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Chern classes of linear submanifolds with application to spaces of k-differentials and ball quotients

Matteo Costantini, Martin Möller, and Johannes Schwab

Abstract. We provide formulas for the Chern classes of linear submanifolds of the moduli spaces of Abelian differentials and hence for their Euler characteristic. This includes as special case the moduli spaces of k-differentials, for which we set up the full intersection theory package and implement it in the SageMath package `diffstrata`. As an application, we give an algebraic proof of the theorems of Deligne–Mostow and Thurston that suitable compactifications of moduli spaces of k-differentials on the 5-punctured projective line with weights satisfying the INT-condition are quotients of the complex two-ball.

Contents

1. Introduction

Linear submanifolds are the most interesting and well-studied subvarieties of moduli spaces of Abelian differentials $\Omega\mathcal{M}_{g,n}(\mu)$ and their classification seems far from complete at present. They are defined as the normalization of algebraic substacks of $\Omega\mathcal{M}_{g,n}(\mu)$ that are locally a union of linear subspaces in period coordinates. In the

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holomorphic case, linear submanifolds defined by real linear equations are precisely the closures of $\mathrm{GL}_2^+(\mathbb{R})$-orbits by the fundamental theorems of Eskin–Mirzakhani–Mohammadi [21, 22]. These orbit closures are automatically algebraic subvarieties by Filip's theorem [23]. Our results require algebraicity, but they work as well for meromorphic differentials and for subvarieties whose equations are only $\mathbb{C}$-linear.

Linear submanifolds include:

- spaces of quadratic differentials;
- Teichmüller curves;
- eigenform loci and Prym loci;
- the recent sporadic examples from [45] and [20]; but also
- spaces defined by covering constructions; and
- in the meromorphic case, spaces defined by residue conditions.

These examples are $\mathbb{R}$-linear. Spaces of k-differentials for $k \geq 2$, and in particular the ball quotients in Section 8 are prominent examples that are only $\mathbb{C}$-linear.

Our primary goal is a formula for the Chern classes of the cotangent bundle of any linear submanifold or rather of its compactification. The Euler characteristic is an intrinsic compactification-independent application. Knowing the Chern classes is a prerequisite for understanding the birational geometry of linear submanifolds, such as computations of the Kodaira dimension, see [9].

This goal was achieved in [13] for the full projectivized strata of Abelian differentials $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ themselves, taking the modular smooth normal crossing compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of multi-scale differentials from [5] as point of departure. In the inextricable zoo of linear manifolds we are not aware of any intrinsic way to construct a smooth compactification with modular properties. Working with the normalization of the closure in some ambient compactification is usually unsuitable for intersection theory computations. Here, however, thanks to the work of Benirschke–Dozier–Grushevsky [7] and some minor upgrades we are able to work with this closure.

We now introduce more notation to state the general results and then apply them to specific linear submanifolds. Let $\Omega\mathcal{H} \to \Omega\mathcal{M}_{g,n}(\mu)$ be a linear submanifold. Let moreover $\mathcal{H} \to \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ be its projectivization and let $\overline{\mathcal{H}} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ denote the normalization of its closure into the space of multi-scale differentials. The boundary strata D_Γ of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are indexed by level graphs Γ as we recall in Section 3.2. By [7, Theorem 1.5] the boundary of $\overline{\mathcal{H}}$ is divisorial and consists two types of divisors: First there are the divisors $D_{\mathrm{h}}^{\mathcal{H}}$ of curves whose level graphs have only horizontal edges (i.e., joining vertices of the same level). Second there are the divisors $D_\Gamma^{\mathcal{H}}$ parametrized by level graphs $\Gamma \in \mathrm{LG}_1(\mathcal{H})$ that have one level below the zero level and no horizontal edges and such that the intersection of $\overline{\mathcal{H}}$ with the interior of the boundary divisor D_Γ is non-empty. Those boundary divisors $D_\Gamma^{\mathcal{H}}$ come with the integer ℓ_Γ, the least common multiple of the prongs κ_e along the edges. The interior of D_Γ can intersect the linear submanifold $\overline{\mathcal{H}}$ in finitely many irreducible components, whose number we denote by $n_\Gamma(\overline{\mathcal{H}})$. We denote by $\mathrm{LG}_1^+(\mathcal{H})$ the set of pairs $\Gamma^+ = (\Gamma, i)$, where $\Gamma \in \mathrm{LG}_1(\mathcal{H})$ and $i \in \{1, \ldots, n_\Gamma(\overline{\mathcal{H}})\}$ is the index set of irreducible components of $D_\Gamma^\circ \cap \overline{\mathcal{H}}$. We refer to $\Gamma^+ = (\Gamma, i)$ as a *refined level graph* and set $\ell_{(\Gamma,i)} := \ell_\Gamma$. This extra notational complexity is necessary since a priori it is possible that a linear submanifold intersects a boundary component in irreducible components with different level dimensions. This is clearly possible since for example we do not require our linear manifolds to be irreducible, which is convenient in order to include entire strata of k-differentials, but it could be possible also in the case of irreducible linear submanifolds. We let $\xi = c_1(\mathcal{O}(-1))$ be the first Chern class of the tautological bundle on $\overline{\mathcal{H}}$.

