



Partial Differential Equations. – *Local regularity of very weak s -harmonic functions via fractional difference quotients*, by ALESSANDRO CARBOTTI, SIMONE CITO, DOMENICO ANGELO LA MANNA and DIEGO PALLARA, communicated on 13 December 2024.

ABSTRACT. – The aim of this paper is to give a new proof that any very weak s -harmonic function u in the unit ball B is smooth. As a first step, we improve the local summability properties of u . Then, we exploit a suitable version of the difference quotient method tailored to get rid of the singularity of the integral kernel and gain Sobolev regularity and local linear estimates of the H_{loc}^s norm of u . Finally, by applying more standard methods, such as elliptic regularity and Schauder estimates, we reach the real analyticity of u . Up to the authors' knowledge, the difference quotient techniques are new.

KEYWORDS. – fractional operators, Sobolev regularity, s -harmonic functions.

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1. INTRODUCTION

This paper comes from our attempt to generalize the by now classical difference quotient method due to L. Nirenberg to nonlocal operators. It has been introduced in [38] and is now presented in all the textbooks dealing with the regularity properties of solutions of elliptic equations. After the introduction of weak, or even distributional, solutions of partial differential equations, the problem of their regularity has been tackled by various techniques. Probably the first result in this direction has been the proof of regularity of weakly harmonic functions, obtained in the fifties by Hermann Weyl in [46] and by Renato Caccioppoli, see [36, p. 122]. Subsequently, much more general operators have been considered and one of the most fruitful and flexible techniques has proved to be that of difference quotients, which – as it is – appears to be strictly depending on the *local* character of differential operators. We refer e.g. to [34, 47] and the references therein for recent regularity results relying on Nirenberg method.

On the other hand, the notion of distributional solution is well established also for equations coming from *nonlocal* operators and the question on the regularity of such solutions is in turn quite natural. One of the first examples of nonlocal operators, and probably the simplest one, is the fractional power of the Laplacian, and the solutions of

the equation $(-\Delta)^s u = 0$, $s \in (0, 1)$, are called s -harmonic. The class of s -harmonic functions has been broadly studied in the last years. Though they have many features relating them with harmonic functions, see for instance [23, 25], s -harmonic functions exhibit a different behaviour in other aspects, due to the nonlocal nature of the fractional Laplacian. Among these facts we mention the local density of s -harmonic functions among smooth functions [24], a purely nonlocal phenomenon that has some interesting consequences, such as the failure of the classical Harnack inequality and a quantitative version of an inverse mean value formula in the fractional case. See [15, 30] for more precise statements, [18] for more general density results and [2, 16] for other applications.

There are several equivalent ways of defining $(-\Delta)^s$, see [33], and the first proof of the local regularity of s -harmonic distributions has been obtained via pseudodifferential techniques by R. T. Seeley [40]. See [41, 42] for more general operators. The Dirichlet problem for the fractional Laplacian

$$(1.1) \quad \begin{cases} (-\Delta)^s w = f & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^c \end{cases}$$

has been studied in [35, 39] providing basic summability estimates according to the summability of the source f and boundary regularity, respectively. We notice that when Ω is the whole of \mathbb{R}^N more than $H^s(\mathbb{R}^N)$, regularity is available for the weak solution of $(-\Delta)^s w = f$, see [32].

It seemed to us that a natural question about the regularity of very weak solutions of nonlocal equations, see (2.4) for the precise definition in the case of the fractional Laplacian, is if it is possible to extend the difference quotient method to such solutions. As a first check, we have considered the operator $(-\Delta)^s$, where the difficulty of getting rid of the singularity of the kernel in its definition, see (2.2) below, already appears.

Very weak solutions of (1.1) have been treated in [4], where the authors observe that such solutions, when Ω is a bounded smooth domain and $f \in L^1(\Omega)$, are actually pointwise solutions; that is, they are given in terms of the Green operator applied to the source f . See [14], where explicit representation formulae when Ω is a ball are given. We also mention that in the fractional setting, maximal regularity estimates for the weak solutions of (1.1) are also available: in [27] the author proves that the solution of (1.1) with $f \in L^p(\Omega)$ belongs to the fractional Sobolev space $W_{\text{loc}}^{2s,p}(\Omega)$, or to the Besov space $B_{p,2,\text{loc}}^{2s}(\Omega)$ according to the values of s and p , and this regularity is sharp since it does not hold true up to the boundary. See also [7, 8]. Anyway, our arguments do not rely on the estimates proved there.

We consider very weak solutions of the following problem:

$$(1.2) \quad \begin{cases} (-\Delta)^s w = 0 & \text{in } B, \\ w = g & \text{in } B^c, \end{cases}$$

where B is the unit ball in \mathbb{R}^N and the outer datum g belongs to the weighted Lebesgue space $L_s^1(\mathbb{R}^N)$ that is defined in (2.1). See formula (2.4) for the precise definition. Notice that our assumptions allow us to deal with s -harmonic functions that blow up at the boundary of B , see [1, 10]. If the external datum is assumed to be bounded, one can also exploit the explicit representation of the solution in terms of the fractional Poisson kernel, see [9, 14, 26].

Our main result is the following theorem.

MAIN THEOREM. *Let u be a very weak solution of (1.2). Then,*

(1) (Sobolev regularity) $u \in H_{\text{loc}}^s(B)$ and the estimate

$$\|u\|_{H^s(B')} \leq c(B') \|u\|_{L_s^1(\mathbb{R}^N)}$$

holds for every $B' \Subset B$.

(2) (Classical regularity) u is real analytic in B and the estimate

$$\|D^\iota u\|_{L^\infty(B_{r_0})} \leq c^{|\iota|} \iota! C(R, r_0, N, s) (\|u\|_{L^\infty(B_R)} + \|u\|_{L_s^1(\mathbb{R}^N)})$$

holds for any $\iota \in \mathbb{N}_0^N$ and $0 < r_0 < R < 1$.

We prove our Main Theorem in several steps. In Theorem 3.1, we prove that the solution u , which is by definition in $L^1(B)$, is actually in $L_{\text{loc}}^2(B)$. This local improvement of summability is done by suitable localisation methods joint with regularity results for the fractional Poisson equation with homogeneous external condition proved in [3, 39]. The second step is done in Theorem 4.1, where we prove that the $L_{\text{loc}}^2(B)$ solution u belongs to the fractional Sobolev space $H_{\text{loc}}^{2s}(B)$, see Section 2 for the relevant definitions. The main point in the proof of this result consists in showing the $H_{\text{loc}}^s(B)$ regularity, as the final step from $H_{\text{loc}}^s(B)$ to $H_{\text{loc}}^{2s}(B)$ follows from L^2 estimates on the operator I_s which is the *carré du champ* of the fractional Laplacian that arises using the relevant fractional Leibniz rule. This kind of estimates, which we also prove for every $p \geq 1$, is different with respect to the one proved in [31]. We exploit a suitable variant of the classical Nirenberg difference quotient method: we introduce a cut-off function that vanishes near the origin and allows us to get rid of the singularity of the kernel and to obtain the fractional Sobolev regularity $H_{\text{loc}}^s(B)$. Difference quotient methods have been used in [6, 12, 17, 19] in a different fashion to improve the regularity of solutions to more general nonlocal equations. We point out once more that the core of the paper is the linear estimates and the new techniques introduced to prove claim (1) in the Main Theorem.

In the third step, we prove that for a cut-off function η , the function $\eta^2 u$ solves the equation $(-\Delta)^s w = f$ in the whole space, with $f \in L^2(\mathbb{R}^N)$, and as a consequence, u belongs to $H_{\text{loc}}^{2s}(B)$.

The fourth step relies on fractional De Giorgi estimates proved in [13] that allow us to gain local boundedness of u in B_1 and also local Hölder regularity. The usefulness of those estimates with respect to the previous literature (see e.g. [28, 29]) leans on the fact that the Hölder continuity exponent is quantified. Namely, the authors prove that $u \in C_{\text{loc}}^{0,\gamma}(B_1)$ for every $\gamma \in (0, \min\{2s, 1\})$.

In the fifth step, we use again that $\eta^2 u$ solves the equation $(-\Delta)^s w = f$ in \mathbb{R}^N but with $f \in C^{0,\gamma}(\mathbb{R}^N)$ and as a consequence, u belongs to $C_{\text{loc}}^{\gamma+2s}(B)$ and is s -harmonic in the classical sense in B . To do this, we also exploit suitable Hölder continuity properties of the operator I_s .

In the last step, we use the explicit Poisson representation formula to give a pointwise expression for u in a small ball. By well-known estimates on the derivatives of the Poisson kernel, we conclude our proof by proving the real analyticity of u .

Finally, we notice that all our results are stated and proved using the unit ball as reference domain in order to avoid technical issues and to focus on the core of our strategy though the same results also hold true for every bounded and sufficiently smooth domain.

2. NOTATIONS AND PRELIMINARY RESULTS

In the whole paper, we always assume that $N \geq 2$.

The space $L_s^1(\mathbb{R}^N)$ is the weighted Lebesgue space defined as

$$(2.1) \quad L_s^1(\mathbb{R}^N) := \{u \in \mathcal{M}(\mathbb{R}^N); \|u\|_{L_s^1(\mathbb{R}^N)} < \infty\},$$

where $\mathcal{M}(\mathbb{R}^N)$ denotes the space of Lebesgue measurable functions on \mathbb{R}^N and

$$\|u\|_{L_s^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} dx.$$

It is very easy to check that

$$L^p(\mathbb{R}^N) \subsetneq L_s^1(\mathbb{R}^N) \subsetneq L_{\text{loc}}^1(\mathbb{R}^N) \quad \text{for every } p \geq 1.$$

The space $L_s^1(\mathbb{R}^N)$ is a natural setting for very weak s -harmonic functions. Indeed, it encodes local integrability and a growth condition at infinity. This is equivalent to requiring that the nonlocal tail of u

$$\text{Tail}(u; x_0, R) := R^{2s} \int_{B_R^c(x_0)} \frac{|u(x)|}{|x - x_0|^{N+2s}} dx$$

is finite for every $x_0 \in \mathbb{R}^N$ and $R > 0$. See [20, 21] for more precise references.

