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Cylinder counts and spin refinement of area Siegel–Veech constants

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Abstract. We study the area Siegel–Veech constants of components of strata of abelian differentials with even or odd spin parity. We prove that these constants may be computed using either: (I) quasimodular forms, or (II) intersection theory. These results refine the main theorems of Chen–Möller–Zagier (2018) and Chen–Möller–Sauvaget–Zagier (2020), which described the area Siegel–Veech constants of the full strata. Along the proof of (II), we establish a new identity for Siegel–Veech constants of cylinders.

1. Introduction

1.1. Relations between cylinder Siegel–Veech constants

Let g, n be non-negative integers satisfying $2g-2+n>0$. Let $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ be the moduli spaces of smooth and stable complex curves of genus g with n distinct markings. We denote by $p\colon \overline{\mathcal{H}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the *Hodge bundle*, i.e., the vector bundle whose fiber at the point $(C, x=\{x_1,\dots,x_n\})$ is the space $H^0(C,\omega_C)$ of abelian differentials on C. If $\mu=(m_1,\dots,m_n)$ is a vector of positive integers satisfying

$$|\mu| \overset{\text{def}}{=} \sum_{i=1}^{n} m_i = 2g-2+n,$$

then we denote by $\mathcal{H}(\mu)$ the *stratum of abelian differentials of type* μ, i.e., the subspace of $\overline{\mathcal{H}}_{g,n}$ of differentials (C,x,η) satisfying: C is smooth, and $\mathrm{ord}_{x_i}(\eta)=m_i-1$ for all $1\le i\le n$. This space is equipped with a canonical measure ν, called the *Masur–Veech measure*. If X is a component of $\mathcal{H}(\mu)$, then we denote by

$$\mathrm{Vol}(X) = (4g-2+2n)\times \nu\left\{(C,x,\eta)\in X,\ \text{s.t. } \frac{\mathrm{i}}{2}\int_C \eta\wedge\overline{\eta}<1\right\},$$

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the *Masur–Veech* volume of X. This volume is finite and rational up to a power of π by [13, 14, 25, 34].

Let (C, x, η) be a differential of type μ. The differential η defines a flat metric with trivial holonomy on $C \setminus \{x_1, \ldots, x_n\}$. Each x_i is a conical singularity of this metric with angle $m_i(2\pi)$. The union of closed geodesics of a given homotopy type in this open surface forms a cylinder Z whose *width* $w(Z)$ is the length of any geodesics in the homotopy class. This cylinder is bounded on each side by at least one singularity. We denote

$$\mathcal{N}(C, \eta, L)_0 = \sum_{Z \text{ s.t. } w(Z) < L} \frac{\text{area}(Z)}{\text{area}(C)},$$

$$\mathcal{N}(C, \eta, L)_{\text{cyl}} = \sum_{Z \text{ s.t. } w(Z) < L} \frac{1}{\text{area}(C)}.$$

Here we consider a refinement of the count of cylinders: for all $1 \leq i \leq n$, we denote

$$\mathcal{N}(C, \eta, L)_{\text{cyl},i} = \frac{1}{2} \sum_{Z \text{ s.t. } w(Z) < L} \frac{f_i(Z)}{\text{area}(C)},$$

where $f_i(Z) = 0, 1$, or 2 if x_i bounds Z on 0, 1, or 2 sides. For a connected component X of $\mathcal{H}(\mu)$, there exist constants $c_0(X), c_{\text{cyl}}(X), c_{\text{cyl},1}(X), \ldots, c_{\text{cyl},n}(X)$ satisfying

$$\mathcal{N}(C, \eta, L)_\star \underset{L \to \infty}{\sim} c_\star(X) \cdot \pi L^2$$

for almost all abelian differentials in X (see [12, 35]). These constants are elements of $\pi^{-2}\mathbb{Q}$ and may be expressed as ratios between volumes of connected components of strata (see Section 4). In [36] Vorobets showed the following identity holds:

$$c_{\text{cyl}}(X) = (2g - 2 + n) c_0(X). \tag{1}$$

Our first theorem is a refinement of this identity.

Theorem 1.1. *For all $1 \leq i \leq n$, we have $c_{\text{cyl},i}(X) = m_i \cdot c_0(X)$.*

Example 1.2. A standard class of abelian differentials is provided by coverings of the square torus ramified above a single point – *square-tiled surfaces*. The differential on the covering surface is the pull-back of the unique (up to scalar) differential on the torus, and its singularities are determined by the orders of ramification. The polygon of Figure 1 is a square-tiled surface in $\mathcal{H}(4, 2)$: the surface is obtained by identifying the edges of the polygon with the same labels. The red/blue vertices of the polygon are identified along this process to produce conical singularities of angles 8π and 4π respectively. In this example, the red cylinder is bounded twice by the red singularity, while the green one is bounded once by each singularity.

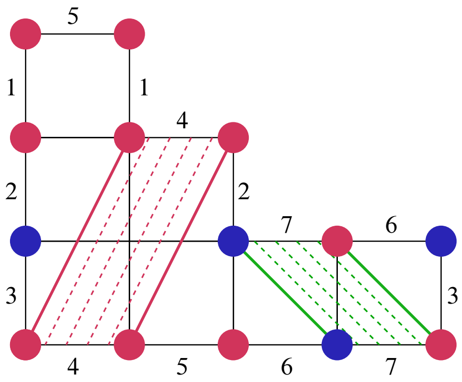


Figure 1. Square-tiled surface in $\mathcal{H}(4, 2)$.

Theorem 1.1 implies that if we choose a random boundary of a cylinder of large width for a generic deformation of this square-tiled surface, then the probability that it contains the red singularity is approximately $2/3$.

1.2. Spin parity of abelian differentials

Here we assume that the entries of μ are all odd. An element (C, x, η) in $\mathcal{H}(\mu)$ determines a canonical *spin structure*, i.e., a line bundle $L \to C$ such that $L^{\otimes 2} \simeq \omega_C$. This line bundle is defined as

$$L = \mathcal{O}_C\left(\frac{m_1 - 1}{2}x_1 + \cdots + \frac{m_n - 1}{2}x_n\right).$$

The *sign*—or *Arf invariant*—of an abelian differential in $\mathcal{H}(\mu)$ equals $(-1)^{h^0(C,L)}$. By classical results of Mumford and Atiyah, this sign is constant in connected families of spin structures [1, 27]. Thus we denote by $\mathcal{H}(\mu)^+/\mathcal{H}(\mu)^-$ the components of $\mathcal{H}(\mu)$ of even/odd differentials.

Remark 1.3. The components $\mathcal{H}(\mu)^+$ and $\mathcal{H}(\mu)^-$ may be disconnected. Indeed, when $\mu = (2g - 1)$ or (g, g), then $\mathcal{H}(\mu)$ contains a connected component of abelian differentials supported on hyperelliptic curves. This connected component may be included in the even or odd component depending on the value of g (see [22]). Hence, the space $\mathcal{H}(\mu)$ may have up to 3 connected components. As Theorem 1.1 holds trivially on hyperelliptic components, it suffices to study $c_i(X)$ for $X = \mathcal{H}(\mu)^+$ and $X = \mathcal{H}(\mu)^-$.

If X is a union of components of $\mathcal{H}(\mu)$, and $\star$ is one of the counting types above then $c_\star(X)$ stands for the average of the Siegel–Veech constants of its connected components (weighted by the Masur–Veech volume). We denote by $c_\star(\mu)$ the Siegel–Veech constants of $\mathcal{H}(\mu)$. Moreover, we write

$$\begin{aligned}
\mathrm{Vol}(\mu) &= \mathrm{Vol}(\mathcal{H}(\mu)),\\
\mathrm{Vol}^{\pm}(\mu) &= \mathrm{Vol}(\mathcal{H}(\mu)^{+}) - \mathrm{Vol}(\mathcal{H}(\mu)^{-}),\\
c_\star(\mu) &= \mathrm{Vol}(\mathcal{H}(\mu)),\\
c_\star^{\pm}(\mu) &= c_\star(\mathcal{H}(\mu)^{+})\frac{\mathrm{Vol}(\mathcal{H}(\mu)^{+})}{\mathrm{Vol}(\mathcal{H}(\mu))} - c_\star(\mathcal{H}(\mu)^{-})\frac{\mathrm{Vol}(\mathcal{H}(\mu)^{-})}{\mathrm{Vol}(\mathcal{H}(\mu))}.
\end{aligned}$$

The functions Vol, c_0 and $\mathrm{Vol}^{\pm}$ may be expressed either

(I) as the value $q \to 1$ of the q-expansion of quasimodular forms (see [7,13,14]); or

(II) as intersection numbers on $\mathbb{P}\overline{\mathcal{H}}(\mu)$, the Zariski closure of the projectivization of $\mathcal{H}(\mu)$ in the projectivized Hodge bundle (see [6]).

We extend these two results to the function $c_0^{\pm}$ in Theorems 2.23 and 3.2 respectively.

Remark 1.4. Several results for the function c_0 may be transposed to $c_0^{\pm}$ with parallel arguments. However, certain ingredients were missing in the previous works to obtain the complete description of $c_0^{\pm}$. We emphasize two arguments that play an important role.

(1) The quasimodular forms approach relies on the description of the character table of the Sergeev group (Propositions 2.16 and 2.18), while the computation of $\mathrm{Vol}^{\pm}$ only relied on the description of the irreducible spin *super*representations. As a result, the expression of the differential operator ∂_2 appearing in Theorem 2.23 is different from the conjectural expression of [6, Section 10.3].

(2) The geometric counterpart relies on the original description of cylinder configurations by Eskin–Masur–Zorich. We show that the expression of the area Siegel–Veech constants as intersection numbers is essentially equivalent to the statement of Theorem 1.1 above. It is interesting to remark that this approach also recovers the result of [6] on c_0 without using quasimodular forms.

This remark is essential as it paves the way for an expression of area Siegel–Veech constants (and sums of Lyapunov exponents) in terms of intersection numbers for (more) general affine invariant manifolds. Indeed, this new approach only relies on the expression of volumes as intersection numbers and could extend the present results to affine invariants manifolds where the expression of volumes as integrals of tautological classes is proved or conjectured.

2. Weighted spin Hurwitz numbers and quasimodular forms

We compute $c_0^{\pm}$ as the limit of weighted spin Hurwitz numbers of the torus of large degree. By the work of [14] generating series of spin Hurwitz numbers can be expressed in terms of quasimodular forms. We extend the approach of [7] for c_0 to compute $c_0^{\pm}$ as the limiting value $q \to 1$ in the q-expansion of a quasimodular form, or, more precisely, as the leading term of the growth polynomial associated to a quasimodular form. We proceed as follows: In Section 2.1 we introduce the spin q-bracket and the symmetric functions p_k. These functions serve as an analogue of both the shifted symmetric functions Q_k and the hook-length moments T_p. In particular, its brackets are quasimodular forms and the action of the p_k inside strict brackets can be encoded by differential operators.

In Section 2.2 we recall the growth polynomials of quasimodular forms and relate this growth to the aforementioned differential operators.

In Section 2.3 we recall the representation theory of the spin symmetric group, and explain the representation theory of the so-called Sergeev group.

In Section 2.4 we refine the aforementioned result of [14]. More precisely, we compute the generating series of *weighted* spin Hurwitz numbers. Here, we extensively make use of the character table of the Sergeev group. We find that strict brackets of the p_k compute weighted Hurwitz numbers.

In Section 2.5 we take all these results together to prove the evaluation of $c_0^{\pm}(\mu)$ as the value $q = 1$ of the q-expansion of quasimodular forms.

2.1. Spin bracket and quasimodular forms

Denote by SP the set of all *strict* partitions of integers (i.e., partitions where all part sizes are different) and by OP the set of all *odd* partitions (i.e., partitions where all part sizes are odd). For $f\colon \mathrm{SP} \to \mathbb{C}$, denote the *spin q-bracket* $\langle f \rangle \in \mathbb{C}[\![q]\!]$ by

$$\langle f \rangle := \frac{\sum_{\lambda \in \mathrm{SP}} (-1)^{\ell(\lambda)} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathrm{SP}} (-1)^{\ell(\lambda)} q^{|\lambda|}},$$

where $|\lambda| := \sum_i \lambda_i$ denotes the size of λ. The denominator is given by

$$\sum_{\lambda \in \mathrm{SP}} (-1)^{\ell(\lambda)} q^{|\lambda|} = \prod_{m \geq 1} (1 - q^m).$$

For a partition λ and a positive integer m, we denote by $r_m(\lambda)$ the number of parts of λ equal to m. Observe that for strict partitions λ we have $r_m(\lambda) \in \{0, 1\}$. Moreover, we let

$$p_k(\lambda) := \sum_i \lambda_i^k = \sum_{m=1}^{\infty} m^k\, r_m(\lambda) \quad (k \in \mathbb{Z}),$$
$$\mathbf{p}_k := -\frac{1}{2}\zeta(-k) + p_k \qquad (k \in \mathbb{Z}_{\geq 0})$$

be the *symmetric power sums* (with an additional constant). Note that this constant equals

$$-\frac{1}{2}\zeta(-k) = \frac{B_{k+1}}{2(k+1)},$$

with B_{k+1} the $(k+1)$th Bernoulli number. Write $\Lambda = \mathbb{C}[\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_5, \ldots]$ for the *symmetric algebra* and assign to $\mathbf{p}_i$ weight $i+1$. Then, for all $f \in \Lambda$, the spin bracket $\langle f \rangle$ is known to be a quasimodular form [14, Section 3.2.2], which is made precise in the following result. Recall that the *Eisenstein series* G_k, given by

$$G_k := -\frac{B_k}{2k} + \sum_{m,r \geq 1} m^{k-1} q^{mr} \quad (k \geq 2 \text{ even}), \tag{2}$$

is an example of a quasimodular form of weight k, and that every quasimodular form is a polynomial in these series: the space of quasimodular forms $\widetilde{M}$ (for the full modular group $\mathrm{SL}_2(\mathbb{Z})$) is given by $\widetilde{M} = \mathbb{Q}[G_2, G_4, G_6]$. This space admits a natural action of $\mathfrak{sl}_2$ by the derivations $D = q\frac{\partial}{\partial q}, -\frac{1}{2}\frac{\partial}{\partial G_2}$ and the diagonal operator multiplying a form by its weight. In particular, D increases the weight of a modular form by 2. For an introduction to quasimodular forms, see [37].

In order to state this result we introduce *connected* brackets, following [7, Section 11]. The *connected spin q-bracket* is the multilinear map $(\mathbb{C}^{\mathrm{SP}})^{\otimes n} \to \mathbb{C}[\![q]\!]$, defined by

$$\langle f_1 \mid \cdots \mid f_n \rangle := \sum_{\alpha \in \Pi(n)} \mu(\alpha) \prod_{A \in \alpha} \Big\langle \prod_{a \in A} f_a \Big\rangle, \tag{3}$$

where $\Pi(n)$ is the set of set partitions of $[\![1, n]\!] := \{1, \ldots, n\}$, and μ is the corresponding Möbius function $\mu(\alpha) = (-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)!$ with $\ell(\alpha)$ the length (cardinality) of the partition α. Let $u_1, u_3, \ldots$ be formal variables and for $\rho \in \mathrm{OP}$, write

$$u_\rho = \prod_i u_{\rho_i}.$$

Similarly, write

$$\mathbf{p}_\rho = \prod_i \mathbf{p}_{\rho_i}.$$

We introduce the generating series Ψ (and Ψ°) of (connected) brackets of symmetric power sums by

$$\Psi(u_1,u_3,\ldots) := \left\langle \exp\Big(\sum_k \mathbf{p}_k u_k\Big)\right\rangle = \sum_{n\geq 0}\frac{1}{n!}\sum_{\ell_1,\ldots,\ell_n}\langle \mathbf{p}_{\ell_1}\cdots\mathbf{p}_{\ell_n}\rangle u_{\ell_1}\cdots u_{\ell_n},$$
$$\Psi^\circ(u_1,u_3,\ldots) := \sum_{n>0}\frac{1}{n!}\sum_{\ell_1,\ldots,\ell_n}\langle \mathbf{p}_{\ell_1}|\cdots|\mathbf{p}_{\ell_n}\rangle u_{\ell_1}\cdots u_{\ell_n},$$

where the sum is over all *odd* integers k, and *odd* $\ell_1,\ldots,\ell_n$ respectively. In this notation, the aforementioned result of [14] is as follows.

Proposition 2.1. *We have*

$$\Psi^\circ(u_1,u_3,\ldots) = -\sum_{\substack{\rho\in\mathrm{OP}\\ \rho\neq\emptyset}}\frac{1}{\mathrm{Aut}(\rho)}D^{\ell(\rho)-1}G_{|\rho|-\ell(\rho)+2}u_\rho,$$

where $D = q\frac{\partial}{\partial q}$, $\mathrm{Aut}(\rho) = \prod_{m\geq 1} r_m(\rho)!$ *and* G_k *is the Eisenstein series of weight* k *(see (2)).*

Proof. We have

$$\begin{aligned}\Psi(u_1,u_3,\ldots) &= \frac{1}{\prod_{m>0}(1-q^m)}\sum_{\lambda\in\mathrm{SP}}(-1)^{\ell(\lambda)}\exp\Big(\sum_k \mathbf{p}_k(\lambda)\,u_k\Big)\,q^{|\lambda|}\\ &= \exp\Big(-\frac{1}{2}\sum_k\zeta(-k)\,u_k\Big)\prod_{m>0}\frac{1-\exp(\sum_k m^k\,u_k)\,q^m}{1-q^m},\end{aligned}$$

where the sums over k are restricted to odd positive integers. Then, by the properties of the connected bracket we have $\exp\Psi^\circ = \Psi$, i.e.,

$$\begin{aligned}&\Psi^\circ(u_1,u_3,\ldots)\\ &\quad= -\frac{1}{2}\sum_k\zeta(-k)\,u_k + \sum_{m,r\geq 1}\Big(1-\exp\Big(r\sum_k m^k u_k\Big)\Big)\frac{q^{mr}}{r}\\ &\quad= -\sum_{\substack{\rho\in\mathrm{OP}\\ \rho\neq\emptyset}}\Big(\frac{1}{2}\sum_k\zeta(-\rho_1)\,\delta_{\ell(\rho)=1} + \frac{1}{\mathrm{Aut}(\rho)}\sum_{m,r\geq 1} m^{|\rho|}\,r^{\ell(\rho)-1}\,q^{mr}\Big)u_\rho,\end{aligned}$$

which is easily seen to match the stated result. ■

Note that this result determines $\langle f\rangle$ for all $f\in\Lambda$. Recall D increases the weight of a quasimodular form by 2. Hence, if f is of weight k, the spin bracket $\langle f\rangle$ is a quasimodular form of weight k.

Two consequences of this result will be important for us: (i) a recursive formula for spin brackets, and (ii) the definition of a modified q-bracket for functions of the form $p_{-1}f$ with $f \in \Lambda$. This recursive formula should be compared with the recursive formula for the hook-length moments in [7, Theorem 16.1] and to a similar result for symmetric functions in the non-spin setting in [33, Proposition 6.2.1].

Corollary 2.2. *For all $i, j \geq 0$ with i even, the differential operators $\varrho_{i,j} \colon \Lambda \to \Lambda$ given by*

$$\varrho_{i,j} = \sum_{\substack{\rho \in \mathrm{OP} \\ \ell(\rho)=j,\, |\rho|=i+j}} \frac{1}{\mathrm{Aut}(\rho)} \frac{\partial^{\ell(\rho)}}{\partial \mathbf{p}_\rho},$$

are such that for all odd $k \geq 1$ and $f \in \Lambda$ one has

$$\langle \mathbf{p}_k f \rangle = - \sum_{i,j \geq 0} \langle \varrho_{i,j}(f) \rangle D^j G_{k+i+1}.$$

Proof. Without loss of generality, we assume that f is a monomial, i.e., $f = \mathbf{p}_\nu$ for some odd partition ν of length n. Then, by applying Möbius inversion to (3), we find

$$\begin{aligned}
\langle \mathbf{p}_k \, \mathbf{p}_\nu \rangle &= \sum_{\alpha \in \Pi(n+1)} \prod_{\{a_1,\ldots,a_r\} \in \alpha} \langle \mathbf{p}_{a_1} | \cdots | \mathbf{p}_{a_r} \rangle \\
&= \sum_{\rho \in \mathrm{OP}} \left(\prod_{m>0} \binom{r_m(\nu)}{r_m(\rho)} \right) \langle \mathbf{p}_{\nu \setminus \rho} \rangle \langle \mathbf{p}_k | \mathbf{p}_{\rho_1} | \mathbf{p}_{\rho_2} | \cdots \rangle \\
&= - \sum_{\rho \in \mathrm{OP}} \left\langle \frac{1}{\mathrm{Aut}(\rho)} \frac{\partial^{\ell(\rho)}}{\partial \mathbf{p}_\rho} \mathbf{p}_\nu \right\rangle D^{\ell(\rho)} G_{k+|\rho|-\ell(\rho)+1}.
\end{aligned}$$
■

Note that

$$\varrho_{i,0} = \delta_{i,0} \cdot \mathrm{Id} \quad \text{and} \quad \varrho_{0,j} = \frac{1}{j!} \frac{\partial^j}{\partial \mathbf{p}_1^j} \quad (i, j \geq 0).$$

Following the proof of [7, Theorem 16.1] in the spin setting, we deduce that a certain linear combination of brackets involving p_{-1} is quasimodular.