Theorem 1.1. *The first Chern class of the logarithmic cotangent bundle of a projectivized compactified linear submanifold $\overline{\mathcal{H}}$ is*

$$\mathrm{c}_1\big(\Omega^1_{\overline{\mathcal{H}}}(\log \partial\mathcal{H})\big) = N \cdot \xi + \sum_{\Gamma^+ \in \mathrm{LG}_1^+(\mathcal{H})} (N - N^\top_{\Gamma^+})\ell_{\Gamma^+}[D^{\mathcal{H}}_{\Gamma^+}] \in \mathrm{CH}^1(\overline{\mathcal{H}}),$$

where $N := \dim(\Omega\mathcal{H})$, and $N^\top_{\Gamma^+} := \dim(D^{\mathcal{H},\top}_{\Gamma^+}) + 1$ is the dimension of the unprojectivized top level stratum in $D^{\mathcal{H}}_{\Gamma^+}$.

To state a formula for the full Chern character we need to recall a procedure that also determines adjacency of boundary strata. It is given by undegeneration maps δ_i that contract all the edges except those that cross from level $-i+1$ to level $-i$, see Section 3.2. This construction can obviously be generalized so that a larger subset of levels remains. For example, the undegeneration map $\delta_i^{\complement}$ contracts only the edges crossing from level $-i+1$ to level $-i$. For any element Γ of the set $\mathrm{LG}_L(\mathcal{H})$ of graphs with L levels below zero and without horizontal edges, we can now define the boundary component $D^{\mathcal{H}}_\Gamma$ of codimension L and the quantity $\ell_\Gamma = \prod_{i=1}^L \ell_{\delta_i(\Gamma)}$. We also extend the undegeneration maps at the level of refined level graphs, i.e., for elements in $\mathrm{LG}_L^+(\mathcal{H})$, which we define analogously to $\mathrm{LG}_1^+(\mathcal{H})$, and we still denote them by the same letter.

Theorem 1.2. *The Chern character of the logarithmic cotangent bundle is*

$$\mathrm{ch}\big(\Omega^1_{\overline{\mathcal{H}}}(\log \partial\mathcal{H})\big) = e^\xi \cdot \sum_{L=0}^{N-1} \sum_{\Gamma^+ \in \mathrm{LG}_L^+(\mathcal{H})} \ell_{\Gamma^+}\big(N - N^\top_{\delta_L(\Gamma^+)}\big)\mathrm{i}_{\Gamma^+ *}$$
$$\cdot \prod_{i=1}^L \mathrm{td}\Big(\mathcal{N}^{\otimes -\ell_{\delta_i(\Gamma^+)}}_{\Gamma^+/\delta_i^{\complement}(\Gamma^+)}\Big)^{-1},$$

where $\mathcal{N}_{\Gamma+/\delta_i^{\complement}(\Gamma+)}$ *denotes the normal bundle of* $D^{\mathcal{H}}_{\Gamma+}$ *in* $D^{\mathcal{H}}_{\delta_i^{\complement}(\Gamma+)}$*, and where* td *is the Todd class and* $\mathrm{i}_{\Gamma+}\colon D^{\mathcal{H}}_{\Gamma+} \hookrightarrow \overline{\mathcal{H}}$ *is the inclusion map.*

So far the results have been stated to parallel exactly those in [13]. The ambient spaces can be mildly singular (see Section 3), but the maps $i_{\Gamma+}$ are regular embeddings (see Section 4.3) which allows us to work tacitly with operational Chow groups just as in the case of the smooth DM stack in [13]. We start explaining the difference in evaluating this along with the next result, a closed formula for the Euler characteristic.

Theorem 1.3. *Let* $\mathcal{H} \to \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ *be a projectivized linear submanifold. The orbifold Euler characteristic of* $\mathcal{H}$ *is given by*

$$\chi(\mathcal{H}) = (-1)^d \sum_{L=0}^{d} \sum_{\Gamma+\in \mathrm{LG}_L^+(\mathcal{H})} \frac{K^{\mathcal{H}}_{\Gamma+}\cdot N^{\top}_{\Gamma+}}{|\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)|} \cdot \prod_{i=0}^{-L} \int_{\mathcal{H}^{[i]}_{\Gamma+}} \xi^{d^{[i]}_{\Gamma+}}_{\mathcal{H}^{[i]}_{\Gamma+}},$$

where the integrals are over the normalization of the closure $\overline{\mathcal{H}} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ *inside the moduli space of multi-scale differentials and similar integrals over boundary strata, where*

- $\mathcal{H}^{[i]}_{\Gamma+}$ *are the linear submanifolds at level* i *of* Γ^+ *as defined in Section 3.5;*
- $d^{[i]}_{\Gamma+} := \dim(\mathcal{H}^{[i]}_{\Gamma+})$ *is the projectivized dimension;*
- $K^{\mathcal{H}}_{\Gamma+}$ *is the product of the number of prong-matchings on each edge of* Γ *that are actually contained in* $D^{\mathcal{H}}_{\Gamma+}$*;*
- $\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)$ *is the set of automorphism of the graph* Γ *whose induced action on a neighborhood of* $D^{\mathcal{H}}_{\Gamma+}$ *preserves* $\overline{\mathcal{H}}$*;*
- $d := \dim(\mathcal{H})$ *is the projectivized dimension.*

The number of *reachable prong matchings* $K^{\mathcal{H}}_{\Gamma+}$ and the number $|\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)|$, as defined in the theorem, are in general non-trivial to determine. Also the description of $\mathcal{H}^{[i]}_{\Gamma+}$ requires specific investigation. For example, for strata of k-differentials, these $\mathcal{H}^{[i]}_{\Gamma+}$ are again some strata of k-differentials, but the markings of the edges have to be counted correctly.

