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The Radul cocycle, the Chern–Connes character, and manifolds with conical singularities

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Abstract. The present work is a continuation of a previous article of ours. First, we aim to explain how the residue index cocycle we had obtained, via pseudodifferential extensions, zeta functions and the boundary map in periodic cyclic cohomology, relates to the Connes–Moscovici residue cocycle. On the other hand, we explore the case of manifolds with conical singularities, and explain why J.-M. Lescure's construction of a regular spectral triple in this situation cannot be significantly improved.

Introduction

In a former article [12], we established a formula calculating the Chern character (in K-homology) of an abstract pseudodifferential extension in terms of residues of zeta functions via a so-called *Radul cocycle*. We direct the reader to the aforementioned article for further references on the history of such a formula (see, e.g., [8]), and recent applications (see, e.g., [9–11]). The present note is a modest complement of this work, where we explain the relationship between our formula and Connes–Moscovici's residue cocycle [1], and also expand on the motivations behind our work. More precisely, we establish the following results:

- We prove that the Connes–Moscovici residue cocycle is a pull-back of the Radul cocycle via a morphism of extensions that involve a pseudodifferential extension and a certain 'truncation' of it. See Theorem 3.1.
- We prove that zeta functions associated to pseudodifferential calculus in a Melrose-type calculus on conical manifolds may exhibit triple poles, slightly expanding a result of J.-M. Lescure [5] that provides a regular spectral triple in this context. We also explain why his result cannot really be improved, in the sense that enlarging the algebra of his spectral triple destroys regularity. We then discuss the modification of the Radul cocycle in this context. This is the content of Theorem 4.11, Corollary 4.12 and Sections 4.2, 4.3.
- Example 2.3 corrects a sign mistake contained in [12] for the operator F.

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In Section 1, we recall some material from an article by Higson [4], which introduces a formalism of abstract differential operators based on the work of Connes–Moscovici, and proposes a conceptual view of the notion of regular spectral triple. This naturally yields an abstract pseudodifferential extension, and we recall in Section 2 the derivation of its Chern character in the form of a cyclic 1-cocycle generalizing the Radul cocycle, applicable to contexts where the zeta function exhibits multiple poles. After mentioning basic examples, we establish in Section 3 the first aforementioned main result, and the second one in the final section of the paper. Background material on the subject, notably on Melrose's pseudodifferential calculus, is provided throughout.

1. Abstract differential operators and traces

In this section, we review the abstract differential operators formalism developed by Higson [4] (see also [13]), introduced in order to conceptualize the proof of the Connes–Moscovici local index formula [1]. Using this language, we then explain how the latter is connected to the calculation of a boundary map in periodic cyclic cohomology associated to a pseudodifferential extension.

1.1. Abstract differential operators

Let H be a complex Hilbert space and let Δ be an unbounded, positive and self-adjoint operator acting on it, with domain $\mathrm{dom}(\Delta)$. Suppose that Δ has a compact resolvent.

Introduce the following space H^∞:

$$H^\infty = \bigcap_{k=0}^{\infty} \mathrm{dom}(\Delta^k)$$

which, according to Sobolev space theory, plays the role of smooth functions.

Definition 1.1. An algebra $\mathcal{D}(\Delta)$ of *abstract differential operators* associated to Δ is an algebra of operators on H^∞ satisfying the following conditions:

(i) The algebra $\mathcal{D}(\Delta)$ is filtered, i.e., it comes with an increasing sequence of vector spaces $(\mathcal{D}_q(\Delta))_{q\in\mathbb{N}}$ such that

$$\mathcal{D}(\Delta) = \bigcup_{q=0}^{\infty} \mathcal{D}_q(\Delta)$$

and $\mathcal{D}_p(\Delta) \cdot \mathcal{D}_q(\Delta) \subset \mathcal{D}_{p+q}(\Delta)$, for every $p, q \in \mathbb{N}$. An element $X \in \mathcal{D}_q(\Delta)$ is an *abstract differential operator of order at most q*.

(ii) There is an $r > 0$ (the order of Δ) such that for every $X \in \mathcal{D}_q(\Delta)$, $[\Delta, X] \in \mathcal{D}_{r+q-1}(\Delta)$.

(iii) *Elliptic estimate.* For $s \geq 0$, define the *s-Sobolev space* H^s as the subspace $\operatorname{dom}(\Delta^{s/r})$ of H, which is a Hilbert space when equipped with the norm

$$\|v\|_s = (\|v\|^2 + \|\Delta^{s/r} v\|^2)^{1/2}.$$

If $X \in \mathcal{D}_q(\Delta)$, then there is a constant $\varepsilon > 0$ such that

$$\|v\|_q + \|v\| \geq \varepsilon \|Xv\|, \quad \forall v \in H^\infty.$$

Following Sobolev space theory, the elliptic estimate tells that $\Delta^{1/r}$ should be thought as an 'abstract elliptic operator' of order 1. It also means that a differential operator X of order q extends to a bounded operator from H^{s+q} to H^s, which leads to a pseudodifferential calculus. The main example to keep in mind is of course the case in which Δ is a Laplace-type operator on a closed Riemannian manifold M.

1.2. Correspondence with spectral triples

Let (A, H, D) a spectral triple (cf. [1] or [4]). The associated *algebra of abstract differential operators* $\mathcal{D} = \mathcal{D}(A, D)$ is the smallest algebra containing A, $[D, A]$, and which is closed under the operation $\operatorname{ad}(\Delta) = [\Delta, \bullet]$. A natural filtration is defined as follows:

$$\begin{aligned}
\mathcal{D}_0 &= \text{algebra generated by } A \text{ and } [D, A], \\
\mathcal{D}_1 &= \mathcal{D}_0 + [\Delta, \mathcal{D}_0] + \mathcal{D}_0[\Delta, \mathcal{D}_0], \\
&\vdots \\
\mathcal{D}_k &= \mathcal{D}^{k-1} + \sum_{j=1}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{k-j} + [\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_0[\Delta, \mathcal{D}_{k-1}].
\end{aligned}$$

Let δ be the unbounded derivation $\operatorname{ad}|D| = [|D|, \,.\,]$ on $\mathcal{B}(H)$. The spectral triple (A, H, D) is said *regular* if A and $[D, A]$ are contained in $\bigcap_{n=1}^\infty \operatorname{dom} \delta^n$. The following result relates algebras of abstract differential operators and spectral triples.

Theorem 1.2 (Higson [4]). *Suppose that A (seen as a subalgebra of bounded operators on H) maps H^∞ into itself. Then, the spectral triple (A, H, D) is regular if and only if the elliptic estimate of Definition 1.1 holds.*

1.3. Zeta functions

Let $\mathcal{D}(\Delta)$ be an algebra of abstract differential operators. To keep the exposition simple, suppose throughout that Δ is invertible; we refer the reader to [4, Section 6] for details about how to remove this hypothesis. For $z \in \mathbb{C}$, one defines the *complex power* Δ^{-z} of Δ using functional calculus,

$$\Delta^{-z} = \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - \Delta)^{-1} \, d\lambda,$$

where the contour of integration is a vertical line pointing downwards separating 0 and the (discrete) spectrum of Δ. This converges in the operator norm when $\mathrm{Re}(z) > 0$, and using the semi-group property, all the complex powers can be defined via multiplication with Δ^k, for $k \in \mathbb{N}$ large enough.

