g_i(y')\big) \le r_i'\rho_2(y,y'),$$

where $c_i, c_i', r_i, r_i' \in \mathbb{R}$, $0 < c_i \le c_i' < 1$ and $0 < r_i \le r_i' < 1$. Assume that $\mathcal{F}$ satisfies SOSC and $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$. Then $s_1 \le \dim_H G(T_{\mathcal{F}\mathcal{G}}) \le s_2$, where s_1 and s_2 are uniquely determined by $\sum_{i=1}^{N} \min\{c_i, r_i\}^{s_1} = 1$ and $\sum_{i=1}^{N} \max\{c_i', r_i'\}^{s_2} = 1$, respectively.

Proof. Since $\mathcal{F}$ satisfies SOSC and $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$, by [19, Theorem 6.8], the graph of the fractal transformation $T_{\mathcal{F}\mathcal{G}}$ is the same as the attractor of the iterated function system $\mathcal{H} = \{X \times Y; h_1, h_2, \dots, h_N\}$, where $h_i(x,y) = (f_i(x), g_i(y))$ for all indices $i \in \{1, 2, \dots, N\}$. In the light of part (2) of Lemma 3.3, part (3) of Lemma 3.2 and Theorem 3.8, we get our required result. ∎

Corollary 3.11. *Let $\mathcal{F} = \{(X,\rho_1); f_1, f_2, \dots, f_N\}$ and $\mathcal{G} = \{(Y,\rho_2); g_1, g_2, \dots, g_N\}$ be two IFSs such that f_i and g_i are similarity transformation as follows*

$$\rho_1(f_i(x), f_i(x')) = c_i\rho_1(x,x'), \quad \rho_2(g_i(y), g_i(y')) = c_i\rho_2(y,y'),$$

where $c_i \in (0,1)$. If $\mathcal{F}$ satisfies SOSC and $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$, then $\dim_H G(T_{\mathcal{F}\mathcal{G}}) = s_0$, where s_0 is uniquely determined by $\sum_{i=1}^{N} {c_i}^{s_0} = 1$.

Proof. This is a direct consequence of Proposition 3.10. ∎

Next, we discuss some aspects of the invariant measure corresponding to an IFS. Consider an IFS $\mathcal{I} = \{[0,1]; f_1, f_2\}$, where f_1 and f_2 are defined as follows

$$f_1(x) = \frac{x}{2}, \quad f_2(x) = \frac{x}{2} + \frac{1}{2}.$$

Then, $[0,1]$ is the attractor of IFS $\mathcal{I}$. If we take two probability vectors $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $q = \left(\frac{1}{3}, \frac{2}{3}\right)$, then the invariant measures μ_p and μ_q corresponding to IFS $\mathcal{I}$ with probability vectors p and q, respectively, satisfy

$$\mu_p = \frac{1}{2}\mu_p \circ f_1^{-1} + \frac{1}{2}\mu_p \circ f_2^{-1}, \quad \mu_p[0,1] = 1,$$

$$\mu_q = \frac{1}{3}\mu_q \circ f_1^{-1} + \frac{2}{3}\mu_q \circ f_2^{-1}, \quad \mu_q[0,1] = 1.$$

Since IFS $\mathcal{I}$ satisfies the OSC and $\partial[0,1] = \{0,1\}$, by using [1, Theorem 2.2], we get $\mu_p(\{0,1\}) = \mu_q(\{0,1\}) = 0$. Therefore,

$$\mu_p\Big[0,\frac{1}{2}\Big] = \frac{1}{2}\mu_p \circ f_1^{-1}\Big[0,\frac{1}{2}\Big] + \frac{1}{2}\mu_p \circ f_2^{-1}\Big[0,\frac{1}{2}\Big] = \frac{1}{2}\mu_p[0,1] + \frac{1}{2}\mu_p\big(\{0\}\big) = \frac{1}{2}.$$

This implies that $\mu_p(\{0,1\}) = \mathcal{L}^1_{[0,1]}\big[0,\frac{1}{2}\big]$. In fact, it is not difficult to show that $\mu_p = \mathcal{L}^1_{[0,1]}$, where $\mathcal{L}^1_{[0,1]}$ is the normalized Lebesgue measure on $[0,1]$. And,

$$\mu_q\Big[0,\frac{1}{2}\Big] = \frac{1}{3}\mu_q \circ f_1^{-1}\Big[0,\frac{1}{2}\Big] + \frac{2}{3}\mu_q \circ f_2^{-1}\Big[0,\frac{1}{2}\Big] = \frac{1}{3}\mu_q[0,1] + \frac{2}{3}\mu_q(\{0\}) = \frac{1}{3}.$$

This implies that $\mu_q\big[0,\frac{1}{2}\big] \neq \mathcal{L}^1_{[0,1]}\big[0,\frac{1}{2}\big]$. So, for different probability vectors, we get different invariant Borel probability measures supported on the attractor of the IFS.

In the following theorem, we determine a relation between the invariant measures of the IFS $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$.

Theorem 3.12. *Let $\mathcal{F} = \{(X,\rho_1); f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{(Y,\rho_2); g_1, g_2, \ldots, g_N\}$ be two IFSs with a probability vector $(p_1, p_2, \ldots, p_N)$. Define the WIFS*

$$\mathcal{F} \times \mathcal{G} := \{X \times Y; \Psi_{ij}; p_i p_j : 1 \leq i, j \leq N\},$$

where $\Psi_{ij}(x,y) = (f_i(x), g_j(y))$. We denote the invariant measures by $\mu_{\mathcal{F}}$, $\mu_{\mathcal{G}}$ and $\mu_{\mathcal{F}\times\mathcal{G}}$ associated with IFSs $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$, respectively. We have the following

$$\mu_{\mathcal{F}\times\mathcal{G}} = \mu_{\mathcal{F}} \times \mu_{\mathcal{G}},$$

where $(\mu_{\mathcal{F}} \times \mu_{\mathcal{G}})(A \times B) = \mu_{\mathcal{F}}(A)\mu_{\mathcal{G}}(B)$.

Proof. With $p_{ij} = p_i p_j$, we have

$$\mu_{\mathcal{F}} = \sum_{i=1}^{N} p_i \mu_{\mathcal{F}} \circ f_i^{-1}, \mu_{\mathcal{G}} = \sum_{i=1}^{N} p_i \mu_{\mathcal{G}} \circ g_i^{-1} \text{ and } \mu_{\mathcal{F}\times\mathcal{G}} = \sum_{i=1,j=1}^{N} p_{ij} \mu_{\mathcal{F}\times\mathcal{G}} \circ \Psi_{ij}^{-1}.$$

Now,

$$\begin{aligned}\mu_{\mathcal{F}} \times \mu_{\mathcal{G}} &= \sum_{i=1}^{N} p_i \mu_{\mathcal{F}} \circ f_i^{-1} \times \sum_{j=1}^{N} p_j \mu_{\mathcal{G}} \circ g_j^{-1} \\ &= \sum_{i=1,j=1}^{N} p_{ij} \mu_{\mathcal{F}} \circ f_i^{-1} \times \mu_{\mathcal{G}} \circ g_j^{-1} \\ &= \sum_{i=1,j=1}^{N} p_{ij} (\mu_{\mathcal{F}} \times \mu_{\mathcal{G}})(f_i^{-1}, g_j^{-1}) \\ &= \sum_{i=1,j=1}^{N} p_{ij} (\mu_{\mathcal{F}} \times \mu_{\mathcal{G}}) \circ \Psi_{ij}^{-1}.\end{aligned}$$

Since $\mu_{\mathcal{F}\times\mathcal{G}}$ is the unique measure satisfying the equation

$$\mu_{\mathcal{F}\times\mathcal{G}} = \sum_{i=1,j=1}^{N} p_{ij} \mu_{\mathcal{F}\times\mathcal{G}} \circ \Psi_{ij}^{-1},$$

the above equation yields that $\mu_{\mathcal{F}\times\mathcal{G}} = \mu_{\mathcal{F}} \times \mu_{\mathcal{G}}$. Thus, the proof of the theorem is complete. ■

In the next result, we obtain a relationship between the quantization dimension of the invariant measures of the IFSs $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$.