Corollary 2.3. *For all $f \in \Lambda_k$,* the modified spin q-bracket

$$\langle f \rangle^* := \langle p_{-1} f \rangle - \langle p_{-1} \rangle \langle f \rangle + \frac{1}{24} \langle \partial_2(f) \rangle$$

is a quasimodular form of weight k, where $\partial_2 = \varrho_{0,1} = \frac{\partial}{\partial \mathbf{p}_1}$. More precisely, we have

$$\langle f \rangle^* = - \sum_{i \geq 2, j \geq 0} \langle \varrho^*_{i,j}(f) \rangle D^j G_i,$$

where $\rho^*_{i,j} = \rho_{i,j} + \delta_{i,2} \rho_{0,j+1}$ and $D = q \frac{\partial}{\partial q}$.

Proof. Note that determining $\langle \mathbf{p}_k f\rangle$ for all positive odd k (in the previous corollary) uniquely determines

$$\langle r_m f\rangle = -\sum_{i,j\geq 0}\langle \varrho_{i,j}(f)\rangle \sum_{r\geq 1} m^{i+j} r^j q^{mr}.$$

As $\rho_{i,0} = \delta_{i,0}\cdot \mathrm{Id}$, we find that $\langle p_{-1} f\rangle$ equals

$$\begin{aligned}
-\sum_{i,j}&\langle \varrho_{i,j}(f)\rangle\Big(\sum_{m,r\geq 1} m^{i+j-1} r^j q^{mr}\Big)\\
&= -\langle f\rangle\Big(\sum_{m,r} m^{-1}q^{mr}\Big)\\
&\qquad -\sum_{j\geq 1}\langle \varrho_{0,j}(f)\rangle D^{j-1}\Big(\frac{1}{24}+G_2\Big) - \sum_{i\geq 2, j\geq 1}\langle \varrho_{i,j}(f)\rangle D^j G_i\\
&= \langle f\rangle\langle p_{-1}\rangle - \frac{1}{24}\langle \partial_2(f)\rangle - \sum_{j\geq 1}\langle \varrho_{0,j}(f)\rangle D^{j-1}G_2 - \sum_{i\geq 2, j\geq 1}\langle \varrho_{i,j}(f)\rangle D^j G_i.
\end{aligned}$$

As $\langle \varrho_{i,j}(f)\rangle$ is quasimodular of weight $k-i-2j$ the result follows. ∎

2.2. Growth polynomials of quasimodular forms

Recall $\widetilde{M} = \mathbb{Q}[G_2, G_4, G_6]$ is the space of quasimodular forms. Write $M = \mathbb{Q}[G_4, G_6]$ for the space of modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Following [7, Section 9], there exists a unique algebra homomorphism ev: $\widetilde{M} \to \mathbb{Q}[\pi^2][1/h]$ such that

$$F(\tau) = \mathrm{ev}[F](h) + O(e^{-h}) \quad (q = e^{2\pi i\tau} = e^{-h})$$

as $h\to 0$ (i.e., $q\to 1$). We call $\mathrm{ev}[F](h)$ the *growth polynomial* of F—it is a polynomial in $1/h$; for more details see the aforementioned paper by Chen, Möller and Zagier. In particular, this morphism is characterized by the following three properties:

(i) $\mathrm{ev}[F](h) = a_0(f)(2\pi \mathrm{i}/h)^k$ for $F\in M_k$;

(ii) $\mathrm{ev}[G_2](h) = \zeta(2)/h^2 - 1/2h$;

(iii) $\mathrm{ev}[DF](h) = -\frac{\partial}{\partial h}\,\mathrm{ev}[F]$ for $F\in \widetilde{M}_k$.

We will be interested in the leading coefficient of the growth polynomial. For $f_1,\ldots,f_r\in\Lambda$, we define the *$\hbar$-bracket* and its *leading term* by

$$\begin{aligned}
\langle f_1\mid\cdots\mid f_r\rangle_\hbar &:= \mathrm{ev}[\langle f_1\mid\cdots\mid f_r\rangle](\hbar),\\
\langle f_1\mid\cdots\mid f_r\rangle_L &:= \lim_{\hbar\to 0}\frac{\hbar^{k-r+1}}{(2\pi\mathrm{i})^{k-2r+2}}\langle f_1\mid\cdots\mid f_r\rangle_\hbar,
\end{aligned}$$

where k is the sum of the weights of the f_i. Note that by [7, Proposition 11.1] this limit is well defined. By Corollary 2.3 we can extend the notation and allow an insertion of p_{-1}:

$$\langle p_{-1} \mid f_1 \mid \cdots \mid f_r \rangle_\hbar := \mathrm{ev}[\langle p_{-1} \mid f_1 \mid \cdots \mid f_r \rangle](\hbar),$$
$$\langle p_{-1} \mid f_1 \mid \cdots \mid f_r \rangle_L := \lim_{\hbar \to 0} \frac{\hbar^{k-r+1}}{(2\pi \mathrm{i})^{k-2r+2}} \langle p_{-1} \mid f_1 \mid \cdots \mid f_r \rangle_\hbar. \tag{4}$$

The behavior as $\hbar \to 0$ also determines the growth of the first N Fourier coefficients. That is, the N*-bracket*, which we define by

$$[f_1 \mid \cdots \mid f_r]_N := \sum_{n=1}^{N} a_n(f_1, \ldots, f_r),$$

where we wrote

$$\langle f_1 | \cdots | f_r \rangle = \sum_{n \geq 0} a_n(f_1, \ldots, f_r) q^n,$$

admits the following growth [7, Proposition 9.4].

Proposition 2.4. *For $f_1, \ldots, f_r \in \Lambda$ of weights $k_1, \ldots, k_r$ with $k = \sum k_i$, we have*

$$[f_1 \mid \cdots \mid f_r]_N = \langle f_1 \mid \cdots \mid f_r \rangle_L \frac{N^{k-r+1} (2\pi \mathrm{i})^{k-2r+2}}{(k-r+1)!} + O\big(N^{k-r} \log N\big)$$
$$[p_{-1} \mid f_1 \mid \cdots \mid f_r]_N = \langle p_{-1} \mid f_1 \mid \cdots \mid f_r \rangle_L \frac{N^{k-r+1} (2\pi \mathrm{i})^{k-2r+2}}{(k-r+1)!} + O\big(N^{k-r} \log N\big)$$

as $N \to \infty$.

In the rest of this subsection, we will now state two lemmas we need in the sequel. First of all, as a corollary of Corollary 2.3, we determine the leading terms of p_{-1} insertions. Secondly, we discuss the relationship between growth polynomials and differential operators.

For all $\ell \geq 1$ odd, we define

$$\mathbf{h}_\ell := \frac{-1}{2\ell} [u^{\ell+1}] \mathbf{P}(u)^\ell, \quad \text{where } \mathbf{P}(u) := \exp\Big(- \sum_{\substack{k \geq 0 \\ k \text{ odd}}} 2 \mathbf{p}_k \, u^{k+1} \Big). \tag{5}$$

These functions are the highest weight part of the central characters $\mathbf{f}_\ell$ introduced in (7), i.e., for all odd $\ell \geq 1$ the function $\mathbf{h}_\ell$ is homogeneous, and $\mathbf{f}_\ell - \mathbf{h}_\ell/\ell$ is of weight less than $\ell + 1$ [6, Theorem 6.7]. We will be interested in the growth of the coefficients in the following sequences

$$\Psi^{H}(u_1, u_3, \ldots) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\ell_1,\ldots,\ell_n} \langle \mathbf{h}_{\ell_1} \cdots \mathbf{h}_{\ell_n} \rangle u_{\ell_1} \cdots u_{\ell_n},$$

$$\Psi^{H,\circ}(u_1, u_3, \ldots) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\ell_1,\ldots,\ell_n} \langle \mathbf{h}_{\ell_1} | \cdots | \mathbf{h}_{\ell_n} \rangle u_{\ell_1} \cdots u_{\ell_n},$$

$$\Psi^{H,\circ}_{-1}(u_1, u_3, \ldots) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\ell_1,\ldots,\ell_n} \langle p_{-1} | \mathbf{h}_{\ell_1} | \cdots | \mathbf{h}_{\ell_n} \rangle u_{\ell_1} \cdots u_{\ell_n},$$

$$\mathcal{C}^{H,\circ}_{-1}(u_1, u_3, \ldots) := \frac{-1}{24\Psi^{H}} \sum_{n\geq 0} \frac{1}{n!} \sum_{\ell_1,\ldots,\ell_n} \sum_{j\geq 1} \langle \partial_2^j (\mathbf{h}_{\ell_1} \cdots \mathbf{h}_{\ell_n}) \rangle \, u_{\ell_1} \cdots u_{\ell_n}.$$

By Corollary 2.3, in the spin setting [6, Lemma 10.4] reads as follows.

Corollary 2.5. *The leading terms of* $\Psi^{H,\circ}_{-1}$ *and* $\mathcal{C}^{H,\circ}_{-1}$ *agree.*

Let the differential operator $\mathcal{D} \colon \Lambda \to \Lambda$ be given by

$$2\mathcal{D} = -\frac{\partial}{\partial \mathbf{p}_1} + \sum_{\ell_1,\ell_2\geq 1} (\ell_1 + \ell_2) \mathbf{p}_{\ell_1+\ell_2-1} \frac{\partial^2}{\partial \mathbf{p}_{\ell_1} \, \partial \mathbf{p}_{\ell_2}}.$$

In [6, Proof of Proposition 6.10] it was shown that (we correct their formula by a factor of $1/2$)

$$\mathfrak{d}\langle f \rangle = \langle \mathcal{D}(f) \rangle$$

for all $f \in \Lambda$, where $\mathfrak{d}$ is the unique derivation on quasimodular forms given by $\mathfrak{d}(G_2) = -1/2$ and $\mathfrak{d}(f) = 0$ if f is modular. This operator $\mathcal{D}$ is extremely useful in determining the growth of the coefficients of $F = \langle f \rangle_q$ for $f \in \Lambda$. Namely, by [6, Proposition 6.10], for all $f \in \Lambda_k$ we have

$$\langle f \rangle_\hbar = \frac{(2\pi \mathrm{i})^k}{\hbar^k} \big(e^{(2\pi \mathrm{i})^{-2} \hbar \mathcal{D}} f \big)(\emptyset). \tag{6}$$

(Be careful that the definition of the $\hbar$-bracket in [6, equation (34)] differs by a power of $2\pi\mathrm{i}$ of the $\hbar$-bracket in [7] and in this work.) Observe that the evaluation at the partition $\emptyset$ of 0 is explicitly given by $\mathbf{p}_k(\emptyset) = -\frac{1}{2}\zeta(-k)$.

Later, we will make use of the following commutation relation, which is the spin analogue of the commutation relation in [6, Lemma 10.5].

Lemma 2.6. *The commutation relation*

$$\partial_2 \circ e^{\mathcal{D}}(f) = e^{\mathcal{D}} \sum_{j\geq 1} \partial_2^j (f) \quad \Big(\partial_2 = \frac{\partial}{\partial \mathbf{p}_1} \Big)$$

holds for all $f \in \Lambda$.

Proof. First, observe that $[\partial_2, \mathcal{D}] = \partial_2^2$. Hence, by induction we find

$$[\partial_2^i, \mathcal{D}] = i\,\partial_2^{i+1} \quad (i \geq 0).$$

Next, again by induction, we show that

$$\partial_2 \mathcal{D}^i = \sum_{k=0}^{i} \frac{i!}{k!} \mathcal{D}^k \partial_2^{i-k+1}$$

for all $i \geq 0$. For $i = 0$ this is trivial. By assuming the result for $i = j$, we obtain

$$\begin{aligned}
\partial_2 \mathcal{D}^{j+1} &= \sum_{k=0}^{j} \frac{j!}{k!} \mathcal{D}^k \partial_2^{j-k+1} \mathcal{D} \\
&= \sum_{k=0}^{j} \frac{j!}{k!} \left(\mathcal{D}^{k+1} \partial_2^{j-k+1} + (j-k+1)\, \mathcal{D}^k \partial_2^{j-k+2}\right) \\
&= \sum_{k=1}^{j+1} \frac{j!k}{k!} \mathcal{D}^k \partial_2^{j-k+2} + \sum_{k=0}^{j} \frac{j!(j-k+1)}{k!} \mathcal{D}^k \partial_2^{j-k+2} \\
&= \sum_{k=0}^{j+1} \frac{(j+1)!}{k!} \mathcal{D}^k \partial_2^{(j+1)-k+1},
\end{aligned}$$

proving the claim. We conclude that

$$\partial_2 \circ e^{\mathcal{D}} = \sum_{i \geq 0} \frac{\partial_2 \mathcal{D}^i}{i!} = \sum_{i \geq 0} \sum_{k=0}^{i} \frac{1}{k!} \mathcal{D}^k \partial_2^{i-k+1} = e^{\mathcal{D}} \sum_{j \geq 1} \partial_2^j. \qquad \blacksquare$$

2.3. Representations of the Sergeev group

The Siegel–Veech constants are computed as the limit of (weighted) Hurwitz numbers of the torus of large degree [7]. These Hurwitz numbers can be expressed in terms of central characters of the symmetric group (the *Burnside Character Formula*; see, e.g., [4]). Analogously, spin Siegel–Veech constants and spin Hurwitz numbers can be expressed in terms of central characters of the spin-symmetric group [14,17,24]. More precisely, spin Siegel–Veech constants can be expressed in terms of central characters corresponding to representations of the Sergeev group, which is closely related to the spin symmetric group. Following [16, 18, 19, 26], we recall some results about the representation theory of both groups and explain how they are related. Though most of the results are not new, the character table of the Sergeev group seems not to be available in the literature. We explain how to derive this table from the known results in the literature.

The spin symmetric group $\widetilde{\mathfrak{S}}_d$ is one of the two representation groups of the symmetric group (for $d \geq 4$), meaning that every projective representation of the symmetric group $\mathfrak{S}_d$ lifts to a (linear) representation of $\widetilde{\mathfrak{S}}_d$. Explicitly, it is defined by the central extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathfrak{S}}_d \xrightarrow{\pi} \mathfrak{S}_d \to 1$$

and can be presented by

$$\widetilde{\mathfrak{S}}_d = \langle t_1, \dots, t_{d-1}, \varepsilon \mid \varepsilon^2 = 1, t_j^2 = \varepsilon, (t_j t_{j+1})^3 = \varepsilon, (t_j t_k)^2 = \varepsilon \text{ for } |j-k| \geq 2 \rangle.$$

The projection π to $\mathfrak{S}_d$ is given by sending ε to the neutral element and t_j to the transposition $(j, j+1)$.

Note that the element ε is central. Hence, ε acts by ± 1 in every representation of $\widetilde{\mathfrak{S}}_d$. We call representations for which ε acts by -1 *spin representations*. These correspond to the projective representation of $\mathfrak{S}_d$, whereas representation for which ε acts by $+1$ correspond to ordinary (linear) representations of $\mathfrak{S}_d$.

Proposition 2.7 ([31]; see also [18, Theorem 8.7]). *The irreducible spin representations $V_\pm^\lambda$ of $\widetilde{\mathfrak{S}}_d$ are parametrized by pairs $(\lambda, (-1)^{d-\ell(\lambda)})$ for strict partitions λ. Moreover, the character values $\varphi_\pm^\lambda(x)$ are determined recursively by*

(i) *an analogue of the* Murnaghan–Nakayama rule *when $\pi(x)$ has only odd cycles in its cycle type;*

(ii) $\varphi_\pm^\lambda(x) = \pm \frac{\mathrm{i}}{\sqrt{2}} \sqrt{\prod \lambda_i}$ *when $d - \ell(\lambda)$ is odd and $\pi(x)$ has cycle type λ;*

(iii) $\varphi_\pm^\lambda(x) = 0$ *in all other cases.*

The *Murnaghan–Nakayama rule* for $\varphi_\pm^\lambda$ is [18, Theorem 10.1]. There exists an amusing way to describe the Murnaghan–Nakayama rule uniformly for both $\mathfrak{S}_d$ and $\widetilde{\mathfrak{S}}_d$, for which we refer to the survey by Morris [26]. Note that in case $d - \ell(\lambda)$ is odd, both cases (i) and (ii) contribute one-half to the character inner product.

Corollary 2.8. *For all $\lambda \in \mathrm{SP}(d)$ with $d - \ell(\lambda)$ odd*

$$\frac{1}{|\widetilde{\mathfrak{S}}_d|} \sum_{\text{(i)}} |\varphi_\pm^\lambda(x)|^2 = \frac{1}{2} = \frac{1}{|\widetilde{\mathfrak{S}}_d|} \sum_{\text{(ii)}} |\varphi_\pm^\lambda(x)|^2,$$

where the first sum is over all x for which $\pi(x)$ has only odd cycles in its cycle type and the second over all x for which $\pi(x)$ has cycle type λ.

Proof. This follows from the fact that the character inner product, which is the sum of the left and right side, is one, and that the conjugacy class of elements for which $\pi(x)$ is of type λ is of size $2d!/\prod \lambda_i$. ∎

The sign of a permutation in the symmetric group determines a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathfrak{S}_d$, which lifts to a $\mathbb{Z}/2\mathbb{Z}$-grading on $\widetilde{\mathfrak{S}}_d$. Explicitly, $\deg(\varepsilon) = 0$ and $\deg(t_i) = 1$. The elements of degree 0 in $\widetilde{\mathfrak{S}}_d$ form the group $\widetilde{\mathrm{A}}_d$, which is a central extension of the alternating group. Given a partition λ, we write $\epsilon(\lambda)$ for the parity of $d - \ell(\lambda)$.

Proposition 2.9 ([31]; see also [18, Theorem 8.7]). *The irreducible spin representations of $\widetilde{\mathrm{A}}_d$ are parametrized by pairs $(\lambda, (\pm 1)^{d-\ell(\lambda)+1})$ for strict partitions λ. More precisely, if $d - \ell(\lambda)$ is odd, then V^λ_+ and V^λ_- are isomorphic irreducible representations of $\widetilde{\mathrm{A}}_d$. If $d - \ell(\lambda)$ is even, the representation V^λ_+ splits as a sum of two irreducible representations of $\widetilde{\mathrm{A}}_d$. The corresponding characters $\alpha^\lambda_\pm$ satisfy*

(i) $\alpha^\lambda_\pm(x) = 2^{\epsilon(\rho)-1}\varphi^\lambda_+(x)$ *when $\pi(x)$ is of cycle type $\rho \in \mathrm{OP}$ and $\rho \neq \lambda$;*

(ii) $\alpha^\lambda_\pm(x) = \frac{1}{2}\varphi^\lambda_+(x) \pm \frac{\mathrm{i}}{2}\sqrt{\prod \lambda_i}$ *when $d - \ell(\lambda)$ is even and $\pi(x)$ has cycle type λ;*

(iii) $\alpha^\pm_\lambda(x) = \varphi^\lambda_+(x) = 0$ *in all other cases.*

Note that the values of $\varphi^\lambda_+(x)$ determined by the Murnaghan–Nakayama rule are real; hence, $\varphi^\lambda_+(x)$ is real if x is as in case (i) and purely imaginary if x is as in case (ii) in Proposition 2.7. Also, note that in $\widetilde{\mathrm{A}}_d$ the conjugacy class of elements for which $\pi(x)$ is of type λ is of size $2d!/\prod \lambda_i$. The analogue of Corollary 2.8 is therefore the following result.

Corollary 2.10. *For all $\lambda \in \mathrm{SP}(d)$ with $d - \ell(\lambda)$ even,*

$$\frac{1}{|\widetilde{\mathrm{A}}_d|}\sum_{x \in \widetilde{\mathrm{A}}_d} \big(\mathrm{Re}\,\alpha^\lambda_\pm(x)\big)^2 = \frac{1}{2} = \frac{1}{|\widetilde{\mathrm{A}}_d|}\sum_{x \in \widetilde{\mathrm{A}}_d} \big(\mathrm{Im}\,\alpha^\lambda_\pm(x)\big)^2.$$

We now return to the spin symmetric group. Spin representations are representations of the *twisted group algebra*

$$\mathcal{T}_d := \mathbb{C}[\widetilde{\mathfrak{S}}_d]/(\varepsilon + 1).$$

Note that $\mathcal{T}_d$ inherits the grading, and hence is a superalgebra. The irreducible supermodules of $\mathcal{T}_d$ (i.e., irreducible $\mathbb{Z}/2\mathbb{Z}$-graded modules) can easily be determined in terms of the irreducible modules of $\widetilde{\mathfrak{S}}_d$.

Proposition 2.11. *The irreducible supermodules of $\mathcal{T}_d$ are given by*

$$V^\lambda = \begin{cases} V^\lambda_+ & d - \ell(\lambda) \text{ is even,} \\ V^\lambda_+ \oplus V^\lambda_- & d - \ell(\lambda) \text{ is odd.} \end{cases}$$

If we were only to know the supermodules of $\mathcal{T}_d$, generalities on superalgebras (see, e.g., [20, Chapter 12]) would allow us to obtain the irreducible supermodules of $\widetilde{\mathfrak{S}}_d$ and $\widetilde{\mathrm{A}}_d$. By doing so, we would almost be able to recover Proposition 2.7

and Proposition 2.9. There is, however, an important subtlety: by doing so we lose information about the characters $\varphi_{\pm}^{\lambda}$ and $\alpha_{\pm}^{\lambda}$ at x for which $\pi(x)$ has cycle type λ, and it would require more specific information about the supermodules to recover part (ii) in these results. Soon, we will come across the same subtlety for a different supermodule.