The most important obstacle to evaluate this formula however is to compute the fundamental classes of linear submanifolds, or to use tricks to avoid this. For strata of Abelian differentials, this step was provided by the recent advances in relating fundamental classes to Pixton's formula [1, 34]. Whenever we have the fundamental classes at our disposal, we can evaluate expressions in the tautological ring, as we briefly summarize in Section 4.

Applications: Teichmüller curves in genus two. As an example where fundamental class considerations can be avoided, we give in Section 6 an alternative quick proof of one of the first computations of Euler characteristics of Teichmüller curves, initially proven in [2], see also [46] for a proof via theta derivatives.

Theorem 1.4 (Bainbridge). *The orbifold Euler characteristic of the Teichmüller curve $W_D \subset \mathbb{P}\Omega\mathcal{M}_{2,1}(2)$ in the eigenform locus for real multiplication by a non-square discriminant D is $\chi(W_D) = -9\zeta(-1)$, where $\zeta = \zeta_{\mathbb{Q}(\sqrt{D})}$ is the Dedekind zeta function.*

Strata of k-differentials. The space of quadratic differentials is the cotangent space to moduli space of curves and thus fundamental in Teichmüller dynamics. We give formulas for Chern classes, Euler characteristics and for the intersection theory in these spaces. In fact, our formulas work uniformly for spaces of k-differentials for all $k \geq 1$. Having the quadratic case in mind, we write

$$\overline{\mathcal{Q}} = \mathbb{P}\,\Xi^k \overline{\mathcal{M}}_{g,n}(\mu)$$

for the space of multi-scale k-differentials defined in [15]. The space $\overline{\mathcal{Q}}$ is the disjoint union over all divisors d of k of the subspaces parametrizing powers of k/d-differentials. We write

$$\overline{\mathcal{Q}}_{\mathrm{pr}} = \mathbb{P}\,\Xi^k_{\mathrm{pr}} \overline{\mathcal{M}}_{g,n}(\mu)$$

for the (still possibly disconnected) subspace of *primitive multi-scale k-differentials*, the closure of the components corresponding to $d = 1$. The space $\overline{\mathcal{Q}}$ coincides (up to explicit isotropy groups, see Lemma 7.2) with the compactification as above of the linear submanifolds associated to its connected components obtained via the canonical covering construction.

The formulas in Theorem 1.2 apply to the connected components of $\overline{\mathcal{Q}}$ viewed as linear submanifolds in some higher genus stratum $\mathcal{M}_{\widehat{g},\widehat{n}}(\widehat{\mu})$. However the fundamental class of these submanifolds is not known, conceivably it is not even a tautological class. The main challenge here is to convert these formulas into formulas that can be evaluated on $\overline{\mathcal{Q}}$ viewed as a submanifold in $\overline{\mathcal{M}}_{g,n}$ where the fundamental class is given by Pixton's formula.

While the boundary strata of the moduli space $\mathbb{P}\,\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ are indexed by level graphs, the boundary strata of the moduli space of multi-scale k-differentials $\overline{\mathcal{Q}}$ are indexed by k-coverings of level graphs $\pi\colon \widehat{\Gamma}_{\mathrm{mp}} \to \Gamma$, where the legs of $\widehat{\Gamma}_{\mathrm{mp}}$ are marked only partially, see Section 7 or also [15, Section 2] for the definitions of these objects and the labeling conventions of those covers. The k-coverings appearing in the boundary of $\overline{\mathcal{Q}}_{\mathrm{pr}}$ are precisely those with $\widehat{\Gamma}$ connected. Each edge $e \in \Gamma$ has an associated k-enhancement κ_e given by $|\operatorname{ord}_e \omega + k|$, where ω is the k-differential on a generic point of the associated boundary stratum D_π. We let $\zeta = c_1(\mathcal{O}(-1))$ be the first Chern class of the tautological bundle on $\overline{\mathcal{Q}}$. Via the canonical cover construction, Theorem 1.3 implies the following formula for the Euler characteristic of strata of k-differentials. Note that if $\mathcal{H}_k$ is the linear submanifold associated to a connected component of a stratum of k-differentials, the information of π is enough to uniquely determine the relevant information of the irreducible components D_{Γ^+} of D_π, i.e., the level strata dimensions $d_{\Gamma^+}^{[i]}$, the number of reachable prong-matchings $K_{\Gamma^+}^{\mathcal{H}_k}$ and $|\operatorname{Aut}_{\mathcal{H}_k}(\Gamma^+)|$. So in the applications of the formulas of Theorem 1.2 and Theorem 1.3 to strata of k-differentials, we can group together all irreducible components of D_π.

Corollary 1.5. *The orbifold Euler characteristic of a projectivized stratum of k-differentials $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ is given by*

$$\chi\big(\mathbb{P}\Omega^k_{\mathrm{pr}}\mathcal{M}_{g,n}(\mu)\big) = \Big(\frac{-1}{k}\Big)^d \sum_{L=0}^{d} \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_L(\mathcal{Q}_{\mathrm{pr}})} S(\pi)\cdot \frac{N_\pi^\top \cdot \prod_{e\in E(\Gamma)}\kappa_e}{|\operatorname{Aut}(\Gamma)|} \cdot \prod_{i=0}^{-L} \int_{\mathcal{Q}_\pi^{[i]}} \zeta_{\mathcal{Q}_\pi^{[i]}}^{d_\pi^{[i]}},$$

where $S(\pi)$ is the normalized size of a stabilizer of a totally labeled version of the graph $\widehat{\Gamma}_{\mathrm{mp}}$ and $\mathcal{Q}_\pi^{[i]}$ are the strata of k-differentials of D_π at level i.