For $u \in C_{\text{loc}}^{2s+\gamma}(B) \cap L_s^1(\mathbb{R}^N)$, $\gamma \in (0, 1)$, the s -Laplacian $(-\Delta)^s u$ is pointwise defined for every $x \in B$ and the following representation formula holds:

$$(2.2) \quad \begin{aligned} (-\Delta)^s u(x) &= C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \\ &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} dy, \end{aligned}$$

where $C_{N,s} := \frac{s2^{2s}\Gamma(\frac{N}{2}+s)}{\pi^{N/2}\Gamma(1-s)}$ and Γ denotes the Euler Gamma function. This choice of the normalisation constant makes the fractional Laplacian a Fourier multiplier with symbol $|\cdot|^{2s}$ whenever for $u \in L^1(\mathbb{R}^N)$ the Fourier transform \mathcal{F} is defined as

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^N} u(x) e^{-2\pi i x \cdot \xi} dx.$$

Notice that if u only belongs to $L_s^1(\mathbb{R}^N)$, formula (2.2) still holds true by taking the integrals in the Cauchy principal value sense.

For $s \in (0, 1)$, $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^N$, we define the fractional Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < \infty\},$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega} dx \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy \right)^{1/p},$$

endowed with the norm $\|\cdot\|_{W^{s,p}(\Omega)} := (\|\cdot\|_{L^p(\Omega)}^p + [\cdot]_{W^{s,p}(\Omega)}^p)^{1/p}$.

When $p = \infty$, any $f \in W^{s,\infty}(\Omega)$ has a representative $\tilde{f} \in C^s(\bar{\Omega})$.

As usual, when $p = 2$, we use the notation $H^s(\Omega)$ to indicate the Hilbert space $W^{s,2}(\Omega)$. See [22] for a gentle introduction to the fractional Sobolev spaces.

Let us also define higher-order fractional Sobolev spaces, confining to the non-integer case: for $\sigma \in (1, \infty)$, $\sigma = k + s$, $k \in \mathbb{N}$, $s \in (0, 1)$ and $1 \leq p < \infty$, the fractional Sobolev space $W^{\sigma,p}(\Omega)$ is defined as follows:

$$W^{\sigma,p}(\Omega) = \{u \in W^{k,p}(\Omega) : D^\alpha u \in W^{s,p}(\Omega), \forall \alpha \in \mathbb{N}_0^N, |\alpha| \leq k\}.$$

Set $\mathcal{Q}_A := (A \times A) \cup (A \times A^c) \cup (A^c \times A)$ for every open set A . We define

$$(2.3) \quad \mathbb{H}^s(B) = \{u \in L^2(B) : [u]_{\mathbb{H}^s(B)} < \infty\},$$

where

$$[u]_{\mathbb{H}^s(B)} := \left(\iint_{\mathcal{Q}_B} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

We say that $u \in \mathbb{H}^s(B)$ is a weak solution for (1.2) if for every $\varphi \in \mathbb{H}_0^s(B) = H_0^s(B) = \overline{C_c^\infty(B)}^{\|\cdot\|_{H^s(B)}}$, it holds that

$$\begin{cases} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dy = 0, \\ u = g \quad \text{in } B^c. \end{cases}$$

Notice that for $u \in \mathbb{H}^s(B)$ and $\varphi \in \mathbb{H}_0^s(B)$, the definition is well posed. Indeed, let $A \Subset B$, $A \supset \text{supp } \varphi$

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dy \right| \\ & \leq \iint_{Q_A} \frac{|u(x) - u(y)| |\varphi(x) - \varphi(y)|}{|x - y|^{N+2s}} dx dy \leq [u]_{\mathbb{H}^s(A)} [\varphi]_{\mathbb{H}^s(A)}. \end{aligned}$$

We notice that if $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u \in \mathbb{H}^s(B) \cap L^\infty(\mathbb{R}^N)$ is a weak solution of (1.2), then u is also a solution in the viscosity sense for (1.2), as proved in [43, Theorem 1] in the inhomogeneous case for continuous sources.

For $s \in (0, 1)$, we also introduce the space

$$L_s^\infty(\mathbb{R}^N) := \left\{ u \in L^\infty(\mathbb{R}^N); \sup_{x \in \mathbb{R}^N} (1 + |x|^{N+2s}) |u(x)| < \infty \right\},$$

equipped with the norm

$$\|u\|_{L_s^\infty(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N} (1 + |x|^{N+2s}) |u(x)|.$$

We say that $u \in L_s^1(\mathbb{R}^N)$ is a very weak solution of (1.2) if, for every φ compactly supported in B such that $(-\Delta)^s \varphi \in L_s^\infty(\mathbb{R}^N)$, it holds that

$$(2.4) \quad \begin{cases} \int_{\mathbb{R}^N} u(-\Delta)^s \varphi dx = 0, \\ u = g \quad \text{in } B^c. \end{cases}$$

Notice that the chosen class of test functions is not empty. Indeed, let $\varphi \in C_c^{2s+\gamma}(B)$ for some $\gamma > 0$. We have

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} u(-\Delta)^s \varphi dx \right| \\ & \leq \int_{B_2} |u(-\Delta)^s \varphi| dx + \int_{B_2^c} |u(-\Delta)^s \varphi| dx \\ & \leq \|u\|_{L^1(B_2)} \|\varphi\|_{C^{2s+\gamma}(B)} + \int_{B_2^c} \frac{|u(x)|}{1 + |x|^{N+2s}} dx \int_B \frac{1 + |x|^{N+2s}}{|x - y|^{N+2s}} |\varphi(y)| dy \end{aligned}$$

$$\begin{aligned}
&\leq \|u\|_{L^1(B_2)} \|\varphi\|_{C^{2s+\gamma}(B)} + \int_{B_2^c} \frac{|u(x)|}{1+|x|^{N+2s}} dx \int_B \left(1 + \frac{1+|y|}{|x-y|}\right)^{N+2s} |\varphi(y)| dy \\
&\leq \|u\|_{L^1(B_2)} \|\varphi\|_{C^{2s+\gamma}(B)} + 3^{N+2s} \|u\|_{L_s^1(\mathbb{R}^N)} \|\varphi\|_{L^1(B)} \\
&\leq C_{N,s} \|\varphi\|_{C^{2s+\gamma}(B)} \|u\|_{L_s^1(\mathbb{R}^N)},
\end{aligned}$$

and then φ is a test function.

We notice that a weak solution is a very weak solution. Indeed, using the symmetry of the double integral in x and y ,

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dx dy = \int_{\mathbb{R}^N} u(x) dx \int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}} dy$$

for every $\varphi \in C_c^\infty(B)$.

Now, we recall some useful results. From now on, for $r \geq 1$, we denote with $r' := \frac{r}{r-1}$ the Hölder conjugate of r , and for $\tau > 0$ such that $r\tau < N$, we denote with $r_\tau^* := \frac{Nr}{N-r\tau}$ the Sobolev conjugate of r with respect to τ . First, we state the Sobolev embedding theorem.

PROPOSITION 2.1 ([5, Theorem 7.63]). *Let $\Omega \subseteq \mathbb{R}^N$ an open and smooth set and let $k, h \geq 0$, $p \geq 1$. If $u \in W^{k,p}(\Omega)$, $k > h$ and $N > (k-h)p$, then the following continuity estimate holds:*

$$\|u\|_{W^{h,q}(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}$$

- for every $1 \leq q \leq \frac{Np}{N-(k-h)p}$ if Ω has finite measure
- for every $p \leq q \leq \frac{Np}{N-(k-h)p}$ if Ω has infinite measure.

We notice that if $k = h \in \mathbb{N}_0$ and if Ω has finite measure, then the statement of Proposition 2.1 still holds true, but if $k = h \in (0, \infty) \setminus \mathbb{N}$, then the embedding may fail in general even if Ω is a ball, see [37].

The following results give the regularity properties of the weak solutions of (1.1) under suitable assumptions on f .

THEOREM 2.2 ([3, Corollary 1.7]). *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ a bounded C^2 domain, $s \in (0, 1)$ and let u be the unique solution of*

$$\begin{cases} (-\Delta)^s w = f & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^c \end{cases}$$

with $f \in L^m(\Omega)$.

(i) If $1 \leq m < \frac{N}{s}$, then for all $1 < p < m_s^*$, there exists $C > 0$ such that

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq C \|f\|_{L^m(\Omega)}.$$

(ii) If $m > \frac{N}{s}$, then for all $1 < p < \infty$, there exists $C > 0$ such that

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq C \|f\|_{L^m(\Omega)}.$$

THEOREM 2.3 ([39, Proposition 1.4]). Let $s \in (0, 1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ a bounded $C^{1,1}$ domain, $f \in C(\bar{\Omega})$ and let u be the weak solution of

$$\begin{cases} (-\Delta)^s w = f & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^c. \end{cases}$$

(i) For each $1 \leq r < (\frac{N}{2s})'$, there exists $C = C(n, r, s, |\Omega|) > 0$ such that

$$\|u\|_{L^r(\Omega)} \leq C \|f\|_{L^1(\Omega)}.$$

(ii) Let $1 < p < \frac{N}{2s}$ and $p_{2s}^* = \frac{Np}{N-2sp}$. Then, there exists $C = C(n, s, p) > 0$ such that for any $1 \leq q \leq p_{2s}^*$,

$$\|u\|_{L^q(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

(iii) Let $\frac{N}{2s} < p < \infty$. Then, there exists $C = C(n, s, p, \Omega) > 0$ such that

$$\|u\|_{C^\beta(\mathbb{R}^N)} \leq C \|f\|_{L^p(\Omega)},$$

where $\beta := \min\{s, 2s - \frac{N}{p}\}$.

3. IMPROVEMENT OF SUMMABILITY

Now we are ready to state and prove the following theorem.

THEOREM 3.1. Let $u \in L_s^1(\mathbb{R}^N)$ a very weak solution of (1.2). Then, $u \in L_{\text{loc}}^2(B)$.