Theorem 3.13. *Let* $\mathcal{F} = \{\mathbb{R}^d\ ; f_1, f_2, \dots, f_N\}$ *and* $\mathcal{G} = \{\mathbb{R}^d\ ; g_1, g_2, \dots, g_N\}$ *be two IFSs such that*

$$\|f_i(x) - f_i(x')\| = c\ \|x - x'\| \ \text{ and } \ \|g_i(y) - g_i(y')\| = c\ \|y - y'\|$$

for all $i \in \{1, 2, \dots, N\}$*, where* $x, x' \in X$*,* $y, y' \in Y$ *and* $0 < c < 1$*. Also, assume that* $(p_1, p_2, \dots, p_N)$*, and* $(q_1, q_2, \dots, q_N)$ *are the probability vectors corresponding to IFS* $\mathcal{F}$ *and* $\mathcal{G}$*, respectively. Define the WIFS*

$$\mathcal{F} \times \mathcal{G} := \{\mathbb{R}^d \times \mathbb{R}^d\ ; \Psi_{ij}; p_i q_j : 1 \le i, j \le N\},$$

where $\Psi_{ij}(x, y) = (f_i(x), g_j(y))$*. We denote the invariant measures by* $\mu_{\mathcal{F}}, \mu_{\mathcal{G}}$ *and* $\mu_{\mathcal{F}\times\mathcal{G}}$ *associated with IFSs* $\mathcal{F}, \mathcal{G}$ *and* $\mathcal{F} \times \mathcal{G}$*, respectively. If* $\mathcal{F}$ *and* $\mathcal{G}$ *satisfy OSC, then*

$$D_r > \max\{{D_r}^*, D_r'\},$$

where ${D_r}^*$*,* D_r' *and* D_r *denotes the quantization dimension of order* $0 < r < \infty$ *of* $\mu_{\mathcal{F}}, \mu_{\mathcal{G}}$ *and* $\mu_{\mathcal{F}\times\mathcal{G}}$*, respectively.*

Proof. Since the IFSs $\mathcal{F}$ and $\mathcal{G}$ satisfy OSC and each f_i and g_i are similarity maps with similarity constant c, $D_r{}^*$ and D_r' are uniquely determined by

$$\sum_{i=1}^{N}(p_i c^r)^{\frac{D_r{}^*}{r+D_r{}^*}} = 1 \tag{3.3}$$

and

$$\sum_{i=1}^{N}(q_i c^r)^{\frac{D_r'}{r+D_r'}} = 1. \tag{3.4}$$

Since each f_i and g_i are similarity maps with similarity constant c, we have

$$\begin{aligned}\|\Psi_{ij}(x,y) - \Psi_{ij}(x',y')\| &= \|(f_i(x), g_j(y)) - (f_i(x'), g_j(y'))\| \\ &= \sqrt{\|f_i(x) - f_i(x')\|^2 + \|g_i(y) - g_i(y')\|^2} \\ &= c\,\sqrt{\|x - x'\|^2 + \|y - y'\|^2} \\ &= c\|(x,y) - (x',y')\|.\end{aligned}$$

Thus, Ψ_{ij} is a similarity map with similarity constant c. Since the IFSs $\mathcal{F}$ and $\mathcal{G}$ satisfy OSC, there exist open sets U and V such that

$$\bigcup_{i=1}^{N} f_i(U) \subset U, \quad f_i(U) \cap f_j(U) = \emptyset \ \text{ for all } i \neq j,$$

and

$$\bigcup_{i=1}^{N} g_i(V) \subset V, \quad g_i(V) \cap g_j(V) = \emptyset \ \text{ for all } i \neq j.$$

This further yields,

$$\Psi_{ij}(U \times V) = f_i(U) \times g_j(V) \subset U \times V,$$

$$\Psi_{ij}(U \times V) \cap \Psi_{i'j'}(U \times V) = (f_i(U) \times g_j(V)) \cap (f_{i'}(U) \times g_{j'}(V)) = \emptyset$$

for all $(i,j) \neq (i',j')$. Therefore, the IFS $\mathcal{F} \times \mathcal{G}$ satisfies OSC. The probability $p_i q_j$ corresponds to similarity Ψ_{ij}. So, the quantization dimension D_r of order $0 < r < \infty$ of invariant measure $\mu_{\mathcal{F}\times\mathcal{G}}$ is given by

$$\sum_{i=1,j=1}^{N} (p_i q_j c^r)^{\frac{D_r}{r+D_r}} = 1.$$

This implies that

$$\sum_{i=1}^{N}(p_i c^r)^{\frac{D_r}{r+D_r}} \sum_{j=1}^{N}(q_j)^{\frac{D_r}{r+D_r}} = 1.$$

Since $\sum_{j=1}^{N}(q_j)^{\frac{D_r}{r+D_r}} > \sum_{i=1}^{N} q_j = 1$, we get

$$\sum_{i=1}^{N}(p_i c^r)^{\frac{D_r}{r+D_r}} < 1.$$

Since $t \to \sum_{i=1}^{N}(p_i c^r)^t$ is a strictly decreasing continuous function, by the above and equation (3.3), we get

$$\frac{D_r}{r+D_r} > \frac{{D_r}^*}{r+{D_r}^*}$$
$$rD_r + D_r{D_r}^* > r{D_r}^* + D_r{D_r}^*$$
$$D_r > {D_r}^*.$$

Similarly, we can also prove that $D_r > D'_r$. Therefore,

$$D_r > \max\{{D_r}^*, D'_r\}.$$

This completes the proof. ∎

Remark 3.14. Let $\mathcal{F}$ and $\mathcal{G}$ be two IFSs, $\mathcal{F} = \{\mathbb{R}; f_1(x) = \frac{x}{2}, f_2(x) = \frac{x}{2} + \frac{1}{2}\}$ and $\mathcal{G} = \{\mathbb{R}; g_1(x) = \frac{x}{2}, g_2(x) = \frac{x}{2} + \frac{1}{2}\}$, with the same probability vector $(\frac{1}{2}, \frac{1}{2})$. Define the IFS $\mathcal{H} := \{\mathbb{R}^2; h_1, h_2\}$, where $h_i(x, y) = (f_i(x), g_i(y))$. In this case, we have

(1) $A_{\mathcal{F}} = A_{\mathcal{G}} = [0, 1]$ and $A_{\mathcal{H}} = \{(x, x) : x \in [0, 1]\}$.

(2) $(\mu_{\mathcal{F}} \times \mu_{\mathcal{G}})(A_{\mathcal{H}}) = 0$.

By the above remark, it is clear that we may not obtain the invariant measure of sub IFS by restricting the invariant measure of the super IFS.

Next, we give an example of a measure supported on a countable set and also compute its quantization dimension.