We suppose that there exists $d \geq 0$ such that for every $X \in \mathcal{D}_q(\Delta)$, the operator $X\Delta^{-z}$ extends to a trace-class operator on H for z on the half-plane $\mathrm{Re}(z) > (q+d)/r$, where $r > 0$ is the order of Δ. The *zeta function* of X is

$$\zeta_X(z) = \mathrm{Tr}(X\Delta^{-z/r}).$$

The smallest d verifying the above property is called the *analytic dimension* of $\mathcal{D}(\Delta)$. In this case, the zeta function is holomorphic on the half-plane $\mathrm{Re}(z) > q + d$. We shall say that $\mathcal{D}(\Delta)$ has the *analytic continuation property* if for every $X \in \mathcal{D}(\Delta)$, the associated zeta function extends to a meromorphic function of the whole complex plane.

Example 1.3. These notions come from properties of the zeta function on a closed Riemannian manifold M: it is well known that the algebra of differential operators on M has analytic dimension $\dim(M)$ and the analytic continuation property. Its extension to a meromorphic function has at most simple poles at the integers less or equal than $\dim(M)$.

1.4. Abstract pseudodifferential operators

Suppose that the algebra of abstract differential operators $\mathcal{D}(\Delta)$ has analytic dimension d. To define a notion of pseudodifferential operator, we need a general notion of order that covers the one induced by the filtration of $\mathcal{D}(\Delta)$.

Definition 1.4. An operator $T : H^\infty \to H^\infty$ is said to have *pseudodifferential order* $m \in \mathbb{R}$ if for every $s \geq 0$, it extends to a bounded operator from H^{m+s} to H^s. In addition, we require that operators of analytic order strictly less than $-d$ are trace-class operators.

That this notion of order covers the differential order is due to the elliptic estimate, as already remarked in Section 1.1. The space of such operators, denoted $\mathrm{Op}(\Delta)$, forms an $\mathbb{R}$-filtered algebra. As in standard pseudodifferential calculus, one defines smoothing operators as the elements of the (two-sided) ideal of operators of all order.

Example 1.5. For every $\lambda \in \mathbb{C}$ not contained in the spectrum of Δ, the resolvent $(\lambda - \Delta)^{-1}$ has analytic order r. Moreover, by spectral theory, its norm as an operator between Sobolev spaces is a $O(|\lambda|^{-1})$.

The following notion is due to Uuye, cf. [13]. We just added an assumption on the zeta function which is necessary for what we do.

Definition 1.6. An algebra of abstract pseudodifferential operators is an $\mathbb{R}$-filtered subalgebra $\Psi(\Delta)$ of $\mathrm{Op}(\Delta)$, also denoted Ψ when the context is clear, satisfying

$$\Delta^{z/r}\Psi^m \subset \Psi^{\mathrm{Re}(z)+m}, \quad \Psi^m\Delta^{z/r} \subset \Psi^{\mathrm{Re}(z)+m}$$

and which commutes, up to operators of lower order, with the complex powers of $\Delta^{1/r}$, that is, for all $m \in \mathbb{R}$, $z \in \mathbb{C}$,

$$[\Delta^{z/r}, \Psi^m] \subset \Psi^{\mathrm{Re}(z)+m-1}.$$

Moreover, we suppose that for every $P \in \Psi^m(\Delta)$, the zeta function

$$\zeta_P(z) = \mathrm{Tr}(P\Delta^{-z/r})$$

is holomorphic on the half-plane $\mathrm{Re}(z) > m + d$, and extends to a meromorphic function of the whole complex plane. The ideal of (abstract) smoothing operators is

$$\Psi^{-\infty} = \bigcap_{m\in\mathbb{R}} \Psi^m.$$

We end this part with a notion of asymptotic expansion for abstract pseudodifferential operators.

Definition 1.7. Let T and T_α (α in a set A) be operators on Ψ. We shall write

$$T \sim \sum_{\alpha\in A} T_\alpha$$

if there exist $c > 0$ and a finite subset $F \subset A$ such that for all finite sets $F' \subset A$ containing F, the map

$$z \mapsto \mathrm{Tr}\Big(\Big(T - \sum_{\alpha\in F'} T_\alpha\Big)\Delta^{z/r}\Big)$$

is holomorphic in a half-plane $\mathrm{Re}(z) > -c$ (which contains $z = 0$).

Therefore, being asymptotic simply means only a finite number of terms survive under the residue. The observation that follows is basically the cornerstone of Connes–Moscovici's local index formula.

Lemma 1.8 (Connes–Moscovici's trick, [1, 4]). *Let $Q \in \Psi(\Delta)$ be an abstract pseudodifferential operator. Then, for any $z \in \mathbb{C}$, we have*

$$[\Delta^{-z}, Q] \sim \sum_{k\geq 1} \binom{-z}{k} Q^{(k)}\Delta^{-z-k}$$

where we denote $Q^{(k)} = \mathrm{ad}(\Delta)^k(Q)$, $\mathrm{ad}(\Delta) = [\Delta, \bullet]$.

The proof relies on the following identity, for $z \in \mathbb{C}$ with $\mathrm{Re}(z)$ large enough (cf. [4, Lemma 4.20]):

$$\Delta^{-z}Q - Q\Delta^{-z} = \sum_{k=1}^{N} \binom{-z}{k} Q^{(k)}\Delta^{-z-k} + \frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}Q^{(N+1)}(\lambda-\Delta)^{-N-1}\,d\lambda. \tag{1.1}$$

1.5. Higher traces on the algebra of abstract pseudodifferential operators

We review in this paragraph a simple generalization of the Wodzicki residue trace in the case where the zeta function of the algebra $\mathscr{D}(\Delta)$ has poles of arbitrary order, introduced in [1, 12].

Proposition 1.9. *Let $\Psi(\Delta)$ an algebra of abstract pseudodifferential operators, following the context of the previous paragraphs. Suppose that all the associated zeta functions have a pole of order at most $p \geq 1$ at 0. Then, the functional*

$$⨍^p P = \mathrm{Res}_{z=0} z^{p-1} \mathrm{Tr}(P\Delta^{-z/r})$$

defines a trace on $\Psi(\Delta)$.