PROOF. To ease the presentation, we divide the proof in three steps.

Step 1: first summability improvement. In this first step, we prove

$$(3.1) \quad u \in L_{\text{loc}}^r(B) \quad \text{for all } r < \frac{N}{N-s}.$$

Let $p > \frac{N}{s}$ and $\psi \in C^\infty(B) \cap C(\bar{B})$. Now, let v be the unique solution of the Dirichlet problem

$$(3.2) \quad \begin{cases} (-\Delta)^s w = \psi & \text{in } B_{1-\delta}, \\ w = 0 & \text{in } B_{1-\delta}^c \end{cases}$$

for some $\delta > 0$ sufficiently small to be conveniently chosen in the sequel. Now let $\eta \in C_c^\infty(B)$ be such that $\eta = 1$ in $B_{1-4\delta}$, $\eta = 0$ in $B_{1-2\delta}^c$ and $|\nabla \eta| \leq \frac{1}{\delta}$. Notice that by Theorem 2.3, we have that $v \in C^s(\mathbb{R}^N)$, and this easily implies that

$$(-\Delta)^s(\eta^2 v) \in L_s^\infty(\mathbb{R}^N) \quad \text{and} \quad \text{supp } \eta^2 v \subset B_{1-2\delta}.$$

Indeed, for $|x| > 2$, it holds that

$$|(-\Delta)^s(\eta^2 v)(x)| \leq \left| \int_B \frac{\eta^2(y)v(y)}{|x-y|^{N+2s}} dy \right| \leq \frac{3^{N+2s}\|v\|_{L^1(B)}}{1+|x|^{N+2s}},$$

while the boundedness of $(-\Delta)^s(\eta^2 v)$ in B_2 is an immediate consequence of Theorem 2.3.

Therefore, we can use $\varphi = \eta^2 v$ as a test function in the definition of very weak solution (2.4). Then,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \eta^2 u \psi dx + 2 \int_{\mathbb{R}^N} uv \eta (-\Delta)^s \eta dx \\ &\quad - \int_{\mathbb{R}^N} u \eta I_s(\eta, v) dx - \int_{\mathbb{R}^N} u I_s(\eta, \eta v) dx, \end{aligned}$$

where for any f_1, f_2 measurable, we have set

$$(3.3) \quad I_s(f_1, f_2)(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{(f_1(x) - f_1(y))(f_2(x) - f_2(y))}{|x-y|^{N+2s}} dy,$$

and the constant $C_{N,s}$ is that one in the definition of $(-\Delta)^s$.

Then,

$$\begin{aligned} (3.4) \quad &\left| \int_{\mathbb{R}^N} \eta^2 u \psi dx \right| \\ &\leq \int_{\mathbb{R}^N} |u I_s(\eta, \eta v)| dx + 2 \int_{\mathbb{R}^N} |uv \eta (-\Delta)^s \eta| dx + \int_{\mathbb{R}^N} |u \eta I_s(\eta, v)| dx \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

We start by estimating the term A_1 . Then,

$$\begin{aligned} &\int_{\mathbb{R}^N} |u I_s(\eta, \eta v)| dx \\ &\leq \|u\|_{L_s^1(\mathbb{R}^N)} \|I_s(\eta, \eta v)\|_{L_s^\infty(\mathbb{R}^N)} \leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|\eta v\|_{C^s(\mathbb{R}^N)} \\ &\leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|v\|_{C^s(\mathbb{R}^N)} \leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^p(B)}, \end{aligned}$$

where the last inequality exploits item (iii) in Theorem 2.3, which holds true since $p > \frac{N}{s}$. For the second inequality, we notice that for any $x \in \mathbb{R}^N$ and $y \in B_{1-2\delta}$, we

have

$$\begin{aligned}
 (3.5) \quad & (1 + |x|^{N+2s}) |\eta(x) - \eta(y)| \\
 & \leq C(\delta) \chi_{B_{1-\delta}}(x) \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} |x - y| \\
 & \quad + 2^{N+2s-1} \chi_{B_{1-\delta}^c}(x) (1 + |x - y|^{N+2s} + |y|^{N+2s}) \\
 & \leq C(\delta) \chi_{B_{1-\delta}}(x) \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} |x - y| + C(N, s, \delta) \chi_{B_{1-\delta}^c}(x) |x - y|^{N+2s},
 \end{aligned}$$

while for any $x \in \mathbb{R}^N$ and $y \in B_{1-2\delta}^c$,

$$(1 + |x|^{N+2s}) |\eta(x) - \eta(y)| \leq C(\delta) \chi_{B_{1-\delta}}(x).$$

Therefore, for any $x \in \mathbb{R}^N$,

$$\begin{aligned}
 (3.6) \quad & (1 + |x|^{N+2s}) |I_s(\eta, \eta v)(x)| \\
 & \leq (1 + |x|^{N+2s}) \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 & \quad + (1 + |x|^{N+2s}) \int_{B_{1-2\delta}^c} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 & \leq (1 + |x|^{N+2s}) \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 & \quad + (1 + |x|^{N+2s}) |v(x)| \int_{B_{1-2\delta}^c} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy \\
 & \leq \chi_{B_{1-\delta}}(x) \|\eta v\|_{C^s(\mathbb{R}^N)} \int_{B_{1-2\delta}} \frac{dy}{|x - y|^{N+s-1}} \\
 & \quad + \chi_{B_{1-2\delta}}(x) C'(\delta) |v(x)| \left(\int_{B_{\delta}^c(x)} \frac{dy}{|x - y|^{N+2s}} + \int_{B_{2-3\delta}(x)} \frac{dy}{|x - y|^{N+2s-2}} \right) \\
 & \quad + C(N, s, \delta) \|\eta v\|_{L^\infty(\mathbb{R}^N)} \chi_{B_{1-\delta}^c}(x) \\
 & \leq C(N, s, \delta) \|v\|_{C^s(\mathbb{R}^N)} \leq C(N, s, \delta) \|\psi\|_{L^p(B)},
 \end{aligned}$$

where in the third inequality, we have used (3.5), the equality

$$|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)| = |v(x)| |\eta(x) - \eta(y)|^2$$

that holds for any $x \in B_{1-\delta}$ and $y \in B_{1-2\delta}^c$ and also that the integral term in the fourth line of (3.6) is nonzero if and only if $x \in B_{1-2\delta}$ and it can be split in the sum of two integrals over $B_{1-\delta}^c$ and $B_{1-\delta} \setminus B_{1-2\delta}$.

For A_2 , we have

$$\int_{\mathbb{R}^N} |u v \eta(-\Delta)^s \eta| dx \leq C \|u\|_{L^1(B_{1-2\delta})} \|v\|_{L^\infty(B_{1-2\delta})} \leq C \|u\|_{L^1(B_{1-2\delta})} \|\psi\|_{L^p(B)},$$

where we used again Theorem 2.3.

To estimate A_3 , we notice that

$$\int_{\mathbb{R}^N} |u \eta I_s(\eta, v)| dx = \int_{B_{1-2\delta}} |u \eta I_s(\eta, v)| dx \leq \|u\|_{L^1(B_{1-2\delta})} \|I_s(\eta, v)\|_{L^\infty(B_{1-2\delta})},$$

and for almost any $x \in B_{1-2\delta}$, if we split A_3 into the sum of the integrals over $B_{1-\delta}$ and $B_{1-\delta}^c$, we have

$$\begin{aligned} |I_s(\eta, v)(x)| &\leq C \|v\|_{C^s(\mathbb{R}^N)} \left(\int_{B_{2-3\delta}(x)} \frac{dy}{|x-y|^{N+s-1}} + \int_{B_\delta^c(x)} \frac{dy}{|x-y|^{N+2s}} \right) \\ &\leq C \|\psi\|_{L^p(B)}, \end{aligned}$$

where we have used again Theorem 2.3 (iii), which holds true for $p > \frac{N}{s}$ with $\beta = s$. Now we have

$$\begin{aligned} (3.7) \quad \left| \int_{\mathbb{R}^N} u \eta^2 \psi dx \right| &\leq C \|u\|_{L_s^1(\mathbb{R}^N)} (\|v\|_{W^{s,p}(\mathbb{R}^N)} + \|v\|_{C^s(\mathbb{R}^N)}) \\ &\leq C \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^p(B)} \end{aligned}$$

for all $\psi \in C^\infty(B) \cap C(\bar{B})$, with $p > \frac{N}{s}$. By the density of $C^\infty(B) \cap C(\bar{B})$ in $L^p(B)$, we have that (3.7) holds for all $\psi \in L^p(B)$, which implies that $\eta^2 u \in L^{p'}(B)$; hence, $\eta^2 u \in L^r(B)$ for all $r < \frac{N}{N-s}$. The arbitrariness of δ gives the claim.

Step 2: higher summability. Our next goal is to show

$$(3.8) \quad u \in L_{\text{loc}}^r(B) \quad \text{for } r \in \left(\frac{N}{N-s}, \frac{N}{N-2s} \right).$$

In order to improve the summability of the solution u , we still use a duality argument, but in a bit different way. Let $\psi \in C^\infty(B) \cap C(\bar{B})$, take $m \in (\frac{N}{(1+\alpha)s}, \frac{N}{s})$ for some $\alpha \in (0, 1)$ and let v, η as before, where now δ is the double of the previous one. Since we know that $u \in L_{\text{loc}}^r(B)$ for all $r < \frac{N}{N-s}$, let $p' \in (\frac{N}{N-\alpha s}, \frac{N}{N-s})$. It is very easy to check that the function $v \eta^2$ is admissible as test function in definition (2.4). We estimate again the three terms appearing in (3.4), but this time we can use the higher summability of u proved in Step 1 to estimate

$$\begin{aligned} (3.9) \quad A_2 &\leq C \|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{L^p(B_{1-2\delta})} \leq C \|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{W^{s,q}(B_{1-2\delta})} \\ &\leq C \|u\|_{L^{p'}(B_{1-2\delta})} \|\psi\|_{L^m(B)}, \end{aligned}$$

where the second inequality exploits the Sobolev embedding theorem for $q \geq \frac{Np}{N+sp} \geq \frac{N}{2s}$ and the third inequality exploits Theorem 2.2 and it holds for all $q \in (1, m_s^*)$ (recall that $m_s^* = \frac{Nm}{N-ms}$) whenever $1 \leq m < \frac{N}{s}$.