Example 3.15. Let $E = \{x_m : x_m = \frac{1}{2}(\frac{1}{m} + \frac{1}{m+1})$ for all $m \in \mathbb{N}\}$ and let μ be the measure supported on E such that $\mu(x_m) = \frac{\gamma}{m^2}$, where $\gamma = (\sum_{m=1}^{\infty} m^{-2})^{-1}$. We show that

$$D_r(\mu) = \frac{1}{2 + \frac{1}{r}}.$$

Let A be a subset of $\mathbb{R}$ with $\text{Card}(A) = n$. We define a set

$$\Delta = \Big\{m \in \mathbb{N} : A \cap \Big[\frac{1}{m+1}, \frac{1}{m}\Big] = \emptyset\Big\}.$$

Then, for each $m \in \Delta$, we have

$$\min_{a \in A} |x_m - a|^r \geq \left(\frac{\frac{1}{m} - \frac{1}{m+1}}{2}\right)^r = \frac{1}{2^r m^r (m+1)^r}.$$

So that,

$$\begin{aligned}\int \min_{a\in A}|x-a|^r\,d\mu(x) &\geq \sum_{m\in\Delta}\min_{a\in A}|x_m-a|^r\frac{\gamma}{m^2}\\ &\geq \sum_{m\in\Delta}\frac{1}{2^r m^r(m+1)^r}\frac{\gamma}{m^2}\\ &\geq \frac{\gamma}{2^r}\sum_{m=n+1}^{\infty}\frac{1}{(m+1)^{2r+2}}\\ &\geq \frac{\gamma}{2^r}\int_{n+1}^{\infty}\frac{1}{(x+1)^{2r+2}}dx\\ &= \frac{\gamma}{2^r}\frac{(n+1)^{-2r-1}}{(2r+1)}.\end{aligned}$$

From the above, we deduce that $V_{n,r}(\mu)\geq \gamma_1(n+1)^{-2r-1}$, where $\gamma_1=\frac{\gamma}{2^r(2r+1)}$. Since $e_{n,r}(\mu)=V_{n,r}^{\frac{1}{r}}\mu$, we get

$$e_{n,r}(\mu)\geq \gamma_1{}^{\frac{1}{r}}(n+1)^{-2-\frac{1}{r}}.$$

From the above inequality, we conclude that

$$\underline{D}_r(\mu)=\liminf_{n\to\infty}\frac{\log n}{-\log e_{n,r}(\mu)}\geq \lim_{n\to\infty}\frac{\log n}{-\frac{1}{r}\log\gamma_1+(2+\frac{1}{r})\log(n+1)}=\frac{1}{2+\frac{1}{r}}.$$

Using the same technique of [7, Example 3.5], it is not difficult to compute that $\dim_B(E)=\frac{1}{2}$. By using Lemma 1 in [21], for any real number $t>\dim_B(E)$, we can choose a set $A_n\subset\mathbb{R}$ with $\mathrm{Card}(A_n)=n$ such that

$$\min_{a\in A_n}|x-a|^r\leq(\gamma_2)^r n^{\frac{-r}{t}}\ \text{for all } x\in E,$$

where γ_2 is some constant. We define a set $B_{2n}=\{x_m:1\leq m\leq n\}\cup A_n$. Thus, by using the above inequality and the definition of $V_{2n,r}(\mu)$, we have

$$\begin{aligned}V_{2n,r}(\mu) &\leq \int\min_{b\in B_{2n}}|x-b|^r\,d\mu(x)\\ &= \sum_{m=1}^{\infty}\min_{b\in B_{2n}}|x_m-b|^r\frac{\gamma}{m^2}\\ &\leq \gamma(\gamma_2)^r n^{\frac{-r}{t}}\sum_{m=n+1}^{\infty}\frac{1}{m^2}\\ &\leq \gamma(\gamma_2)^r n^{\frac{-r}{t}}\int_n^{\infty}\frac{1}{x^2}dx\\ &= \gamma_0\, n^{\frac{-r}{t}-1},\end{aligned}$$

where $\gamma_0 = \gamma(\gamma_2)^r$. By the above inequality and the definition of $e_{2n,r}(\mu)$, we deduce that

$$e_{2n,r}(\mu) \leq \gamma_0^{\frac{1}{r}} n^{\frac{-1}{t}-\frac{1}{r}}.$$

Therefore, by the above inequality, we determine that

$$\overline{D}_r(\mu) = \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)} \leq \lim_{n\to\infty} \frac{\log 2n}{-\frac{1}{r}\log\gamma_0 + (\frac{1}{t}+\frac{1}{r})\log n} = \frac{1}{\frac{1}{t}+\frac{1}{r}}.$$

The above inequality holds for any $t > \frac{1}{2}$. Therefore, $\overline{D}_r(\mu) \leq \frac{1}{2+\frac{1}{r}}$. This proves our claim.

4. Bounds on the quantization dimension of the invariant probability measure supported on the attractor of a bi-Lipschitz WIFS

In this section, we establish bounds for the quantization dimension of the invariant Borel probability measure supported on the attractor of a bi-Lipschitz IFS. We should admit that we use the similar technique of Graf and Luschgy [8] to determine the upper bound. However, we use a different technique for the lower bound.

Firstly, we give some lemmas and propositions for determining an upper bound of the quantization dimension for the invariant Borel probability measure supported on the attractor of a bi-Lipschitz WIFS.

Lemma 4.1. *Let μ be a Borel probability measure on $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz map such that $s\|x-y\| \leq \|f(x)-f(y)\| \leq c\|x-y\|$, where $x, y \in \mathbb{R}^d$ and $s, c \in (0,1)$. Then*

$$V_{n,r}(\mu \circ f^{-1}) \leq c^r V_{n,r}(\mu).$$

Proof. Let $\nu = \mu \circ f^{-1}$. It can be easily observed that ν is a Borel probability measure on $\mathbb{R}^d$. We begin with the definition of $V_{n,r}(\nu)$. We have

$$\begin{aligned}
V_{n,r}(\nu) &= \inf\Big\{\int \min_{a\in A} \|x-a\|^r d\nu(x) : A \subset \mathbb{R}^d, \text{ Card}(A) \leq n\Big\} \\
&= \inf\Big\{\int \min_{a\in A} \|x-a\|^r d(\mu\circ f^{-1})(x) : A \subset \mathbb{R}^d, \text{ Card}(A) \leq n\Big\} \\
&= \inf\Big\{\int \min_{a\in A} \|f(y)-a\|^r d\mu(y) : A \subset \mathbb{R}^d, \text{ Card}(A) \leq n\Big\} \\
&\leq \inf\Big\{\int \min_{f(b)\in f(B)} \|f(y)-f(b)\|^r d\mu(y) : B \subset \mathbb{R}^d, \text{ Card}(B) \leq n\Big\} \\
&\leq c^r \inf\Big\{\int \min_{b\in B} \|y-b\|^r d\mu(y) : B \subset \mathbb{R}^d, \text{ Card}(B) \leq n\Big\} \\
&= c^r V_{n,r}(\mu).
\end{aligned}$$

Thus, the proof of the lemma is established. ∎

Lemma 4.2. *Let μ_i be a Borel probability measure on $\mathbb{R}^d$ for $i = 1, 2, \ldots, K$. We define $\mu = \sum_{i=1}^{K} s_i \mu_i$, where $s_i \geq 0$ and $\sum_{i=1}^{K} s_i = 1$. If $\sum_{i=1}^{K} n_i \leq n$, where $n_i \in \mathbb{N}$ and $\int \|x\|^r d\mu_i(x) < \infty$ for $i = 1, 2, \ldots, K$. Then*

$$V_{n,r}(\mu) \leq \sum_{i=1}^{K} s_i V_{n_i,r}(\mu_i).$$

Proof. Since $\int \|x\|^r d\mu_i(x) < \infty$ for $i = 1, 2, \ldots, K$, we get n_i-optimal set $A_{n_i} \subset \mathbb{R}^d$ for measure μ_i with $\sum_{i=1}^{K} n_i \leq n$. Let $A = \cup_{i=1}^{K} A_{n_i}$. Then Card$(A) \leq n$. By the definition of $V_{n,r}(\mu)$, we have

$$\begin{aligned} V_{n,r}(\mu) &\leq \int \min_{a \in A} \|x - a\|^r d\mu(x) \\ &= \sum_{i=1}^{K} s_i \int \min_{a \in A} \|x - a\|^r d\mu_i(x) \\ &\leq \sum_{i=1}^{K} s_i \int \min_{a \in A_{n_i}} \|x - a\|^r d\mu_i(x) \\ &= \sum_{i=1}^{K} s_i V_{n_i,r}(\mu_i). \end{aligned}$$