Example 1.10. We recall the example of the usual Wodzicki residue trace [14] on the algebra of (classical) pseudodifferential operators $\Psi_{\mathrm{cl}}(M)$ on a closed (Riemannian) manifold M; here Δ is the Laplace operator and $r = 2$. (In fact, in the zeta function, one may replace $\Delta^{1/2}$ by any elliptic operator of order 1.) In this case, the zeta function has simple poles. Recall also that when the classical symbol σ_P of $P \in \Psi^m_{\mathrm{cl}}(M)$ has the following asymptotic expansion:

$$\sigma_P(x,\xi) \sim \sum_{j \geq 0} \sigma_{m-j}(x,\xi),$$

where $\sigma_k(x,\xi)$ are homogeneous of order $k \in \mathbb{Z}$ in ξ (for $|\xi|$ large enough), then the residue at $z = 0$ has the following expression:

$$⨍ P = \mathrm{Res}_{z=0} \mathrm{Tr}(P\Delta^{-z/2}) = \frac{1}{(2\pi)^n} \int_{S^*M} \iota_L \left(\sigma_{-n}(x,\xi) \frac{\omega^n}{n!} \right)$$

where S^*M is the cosphere bundle of M, ω is the standard symplectic form of $T^*\mathbb{R}^n$, and L is the generator of the dilations (on covectors). The quantity $⨍ P$ is called the *Wodzicki residue* of P. The latter formula makes it clear that the residue of a smoothing operator vanishes, therefore, $⨍$ passes to the quotient $S_{\mathrm{cl}}(M) = \psi_{\mathrm{cl}}(M)/\Psi^{-\infty}(M)$, also called the algebra of formal (or full) symbols.

Examples involving higher order poles will be discussed in the last section of the paper.

2. The Radul cocycle for abstract pseudodifferential operators

2.1. Abstract index theorems

Let A be an associative algebra over $\mathbb{C}$, possibly without unit, and let I be a two-sided ideal in A. The extension

$$0 \to I \to A \to A/I \to 0$$

gives rise to the following diagram relating algebraic K-theory and periodic cyclic homology:

$$\begin{array}{ccc} K_1^{\mathrm{alg}}(A/I) & \xrightarrow{\mathrm{Ind}} & K_0^{\mathrm{alg}}(I) \\ \Big\downarrow \mathrm{ch}_1 & & \Big\downarrow \mathrm{ch}_0 \\ \mathrm{HP}_1(A/I) & \xrightarrow{\partial} & \mathrm{HP}_0(I). \end{array}$$

The vertical arrows are respectively the odd and even Chern character.

By a slight abuse of language, denote $\partial : \mathrm{HP}^0(I) \to \mathrm{HP}^1(A/I)$ the boundary map in cohomology. As mentioned in [8], for $[\tau] \in \mathrm{HP}^0(I)$, $[u] \in K^1(A/I)$, one has the equality

$$\langle [\tau], \mathrm{ch}_0 \mathrm{Ind}[u] \rangle = \langle \partial[\tau], \mathrm{ch}_1[u] \rangle. \tag{2.1}$$

A standard procedure to express the boundary map ∂ goes as follows. If $[\tau] \in \mathrm{HP}^0(I)$ is represented by a hypertrace $\tau : I \to \mathbb{C}$, i.e., a linear map satisfying the condition $\tau([A, I]) = 0$, then choose a lift $\tilde{\tau} : A \to \mathbb{C}$ of τ, such that $\tilde{\tau}$ is linear (in general, this is not a trace), and a linear section $\sigma : A/I \to A$ such that $\sigma(1) = 1$, after eventually adjoining a unit whenever necessary. Then, $\partial[\tau]$ is represented by the following cyclic 1-cocycle:

$$c(a_0, a_1) = b\tilde{\tau}(\sigma(a_0), \sigma(a_1)) = \tilde{\tau}([\sigma(a_0), \sigma(a_1)])$$

where b is the Hochschild coboundary.

2.2. The generalized Radul cocycle

Let $\mathcal{D}(\Delta)$ be an algebra of abstract differential operators and Ψ be an algebra of abstract pseudodifferential operators. We consider the extension

$$0 \to \Psi^{-\infty} \to \Psi \to S \to 0$$

where S is the quotient $\Psi/\Psi^{-\infty}$. The operator trace on $\Psi^{-\infty}$ is well defined, and $\mathrm{Tr}([\Psi^{-\infty}, \Psi]) = 0$.

Theorem 2.1 (Cf. [12]). *Suppose that the pole in zero of the zeta function is of order $p \geq 1$. Then, the cyclic 1-cocycle $\partial[\mathrm{Tr}] \in \mathrm{HP}^1(S)$ is represented by the following functional:*

$$c(a_0, a_1) = {-\!\!\!\!\!\!\int}^1 a_0 \delta(a_1) - \frac{1}{2!} {-\!\!\!\!\!\!\int}^2 a_0 \delta^2(a_1) + \cdots + \frac{(-1)^{p-1}}{p!} {-\!\!\!\!\!\!\int}^p a_0 \delta^p(a_1)$$

where $\delta(a) = [\log \Delta^{1/r}, a]$ and $\delta^k(a) = \delta^{k-1}(\delta(a))$ for every $k \in \mathbb{N}$. We call this cocycle the generalized Radul cocycle.

Here, the commutator $[\log \Delta^{1/r}, a]$ is defined as the non-convergent asymptotic expansion

$$[\log \Delta^{1/r}, a] \sim \frac{1}{r} \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} a^{(k)} \Delta^{-k}$$

where $a^{(k)}$ has the same meaning as in Lemma 1.8. This expansion arises by first using functional calculus,

$$\log \Delta^{1/r} = \frac{1}{2\pi \mathbf{i}} \int \log \lambda^{1/r} (\lambda - \Delta)^{-1}\, d\lambda,$$

and then, reproducing the same calculations made in the proof of Lemma 1.8 to obtain the formula (cf. [4] for details). In particular, note that $\log \Delta^{1/r} = \frac{1}{r} \log \Delta$.

Another way to write the expansion is as follows:

$$[\log \Delta^{1/r}, a] \sim \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} a^{[k]} \Delta^{-k/r}$$

where $a^{[1]} = [\Delta^{1/r}, a]$, and $a^{[k+1]} = [\Delta^{1/r}, a^{[k]}]$. The lifting of the operator trace on $\Psi^{-\infty}$ to a linear map $\widetilde{\tau}$ on Ψ uses a zeta function regularization by 'Partie Finie',

$$\widetilde{\tau}(P) = \mathrm{Pf}_{z=0} \mathrm{Tr}(P \Delta^{-z/r})$$

for any $P \in \Psi$. The 'Partie Finie' Pf is defined as the constant term in the Laurent expansion of a meromorphic function.

Example 2.2. As a first concrete example, let us reprove the Noether index theorem. Let $M = S^1$ be the unit circle. Consider the operators $D = \frac{1}{\mathbf{i}} \frac{d}{dt}$, $F = D|D|^{-1}$ and $P = \frac{1+F}{2}$ acting on the Hardy space $H^2(S^1)$. The cosphere bundle of S^1 is $S^*S^1 = S^1 \times \{1\} \cup S^1 \times \{-1\}$. Then, observe that P is a pseudodifferential operator of order 0, its symbol defined on T^*S^1 is $\sigma_F(t, \xi) = \frac{1+\xi|\xi|^{-1}}{2}$, where $|.|$ denotes the Euclidean norm.

