J. Comb. Algebra 9 (2025), 121–127
DOI 10.4171/JCA/93

On eventually injective endomorphisms

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Abstract. We describe a property of linear groups that strengthens the Hopf property of finitely generated residually finite groups.

This is an updated version of a note written in 1999 – just around the time I wrote a couple of papers with Mark Sapir on related topics. A few years before that I was curious about the question of whether ascending HNN extensions of free groups were residually finite – a problem Mark brilliantly solved later with Borisov in [1, 2]. We wrote a paper giving two examples of an ascending HNN extension of a residually finite group that is not residually finite [13]. We also proved that ascending HNN extensions of f.g. free groups are Hopfian [5]. That latter paper had an elementary proof that was superseded by Borisov–Sapir. Surely emerging from his ascending HNN extension inquiry, Druţu–Sapir provided the one-relator group

$$\langle a, t \mid t^{-2}at^2 = a^2\rangle,$$

which is a strikingly simple example of a group that is residually finite but not linear [4]. Both examples in [13] are ascending HNN extensions of residually finite groups that are not linear. One is Grigorchuk's group, which is not linear since it is an infinite torsion group. Schur proved this over $\mathbb{R}$ and Kaplansky proved this over arbitrary fields [9], but we know this from Tits' alternative [17], as a virtually solvable torsion group is obviously finite. The base group $F *_H F$ of the other ascending HNN extension is also non-linear by Theorem 6. The non-linearity in these examples suggests Problem 11 below.

Mark had a great attitude about mathematics and a very "can-do attitude" about life. Working and interacting with him was always fun. He shared his opinions and his values. If he didn't like the way I handled something he would say "this isn't rocket science" and give his own version (that he thought was simpler). He had strong opinions about things and stood up for what he believed in. Though he seemed to

Mathematics Subject Classification 2020: 20H20 (primary); 20E06, 20E26 (secondary).
Keywords: linear group, Hopfian, endomorphism.

savour the word “no”, and he enjoyed and celebrated a skeptical spirit, fundamentally he really was quite an optimist about both mathematics and life. Every time I met him, Mark was learning and sharing new things – math, music, or culture. Mark Sapir was a unique and interesting person and we miss him sorely.

Definition 1. An endomorphism $\phi: G \to G$ is *eventually injective* if for some n, the restriction of ϕ to $\phi^n(G)$ is an injection.

It then follows that the restriction of ϕ to $\phi^m(G)$ is injective for all $m \geq n$. Eventual injectivity of ϕ is also equivalent to requiring that the sequence of kernels $\ker(\phi^n)$ terminates. Note that $\phi(\phi^n(G)) = \phi^n(\phi(G)) \subset \phi^n(G)$.

The following easy proposition gives examples of familiar endomorphisms which are eventually injective.

Proposition 2. *Let G be either an f.g. free group, or an f.g. abelian group. Every endomorphism of G is eventually injective.*

Proof. Observe that $\text{rank}(\phi^n(G))$ is a non-increasing sequence, and consequently, for some N,

$$\text{rank}(\phi^N(G)) = \text{rank}(\phi^n(G))$$

for $n \geq N$. In case G is free, this implies that $\phi^N(G) \cong \phi^{N+1}(G)$ and so since free groups are Hopfian (recalled below), the restriction of ϕ to $\phi^N(G)$ is injective.

If G is abelian, define $\text{comp}(G)$ to be the ordered set (f, t) where f is the rank of the maximal torsion-free subgroup of G, and t is the order of the largest finite subgroup of G. We say $(t_1, f_1) > (t_2, f_2)$ provided that $t_1 > t_2$, or $t_1 = t_2$ and $f_1 > f_2$. It is easy to see that if $G \to Q$ is an epimorphism which is not a monomorphism then $\text{comp}(G) > \text{comp}(Q)$, and that the set of complexities is well ordered. Thus $\phi^n(G)$ has the same complexity as $\phi^{n+1}(G)$ for some n, and hence the restriction of ϕ to $\phi^n(G)$ is injective. ■

A group G is *residually finite* if $\{1_G\}$ is the intersection of finite index subgroups of G. A group G is *Hopfian* if every surjective endomorphism of G is injective. Mal’cev proved that f.g. residually finite groups are Hopfian [11]. Hirshon generalized Mal’cev’s result in [7] by showing that if $\phi: G \to G$ is an endomorphism of an f.g. residually finite group, and $[G : \phi(G)] < \infty$ then ϕ is eventually injective. Sela proved in [14] that every endomorphism of a torsion-free word-hyperbolic group is eventually injective. We gave an example in [18] of an f.g. residually finite group G with an endomorphism $\phi: G \to G$ that is *not* eventually injective.