This completes the proof. ∎

Lemma 4.3. *Let $\{\mathbb{R}^d; f_1, f_2, \ldots, f_N\}$ be an IFS such that each f_i satisfies*

$$s_i \|x - y\| \leq \|f_i(x) - f_i(y)\| \leq c_i \|x - y\|,$$

where $x, y \in \mathbb{R}^d$ and $0 < s_i \leq c_i < 1$. Also, assume that $(p_1, p_2, \ldots, p_N)$ be a probability vector corresponding to the IFS. Let $r \in (0, \infty)$ be a fixed number. Then there exists a unique $l_r \in (0, \infty)$, satisfying

$$\sum_{i=1}^{N} (p_i {c_i}^r)^{\frac{l_r}{r + l_r}} = 1.$$

Proof. We define a function $F : [0, \infty) \to \mathbb{R}$ by

$$F(t) = \sum_{i=1}^{N} (p_i {c_i}^r)^t.$$

We see that F is a strictly decreasing continuous function as $0 < p_i, c_i < 1$ and $r \in (0, \infty)$. Furthermore,

$$F(0) = N \geq 2 \text{ and } F(1) = \sum_{i=1}^{N} p_i c_i^r < \sum_{i=1}^{N} p_i = 1.$$

Therefore, by the intermediate value theorem and the strictly decreasing property of F, there exists a unique number $t_0 \in (0, 1)$ such that $F(t_0) = \sum_{i=1}^{N} (p_i {c_i}^r)^{t_0} = 1$. Since $t_0 \in (0, 1)$, there is a unique $l_r \in (0, \infty)$ such that $\frac{l_r}{r+l_r} = t_0$. Hence, there is a unique number $l_r \in (0, \infty)$ such that $\sum_{i=1}^{N} (p_i {c_i}^r)^{\frac{l_r}{r+l_r}} = 1$. This completes the proof. ∎

Proposition 4.4. *Let $\mathcal{W} = \{\mathbb{R}^d;\ f_1, f_2, \ldots, f_N;\ \ p_1, p_2, \ldots, p_N\}$ be a weighted IFS (WIFS) such that each f_i satisfies $s_i \|x - y\| \leq \|f_i(x) - f_i(y)\| \leq c_i \|x - y\|$, where $x, y \in \mathbb{R}^d$ and $0 < s_i \leq c_i < 1$. Let μ be the invariant Borel probability measure corresponding to the WIFS $\mathcal{W}$. Then, for every $n \in \mathbb{N}$ and $r \in (0, \infty)$*

$$V_{n,r}(\mu) \leq \min\left\{ \sum_{i=1}^{N} p_i {c_i}^r V_{n_i,r}(\mu) : n_i \in \mathbb{N}, \sum_{i=1}^{N} n_i \leq n \right\}.$$

Proof. Since μ is the invariant measure corresponding to the WIFS $\mathcal{W}$,

$$\mu = \sum_{i=1}^{N} p_i \mu \circ {f_i}^{-1}.$$

Using Lemma 4.1, for each $n \in \mathbb{N}$, we get

$$V_{n,r}(\mu \circ {f_i}^{-1}) \leq c_i^r V_{n,r}(\mu) \text{ for all } i = 1, 2, \ldots, N.$$

Since $\int \|x\|^r d\mu(x) < \infty$, $\int \|x\|^r d(\mu \circ {f_i}^{-1})(x) < \infty$. Thus, in view of Lemma 4.2 and by the above inequality, we get

$$V_{n,r}(\mu) \leq \min\left\{ \sum_{i=1}^{N} p_i {c_i}^r V_{n_i,r}(\mu) : n_i \in \mathbb{N}, \sum_{i=1}^{N} n_i \leq n \right\}.$$

Thus, the proof is complete. ∎

Corollary 4.5. *Let $\mathcal{W} = \{\mathbb{R}^d;\ f_1, f_2, \ldots, f_N; p_1, p_2, \ldots, p_N\}$ be a WIFS such that each f_i satisfies $s_i \|x - y\| \leq \|f_i(x) - f_i(y)\| \leq c_i \|x - y\|$, where $x, y \in \mathbb{R}^d$ and $0 < s_i \leq c_i < 1$ and each $p_i > 0$. Let μ be the invariant Borel probability measure corresponding to the WIFS $\mathcal{W}$. Let $\Gamma \subset \{1, 2, \ldots, N\}^*$ be a finite maximal antichain. Then, for any $n \in \mathbb{N}$ with $n \geq |\Gamma|$, $\sigma \in \Gamma$ and $r \in (0, \infty)$*

$$V_{n,r}(\mu) \leq \min\left\{ \sum_{\sigma \in \Gamma} p_\sigma c_\sigma^r V_{n_\sigma,r}(\mu) : n_\sigma \in \mathbb{N}, \sum_{\sigma \in \Gamma} n_\sigma \leq n \right\}.$$

Proof. Since Γ is a finite maximal antichain, by Lemma 2.15 measure μ satisfies

$$\mu = \sum_{\sigma \in \Gamma} p_\sigma \mu \circ f_\sigma{}^{-1}.$$

The rest of the proof of this corollary follows from the proof of Proposition 4.4. ∎

In the upcoming theorem, we give an upper bound of the quantization dimension for the invariant Borel probability measure supported on the attractor of a bi-Lipschitz WIFS without any separation condition on the IFS.

Theorem 4.6. *Let $\mathcal{W} = \{\mathbb{R}^d;\ f_1, f_2, \dots, f_N;\ \ p_1, p_2, \dots, p_N\}$ be a WIFS such that each f_i satisfies $s_i\|x - y\| \leq \|f_i(x) - f_i(y)\| \leq c_i\|x - y\|$, where $x, y \in \mathbb{R}^d$ and $0 < s_i \leq c_i < 1$ and each $p_i > 0$. Let μ be the invariant Borel probability measure corresponding to the WIFS $\mathcal{W}$. Let $r \in (0, \infty)$ and $l_r \in (0, \infty)$ be the unique number such that $\sum_{i=1}^N (p_i c_i{}^r)^{\frac{l_r}{r+l_r}} = 1$. Then*

$$\limsup_{n\to\infty} n \cdot e_{n,r}^{l_r}(\mu) < \infty,$$

where $e_{n,r}(\mu) = V_{n,r}(\mu)^{\frac{1}{r}}$. Moreover, $\overline{D_r}(\mu) \leq l_r$, where $\overline{D_r}(\mu)$ denotes the upper quantization dimension of order r of μ.