Then, let $u \in C^\infty(S^1)$ be a nowhere vanishing smooth function. We extend the associated Toeplitz operator PuP to $L^2(S^1)$ by considering the (Toeplitz) operator $T_u = PuP - (1 - P)$, which is an elliptic pseudodifferential operator of order 0 with symbol given by

$$\begin{cases} u(t) & \text{on } S^1 \times (0, \infty), \\ 1 & \text{on } S^1 \times (-\infty, 0). \end{cases}$$

Then, using the star-product formula, one sees that the part of order -1 in the symbol of $T_{u^{-1}}[\log D, T_u]$ is

$$\begin{cases} \frac{1}{\mathbf{i}\xi} \frac{u'(t)}{u(t)} & \text{on } S^1 \times (0, \infty), \\ 0 & \text{on } S^1 \times (-\infty, 0). \end{cases}$$

Using the symbol formula for the Wodzicki residue yields

$$\mathrm{Ind}(T_u) = -\frac{1}{2\pi \mathbf{i}} \int_{S^1} u^{-1}\, du.$$

Example 2.3. We now consider the extension associated to the classical pseudodifferential calculus on $\mathbb{R}^n$,

$$0 \to \Psi_c^{-\infty}(\mathbb{R}^n) \to \Psi^0_{\mathrm{cl},c}(\mathbb{R}^n) \to S_{\mathrm{cl},c}(\mathbb{R}_n) \to 0;$$

where the subscript 'c' stands for 'compact support'; $\Psi_c^{-\infty}(\mathbb{R}^n)$ is the ideal of smoothing operators in the algebra of classical pseudodifferential operators $\Psi^0_{\mathrm{cl},c}(\mathbb{R}^n)$ (of order ≤ 0), and $S^0_{\mathrm{cl},c}(\mathbb{R}^n)$ is the algebra of formal symbols (of order ≤ 0), i.e., the quotient $\Psi^0_{\mathrm{cl},c}(\mathbb{R}^n)/\Psi_c^{-\infty}(\mathbb{R}^n)$.

Since zeta functions also exhibit simple poles in this context, the Radul cocycle is also given by the following cyclic 1-cocycle on $S^0_{\mathrm{cl},c}(\mathbb{R}^n)$:

$$c(a_0, a_1) = ⨍ a_0[\log|\xi|, a_1] = ⨍ a_0 \delta a_1.$$

Then, proceeding exactly as in [12] with the classical Wodzicki residue and the generator of dilations $L = \sum_{j=1}^n \xi_j \partial_{\xi_j}$ on $T^*\mathbb{R}^n$ instead, and correcting the sign mistake in the definition of the operator F by defining it instead as $F = \sum_{j=1}^n (\xi_j dx_j - x_j d\xi_j)$, one shows that in the (B, b)-complex, the Radul cocycle is cohomologous to the fundamental class on $S^*\mathbb{R}^n$ (up to a sign depending on the choice of orientation),

$$\psi_{2n-1}(a_0, \ldots, a_{2n-1}) = -\frac{1}{(2\pi \mathbf{i})^n (2n-1)!} \int_{S^*\mathbb{R}^n} \sigma(a_0)\, d\sigma(a_1) \cdots d\sigma(a_{2n-1}).$$

More generally, a generalization of the above calculation (requiring a substantial amount of work) shows that on a closed manifold, the Radul cocycle identifies the Poincaré dual of the Todd class. This can be found in the work of Perrot [9]. Further generalizations of this result may be found in [10, 11].

3. Relation to the Chern–Connes character

In this section, we establish a relationship between the Radul cocycle and the Chern–Connes character of a spectral triple.

Let (A, H, F) be a (trivially graded) p-summable Fredholm module. In addition, let $\Psi = \Psi(\Delta)$ be an abstract algebra of pseudodifferential operators, such that

(1) Ψ^0 is an algebra of bounded operators on H containing the representation of A,

(2) Ψ^{-1} is a two-sided ideal consisting of p-summable operators on H,

(3) F is a multiplier of Ψ^0 and $[F, \Psi^0] \subset \Psi^{-1}$.

We have an abstract principal symbol exact sequence,

$$0 \to \Psi^{-1} \to \Psi^0 \to \Psi^0/\Psi^{-1} \to 0, \tag{3.1}$$

Ψ^0/Ψ^{-1} should be viewed as an 'abstract cosphere bundle'. This extension is related to the one involving smoothing operators, as the inclusion of ideals $\psi^{-\infty} \subset \psi^{-1}$ yields the following morphism of extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Psi^{-\infty} & \longrightarrow & \Psi^0 & \longrightarrow & S^0 = \Psi^0/\Psi^{-\infty} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \Psi^{-1} & \longrightarrow & \Psi^0 & \longrightarrow & \Psi^0/\Psi^{-1} & \longrightarrow & 0.
\end{array}$$

Then, the cyclic cohomology class of the operator trace $[\mathrm{Tr}] \in \mathrm{HP}^0(\Psi^{-\infty})$ extends to a cyclic cohomology class $[\tau] \in \mathrm{HP}^0(\Psi^{-1})$, represented for any choice of integer $k > p$ by the following cyclic k-cocycle on ψ^{-1}:

$$\tau_k(x_0, \dots, x_k) = \mathrm{Tr}(x_0 \cdots x_k).$$

By naturality of excision, the image of the trace $\partial[\mathrm{Tr}] \in \mathrm{HP}^1(S^0)$ by excision is the pull-back of the class $\partial[\tau] \in \mathrm{HP}^1(\Psi^0/\Psi^{-1})$. We shall then make a slight abuse of notation by identifying both.

Let $P = \frac{1}{2}(1 + F)$. Then $[P, a] \in \Psi^{-1}$ for every $a \in A$. The linear map

$$\rho_F : A \to \Psi^0/\Psi^{-1}, \qquad \rho_F(a) = PaP \mod \Psi^{-1},$$

is an algebra homomorphism since $Pa_1Pa_2P = Pa_1a_2P \mod \Psi^{-1}$ for all $a_1, a_2 \in A$.

Theorem 3.1. *The Chern–Connes character of the Fredholm module (H, F) is given by the odd cyclic cohomology class over A*

$$\mathrm{ch}(H, F) = \rho_F^* \circ \partial([\mathrm{Tr}])$$

where $[\mathrm{Tr}] \in \mathrm{HP}^0(\Psi^{-1})$ is the class of the operator trace, $\partial : \mathrm{HP}^0(\Psi^{-1}) \to \mathrm{HP}^1(\Psi^0/\Psi^{-1})$ is the excision map associated to extension (3.1), and $\rho_F^* : \mathrm{HP}^1(\Psi^0/\Psi^{-1}) \to \mathrm{HP}^1(A)$ is induced by the homomorphism ρ_F.