The method employed in [18] is as follows: First produce an endomorphism $\phi: F \to F$ with a subgroup $H \subset F$ such that $\phi(H) \subset H$. Let $G = F *_{H=H} F$ be the double of F along H, then the endomorphism ϕ of F extends to an endomorphism ϕ of G. If the sequence $\phi^{-n}(H)$ does not terminate, then $\phi: G \to G$ is not eventually injective. Finally, if F is residually finite, and H is separable, then $G = F *_H F$ is residually finite.

For instance, let $F = \langle a, b\rangle$ and $H = \langle b^n a^{2^n} b^{-n} : n \geq 0\rangle$, and let $\phi: F \to F$ be induced by $b \mapsto b$ and $a \mapsto a^2$. Note $\phi(H) \subset H$ since ϕ maps each $b^n a^{2^n} b^{-n}$ to its square. Letting the second copy of F be generated by $\langle \overline{a}, \overline{b}\rangle$, the group $G = F *_H F$ is presented by:

$$\langle a, b, \overline{a}, \overline{b} \mid b^n a^{2^n} b^{-n} = \overline{b}^n \overline{a}^{2^n} \overline{b}^{-n} : n \geq 0 \,\rangle \tag{1}$$

The sequence $\phi^{-n}(H)$ does not terminate as $\phi^{n-1}(b^n a b^{-n}) = (b^n a^{2^{n-1}} b^{-n}) \notin H$, but $\phi^n(b^n a b^{-n}) = (b^n a^{2^n} b^{-n}) \in H$.

Problem 3. Give an example of a finitely presented residually finite group with an endomorphism that is not eventually injective.

Note that for $G = F *_H F$ of equation (1), the group F is a free group of rank 2, but H is an infinite rank subgroup. Consequently, the double $F *_H F$ is not finitely presented.

A crucial property of $\phi: F \to F$ above is that the sequence $\phi^{-n}(H)$ does not terminate. The following shows we cannot maintain this property with H replaced by an f.g. subgroup of F.

Proposition 4. *Let $\phi: F \to F$ be an endomorphism of a free group, and let $H \subset F$ be a f.g. subgroup with $\phi(H) \subset H$. Then the ascending sequence $\phi^{-n}(H)$ terminates.*

Proof. By Proposition 2, for some r, the restriction of ϕ to $\phi^r(F)$ is an injection. By [8], for each n, the rank of $\phi^n(F) \cap H$ is bounded by $2\,\mathrm{rank}(F)\,\mathrm{rank}(H)$. Consequently, the preimages of H under the maps $\phi^m: \phi^r(F) \to F$ form an increasing sequence K_m of subgroups of $\phi^r(F)$, and each of these subgroups has

$$\mathrm{rank} < 2\,\mathrm{rank}(F)\,\mathrm{rank}(H).$$

By a result of Takahasi [16], such a sequence must terminate. Indeed, think in terms of immersions of graphs, and observe that the subgraphs corresponding to K_m must stabilize but cannot continue growing because the rank is bounded.

For each m, observe that $\phi^{r+m}(x) \in H$ iff $\phi^m(\phi^r(x)) \in H$ iff $\phi^r(x) \in K_m$. Thus,

$$\phi^{-r-m}(H) = \phi^{-r}(K_m).$$

Since for some M we have $K_i = K_j$ for $i, j \geq M$, we conclude that

$$\phi^{-r-i}(H) = \phi^{-r}(K_i) = \phi^{-r}(K_j) = \phi^{-r-j}(H). \qquad \blacksquare$$

Problem 5. Let $\phi: F \to F$ be an endomorphism of an f.g. free group, and let H be an f.g. subgroup of F. Is the collection of preimages $\{\phi^{-n}(H) : n \geq 0\}$ actually a finite collection?

Our main goal is the following generalization of Proposition 2. We give two proofs. The first applies a result of Stallings, and the second is direct.

Theorem 6. *Let G be an f.g. linear group. Then every endomorphism of G is eventually injective.*

Proof. Let $\mu: A \to B$ and $\nu: A \to B$ be homomorphisms. The *equalizer* Equal(μ, ν) is the subgroup

$$\{a \in A : \mu(a) = \nu(a)\}.$$

Stallings proved in [15, Corollary 3.5] that if G is an f.g. linear group, then the collection of equalizer subgroups of G satisfies the ascending chain condition.

Let $\theta: G \to G$ be the homomorphism with trivial image, and let $\rho: G \to G$ be an arbitrary endomorphism. Observe that $\ker(\rho^n) = \text{Equal}(\rho^n, \theta)$ is an ascending sequence of equalizers. By Stallings' theorem, for some N, we have

$$\ker(\rho^n) = \ker(\rho^N)$$

for all $n \geq N$. ■

For $G \subset SL_d(\mathbb{F})$, we consider the sequence of kernels of homomorphisms

$$\rho^n: G \to G \subset SL_d(\mathbb{F}).$$

The sequence must terminate because of the following, and so we obtain another proof of Theorem 6.