The full definition of $S(\pi)$ is presented in (29). It equals one for many π, e.g., if all vertices in Γ have only one preimage in $\widehat{\Gamma}_{\mathrm{mp}}$. See Remark 7.6 for values of this combinatorial constant.

k	1	2	3	4	5	6	7	8	9
$\chi(\mathbb{P}\Omega^k_{\mathrm{pr}}\mathcal{M}_{2,1}(2k))$	$-\frac{1}{40}$	0	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{21}{5}$	9	18	30	51

Table 1. Euler characteristics of some minimal strata of primitive k-differentials. Note that in the case of $k = 2$ the stratum of primitive minimal k-differentials is empty, see [41, Theorem 2(c)].

Table 1 gives the Euler characteristics of some strata of primitive quadratic differentials, for more examples and cross-checks see Section 7.5.

All the formulas for evaluations in the tautological ring of strata of k-differentials have been coded in an extension of the SageMath package `diffstrata` (an extension of `admcycles` by [16]) that initially had this functionality for Abelian differentials only (see [13, 14]). See Section 4 for generalities on tautological ring computations, and in particular Section 7, for the application to k-differentials. The program `diffstrata` has been used to verify the Hodge-DR-conjecture from [10] in low genus. Moreover, `diffstrata` confirms that the values of the tables in [28] can be obtained via intersection theory computations.

Proposition 1.6. *A conjecture of Chen–Möller–Sauvaget* [11, Conjecture 1.1], *which expresses Masur–Veech volumes for strata of quadratic differentials as intersection* *numbers, holds true for strata of projectivized dimension up to six. For example,* $\mathcal{Q}(12) = 5614/6075 \cdot \pi^6$.

Ball quotients. Deligne–Mostow [17] and Thurston [54] constructed compactifications of strata of k-differentials on $\mathcal{M}_{0,n}$ for very specific choices of μ and showed that these compactified strata are quotients of the complex $(n-3)$-ball. These results were celebrated as they give a list of non-arithmetic ball quotients, of which there are still only finitely many sporadic examples today, see [19] and [18] for recent progress. The compactifications are given as GIT quotients (in [17]) or in the language of cone manifolds (in [54]) and the proof of the discreteness of the monodromy representation requires delicate arguments for extension of the period at the boundary, resp. surgeries for the cone manifold completion.

As application of our Chern class formulas we give a purely algebraic proof that these compactifications are ball quotients, based on the fact that the equality case in the Bogomolov–Miyaoka–Yau inequality implies a ball quotient structure, see Proposition 8.1. Since this is a proof of concept, we restrict to the case $n = 5$, i.e., to quotients of the complex two-ball, and to the condition INT in (1), leaving the analog for Mostow's generalized ΣINT-condition [47] for the reader.

The computation of the hyperbolic volume of these ball quotients had been open for a long time. A solution has been given by McMullen in [44] and Koziarz–Nguyen in [39], see also [35]. Since computing the hyperbolic volume is equivalent to computing the Euler characteristic by Gauss–Bonnet, our results provide alternative approach to this question, too.

For spaces $\overline{\mathcal{Q}}$ of multi-scale k-differentials in $g = 0, n = 5$ with these conditions, there are only four kinds of boundary divisors:

- the divisors Γ_{ij} where two points with $a_i + a_j < k$ collide;
- the divisors L_{ij} where two points with $a_i + a_j > k$ collide;
- the 'horizontal' boundary divisor D_{hor} consisting of all components where two points with $a_i + a_j = k$ collide;
- the 'cherry' boundary divisors ${}_{ij}\Lambda_{kl}$.

Theorem 1.7. *Suppose that* $\mu = (-a_1, \ldots, -a_5)$ *is a tuple with* $a_i \geq 0$ *and with the condition*

$$\left(1 - \frac{a_i}{k} - \frac{a_j}{k}\right)^{-1} \in \mathbb{Z} \text{ if } a_i + a_k < k \qquad \text{(INT)} \tag{1}$$

for all $i \neq j$. *Then there exists a birational contraction morphism* $\overline{\mathcal{Q}} \to \overline{\mathfrak{B}}$ *onto a smooth proper DM-stack* $\overline{\mathfrak{B}}$ *that contracts precisely all the divisors* L_{ij} *and* ${}_{ij}\Lambda_{kl}$. *The target* $\overline{\mathfrak{B}}$ *satisfies the Bogomolov–Miyaoka–Yau equality for* $\Omega^1_{\overline{\mathfrak{B}}}(\log D_{\mathrm{hor}})$.

As a consequence $\mathfrak{B} = \overline{\mathfrak{B}} \setminus D_{\mathrm{hor}}$ *is a ball quotient.*

The signature of the intersection form on the eigenspace that k-differentials are modeled on has been computed by Veech [57]. The only other case where the signature is $(1,2)$ are strata in $\mathcal{M}_{1,3}$. As observed by Ghazouani–Pirio in [26] (see also [27]), there are only few cases where the metric completion of the strata can be a ball quotient. However, they also find additional cases where the monodromy of the stratum is discrete. This implies that the period map descends to a map from the compactified stratum to a ball quotient. It would be interesting to investigate if there are more such cases, possibly with non-arithmetic monodromy.