Proof. Consider the algebra $\mathcal{E} = \{(Q, a) \in \Psi^0 \oplus A;\ Q = PaP \mod \Psi^{-1}\}$. The homomorphism

$$(Q, a) \in \mathcal{E} \to a \in A$$

yields an extension

$$0 \to \Psi^{-1} \to \mathcal{E} \to A \to 0.$$

The Chern–Connes character $\mathrm{ch}(H, F) \in \mathrm{HP}^1(A)$ is the image of the operator trace by the boundary map of this extension (cf., for instance, Cuntz' survey in [2]). On the other hand, the homomorphism $\mathcal{E} \to \Psi^0$, $(Q, a) \mapsto Q$ yields a morphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Psi^{-1} & \longrightarrow & \mathcal{E} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \rho_F & & \\
0 & \longrightarrow & \Psi^{-1} & \longrightarrow & \Psi^0 & \longrightarrow & \Psi^0/\Psi^{-1} & \longrightarrow & 0.
\end{array}$$

The conclusion then follows from the naturality of the boundary map. ∎

4. Discussion on manifolds with conical singularities

Manifolds with conical singularities were actually the main motivation for studying cases when the zeta function exhibits multiple poles, which is well known, for instance, since the work of Lescure [5]. We first start with relevant background material on the analysis of manifolds with conical singularities, i.e., pseudodifferential calculus, residues and results on the associated zeta functions.

4.1. Generalities on b-calculus and cone pseudodifferential operators

We review some elements of Melrose's b-calculus and its application to conical manifolds. The standard reference for Melrose's work on manifolds with boundary is [6]. As for the adaptation to manifolds with conical singularities, we essentially follow the presentation of [3].

In our context, manifolds with conical singularities are manifolds with boundary with an additional structure given by a suitable algebra of differential operators.

More precisely, let M be a compact n-manifold with (connected) boundary, and let $r : M \to \mathbb{R}_+$ be a boundary defining function, i.e., a smooth function vanishing on ∂M whose differential is non-zero at every point of ∂M. We work in a collar neighborhood $[0,1)_r \times \partial M_x$ of the boundary, subscripts referred to symbols used for local coordinates.

Definition 4.1. A *Fuchs-type differential operator* P of order $m \in \mathbb{N}$ and weight $-p \leq 0$ is a differential operator on M which can be written in the form

$$P(r,x) = r^{-p} \sum_{j+|\alpha|\leq m} a_{j,\alpha}(r,x)(r\partial_r)^j \partial_x^\alpha$$

in the collar $[0,1)_r \times \partial M_x$. The space of such operators will be denoted $r^{-p}\mathrm{Diff}_b^m(M)$.

$\mathrm{Diff}_b^m(M)$ denotes the algebra of b-differential operators in Melrose's b-calculus for manifolds with boundary, which consists of differential operators that can be written as follows in the collar $[0,1)_r \times \partial M_x$:

$$\sum_{j+|\alpha|\leq m} a_{j,\alpha}(r,x)(r\partial_r)^j \partial_x^\alpha.$$

Geometrically, this means that differential operators of order 1 in this calculus consist of vector fields that are tangent to the boundary when $r \to 0$. We now recall the associated small b-pseudodifferential calculus $\Psi_b(M)$.

The *b-stretched product M_b^2 of M* is the manifold with corners obtained by blowing-up the corner ∂M^2 within M^2. See Figure 1 for an illustration. Locally, we can describe M_b^2 as follows: on $M^2 \setminus \partial M^2$, its local charts are the usual ones, whereas near the corner ∂M^2 in M^2, it is parametrized in polar coordinates as follows: writing $M \times M$ near $r = r' = 0$ as

$$M^2 = [0,1)_r \times \partial M_x \times [0,1)_{r'} \times \partial M_{x'} \simeq [0,1)_r \times [0,1)_{r'} \times \partial M_x \times \partial M_{x'},$$

the factor $[0,1)_r \times [0,1)_{r'}$ is parametrized by

$$r = \rho \cos\theta, \quad r' = \rho \sin\theta$$

for $\rho > 0$, $\theta \in [0, \pi/2]$. The right and left boundary faces are respectively the points where $\theta = 0$ and $\theta = \pi/2$. Hence, the b-stretched product clearly appears as a smooth version of the algebraic blow-up, in which one would replace the corner ∂M^2 by its projective normal bundle in M^2.

Let Δ_b be the b-diagonal of M_b^2, that is, the lift of the diagonal in M^2. In the aforementioned coordinate system, it corresponds near the corner $\partial M^2 \subset M^2$ to the slice at $\theta = \pi/4$.

Naturally, Δ_b is diffeomorphic to M, and any local chart on Δ_b can be considered as a local chart on M: a convenient way to describe this identification is to replace the factor $[0,1)_r \times [0,1)_{r'}$ with $[0,1)_r \times \mathbb{R}_z$ and $z = \log(r/r') = \log(\mathrm{cotan}^{-1}\theta)$ (i.e., a logarithmic version of projective coordinates). In this description, we can locally write

$$M^2 \simeq [0,1)_r \times \partial M_x \times \mathbb{R}_z \times \partial M_{x'}$$

and Δ_b corresponds near the corner to $[0,1)_r \times \{0\}_z \times \partial M$, which is diffeomorphic to the collar $[0,1)_r \times \partial M$.

Definition 4.2. The *algebra of b-pseudodifferential operators of order m*, denoted $\Psi_b^m(M)$, consists of operators $P : C^\infty(M) \to C^\infty(M)$ having a Schwartz kernel K_P such that

(i) away from Δ_b, K_P is a smooth kernel vanishing to infinite order on the right and left boundary faces;

(ii) in an open neighborhood $U_{(r,x)} \times \mathbb{R}_z \times \partial M_{x'}$ of $\Delta_b \subset M_b^2$ such that $U_{(r,x)} \times \{0\}_{(z,x')} \subset \Delta_b$, we have

$$K_P(r,x,r',x') = \frac{1}{(2\pi)^n} \int e^{\mathbf{i}(\log(r/r')\cdot\tau + (x-x')\cdot\xi)} a(r,x,\tau,\xi)\, d\tau\, d\xi$$

where $a(y,\eta)$, with $y = (r,x)$ and $\eta = (\tau,\xi)$, is a classical pseudodifferential symbol of order m.

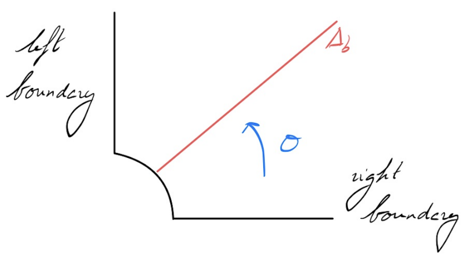

Figure 1. The b-stretched product M_b^2

Technically, to involve $x - x'$ or even write $\{0\}_{(z,x)}$, we should consider that x, x' are located in coordinate patches $\mathbb{R}^{n-1}$ for ∂M; we tacitly presuppose this is the case whenever necessary for notational simplicity. On the other hand, observe that $\log(r/r')$ is singular at $r = r' = 0$; the role of the b-stretch product M_b^2 is to blow-up this singularity.