Proof. Let $\xi_i = (p_i c_i{}^r)^{\frac{l_r}{r+l_r}}$. Notice that each $\xi_i > 0$ and $\sum_{i=1}^N \xi_i = 1$. Therefore, $(\xi_1, \xi_2, \dots, \xi_N)$ is a probability vector. Let $\xi_{min} = \min\{\xi_1, \xi_2, \dots, \xi_N\}$. Then, we have $\xi_{min} > 0$. Let $m_0 \in \mathbb{N}$ be fixed. Choose $n \in \mathbb{N}$ with the property $\frac{m_0}{n} < \xi_{min}^2$, this property holds for all but finitely many values of $n \in \mathbb{N}$. Set $\epsilon = \xi_{min}^{-1} \frac{m_0}{n}$. We define a set

$$\Gamma_\epsilon = \left\{\sigma \in \{1, 2, \dots, N\}^*;\ \ \xi_{\sigma^-} \geq \epsilon > \xi_\sigma\right\}.$$

Lemma 2.16 yields that Γ_ϵ is a finite maximal antichain. Therefore, in the light of Lemma 2.15, we obtain

$$1 = \sum_{\sigma\in\Gamma_\epsilon} \xi_\sigma = \sum_{\sigma\in\Gamma_\epsilon} \xi_{\sigma^-} \cdot \xi_{\sigma_{|\sigma|}} \geq \sum_{\sigma\in\Gamma_\epsilon} \epsilon \cdot \xi_{\sigma_{|\sigma|}} \geq \sum_{\sigma\in\Gamma_\epsilon} \epsilon \cdot \xi_{min} = \epsilon \cdot \xi_{min} \cdot |\Gamma_\epsilon|.$$

Therefore, by the above, we get $|\Gamma_\epsilon| \leq (\epsilon \cdot \xi_{min})^{-1} = \frac{n}{m_0}$. This can be also written as $\sum_{\sigma\in\Gamma_\epsilon} m_0 \leq n$. So, by using Corollary 4.5, we get the following inequality

$$\begin{aligned} V_{n,r}(\mu) &\leq \sum_{\sigma\in\Gamma_\epsilon} p_\sigma c_\sigma^r V_{m_0,r}(\mu) \\ &= \sum_{\sigma\in\Gamma_\epsilon} (p_\sigma c_\sigma^r)^{\frac{l_r}{r+l_r}} (p_\sigma c_\sigma^r)^{\frac{r}{r+l_r}} V_{m_0,r}(\mu) \\ &= \sum_{\sigma\in\Gamma_\epsilon} \xi_\sigma (p_\sigma c_\sigma^r)^{\frac{r}{r+l_r}} V_{m_0,r}(\mu). \end{aligned}$$

Since $\epsilon > \xi_\sigma = (p_\sigma {c_\sigma}^r)^{\frac{l_r}{r+l_r}}$ for $\sigma \in \Gamma_\epsilon$, by the above inequality, we conclude that

$$\begin{aligned} V_{n,r}(\mu) &\leq \sum_{\sigma\in\Gamma_\epsilon} \xi_\sigma \left(\epsilon^{\frac{r+l_r}{l_r}}\right)^{\frac{r}{r+l_r}} V_{m_0,r}(\mu) \\ &= \sum_{\sigma\in\Gamma_\epsilon} \xi_\sigma \cdot \epsilon^{\frac{r}{l_r}} \cdot V_{m_0,r}(\mu) \\ &= \epsilon^{\frac{r}{l_r}} \cdot V_{m_0,r}(\mu) \sum_{\sigma\in\Gamma_\epsilon} \xi_\sigma. \end{aligned}$$

Since $\epsilon = {\xi_{\min}}^{-1} \frac{m_0}{n}$, we have

$$\epsilon^{\frac{r}{l_r}} = \xi_{min}^{\frac{-r}{l_r}} \left(\frac{m_0}{n}\right)^{\frac{r}{l_r}}.$$

By Lemma 2.15, we get $\sum_{\sigma\in\Gamma_\epsilon} \xi_\sigma = 1$. Thus, by the previous inequality, we obtain

$$\begin{aligned} V_{n,r}(\mu) &\leq \xi_{min}^{\frac{-r}{l_r}} \left(\frac{m_0}{n}\right)^{\frac{r}{l_r}} V_{m_0,r}(\mu) \\ n^{\frac{r}{l_r}} \cdot V_{n,r}(\mu) &\leq \xi_{min}^{\frac{-r}{l_r}} \cdot {m_0}^{\frac{r}{l_r}} \cdot V_{m_0,r}(\mu) \\ n \cdot e_{n,r}^{l_r}(\mu) &\leq \xi_{min}^{-1} \cdot m_0 \cdot e_{m_0,r}^{l_r}(\mu). \end{aligned}$$

The above inequality holds for all but finitely many values of n. Therefore, we get

$$\limsup_{n\to\infty} n \cdot e_{n,r}^{l_r}(\mu) \leq \xi_{min}^{-1} \cdot m_0 \cdot e_{m_0,r}^{l_r}(\mu) < \infty.$$

By using Proposition 2.14, we get $\overline{D}_r(\mu) \leq l_r$. Thus, the proof of the theorem is complete. ■

Next, for obtaining a lower bound of the quantization dimension of invariant Borel probability measure corresponding to a WIFS, we give some lemmas and propositions.

Lemma 4.7. *Let $\mathcal{W} = \{\mathbb{R}^d;\ f_1, f_2, \dots, f_N;\ \ p_1, p_2, \dots, p_N\}$ be a WIFS such that each f_i satisfies $s_i\|x - y\| \leq \|f_i(x) - f_i(y)\| \leq c_i\|x - y\|$, where $x, y \in \mathbb{R}^d$ and $0 < s_i \leq c_i < 1$. Let μ be the invariant Borel probability measure and $A_{\mathcal{W}}$ be the invariant attractor corresponding to the WIFS $\mathcal{W}$. Then, for every $\epsilon > 0$*

$$\inf\{\mu(B(x, \epsilon)) : x \in A_{\mathcal{W}}\} > 0,$$

where $B(x, \epsilon)$ is the open ball of radius ϵ and centre x.

Proof. It is well known that there is a unique $m \in \mathbb{R}$ such that $\sum_{i=1}^N {c_i}^m = 1$. Set $q_i = {c_i}^m$. Then $(q_1, q_2, \dots, q_N)$ is a probability vector. Let $\epsilon_0 = \left(\frac{\epsilon}{\operatorname{diam} A_{\mathcal{W}}}\right)^m$. Then

$\Gamma_{\epsilon_0} = \{\sigma \in \{1,2,\ldots,N\}^* : q_{\sigma^-} \geq \epsilon_0 > q_\sigma\}$ is a finite maximal antichain. Next, let $x \in A_{\mathcal{W}}$. Since $A_{\mathcal{W}} = \cup_{\sigma \in \Gamma_{\epsilon_0}} f_\sigma(A_{\mathcal{W}})$, there is $\sigma_0 \in \Gamma_{\epsilon_0}$ such that $x \in f_{\sigma_0}(A_{\mathcal{W}})$. Let $x_1, x_2 \in A_{\mathcal{W}}$. Then

$$\|f_{\sigma_0}(x_1) - f_{\sigma_0}(x_2)\| \leq c_{\sigma_0}\|x_1 - x_2\|.$$

From the above inequality, we can conclude that $\operatorname{diam}(f_{\sigma_0}(A_{\mathcal{W}})) \leq c_{\sigma_0} \operatorname{diam}(A_{\mathcal{W}})$. Thus, $\operatorname{diam}(f_{\sigma_0}(A_{\mathcal{W}})) \leq \epsilon$, which further yields that $f_{\sigma_0}(A_{\mathcal{W}}) \subseteq B(x,\epsilon)$. Since μ satisfies $\mu = \sum_{\sigma \in \Gamma_{\epsilon_0}} p_\sigma \mu \circ {f_\sigma}^{-1}$, we have

$$\begin{aligned} \mu\big(f_{\sigma_0}(A_{\mathcal{W}})\big) &= \sum_{\sigma \in \Gamma_{\epsilon_0}} p_\sigma \mu \circ {f_\sigma}^{-1}\big(f_{\sigma_0}(A_{\mathcal{W}})\big) \\ &\geq p_{\sigma_0}\mu \circ {f_{\sigma_0}}^{-1}\big(f_{\sigma_0}(A_{\mathcal{W}})\big) \\ &= p_{\sigma_0}. \end{aligned}$$