2. Logarithmic differential forms and toric varieties

This section connects the Euler characteristic to integrals of characteristic classes of the sheaf of logarithmic differential forms. We work on a possibly singular but normal, proper and irreducible variety $\overline{\mathcal{H}}$ of dimension d, whose singularities are toric and contained in some boundary divisor $\partial\mathcal{H}$. We are interested in the Euler characteristic of the (Zariski) open subvariety $\mathcal{H} = \overline{\mathcal{H}} \setminus \partial\mathcal{H}$ given by the complement of $\partial\mathcal{H}$, in the situation where the inclusion $\mathcal{H} \hookrightarrow \overline{\mathcal{H}}$ is a toroidal embedding. In particular, in this case, the boundary divisor $\partial\mathcal{H}$ is locally on open subsets U_α a torus-invariant divisor.

In this situation we define locally $\Omega^1_{U_\alpha}(\log)$ to be the sheaf generated by $(\mathbb{C}^*)^d$-invariant meromorphic differential forms. These glue to sheaf $\Omega^1_{\overline{\mathcal{H}}}(\log\partial\mathcal{H})$, that is called *logarithmic differential sheaf*. This terminology is justified by the following idea from [49, Section 4], the details and definitions being given in [36]. For any 'allowable' smooth modification $p\colon \overline{W} \to \overline{\mathcal{H}}$ that maps a normal crossing boundary divisor $\partial W \subset \overline{W}$ onto $\partial\mathcal{H}$, we have

$$p^*\Omega^1_{\overline{\mathcal{H}}}(\log\partial\mathcal{H}) = \Omega^1_{\overline{W}}(\log\partial W)$$

for the usual definition of the logarithmic sheaf on $\overline{W}$. Moreover, such an 'allowable' smooth modification always exists. The previous situation can be generalized verbatim to the case where $\overline{\mathcal{H}}$ is a Deligne–Mumford stack and this is the setup we are interested in.

Proposition 2.1. *Let $\overline{\mathcal{H}}$ be a proper irreducible Deligne–Mumford stack of dimension d with toric singularities. Assume moreover that the coarse moduli space of $\overline{\mathcal{H}}$ is projective. Let $\mathcal{H} \hookrightarrow \overline{\mathcal{H}}$ be a toroidal embedding and $\partial\mathcal{H} = \overline{\mathcal{H}} \setminus \mathcal{H}$. Then the Euler characteristic of $\mathcal{H}$ can be computed as the integral*

$$\chi(\mathcal{H}) = (-1)^d \int_{\overline{\mathcal{H}}} c_d\big(\Omega^1_{\overline{\mathcal{H}}}(\log\partial\mathcal{H})\big)$$

over the top Chern class of the logarithmic differential sheaf.

Proof. If $\overline{\mathcal{H}}$ is a smooth Deligne–Mumford stack and $\partial\mathcal{H} = \emptyset$, this is well known (see, e.g., [55, Corollary 4.16]). In the case where $\overline{\mathcal{H}}$ is still smooth but $\partial\mathcal{H}$ is not empty, a self-contained proof of the statement was given in [13, Proposition 2.1] (the proof was given in the case where $\overline{\mathcal{H}}$ is a smooth variety, but it works verbatim for the more general case of smooth DM stack).

In general, we use an allowable modification. By definition this restricts to an isomorphism $W \to \mathcal{H}$, hence does not change the left-hand side. The right-hand side also stays the same by push-pull and the pullback formula along an allowable smooth modification. ∎

3. The closure of linear submanifolds

The compactification of a linear submanifold we work with has (currently) no intrinsic definition. Rather we consider the normalization of the closure of a linear submanifold inside the moduli space of multi-scale differentials $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. We recall from [7] the basic properties of such closures. The goal of this section is to make precise and to explain the following two slogans:

(i) Near boundary points without horizontal edges, the closure is determined as for the ambient Abelian stratum by the combinatorics of the level graph and it is smooth. The *ghost automorphisms*, the stack structure at the boundary that stems from twist groups, agrees with the ghost automorphisms of the ambient stratum and the intersection pattern is essentially determined by the *profiles* of the level graph, a subset of the profiles of the ambient stratum.

(ii) In the presence of horizontal edges there are toric singularities. Working with the appropriate definition of the logarithmic cotangent sheaf these singularities do not matter. This sheaf decomposes into summands from horizontal nodes, from the level structure, and the deformation of the differentials at the various levels, just as in the ambient stratum.

3.1. Linear submanifolds in generalized strata

Let $\Omega\mathcal{M}_{g,n}(\mu)$ denote the moduli space of Abelian differentials of possibly meromorphic signature μ. Despite calling them 'moduli space' or 'strata', we always think of them as quotient stacks or orbifolds and intersection numbers etc. are always understood in that sense. These strata come with a linear structure given by period coordinates (see, e.g., [58] for an introduction). A *linear submanifold* $\Omega\mathcal{H}$ of $\Omega\mathcal{M}_{g,n}(\mu)$ is an algebraic stack with a map $\Omega\mathcal{H} \to \Omega\mathcal{M}_{g,n}(\mu)$ which is the normalization of its image and whose image is locally given as a finite union of linear subspaces in period coordinate charts. See [24, Example 4.1.10] for an example that illustrates why we need to pass to the normalization for $\Omega\mathcal{H}$ to be a smooth stack. In the context of holomorphic signatures and $\mathrm{GL}_2(\mathbb{R})$-orbit closures, the linear manifolds obtained in this way can locally be defined by equations with $\mathbb{R}$-coefficients [21, 22]. We refer to them as *$\mathbb{R}$-linear submanifolds*. In this context, the algebraicity follows from being closed by the result of Filip [23], but in general algebraicity is an extra hypothesis.