Now we estimate again the term A_1 in (3.4):

$$\begin{aligned}
 (3.10) \quad A_1 &\leq \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}^c} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 &\quad + \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 &\quad + \int_{B_{1-\delta}^c} |u(x)| dx \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 &=: B_1 + B_2 + B_3.
 \end{aligned}$$

To bound B_1 , we first observe that since $\eta(y) = 0$ for $y \in B_{1-2\delta}^c$, we have

$$\begin{aligned}
 (3.11) \quad B_1 &= \int_{B_{1-2\delta}} |u(x)| |v(x)| \eta^2(x) dx \int_{B_{1-2\delta}^c} \frac{dy}{|x - y|^{N+2s}} \\
 &= \int_{B_{1-2\delta}} |u(x)| |v(x)| dx \int_{B_{1-2\delta}^c} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy \\
 &\leq \int_{B_{1-2\delta}} |u(x)| |v(x)| dx \int_{B_{1-\delta}^c} \frac{dy}{|x - y|^{N+2s}} \\
 &\quad + \int_{B_{1-2\delta}} |u(x)| |v(x)| dx \int_{B_{1-\delta} \setminus B_{1-2\delta}} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy \\
 &\leq C(\delta) \left(\|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{L^p(B_{1-2\delta})} \right. \\
 &\quad \left. + \int_{B_{1-2\delta}} |u(x)| |v(x)| dx \int_{B_{1-\delta}} \frac{1}{|x - y|^{N+2s-2}} dy \right) \\
 &\leq C(\delta) \|u\|_{L^{p'}(B_{1-2\delta})} \|\psi\|_{L^m(B)}.
 \end{aligned}$$

Analogously for B_3 , we use that $\eta(x) = 0$ for $x \in B_{1-\delta}^c$ and that $(1 + |x|^{N+2s}) \leq C_\delta |x - y|^{N+2s}$ for any $x \in B_{1-\delta}^c$ and $y \in B_{1-2\delta}$ to find

$$\begin{aligned}
 (3.12) \quad B_3 &\leq \int_{B_{1-\delta}^c} \frac{|u(x)|}{1 + |x|^{N+2s}} dx \int_{B_{1-2\delta}} (1 + |x|^{N+2s}) \frac{|v(y)| \eta^2(y)}{|x - y|^{N+2s}} dy \\
 &\leq C_\delta \|u\|_{L^1_s(\mathbb{R}^N)} \|v\|_{L^1(B_{1-2\delta})} \leq C_\delta \|u\|_{L^1_s(\mathbb{R}^N)} \|\psi\|_{L^m(B)}
 \end{aligned}$$

and again the Sobolev embedding Theorem holds true for any $q \geq \frac{N}{2s}$. To bound B_2 , we use the Hölder inequality in the following way:

$$\begin{aligned}
 (3.13) \quad B_2 &\leq \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 &\leq \|u\|_{L^{p'}(B_{1-\delta})} \left(\int_{B_{1-\delta}} \left(\int_{B_{1-2\delta}} \frac{|\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s-1}} dy \right)^p dx \right)^{1/p}
 \end{aligned}$$

$$\begin{aligned}
&\leq \|u\|_{L^{p'}(B_{1-\delta})} \left(\int_{B_{1-\delta}} \int_{B_{1-\delta}} \frac{|\eta(x)v(x) - \eta(y)v(y)|^p}{|x-y|^{N+sp}} dy \right. \\
&\quad \left. \cdot \left(\int_{B_{1-\delta}} \frac{dy}{|x-y|^{N-(1-s)p'}} \right)^{p-1} dx \right)^{1/p} \\
&\leq C \|u\|_{L^{p'}(B_{1-\delta})} \|\eta v\|_{W^{s,p}(\mathbb{R}^N)} \\
&\leq C \|u\|_{L^{p'}(B_{1-\delta})} \|\psi\|_{L^m(B)},
\end{aligned}$$

provided that $p < \frac{Nm}{N-sm}$. Since $p' > \frac{N}{N-\alpha s}$, we get $p < \frac{N}{\alpha s}$; hence, $\|v\|_{W^{s,p}(B)} \leq \|\psi\|_{L^m(B)}$ if $m > \frac{N}{(1+\alpha)s}$. Inequalities (3.11), (3.12), (3.13) give

$$(3.14) \quad |A_2| \leq C(\delta) \|\psi\|_{L^m(B)} \|u\|_{L^{p'}(B_{1-2\delta})}.$$

To estimate A_3 , we proceed as in the previous estimate:

$$\begin{aligned}
A_3 &= \int_{B_{1-2\delta}} |u \eta I_s(\eta, v)| dx \\
&\leq \int_{B_{1-2\delta}} |\eta(x)u(x)| \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)| |v(x) - v(y)|}{|x-y|^{N+2s}} dy dx.
\end{aligned}$$

We split the integral on \mathbb{R}^N in $B_{1-\delta}$ and $B_{1-\delta}^c$ and we use again the Hölder inequality and Theorem 2.2 to infer

$$\begin{aligned}
(3.15) \quad &\int_{B_{1-2\delta}} |\eta(x)u(x)| \int_{B_{1-\delta}} \frac{|\eta(x) - \eta(y)| |v(x) - v(y)|}{|x-y|^{N+2s}} dy dx \\
&\leq \left(\int_B |\eta u|^{p'} dx \right)^{1/p'} \left(\int_{B_{1-\delta}} \left(\int_{B_{1-\delta}} \frac{|v(x) - v(y)|}{|x-y|^{N+2s-1}} dy \right)^p dx \right)^{1/p} \\
&\leq C \|\eta u\|_{L^{p'}(B)} \left(\int_{B_{1-\delta}} dx \int_{B_{1-\delta}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} dy \right)^{1/p} \\
&\leq C \|\eta u\|_{L^{p'}(B)} \|v\|_{W^{s,p}(\mathbb{R}^N)} \\
&\leq C \|\eta u\|_{L^{p'}(B)} \|\psi\|_{L^m(B)}.
\end{aligned}$$

Concerning the second integral, since $v(y) = \eta(y) = 0$ for $y \in B_{1-\delta}^c$, we have

$$\begin{aligned}
(3.16) \quad &\int_B |\eta(x)u(x)| dx \int_{B_{1-\delta}^c} \frac{|\eta(x) - \eta(y)| |v(x) - v(y)|}{|x-y|^{N+2s}} dy \\
&= \int_{B_{1-2\delta}} |\eta(x)u(x)v(x)| dx \int_{B_{1-\delta}^c} \frac{dy}{|x-y|^{N+2s}} \\
&\leq C(\delta) \|\eta u\|_{L^{p'}(B)} \|v\|_{L^p(B)} \\
&\leq C(\delta) \|\eta u\|_{L^{p'}(B)} \|\psi\|_{L^m(B)}
\end{aligned}$$

provided that $m > \frac{N}{(1+\alpha)s}$. Inequalities (3.15) and (3.16) give

$$|A_3| \leq C(\delta) \|\eta u\|_{L^{p'}(B)} \|\psi\|_{L^m(B)}.$$

Hence, using the latter and inequalities (3.10), (3.14), we deduce

$$\left| \int_B u \eta^2 \psi \, dx \right| \leq C(\delta) \|\eta u\|_{L^{p'}(B)} \|\psi\|_{L^m(B)}$$

for all $\psi \in C^\infty(B) \cap L^\infty(B)$. By density, the inequality holds for $\psi \in L^m(B)$ and hence $\eta^2 u \in L^{m'}(B)$ with $m' \in (\frac{N}{N-s}, \frac{N}{N-(1+\alpha)s})$. Since this is true for any $\alpha \in (0, 1)$, we get our claim.

Step 3. We finally show that $u \in L_{\text{loc}}^p(B)$ for $p < \frac{N}{s}$.

We first prove recursively that

$$u \in L_{\text{loc}}^r(B) \quad \text{for } r < \frac{N}{N - ks}$$

for all $k \in \mathbb{N}$ such that $k < \frac{N}{s}$. We notice that we already proved the claim for $k = 1, 2$. Hence, let us assume that $u \in L_{\text{loc}}^{p'}(B)$ for $p' \in (\frac{N}{N-(k-\alpha)s}, \frac{N}{N-ks})$ for some $\alpha \in (0, 1)$. Fix $\delta > 0$ to be chosen again as the double of the one selected in the previous step, let $\psi \in C^\infty(B) \cap L^\infty(B)$, take $m \in (\frac{N}{(k+1-\alpha)s}, \frac{N}{ks})$ and let v be the unique solution of (1.1). For a cut-off function η supported in $B_{1-2\delta}$, we use $\eta^2 v$ as a test function in (2.4) to find again

$$\left| \int_B \eta^2 u \psi \, dx \right| \leq A_1 + A_2 + A_3,$$

with A_1, A_2, A_3 defined as in (3.4). As before, to estimate A_1 , we split it in three terms:

$$\begin{aligned} (3.17) \quad A_1 &\leq \int_{B_{1-\delta}} |u(x)| \, dx \int_{B_{1-2\delta}^c} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} \, dy \\ &\quad + \int_{B_{1-\delta}} |u(x)| \, dx \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} \, dy \\ &\quad + \int_{B_{1-\delta}^c} |u(x)| \, dx \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} \, dy \\ &=: B_1 + B_2 + B_3. \end{aligned}$$