Instead of working with $\mathcal{T}_d$, often it is more convenient to work with the Sergeev superalgebra. Both algebras capture the same information. Before introducing this algebra, we introduce two more groups. Let

$$\mathrm{Cl}_d := \{\xi_1, \ldots, \xi_d, \varepsilon \mid \varepsilon^2 = 1, \xi_i^2 = \varepsilon, \varepsilon\xi_i = \xi_i\varepsilon, \xi_i\xi_j = \varepsilon\xi_j\xi_i \text{ for all } i \neq j\}$$

be the *Clifford group*, which is a central extension of $(\mathbb{Z}/2\mathbb{Z})^d$. Following Sergeev in [32], we define the *Sergeev group* Se_d as the semidirect product

$$\mathrm{Se}_d := \mathfrak{S}_d \ltimes \mathrm{Cl}_d,$$

where $\mathfrak{S}_d$ acts on Cl_d by permuting the ξ_i. (Some authors define the Sergeev group slightly differently by setting $\xi_i^2 = 1$ instead of $\xi_i^2 = \varepsilon$. The representation theory of these two groups is the same; note, however, that the two Sergeev groups are non-isomorphic.)

Before we study the representation theory of Se_d, we describe the conjugacy classes C such that $C \cap \varepsilon C = \emptyset$. Namely, if the latter condition is not satisfied, every spin representation is trivial on C.

Lemma 2.12 ([32, Lemma 5]). *Let $g = (\sigma, \xi) \in \mathrm{Se}_d$ and write $\xi = \varepsilon^{a_0}\xi_1^{a_1}\cdots\xi_n^{a_n}$ for $a_i \in \{0,1\}$. Then, g is not conjugate to εg if and only if*

(1) $\deg(\xi) = 0$, *the cycle type of σ is in* OP, *and $\sum_{j\in\tau} a_j$ is even for all disjoint cycles τ of σ;*

(2) $\deg(\xi) = 1$, *the cycle type of σ is $\lambda \in$ SP with $\ell(\lambda)$ odd and $\sum_{j\in\tau} a_j$ is odd for all disjoint cycles τ of σ.*

The first case corresponds to the conjugacy class of a pure permutation $(\sigma, 1) \in \mathrm{Se}_d$. Note that this conjugacy class only depends on the cycle type $\rho \in \mathrm{OP}$ of σ; we denote this conjugacy class by C_ρ. In the second case, we write $C_{\lambda,1}$ for the corresponding conjugacy class.

On Cl_d we introduce the $\mathbb{Z}/2\mathbb{Z}$-grading by setting $\deg\xi_i = 1$. This grading extends to Se_d by additionally letting $\deg\sigma = \deg\varepsilon = 0$ for $\sigma \in \mathfrak{S}_d$. Define the corresponding *twisted algebras* by

$$\mathcal{C}_d := \mathbb{C}[\mathrm{Cl}_d]/(\varepsilon + 1), \quad \mathcal{X}_d := \mathbb{C}[\mathrm{Se}_d]/(\varepsilon + 1),$$

which both are superalgebras. The following results determine the corresponding irreducible supermodules. In these results, it is important to recall that the tensor product of two superalgebras $\mathcal{A}$ and $\mathcal{B}$ is not the same as the tensor product of two algebras. Instead, the multiplication is defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')}(aa') \otimes (bb') \quad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}).$$

The following two propositions are [20, Example 12.1.3 and Lemma 13.2.3].

Proposition 2.13. *The Clifford superalgebra, denoted $\mathcal{C}_d$, is irreducible of dimension $2^{\lfloor (d+1)/2 \rfloor}$.*

In particular, the character ζ corresponding to $\mathcal{C}_d$ satisfies $\zeta(1) = 2^{\lfloor (d+1/2) \rfloor}$ and $\zeta(x) = 0$ if $x \neq 1$.

Proposition 2.14. *The map $\vartheta_d \colon \mathcal{T}_d \otimes \mathcal{C}_d \to \mathcal{X}_d$ given by*

$$\begin{aligned} t_j \otimes 1 &\mapsto \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1})(j, j+1), \\ 1 \otimes \xi_j &\mapsto \xi_j \end{aligned}$$

is an isomorphism of superalgebras.

We write $\delta(\lambda)$ and $\epsilon(\lambda)$ for the parity of $\ell(\lambda)$ and $d - \ell(\lambda)$ respectively.

Corollary 2.15. *The irreducible supermodules of $\mathcal{X}_d$ are given by $V_\lambda \otimes \mathcal{C}_d$, where λ goes over all strict partitions. The corresponding character θ^λ is given by*

$$\theta^\lambda(x) = \begin{cases} 2^{(\ell(\rho)+\delta(\lambda)-\epsilon(\lambda))/2}\varphi^\lambda(\rho), & x \in C_\rho \textit{ for some } \rho \in \mathrm{OP}, \\ 0, & x \notin C_\rho \cup \varepsilon C_\rho \textit{ for some } \rho \in \mathrm{OP}, \end{cases}$$

where $\varphi^\lambda(\rho)$ is the character of V_λ evaluated at a permutation of type $\rho \in \mathrm{OP}$.

The structure of *super*modules of $\mathcal{X}_d$ does not directly imply the character table of the Sergeev group, nor does seem to be contained in the literature. However, the irreducible representations of Se_d were constructed by Maxim Nazarov in [28, Section 1]. We deduce the following proposition from his result.

Proposition 2.16. *The irreducible spin representations of Se_d are parametrized by pairs $(\lambda, (\pm 1)^{\ell(\lambda)})$ for strict partitions λ. Moreover, the character values $\chi^\lambda_\pm(x)$ are determined recursively by*

(i) $\chi^\lambda_\pm(x) = 2^{-\delta(\lambda)}\theta^\lambda(\rho)$ *if $x \in C_\rho$;*

(ii) $\chi^\lambda_\pm(x) = \pm 2^{(\ell(\lambda)-1)/2}\mathrm{i}\sqrt{\prod \lambda_i}$ *when $\ell(\lambda)$ is odd and $x \in C_{\lambda,1}$;*

(iii) $\chi^\lambda_\pm(x) = 0$ *in all other cases.*

Proof. First of all, observe that by the previous corollary the result holds if $\ell(\lambda)$ is even. Namely, in that case, the irreducible representation corresponding to λ equals the *super*representation corresponding to λ. We now follow Nazarov's construction in the case $\ell(\lambda)$ is odd.

First, assume d is even. Write $\tau^\lambda_\pm$ for the representations corresponding to $V^\lambda_\pm$. Then the irreducible representation of Se_d corresponding the pair $(\lambda,\pm)$ in the space $\mathcal{C}_d \otimes V^\lambda_\pm$ is given by ($j = 1,\dots,d-1$)

$$(j,j+1) \mapsto \mathrm{i}\xi_0 \frac{\xi_j - \xi_{j+1}}{\sqrt{2}} \otimes \tau^\lambda_\pm(t_j), \quad \xi_j \mapsto \xi_j \otimes \mathrm{Id}, \quad \varepsilon \mapsto -1.$$

Let $x \in \mathrm{Se}_d$. Then, as the trace of the Clifford algebra only takes values on the identity, we find

$$\chi^\lambda_\pm(x) = C(x)\frac{2^{d/2}}{2^{(d-\ell(\lambda))/2}}\varphi^\lambda_\pm(\rho),$$

where $C(x)$ denotes the multiplicity of the identity in

$$\xi_0(\xi_1-\xi_2)\cdots\xi_0(\xi_{\lambda_{j_1}-1}-\xi_{\lambda_{j_r}})\xi$$

if we write $x = (j_1, j_1+1)\cdots(j_r, j_r+1)\xi$, and where ρ is the cycle type of the projection of x to the symmetric group. Note that this multiplicity takes values in $\{-1,0,1\}$. In case $\rho \in \mathrm{OP}$, we obtain

$$\chi^\lambda_\pm(x) = 2^{\ell(\lambda)/2}\varphi^\lambda_\pm(\rho) = 2^{(\ell(\lambda)-1)/2}\theta^\lambda(\rho),$$

which we could also have deduced directly from Corollary 2.15. More interestingly, if $x \in C_{\lambda,1}$, we obtain

$$\chi^\lambda_\pm(x) = \pm 2^{(\ell(\lambda)-1)/2}\mathrm{i}\sqrt{\prod \lambda_i}.$$

That $\chi^\lambda_\pm(x) = 0$ in all other cases, now easily follows from the orthogonality relations, or by a similar computation as above.

Next, suppose d is odd. Note that $\mathcal{C}_d$, considered as an ordinary module, rather than a super module, splits as a sum $\mathcal{C}_d \simeq \mathcal{C}_d^+ \oplus \mathcal{C}_d^-$ of irreducible (non-super) modules $\mathcal{C}_d^\pm$ of dimension $2^{(d-1)/2}$. Let $I = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0\end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$. Nazarov now first constructs a reducible representation of Se_d corresponding to λ in the space $\mathbb{C}^2 \otimes \mathcal{C}_d^+ \otimes V^\lambda$ by

$$(j,j+1) \mapsto I \otimes \frac{\xi_j - \xi_{j+1}}{\sqrt{2}} \otimes \tau_\lambda(t_j), \quad \xi_j \mapsto J \otimes \xi_k \otimes \mathrm{Id}\,.$$

Write ω_λ for the endomorphism of V_λ defined by $v \mapsto (-1)^{\deg(v)}v$. Then, the ± 1-eigenspaces with respect to the involution $J \otimes \mathrm{Id} \otimes \omega_\lambda$ form two irreducible representations corresponding to $(\lambda,\pm)$. In particular, in this case, we obtain

$$\chi^\lambda_\pm(x) = D(x)\frac{2^{(d-1)/2}}{2^{(d-\ell(\rho))/2}} \cdot \begin{cases} \alpha^\lambda_+(\rho) + \alpha^\lambda_-(\rho), & \deg x = 0, \\ \alpha^\lambda_+(\rho) - \alpha^\lambda_-(\rho), & \deg x = 1 \end{cases}$$

with $D(x)$ the multiplicity of the identity in

$$(\xi_{j_1}-\xi_{j_1+1})\cdots(\xi_{j_2}-\xi_{\lambda_{j_2+1}})\xi$$

if we write $x=(j_1,j_1+1)\cdots(j_r,j_r+1)$, and where the product $(j_1,j_1+1)\cdots(j_r,j_r+1)$ is of cycle type ρ (i.e., $x\in C_\rho\xi$). In other words, if $x\in C_\rho$, we obtain

$$\chi^\lambda_\pm(x)=2^{(\ell(\lambda)+1)/2}\alpha^\lambda_\pm(\rho)=2^{(\ell(\lambda)+1)/2}\alpha^\lambda_\pm(\rho)=\frac{1}{2}\theta^\lambda(\rho).$$

Moreover, if $x\in C_{\lambda,1}$, we obtain

$$\chi^\lambda_\pm(x)=\pm 2^{(\ell(\lambda)-1)/2}\mathrm{i}\sqrt{\prod\lambda_i}. \qquad \blacksquare$$

Similar to the spin symmetric group, we conclude that both cases contribute one-half to the inner product of characters.

Corollary 2.17. *For all $\lambda\in\mathrm{SP}$ with $\ell(\lambda)$ odd,*

$$\frac{1}{|\,\mathrm{Se}_d\,|}\sum_{\rho\in\mathrm{OP}}\sum_{x\in C_\rho}|\chi^\lambda_\pm(x)|^2=\frac{1}{2}=\frac{1}{|\,\mathrm{Se}_d\,|}\sum_{x\in C_{\lambda,1}}|\chi^\lambda_\pm(x)|^2.$$

Write Se_d^0 for the subgroup of Se_d consisting of elements of even degree. Similarly to Proposition 2.16, we obtain the character table.

Proposition 2.18. *The irreducible spin representations of Se_d^0 are parametrized by pairs $(\lambda,(\pm1)^{\ell(\lambda)+1})$ for strict partitions λ. Moreover, the character values $\psi^\lambda_\pm(x)$ are determined recursively by*

(i) $\psi^\lambda_\pm(x)=\frac{1}{2}\theta^\lambda(\rho)$ *if $x\in C_\rho$ and $\rho\neq\lambda$;*

(ii) $\psi^\lambda_\pm(x)=\frac{1}{2}\theta^\lambda(\rho)\pm\frac{1}{2}2^{\ell(\lambda)/2}\mathrm{i}\sqrt{\prod\lambda_i}$ *when $\ell(\lambda)$ is even and $x\in C_\lambda$;*

(iii) $\psi^\lambda_\pm(x)=0$ *in all other cases.*

Corollary 2.19. *For all $\lambda\in\mathrm{SP}(d)$ with $\ell(\lambda)$ even,*

$$\frac{1}{|\,\mathrm{Se}_d^0\,|}\sum_{x\in\mathrm{Se}_d^0}\big(\mathrm{Re}\,\psi^\lambda_\pm(x)\big)^2=\frac{1}{2}=\frac{1}{|\,\mathrm{Se}_d^0\,|}\sum_{x\in\mathrm{Se}_d^0}\big(\mathrm{Im}\,\psi^\lambda_\pm(x)\big)^2.$$

Finally, we end our discussion by introducing the central characters associated to the Sergeev group. Given $\rho\in\mathrm{OP}(d)$, the class sum $\overline{C}_\rho$, which is the sum of all permutations in C_ρ, acts by Schur's lemma as a constant on a supermodule $V_\lambda\otimes\mathcal{C}_d$. We call this constant *the central character* and denote it by $\mathbf{f}_\rho(\lambda)$. In fact,

$$\mathbf{f}_\rho(\lambda)=|C_\rho|\frac{\theta^\lambda(\rho)}{\dim\theta^\lambda}=|C_\rho|\frac{\chi^\lambda_\pm(\rho)}{\dim\chi^\lambda_\pm}=|C_\rho|\frac{\psi^\lambda(\rho)}{\dim\psi^\lambda}, \tag{7}$$

where ψ^λ is the average of ψ^λ_+ and ψ^λ_- (or equal to ψ^λ_+ if $\ell(\lambda)$ is odd). In particular, the central characters associated to Se_d and Se^0_d agree. (Note that we restricted ourselves to $\rho \in \mathrm{OP}$ in the definition of central characters.)

We extend this notation to all $\rho \in \mathrm{OP}$ and $\lambda \in \mathrm{SP}$, even if ρ and λ are not of the same size. Write $\rho \sim \rho'$ if the partitions ρ and ρ' only differ in the number of parts equal to 1, and set

$$\mathbf{f}_\rho(\lambda) = \begin{cases} \mathbf{f}_{\rho'}(\lambda), & |\rho| < |\lambda|, \\ 0, & |\rho| > |\lambda|, \end{cases} \tag{8}$$

where $\rho \sim \rho'$ and $|\rho'| = |\lambda|$. Then, by [19] we have $\mathbf{f}_\rho \in \Lambda$, where Λ is the symmetric algebra introduced before. More concretely, if $\rho = (\ell)$, then by [6, Theorem 6.7]

$$\ell \mathbf{f}_\ell(\lambda) = \frac{-1}{2\ell}[t^{\ell+1}]\biggl(\prod_{j=1}^{\ell-1}(1-jt)\cdot\exp\biggl(\sum_{\substack{k\geq 1\\ k \text{ odd}}}\frac{2}{k}p_k(\lambda)\,t^k\bigl(1-(1-\ell t)^{-k}\bigr)\biggr)\biggr).$$

(The central characters in our work agree with those in [14], but differ by a factor ℓ of those in [6, Theorem 6.7] and in [19, Definition 6.3].) By comparing this formula with the definition of $\mathbf{h}_\ell$, we see that $\mathbf{f}_\ell - \mathbf{h}_\ell/\ell$ is of weight less than $\ell + 1$.

2.4. Weighted spin Hurwitz numbers

Let $\Pi = (\mu^{(1)}, \ldots, \mu^{(n)})$ with $\mu^{(i)} \in \mathrm{OP}(d)$. A *Hurwitz tuple of degree d and ramification type* Π is an element

$$(\alpha, \beta, \gamma_1, \ldots, \gamma_n) \in (\mathfrak{S}_d)^{n+2}$$

such that $[\alpha, \beta]\gamma_1 \ldots \gamma_n = 1$ and the type of γ_i is equivalent to the partition $\mu^{(i)}$ (i.e., they only differ in the amount of 1's).

Write $\mathrm{Hur}_d(\Pi)$ for the set of all Hurwitz tuples h of degree d and ramification type Π. Every such Hurwitz tuple corresponds to a (ramified) covering of the torus, which, after pulling back the flat metric on the torus, yields a differential of type μ. This induces a spin parity $s\colon \mathrm{Hur}(\Pi) \to \mathbb{Z}/2\mathbb{Z}$. By [14, Theorem 2] the *spin Hurwitz number* of degree d and ramification profile Π is given by

$$\frac{1}{d!}\sum_{h \in \mathrm{Hur}_d(\Pi)} (-1)^{s(h)} = 2^{\chi/2} \sum_{\lambda \in \mathrm{SP}(d)} (-1)^{\ell(\lambda)}\, \mathbf{f}_\Pi(\lambda),$$

where $\mathbf{f}_\Pi = \prod_i \mathbf{f}_{\mu^{(i)}}$ with $\mathbf{f}_{\mu^{(i)}}$ a central character, defined by (8), and χ is the Euler characteristic of the cover, i.e.,

$$\chi = \sum_i \bigl(\ell(\mu^{(i)}) - |\mu^{(i)}|\bigr).$$

We generalize this result, by finding the following expression for a weighted count. Given a Hurwitz tuple $h = (\alpha, \beta, \gamma_1, \ldots, \gamma_n)$ and $f \in \Lambda$, write $f(\alpha)$ to denote the value of f applied to the partition corresponding to the conjugacy class of α.

Proposition 2.20. *For any $f \in \Lambda$, we have*

$$\frac{1}{d!} \sum_{h \in \mathrm{Hur}_d(\Pi)} (-1)^{s(h)} f(\alpha) = 2^{\chi/2} \sum_{\lambda \in \mathrm{SP}(d)} (-1)^{\ell(\lambda)} \mathbf{f}_\Pi(\lambda)\, f(\lambda).$$

Proof. Given conjugacy classes $C_1, \ldots, C_n$ in a finite group G, we define *a Hurwitz tuple for G with ramification type $\mathcal{C} = (C_1, \ldots, C_n)$* to be an element

$$(\alpha, \beta, \gamma_1, \ldots, \gamma_n) \in G^{n+2}$$

such that $[\alpha, \beta]\gamma_1 \cdots \gamma_n = 1$ and $\gamma_i \in C_i$ for all i. Denote by $\mathrm{Hur}_G(\mathcal{C})$ the set of all Hurwitz tuples for G with ramification type $\mathcal{C}$. The sum of all elements of a conjugacy class C in the group algebra of G acts by Schur's lemma by a constant; this constant is the central character f_C, which we consider as a function $f_C\colon G^\wedge \to \mathbb{C}$, where $G^\wedge$ denotes the set of irreducible representations of G. We write $f_\mathcal{C} = \prod_{C \in \mathcal{C}} f_C$. Given an irreducible representation π of G and a class function $f\colon G \to \mathbb{C}$, we let

$$M_G(f, \pi) := \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g)\, \overline{\chi_\pi(g)}\, f(g) = \frac{1}{|G|} \sum_{[g]} |[g]| \chi_\pi(g) \overline{\chi_\pi(g)}\, f(g),$$

where the last sum is over all conjugacy classes $[g]$ of G, and with χ_π the character of the representation π and $|[g]|$ the size of the conjugacy class of g in G.

Mutatis mutandis, the proof of [23, Theorem A.1.10] implies that for all class functions $f\colon G \to \mathbb{C}$, we have that

$$H_{f,\mathcal{C}}(G) := \frac{1}{|G|} \sum_{h \in \mathrm{Hur}_G(\mathcal{C})} f(\alpha) = \sum_{\pi \in G^\wedge} f_\mathcal{C}(\pi) M_G(f, \pi),$$

where $h = (\alpha, \beta, \gamma_1, \ldots, \gamma_n)$. In the non-spin setting, this result for $G = \mathfrak{S}_d$ suffices to prove the non-spin variant of this proposition (see [7, Proposition 6.3]). Here, it does not suffice to let $G = \mathrm{Se}_d$, as we have to take care of the sign of the Hurwitz tuple. Hence, we complete the proof along the same lines as [14, Sections 3.1.6–3.1.10].

Given a group homomorphism $\phi\colon G' \to G$, $h \in \mathrm{Hur}_G(\mathcal{C})$ and a conjugacy class $\mathcal{C}'$ of G' such that $\phi(\mathcal{C}') = \mathcal{C}$, we write $\mathrm{Hur}_{G'}(h, \phi) = \mathrm{Hur}_{G'}(h)$ for the set of Hurwitz tuples $h' \in \mathrm{Hur}_{G'}(\mathcal{C}')$ such that $\phi(h') = h$. For all $d \geq 1$, write B_d for the group of signed permutations, obtained by setting $\varepsilon = 1$ in Se_d. Moreover, write B_d^0 and Se_d^0 for the subgroups of even elements with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on Se_d. Note that all these four groups B_d, B_d^0, Se_d^0 and Se_d^0 admit a natural projection homomorphism to $\mathfrak{S}_d$.