To set up for clutching morphisms and a recursive description of the boundary of compactified linear submanifolds, we now define *generalized strata*, compare [13, Section 4]. For a tuple $\mathbf{g} = (g_1, \dots, g_k)$ of genera and a tuple $\mathbf{n} = (n_1, \dots, n_k)$, together with a collection of types $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ with $|\mu_i| = n_i$, we first define the disconnected stratum

$$\Omega\mathcal{M}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu}) = \prod_{i=1}^{k} \Omega\mathcal{M}_{g_i,n_i}(\mu_i).$$

Then, for a linear subspace $\mathfrak{R}$ inside the space of the residues at all poles of $\boldsymbol{\mu}$ we define the generalized stratum $\Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ to be the subvariety with residues lying in $\mathfrak{R}$. Generalized strata obviously come with period coordinates and we thus define a *generalized linear submanifold* $\Omega\mathcal{H}$ to be an algebraic stack together with a map to $\Omega\mathcal{M}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$, whose image is locally linear in period coordinates, and where $\Omega\mathcal{H}$ is the normalization of its image.

Rescaling the differential gives an action of $\mathbb{C}^*$ on strata and the quotient are projectivized strata $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. The image of a linear submanifold in $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is called *projectivized linear manifold* $\mathcal{H}$, but we usually omit the 'projectivized'.

We refer with an index B to quantities of the ambient projectivized stratum, such as its dimension d_B and the unprojectivized dimension $N_B = d_B + 1$. The same letters without additional index are used for the linear submanifold, e.g., $N = d + 1$, and we write $d_{\mathcal{H}}$ and $N_{\mathcal{H}}$ only if ambiguities may arise.

3.2. Multi-scale differentials: Boundary combinatorics

We will work within the moduli stack of multi-scale differentials $\overline{B} := \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, which provides a compactification of the stratum $B := \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ and was constructed in [5]. We recall some of its key properties, as detailed in [5] and further discussed in [13, Section 3]. Everything carries over with obvious modifications to the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}^{\mathfrak{R}}_{\mathbf{g},\mathbf{n}}(\boldsymbol{\mu})$ of generalized strata, see [13, Proposition 4.1].

Each boundary stratum of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ has its associated level graph Γ, a stable graph of the underlying pointed stable curve together with a weak total order on the vertices, usually given by a level function normalized to have top level zero, and an enhancement $\kappa_e \geq 0$ associated to the edges. Edges are called *horizontal*, if they start and end at the same level, and *vertical otherwise*. Moreover, $\kappa_e = 0$ if and only if the edge is horizontal. We denote the closure of the boundary stratum of points with level graph Γ by D_Γ^B and, for any level graph Δ that is a degeneration of Γ, we let $D_{\Gamma,\Delta}^{B,\circ} \subset D_\Gamma^B$ be the open subset parametrizing multi-scale differentials compatible with an undegeneration of Δ. In particular, the points of $D_\Gamma^{B,\circ} := D_{\Gamma,\Gamma}^{B,\circ}$ represent multi-scale differentials with level graph exactly Γ. These D_Γ^B are in general not connected, and might be empty (e.g., for unsuitably large κ_e).

We let $\mathrm{LG}_L(B)$ be the set of all enhanced $(L+1)$-level graphs without horizontal edges. The structure of the normal crossing boundary of $\mathbb{P}\Xi\bar{\mathcal{M}}_{g,n}(\mu)$ is encoded by *undegenerations*. For any subset $I = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, L\}$, there are undegeneration map

$$\delta_{i_1,\ldots,i_n} \colon \mathrm{LG}_L(B) \to \mathrm{LG}_n(B),$$

that preserves the level passage given as a horizontal line just above level $-i$ and contracts the remaining level passages. We define $\delta_I^\complement = \delta_{I^\complement}$.

The boundary strata D_Γ^B for $\Gamma \in \mathrm{LG}_L(B)$ are commensurable to a product of generalized strata $B_\Gamma^{[i]} = \mathbb{P}\Xi\bar{\mathcal{M}}^{\mathfrak{R}_i}_{\mathbf{g}_i,\mathbf{n}_i}(\boldsymbol{\mu}_i)$ defined via the following diagram:

$$\begin{array}{ccccccc}
 & & c_\Gamma^{-1}(D_{\Gamma,\Delta}^{B,\circ}) & \xrightarrow{\subset} & D_\Gamma^{B,s} & & \\
 & \swarrow^{p_\Gamma} & & \searrow^{q_\Delta} & & \searrow^{c_\Gamma} & \\
 & & & & B_{\Gamma,\Delta}^s & & \\
 & & & \swarrow^{p_\Gamma^\Delta} & & \searrow^{c_\Gamma^\Delta} & \\
\prod_{i=-L}^0 B_\Gamma^{[i]} =: B_\Gamma & \xleftarrow{\supset} & B_{\Gamma,\Delta} & & & & D_{\Gamma,\Delta}^{B,\circ} \xrightarrow{\subset} D_\Gamma^B .
\end{array} \tag{2}$$

Here $\mathbf{g}_i$, $\mathbf{n}_i$ and $\boldsymbol{\mu}_i$ are the tuples of the genera, marked points and signatures of the components at level i of the level graph and $\mathfrak{R}_i$ is the global residue condition induced by the levels above. The covering space $D_\Gamma^{B,s}$ and the moduli stack $B_{\Gamma,\Delta}^s$ of *simple multi-scale differentials compatible with an undegeneration of* Δ were constructed in [13, Section 4.2].