The same argument used to bound B_1 and B_3 in Step 2 provides

$$|B_1| \leq C(\delta) \|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{W^{s,p}(\mathbb{R}^N)},$$

$$|B_3| \leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|v\|_{W^{s,p}(\mathbb{R}^N)},$$

where at this stage, $p \in (\frac{N}{ks}, \frac{N}{(k-\alpha)s})$. For B_2 , we use the Hölder inequality to have

$$\begin{aligned}
 |B_2| &\leq \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)| |\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \\
 &\leq \|u\|_{L^{p'}(B_{1-\delta})} \left(\int_{B_{1-\delta}} \left(\int_{B_{1-\delta}} \frac{|\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s-1}} dy \right)^p dx \right)^{1/p} \\
 &\leq \|u\|_{L^{p'}(B_{1-\delta})} \left(\int_{B_{1-\delta}} \int_{B_{1-\delta}} \frac{|\eta(x)v(x) - \eta(y)v(y)|^p}{|x - y|^{N+sp}} dy \right. \\
 &\quad \cdot \left. \left(\int_{B_{1-\delta}} \frac{dy}{|x - y|^{N+(s-1)p'}} \right)^{p-1} dx \right)^{1/p} \\
 &\leq C \|u\|_{L^{p'}(B_{1-\delta})} \|\eta v\|_{W^{s,p}(\mathbb{R}^N)}.
 \end{aligned}$$

Since $p > 1$, we have

$$|A_2| \leq C(\delta) \|u\|_{L^{p'}(B_{1-\delta})} \|v\|_{W^{s,p}(\mathbb{R}^N)},$$

and using Theorem 2.2, we get

$$\|v\|_{W^{s,p}(\mathbb{R}^N)} \leq C \|\psi\|_{L^m(B)}$$

whenever

$$p < \frac{Nm}{N - sm} \quad \text{i.e.,} \quad m > \frac{Np}{N + sp}.$$

Since $p < \frac{N}{(k-\alpha)s}$, we get

$$|A_2| \leq C(\delta) \|u\|_{L^{p'}(B_{1-\delta})} \|\psi\|_{L^m(B)}$$

if $m > \frac{N}{(k+1-\alpha)s}$. The estimate for A_3 follows from the same argument and gives

$$|A_3| \leq C(\delta) \|u\|_{L^{p'}(B_{1-\delta})} \|\psi\|_{L^m(B)}.$$

Thus, we arrive at

$$\left| \int_B \eta^2 u \psi dx \right| \leq C(\delta) \|u\|_{L^{p'}(B_{1-\delta})} \|\psi\|_{L^m(B)}$$

for any $m > \frac{N}{(k+1-\alpha)s}$. Using again a duality argument and since the latter is true for all $\alpha > 0$, we get

$$u \in L_{\text{loc}}^{m'}(B)$$

for any $m' < \frac{N}{N-(k+1)s}$.

Hence, we now run this argument k_0 times, where $k_0 := \max\{d \in \mathbb{N}; d \leq \bar{k}\}$ and $\bar{k} := \frac{N}{s} - 1$ to find

$$u \in L_{\text{loc}}^r(B)$$

for all $r < \frac{N}{N-k_0s} \leq \frac{N}{s}$. Since in particular $2 < \frac{2}{s} \leq \frac{N}{s}$ by our assumptions in Section 2, the proof is complete. \blacksquare

REMARK 3.2. In each step k of the proof of Theorem 3.1, we choose $\delta_k > 0$ such that $\delta_k < \frac{1}{4}$ and $\delta_k = 2\delta_{k-1} < 1$, for $k \in \{1, \dots, k_0\}$, and these conditions imply that in Step 1, we have to fix $\delta_1 := \delta < \frac{1}{2k_0+2}$.

REMARK 3.3. We notice that from estimates (3.11), (3.12), (3.13) and (3.15), we deduce that for any $w \in L_s^1(\mathbb{R}^N) \cap W_{\text{loc}}^{s,p}(B)$ for some $1 < p < \infty$, by definition, $\eta w \in W^{s,p}(\mathbb{R}^N)$ for any $\eta \in C_c^\infty(B)$ cut-off function and, if $\text{supp } \eta = B_{1-2\delta} \Subset B_{1-\delta} \Subset B$, there exists C , which depends only on δ, s, N, p but independent of w , such that

$$(3.18) \quad \|I_s(\eta, \eta w)\|_{L^p(\mathbb{R}^N)} \leq C \| \eta w \|_{W^{s,p}(\mathbb{R}^N)}.$$

Moreover, for any $x \in \mathbb{R}^N$, we have

$$(3.19) \quad \begin{aligned} & |\eta(x) I_s(\eta, w)(x)| \\ & \leq \eta(x) \int_{B_{1-\delta}} \frac{|\eta(x) - \eta(y)| |w(x) - w(y)|}{|x - y|^{N+2s}} dy \\ & \quad + \eta(x) \int_{B_{1-\delta}^c} \frac{|\eta(x) - \eta(y)| |w(x) - w(y)|}{|x - y|^{N+2s}} dy \\ & \leq C(\delta) \eta(x) \int_{B_{1-\delta}} \frac{|w(x) - w(y)|}{|x - y|^{N+2s-1}} dy + 2\eta(x) \int_{B_{1-\delta}^c} \frac{|w(x) - w(y)|}{|x - y|^{N+2s}} dy \\ & \leq C(\delta) \eta(x) \left(\int_{B_{1-\delta}} \frac{|w(x) - w(y)|^p}{|x - y|^{N+sp}} dy \right)^{1/p} \left(\int_{B_{1-\delta}} \frac{dy}{|x - y|^{N-p'(1-s)}} \right)^{1/p'} \\ & \quad + 2\eta(x) \left(|w(x)| \int_{B_{1-\delta}^c(x)} \frac{dy}{|x - y|^{N+2s}} + \int_{\mathbb{R}^N} \frac{|w(y)|}{1 + |y|^{N+2s}} dy \right). \end{aligned}$$

From (3.19), we deduce that there exists a positive constant $C = C(N, s, \delta, p)$ such that

$$(3.20) \quad \|\eta I_s(\eta, w)\|_{L^p(\mathbb{R}^N)} \leq C(N, s, \delta, p) (\|w\|_{W^{s,p}(B_{1-\delta})} + \|w\|_{L_s^1(\mathbb{R}^N)}).$$

The estimates in the cases $p = 1$ and $p = \infty$ also hold true with analogous computations. In the case $p = \infty$, we recall that for any Ω open and smooth set $W^{s,\infty}(\Omega) = C^{0,s}(\bar{\Omega})$, see e.g. [22, p. 59].

4. FRACTIONAL SOBOLEV REGULARITY

In this section, we prove the local H^s regularity of very weak solutions, which we know to be in L^2_{loc} . In the classical case, Sobolev regularity is usually obtained via the Nirenberg difference quotients method, but in the nonlocal case, this method does not work directly because of the presence of a divergent kernel. Therefore, we have devised a different approach, which consists in using another cut-off function (the η_τ below) that eliminates the singularity and makes the relevant integrals convergent.

THEOREM 4.1. *Let u be a very weak solution of (1.2) in $L^2_{\text{loc}}(B)$. Then, $u \in H^{2s}_{\text{loc}}(B)$.*

PROOF. Let u be given as in the statement. Let $\tau \in (0, 1/2)$ and let $\eta_\tau : [0, +\infty) \rightarrow [0, 1]$ be the cut-off function defined as

$$\eta_\tau(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \tau/2, \\ \frac{2}{\tau}t - 1 & \text{if } \tau/2 \leq t \leq \tau, \\ 1 & \text{if } \tau \leq t. \end{cases}$$

For $\delta \in (0, \frac{1}{4})$, let us consider another cut-off function $\eta : \mathbb{R}^N \rightarrow [0, 1]$ such that

$$\eta = 1 \text{ in } B_{1-4\delta}, \quad \eta = 0 \text{ in } B^c_{1-2\delta}, \quad |\nabla \eta| \leq \frac{1}{\delta}.$$

For any $x \in \mathbb{R}^N$, let us define the function

$$(4.1) \quad D^s_{\eta_\tau, \eta} u(x) := \int_{\mathbb{R}^N} \eta_\tau(|x-y|) \frac{\eta(x)u(x) - \eta(y)u(y)}{|x-y|^{N+2s}} dy.$$

In order to prove the required regularity, as a test function in (2.4), we choose

$$\varphi(x) := \eta(x)v(x),$$

where v is the solution of the problem

$$(4.2) \quad \begin{cases} (-\Delta)^s w = D^s_{\eta_\tau, \eta} u & \text{in } B_{1-\delta}, \\ w = 0 & \text{in } B^c_{1-\delta}. \end{cases}$$

We notice that φ is an admissible test function since $(-\Delta)^s \varphi \in L^\infty_s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$. Since η is supported in $B_{1-2\delta}$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx = \int_{\mathbb{R}^N} u(-\Delta)^s (\eta v) \, dx \\ &= \int_{B_{1-\delta}} uv(-\Delta)^s \eta \, dx + \int_{B_{1-2\delta}} u\eta(-\Delta)^s v \, dx - \int_{\mathbb{R}^N} uI_s(\eta, v) \, dx, \end{aligned}$$

where I_s is defined in (3.3). It follows

$$(4.3) \quad \left| \int_{B_{1-2\delta}} u \eta D_{\eta_\tau, \eta}^s u \, dx \right| \leq \left| \int_{B_{1-\delta}} u v (-\Delta)^s \eta \, dx \right| + \left| \int_{\mathbb{R}^N} u I_s(\eta, v) \, dx \right| \\ =: |C_1| + |C_2|.$$

First of all, rewrite the left-hand side of (4.3) as

$$(4.4) \quad \int_{\mathbb{R}^N} u \eta D_{\eta_\tau, \eta}^s u \, dx \\ = \int_{\mathbb{R}^N} u(x) \eta(x) \, dx \int_{\mathbb{R}^N} \eta_\tau(|x-y|) \frac{\eta(x)u(x) - \eta(y)u(y)}{|x-y|^{N+2s}} \, dy \\ = \frac{1}{2} \int_{\mathbb{R}^N} \, dx \int_{\mathbb{R}^N} \eta_\tau(|x-y|) \frac{(\eta(x)u(x) - \eta(y)u(y))^2}{|x-y|^{N+2s}} \, dy \\ =: \frac{1}{2} G_{\eta_\tau, \eta}^s(u).$$