Write $\mathrm{Hur}_d(\mathcal{C})$ for $\mathrm{Hur}_{\mathfrak{S}_d}(\mathcal{C})$ and let $h \in \mathrm{Hur}_d(\mathcal{C})$ be given. Now, in terms of Hurwitz tuples, [14, Proposition 5] reads

$$\frac{(-1)^{s(h)}}{d!} = 2^{\chi/2}\Big(\frac{|\mathrm{Hur}_{\mathrm{B}_d}(h)|}{|\mathrm{B}_d|} - \frac{|\mathrm{Hur}_{\mathrm{B}_d^0}(h)|}{|\mathrm{B}_d^0|} - \frac{|\mathrm{Hur}_{\mathrm{Se}_d}(h)|}{|\mathrm{Se}_d|} + \frac{|\mathrm{Hur}_{\mathrm{Se}_d^0}(h)|}{|\mathrm{Se}_d^0|}\Big).$$

Hence, summing over all $h \in \mathrm{Hur}_d(\mathcal{C})$ and multiplying by $f(\alpha)$, we obtain

$$\sum_{h\in\mathrm{Hur}_d(\mathcal{C})} \frac{(-1)^{s(h)}}{d!} f(\alpha) = 2^{\chi/2}\big(H_{f,\mathcal{C}}(\mathrm{B}_d) - H_{f,\mathcal{C}}(\mathrm{B}_d^0) - H_{f,\mathcal{C}}(\mathrm{Se}_d) + H_{f,\mathcal{C}}(\mathrm{Se}_d^0)\big),$$

where in $H_{f,\mathcal{C}}(G)$ the conjugacy class $\mathcal{C}$ should be interpreted as the conjugacy class of pure permutations in G.

Observe that the irreducible representations π of Se_d such that $\pi(\varepsilon) = 1$ correspond to the representations of B_d. In particular, $M_{\mathrm{B}_d}(f,\pi) = M_{\mathrm{Se}_d}(f,\pi)$ for such π. Hence,

$$H_{f,\mathcal{C}}(\mathrm{Se}_d) - H_{f,\mathcal{C}}(\mathrm{B}_d) = \sum_{\pi\in(\mathrm{Se}_d)^{\wedge}_{-}} f_{\mathcal{C}}(\pi)\, M_{\mathrm{Se}_d}(f,\pi),$$

where $(\mathrm{Se}_d)^{\wedge}_{-}$ denotes the set of irreducible spin representations of Se_d. Similarly, the result holds after replacing Se_d and B_d with Se_d^0 and B_d^0, so that

$$\sum_{h\in\mathrm{Hur}_d(\mathcal{C})} \frac{(-1)^{s(h)}}{d!} f(\alpha) = \sum_{\pi\in(\mathrm{Se}_d^0)^{\wedge}_{-}} f_{\mathcal{C}}(\pi)\, M_{\mathrm{Se}_d^0}(f,\pi) - \sum_{\pi\in(\mathrm{Se}_d)^{\wedge}_{-}} f_{\mathcal{C}}(\pi)\, M_{\mathrm{Se}_d}(f,\pi).$$

At this point, we can no longer follow [14] closely. Instead, we will compute $M_{\mathrm{Se}_d}(f,\pi)$ using Proposition 2.16. Recall the conjugacy class of cycle type ρ in the symmetric group $\mathfrak{S}_d$ is of size $d!/z_\rho$ with

$$z_\rho = \prod_{m>0} m^{r_m(\rho)} r_m(\rho)! = \mathrm{Aut}(\rho)\prod_i \rho_i.$$

First assume that $\pi = \pi_\lambda$ is associated to $\lambda \in \mathrm{SP}$ and $\ell(\lambda)$ is odd. Then, π_λ only takes values on the conjugacy classes C_ρ and εC_ρ. Both conjugacy classes consist of $2^{d-\ell(\rho)} d!/z_\rho$ elements. Hence, we obtain

$$\begin{aligned} M_{\mathrm{Se}_d}(f,\pi_\lambda) &= \frac{1}{2^{d+1}d!} \sum_{\rho\in\mathrm{OP}(d)} 2\cdot 2^{d-\ell(\rho)} \frac{d!}{z_\rho} |\chi_\lambda(\rho)|^2 f(\rho) \\ &= \sum_{\rho\in\mathrm{OP}(d)} \frac{2^{-\ell(\rho)}}{z_\rho} |\chi_\lambda(\rho)|^2 f(\rho) =: \mathrm{M}(f)(\lambda), \end{aligned}$$

where $\mathrm{M}(f)$ can be thought of as the spin analogue of the Möller transform in [7, Corollary 13.2]. Next, let $\pi_\lambda^\pm$ be associated to $\lambda \in \mathrm{SP}$ with $\ell(\lambda)$ odd. By Corollary 2.17, we find that

$$M_{\mathrm{Se}_d}(f, \pi_\lambda^\pm) = \frac{1}{2} f(\lambda) + \sum_{\rho \in \mathrm{OP}(d)} \frac{2^{-\ell(\rho)}}{z_\rho} |\chi_\lambda^\pm(\rho)|^2 f(\rho) =: \frac{1}{2} f(\lambda) + \frac{1}{2}\mathrm{M}(f)(\lambda),$$

where we now also defined the Möller transform if $\ell(\lambda)$ is odd.

Similarly, we compute $M_{\mathrm{Se}_d^0}(f, \pi)$ for all $\pi \in (\mathrm{Se}_d^0)^\wedge$ using Proposition 2.18. First, suppose $\pi = \pi_\lambda$ and $\ell(\lambda)$ is odd. As $\psi_+^\lambda = \chi_+^\lambda$, we find

$$M_{\mathrm{Se}_d^0}(f, \pi_\lambda) = \frac{1}{2^d d!} \sum_{\rho \in \mathrm{OP}(d)} 2 \cdot 2^{d-\ell(\rho)} \frac{d!}{z_\rho} |\chi_+^\lambda(\rho)|^2 f(\rho) = \mathrm{M}(f)(\lambda).$$

Finally, let $\pi_\lambda^\pm \in (\mathrm{Se}_d^0)^\wedge$ with $\ell(\lambda)$ even. By Corollary 2.19, we obtain

$$\begin{aligned} M_{\mathrm{Se}_d^0}(f, \pi_\lambda^\pm) &= \frac{1}{2} f(\lambda) + 2 \sum_{\rho \in \mathrm{OP}(d)} \frac{2^{-\ell(\rho)}}{z_\rho} \left| \frac{1}{2} \chi_\lambda(\rho) \right|^2 f(\rho) \\ &= \frac{1}{2} f(\lambda) + \frac{1}{2}\mathrm{M}(f)(\lambda). \end{aligned}$$

Recall that the central characters corresponding to all representations of Se_d and Se_d^0 agree on odd partitions. Hence, we conclude

$$\begin{aligned} &\sum_{h \in \mathrm{Hur}_d(\mathcal{C})} \frac{(-1)^{s(h)}}{d!} f(\alpha) \\ &\quad = 2^{\chi/2} \sum_{\lambda \in \mathrm{SP}(d)} \mathbf{f}_\Pi(\lambda) (-1)^{\ell(\lambda)} \Big(M(f)(\lambda) - 2\Big(\frac{1}{2} f(\lambda) + \frac{1}{2} M(f)(\lambda)\Big)\Big) \\ &\quad = 2^{\chi/2} \sum_{\lambda \in \mathrm{SP}(d)} (-1)^{\ell(\lambda)} \mathbf{f}_\Pi(\lambda) f(\lambda). \end{aligned}$$

∎

Define the *(combinatorial) ℓ-weighted spin Siegel–Veech constant* $c_\ell^\pm(d, \Pi)$ to be

$$c_\ell^\pm(d, \Pi) := \frac{1}{d!} \sum_{h \in \mathrm{Hur}_d(\Pi)} (-1)^{s(h)} \, p_\ell(\alpha),$$

where $h = (\alpha, \beta, \gamma_1, \ldots, \gamma_n)$.

Corollary 2.21. *For all odd ℓ,*

$$c_\ell^\pm(d, \Pi) = 2^{\chi/2} \sum_{\lambda \in \mathrm{SP}(d)} (-1)^{\ell(\lambda)} \mathbf{f}_\Pi(\lambda) p_\ell(\lambda).$$

2.5. Recursion relation for $c_0^{\pm}(\mu)$

The symmetric algebra Λ is canonically identified with $R = \mathbb{Q}[\mathbf{h}_1, \mathbf{h}_3, \ldots]$ (see (5) for the definition of the $\mathbf{h}_i$). As in [6, equation (56)], for all non-empty sets $I \subset \mathbb{N}$ of cardinality n, we define a function $\mathcal{A}_I \in R[\![(z_i)_{i \in I}]\!]$ as follows:

$$\mathcal{A}_{\{i\}} := z_i^{-1} + \sum_{s \geq 0} \mathbf{h}_{2s+1} z_i^{2s+1},$$

$$\mathcal{A}_{\{i,j\}} := \frac{z_i \mathcal{A}'(z_i) - z_j \mathcal{A}'(z_j)}{\mathcal{A}(z_j) - \mathcal{A}(z_i)} - 1,$$

$$\mathcal{A}_I := \frac{1}{(n-1)} \sum_{\substack{k \geq 0 \\ \underline{\ell} = (\ell_1 \ldots, \ell_k)}} \sum_{I = \{r,s\} \sqcup I_1 \sqcup \cdots \sqcup I_k} \frac{1}{k!} \mathcal{A}^{\underline{\ell}}_{\{r,s\}} \prod_{i=1}^{k} \mathcal{A}_{I_i}^{[\ell_i]} \quad (n \geq 3).$$

In the last line, the first sum on the right-hand side is over all vectors of odd positive integers of length k, while the second sum is over partitions of I into $k+1$ non-empty sets, and we let

$$\mathcal{A}^{\underline{\ell}}_{\{i,j\}} := \frac{\partial^k}{\partial_{\ell_1} \ldots \partial_{\ell_k}} \mathcal{A}_{\{i,j\}} \quad \text{and} \quad \mathcal{A}_I^{[\ell]} := [z_i^{\ell}] \mathcal{A}_{I \cup \{i\}}.$$

Set $\mathcal{A}_n = \mathcal{A}_{\{1,\ldots,n\}}$. Then for any element $\mathbf{h} \in R$ we denote by $\mathbf{h}|_{\mathbf{h}_\ell \mapsto \boldsymbol{\alpha}_\ell} \in \mathbb{Q}$ the image of $\mathbf{h}$ under the unique ring morphism $\Lambda \to \mathbb{Q}$ mapping $\mathbf{h}_\ell$ to the rational number $\boldsymbol{\alpha}_\ell$ defined by

$$\boldsymbol{\alpha}_\ell := \frac{-1}{2\ell} [u^\ell]\, \mathbf{P}(u)^\ell, \quad \text{where } \mathbf{P}(u) := \exp\Bigg(\sum_{s \geq 1} \zeta(-s)\, u^{s+1} \Bigg).$$

Notation 2.22. If X is a connected component of a stratum $\mathcal{H}(\mu)$, the *normalized Masur–Veech volume* of X is defined as

$$\mathrm{vol}(X) := \frac{(|\mu| - 1)! \prod_{i=1}^{n} m_i}{2(2\pi \mathrm{i})^{2g}} \mathrm{Vol}(X).$$

We denote $\mathrm{vol}(\mu) = \mathrm{vol}(\mathcal{H}(\mu))$ and $\mathrm{vol}^{\pm}(\mu) = \mathrm{vol}(\mathcal{H}(\mu)^+) - \mathrm{vol}(\mathcal{H}(\mu)^-)$.

Recall that the leading term $\langle \cdots \rangle_L$ of the growth polynomial is defined by (4), the elements $\mathbf{h}_\ell \in \Lambda$ are defined by (5) and recall that $\partial_2 = \frac{\partial}{\partial \mathbf{p}_1}$.

Theorem 2.23. *We have*

$$\begin{aligned} c_0^{\pm}(\mu) &= \frac{3 \cdot 2^{\chi/2}}{2\pi^2 \cdot |\mu| \cdot \mathrm{vol}(\mu)} \langle p_{-1} | \mathbf{h}_{m_1} | \cdots | \mathbf{h}_{m_n} \rangle_L \\ &= \frac{-2^{\chi/2}}{16\pi^2 \cdot |\mu| \cdot \mathrm{vol}(\mu)} [z_1^{m_1} \cdots z_n^{m_n}]\, \partial_2 \mathcal{A}_n |_{\mathbf{h}_\ell \mapsto \boldsymbol{\alpha}_\ell}. \end{aligned}$$

Proof. We compute $c_0^{\pm}$ as the limit of Hurwitz numbers of the torus of large degree:

$$c_0^{\pm}(\mu) = \lim_{D\to\infty} \frac{3}{\pi^2} \frac{\sum_{d=1}^{D} c_{-1}^{\pm,\circ}(d,\Pi)}{\sum_{d=1}^{D} N_d^{\circ}(\Pi)},$$

where $\Pi = ((m_1, 1, \ldots, 1), \ldots, (m_n, 1, \ldots, 1))$, $N_d^{\circ}(\Pi)$ is the number of *connected* torus covers of degree d with ramification profile Π, while $c_{-1}^{\pm,\circ}(d,\Pi)$ is the sum over those covers with -1st Siegel–Veech weight and with sign given by the parity. The proof of this formula is obtained from a direct transposition of the proof of the analogue result for $c_0(\mu)$ given by [7, Proposition 17.1]. Note that by [13] we have that

$$\sum_{d=1}^{D} N_d^{\circ}(\Pi) \underset{D\to\infty}{\sim} \frac{D^{|\mu|+1}}{|\mu|+1} \operatorname{Vol}(\mu).$$

By Corollary 2.21 and the inclusion-exclusion principle used to obtain connected cover counts from possibly disconnected ones, we have:

$$c_\ell^{\pm,\circ}(d,\Pi) = 2^{\chi/2}[q^d]\langle p_\ell|\mathbf{f}_{m_1}|\cdots|\mathbf{f}_{m_n}\rangle,$$

where the connected q-brackets were defined by (3). The growth rate of the connected bracket is determined by Proposition 2.4. Note $2-2g = \chi = n - |\mu|$. Therefore,

$$\begin{aligned}
c_0^{\pm}(\mu) &= \frac{3}{\pi^2} \lim_{D\to\infty} \frac{2^{\chi/2}[p_{-1}|\mathbf{f}_{m_1}|\cdots|\mathbf{f}_{m_n}]_D}{\frac{D^{|\mu|+1}}{|\mu|+1}\operatorname{Vol}(\mu)} \\
&= \frac{3}{\pi^2} \lim_{D\to\infty} \frac{2^{\chi/2}\langle p_{-1}|\mathbf{f}_{m_1}|\cdots|\mathbf{f}_{m_n}\rangle_L \frac{D^{|\mu|+1}}{(|\mu|+1)!}(2\pi \mathrm{i})^{2g}}{\frac{D^{|\mu|+1}}{|\mu|+1}\operatorname{Vol}(\mu)} \\
&= \frac{3}{\pi^2} \frac{2^{\chi/2}\langle p_{-1}|\mathbf{f}_{m_1}|\cdots|\mathbf{f}_{m_n}\rangle_L}{2|\mu|\ \operatorname{vol}(\mu)} \prod_{i=1}^{n} m_i \\
&= \frac{3}{\pi^2} \frac{2^{\chi/2}\langle p_{-1}|\mathbf{h}_{m_1}|\cdots|\mathbf{h}_{m_n}\rangle_L}{2|\mu|\ \operatorname{vol}(\mu)},
\end{aligned}$$

where the last equation follows as $\mathbf{f}_\ell - \mathbf{h}_\ell/\ell$ is of weight less than $\ell+1$.

We denote by $\operatorname{Aut}(\mathbf{m})$ the cardinality of the stabilizer of the action of $\mathfrak{S}_n$ on $(m_1, \ldots, m_n)$, and recall $\Psi_{-1}^{H,\circ}(\mathbf{u})$, $\mathcal{C}_{-1}^{H,\circ}(\mathbf{u})$, $\Psi^H(\mathbf{u})$ and $\Psi^{H,\circ}(\mathbf{u})$ defined in Section 2.2 and their corresponding L-brackets. Then, by using Corollary 2.5, we find

$$\begin{aligned}
\langle p_{-1}|\mathbf{h}_{m_1}|\cdots|\mathbf{h}_{m_n}\rangle_L &= \frac{1}{\operatorname{Aut}(\mathbf{m})}[u_{m_1}\cdots u_{m_n}]\Psi_{-1}^{H,\circ}(\mathbf{u})_L \\
&= \frac{1}{\operatorname{Aut}(\mathbf{m})}[u_{m_1}\cdots u_{m_n}]\mathcal{C}_{-1}^{H,\circ}(\mathbf{u})_L.
\end{aligned}$$

By (6) and Lemma 2.6, we see that the leading term in $\hbar$ of $\mathcal{C}_{-1}^{H,\circ}(\mathbf{u})_\hbar$ and $\frac{\partial_2\Psi^H(\mathbf{u})_\hbar}{\Psi^H(\mathbf{u})_\hbar}$ agree, i.e.,

$$\begin{aligned}\langle p_{-1}|\mathbf{h}_{m_1}|\cdots|\mathbf{h}_{m_n}\rangle_L &= \frac{-1}{24}\frac{1}{\mathrm{Aut}(\mathbf{m})}[u_{m_1}\cdots u_{m_n}]\frac{\partial_2\Psi^H(\mathbf{u})_L}{\Psi^H(\mathbf{u})_L}\\ &= \frac{-1}{24}\frac{1}{\mathrm{Aut}(\mathbf{m})}[u_{m_1}\cdots u_{m_n}]\,\partial_2\Psi^{H,\circ}(\mathbf{u})_L\\ &= \frac{-1}{24}[z_1^{m_1}\cdots z_n^{m_n}]\,\partial_2\mathcal{A}_n|_{\mathbf{h}_\ell\mapsto\boldsymbol{\alpha}_\ell},\end{aligned}$$

where, for the last equality, see [6, Proposition 6.9 and Corollary 6.11]. ■

3. Twisted graphs and the boundary of the Hodge bundle

Fix μ, g, n as in the introduction. We will consider the following cohomology classes of the projectivized Hodge bundle:

- the tautological class $\xi = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}\overline{\mathcal{H}}_{g,n}, \mathbb{Q})$;
- the Poincaré-dual class of the locus of curves with a non-separating node $\delta_0 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$;
- the Chern class of the cotangent line at the ith marking $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ for all $1 \le i \le n$.

For all $1 \le i \le n$, we denote

$$\beta_i = \xi^{2g-2}\cdot\Big(\prod_{j\neq i} m_i\cdot\psi_i\Big). \tag{9}$$

Let X be a component of $\mathcal{H}(\mu)$. We denote by $\mathbb{P}\overline{X}$ the Zariski closure of the projectivization of X. In [6, 9, 29], it was shown that

$$\mathrm{vol}(X) = \int_{\mathbb{P}\overline{X}} \beta_i\cdot\xi$$

for all $1 \le i \le n$ (see Notation 2.22 for the definition of vol and $\mathrm{vol}^{\pm}$). Here, we will consider the following intersection numbers for all $1 \le i \le n$:

$$d_i(X) = \int_{\mathbb{P}\overline{X}} \beta_i\cdot\delta_0.$$

The purpose of the rest of the paper is to prove the following result.

Theorem 3.1. *For all connected components X of $\mathcal{H}(\mu)$, and $1 \le i \le n$, we have*

$$c_{\mathrm{cyl},i}(X) = \frac{-m_i}{4\pi^2}\cdot\frac{d_i(X)}{\mathrm{vol}(X)}.$$

For all X, the number $d_i(X)$ is independent of the choice of i (see [6]). Besides, we recall that $c_{\mathrm{cyl}}(X) = \sum_{i=1}^{n} c_{\mathrm{cyl},i}(X)$ is related to $c_0(X)$ via the Vorobets relation (1). Thus, Theorem 3.1 directly implies Theorem 1.1 and the following expression for the area Siegel–Veech (SV) constants of connected components, which is a new check that the class β_i represents the Kontsevich–Zorich cocycle (see [11, 21]).

Theorem 3.2. *For all connected component of a stratum of abelian differentials X, and all $1 \leq i \leq n$, we have*

$$c_0(X) = \frac{-1}{4\pi^2} \cdot \frac{d_i(X)}{\mathrm{vol}(X)}.$$

As SV constants of hyperelliptic components are explicit and Theorem 1.1 holds trivially for these components, we only need to consider the SV constants of strata and their weighting by their spin sign

$$d_i(\mu) = d_i(\mathcal{H}(\mu)) \quad \text{and} \quad d_i^{\pm}(\mu) = d_i(\mathcal{H}(\mu)^+) - d_i(\mathcal{H}(\mu)^-).$$

Then, to prove Theorem 3.1 we need to prove that the identities

$$c_{\mathrm{cyl},i}(\mu) = \frac{-m_i}{4\pi^2} \cdot \frac{d_i(\mu)}{\mathrm{vol}(\mu)} \quad \text{and} \quad c_{\mathrm{cyl},i}^{\pm}(\mu) = \frac{-m_i}{4\pi^2} \cdot \frac{d_i^{\pm}(\mu)}{\mathrm{vol}(\mu)}$$

hold for all (odd) partitions μ and for all $1 \leq i \leq n$. Besides, we will show that Theorem 3.1 holds when we set $i = 1$ (the general case follows immediately by permuting the entries of μ). We proceed in two steps:

(1) In the present section we use intersection theory to express the numbers $d_1(\mu)$ and $d_1^{\pm}(\mu)$ in terms of the functions vol, $\mathrm{vol}^{\pm}$ and intersection numbers in genus 0 (see Proposition 3.11).