To such an operator $P \in r^{-p}\Psi_b^m$, we define a local density on the chart $U \subset \Delta_b$ (which is seen as a chart in M via the identification $\Delta_b \simeq M$ described above),

$$\omega(P)(r,x) = \left(\int_{|\nu|=1} a_{-n}(r,x,\tau,\xi)\iota_L d\tau d\xi \right) \cdot \frac{dr}{r}\, dx$$

where $\eta = (\tau, \xi)$ any covector at $(r,x) \in [0,1) \times \partial M$, and L is the generator of the dilations.

Let Ω_b denote the bundle of b-densities on M, that is, the trivial line bundle with local basis $(dr/r)dx$ over a coordinate chart of the collar of ∂M. It turns out (but this is not obvious) that $\omega(P)(r,x)$, which is a priori defined only locally, does not depend on the choice of coordinates on M. Therefore, it defines a globally defined smooth b-density $\omega(P) \in C^\infty(M, \Omega_b)$ that we call the *Wodzicki residue density*. Note that $\omega(P) = 0$ when $p \notin \mathbb{Z}$.

The integral on M of this density does not converge in general because of the factor $1/r$, $r \in [0,1)$, but it can be regularized thanks to the following lemma (see [3]).

Lemma 4.3. *Let $u \in r^{-p}C^\infty(M, \Omega_b)$, and $p \in \mathbb{R}$. Then, the function*

$$z \in \mathbb{C} \mapsto \int_M r^z u$$

is holomorphic on the half plane $\mathrm{Re}(z) > p$*, and extends to a meromorphic function with only simple poles at $z = p, p-1, \ldots$ If $p \in \mathbb{N}$, its residue at $z = 0$ is given by*

$$\mathrm{Res}_{z=0} \int_M r^z u = \frac{1}{p!} \int_{\partial M} \partial_r^p (r^p u)_{r=0}.$$

The expression $\partial_r^p (r^p u)_{r=0}$ is defined as follows: in a collar $[0,1)_r \times \partial M_x$, $r^p \omega(P) = f(r)dr/r$, where $(f(r))_{r\in[0,1)}$ is a smooth family of densities on ∂M; we then set $\partial_r^p (r^p u)_{r=0} = \partial_r^p f(0)$.

Traces on conical pseudodifferential operators

We define different algebras of pseudodifferential operators, introduced by Melrose and Nistor in [7]. The main algebra that we shall consider is the *algebra of conic pseudodifferential operators*,

$$A = r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M) = \bigcup_{p\in\mathbb{Z}} \bigcup_{m\in\mathbb{Z}} r^{-p}\Psi^m(M)$$

which clearly contains the algebra of Fuchs-type operators. The opposite signs in the exponents are here to emphasize that

$$I = r^{\infty}\Psi_b^{-\infty}(M) = \bigcap_{p\in\mathbb{Z}} \bigcap_{m\in\mathbb{Z}} r^{-p}\Psi^m(M)$$

is the *ideal of smoothing operators*. In a similar vein, define

$$r^{\infty}\Psi_b^{\mathbb{Z}}(M) = \bigcap_{p\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} r^{-p}\Psi_b^m(M); \quad r^{\mathbb{Z}}\Psi_b^{-\infty}(M) = \bigcup_{p\in\mathbb{Z}}\bigcap_{m\in\mathbb{Z}} r^{-p}\Psi_b^m(M)$$

and consider the following quotients:

$$I_\sigma = r^{\infty}\Psi_b^{\mathbb{Z}}(M)/I; \quad I_\partial = r^{\mathbb{Z}}\Psi_b^{-\infty}(M)/I.$$

Loosely speaking, I_σ may be viewed of as an extension of the algebra of symbols in the interior of M, more precisely as symbols on M that vanish to infinite order on the boundary. The algebra I_∂ has the following interpretation: if we identify the collar $(0,1)_r \times \partial M_x$ to $\mathbb{R}_{1/r} \times \partial M_x$, a Taylor expansion calculation identifies I_∂ to the space of Laurent series in r, whose coefficients are $\mathbb{R}$-translation-invariant smoothing operators on $\mathbb{R} \times \partial M$ (cf. the appendix of [7] for further details on these interpretations). We finally define

$$A_\sigma = A/I_\partial; \quad A_\partial = A/I_\sigma; \quad A_{\partial,\sigma} = A/(I_\partial + I_\sigma),$$

which have analogous interpretations to their counterparts I_σ and I_∂. To expand slightly further, A_σ may basically be identified to symbols on M containing factors that are powers of $1/r$; on the other hand, A_∂ may be identified to the space of Laurent series in r, whose coefficients are $\mathbb{R}$-translation-invariant pseudodifferential operators on $\mathbb{R} \times \partial M$.

We now define diverse functionals on these algebras.

Definition 4.4. Let $P \in r^{-p}\Psi_b^m(M)$ be a conical pseudodifferential operator, with $p, m \in \mathbb{Z}$. According to Lemma 4.3, define the functionals $\mathrm{Tr}_{\partial,\sigma}$, Tr_σ to be

$$\mathrm{Tr}_\sigma(P) = \mathrm{Pf}_{z=0}\int_M r^z \omega(P),$$

$$\mathrm{Tr}_{\partial,\sigma}(P) = \mathrm{Res}_{z=0}\int_M r^z\omega(P) = \frac{1}{p!}\int_{\partial M} \partial_r^p(r^p\omega(P))_{r=0},$$

where Pf denotes the constant term in the Laurent expansion of a meromorphic function.

Remark 4.5. Using Lemma 4.3, one can show that $\mathrm{Tr}_{\partial,\sigma}(P)$ does not depend on the choice of the boundary defining function r. This is not the case for $\mathrm{Tr}_\sigma(P)$, but its dependence on r can be determined explicitly, cf. [3].

The Partie Finie regularization of a trace does not give in general a trace, and this is indeed the same for the functional $\mathrm{Tr}_\sigma(P)$ acting on these algebras, the obstruction to that is precisely the presence of the boundary. However, by definition, $\mathrm{Tr}_\sigma(P)$ clearly defines an extension of the Wodzicki residue, so one can expect it to be a trace on $I_\sigma = r^{\infty}\Psi^{\mathbb{Z}}(M)/I$.

Theorem 4.6 (Melrose-Nistor, [3, 7]). Tr_σ *is, up to a multiplicative constant, the unique trace on the algebra* I_σ.

By Lemma 4.3 and a standard commutator calculation, one sees without much difficulty that the defect of Tr_σ to be a trace is precisely measured by $\mathrm{Tr}_{\partial,\sigma}(P)$, which can therefore be viewed as a restriction of the Wodzicki residue to the boundary ∂M. Another way to see this is to Taylor-expand $\omega(P)$ near the boundary. Then, the following proposition seems natural.