Let $p = \min\{p_\sigma; \sigma \in \Gamma_{\epsilon_0}\} > 0$. Then, by the above inequality

$$\mu\big(B(x,\epsilon)\big) \geq \mu\big(f_{\sigma_0}(A_{\mathcal{W}})\big) \geq p > 0.$$

This holds for arbitrary $x \in A_{\mathcal{W}}$. Therefore, we get our assertion. ∎

Lemma 4.8. *Let $\mathcal{W}$, as in the above lemma, satisfy strong open set condition and μ be the invariant measure corresponding to $\mathcal{W}$. Then, for each $m \in \mathbb{N}$, there is a WIFS $\mathcal{L}_m = \{\mathbb{R}^d; g_i; p_i \text{ for all } i \in \{1,2,\ldots,N\}^m\}$ such that*

$$S_i\|x - y\| \leq \|g_i(x) - g_i(y)\| \leq C_i\|x - y\|,$$

where $x, y \in \mathbb{R}^d$ and $0 < S_i \leq C_i < 1$ and the WIFS $\mathcal{L}_m$ satisfies SSC. Furthermore, if μ_m^* is the invariant Borel probability measure corresponding to the IFS $\mathcal{L}_m$, then there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists $n_i \in \mathbb{N}$ such that $\sum_{i \in \{1,2,\ldots,N\}^m} n_i \leq n$ and

$$V_{n,r}(\mu_m^*) \geq \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r V_{n_i,r}(\mu_m^*),$$

for any $r \in (0,\infty)$.

Proof. Since $\mathcal{W}$ satisfies the strong open set condition, there exists an open set U such that

$$\bigcup_{i=1}^N f_i(U) \subset U, \quad U \cap A_{\mathcal{W}} \neq \emptyset, \; f_i(U) \cap f_j(U) = \emptyset \; \text{ for all } i \neq j.$$

Since $U \cap A_{\mathcal{W}} \neq \emptyset$, there exists $\sigma \in \{1, 2, \ldots, N\}^*$ such that $f_\sigma(A_{\mathcal{W}}) \subseteq U$. Set $g_i = f_i \circ f_\sigma$ for $i \in \{1, 2, \ldots, N\}^m$. For $x, y \in \mathbb{R}^d$, we have

$$s_i s_\sigma \|x - y\| \leq \|g_i(x) - g_i(y)\| \leq c_i c_\sigma \|x - y\|.$$

Hence

$$S_i \|x - y\| \leq \|g_i(x) - g_i(y)\| \leq C_i \|x - y\|,$$

where $S_i = s_i s_\sigma \in (0, 1)$ and $C_i = c_i c_\sigma \in (0, 1)$. Therefore,

$$\mathcal{L}_m = \left\{\mathbb{R}^d; g_i \text{ for all } i \in \{1, 2, \ldots, N\}^m\right\}$$

is an IFS. Since $\sum_{i \in \{1,2,\ldots,N\}^m} p_{i\sigma} = p_\sigma \neq 1$, we have

$$\sum_{i \in \{1,2,\ldots,N\}^m} \frac{p_{i\sigma}}{p_\sigma} = \sum_{i \in \{1,2,\ldots,N\}^m} p_i = 1.$$

Therefore, $\mathcal{L}_m = \{\mathbb{R}^d; g_i; p_i \text{ for all } i \in \{1, 2, \ldots, N\}^m\}$ is a WIFS. Let A_m^* be the invariant attractor and let μ_m^* be the invariant Borel probability measure corresponding to WIFS $\mathcal{L}_m$. Analysing the code space of both IFS $\mathcal{W}$ and $\mathcal{L}_m$, one can easily show that $A_m^* \subseteq A_{\mathcal{W}}$. Since $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j \in \{1, 2, \ldots, N\}^m$ and $f_\sigma(A_{\mathcal{W}}) \subseteq U$, we get $g_i(A_m^*) \cap g_j(A_m^*) = \emptyset$ for all $i \neq j \in \{1, 2, \ldots, N\}^m$. Thus, the WIFS $\mathcal{L}_m = \{\mathbb{R}^d; g_i; p_i \text{ for all } i \in \{1, 2, \ldots, N\}^m\}$ satisfies strong separation condition. Set

$$\delta_0 = \min_{i \neq j \in \{1,2,\ldots,N\}^m} \left\{\min_{a \in g_i(A_m^*)} \min_{b \in g_j(A_m^*)} \|a - b\|\right\}.$$

Since the WIFS $\mathcal{L}_m$ satisfies strong separation condition, we have $\delta_0 > 0$. Lemma 4.7 yields that

$$\delta = \inf\left\{\left(\frac{\delta_0}{8}\right)^r \mu_m^*\left(B\left(x, \frac{\delta_0}{8}\right)\right) : \ x \in A_m^*\right\} > 0.$$

By using Lemma 2.11, we have $V_{n,r}(\mu_m^*) \to 0$ as $n \to \infty$. Then there is an $n_0 \in \mathbb{N}$ such that $V_{n,r}(\mu_m^*) < \delta$ for all $n \geq n_0$. Let A_n be an n-optimal set for measure μ_m^* of order r. In the light of Lemma 2.13, for all $n \geq n_0$, we have

$$\left(\frac{\|A_n\|_\infty}{2}\right)^r \min_{x \in A_m^*} \mu_m^*\left(B\left(x, \frac{\|A_n\|_\infty}{2}\right)\right) < \delta.$$

It can be easily shown that for $r \in (0, \infty)$, the function $\phi : (0, \infty) \to \mathbb{R}$ defined by $\phi(t) = t^r \min_{x \in A_m^*} \mu_m^*(B(x, t))$ is increasing. This further yields that

$$\frac{\|A_n\|_\infty}{2} < \frac{\delta_0}{8}$$

and hence

$$\|A_n\|_\infty < \frac{\delta_0}{4}$$

for all $n \geq n_0$. We define

$$A_{n_i} = \{a \in A_n;\ \ W(a|A_n) \cap g_i(A_m^*) \neq \emptyset\} \text{ for } i \in \{1, 2, \ldots, N\}^m.$$

Let n_i be the cardinality of A_{n_i}. By the definition of $\|A_n\|_\infty$ and δ_0, we conclude that for all $n \geq n_0$, $A_{n_i} \cap A_{n_j} = \emptyset$ for $i \neq j \in \{1, 2, \ldots, N\}^m$ and $\sum_{i \in \{1,2,\ldots,N\}^m} n_i \leq n$. Now, for $n \geq n_0$, we have

$$\begin{aligned}
V_{n,r}(\mu_m^*) &= \int \min_{a \in A_n} \|x - a\|^r d\mu_m^*(x) \\
&= \sum_{i \in \{1,2,\ldots,N\}^m} p_i \int \min_{a \in A_n} \|g_i(x) - a\|^r d\mu_m^*(x) \\
&= \sum_{i \in \{1,2,\ldots,N\}^m} p_i \int \min_{a \in A_{n_i}} \|g_i(x) - a\|^r d\mu_m^*(x) \\
&\geq \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r \int \min_{b \in {g_i}^{-1}(A_{n_i})} \|x - b\|^r d\mu_m^*(x) \\
&\geq \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r V_{n_i,r}(\mu_m^*).
\end{aligned}$$

This completes the result. ∎

Proposition 4.9. *Let $\mathcal{W}$ and $\mathcal{L}_m$ be two WIFSs as defined in the above lemma. For $r \in (0, \infty)$, let $k_{m,r} \in (0, \infty)$ be the unique number such that*

$$\sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^{\frac{k_{m,r}}{r + k_{m,r}}} = 1.$$

Then

$$\liminf_{n \to \infty} n \cdot e_{n,r}^{l_0}(\mu_m^*) > 0$$

for any $l_0 \in (0, k_{m,r})$. Moreover, $k_{m,r} \leq \underline{D}_r(\mu_m^*)$, where $\underline{D}_r(\mu_m^*)$ denotes the lower quantization dimension of measure μ_m^* of order r.