(2) In the next section we use arguments of combinatorics to show that Proposition 3.11 may be rewritten as the sum of the contributions of cylinder configurations in the sense of [12] thus finishing the proof of Theorem 3.1.

3.1. Twisted graphs

We recall here the definition of twisted graphs of [15]. A *stable graph* is the data of

$$\Gamma = \left(V, H, g\colon V \to \mathbb{Z}_{\geq 0},\ \iota\colon H \to H,\ \phi\colon H \to V, H^{\iota} \simeq [\![1, n]\!]\right),$$

where

- an element $v \in V$ is called a *vertex*. We denote by $g(v)$ the *genus* of v.
- an element $h \in H$ is called an *half-edge*. We say h is incident to $\phi(h)$, and write $h \mapsto v$ if $\phi(h) = v$. Moreover, we denote by $n(v)$ the *valency* of the vertex v, i.e., the number of half-edges incident to v.

- the function ι is an involution. The set E consist of cycles of length 2 for ι, which are called *edges*.
- the fixed points of ι are called *legs*. We write n for the number of legs, and identify the set of legs with the set $[\![1,n]\!] := \{1,2,\dots,n\}$.
- for all vertices v we have $2g(v)-2+n(v)>0$.
- the graph (V,E) is connected.

The *genus* of Γ is defined as

$$g(\Gamma) = h^1(\Gamma) + \sum_{v\in V} g(v) \quad \text{with } h^1(\Gamma) = |E| - |V| + 1.$$

An *automorphism* of Γ consists of automorphisms of the sets V and H that leave invariant the data g,ι and ϕ. A stable graph is said to be *of compact type* if $h^1(\Gamma)=0$, i.e., if the graph is a tree.

Definition 3.3. A *twist* on a stable graph Γ is a function $m : H \to \mathbb{Z}$ satisfying the following conditions:

- For all $v \in V$, we denote by $\mu(v) = (m(h))_{h\mapsto v}$ the vector of twists at half-edges incident to v. We impose

$$|\mu(v)|\left(\stackrel{\text{def}}{=} \sum_{h\mapsto v} m(h)\right) = 2g(v) - 2 + n(v).$$

- If $e=(h,h')$ is an edge of Γ from v to v', then we have $m(h) = -m(h')$.
- There exists a partial order $\geq$ on V such that for all vertices v,v' connected by an edge (h,h') we have $(v \geq v') \Leftrightarrow (m(h) \geq 0)$.

For an edge $e=(h,h')$ from v to v', we call $m_e = |m(h)|$ the *twist at the edge*. If $m_e = 0$ (or, equivalently, if $v \geq v'$ and $v' \geq v$) then we will say that the edge is *horizontal*.

A *twisted graph* is a pair (Γ, m), where m is a twist on Γ. It is said *compatible* with an integral vector μ of length n, if the twist at the ith leg is equal to m_i.

Most of the twisted graphs that will be considered will be in the following set.

Definition 3.4. A twisted graph (Γ, m) is a *graph of rational type* if it is of compact type and there exists a partition of the set of vertices $V(\Gamma) = R(\Gamma) \sqcup D(\Gamma)$ satisfying:

- the twists at the legs are non-negative.
- $g(v) = 0 \Leftrightarrow v \in R(\Gamma)$ (the set of *rational vertices*).
- if $v \in D(\Gamma)$ (the set of *decorations*), then all half-edges incident to v have positive twist.

Remark 3.5. In this definition, the twist and the partition of the set of vertices are uniquely determined by the underlying stable graph and the twists at the legs in $[\![1,n]\!]$. In particular, an automorphism of Γ automatically respects the twist function. Thus, to keep the notation simple, we denote by Γ a graph of rational type.

Definition 3.6. A twisted graph (Γ, m) is a *bicolored graph* if there is a partition of the set of vertices

$$V(\Gamma) = V_0 \sqcup V_{-1}$$

such that all edges connect a vertex $v \in V_0$ to a vertex $v' \in V_{-1}$ with $v > v'$.

A bicolored graph is a *rational backbone graph* if it is of compact type, has a unique vertex in V_{-1} of genus 0 which carries the first marking, and all vertices of V_0 have positive genus (note that a rational backbone graph is of rational type and satisfies $D(\Gamma) = V_0$ and $R(\Gamma) = V_{-1}$).

3.2. Boundary of strata of differentials

If μ is a vector of (not necessarily positive) integers of length n with $|\mu| = 2g-2+n$, then we denote by $\mathcal{H}(\mu)$ the moduli space of objects (C, x, η), where C is smooth and η is a meromorphic differential with $\mathrm{ord}_{x_i}(\eta) = m_i - 1$ for all $1 \leq i \leq n$. This space is canonically embedded in the vector bundle

$$\pi_* \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(p_1 \cdot D_1 + \cdots + p_n \cdot D_n),$$

where $\pi \colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the universal curve, D_i is the divisor associated to the ith marking, and p_i is a positive integer bigger than $-m_i$ for all $1 \leq i \leq n$. The *incidence variety compactification* $\mathbb{P}\overline{\mathcal{H}}(\mu)$ is the Zariski closure of $\mathbb{P}\mathcal{H}(\mu)$ in the projectivization of the above vector bundle. The geometry of $\mathbb{P}\overline{\mathcal{H}}(\mu)$ does not depend on the choice of the p_i's and was described in [2].

3.2.1. Residue conditions. If μ has r non-positive entries then we denote by $\mathfrak{R}(\mu)$ the subspace of $\mathbb{C}^r$ with sum equal to 0. If R is a linear subspace of $\mathfrak{R}(\mu)$, then we denote by $\mathcal{H}(\mu, R) \subset \mathcal{H}(\mu)$ and $\mathbb{P}\overline{\mathcal{H}}(\mu, R) \subset \mathbb{P}\overline{\mathcal{H}}(\mu)$ the space of differentials with residues in R (up to a scalar in the projectivized case).

3.2.2. Boundary components of $\mathbb{P}\overline{\mathcal{H}}(\mu)$. We recall that a non-trivial stable graph Γ determines a boundary component of the moduli space of curves

$$\Gamma \colon \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}$$

with $g = g(\Gamma)$ and n the number of markings. A bicolored graph (Γ, m) determines two moduli spaces:

$$\mathcal{H}(\Gamma, m, R)_0 = \prod_{v \in V_0} \mathcal{H}(\mu(v), R_v),$$
$$\mathcal{H}(\Gamma, m, R)_{-1} = \prod_{v \in V_{-1}} \mathcal{H}(\mu(v), R_v),$$

where for all v the vector space R_v is defined by the so-called *global residue condition* defined in [2, Definition 1.2]. Moreover, it determines a morphism

$$\zeta_{(\Gamma,m)} \colon \mathbb{P}\mathcal{H}(\Gamma, m, R)_0 \times \mathbb{P}\mathcal{H}(\Gamma, m, R)_{-1} \to \mathbb{P}\overline{\mathcal{H}}(\mu, R).$$

Denote by $\mathbb{P}\overline{\mathcal{H}}(\Gamma, m, R)$ the Zariski closure of the image of this morphism.

Besides, if (Γ, m) is a twisted graph with exactly one horizontal edge, then we denote by $\mathbb{P}\overline{\mathcal{H}}(\Gamma, m, R) \subset \mathbb{P}\overline{\mathcal{H}}(\mu, R)$ the subspace of differentials whose underlying curve sits in the image of $\overline{\mathcal{M}}_\Gamma$. With these notations at hand, $\mathbb{P}\overline{\mathcal{H}}(\mu, R)$ is the union of the $\mathbb{P}\overline{\mathcal{H}}(\Gamma, m, R)$ for (Γ, m) bicolored graphs and twisted graphs with exactly one horizontal edge compatible with μ.

3.2.3. Generalization of spin parity. If μ is odd, then the parity of a point $(C, x, \eta) \in \mathcal{H}(\mu)$ is the parity of $h^0(C, \mathcal{O}((m_1 - 1)/2 + \cdots + (m_n - 1)/2))$.

Besides, if μ has only odd entries apart from the first two which are equal to 0, and $R \subset \mathfrak{R}(\mu)$ is the vector space defined by $r_1 + r_2 = 0$, then the parity of a point in $(C, x, \eta) \in \mathcal{H}(\mu, R)$ is defined as the parity of the differentials in the desingularization of the node created by attaching the two poles of order 1 (see [3] for the details of this construction). Then, as in the holomorphic case, we denote by

$$[\mathbb{P}\overline{\mathcal{H}}(\mu, R)]^{\pm} = [\mathbb{P}\overline{\mathcal{H}}(\mu, R)]^{+} - [\mathbb{P}\overline{\mathcal{H}}(\mu, R)]^{-},$$

where $[\cdot]$ stands for the Poincaré dual class of a subspace (here we consider $\mathbb{P}\overline{\mathcal{H}}(\mu, R)$ as a subspace of a projectivized vector bundle of differentials with large enough poles).

3.3. Functions defined by intersection numbers in genus 0

Here, we define three functions f, φ, and $\varphi^{\pm}$ as intersection numbers on strata of differentials of genus 0. We also show how to compute these functions recursively. The results collected in this section will only be used in Section 4 to manipulate the sums indexed by different families of graphs of rational type.

3.3.1. The f-function. Let $\mu = (m_1, m_2, \dots) \in \mathbb{Z}^n$ be vector of length $n \geq 3$ such that: no entry of μ is equal to 0, m_1 and m_2 are positive, and $|\mu| = n-2$, where $|\mu| = \sum_{i=1}^n m_i$. Then, we define

$$f(\mu) = \int_{\mathbb{P}\bar{\mathcal{H}}(\mu,\{0\})} \prod_{\substack{i\geq 3\\ \text{s.t. } m_i>0}} m_i \psi_i,$$

where $\{0\}$ stands for the trivial vector space.[1] This function may be computed inductively by using twisted graphs. We fix an index i in $[\![2,n]\!] \setminus \{3\}$. We say that a twisted graph Γ of genus 0 is of *type* f_i if it has exactly two vertices $v_0 > v_{-1}$, one (non-horizontal) edge e and satisfies:

- the legs i and 1 are on one of the vertices, and the leg 3 is on the other one.
- the vertex v_0 (respectively v_{-1}) has exactly 1 (respectively 2) of the legs with index in $\{1,2,3\}$.

In Table 1 we represented the graphs of type f_2 and f_i with $m_i < 0$ below (the last case $m_i > 0$ and $i > 2$ will not be used in the rest of the text). Moreover, we write $\Gamma \vdash f_i$ to denote that Γ is of type f_i and compatible with μ. Then we set

$$f(\Gamma) = f(\mu(v_{-1})) \cdot f(\mu(v_0)),$$

where we order the entries of $\mu(v_{-1})$ and $\mu(v_0)$ in such a way that the first entries of $\mu(v_{-1})$ are the m_j's for the values $j \in \{1,2,3\}$ and incident to v_{-1}, while the first entries of $\mu(v_0)$ are m_e and m_j for the last value $j \in \{1,2,3\}$.

Lemma 3.7. *If $m_i < 0$ for $i > 2$, then we have*

$$f(\mu) = (n-3)!\,[t^{m_2}] \prod_{i>2} t\,\frac{1-t^{-m_i}}{1-t},$$

where $[t^{m_2}]$ stands for m_2-coefficient in the variable t. Moreover, if $m_3 > 0$, then for all $i \in [\![2,n]\!] \setminus \{3\}$ we have the following relation:

$$f(\mu) = m_3 \sum_{\Gamma \vdash f_i} f(\Gamma),$$

where the sum is over all twisted graphs of type f_i compatible with μ.

This lemma generalizes [6, Proposition 2.1] where the induction formula is proved for the case $i = 2$ (i.e., the third marking is separated from markings 1 and 2). In the present work, we will apply this lemma to cases where i is a marking with $m_i < 0$.

---

[1] Note that this definition differs from the function $h_{\mathbb{P}^1}$ in [6, Section 2.2] by a product of the m_i.

Proof. We use the same approach as in [6]. For all i, we have

$$\psi_3 = \sum_{\Sigma \subset \{1,\dots,n\} \setminus \{1,i,3\}} (\zeta_{\Gamma_\Sigma})_*(1) \in H^2(\bar{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

where Γ_Σ is the graph with one edge and one of the vertices carries the markings in $\Sigma \cup \{1, i\}$ while the other vertex carries the other markings. There is a unique twisted graph structure m on Γ which is compatible with μ. Then the intersection of $(\zeta_{\Gamma_\Sigma})_*(1)$ with $p_*[\mathbb{P}\bar{\mathcal{H}}(\mu, \{0\})]$ is transverse (we recall that $p\colon \mathbb{P}\bar{\mathcal{H}}(\mu) \to \bar{\mathcal{M}}_{g,n}$ is the forgetful morphism of the differential) and given by $p_*[\mathbb{P}\bar{\mathcal{H}}(\Gamma, m, \{0\})]$. The integral of

$$\prod_{\substack{i \geq 4 \\ \text{s.t. } m_i > 0}} m_i \psi_i$$

is non-trivial if and only if the unique structure of twisted graph on the stable graph defining δ_Σ satisfies the constraints of the type f_i for dimension reasons. In that case, the space $\mathbb{P}\bar{\mathcal{H}}(\Gamma, m, \{0\})$ is isomorphic to

$$\mathbb{P}\bar{\mathcal{H}}(\mu_0, \{0\}) \times \mathbb{P}\bar{\mathcal{H}}(\mu_{-1}, \{0\})$$

the integral is given by $f(\Gamma_\Sigma)$ (see [6]). ∎

We use this lemma to show another identity satisfied by the function f. For $i \in [\![3, n]\!]$, we say a twisted graph is of *type* f_i' if it satisfies the same conditions as for the type f_i, but interchanging the roles of the markings 2 and 3. Its contribution is also determined by the same formula as for graphs of type f_i by interchanging the roles of the markings 2 and 3.

Lemma 3.8. *If m_1, m_2, m_3 are positive, then we have*

$$\sum_{\Gamma \vdash f_2} (m_e + m_3) f(\Gamma) = \sum_{\Gamma \vdash f_3'} (m_e + m_2) f(\Gamma).$$

Proof. We have the following identity in $H^*(\bar{\mathcal{M}}_{0,n+2}, \mathbb{Q})$:

$$\begin{aligned}(m_3\psi_3 - m_2\psi_2)\, & p_*\big[\mathbb{P}\bar{\mathcal{H}}(\mu, \{0\})\big] \\ &= \sum_{\Gamma \in \mathrm{Bic}_{2,3}} m_{E(\Gamma)} p_*\big[\mathbb{P}\bar{\mathcal{H}}(\Gamma, m, \{0\})\big] - \sum_{\Gamma \in \mathrm{Bic}_{3,2}} m_{E(\Gamma)} p_*\big[\mathbb{P}\bar{\mathcal{H}}(\Gamma, m, \{0\})\big],\end{aligned}$$

where $\mathrm{Bic}_{j,j'}$ is the set of bicolored graphs with the leg j incident to a vertex in V_{-1} and j' is incident to a vertex in V_0 (see [30, Theorem 6]), and $m_{E(\Gamma)} = \prod_{e \in E(\Gamma)} m_e$ (and we recall that $p\colon \mathbb{P}\bar{\mathcal{H}}(\mu) \to \bar{\mathcal{M}}_{g,n}$ is the forgetful morphism of the differential). We intersect this identity with

$$\prod_{\substack{i \geq 4 \\ \text{s.t. } m_i > 0}} m_i \psi_i.$$

On the left-hand side, we use Lemma 3.7 as sums over graphs of type f_2 or f_3'. On the right-hand side, only the graphs of type f_2 and f_3' give a non-trivial contribution on the right-hand side (see [6, Section 2.4]. All-in-all, we obtain the following expression:

$$\sum_{\Gamma \vdash f_2} m_3 f(\Gamma) - \sum_{\Gamma \vdash f_3'} m_2 f(\Gamma) = \sum_{\Gamma \vdash f_3'} m_e f(\Gamma) - \sum_{\Gamma \vdash f_2} m_e f(\Gamma),$$

which is the desired identity. ∎

3.3.2. The φ-function. Now, let μ be a vector of length $n+2$ satisfying: m_1 is positive, m_{n+1} and m_{n+2} are non-positive, no other entry is zero, and $|\mu| = n$. Then, we denote

$$\varphi(\mu) = \int_{\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})} \prod_{\substack{i \geq 2 \\ \text{s.t. } m_i > 0}} m_i \, \psi_i,$$

where $R_{1,2}$ is the vector space defined by $r_{n+1} + r_{n+2} = 0$ while all other residues are equal to 0. Moreover, if all entries of μ are odd apart from $m_{n+1} = m_{n+2} = 0$, we denote

$$\varphi^{\pm}(\mu) = \int_{[\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})]^{\pm}} \prod_{\substack{i \geq 2 \\ \text{s.t. } m_i > 0}} m_i \, \psi_i.$$

We will need three types of graphs of rational type of genus 0 to compute φ and $\varphi^{\pm}$ inductively (see also Table 1). A graph of rational type Γ is of:

Type φ_1 if it has one edge e, the legs $1, n+1$, and $n+2$ are incident to a vertex v_{-1} which is lower than the vertex v_0 carrying the leg 2. Then, we set

$$\varphi(\Gamma) = \varphi(\mu(v_{-1})) \cdot f(\mu(v_0)),$$
$$\varphi^{\pm}(\Gamma) = \varphi^{\pm}(\mu(v_{-1})) \cdot f(\mu(v_0)),$$

where $\mu(v_{-1}) = (m_1, \ldots, m_{n+1}, m_{n+2})$ and $\mu(v_0) = (m_2, m_e, \ldots)$.

Type φ_1' is defined similarly but we impose that the leg $n+2$ is incident to vertex v_0. Then, we set

$$\varphi(\Gamma) = \varphi(\mu(v_{-1})) \cdot f(\mu(v_0)),$$

where $\mu(v_{-1}) = (m_1, \ldots, m_{n+1}, -m_e)$ and $\mu(v_0) = (m_2, m_e, \ldots)$ (note that this configuration may only occur if m_{n+2} is negative so we do not need to define $\varphi^{\pm}$ for such a graph).

Type φ_1'' if it has one edge e, the legs 2 and $n+2$ are incident to a vertex v_{-1} which is lower than the vertex v_0 carrying the legs 1 and $n+1$. Then, we set

$$\varphi(\Gamma) = \varphi(\mu(v_{-1})) \cdot f(\mu(v_0)),$$

where $\mu(v_{-1}) = (m_2, \dots, m_{n+2}, -m_e)$ and $\mu(v_0) = (m_1, m_e, \dots)$ (here m_{n+1} is negative thus we do not need to define $\varphi^{\pm}$ for such a graph).

Type φ_2 if it has one edge, which is horizontal, the legs 1 and $n+1$ are incident to a vertex v_1 and the legs 2 and $n+2$ to the vertex v_2. Then, we set

$$\varphi(\Gamma) = \varphi(\mu(v_1)) \cdot \varphi(\mu(v_2)),$$
$$\varphi^{\pm}(\Gamma) = -\varphi^{\pm}(\mu(v_1)) \cdot \varphi^{\pm}(\mu(v_2)),$$

where $\mu(v_1) = (m_1, \dots, m_{n+1}, 0)$ and $\mu(v_0) = (m_2, \dots, 0, m_{n+2})$.

Type φ_3 if it has two edges, both not horizontal, and the legs 1 and $n+1$ are incident to a vertex v_1, the legs 2 and $n+2$ to a vertex v_2 which are both connected to an upper vertex v_0 with edges e_1 and e_2. Then, we set

$$\varphi(\Gamma) = (m_{e_1} + m_{e_2}) \cdot \varphi(\mu(v_1)) \cdot f(\mu(v_0)) \cdot \varphi(\mu(v_2)),$$

where $\mu(v_1) = (m_1, \dots, m_{n+1}, -m_{e_1})$, $\mu(v_2) = (m_2, \dots, -m_{e_2}, m_{n+2})$ and $\mu(v_{-1}) = (m_{e_1}, m_{e_2}, \dots)$.

The function φ may be computed by the following lemma.

Lemma 3.9. *If $m_i < 0$ for all $1 < i \le n$, then*

$$\varphi(\mu) = (n-1)! \prod_{i=2}^{n} (-m_i) \quad \text{and} \quad \varphi^{\pm}(\mu) = -(n-1)!.$$

If $m_2 > 0$, then we may compute φ and $\varphi^{\pm}$ recursively

$$\varphi(\mu) = m_2 \sum_{\Gamma \vdash \varphi_1, \varphi_1', \varphi_1'', \varphi_2, \varphi_3} \varphi(\Gamma),$$
$$\varphi^{\pm}(\mu) = m_2 \sum_{\Gamma \vdash \varphi_1, \varphi_2} \varphi^{\pm}(\Gamma),$$

where the sums are over graphs of type $\varphi_1, \varphi_1', \varphi_1'', \varphi_2$, or φ_3 compatible with μ.

Proof. The base cases of the lemma will be established in Section 4 with the language of chains and cylinder configurations.