Let us estimate the term C_2 . We write $C_2 = C_3 + C_4$, where

$$C_3 := \int_{B_{1-\delta}} u(x) \, dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy, \\ C_4 := \int_{B_{1-\delta}^c} u(x) \, dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy.$$

We have

$$(4.5) \quad |C_3| = \left| \int_{B_{1-\delta}} u(x) \, dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy \right| \\ \leq \|u\|_{L^2(B_{1-\delta})} \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy \right)^2 \, dx \right)^{1/2} \\ \leq \|u\|_{L^2(B_{1-\delta})} \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \, dy \right) \right. \\ \quad \cdot \left. \left(\int_{\mathbb{R}^N} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{N+2s}} \, dy \right) \, dx \right)^{1/2} \\ \leq C(\eta) \|u\|_{L^2(B_{1-\delta})} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \, dy \, dx \right)^{1/2} \\ = C(\eta) \|u\|_{L^2(B_{1-\delta})} [v]_{H^s(\mathbb{R}^N)}.$$

Now, in order to estimate the right-hand side in (4.5), we use that v is a weak solution of (4.2); by testing against v itself, we obtain

$$\begin{aligned}
 (4.6) \quad [v]_{H^s(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x - y|^{N+2s}} dy \\
 &= 2 \int_{\mathbb{R}^N} v(-\Delta)^s v \, dx = 2 \int_{\mathbb{R}^N} v D_{\eta_\tau, \eta}^s u \, dx \\
 &= 2 \int_{\mathbb{R}^N} v(x) dx \int_{\mathbb{R}^N} \frac{(\eta(x)u(x) - \eta(y)u(y))}{|x - y|^{N+2s}} \eta_\tau(|x - y|) dy \\
 &= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \left(\frac{(v(x) - v(y))}{|x - y|^{\frac{N+2s}{2}}} \sqrt{\eta_\tau(|x - y|)} \right) \\
 &\quad \cdot \left(\frac{(\eta(x)u(x) - \eta(y)u(y))}{|x - y|^{\frac{N+2s}{2}}} \sqrt{\eta_\tau(|x - y|)} \right) dy \\
 &\leq \left(\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x - y|^{N+2s}} \eta_\tau(|x - y|) dy \right)^{1/2} \\
 &\quad \cdot \left(\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(\eta(x)u(x) - \eta(y)u(y))^2}{|x - y|^{N+2s}} \eta_\tau(|x - y|) dy \right)^{1/2} \\
 &\leq [v]_{H^s(\mathbb{R}^N)} \sqrt{G_{\eta_\tau, \eta}^s(u)},
 \end{aligned}$$

where in the first estimate we applied the Hölder inequality with exponent 2 and in the second one we took into account that $\|\eta_\tau\|_{L^\infty((0, \infty))} = 1$. Summarising,

$$(4.7) \quad [v]_{H^s(\mathbb{R}^N)} \leq \sqrt{G_{\eta_\tau, \eta}^s(u)},$$

and thus by (4.5), we get

$$(4.8) \quad |C_3| \leq C(\eta) \|u\|_{L^2(B_{1-\delta})} \sqrt{G_{\eta_\tau, \eta}^s(u)}.$$

Let us estimate C_4 . Since $\eta(x) = \eta(y) = 0$ for $x \in B_{1-\delta}^c$ and $y \in B_{1-2\delta}^c$, we have

$$\begin{aligned}
 |C_4| &= \left| \int_{B_{1-\delta}^c} u(x) dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \right| \\
 &= \left| \int_{B_{1-\delta}^c} u(x) dx \int_{B_{1-2\delta}} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \right|,
 \end{aligned}$$

and then

$$\begin{aligned}
 (4.9) \quad |C_4| &= \left| \int_{B_{1-\delta}^c} \frac{u(x)}{1+|x|^{N+2s}} dx \int_{B_{1-2\delta}} \left(\frac{1+|x|^{N+2s}}{|x-y|^{N+2s}} \right) v(y) \eta(y) dy \right| \\
 &\leq \int_{B_{1-\delta}^c} \frac{|u(x)|}{1+|x|^{N+2s}} dx \int_{B_{1-2\delta}} \left(1 + \frac{1+|y|}{|x-y|} \right)^{N+2s} |v(y)| dy \\
 &\leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} [v]_{H^s(\mathbb{R}^N)} \\
 &\leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \sqrt{G_{\eta_\tau, \eta}^s(u)}.
 \end{aligned}$$

In the third inequality, we exploited (4.7), and the second one follows from the Hölder inequality and the fractional Sobolev inequality, see [22, Theorem 6.5]; taking into account that v has compact support in $B_{1-\delta}$, we have

$$\|v\|_{L^1(B_{1-2\delta})} \leq |B_{1-2\delta}|^{\frac{2_s^*-1}{2_s^*}} \|v\|_{L^{2_s^*}(\mathbb{R}^N)} \leq C(\delta, N, s) [v]_{H^s(\mathbb{R}^N)}.$$

Now, let us estimate the term C_1 :

$$\begin{aligned}
 (4.10) \quad |C_1| &= \left| \int_{B_{1-\delta}} u v (-\Delta)^s \eta dx \right| \\
 &\leq \|u\|_{L^{(2_s^*)'}(B_{1-\delta})} \|v\|_{L^{2_s^*}(\mathbb{R}^N)} \|(-\Delta)^s \eta\|_{L^\infty(\mathbb{R}^N)} \\
 &\leq C(\delta) [v]_{H^s(\mathbb{R}^N)} \leq C(\delta) \sqrt{G_{\eta_\tau, \eta}^s(u)},
 \end{aligned}$$

where $C(\delta) > 0$ can be explicitly computed, and in the second and third inequalities, we exploited again the fractional Sobolev inequality and (4.7), respectively.

By putting together (4.5), (4.9) and (4.10) in (4.3) and using equality (4.4), we have

$$(4.11) \quad G_{\eta_\tau, \eta}^s(u) \leq C(\|u\|_{L_s^1(\mathbb{R}^N)}, \|u\|_{L^2(B_{1-\delta})}, \delta, \eta).$$

Now, we recall that η_τ depends on the parameter $\tau \in (0, 1/2)$, but the estimate (4.11) is uniform with respect to τ because the right-hand side is independent of τ . Therefore, estimate (4.11) finally yields

$$(4.12) \quad [\eta u]_{H^s(\mathbb{R}^N)}^2 \leq \sup_{\tau \in (0, 1/2)} G_{\eta_\tau, \eta}^s(u) < \infty,$$

where the first inequality holds true in view of Fatou's lemma. Thus, $\eta u \in H^s(\mathbb{R}^N)$ and then $u \in H_{\text{loc}}^s(B)$.

In order to complete the proof, let us show that $\eta^2 u$ is a compactly supported weak solution of $(-\Delta)^s w = f$, with $f \in L^2(\mathbb{R}^N)$. This implies that $\eta^2 u \in H^{2s}(\mathbb{R}^N)$.

For any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(\eta^2(x)u(x) - \eta^2(y)u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dy \\
&= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{\eta^2(x)u(x)(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dy \\
&= C_{N,s}^{-1} \int_{\mathbb{R}^N} u(x)(-\Delta)^s(\eta^2\varphi)(x)dx + \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{u(x)\varphi(y)(\eta^2(y) - \eta^2(x))}{|x - y|^{N+2s}} dy \\
&= \int_{\mathbb{R}^N} \varphi(x)dx \int_{\mathbb{R}^N} \frac{u(y)(\eta^2(x) - \eta^2(y))}{|x - y|^{N+2s}} dy,
\end{aligned}$$

where in the last equality we used that $\eta^2\varphi \in C_c^\infty(B)$ and that u is a very weak solution of (1.2). To conclude, we show that the function f defined a.e. by

$$(4.13) \quad f(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{u(y)(\eta^2(x) - \eta^2(y))}{|x - y|^{N+2s}} dy$$

belongs to $L^2(\mathbb{R}^N)$. We point out that

$$\begin{aligned}
f(x) &= C_{N,s} \int_{\mathbb{R}^N} \frac{u(y)(\eta(x) + \eta(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \\
&= C_{N,s} \int_{\mathbb{R}^N} \frac{[2u(x)\eta(x) + u(y)\eta(y) - u(x)\eta(x) + \eta(x)(u(y) - u(x))](\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \\
&= 2\eta(x)u(x)(-\Delta)^s\eta(x) - I_s(\eta, \eta u)(x) - \eta(x)I_s(\eta, u)(x).
\end{aligned}$$

The last equality shows that $f \in L^2(\mathbb{R}^N)$, as $I_s(\eta, \eta u)$ and $\eta I_s(\eta, u)$ belong to $L^2(\mathbb{R}^N)$ in view of Remark 3.3 and $\eta u(-\Delta)^s\eta$, supported in $B_{1-2\delta}$, belongs to $L^2(\mathbb{R}^N)$ since $u \in L_{\text{loc}}^2(B)$.

We thus obtain that $\eta^2u \in H^{2s}(\mathbb{R}^N)$, and so $u \in H_{\text{loc}}^{2s}(B)$. ■

5. FULL REGULARITY

In this section, we prove that a very weak s -harmonic function u is actually a classical s -harmonic function hence locally smooth. To do this, we use the fact that u is locally bounded and by fractional De Giorgi estimates proved in [13, Theorem 1.4] also belongs to $C_{\text{loc}}^{0,\gamma}(B)$ for every $\gamma \in (0, \min\{2s, 1\})$. Then, in Propositions 5.2 and 5.3, we prove that the operator I_s enjoys useful Hölder continuity properties that allow us to exploit that the function η^2u solves the equation $(-\Delta)^s w = f \in C^{0,\gamma}(\mathbb{R}^N)$, and then $u \in C_{\text{loc}}^{\gamma+2s}(B) \cap L_s^1(\mathbb{R}^N)$. Therefore, u is s -harmonic in B in the classical sense and, by Theorem 5.6, is real analytic in B .