Proof. Using the fact that $t \mapsto \sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^t$ is a strictly decreasing function and $\sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^{\frac{k_{m,r}}{r + k_{m,r}}} = 1$, then for $0 < l_0 < k_{m,r}$, we have

$$\sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^{\frac{l_0}{r + l_0}} > \sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^{\frac{k_{m,r}}{r + k_{m,r}}} = 1.$$

Let n_0 be as in the above lemma. Let $C_* = \min\{n^{\frac{r}{l_0}} \cdot V_{n,r}(\mu_m^*); n < n_0\}$. It is easy to show that $V_{n,r}(\mu_m^*) > 0$, from which we deduce that $C_* > 0$. Our aim is to show that $n^{\frac{r}{l_0}} \cdot V_{n,r}(\mu_m^*) \geq C_*$ for all $n \in \mathbb{N}$. Clearly the inequality holds for $n < n_0$ by the definition of C_*. For $n \geq n_0$, we prove it by induction on $n \in \mathbb{N}$. Let $n \geq n_0$ and $\eta^{\frac{r}{l_0}} \cdot V_{\eta,r}(\mu_m^*) \geq C_*$ hold for all $\eta < n$. Lemma 4.8 yields that there are numbers $n_i \in \mathbb{N}$ such that $\sum_{i \in \{1,2,\ldots,N\}^m} n_i \leq n$ and

$$\begin{aligned} n^{\frac{r}{l_0}} V_{n,r}(\mu_m^*) &\geq n^{\frac{r}{l_0}} \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r V_{n_i,r}(\mu_m^*) \\ &= n^{\frac{r}{l_0}} \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r {n_i}^{\frac{-r}{l_0}} {n_i}^{\frac{r}{l_0}} V_{n_i,r}(\mu_m^*) \\ &\geq C_* n^{\frac{r}{l_0}} \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r {n_i}^{\frac{-r}{l_0}} \\ &= C_* \sum_{i \in \{1,2,\ldots,N\}^m} p_i {S_i}^r \left(\frac{n_i}{n}\right)^{\frac{-r}{l_0}}. \end{aligned}$$

Using Hölder's inequality for negative exponents, we get

$$n^{\frac{r}{l_0}} V_{n,r}(\mu_m^*) \geq C_* \Bigg(\sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^{\frac{l_0}{r+l_0}} \Bigg)^{1+\frac{r}{l_0}} \Bigg(\sum_{i \in \{1,2,\ldots,N\}^m} \left(\frac{n_i}{n}\right)^{\frac{-r}{l_0} \cdot \frac{-l_0}{r}} \Bigg)^{\frac{-r}{l_0}}.$$

Since $\sum_{i \in \{1,2,\ldots,N\}^m} (p_i {S_i}^r)^{\frac{l_0}{r+l_0}} > 1$ and $\sum_{i \in \{1,2,\ldots,N\}^m} \frac{n_i}{n} \leq 1$, we have

$$n^{\frac{r}{l_0}} V_{n,r}(\mu_m^*) \geq C_*.$$

Therefore, by induction on $n \in \mathbb{N}$, $n^{\frac{r}{l_0}} V_{n,r}(\mu_m^*) \geq C_*$ hold for all $n \in \mathbb{N}$. Thus, we deduce that

$$\liminf_{n \to \infty} n \cdot e^{l_0}_{n,r}(\mu_m^*) \geq {C_*}^{\frac{l_0}{r}} > 0.$$

Proposition 2.14 implies that $k_{m,r} \leq \underline{D}_r(\mu_m^*)$. The proof is complete. ∎

In the following theorem, we determine a lower bound of the quantization of the invariant Borel probability measure supported on the attractor of a bi-Lipschitz WIFS under the SOSC.

Theorem 4.10. *Let $\mathcal{W}$ be the WIFS defined as in Lemma 4.7 and assume that it satisfies the SOSC. For $r \in (0, \infty)$, we have $\underline{D}_r(\mu) \geq k_r$, where*

$$\sum_{i=1}^{N} (p_i {s_i}^r)^{\frac{k_r}{r+k_r}} = 1.$$

Proof. Since the WIFS $\mathcal{W}$ satisfies the SOSC, there exists an open set U of $\mathbb{R}^d$ such that

$$\bigcup_{i=1}^{N} f_i(U) \subset U, \quad U \cap A_{\mathcal{W}} \neq \emptyset, f_i(U) \cap f_j(U) = \emptyset \text{ for all } i \neq j, \quad 1 \leq i, j \leq N.$$

Since $U \cap A_{\mathcal{W}} \neq \emptyset$, there exists a $\sigma \in \{1, 2, \ldots, N\}^*$ such that $f_\sigma(A_{\mathcal{W}}) \subset U$. We denote $f_\sigma(A_{\mathcal{W}})$ by $(A_{\mathcal{W}})_\sigma$ for any $\sigma \in \{1, 2, \ldots, N\}^*$. Now, by using the condition $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j, 1 \leq i, j \leq N$, it is clear that for each $m \in \mathbb{N}$, the sets $\{(A_{\mathcal{W}})_{i\sigma} : i \in \{1, 2, \ldots, N\}^m\}$ are pairwise disjoint. We define a WIFS $\mathcal{L}_m = \{\mathbb{R}^d; f_{i\sigma}; p_i \text{ for all } i \in \{1, 2, \ldots, N\}^m\}$ as in Lemma 4.8. Therefore, by the Proposition 4.9, we obtain that $k_{m,r} \leq \underline{D}_r(\mu_m^*)$, where $k_{m,r}$ is given by

$$\sum_{i \in \{1,2,\ldots,N\}^m} (p_i s_{i\sigma}^r)^{\frac{k_{m,r}}{r+k_{m,r}}} = 1.$$

Let A_m^* be the attractor of the WIFS $\mathcal{L}_m$. Note that $0 < \dim_H(\mu) \leq \underline{D}_r(\mu)$ (see [8, Theorem 11.6]). Hence, $\underline{D}_r(\mu) > 0$. Now, we claim that $\underline{D}_r(\mu_m^*) \leq \underline{D}_r(\mu)$. If $\underline{\dim}_B(A_m^*) \leq \underline{D}_r(\mu)$ is not true, then we may choose another word $\tau = \sigma\omega \in \{1, 2, \ldots, N\}^*$ for some $\omega \in \{1, 2, \ldots, N\}^*$ such that the attractor A_m^* of the WIFS $\mathcal{L}_m = \{\mathbb{R}^d : f_{i\tau}; p_i \text{ for all } i \in \{1, 2, \ldots, N\}^m\}$ satisfies $\underline{\dim}_B(A_m^*) \leq \underline{D}_r(\mu)$. By using the inequality $\underline{D}_r(\mu_m^*) \leq \underline{\dim}_B(A_m^*)$ given by [8, Proposition 11.9] and the condition $\underline{\dim}_B(A_m^*) \leq \underline{D}_r(\mu)$, we obtain $\underline{D}_r(\mu_m^*) \leq \underline{D}_r(\mu)$. Since $k_{m,r} \leq \underline{D}_r(\mu_m^*)$, we have $k_{m,r} \leq \underline{D}_r(\mu)$. Now, we show that $k_r \leq \underline{D}_r(\mu)$. Suppose for contradiction that $\underline{D}_r(\mu) < k_r$. Let $t_{\max} = \max\{p_1 s_1^r, p_2 s_2^r, \ldots, p_N s_N^r\}$. Using