To prove the induction formulas, we use the same strategy as for f. We use the following relation in $H^2(\bar{\mathcal{M}}_{0,n+2}, \mathbb{Q})$:

$$\psi_2 = \sum_{\Sigma \subset \{3,\dots,n+2\} \setminus \{n+1\}} \delta_\Sigma \in H^2(\bar{\mathcal{M}}_{0,n+2}, \mathbb{Q}),$$

where $\delta_\Sigma = (\zeta_{\Gamma_\Sigma})_*(1)$, and Γ_Σ for the unique graph with one edge and one vertex carrying the legs in $\{1, n+1\} \cup \Sigma$ (while the other one carries the others).

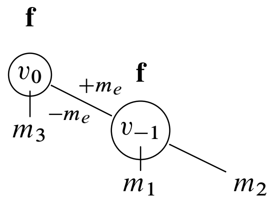
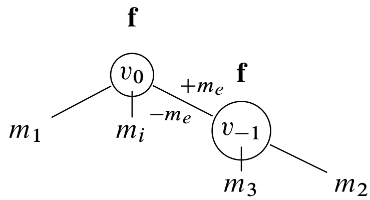
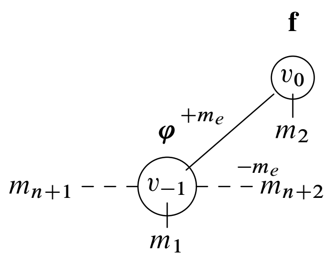
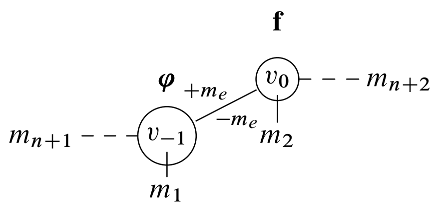
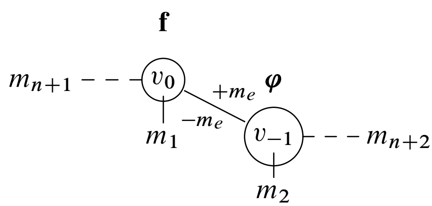
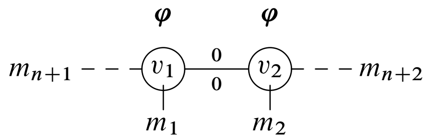
Type of twisted graph	Contribution
Type f_2	
Type f_i (with $m_i < 0$)	
Type φ_1	
Type φ_1'	
Type φ_1''	

Table 1. The types of twisted graphs involved in the induction formulas defining f and φ. Only the twists at the first legs and the half-edges of the edge are indicated. Besides, the letters **f** or $\boldsymbol{\varphi}$ above each vertex indicate which function is used to define the contribution of the graph (cont. on next page).

Type φ_2



Type φ_3

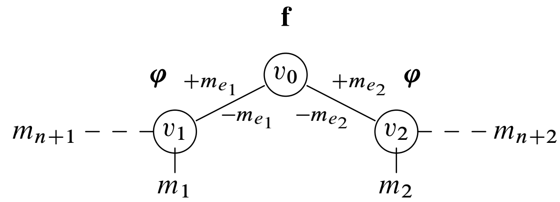


Table 1. The types of twisted graphs involved in the induction formulas defining f and φ. Only the twists at the first legs and the half-edges of the edge are indicated. Besides, the letters **f** or $\boldsymbol{\varphi}$ above each vertex indicate which function is used to define the contribution of the graph (cont.).

Let Σ be a set appearing in the expression of ψ_2. The schematic intersection of δ_Σ with $\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})$ is the union of all divisors of $\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})$ defined by bicolored graphs or twisted graphs with one horizontal edge and which have an edge which separates $\{1, n+1\} \cup \Sigma$ from the other legs. To compute the intersection number of $(\delta_\Sigma \cdot [\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})])$ with

$$\prod_{\substack{i \geq 2 \\ \text{s.t. } m_i > 0}} m_i \psi_i,$$

we only need to consider the twisted graphs that do not vanish once we push forward the class $[\mathbb{P}\overline{\mathcal{H}}(\Gamma, m, R_{1,2})]$ along $p\colon \mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2}) \to \overline{\mathcal{M}}_{0,n+2}$. This is the case only for the types of graphs $\varphi_1, \varphi_1', \varphi_1'', \varphi_2$ and φ_3. Namely, as there are exactly two poles with opposite residues, there may be only two vertices per level. Indeed,

- if (Γ, m) has a unique horizontal edge, then the two poles $n+1$ and $n+2$ should be carried by the two distinct vertices. Otherwise, the residue at the edge would vanish and the graph would define a space of dimension smaller than $\dim(\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})) - 1$.
- if (Γ, m) is a bicolored graph, then V_{-1} may have up to 2 vertices and V_0 only 1. If V_{-1} has two vertices, then necessarily, each vertex carries one of the poles with non-vanishing residue. This gives the type $\varphi_1, \varphi_1', \varphi_1''$ and φ_3.

The multiplicity of the graphs of type $\varphi_1, \varphi_1', \varphi_1''$, and φ_2 is one as for the graphs of type f. Thus we only need to show that a graph (Γ, m) of type φ_3 contributes trivially to the function $\varphi^{\pm}$ and with multiplicity $m_{e_1} + m_{e_2}$ to φ.

To do this we choose $i = 1$ or 2. Then the edge e_i determines a set Σ as: the set of legs incident to the vertex v_1 if $i = 1$, and the set of legs incident to the vertices v_1 and v_0 if $i = 2$. In both cases we will show that

$$\delta_\Sigma \cdot [\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})] = m_{3-i}\,[\mathbb{P}\overline{\mathcal{H}}(\Gamma, m, R_{1,2})],$$

while $\delta_\Sigma \cdot [\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})]^\pm = 0$. Indeed, a generic point $y \in \mathbb{P}\mathcal{H}(\Gamma, m, R_{1,2})$ has neighborhood in $\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})$ given by $\Delta \times G \times U$, where U is neighborhood of y in $\mathbb{P}\mathcal{H}(\Gamma, m, R_{1,2})$, G is a discrete set of cardinality $\gcd(m_{e_1}, m_{e_2})$, and Δ is an open disk of $\mathbb{C}$ containing 0 parametrized by some parameter ϵ (see [30, Lemma 5.6]). Moreover, the neighborhood of the node corresponding to e_i in the universal curve $\mathcal{C} \to \Delta \times G \times U$ is given by $z \times w = \epsilon^{\mathrm{lcm}(m_{e_1}, m_{e_2})/m_{e_i}}$. Therefore, the intersection of $\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})$ with δ_Σ is equal to

$$\gcd(m_{e_1}, m_{e_2}) \cdot \frac{\mathrm{lcm}(m_{e_1}, m_{e_2})}{m_{e_i}} = \frac{m_{e_1} m_{e_2}}{m_{e_i}},$$

which is the expected contribution for the function φ.

Spin parity. To prove the induction formula for $\varphi^\pm$, we need to compute the parity of differentials closed to divisors of each type (as mentioned above we do not need to consider the type φ_1' and φ_1'' here).

The degeneration of type φ_2 separates a genus 1 curve from a genus 0 curve via a separating node, therefore the parity of the limit curve is the sum of the parities on each component, but as the parity in genus 0 is always even, the parity is determined by the vertex carrying the poles of order 1.

For the type φ_1 we compute the parity on a smoothing of the degenerated curve by considering index (or winding number) of the flat structure along two cycles: a cycle A which is a closed curve in the cylinder that is bounded by x_{n+1}, and a cycle B which goes from x_{n+1} to x_{n+2}. Then the parity of the differential is computed as

$$(\mathrm{ind}_B + 1)(\mathrm{ind}_A + 1) (\mathrm{mod}\ 2)$$

(see [12, Section 5.1] for instance). The index along A is 0 as a periodic curve does no rotate a tangent vector. Therefore the parity of the differential is determined as the parity of $\mathrm{ind}_B + 1$. When the curve degenerate this cycle breaks into two cycles B_1 and B_2 for which $\mathrm{ind}_B = \mathrm{ind}_{B_1} + \mathrm{ind}_{B_2}$. Therefore the parity of the nearby differential is equal to

$$(\mathrm{ind}_B + 1) = (\mathrm{ind}_{B_1} + 1) + (\mathrm{ind}_{B_2} + 1) + 1 (\mathrm{mod}\ 2),$$

and thus equal to minus the sum of parities of the limit differentials of each component.

To compute the parity for the situation φ_3 we consider the parity of the generic element in

$$U \times \{\gamma\} \times \big(\Delta \setminus \{0\}\big)$$

for an element in $\gamma \in G$ as described above. As both vertices have even twists, half of the elements of G give even or odd parity (see [10, Proposition 5.2]). Therefore, the intersection of δ_Σ with $[\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})^+] - [\mathbb{P}\overline{\mathcal{H}}(\mu, R_{1,2})^-]$ is trivial. ∎

3.4. Intersection of strata with δ_0

From now on, μ is a vector of *positive* entries. We recall from [6, Theorems 1.2 and 6.2] the following induction formula for the normalized volume (Notation 2.22).

Proposition 3.10. *If $n \geq 2$, then we have the following relation:*

$$\mathrm{vol}(\mu) = \sum_{\Gamma \in \mathrm{BB}(\mu)_2} f(\mu(v_{-1})) \frac{\prod_{e \in E(\Gamma)} m_e}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V_0} \mathrm{vol}(\mu(v)),$$

$$\mathrm{vol}^{\pm}(\mu) = \sum_{\Gamma \in \mathrm{BB}(\mu)_2} f(\mu(v_{-1})) \frac{\prod_{e \in E(\Gamma)} m_e}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V_0} \mathrm{vol}^{\pm}(\mu(v)),$$

where $\mathrm{BB}(\mu)_2$ *is the set of rational backbone graphs compatible with μ such that the second leg is incident to the vertex v_{-1} in V_{-1}, and where $\mu(v_{-1}) = (m_1, m_2, \ldots)$ is the vector of twists at half-edges incident to v_{-1}.*

Remark that the spin part of Proposition 3.10 is slightly different from [6, Theorem 6.2] but both statements are equivalent. Indeed, this theorem expresses the volume of the odd and even components separately, but the difference between the two expressions gives the simpler expression above.

Here we will prove the following expression of the functions d_1 and $d_1^{\pm}$ in terms of the volume function.

Proposition 3.11. *For all μ, we have*

$$d_1(\mu) = \sum_{\Gamma \in \mathrm{BB}(\mu)_0} \varphi(\mu(v_{-1})) \frac{\prod_{e \in E(\Gamma)} m_e}{2\,|\operatorname{Aut}(\Gamma)|} \prod_{v \in V_0} \mathrm{vol}(\mu(v)),$$

$$d_1^{\pm}(\mu) = \sum_{\Gamma \in \mathrm{BB}(\mu)_0} \varphi^{\pm}(\mu(v_{-1})) \frac{\prod_{e \in E(\Gamma)} m_e}{2\,|\operatorname{Aut}(\Gamma)|} \prod_{v \in V_0} \mathrm{vol}^{\pm}(\mu(v)),$$

where $\mathrm{BB}(\mu)_0$ *is the set of rational backbone graphs compatible with* $(m_1, \ldots, m_n, 0, 0)$ *with the legs $n+1$ and $n+2$ incident to v_{-1}, and* $\mu(v_{-1}) = (m_1, \ldots, 0, 0)$ *is the vector of twist at half-edges incident to v_{-1}.*

Proof. To prove this proposition we will proceed in two steps:

(i) we express $\delta_0 \cdot [\mathbb{P}\overline{\mathcal{H}}(\mu)]$ in terms of boundary component of $\mathbb{P}\overline{\mathcal{H}}(\mu)$;

(ii) we use the results of [30] to compute the intersection of ξ with these boundary components.

Intersection of strata with δ_0. Let X be connected component of $\mathbb{P}\overline{\mathcal{H}}(\mu)$. Up to loci of co-dimension 2, the schematic intersection of $\mathbb{P}\overline{X}$ and δ_0 is contained in the union of the spaces $\mathbb{P}\overline{\mathcal{H}}(\Gamma, m)$, where (Γ, m) is either the unique graph with one non-separating horizontal edge, or a bicolored graph with $h^1(\Gamma) > 0$. We will show that graphs of the second type contribute trivially to the integral of β_1 (see (9) for the definition of β_1).

Let (Γ, m) be a bicolored graph with $h^1(\Gamma) > 0$ and let X be a connected component of $\mathcal{H}(\Gamma, m)$. We have

$$\beta_1 \cdot [\mathbb{P}\overline{X}] = p_*\big(\xi^{2g-2} \cdot [\mathbb{P}\overline{X}]\big) \cdot \prod_{i=2}^{n} m_i \psi_i,$$

where $p\colon \mathbb{P}\overline{\mathcal{H}}(\mu) \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism of the differential. We have $p_*(\xi^{2g-2} \cdot [\mathbb{P}\overline{X}]) = 0$ by [6, Proposition 3.10] (actually the proposition is given there for the complete stratum $\mathbb{P}\overline{\mathcal{H}}(\mu)$, but the arguments can be transposed directly for all connected components). Therefore, as δ_0 intersects transversally along (Γ_0, m_0), the unique graph with one non-separating horizontal edge, we have

$$d_1(\mu) = \int_{\mathbb{P}\overline{\mathcal{H}}(\Gamma_0, m_0)} \beta_i = \frac{1}{2} \int_{\mathbb{P}\overline{\mathcal{H}}(\mu+(0,0))} \xi^{2g-2} \prod_{i=2}^{n} m_i \psi_i$$

(the factor $1/2$ comes from the automorphism group of (Γ_0, m_0)). Moreover, we also get the spin analogue

$$d_1^{\pm}(\mu) = \frac{1}{2} \int_{[\mathbb{P}\overline{\mathcal{H}}(\mu+(0,0))]^{\pm}} \xi^{2g-2} \prod_{i=2}^{n} m_i \psi_i.$$

Expression of ξ *on* $\mathbb{P}\overline{\mathcal{H}}(\mu + (0,0))$. We use [30] to write ξ as a linear combination of boundary divisors. Indeed,

$$\xi \cdot \big[\mathbb{P}\overline{\mathcal{H}}(\mu + (0,0))\big] = \big[\mathbb{P}\overline{\mathcal{H}}(\mu + (0,0), \{0\})\big] + \text{boundary terms}.$$

However, $\mathbb{P}\overline{\mathcal{H}}(\mu + (0,0), \{0\})$ is empty as there can be no pole of order exactly one with vanishing residue. The boundary terms are supported on the strata $\overline{\mathcal{H}}(\Gamma, m)$ for all bicolored graphs (Γ, m) in B_0 the set of graphs with the two poles incident to vertices of V_{-1}. Then we write

$$\xi \cdot \big[\mathbb{P}\overline{\mathcal{H}}(\mu + (0,0))\big] = \sum_{\Gamma \in \mathrm{BB}(\mu)_0} \Bigg(\prod_{e \in E(\Gamma)} m_e\Bigg) \cdot \big[\mathbb{P}\overline{\mathcal{H}}(\Gamma)\big] + \Delta,$$

where Δ is supported on the union of strata associated with graphs in $B_0 \setminus \mathrm{BB}(\mu)_0$. We will prove that

$$\int_{\Delta} \xi^{2g-3} \prod_{i=2}^{n} \psi_i = 0. \tag{10}$$

To do so, we chose a connected component X of $\mathcal{H}(\Gamma, m)$ for a twisted graph of $B_0 \setminus \mathrm{BB}(\mu)_0$, then we will show that $\int_{\mathbb{P}\bar{X}} \xi^{2g-3} \prod_{i=2}^{n} \psi_i$ vanishes. First, if we assume that (Γ, m) is not a rational backbone graph, then

$$p_*\big(\xi^{2g-3} \cdot [\mathbb{P}\bar{X}]\big) = 0$$

by the same arguments that were used to prove [6, Proposition 3.10]. Now we assume that (Γ, m) is a rational backbone graph. Then $\mathcal{H}(\Gamma)_{-1}$ has dimension n_{-1}, where n_{-1} is the number of legs in $[\![1, n]\!]$ incident to the vertex of V_{-1}. Thus its projectivization has dimension $n_{-1} - 1$, and we obtain

$$\int_{\mathbb{P}\bar{\mathcal{H}}_{(\Gamma,m)}} \xi^{2g-3} \prod_{i=2}^{n} \psi_i = 0$$

unless 1 is incident to the vertex of V_{-1} which implies the identity (10).

Finally, if $(\Gamma, m) \in \mathrm{BB}(\mu)_0$, then we have

$$\int_{\mathbb{P}\bar{\mathcal{H}}(\Gamma,m)} \xi^{2g-3} \prod_{i=2}^{n} m_i \psi_i = \frac{\varphi(\mu(v_{-1}))}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V_0} \operatorname{vol}(\mu(v)).$$

So we obtained the desired expression of $d_1(\mu)$. Also, if μ is odd, then we have

$$\int_{[\mathbb{P}\bar{\mathcal{H}}(\Gamma,m)]^{\pm}} \xi^{2g-3} \prod_{i=2}^{n} m_i \psi_i = \frac{\varphi^{\pm}(\mu(v_{-1}))}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V_0} \operatorname{vol}^{\pm}(\mu(v)).$$

Indeed, a backbone graph is of compact type thus the parity of a generic element of $\mathcal{H}(\Gamma, I)$ is defined as the product of the parities of the elements in $\mathcal{H}(\mu(v), R_v)$ for all vertices of Γ (see [8]). ∎

4. Chains and cylinder configurations

In this section we use Proposition 3.11 to express $d_1(\mu)$ and $d_1^{\pm}(\mu)$ as sums over cylinder configuration.

4.1. Chains

Here we define a category of graphs called chains to encode cylinder configurations in the spirit of [5].

Definition 4.1. A *chain* is a graph of rational type Γ with $n+2$ legs and a partition $D(\Gamma) = F(\Gamma) \sqcup P(\Gamma)$ satisfying the following constraints:

- $m(n+1) = m(n+2) = 0$, and the other legs have positive twists. Moreover, the first leg is incident to the same vertex as the $(n+1)$-st.
- If $v \in F(\Gamma)$ (set of "figure eights" in the terminology of [12]), then v has exactly 1 incident edge.
- If $v \in P(\Gamma)$ (set of "pairs of holes" in the terminology of [12]), then v has exactly 2 incident edges.
- Each vertex in $R(\Gamma)$ has exactly two of the following half-edges: the leg $(n+1)$ or $(n+2)$, or the half-edge of a horizontal edge or of an edge to a vertex in $P(\Gamma)$.
- Each vertex in $R(\Gamma)$ has exactly one leg with index in $[\![1, n]\!]$.

We say a chain Γ is *odd* if all positive twists are odd. Denote by $\mathrm{CH}(\mu)$ and $\mathrm{CH}(\mu)^{\mathrm{odd}}$ the set of (odd) chains compatible with $(m_1, \ldots, m_n, 0, 0)$.

Observe that each vertex has an even number of incident half-edges with even twists. As $m(n+1) = m(n+2) = 0$, it follows that each vertex in $R(\Gamma)$ has at least two incident half-edges with even twists. Hence, if Γ is odd, then $P(\Gamma)$ is empty.

Definition 4.2. Let $1 \le i \le n$. A *cylinder configuration marked by* x_1 is the data of: a cylinder configuration of a stratum of abelian differentials (in the sense of [12] or [5]), and the choice of a side of one of the cylinders which is bounded by the marking x_1.

The data of a cylinder configurations marked by x_1 is equivalent to the choice of:

(i) a chain in $\mathrm{CH}(\mu)$;

(ii) an order on the vertices in $F(\Gamma)$ connected to v for all v in $R(\Gamma)$;

(iii) an integer $1 \le a_e \le m_e$ for all edges incident to a vertex in $F(\Gamma)$ or $P(\Gamma)$;

(iv) a choice of connected component of the space $\mathcal{H}(\mu(v))$ for all $v \in F(\Gamma)$ and $P(\Gamma)$. We use this fact to express the number $c_{\mathrm{cyl},1}(\mu)$ as a sum over chains.

Example 4.3. In Figure 2, we represent a chain for $\mu = (6, 6, 3, 2, 2, 2, 2)$. The vertices in $R(\Gamma)$, $F(\Gamma)$, and $P(\Gamma)$ are colored green, red, and blue respectively. Using the notation of [12] (in particular the non-logarithmic convention), the configurations

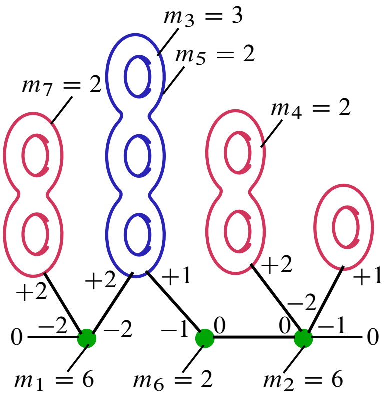


Figure 2. Example of chain in CH(6, 6, 3, 2, 2, 2, 2).

associated to this graph are of the form:

$$\Rightarrow (\overline{\alpha_1 + (1-\alpha_1), 1}) \to (\bar{1}, \bar{0}, 2, 1) \Rightarrow (\overline{\alpha_2 + (1-\alpha_2), 1}) \to (\overline{0+0}) \Rightarrow,$$

or

$$\Rightarrow (\overline{\alpha_1 + (1-\alpha_1), 1}) \to (\bar{1}, \bar{0}, 2, 1) \Rightarrow (\overline{0+0}) \to (\overline{\alpha_2 + (1-\alpha_2), 1}) \Rightarrow,$$

for α_1 and α_2 equal to 0 or 1. Here $\alpha_i = a_i - 1$ in our system of notation.