3.3. Multi-scale differentials: Prong-matchings and stack structure

The notion of a multi-scale differential is based on the following construction. Given a pointed stable curve $(X, \mathbf{z})$, a *twisted differential* is a collection of differentials η_v on each component X_v of X, that is *compatible with a level structure* on the dual graph Γ of X, i.e., vanishes as prescribed by μ at the marked points z, satisfies the matching order condition at vertical nodes, the matching residue condition at horizontal nodes and global residue condition of [3]. A *multi-scale differential of type* μ on a stable curve $(X, \mathbf{z})$ consists of an enhanced level structure $(\Gamma, \ell, \{\kappa_e\})$ on the dual graph Γ of X, a twisted differential $\boldsymbol{\omega}$ of type μ compatible with the enhanced level structure, and a prong-matching for each node of X joining components of non-equal level. Here a *prong-matching* $\boldsymbol{\sigma}$ is a cyclic order-reversing identification of the horizontal (outgoing resp. incoming) real tangent vectors at a zero resp. a pole corresponding to each vertical edge of Γ. Multi-scale differentials are equivalences classes of $(X, \mathbf{z}, \Gamma, \boldsymbol{\sigma})$ up to the action of the level rotation torus that rescales differentials on lower levels and rotates prong-matchings at the same time.

To an enhanced two-level graph we associate the quantity

$$\ell_\Gamma = \operatorname{lcm}\big(\kappa_e\colon e \in E(\Gamma)\big),$$

which appears in several important places of the construction of $\mathbb{P}\Xi\bar{\mathcal{M}}_{g,n}(\mu)$:

(i) It is the size of the orbit of prong-matchings when rotating the lower level differential.

Closely related:

(ii) The local equations of a node are $xy = t_1^{\ell_\Gamma/\kappa_e}$, where t_1 is a local parameter (a *level parameter*) transverse to the boundary. As a consequence a family of differential forms that tends to a generator on top level scales with $t_1^{\ell_\Gamma}$ on the bottom level of Γ.

For graphs with L level passages we define $\ell_i = \ell_{\Gamma,i} = \ell_{\delta_i(\Gamma)}$ to be the lcm of the edges crossing the i-th level passage and $\ell_\Gamma = \prod_{i=1}^{L} \ell_{\Gamma,i}$.

There are two sources of automorphisms of multi-scale differentials: on the one hand, there are automorphism of pointed stable curves that respect the additional structure (differential, prong-matching). On the other hand, there are *ghost automorphisms*, whose group we denote by $\mathrm{Gh}_\Gamma = \mathrm{Tw}_\Gamma/\mathrm{Tw}_\Gamma^s$, that stem from the toric geometry of the compactification. We emphasize that the twist group Tw_Γ and the simple twist group Tw_Γ^s, hence also the ghost group Gh_Γ, depend only on the data of the enhanced level graph and will be inherited by linear submanifolds below. The local isotropy group $\mathrm{Iso}(X, \boldsymbol{\omega})$ of $\Xi\bar{\mathcal{M}}_{g,n}(\mu)$ sits in an exact sequence

$$0 \to \mathrm{Gh}_\Gamma \to \mathrm{Iso}(X, \boldsymbol{\omega}) \to \mathrm{Aut}(X, \boldsymbol{\omega}) \to 0$$

and locally near $(X, \mathbf{z}, \Gamma, \boldsymbol{\sigma})$ the stack of multi-scale differentials is the quotient stack $[U/\operatorname{Iso}(X, \boldsymbol{\omega})]$ for some open $U \subset \mathbb{C}^{N_B}$, see [9, Remark 2.1]. The same holds for $\mathbb{P}\Xi\bar{\mathcal{M}}_{g,n}(\mu)$ where the automorphism group is potentially larger since $\boldsymbol{\omega}$ is only required to be fixed projectively.

3.4. Decomposition of the logarithmic tangent bundle

We now define a Γ-*adapted basis*, combining [7] and [13] with the goal of giving a decomposition of the logarithmic tangent bundle that is inherited by a linear submanifold, if the Γ-adapted basis is suitably chosen.

We work on a neighborhood $U \subseteq B$ of a point $p = (X, [\omega], \mathbf{z}) \in D_{\Gamma}^{B,\circ}$, where Γ is an arbitrary level graph with L levels below zero. On a nearby smooth surface $(\widetilde{X}, \widetilde{\omega}, \widetilde{\mathbf{z}})$ we let $\alpha_j^{[i]}$ for $i = 0, \ldots, -L$ be the vanishing cycles around the horizontal nodes at level i. Let $\beta_j^{[i]}$ be a dual horizontal-crossing cycle, i.e., i is the top level (in the sense of [7]) of this cycle, $\langle \alpha_j^{[i]}, \beta_j^{[i]} \rangle = 1$ and $\beta_j^{[i]}$ does not have non-zero intersection with any other horizontal vanishing cycle at level i. Let $h(i)$ be the number of those horizontal vanishing cycles at level i.