Theorem 7. *Let G be an f.g. group and let $\mathbb{F}$ be a field. Consider an ascending sequence $N_j \subset G$ of normal subgroups whose corresponding quotients are subgroups of $SL_d(\mathbb{F})$. This ascending sequence terminates.*

Proof. Let G be an f.g. group with generators $\{g_1, \dots, g_r\}$. Let $\mathbb{F}$ be field. Let $\text{Rep}_d(G)$ denote the set of homomorphisms from G to $SL_d(\mathbb{F})$. Note that $\text{Rep}_d(G)$ corresponds to a subspace of $M_d^r \cong \mathbb{F}^{d^2 r}$, where M_d is the set of $d \times d$ matrices with coefficients in $\mathbb{F}$. Specifically, each homomorphism $\rho: G \to SL_d(\mathbb{F})$ determines the element $(\rho(g_1), \rho(g_2), \dots, \rho(g_r))$ of M_d^r.

The set $\text{Rep}_d(G)$ is an *affine algebraic set* which means that $\text{Rep}_d(G)$ is the set of zeros $Z(S)$ of a set S of polynomials in the $d^2 r$ variables. Specifically, letting $\mathbb{F}[x_1, \dots, x_{d^2 r}]$ denote the free commutative algebra on $d^2 r$ variables, then the relations of G together with the requirement that G map to $SL_d(\mathbb{F})$ determine a set S

of polynomials in $\{x_1, \ldots, x_{d^2r}\}$ and these polynomials evaluate to zero precisely for an element of $\text{Rep}_d(G)$. The ideal I of $\mathbb{F}[x_1, \ldots, x_{d^2r}]$ generated by S is f.g. by Hilbert's Basis theorem.

Suppose (ρ_j) is an infinite sequence of elements of $\text{Rep}_d(G)$ whose kernels N_j form an ascending sequence of subgroups of G. For each j, there is a quotient $G_j = G/N_j$ and a corresponding representation space $\text{Rep}_d(G_j)$ and a corresponding ideal I_j. In this way we obtain a corresponding ascending sequence of ideals $I_1 \subset I_2 \subset I_3 \cdots$. Similarly, let $N_\infty = \cup_j N_j$ and $G_\infty = G/N_\infty$ and let I_∞ denote the corresponding ideal, and note that $I_\infty = \cup_j I_j$. By Hilbert's Basis theorem, I_∞ is f.g. and so these generators are contained in I_m for some m, and so

$$I_m = I_{m+1} = \cdots = I_\infty.$$

Finally, we show that for $q > m$ we have $N_m = N_q$. Since $I_m = I_q$, we have $\text{Rep}_d(G_m) = \text{Rep}_d(G_q)$. But then G_m is a homomorphic image of G_q, and so we have surjections

$$G_m \to G_q \to G_m.$$

But by Mal'cev's theorems, G_m is f.g. linear (as a subgroup of $SL_d(\mathbb{F})$), and hence f.g. residually finite and hence Hopfian, so the composition $G_m \to G_q$ is injective. ■

Problem 8. Let G be an f.g. 3-manifold group. Is every endomorphism of G eventually injective?

Remark 9. The last two decades have given some advances in recognizing various groups as linear.

In particular, we now know that most 3-manifold groups are linear see [12] and references. The unknown cases are fundamental groups of certain closed graph manifolds [10]. It is hard to believe that such groups might admit endomorphisms that are not eventually injective.

Similarly, any double $F *_H F$ of a free group along an f.g. subgroup H is linear [6]. And this gives an unwieldy indirect proof of Proposition 4. But it also indicates how to prove that other examples of f.g. convex subgroups H of (virtually) special groups F have the property of Proposition 4.

Remark 10. Let $G = F *_H F$ be the double along an infinitely generated subgroup F that we described in (1). A consequence of Theorem 6 is that G is residually finite but not linear. We now understand that there are uncountably many pairwise non-isomorphic groups of this type [3]. Yet there are only countably many isomorphism classes of f.g. linear groups . . .

We close with the following problem motivated by the non-linearity of the base groups of the ascending HNN extensions in [13].

Problem 11. Let H be an f.g. linear group. Let $\phi\colon H \to H$ be a monomorphism, and let $G = H*_t = \langle H, t \mid t^{-1}ht = \phi(h)\rangle$ be the ascending HNN extension of H associated to ϕ. Is every endomorphism of G eventually injective? Is G linear?

Note that Borisov–Sapir proved G is residually finite in this case [1].

Acknowledgements. I am extremely grateful to the referee for many helpful corrections to this text.

Funding. Research supported by NSERC.

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Received 18 May 2023; revised 25 March 2024.

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