Let Γ be a chain. We define the contribution of a vertex v of Γ according to the type of vertex:

- if $v \in R(\Gamma)$, then $\widehat{c}(v) := m_i(n(v)-3)!$, where m_i is the marking of the unique leg incident to v;
- if $v \in F(\Gamma)$, then $\widehat{c}(v) := m_e|\mu(v)|\operatorname{vol}(\mu(v))$, where e is the unique edge incident to v;
- if $v \in P(\Gamma)$, then $\widehat{c}(v) := |\mu(v)|\operatorname{vol}(\mu(v))$.

Then, we define

$$\widehat{\operatorname{cont}}(\Gamma) := \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \widehat{c}(v).$$

If μ is odd and Γ is odd, then we define $\widehat{c}^{\pm}(v) := m_i(n(v)-3)!$ for a vertex $v \in R(\Gamma)$, and $\widehat{c}^{\pm}(v) = -|\mu(v)|\operatorname{vol}^{\pm}(\mu(v))$ for a vertex $v \in F(\Gamma)$, and we set

$$\widehat{\operatorname{cont}}^{\pm}(\Gamma) := \frac{-1}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \widehat{c}^{\pm}(v).$$

Proposition 4.4. *The following identity holds:*

$$c_{\mathrm{cyl},1}(\mu) = \frac{(-4\pi^2)^{-1}}{2\,\mathrm{vol}(\mu)} \sum_{\Gamma\in \mathrm{CH}(\mu)} \widehat{\mathrm{cont}}(\Gamma).$$

Moreover, if μ is odd, we have

$$c^{\pm}_{\mathrm{cyl},1}(\mu) = \frac{(-4\pi^2)^{-1}}{2\,\mathrm{vol}(\mu)} \sum_{\Gamma\in \mathrm{CH}(\mu)^{\mathrm{odd}}} \widehat{\mathrm{cont}}^{\pm}(\Gamma).$$

Proof. Let Γ be a chain graph. We choose a connected component X_v of $\mathcal{H}(\mu(v))$ for each vertex of $F(\Gamma)$ and $P(\Gamma)$. All the marked cylinder configurations with chain graph Γ and with the same choices of (X_v) have the same Siegel–Veech constant, given by

$$\Big(\prod_{e\in E(\Gamma)} m(e)\Big)\frac{\prod_{v\in D(\Gamma)} \frac{1}{2}|\mu(v)|!\,\mathrm{Vol}(\mathcal{H}(X_v))}{(\frac{1}{2}|\mu|-1)!\,\mathrm{Vol}(\mathcal{H}(\mu))},$$

see [12, formula (32)].[2] Now, we can use the expression of the normalized volume vol given in Notation 2.22 to rewrite this Siegel–Veech constant as

$$\frac{(-4\pi^2)^{-1}}{\mathrm{vol}(\mu)}\Big(\prod_{v\in R(\Gamma)} m_i\Big)\cdot\Big(\prod_{v\in F(\Gamma)} |\mu(v)|\cdot \mathrm{vol}(X_v)\Big)\cdot\Big(\prod_{v\in P(\Gamma)} |\mu(v)|\cdot \mathrm{vol}(X_v)\Big),$$

where, in the first product m_i is the twist of the unique leg incident to $v \in R(\Gamma)$ (see [12, formulas (13.1) and (14.4)]). Besides, there are

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\Big(\prod_{v\in R(\Gamma)} (n(v)-3)!\Big)\cdot\Big(\prod_{e\mapsto v\in F(\Gamma)} m_e\Big)$$

such configurations. Indeed the first product accounts for all choices of orders on the vertices of $F(\Gamma)$ connected to the vertices in $R(\Gamma)$, while the second product accounts for the choice of the a_e for edges incident to the vertices in $F(\Gamma)$. Thus the first identity follows as $c_{\mathrm{cyl},1}(\mu)$ is the sum of the Siegel–Veech constants of all configurations.

To obtain the second identity, we recall that if $\Gamma \in \mathrm{CH}(\mu)\setminus \mathrm{CH}(\mu)^{\mathrm{odd}}$ then half of the choices of tuples $(a_e)_{e\mapsto v\in F(\Gamma)}$ contribute to the even or odd component (see [12, Lemma 14.4]). Thus the contribution of the odd and even components compensate

[2]This formula is written with the real dimension d of the stratum or d_v for the stratum of the vertices in $D(\Gamma)$. Here we have used the fact that $d = 2(|\mu|+1)$.

and Γ contributes trivially to $c^{\pm}_{\mathrm{cyl},1}(\mu)$. Besides, if $\Gamma \in \mathrm{CH}(\mu)^{\mathrm{odd}}$, then by [12, Lemma 14.2] the parity of the configuration is the parity of

$$1 + \sum_{e \mapsto v \in F(\Gamma)} a_e + \sum_{v \in F(\Gamma)} \phi(X_v),$$

where $\phi(X_v)$ equals 0 or 1 if X_v is an even or odd component respectively. Thus, for each edge e incident to a vertex in $F(\Gamma)$, we have $(a_e - 1)/2$ terms which give the same parity while the other $(a_e + 1)/2$ produce the inverse parity, thus only one of these choices of a_e contributes. ∎

Base case of Lemma 3.9. First, we remark that if $m_i < 0$ for all $1 < i \leq n$, then the space $\mathbb{P}\mathcal{H}(\mu, R_{1,2})$ is of dimension 0 and thus the integrals φ and $\varphi^{\pm}$ are numbers of points in this space (or numbers of points counted with spin sign). By [5, Proposition 3.8], these points are in bijection with the number of cylinder configurations for a single cylinder with figure eight constructions. As explained in the proof of Proposition 4.4 there are exactly

$$(n-1)! \prod_{i, m_i < 0} |m_i|$$

such configurations (the factorial accounts for the ordering of the figure eights and the m_i terms account for the partitions of the $m_i - 1$ as a sum of two non-negative integers. Finally, by [5, Lemma 4.8] the parity of the meromorphic differential is determined by the parity of the associated configuration. ∎

4.2. Expanded chains

Proposition 4.4 provides an expression of $c_{\mathrm{cyl},1}(\mu)$ as a sum over chains in $\mathrm{CH}(\mu)$, while Proposition 3.11 above provides an expression of $d_1(\mu)$ as a sum over backbone graphs. To compare $c_{\mathrm{cyl},1}$ and d_1, we introduce a family of sets of graphs of rational type $\mathrm{ECH}(\mu)_i$, the *expanded chains of complexity i* for $i = 1, \dots, n$ (see Definition 4.7 below). We will also construct maps between the different sets of graphs of rational type constructed until here:

$$\begin{tikzcd}
\mathrm{BB}(\mu)_0 \simeq \mathrm{ECH}(\mu)_1 & \mathrm{ECH}(\mu)_2 \arrow[l, "F_2"'] & \cdots \arrow[l, "F_3"'] & \mathrm{ECH}(\mu)_n \arrow[l, "F_n"'] \arrow[d, "F"] \\
 & & & \mathrm{CH}(\mu).
\end{tikzcd}$$

These maps will be used to compare the different expressions of $d_1(\mu)$ by applying Lemmas 3.7 and 3.9 repeatedly.

Definition 4.5. A *pre-expanded chain* is a graph of rational type with $n+2$ legs such that there exists a partition $R(\Gamma) = C(\Gamma) \sqcup L(\Gamma)$ satisfying the following constraints:

- $m(n+1) = m(n+2) = 0$, and the other legs have positive twists. Moreover, the first leg is incident to the same vertex as the $(n+1)$-st.
- Let $(v_0, \dots, v_k)$ be the shortest path from the vertex v_0 with the leg $n+1$ to the vertex v_k with leg $n+2$. A vertex is in $C(\Gamma)$ (the *core*) if and only if it appears in this path. Thus we have an ordering on the vertices of the core.
- A vertex v in $C(\Gamma)$ is called a *bottom* or a *top* if for all v' in $C(\Gamma)$ connected to v, we have $v \leq v'$ or $v > v'$ respectively.
- All half-edges with vanishing twists are incident to bottoms of $C(\Gamma)$. In particular, all horizontal edges are between two bottoms.
- Each vertex in $D(\Gamma)$ has exactly one edge.
- Each vertex in $L(\Gamma)$ (the set of *links*), has exactly one edge to a lower vertex (it may have any number of edges to upper vertices).
- If v is a vertex in $L(\Gamma)$ or a vertex in $C(\Gamma)$ which is not a top, then v has at least one leg in $[\![1, n]\!]$.

Definition 4.6. Let Γ be an pre-expanded chain. For all vertices of Γ, we denote by $\mathrm{ind}(v)$ the minimum of the indices of the legs incident to v and $+\infty$ if there are no legs incident to v. Let v be a top in $C(\Gamma)$. It determines a unique subpath of the core $C(\Gamma)$:

$$(v_{k_1}, v_{k_1+1}, \dots, v_N = v, v_{N+1}, \dots, v_{k_2})$$

such that v_N is the unique top of the sequence, and v_{k_1} and v_{k_2} are the only bottoms. We say that v is *admissible* if the minimum of $\mathrm{ind}(v_j)$ for $j = k_1+1, \dots, k_2$ is reached for $j = N+1$. A pre-expanded chain is an *expanded chain* if all tops of $C(\Gamma)$ are admissible.

Definition 4.7. If $1 \leq i \leq n$, then we denote by $\mathrm{ECH}(\mu)_i$ the set of *expanded chains of complexity* i, i.e., the chains satisfying:

- for all $1 \leq j \leq i$, we have: if the jth leg is incident to a vertex v, then v is not a top of $C(\Gamma)$ and either v in $D(\Gamma)$ or $j = \mathrm{ind}(v)$;
- for all v in $R(\Gamma)$, we have $\mathrm{ind}(v) \leq i$ or v is a top of $C(\Gamma)$.

To compare the different sets of graphs we define the functions $F_2, \dots, F_n$ and F (see the above diagram).

4.2.1. Construction of the maps F_i. Let Γ be a graph in $\mathrm{ECH}(\mu)_i$. We construct the image of Γ as follows:

- If the ith leg is incident to a vertex of $D(\Gamma)$, then $F_i(\Gamma) = \Gamma$.

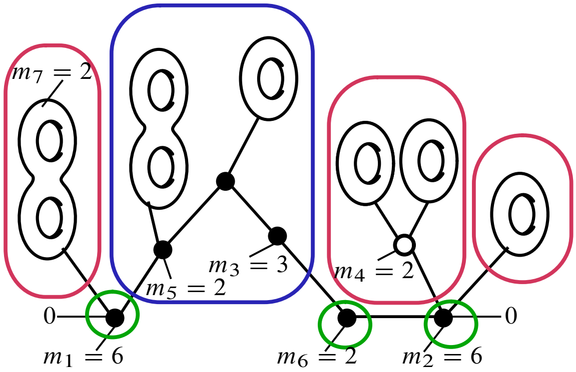


Figure 3. An expanded chain in $\mathrm{ECH}(6, 6, 3, 2, 2, 2, 2)_i$ for $i = 6$ or 7.

- If the ith leg is incident to a vertex of $L(\Gamma)$, then $F_i(\Gamma)$ is obtained by contracting the unique edge to a lower vertex.
- If the ith leg is on a vertex v_j of $C(\Gamma)$ and v_{j-1} is not a top, then $F_i(\Gamma)$ is obtained by contracting the edge between v_{j-1} and v_j; if v_{j-1} is a top, we also contract the edge between v_{j-2} and v_{j-1}.

Note that $F_i(\Gamma)$ then satisfies the first condition of elements in ECH_{i-1} by the admissibility condition, and the second condition because if there is a vertex v in Γ with $\mathrm{ind}(v) = i$, then this vertex is merged with a vertex which has a leg of smaller index incident to it.

4.2.2. Construction of the map F. Let Γ be a graph in $\mathrm{ECH}(\mu)_n$. The image of Γ is defined by contracting all edges which are not incident to at least one bottom vertex of $C(\Gamma)$, where the genus of a merged vertex is the sum of the genera of the previous vertices.

Example 4.8. In Figure 3, we represent an example of an expanded chain in

$$\mathrm{ECH}(6, 6, 3, 2, 2, 2, 2)_i$$

for $i = 6$ or 7 (for simplicity we did not put the twist at the edges as they may be computed from the twists at the legs). The vertices of $C(\Gamma)$ are black dots while the vertex in $L(\Gamma)$ is a white dot. Remark that the admissibility condition is satisfied, as the marking on the vertex following the unique top of $C(\Gamma)$ has index 3 which is smaller than 5 and 6. This expanded chain is mapped to the chain of Example 2 under F so we have surrounded in green, red, or blue the subgraphs that have to be contracted to obtain this chain.

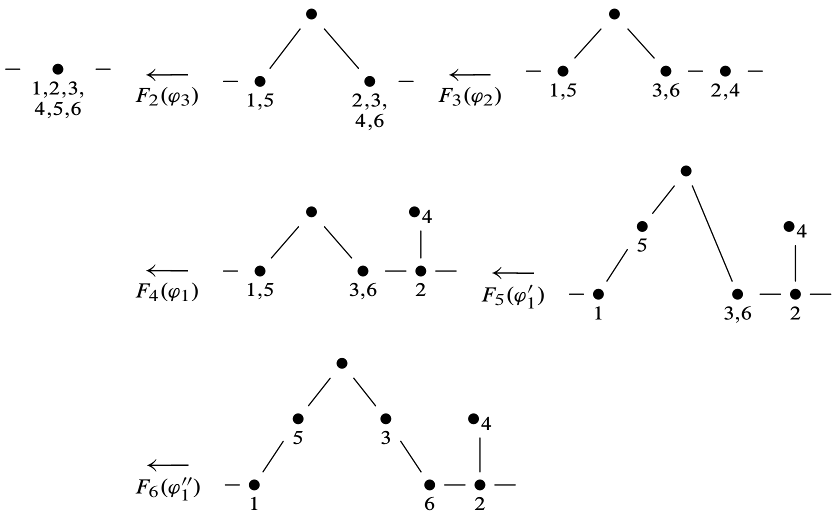


Figure 4. Effect of the functions F_i on the expanded chain of Figure 3. The function F_7 acts trivially on this example so we do not represent it.

To illustrate the effect of the functions F_i on graphs, we represent on Figure 4 the images of this twisted graph in the $\mathrm{ECH}(6,6,3,2,2,2,2)_i$. Here, we only represent the rational part of the graph (in particular the 7th marking is not present) and where the markings sit. At each step, we indicate in the parenthesis which type of φ-degeneration is used to create new vertices.

4.2.3. Odd expanded chains. If μ is odd, then an expanded chain Γ is *odd* if the all positive twists are odd. In particular, $C(\Gamma)$ contains only bottom vertices. Indeed, each vertex has an even number of incident half-edges with even twists, thus each vertex of the core has exactly 2 incident half-edges with even twists (the ones connecting to the previous and next vertex or the legs $n+1$ and $n+2$). As all of these twists are equal to 0, all vertices are bottom and all edges of the core are horizontal. The functions $F_i\colon \mathrm{ECH}_i^{\mathrm{odd}}(\mu) \to \mathrm{ECH}_{i-1}^{\mathrm{odd}}(\mu)$ and $F\colon \mathrm{ECH}_n^{\mathrm{odd}}(\mu) \to \mathrm{CH}^{\mathrm{odd}}(\mu)$ are defined by restricting the functions F and F_i to the sets of odd expanded chains.

4.3. Contribution of expanded chains

Let Γ be an expanded chain. The contribution $c(v)$ of a vertex v of Γ is defined according to the type of vertex:

- If $v \in D(\Gamma)$, then $c(v) := m_e \operatorname{vol}(\mu(v))$, where m_e is the twist at the edge incident to v and $\mu(v)$ is the vector of twists of half-edges incident to v.

- If $v \in L(\Gamma)$, or if $v \in C(\Gamma)$ is neither a top nor a bottom, then we set $c(v) := m_j\, f(\mu(v))$, where $j = \mathrm{ind}(v)$ and $\mu(v) = (m_j, m_e, \ldots)$ with e the unique edge to a lower vertex.
- If v is a top of $C(\Gamma)$, then $c(v) := (m_e + m_{e'})\, f(\mu(v))$ where m_e and $m_{e'}$ are the twists at the two edges e and e' to lower vertices, and $\mu(v) = (m_e, m_{e'}, \ldots)$.
- If v is a bottom of $C(\Gamma)$, then $c(v) := m_j\, \varphi(\mu(v))$, where $j = \mathrm{ind}(v)$, and $\mu(v) = (m_j, \ldots, m_e, m_{e'})$, where e and e' are the edges to the previous and next vertex in $C(\Gamma)$ if there is one, and else, $m_e = 0$ or $m_{e'} = 0$ respectively.

Then, we define

$$\mathrm{cont}(\Gamma) = \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} c(v).$$

Moreover, if μ is odd and Γ is an odd expanded chain, and v is a vertex of Γ, then we define $c^{\pm}(v)$ and $\mathrm{cont}^{\pm}(\Gamma)$ by replacing the function vol by $\mathrm{vol}^{\pm}$ and the function φ by $\varphi^{\pm}$ in the definition of $c(v)$ and $\mathrm{cont}(\Gamma)$.

Proposition 4.9. *Let $1 < i \le n$ and let $\Gamma \in \mathrm{ECH}(\mu)_{i-1}$. We have*

$$\mathrm{cont}(\Gamma) = \sum_{\Gamma' \in F_i^{-1}\{\Gamma\}} \mathrm{cont}(\Gamma').$$

Besides, if Γ is odd, then

$$\mathrm{cont}^{\pm}(\Gamma) = \sum_{\Gamma' \in F_i^{-1}\{\Gamma\}} \mathrm{cont}^{\pm}(\Gamma'),$$

where in the last sum we restrict F_i to the sets of odd *expanded chains.*

Proof. Let $i > 1$ and $\Gamma \in \mathrm{ECH}(\mu)_{i-1}$. Any graph in $F_i^{-1}\{\Gamma\}$ is obtained by modifying the vertex v carrying the leg i and not the others. Thus we show that the proposition holds by studying each possible type of vertex for v.

If $v \in D(\Gamma)$, then $\Gamma \in \mathrm{ECH}(\mu)_i$ and $F_i^{-1}\{\Gamma\} = \{\Gamma\}$. Thus, the proposition holds trivially.

If v is in $L(\Gamma)$, then $c(v)$ is given in terms of the function f. Using Lemma 3.7, we may write this function as a sum over all twisted graphs of type f_2, i.e., as all possible ways to split v into 2 vertices (leaving $\mathrm{ind}(v)$ and the edge towards a lower vertex together on the lowest of the created vertices while i is on the upper one). All the graphs of $F_i^{-1}\{\Gamma\}$ are obtained in this way, and we may check that the induction formula of Lemma 3.7 gives the right contribution of any element of $F_i^{-1}\{\Gamma\}$.

Therefore from now on, we assume that v is in $C(\Gamma)$. First, let us remark that v cannot be a top of $C(\Gamma)$. Namely, v cannot be the result of contracting the edge between a top v_j and a vertex v_{j-1}, as that would violate the first condition of expanded chains of complexity i. Moreover, if v is obtained by contracting the edge between a vertex v_j, a top v_{j-1} and a vertex v_{j-2}, by the admissibility condition, we know that v is a bottom (and not a top).

If v is neither a top nor a bottom, then its contribution is given by the function f. Then we apply the recursion of Lemma 3.7 and we write f as a sum over graphs of type f_h where h is the half-edge of the edge that connects v to the previous vertex of $C(\Gamma)$. Either we obtain two vertices in $C(\Gamma)$, where the closest to v_0 carries the leg $\mathrm{ind}(v)$ while the other carries the leg i; or a vertex in $C(\Gamma)$ with the leg $\mathrm{ind}(v)$ and a vertex in $L(\Gamma)$ carrying the leg i. All the graphs of $F_i^{-1}\{\Gamma\}$ are obtained in this way, and we may check that the induction formula of Lemma 3.7 gives the right contribution of any element of $F_i^{-1}\{\Gamma\}$.

If v is a bottom in $C(\Gamma)$, then its contribution is given by the function φ. Any graph in $F_i^{-1}\{\Gamma\}$ is obtained by splitting v into 2 or 3 vertices of type φ_1, φ_1', φ_1'', φ_2, or φ_3. Then the induction formula of Lemma 3.9 shows that the contribution of Γ may be written as the sum of the contributions of $F_i^{-1}\{\Gamma\}$ as in the previous case. ∎

4.4. Contribution of chains

The purpose of this section is to show the following identities.

Proposition 4.10. *Let $\Gamma \in \mathrm{CH}(\mu)$. We have*

$$\widehat{\mathrm{cont}}(\Gamma) = \sum_{\Gamma' \in F^{-1}\{\Gamma\}} \mathrm{cont}(\Gamma'),$$

and if Γ is odd, then

$$\widehat{\mathrm{cont}}^{\pm}(\Gamma) = \sum_{\Gamma' \in F^{-1}\{\Gamma\}} \mathrm{cont}^{\pm}(\Gamma').$$

To prove these identities we need two extra families of combinatorial objects, called rooted trees and expanded pairs of holes.