THEOREM 5.1. *Let $u \in H_{\text{loc}}^{2s}(B) \cap L_s^1(\mathbb{R}^N)$ a very weak s -harmonic function in B . Then, $u \in C_{\text{loc}}^{0,\alpha}(B)$ for every $\alpha = \alpha(s) \in (0, \min\{2s, 1\})$.*

PROOF. Since $u \in H_{\text{loc}}^{2s}(B) \cap L_s^1(\mathbb{R}^N)$, then u is a local weak solution of (1.2), and by [13, Theorem 3.2, Remark 3.3], one has that $u \in L_{\text{loc}}^\infty(B)$. Then, the claim plainly follows from [13, Theorem 1.4]. \blacksquare

PROPOSITION 5.2. *Let $s \in (0, 1)$ and $\alpha \in (s, \min\{2s, 1\})$. For any $f \in C^{0,\alpha}(\mathbb{R}^N)$ and $g \in C^{0,1}(\mathbb{R}^N)$, we have that*

$$I_s(f, g) \in C^{0,\gamma(\alpha,s)}(\mathbb{R}^N) \quad \text{where } \gamma(\alpha, s) := \begin{cases} 2\alpha - 2s & \text{if } 0 < s \leq \frac{1}{2}, \\ \alpha - 2s + 1 & \text{if } \frac{1}{2} < s < 1 \end{cases}$$

and

$$[I_s(f, g)]_{C^{0,\gamma(\alpha,s)}(\mathbb{R}^N)} \leq C[f]_{C^{0,\alpha}(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}.$$

PROOF. Let $x, x' \in \mathbb{R}^N$, $x \neq x'$, and let $R := |x - x'|$. We estimate

$$(5.1) \quad I_s(f, g)(x) - I_s(f, g)(x') = \int_{\mathbb{R}^N} \frac{(f(x) - f(x-y))(g(x) - g(x-y)) - (f(x') - f(x'-y))(g(x') - g(x'-y))}{|y|^{N+2s}} dy.$$

By adding and subtracting $(f(x) - f(x-y))(g(x') - g(x'-y))$ in the numerator of the integrand in (5.1), we can equivalently write

$$I_s(f, g)(x) - I_s(f, g)(x') = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 &:= \int_{B_R} \frac{(f(x) - f(x-y))[(g(x) - g(x-y)) + (g(x'-y) - g(x'))]}{|y|^{N+2s}} dy, \\ J_2 &:= \int_{B_R^c} \frac{(f(x) - f(x-y))[(g(x) - g(x')) + (g(x'-y) - g(x-y))]}{|y|^{N+2s}} dy, \\ J_3 &:= \int_{B_R} \frac{(g(x') - g(x'-y))[(f(x) - f(x-y)) + (f(x'-y) - f(x'))]}{|y|^{N+2s}} dy, \\ J_4 &:= \int_{B_R^c} \frac{(g(x') - g(x'-y))[(f(x) - f(x')) + (f(x'-y) - f(x-y))]}{|y|^{N+2s}} dy. \end{aligned}$$

Now if $s \leq \frac{1}{2}$, we have

$$\begin{aligned} |J_1| &\leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)} \int_{B_R} \frac{dy}{|y|^{N+2s-\alpha-1}} \\ &= C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)} |x - x'|^{\alpha-2s+1}, \end{aligned}$$

$$\begin{aligned}
|J_2| &\leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'| \int_{B_R^c} \frac{dy}{|y|^{N+2s-\alpha}} \\
&= C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^{\alpha-2s+1}.
\end{aligned}$$

The estimate of J_3 is analogous to the one of J_1 , while for J_4 we have

$$\begin{aligned}
|J_4| &\leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^\alpha(\mathbb{R}^N)}|x - x'|^\alpha \int_{B_R^c} \frac{dy}{|y|^{N+2s-\alpha}} \\
&= C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^\alpha(\mathbb{R}^N)}|x - x'|^{2\alpha-2s}.
\end{aligned}$$

Since $2\alpha - 2s < \alpha - 2s + 1$, we get the thesis.

If $s > \frac{1}{2}$, the estimates of J_1, J_2, J_3 are analogous to the previous case, while for J_4 we can write

$$\begin{aligned}
|J_4| &\leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^\alpha \int_{B_R^c} \frac{dy}{|y|^{N+2s-1}} \\
&= C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^{\alpha-2s+1}.
\end{aligned}$$

Hence, $I_s(f, g) \in C^{\alpha-2s+1}(\mathbb{R}^N)$. ■

PROPOSITION 5.3. *Let $s \in (0, 1)$, $\alpha \in (s, \min\{2s, 1\})$ and $\beta = \beta(\alpha) := \frac{\alpha}{\alpha+1} < \frac{\alpha}{2s}$. If $f \in C_{\text{loc}}^{0,\alpha}(B) \cap L_s^1(\mathbb{R}^N)$ and $\eta \in C_c^\infty(B)$, then $\eta I_s(\eta, f) \in C^{0,\gamma}(\mathbb{R}^N)$ where $\gamma := (\alpha - 2s + 1)\beta$ and*

$$\|\eta I_s(\eta, f)\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq C(\|f\|_{C^{0,\alpha}(B_{r'})} + \|f\|_{L_s^1(\mathbb{R}^N)}),$$

where $r' \in (0, 1)$ is such that $B_{r'} = \text{supp } \eta$.

PROOF. Let $B_r \Subset B$ with $0 < r' < r$. Then, $f \in C^{0,\alpha}(B_r)$. Since

$$\begin{aligned}
\|\eta I_s(\eta, f)\|_{C^{0,\gamma}(\mathbb{R}^N)} &\leq \|\eta\|_{C^{0,\gamma}(\mathbb{R}^N)} \|I_s(\eta, f)\|_{L^\infty(B_{r'})} \\
&\quad + \|\eta\|_{L^\infty(\mathbb{R}^N)} \|I_s(\eta, f)\|_{C^{0,\gamma}(B_{r'})},
\end{aligned}$$

we reduce to prove that $I_s(\eta, f) \in C^{0,\gamma}(B_{r'})$. By (3.20), we already know that $I_s(\eta, f) \in L^\infty(B_{r'})$.

Now, let $x, x' \in B_{r'}$, $x \neq x'$. We set $R := |x - x'|$. We assume that $R < (\frac{1}{2})^{\frac{1}{1-\beta}}$; otherwise the proof is done. We observe that thanks to the required upper bound on R , the following inclusions of sets

$$B_{R^\beta}(x') \subset B_{\frac{3}{2}R^\beta}(x) \quad \text{and} \quad B_{R^\beta}^c(x') \subset B_{\frac{R^\beta}{2}}^c(x)$$

hold true. We write

$$\begin{aligned}
(5.2) \quad & |I_s(\eta, f)(x) - I_s(\eta, f)(x')| \\
&= \left| \int_{\mathbb{R}^N} \frac{(f(x) - f(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy - \int_{\mathbb{R}^N} \frac{(f(x') - f(y))(\eta(x') - \eta(y))}{|x' - y|^{N+2s}} dy \right| \\
&\leq \int_{B_{R^\beta}(x')} \frac{|f(x) - f(y)| |\eta(x) - \eta(y)|}{|x - y|^{N+2s}} dy \\
&\quad + \int_{B_{R^\beta}(x')} \frac{|f(x') - f(y)| |\eta(x') - \eta(y)|}{|x' - y|^{N+2s}} dy \\
&\quad + \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - \frac{|x-y|^{N+2s}}{|x'-y|^{N+2s}} (f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right| \\
&\leq \int_{B_{\frac{3}{2}R^\beta}(x)} \frac{|f(x) - f(y)| |\eta(x) - \eta(y)|}{|x - y|^{N+2s}} dy \\
&\quad + \int_{B_{R^\beta}(x')} \frac{|f(x') - f(y)| |\eta(x') - \eta(y)|}{|x' - y|^{N+2s}} dy \\
&\quad + \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - \frac{|x-y|^{N+2s}}{|x'-y|^{N+2s}} (f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right| \\
&\leq C[f]_{C^{0,\alpha}(B_r)} R^{(\alpha-2s+1)\beta} \\
&\quad + \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - \frac{|x-y|^{N+2s}}{|x'-y|^{N+2s}} (f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right| \\
&=: C[f]_{C^{0,\alpha}(B_r)} R^{(\alpha-2s+1)\beta} + |B|.
\end{aligned}$$

By adding and subtracting $\frac{(f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}}$ to the integrand defining B , we have

$$\begin{aligned}
|B| &\leq \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - (f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right| \\
&\quad + \int_{B_{R^\beta}^c(x')} \left| \frac{1}{|x - y|^{N+2s}} - \frac{1}{|x' - y|^{N+2s}} \right| |f(x') - f(y)| |\eta(x') - \eta(y)| dy \\
&\leq \int_{B_{R^\beta}^c(x')} \frac{|f(x)\eta(x) - f(x')\eta(x')|}{|x - y|^{N+2s}} dy \\
&\quad + \int_{B_{R^\beta}^c(x')} \frac{|f(y)| |\eta(x) - \eta(x')|}{|x - y|^{N+2s}} dy + \int_{B_{R^\beta}^c(x')} \frac{|\eta(y)| |f(x) - f(x')|}{|x - y|^{N+2s}} dy \\
&\quad + \int_{B_{R^\beta}^c(x')} \left| \frac{1}{|x - y|^{N+2s}} - \frac{1}{|x' - y|^{N+2s}} \right| |f(x') - f(y)| |\eta(x') - \eta(y)| dy \\
&\leq \int_{B_{\frac{R^\beta}{2}}^c(x)} \frac{|f(x)\eta(x) - f(x')\eta(x')|}{|x - y|^{N+2s}} dy \\
&\quad + \int_{B_{\frac{R^\beta}{2}}^c(x)} \frac{|f(y)| |\eta(x) - \eta(x')|}{|x - y|^{N+2s}} dy + \int_{B_{\frac{R^\beta}{2}}^c(x)} \frac{|\eta(y)| |f(x) - f(x')|}{|x - y|^{N+2s}} dy \\
&\quad + \int_{B_{R^\beta}^c(x')} \left| \frac{1}{|x - y|^{N+2s}} - \frac{1}{|x' - y|^{N+2s}} \right| |f(x') - f(y)| |\eta(x') - \eta(y)| dy \\
&=: B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