Theorem 4.7 (Melrose-Nistor, [3,7]). $\mathrm{Tr}_{\partial,\sigma}$ *is, up to a multiplicative constant, the unique trace on the algebras* A_∂*,* A_σ *and* $A_{\partial,\sigma}$*.*

These two traces may be seen as 'local' terms, since they only depend on the symbol of the pseudodifferential operator considered. The first can be seen as a trace on the interior of M, the second is related to the boundary ∂M. There is a last trace to introduce, which is not local.

Fix a holomorphic family $Q(z) \in r^{\alpha z}\Psi_b^{\beta z}(M)$, with $\alpha, \beta \in \mathbb{R}$, such that Q is the identity at $z = 0$. Take $P \in r^{-p}\Psi_b^m(M)$, with $p, m \in \mathbb{Z}$ and let $(PQ(z))_\Delta$ be the restriction to the diagonal Δ of M^2 of the Schwartz kernel of $PQ(z)$. Melrose and Nistor prove in [7] that $(PQ(z))_\Delta$ is meromorphic in $\mathbb{C}$, with values in $r^{\alpha z - p}C^\infty(M)$ and eventual simple poles in the set

$$\left\{\frac{-n-m}{\beta}, \frac{-n-m+1}{\beta}, \dots\right\}.$$

Definition 4.8. Let $P \in r^{-p}\Psi_b^m(M)$ be a conical pseudodifferential operator, with $p \in \mathbb{Z}$. Then, we define

$$\mathrm{Tr}_\partial(P) = \frac{1}{p!}\int_{\partial M} \partial_r^p (r^p \mathrm{Pf}_{z=0}(PQ(z))_\Delta)_{r=0}\, dx.$$

If $p \notin \mathbb{Z}$ then define $\mathrm{Tr}_\partial(P)$ to be 0.

Remark 4.9. $\mathrm{Tr}_\partial(P)$ depends on the choice of the operator Q, but the dependence can be explicitly determined, see [7].

There is an interpretation of Tr_∂ analogous to those of $\mathrm{Tr}_{\partial,\sigma}$: if the order of P is less than the dimension of M, then Tr_∂ may be viewed as an L^2-trace on the boundary, as seen from the result below.

Theorem 4.10 (Melrose-Nistor, [3, 7]). $\mathrm{Tr}_\partial(P)$ *is, up to a multiplicative constant, the unique trace on the algebra*

$$I_\partial = r^{\mathbb{Z}}\Psi^{-\infty}(M)/I.$$

Heat kernel expansion and zeta function

Let $\Delta_g \in r^{-2}\mathrm{Diff}_b^2(M)$ be the Laplace–Beltrami operator associated to a cone Riemannian metric g on M, i.e., the metric writes $g = dr^2 + r^2 g_{\partial M}$ in the collar neighborhood $[0,1)_r \times \partial M$ of the boundary, where $g_{\partial M}$ is a Riemannian metric on ∂M. Let $\Delta \in$

$r^{-2}\mathrm{Diff}_b^2(M)$ be the operator defined as follows:

$$\Delta = -r^{n/2-1}\left(\Delta_g + \frac{a^2}{r^2}\right)r^{-n/2+1} = \frac{1}{r^2}\left((r\partial_r)^2 - \Delta_{\partial M} + \frac{(n-2)^2}{4} + a^2\right) \quad (4.1)$$

where $a > 1$ and $n = \dim(M)$.

One shows that the heat kernel $e^{-t\Delta}$ of Δ is well defined for $t > 0$, and that for every $P \in r^{-p}\Psi_b^m(M)$, operators of the type $P\Delta^{-z}$ are trace-class on $r^{1-m}L^2(M, \Omega_b)$, and for every z in the half-plane $\mathrm{Re}(z) > \max\{\frac{m+n}{2}, \frac{p}{2}\}$.

The traces introduced in the previous paragraph allow to express the coefficients of the asymptotic expansion of $\mathrm{Tr}(Pe^{-t\Delta})$ when $t \to 0$.

Theorem 4.11 (Gil–Loya, [3]). *$P \in r^{-p}\Psi_b^m(M)$ and let $\Delta \in r^{-2}\mathrm{Diff}_b^2(M)$ be the Laplace-type operator defined above. Then,*

$$\mathrm{Tr}(Pe^{-t\Delta}) \sim_{t\to 0} \sum_{k\geq 0} a_k t^{(k-p)/2} + (b_k + \beta_k \log t)t^k$$
$$+ (c_k + \gamma_k \log t + \delta_k (\log t)^2)t^{(k-m-n)/2}$$

where $a_k, b_k, c_k, \beta_k, \gamma_k$ and δ_k are constants. The constants β_k, γ_k and δ_k associated to the $\log$-terms are explicit linear functions of Tr_σ, Tr_∂, $\mathrm{Tr}_{\partial,\sigma}$ evaluated on operators of the form $P \cdot$ power of Δ.

The coefficient of $\log t$ is

$$-\frac{1}{2}\mathrm{Tr}_\sigma(P) - \frac{1}{2}\mathrm{Tr}_\partial(P) - \frac{1}{4}\mathrm{Tr}_{\partial,\sigma}(P)$$

and the coefficient of $(\log t)^2$ is

$$-\frac{1}{4}\mathrm{Tr}_{\partial,\sigma}(P).$$

Using a Mellin transform, we can write

$$\mathrm{Tr}(P\Delta^{-z/2}) = \frac{1}{\Gamma(z/2)}\int_0^\infty t^{z-1}\mathrm{Tr}(Pe^{-t\Delta})\,dt,$$

and knowing, that $z \mapsto \int_1^\infty t^{z-1}\mathrm{Tr}(Pe^{-t\Delta})\,dt$ is entire, the asymptotic expansion of the previous proposition gives the following corollary on the zeta function.

Corollary 4.12. *The zeta function $z \mapsto \mathrm{Tr}(P\Delta^{-z/2})$ is holomorphic in the half-plane $\mathrm{Re}(z) > \max\{m+n, p\}$, and extends to a meromorphic function with at most triple poles, whose set is discrete. At $z = 0$, there are simple and double poles only, which are respectively given by the terms of $\log t$ and $(\log t)^2$ in the heat kernel expansion of $\mathrm{Tr}(Pe^{-t\Delta})$.*

4.2. Spectral triple and (non-)regularity

In this paragraph, we will see what the algebra of Fuchs-type operators misses to be an abstract algebra of differential operators (in the sense of the first section), and discuss what kind of information may remain nonetheless.