$$\sum_{i \in \{1,2,\ldots,N\}^m} (p_i s_i^r)^{\frac{k_r}{r+k_r}} = 1 \text{ and } \frac{\underline{D}_r(\mu)}{r + \underline{D}_r(\mu)} - \frac{k_r}{r + k_r} < 0,$$

we have

$$\begin{aligned}
s_\tau^{\frac{-r \cdot k_{m,r}}{r+k_{m,r}}} &= \sum_{i \in \{1,2,\ldots,N\}^m} (p_i s_i^r)^{\frac{k_{m,r}}{r+k_{m,r}}} \\
&\geq \sum_{i \in \{1,2,\ldots,N\}^m} (p_i s_i^r)^{\frac{\underline{D}_r(\mu)}{r+\underline{D}_r(\mu)}} \\
&= \sum_{i \in \{1,2,\ldots,N\}^m} (p_i s_i^r)^{\frac{\underline{D}_r(\mu)}{r+\underline{D}_r(\mu)}} (p_i s_i^r)^{\frac{-k_r}{r+k_r}} (p_i s_i^r)^{\frac{k_r}{r+k_r}} \\
&\geq \sum_{i \in \{1,2,\ldots,N\}^m} (p_i s_i^r)^{\frac{k_r}{r+k_r}} t_{\max}^{m\left(\frac{\underline{D}_r(\mu)}{r+\underline{D}_r(\mu)} - \frac{k_r}{r+k_r}\right)} \\
&= t_{\max}^{m\left(\frac{\underline{D}_r(\mu)}{r+\underline{D}_r(\mu)} - \frac{k_r}{r+k_r}\right)}
\end{aligned}$$

This implies that

$$s_\tau^{\frac{-rk_r}{r+k_r}} \geq t_{\max}^{m\left(\frac{\underline{D}_r(\mu)}{r+\underline{D}_r(\mu)}-\frac{k_r}{r+k_r}\right)}.$$

We have a contradiction for large values of $m \in \mathbb{N}$. Therefore, we get $\underline{D}_r(\mu) \geq k_r$, proving the assertion. ■

In the next theorem, we combine the above results on the quantization dimension.

Theorem 4.11. *Let $\mathcal{W} = \{\mathbb{R}^d; f_1, f_2, \ldots, f_N; p_1, p_2, \ldots, p_N\}$ be a WIFS such that each f_i satisfies*

$$s_i \|x - y\| \leq \|f_i(x) - f_i(y)\| \leq c_i \|x - y\|, \tag{4.1}$$

where $x, y \in \mathbb{R}^d$ and $0 < s_i \leq c_i < 1$. Let μ be the invariant Borel probability measure corresponding to the WIFS $\mathcal{W}$ and assume that $\mathcal{W}$ satisfies SOSC. Then

$$k_r \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq l_r,$$

where k_r and l_r are given by $\sum_{i=1}^N (p_i {s_i}^r)^{\frac{k_r}{r+k_r}} = 1$ and $\sum_{i=1}^N (p_i {c_i}^r)^{\frac{l_r}{r+l_r}} = 1$, respectively.

Proof. By combining Theorems 4.6 and 4.10, we get our required result. ■

Remark 4.12. In [9], Graf and Luschgy gave a formula for the quantization dimension of the invariant self-similar probability measure generated by a self-similar IFS under the open set condition. Here, we give bounds for the quantization dimension of the invariant probability measure generated by a bi-Lipschitz IFS under the strong open set condition. If we choose $s_i = c_i$ in (4.1), then our result gives the formula of Graf and Luschgy [9]. Thus, our result generalizes the result of Graf and Luschgy [9] in a more general setting.

Remark 4.13. In [16], Roychowdhury determined the quantization dimension of the invariant probability measure supported on the limit set generated by a bi-Lipschitz IFS with the strong open set condition and the bi-Lipschitz constants satisfying that $\overline{s}_\sigma \leq K \underline{s}_\sigma$ for all $\sigma \in I^*$. The condition taken by the author is satisfied for similarity mappings but not for the general class of bi-Lipschitz mappings. For example, if we take $\overline{s}_i = \frac{1}{2}, \underline{s}_i = \frac{1}{3}$ for all $1 \leq i \leq N$, then we cannot find any such $K \in \mathbb{R}$. However, our result, Theorem 4.11, gives the quantization dimensions of the invariant probability measures corresponding to a general class of bi-Lipschitz IFSs.

Remark 4.14. Here, we shed some light on mathematical issues that seem to have appeared in [16]. In the proof of [16, Lemma 3.11], the author uses the following incorrect equality

$$\varphi_\sigma(U^c) = \left(\varphi_\sigma(U)\right)^c,$$

where $\sigma \in \Gamma$. Choose an IFS $\{[0,2]; f_1(x) = \frac{x}{2}, f_2(x) = \frac{x}{2} + \frac{1}{2}\}$ and an open set $U = (0,1)$ satisfying SOSC. Now,

$$U^c = \{0\} \cup [1,2],\ f_2(U^c) = \left\{\frac{1}{2}\right\} \cup [1, 1.5],\ \big(f_2(U)\big)^c = \left[0, \frac{1}{2}\right] \cup [1,2],$$

which yields $f_2(U^c) \neq (f_2(U))^c$. After a careful reading, one may notice some other issues in the proofs of some other results in the same paper.

In the next proposition, we estimate the quantization dimension of the invariant Borel probability measures supported on the graph of the fractal transformation.

Proposition 4.15. *Let $\mathcal{F} = \{\mathbb{R}^d; f_1, f_2, \dots, f_N\}$ and $\mathcal{G} = \{\mathbb{R}^d; g_1, g_2, \dots, g_N\}$ be two IFSs such that f_i and g_i are bi-Lipschitz mappings as follows*

$$c_i \|x - x'\| \leq \|f_i(x) - f_i(x')\| \leq c_i' \|x - x'\|,$$

$$r_i \|y - y'\| \leq \|g_i(y) - g_i(y')\| \leq r_i' \|y - y'\|,$$

where $c_i, c_i', r_i, r_i' \in \mathbb{R}$, $0 < c_i \leq c_i' < 1$ and $0 < r_i \leq r_i' < 1$. Assume that $\mathcal{F}$ satisfies SOSC and $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$. Let $\mathcal{H} = \{\mathbb{R}^d \times \mathbb{R}^d; h_1, h_2, \dots, h_N\}$ be the IFS with probability vector $(p_1, p_2, \dots, p_N)$, where $h_i(x,y) = (f_i(x), g_i(y))$. Let μ be the invariant Borel probability measure corresponding to the IFS $\mathcal{H}$ such that $\mu(T_{\mathcal{F}\mathcal{G}}) = 1$. Then

$$s_r \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq t_r,$$

where s_r and t_r are uniquely given by

$$\sum_{i=1}^{N} \big(p_i \min\{c_i, r_i\}^r\big)^{\frac{s_r}{r+s_r}} = 1 \text{ and } \sum_{i=1}^{N} \big(p_i \max\{c_i', r_i'\}^r\big)^{\frac{t_r}{r+t_r}} = 1,$$

respectively.

Proof. Since $\mathcal{F}$ satisfies SOSC and $\mathcal{C}_{\mathcal{F}} \prec \mathcal{C}_{\mathcal{G}}$, by using [19, Theorem 6.8], the graph of the fractal transformation $T_{\mathcal{F}\mathcal{G}}$ is the same as the attractor of the IFS $\mathcal{H}$. Using Theorem 4.11, part (3) of Lemma 3.2 and part (2) of Lemma 3.3, we get our required result. ∎

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