Now we estimate B_i for any $i \in \{1, 2, 3, 4\}$:

$$(5.3) \quad B_1 \leq C[\eta u]_{C^{0,\alpha}(\mathbb{R}^N)} R^\alpha \int_{B_{R\beta/2}^c(x)} \frac{dy}{|x-y|^{N+2s}} \leq C[\eta u]_{C^{0,\alpha}(\mathbb{R}^N)} R^{\alpha-2\beta s},$$

$$(5.4) \quad \begin{aligned} B_2 &\leq CR \left(\int_{B_{R\beta/2}^c(x) \cap B_r} \frac{|f(y)|}{|x-y|^{N+2s}} dy + \int_{B_{R\beta/2}^c(x) \cap B_r^c} \frac{|f(y)|}{|x-y|^{N+2s}} dy \right) \\ &\leq CR \left(\|f\|_{L^\infty(B_r)} \int_{B_{R\beta/2}^c(x)} \frac{dy}{|x-y|^{N+2s}} + C(r, r') \int_{B_r^c} \frac{|f(y)|}{1+|y|^{N+2s}} dy \right) \\ &\leq CR (C_1 \|f\|_{L^\infty(B_r)} R^{-2\beta s} + C(r, r') \|f\|_{L_s^1(\mathbb{R}^N)}) \\ &\leq C(r, r', N, s) (\|f\|_{L^\infty(B_r)} + \|f\|_{L_s^1(\mathbb{R}^N)}) R^{1-2\beta s} \\ &\leq C(r, r', N, s) (\|f\|_{L^\infty(B_r)} + \|f\|_{L_s^1(\mathbb{R}^N)}) R^{\alpha-2\beta s}, \end{aligned}$$

$$(5.5) \quad B_3 \leq C[f]_{C^{0,\alpha}(B_{r'})} R^\alpha \int_{B_{R\beta/2}^c(x)} \frac{dy}{|x-y|^{N+2s}} \leq C[f]_{C^{0,\alpha}(B_{r'})} R^{\alpha-2\beta s}.$$

For B_4 we write

$$\begin{aligned} B_4 &= \int_{B_{R\beta}^c(x') \cap B_r} \left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| |f(x')-f(y)| |\eta(x')-\eta(y)| dy \\ &\quad + \int_{B_{R\beta}^c(x') \cap B_r^c} \left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| |f(x')-f(y)| |\eta(x')-\eta(y)| dy \\ &=: D_1 + D_2. \end{aligned}$$

Using the Fundamental Theorem of Calculus, we write

$$\left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| \leq C_{N,s} |x-x'| |x'-y|^{-N-2s-1}.$$

Therefore,

$$(5.6) \quad \begin{aligned} D_1 &\leq C[f]_{C^{0,\alpha}(B_{r'})} R \int_{B_{R\beta}^c(x')} |x'-y|^{\alpha+1} |x'-y|^{-N-2s-1} dy \\ &= C[f]_{C^{0,\alpha}(B_{r'})} R^{1+\beta(\alpha-2s)} \leq C[f]_{C^{0,\alpha}(B_{r'})} R^{(\alpha-2s+1)\beta}. \end{aligned}$$

To conclude for D_2 , we have

$$(5.7) \quad \begin{aligned} D_2 &\leq CR \int_{B_{R\beta}^c(x') \cap B_r^c} |f(x')-f(y)| |x'-y| |x'-y|^{-N-2s-1} dy \\ &\leq CR \left(\|f\|_{L^\infty(B_{r'})} \int_{B_{R\beta}^c(x')} \frac{dy}{|x'-y|^{N+2s}} + \int_{B_r^c} \frac{|f(y)|}{|x'-y|^{N+2s}} dy \right) \end{aligned}$$

$$\begin{aligned}
&\leq CR \left(C_1 \|f\|_{L^\infty(B_{r'})} R^{-2\beta s} + C(r, r') \int_{\mathbb{R}^N} \frac{|f(y)|}{1 + |y|^{N+2s}} dy \right) \\
&\leq C(r, r', N, s) (\|f\|_{L^\infty(B_{r'})} + \|f\|_{L_s^1(\mathbb{R}^N)}) R^{1-2\beta s} \\
&\leq C(r, r', N, s) (\|f\|_{L^\infty(B_{r'})} + \|f\|_{L_s^1(\mathbb{R}^N)}) R^{\alpha-2\beta s}.
\end{aligned}$$

Putting (5.3), (5.4), (5.5), (5.6) and (5.7) into (5.2) and taking into account that $(\alpha - 2s + 1)\beta = \alpha - 2\beta s$ by the choice of β , our claim is proved. ■

THEOREM 5.4. *Let $u \in C_{\text{loc}}^{0,\gamma}(B) \cap L_s^1(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$ a very weak s -harmonic function in B . Then, $u \in C_{\text{loc}}^{\gamma+2s}(B) \cap L_s^1(\mathbb{R}^N)$; hence, u is s -harmonic in the classical sense in B .*

PROOF. Let $\eta \in C_c^\infty(B)$. By Theorem 4.1, the function $\eta^2 u$ is a weak solution of $(-\Delta)^s w = f$ in \mathbb{R}^N with $f := 2\eta u(-\Delta)^s \eta - I_s(\eta, \eta u) - \eta I_s(\eta, u)$. Moreover by applying Theorem 5.1 with $\alpha \in (s, \min\{2s, 1\})$ and Propositions 5.2 and 5.3, we have that $f \in C^{0,\gamma}(\mathbb{R}^N)$. Therefore, by using Schauder estimates for bounded weak solutions to $(-\Delta)^s w = f$ (see [44, Proposition 2.8] or [45, Theorem 15]), it follows that $\eta^2 u \in C^{0,\gamma+2s}(\mathbb{R}^N)$ if $0 < \gamma + 2s < 1$ or $\eta^2 u \in C^{1,\gamma+2s-1}(\mathbb{R}^N)$ if $1 < \gamma + 2s < 2$ or $\eta^2 u \in C^{2,\gamma+2s-2}(\mathbb{R}^N)$ if $2 < \gamma + 2s < 3$. If $\gamma + 2s \in \mathbb{N}$, apply Propositions 5.2 and 5.3 replacing α with α_1 such that $\alpha_1 + 2s \notin \mathbb{N}$. By the arbitrariness of η , we get the thesis. ■

REMARK 5.5. Notice that when $s \in (0, \frac{1}{2})$, Theorem 5.4 also follows from [11, Theorem 3.5] applied with $p = 2$.

THEOREM 5.6. *Let $u \in C_{\text{loc}}^{\gamma+2s}(B) \cap L_s^1(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$ a very weak s -harmonic function in B . Then, u is real analytic in B .*

PROOF. Let $\delta \in (0, 1)$ and $R = R(\delta) := 1 - \frac{\delta}{4}$, $r = r(\delta) := 1 - \frac{\delta}{2}$, $r_0 = r_0(\delta) := 1 - \delta$. By Theorem 5.4, u is a classical solution of

$$(5.8) \quad \begin{cases} (-\Delta)^s w = 0 & \text{in } B_r, \\ w = h & \text{in } B_r^c, \end{cases}$$

where

$$h := \begin{cases} u & \text{in } B, \\ g & \text{in } B^c. \end{cases}$$

Since $h \in C(\overline{B_R}) \cap L_s^1(\mathbb{R}^N)$ if for $\rho > 0$ we set

$$P_\rho(x, y) := C_{N,s} \left(\frac{\rho^2 - |x|^2}{|y|^2 - \rho^2} \right)^s \frac{1}{|x - y|^N}$$

the fractional Poisson kernel (see [14]) for B_ρ for any $x \in B_\rho$ and $y \in B_\rho^c$, the function

$$(5.9) \quad u_h(x) := \int_{B_r^c} P_r(x, y) h(y) dy$$

is well posed for every $x \in B_r$. Indeed

$$\begin{aligned} (5.10) \quad |u_h(x)| &\leq \int_{B_R \setminus B_r} P_r(x, y) |h(y)| dy + \int_{B \setminus B_R} P_r(x, y) |h(y)| dy \\ &\quad + \int_{B^c} P_r(x, y) |h(y)| dy \\ &= \int_{B_R \setminus B_r} P_r(x, y) |u(y)| dy + \int_{B \setminus B_R} P_r(x, y) |u(y)| dy \\ &\quad + \int_{B^c} P_r(x, y) |g(y)| dy \\ &\leq \|u\|_{L^\infty(B_R)} \int_{B_r^c} P_r(x, y) dy + \frac{r^{2s}}{(R^2 - r^2)^s (R - r)^N} \|u\|_{L^1(B)} \\ &\quad + C_{r, R, N, s} \int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2s}} dy \\ &\leq C(r, R, N, s) (\|u\|_{L^\infty(B_R)} + \|u\|_{L^1_s(\mathbb{R}^N)}). \end{aligned}$$

By [14, Theorem 2.10], the function u_h is a classical solution of (5.8). By the uniqueness of solutions of (5.8) (see [16, Theorem 3.3.2]), we conclude that $u_h = u$ in B_r .

Moreover, for every $y \in B_r^c$, the function

$$B_{r_0} \ni x \mapsto \frac{(r^2 - |x|^2)^s}{|x - y|^N}$$

is smooth, and it is easy to check that

$$(5.11) \quad |(\partial_x^\iota P_r)(x, y)| \leq c^{|\iota|} \iota! C(r, r_0, N, s) P_r(x, y)$$

for every $x \in B_{r_0}$, $y \in B_r^c$ and $\iota \in \mathbb{N}_0^N$. Therefore, by differentiating under integral sign formula (5.9) by estimates (5.10), (5.11), we have

$$\|D^\iota u\|_{L^\infty(B_{r_0})} \leq c^{|\iota|} \iota! C(R, r, r_0, N, s) (\|u\|_{L^\infty(B_R)} + \|u\|_{L^1_s(\mathbb{R}^N)})$$

for any $\iota \in \mathbb{N}_0^N$. From the arbitrariness of $\delta \in (0, 1)$, we get the thesis. \blacksquare

Data availability statement. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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