Definition 4.11. Let $\Sigma \subset [\![1,n]\!]$ and p be a positive integer. A *rooted tree* is a graph of rational type with $1+|\Sigma|$ legs indexed by $\{r\} \cup \Sigma$, which is either the trivial graph (i.e., the stable graph with one vertex and no edges) or is such that:

- no edge is horizontal;
- each vertex in $D(\Gamma)$ has exactly one incident edge;
- the vertex with the leg r is called the *root*. The root has 2 legs and no edges to lower vertices. All other vertices of $R(\Gamma)$ have 1 leg and 1 edge to a lower vertex (they may have any number of edges to upper vertices).

We denote by $\mathrm{RT}(\mu, \Sigma, p)$ the rooted trees compatible with $(p) + (m_i)_{i\in\Sigma}$.

Definition 4.12. Let $\Sigma \subset [\![2, n]\!]$, p_1, p_2 be positive integers and $I \in \mathbb{R} \setminus \Sigma$. A *pre-expanded pair of holes* is a graph of rational type Γ with legs indexed by the set $\Sigma \cup \{h_1, h_2\}$, and with a partition $R(\Gamma) = C(\Gamma) \sqcup L(\Gamma)$ satisfying:

- All legs have positive twists and there are no horizontal edges.
- Let $(v_0, \ldots, v_k)$ be the shortest path from the vertex with the leg h_1 to the vertex with the leg h_2. A vertex is in $C(\Gamma)$ if and only if it appears in this path.
- There is exactly one top in $C(\Gamma)$, i.e., there is precisely one vertex v whose edges to other vertices of $v' \in C(\Gamma)$ satisfy $v > v'$. No leg in Σ is incident to the top.
- Each vertex in $D(\Gamma)$ has exactly one edge.
- Each vertex in $L(\Gamma)$ (the set of *links*), has exactly one edge to a lower vertex (it may have any number of edges to upper vertices).
- If v is a vertex in $L(\Gamma)$ or a vertex in $C(\Gamma)$ which is not the top, then v has exactly one leg in Σ.

It is an *expanded pair of holes* in $\mathrm{EP}(\mu, \Sigma, p_1, p_2, I)$ if it is compatible with

$$(m_i)_{i \in \Sigma} + (p_1, p_2),$$

and the following *admissibility condition* holds:

- either the leg h_2 is incident to the top vertex (i.e., the top is the last vertex of the core), and I is smaller than all legs incident to vertices in $C(\Gamma)$;
- or the smallest index j of a leg incident to a vertex of $C(\Gamma)$ is smaller than I and is incident to the vertex directly after the top.

Example 4.13. In Figure 3, examples of rooted trees or expanded pairs of holes are given by the subgraphs surrounded by red or blue lines respectively.

If Γ is a rooted tree or an expanded pair of holes, and v is a vertex of Γ, then the contribution $c(v)$ of v is defined as for expanded chains. Here, for rooted trees, we set $R(\Gamma) = L(\Gamma)$ and identify m_e with p for the root. Also, for expanded pair of holes, we identify p_1 and p_2 with the twists m_e and $m_{e'}$ at the first and last vertices of the core respectively. Then we define $\mathrm{cont}(\Gamma)$ as $|\operatorname{Aut}(\Gamma)|^{-1} \prod_v c(v)$. If μ is odd and Γ is a rooted tree then we define $\mathrm{cont}^{\pm}(\Gamma)$ in the same way.

Lemma 4.14 ([6, Sections 3.5 and 6.1]). *For all positive integers p and $\Sigma \subset [\![2, n]\!]$, we have*

$$\Big(p + \sum_{i \in \Sigma} m_i\Big) \operatorname{vol}\big((p) + (m_i)_{i \in \Sigma}\big) = \sum_{\Gamma \in \mathrm{RT}(\mu, \Sigma, p)} \mathrm{cont}(\Gamma),$$

and if μ is odd, then we have

$$\Big(p + \sum_{i\in\Sigma} m_i\Big) \operatorname{vol}^{\pm}\big((p) + (m_i)_{i\in\Sigma}\big) = \sum_{\Gamma\in \mathrm{RT}(\mu,\Sigma,p)} \operatorname{cont}^{\pm}(\Gamma).$$

We will show the following analogue result for an expanded pair of holes.

Lemma 4.15. *For all positive integers p_1, p_2, $\Sigma \subset [\![2, n]\!]$ and $I \in \mathbb{R} \setminus \Sigma$, we have*

$$\Big(p_1 + p_2 + \sum_{i\in\Sigma} m_i\Big) \operatorname{vol}\big((m_i)_{i\in\Sigma} + (p_1, p_2)\big) = \sum_{\Gamma\in \mathrm{EP}(\mu,\Sigma,p_1,p_2,I)} \operatorname{cont}(\Gamma).$$

Proof. We fix a choice of μ, Σ, p_1, p_2. We denote by $S(I)$ the sum on the right-hand side of the identity of the Lemma. This function is locally constant on $\mathbb{R} \setminus \Sigma$, thus to prove the lemma we proceed in two steps: first, we show that it is valid when $I = 0$; then we show that $S(i - 1/2) - S(i + 1/2) = 0$ for all $i \in \Sigma$.

The case $I = 0$. We proceed by induction on the size of Σ. If Σ is empty, then a graph in $\mathrm{EP}(\mu, \Sigma, p_1, p_2, 0)$ is a backbone s.t. the legs h_1 and h_2 are incident to v_{-1}. Thus, we may apply Proposition 3.10 for $\mu = (p_1, p_2)$, which implies the lemma in this case.

If Σ is non-empty, then the admissibility condition implies that the marking h_2 is incident to the top. Then a graph in $\mathrm{EP}(\mu, \Sigma, p_1, p_2, 0)$ is determined by:

(1) an element $i \in \Sigma \cup \{h_2\}$;

(2) an integer $k \geq 1$;

(3) a partition $\Sigma \setminus \{i\} = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$, and a partition $p_1 + m_i - k = p'_1 + \cdots + p'_k$, where $m_i = p_2$ if $i = h_2$;

(4) a rooted tree in $\mathrm{RT}(\mu, \Sigma_j, p'_j)$ for $1 \leq j \leq k$ if $h_2 \notin \Sigma_j$, or an element of $\mathrm{EP}(\mu, \Sigma_j, p'_j, p_2, 0)$ otherwise.

Indeed, with this datum we construct a graph in $\mathrm{EP}(\mu, \Sigma, p_1, p_2, 0)$ by attaching the k graphs of the last part of the data to a vertex of genus 0 with the markings h_1 and i. Here, we replace an half-edge with marking p'_j by an edge e_j with twist $m_{e_j} = p'_j$ to this vertex with markings h_1 and i. Using this fact, we may rewrite the sum defining $S(0)$ as follows:

$$\sum_{\substack{i\in\Sigma\\ k\geq 1}} \sum_{\substack{\Sigma_1\sqcup\cdots\sqcup\Sigma_k=(\Sigma\setminus\{i\})\cup\{h_2\}\\ p'_1+\cdots+p'_k=p_1+m_i-k}} \frac{m_i f(p_1, m_i, -p'_1, \ldots, -p'_k)}{(k-1)!}$$
$$\times \Big(\sum_{\Gamma\in \mathrm{EP}(\mu,\Sigma_1,p'_1,p_2,0)} \operatorname{cont}(\Gamma)\Big) \prod_{j=2}^{k} \Big(\sum_{\Gamma\in \mathrm{RT}(\mu,\Sigma_j,p'_j)} \operatorname{cont}(\Gamma)\Big)$$

$$+\sum_{k\geq 1}\sum_{\substack{\Sigma_1\sqcup\cdots\sqcup\Sigma_k=\Sigma\\ p'_1+\cdots+p'_k=p_1+p_2-k}}\frac{(p_1+p_2)f(p_1,p_2,-p'_1,\ldots,-p'_k)}{k!}$$
$$\times\prod_{j=1}^{k}\Bigg(\sum_{\Gamma\in\mathrm{RT}(\mu,\Sigma_j,p'_j)}\mathrm{cont}(\Gamma)\Bigg).$$

The first sum accounts for the contribution of graphs where h_2 is not incident to the same vertex as h_1 (thus one of the descendants of the main vertex of genus 0 is distinguished as it carries the leg h_2) while the second sum accounts for the contribution of graphs with h_1 and h_2 incident to the same vertex.

Therefore, we may compute the sum $S(0)$ recursively by applying Lemma 4.14 and the induction hypothesis to obtain that $S(0)$ is given by

$$\sum_{\substack{i\in\Sigma\\ k\geq 1}}\sum_{\substack{\Sigma_1\sqcup\cdots\sqcup\Sigma_k=(\Sigma\setminus\{i\})\cup\{h_2\}\\ p'_1+\ldots+p'_k=p_1+m_i-k}}\frac{m_i f(p_1,m_i,-p'_1,\ldots,-p'_k)}{k!}$$
$$\times\prod_{j=1}^{k}\Bigg(p'_j+\sum_{i'\in\Sigma_j}m_{i'}\Bigg)\mathrm{vol}\big((p'_j)+(m_{i'})_{i'\in\Sigma_j}\big)$$
$$+\sum_{k\geq 1}\sum_{\substack{\Sigma_1\sqcup\cdots\sqcup\Sigma_k=\Sigma\\ p'_1+\cdots+p'_k=p_1+p_2-k}}\frac{(p_1+p_2)f(p_1,p_2,-p'_1,\ldots,-p'_k)}{k!}$$
$$\times\prod_{j=1}^{k}\Bigg(p'_j+\sum_{i'\in\Sigma_j}m_{i'}\Bigg)\mathrm{vol}\big((p'_j)+(m_{i'})_{i'\in\Sigma_j}\big),$$

where, again, $m_{h_2}=p_2$. We apply [6, Proposition 3.11] to deduce that

$$S(0)=\sum_{i\in\Sigma}\big(m_i\,\mathrm{vol}\big(\Sigma+(p_1,p_2)\big)\big)+(p_1+p_2)\,\mathrm{vol}\big(\Sigma+(p_1,p_2)\big),$$

which is the desired identity.

Crossing an element in Σ. We fix i in Σ. We first remark that the contribution of an element in the sum defining $S(I)$ does not depend on I. Indeed, the dependence of $S(I)$ on I is uniquely given by the set of graphs that contribute. Thus we need to determine which graphs contribute to $S(i+1/2)$ but not to $S(i-1/2)$ and conversely.

We assume that $i=\min\{\mathrm{ind}(v)\mid v\in C(\Gamma)\}$. Namely, if this is not the case then the same graphs contribute to $S(i+1/2)$ and $S(i-1/2)$. Now, a graph contributes to $S(i+1/2)$ but not to $S(i-1/2)$ if and only if the label i belongs to $C(\Gamma)$ and h_2

is incident to the top. We denote by S^+ the sum of the contributions for graphs of this type. Conversely, a graph contributes to $S(i-1/2)$ but not to $S(i+1/2)$ if and only if the label i is incident to the vertex following the top in $C(\Gamma)$. We denote by S^- the sum of the contributions for graphs of this type. We will show that $S^+ = S^-$ to finish the proof of the lemma.

To do so, we introduce a family of sets of pre-expanded pairs of holes $S(i,\ell)$ for all $\ell \geq 0$. A pre-expanded pair of holes $\Gamma \in \mathrm{EP}(\mu, \Sigma, p_1, p_2, I)$ compatible with $(m_i)_{i\in\Sigma} + (p_1, p_2)$ belongs to $S(i,\ell)$ if:

- the leg i is incident to a vertex of $C(\Gamma)$ and if a leg $i' \in \Sigma$ is incident to a vertex of $C(\Gamma)$ then $i' > i$;
- the top vertex is the tth vertex of the core, and the vertex carrying i the sth of the core then either $\ell > 0$ and the top is the last vertex of the core, *or* we have $t - s + 1 = \ell$.

In particular, with this notation we have $S(i,0) = S^-$, while $S(i,\ell) = S^+$ if ℓ is sufficiently large. Then we show that

$$\sum_{\Gamma\in S(i,\ell)} \mathrm{cont}(\Gamma) = \sum_{\Gamma\in S(i,\ell+1)} \mathrm{cont}(\Gamma)$$

for all $\ell \geq 0$. Indeed, for each graph $\Gamma \in S(i,\ell)$: either the top vertex is the last vertex and then it belongs to $S(i,\ell+1)$ too, or we apply Lemma 3.8 to the subgraph made of the top vertex and the next vertex in the $C(\Gamma$. Then Lemma 3.8 exchange the roles of these two vertices and thus the difference between the positions of the top vertex and the vertex carrying the ith leg is augmented by 1. We thus obtain a summation on graphs in $S(i,\ell+1)$. ∎

End of the proof of Proposition 4.10. To finish the proof, we simply remark that the datum of an expanded chain is equivalent to the datum of:

(i) a chain Γ;

(ii) a rooted tree for each vertex of $F(\Gamma)$;

(iii) an expanded pair of holes for each vertex of $P(\Gamma)$.

Moreover, applying Lemmas 4.14 and 4.15 provides the equality between the contribution of a vertex in $F(\Gamma)$ or $P(\Gamma)$ and the sum of the contributions of the rooted trees or expanded pairs of holes associated to this vertex. Note that the factor m_e in the contribution $m_e|\mu(v)|\,\mathrm{vol}(\mu(v))$ of a vertex $v \in F(\Gamma)$ does not occur in Lemma 4.14, but does occur in the function $\varphi(\mu(v'))$ in the contribution of v', where v' is the vertex of the core of the chain connected to v. ∎

4.5. End of the proof of Theorem 3.1

An expanded chain in $\mathrm{ECH}(\mu)_1$ has only one vertex in $R(\Gamma)$, and thus uniquely determines a backbone graph in $\mathrm{BB}(\mu)_0$. Moreover, Proposition 3.11 may be rewritten as

$$d_1(\mu) = \frac{1}{2m_1} \sum_{\Gamma \in \mathrm{ECH}(\mu)_1} \mathrm{cont}(\Gamma).$$

Thus using Propositions 4.9, 4.10 and 4.4 successively, we obtain

$$\begin{aligned} d_1(\mu) &= \frac{1}{2m_1} \sum_{\Gamma \in \mathrm{ECH}(\mu)_n} \mathrm{cont}(\Gamma) = \frac{1}{2m_1} \sum_{\Gamma \in \mathrm{CH}(\mu)} \widehat{\mathrm{cont}}(\Gamma) \\ &= \frac{-4\pi^2}{m_1} c_{\mathrm{cyl},1}(\mu) \cdot \mathrm{vol}(\mu), \end{aligned}$$

which is the first identity stated in Theorem 3.1. If μ is odd, then the second statement of Theorem 3.1 is obtained similarly by applying the spin counterpart of Propositions 3.11, 4.9, 4.10 and 4.4.

A. Character tables of the spin symmetric group and the Sergeev group for $d \leq 5$

For $d \leq 5$, and G being one the groups $\widetilde{\mathfrak{S}}_d, \widetilde{\mathrm{A}}_d, \mathrm{Se}_d$ and Se_d^0, we compute the character table of all irreducible *spin* representations, i.e., we assume the central element $\varepsilon \in G$ acts by -1. If the characters are $\chi_1, \chi_2, \ldots$ and the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \ldots,$ we write

G	$\mathcal{C}_1$	$\cdots$
$\lvert G\rvert$	$\lvert\mathcal{C}_1\rvert$	$\cdots$
χ_1	$\chi_1(\mathcal{C}_1)$	$\cdots$
$\vdots$	$\vdots$	

for the character table of G. Note that $\chi_i(\varepsilon\mathcal{C}_j) = -\chi_i(\mathcal{C}_j)$. Thus, we omit all conjugacy classes for which $\mathcal{C}_j \cap \varepsilon\mathcal{C}_j \neq \emptyset$, and else pick only one of the two conjugacy classes $\mathcal{C}_j$ and $\varepsilon\mathcal{C}_j$. Row and column orthogonality relations are satisfied, i.e.,

$$\begin{aligned} \sum_i |\mathcal{C}_j| \chi_i(\mathcal{C}_j) \overline{\chi_i(\mathcal{C}_{j'})} &= \frac{1}{2}|G|\delta_{j,j'} \quad \text{for all } j, j', \\ \sum_j |\mathcal{C}_j| \chi_i(\mathcal{C}_j) \overline{\chi_{i'}(\mathcal{C}_j)} &= \frac{1}{2}|G|\delta_{i,i'} \quad \text{for all } i, i'. \end{aligned}$$

The first factor $1/2$ appears because we omit half of the conjugacy classes, as explained before. The second factor $1/2$ appears because we omit the non-spin representations, which correspond to representations of the quotient G/ε.

$d = 1$

$\widetilde{\mathfrak{S}}_1$	e
2	1
1	1

$\widetilde{\mathrm{A}}_1 \simeq \widetilde{\mathfrak{S}}_1$

Se_1	e	$C_{e,1}$
4	1	1
1+	1	$+\mathrm{i}$
1−	1	$-\mathrm{i}$

Se_1^0	e
2	1
1	1

$d = 2$

$\widetilde{\mathfrak{S}}_2$	e	(12)
4	1	1
2+	1	$+\mathrm{i}$
2−	1	$-\mathrm{i}$

$\widetilde{\mathrm{A}}_2$	e
2	1
2	1

Se_2	e	$C_{(12),1}$
16	1	2
2+	2	$+\mathrm{i}\sqrt{2}$
2−	2	$-\mathrm{i}\sqrt{2}$

Se_2^0	e
8	1
2	2

$d = 3$

$\widetilde{\mathfrak{S}}_3$	e	(123)	(12)
12	1	2	3
3	2	1	0
2, 1+	1	−1	$+\mathrm{i}$
2, 1−	1	−1	$-\mathrm{i}$

$\widetilde{\mathrm{A}}_3$	e	(123)	(321)
6	1	1	1
3+	1	$\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$	$\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$
3−	1	$\frac{1}{2}-\frac{1}{2}\mathrm{i}\sqrt{3}$	$\frac{1}{2}-\frac{1}{2}\mathrm{i}\sqrt{3}$
2, 1	1	−1	−1

Se_3	e	$C_{(123)}$	$C_{(123),1}$
96	1	8	8
3+	4	1	$+\mathrm{i}\sqrt{3}$
3−	4	1	$-\mathrm{i}\sqrt{3}$
2, 1	4	−2	0

Se_3^0	e	$C_{(123)}$	$C_{(12)}$
48	1	8	6
3	4	1	0
2, 1+	2	−1	$+\mathrm{i}\sqrt{2}$
2, 1−	2	−1	$-\mathrm{i}\sqrt{2}$

$d = 4$

$\widetilde{\mathfrak{S}}_4$	e	(123)	(1234)
48	1	8	6
$4+$	2	1	$+\mathrm{i}\sqrt{2}$
$4-$	2	1	$-\mathrm{i}\sqrt{2}$
$3,1$	4	-1	0

$\widetilde{\mathrm{A}}_4$	e	(123)	(321)
24	1	4	4
4	2	1	1
$3,1+$	2	$-\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$	$-\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$
$3,1-$	2	$-\frac{1}{2}-\frac{1}{2}\mathrm{i}\sqrt{3}$	$-\frac{1}{2}-\frac{1}{2}\mathrm{i}\sqrt{3}$

Se_4	e	$C_{(123)}$	$C_{(1234),1}$
768	1	32	48
$4+$	8	2	$+2\mathrm{i}$
$4-$	8	2	$-2\mathrm{i}$
$3,1$	16	-2	0

Se_4^0	e	$C_{(123)}(a)$	$C_{(123)}(b)$
48	1	16	16
4	8	2	2
$3,1+$	8	$-1+\mathrm{i}\sqrt{3}$	$-1+\mathrm{i}\sqrt{3}$
$3,1-$	8	$-1-\mathrm{i}\sqrt{3}$	$-1-\mathrm{i}\sqrt{3}$

$d = 5$

$\widetilde{\mathfrak{S}}_5$	e	(12345)	(123)	(1234)	$(123)(45)$
240	1	24	20	30	20
5	4	1	2	0	0
$4,1+$	6	-1	0	$+\mathrm{i}\sqrt{2}$	0
$4,1-$	6	-1	0	$-\mathrm{i}\sqrt{2}$	0
$3,2+$	4	1	-1	0	$+\mathrm{i}\sqrt{3}$
$3,2-$	4	1	-1	0	$-\mathrm{i}\sqrt{3}$

$\widetilde{\mathrm{A}}_5$	e	(12345)	(54321)	(123)
120	1	12	12	20
$5+$	2	$\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{5}$	$\frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{5}$	1
$5-$	2	$\frac{1}{2}-\frac{1}{2}\mathrm{i}\sqrt{5}$	$\frac{1}{2}-\frac{1}{2}\mathrm{i}\sqrt{5}$	1
$4,1$	6	-1	-1	0
$3,2$	4	1	1	-1

Se_5	e	$C_{(12345)}$	$C_{(123)}$	$C_{(12345),1}$
7680	1	384	80	384
$5+$	16	1	4	$+\mathrm{i}\sqrt{5}$
$5-$	16	1	4	$-\mathrm{i}\sqrt{5}$
$4,1$	48	-2	0	0
$3,2$	32	2	-4	0

Se_5^0	e	$C_{(12345)}$	$C_{(123)}$	$C_{(1234)}$	$C_{(123)(45)}$
3840	1	384	80	240	160
5	16	1	4	0	0
$4,1+$	24	-1	0	$+2\mathrm{i}$	0
$4,1-$	24	-1	0	$-2\mathrm{i}$	0
$3,2+$	16	1	-2	0	$+\mathrm{i}\sqrt{6}$
$3,2-$	16	1	-2	0	$-\mathrm{i}\sqrt{6}$

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