Let M be a manifold with connected boundary, with boundary defining function r, equipped with the algebra of Fuchs-type differential operators. The points (i), (ii) of Definition 1.1 are verified, if for example we take for Δ the fully-elliptic operator of order 2 given in Example 4.1, and require that the order is given by the differential order. More generally, working locally in a collar neighborhood $[0,1)_r \times \partial M_x$ of the boundary ∂M, one easily observes that

$$[r^p \mathrm{Diff}_b^m(M), r^{p'} \mathrm{Diff}_b^{m'}(M)] \subset r^{p+p'} \mathrm{Diff}_b^{m+m'-1}(M). \tag{4.2}$$

For that reason, the order in r does not decrease in commutators, and cannot be considered as a suitable notion of differential order; this is precisely the main issue when it comes to the elliptic estimate. In the literature, one usually introduces suitable weighted Sobolev spaces to fix this problem, see [3] and references therein for more details.

Consequently, the formalism of abstract differential operators, and in particular the formula of Lemma 1.8 do not apply directly. Indeed, if $b(r,x) = r^p$ with $p \in \mathbb{Z}_{\leq 0}$, observation (4.2) shows that the terms $b^{(k)}$ are in $r^{p-2k}\mathrm{Diff}_b^k(M)$, but by the properties of the zeta function given in Corollary 4.12, the function

$$z \mapsto \mathrm{Tr}(b^{(k)} \Delta^{-k-z})$$

is holomorphic for $\mathrm{Re}(z) + k > \max\{\frac{n+k}{2}, \frac{2k-p}{2}\}$, which is equivalent to $\mathrm{Re}(z) > \max\{\frac{n-k}{2}, -\frac{p}{2}\}$. Hence, if $p \leq 0$, the function above is in general not holomorphic at 0 for large k.

However, we may still recover interesting information on M from higher Wodzicki residues (cf. Proposition 1.9). Note, for instance, that $⨍^2$ is the trace $\mathrm{Tr}_{\partial,\sigma}$ (up to some irrelevant constant); $⨍^1$ is, modulo some constant terms, the sum of the three functionals $\mathrm{Tr}_{\partial,\sigma}$, Tr_σ, Tr_∂, which illustrates that it is no more a trace on the algebra of conical pseudodifferential operators. We discuss this point further in the next subsection.

Let us denote by $r^p C^\infty(\partial M)$ the subalgebra of $C^\infty(M)$ of functions f which have an asymptotic expansion

$$f(r,x) \sim r^p f_p(x) + r^{p+1} f_{p+1}(x) + \cdots$$

in a neighborhood of $r = 0$, where the coefficients f_j are smooth functions on ∂M. Here, the $\sim$ means that the remainder of such an expansion is of the form $r^N f_N(r,x)$, with f_N bounded in the collar $[0,1) \times \partial M$. The case $p = 0$ corresponds to the smooth functions on the collar.

Previous discussion shows that it is not possible to build a regular spectral triple from such an algebra if we consider any weight p. However, we see that the aforementioned problem concerning the zeta function disappears if we consider functions $b \in C_c^\infty(\mathring{M}) \oplus \mathbb{C}$. This is the approach adopted by J.-M. Lescure [5] (the operator he considers is a Dirac-type operator, whose associated Laplacian is similar to Δ). In other words, if we restrict our attention to Fuchs operators whose coefficients are either r-compactly supported in restriction to a collar $(0,1)_r \times \partial M$ or constant with respect to r, then the elliptic estimates hold and we do have an algebra of abstract differential operators. (This is just a tedious way of saying that removing the boundary removes the main issue ...).

4.3. A non-local index formula

We now analyze how to modify Theorem 2.1 for conical manifolds.

Consider the extension

$$0 \to r^{\infty}\Psi_b^{-\infty}(M) \to r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M) \to r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/r^{\infty}\Psi_b^{-\infty}(M) \to 0.$$

Here, by an *elliptic pseudodifferential operator* $P \in r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)$, we shall mean that P is invertible in the quotient $A = r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/r^{\infty}\Psi_b^{-\infty}(M)$. Being *fully elliptic* (in the sense of Melrose) is an extra condition which roughly says that the normal operator is a family of invertible operators, guaranteeing that P is Fredholm (when considering suitable Sobolev spaces). We shall not enter into these details: what we want to investigate is the pairing given in equation (2.1). In particular, if P is fully elliptic, then the pairing really calculates a Fredholm index.

Now, let $P, Q \in r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)$. Going back to the 'Partie Finie' regularization given in the paragraph following Theorem 2.1, we still write

$$\begin{aligned} c(P,Q) &= \mathrm{Pf}_{z=0}\mathrm{Tr}([P,Q]\Delta^{-z}) \\ &= \mathrm{Res}_{z=0}\mathrm{Tr}\Big(P \cdot \Big(\frac{Q - \Delta^{-z}Q\Delta^{z}}{z}\Big)\Delta^{-z}\Big). \end{aligned}$$

The remainder term in Lemma 1.8 is no more negligible, but algebraically formula (1.1) still holds. So, for any integer $N \gg 0$, we have

$$Q - \Delta^{-z}Q\Delta^{z} = \sum_{k=1}^{N} Q^{(k)}\Delta^{-k} + \frac{1}{2\pi\mathbf{i}}\int \lambda^{-z}(\lambda-\Delta)^{-1}Q^{(N+1)}(\lambda-\Delta)^{-N-1}\,d\lambda.$$

Because the traces Tr_σ and $\mathrm{Tr}_{\partial,\sigma}$ vanish when the differential order of the operators is less that the dimension of M, we obtain the following result.

Theorem 4.13. *Then, the Radul cocycle associated to the pseudodifferential extension*

$$0 \to r^{\infty}\Psi_b^{-\infty}(M) \to r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M) \to r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/r^{\infty}\Psi_b^{-\infty}(M) \to 0$$

is given by the following non-local formula:

$$\begin{aligned} c(a_0,a_1) &= (\mathrm{Tr}_{\partial,\sigma} + \mathrm{Tr}_\sigma)(a_0[\log\Delta, a_1]) - \frac{1}{2}\mathrm{Tr}_{\partial,\sigma}(a_0[\log\Delta,[\log\Delta,a_1]]) \\ &\quad + \mathrm{Tr}_\partial\Big(a_0\sum_{k=1}^{N} a_1^{(k)}\Delta^{-k}\Big) \\ &\quad + \frac{1}{2\pi\mathbf{i}}\mathrm{Res}_{z=0}\mathrm{Tr}\Big(\int \lambda^{-z}a_0(\lambda-\Delta)^{-1}a_1^{(N+1)}(\lambda-\Delta)^{-N-1}\Big)\,d\lambda \end{aligned}$$

for $a_0, a_1 \in r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)/r^{\infty}\Psi_b^{-\infty}(M)$.

On the right-hand side, the first line consists of local terms depending only on the symbol of P, the second and third line shows the non-local contributions. Using the techniques in the article [3], one can prove that the last term containing the contour integral is a meromorphic function.

If $P \in r^{-\mathbb{Z}}\Psi_b^{\mathbb{Z}}(M)$ is an elliptic operator, so that P defines an element in the odd K-theory group $K_1^{\mathrm{alg}}(A)$, and Q an inverse of P modulo A, we then obtain a formula for the index of P. The second and third line of the formula above should be a part of the eta invariant, but we do not know how to establish a precise relationship with it.

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