We complement the cycles $\beta_j^{[i]}$ by a collection of relative cycles $\gamma_j^{[i]}$ such that for any fixed level i their top level restrictions form a basis of the cohomology at level i with punctures at the poles and at horizontal nodes and relative to the zeros of ω quotiented by the subspace of global residue conditions. In particular, the span of the $\gamma_j^{[i]}$ contains the $\alpha_j^{[i]}$, and moreover the union

$$\bigcup_{j=-L}^{0} \{\beta_1^{[j]}, \ldots, \beta_{h(j)}^{[j]}, \gamma_1^{[j]}, \ldots, \gamma_{s(j)}^{[j]}\} \quad \text{is a basis of } H_1(\widetilde{X} \setminus \widetilde{P}, \widetilde{Z}, \mathbb{C}),$$

where $\widetilde{Z} \cup \widetilde{P} = \mathbf{z}$ is the set of zeros and poles of $\widetilde{\omega}$". Next, we define the ω-periods of these cycles and exponentiate to kill the monodromy around the vanishing cycles. The functions

$$a_j^{[i]} = \int_{\alpha_j^{[i]}} \omega, \quad b_j^{[i]} = \int_{\beta_j^{[i]}} \omega, \quad q_j^{[i]} = \exp\bigl(2\pi I b_j^{[i]}/a_j^{[i]}\bigr), \quad c_j^{[i]} = \int_{\gamma_j^{[i]}} \omega$$

are however still not defined on U (only on sectors of the boundary complement) due to monodromy around the vertical nodes.

Coordinates on U are given by *perturbed period coordinates* ([5]), which are related to the periods above as follows. For each level passage there is a *level parameter* t_i that stem from the construction of the moduli space via plumbing. On the bottom level passage L we may take $t_L = c_1^{[-L]}$ as a period. For the higher level passage, the t_i are closely related to the periods of a cycle with top level $-i$, but the latter are in general not monodromy invariant. It will be convenient to write

$$t_{\lceil i \rceil} = \prod_{j=1}^{i} t_j^{\ell_j}, \quad i \in \mathbb{N}.$$

There are perturbed periods $\widetilde{c}_j^{[-i]}$ obtained by integrating $\omega/t_{\lceil i \rceil}$ against a cycle with top level $-i$ over the part of level $-i$ to points nearby the nodes, cutting off the lower level part. By construction, on each sector of the boundary complement we have

$$\widetilde{c}_j^{[-i]} - \frac{c_j^{[-i]}}{t_{\lceil i \rceil}} = \sum_{s>i} \frac{t_{\lceil s \rceil}}{t_{\lceil i \rceil}} E_{j,i}^{[-s]} \tag{3}$$

for some linear ('error') forms $E_{j,i}^{[-s]}$ depending on the variables $c_j^{[-s]}$ on the lower level $-s$. Similarly, we can exponentiate the ratio over $a_j^{[-i]}$ of the similarly perturbed $\widetilde{b}_j^{[-i]}$ and obtain perturbed exponentiated periods $\widetilde{q}_j^{[-i]}$, such that on each sector

$$\log \widetilde{q}_j^{[-i]} - \log q_j^{[-i]} = \sum_{s>i} \frac{t_{\lceil s \rceil}}{t_{\lceil i \rceil}} E'^{[-s]}_{j,i} \tag{4}$$

for some linear forms $E'^{[-s]}_{j,i}$. In these coordinates the boundary is given by $\widetilde{q}_j^{[-i]} = 0$ and $t_i = 0$. If we let

$$\Omega^{\mathrm{hor}}_{i,B}(\log) = \langle d\widetilde{q}_1^{[i]}/\widetilde{q}_1^{[i]}, \ldots, d\widetilde{q}_{h(i)}^{[i]}/\widetilde{q}_{h(i)}^{[i]} \rangle, \quad \Omega^{\mathrm{lev}}_{i,B}(\log) = \langle dt_{-i}/t_{-i} \rangle,$$
$$\Omega^{\mathrm{rel}}_{i,B} = \langle d\widetilde{c}_2^{[i]} \ldots, d\widetilde{c}_{N(i)-h(i)}^{[i]} \rangle,$$

with $\Omega^{\mathrm{lev}}_{0,B}(\log) = 0$ by convention, we thus obtain a decomposition

$$\Omega^1_{\overline{B}}(\log \partial B)|_U = \bigoplus_{i=-L}^{0} \big(\Omega^{\mathrm{hor}}_{i,B}(\log) \oplus \Omega^{\mathrm{lev}}_{i,B}(\log) \oplus \Omega^{\mathrm{rel}}_{i,B} \big). \tag{5}$$

3.5. The closure of linear submanifolds

For a linear submanifold $\mathcal{H}$ we denote by $\overline{\mathcal{H}}$ the normalization of the closure of the image of $\mathcal{H}$ as a substack of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. We denote by $D_\Gamma = D_\Gamma^{\mathcal{H}}$ the preimage of the boundary divisor D_Γ^B in $\overline{\mathcal{H}}$. Again, a $\circ$ denotes the complement of more degenerate boundary strata, i.e., $D_\Gamma^{\mathcal{H},\circ}$ is the preimage of $D_\Gamma^{B,\circ}$ in $\overline{\mathcal{H}}$.

We will now give several propositions that explain that $\overline{\mathcal{H}}$ is a compactification of $\mathcal{H}$ almost as nice as the compactification $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of strata. The first statement explains the 'almost'.

Proposition 3.1. *Let Γ be a level graph with only horizontal nodes, i.e., with one level only. Then $D_\Gamma^{\mathcal{H},\circ}$ has at worst toric singularities.*

More precisely, the linear submanifold is cut out by linear and binomial equations, see (8) below.

Second, the intersection with non-horizontal boundary components is transversal in the strong sense that each level actually causes dimension drop.

Proposition 3.2. *Let $\Gamma \in \mathrm{LG}_L(B)$ be a level graph without horizontal nodes. Each point in $D_\Gamma^{\mathcal{H},\circ}$ is smooth and $D_\Gamma^{\mathcal{H},\circ}$ is a normal crossing divisor given as the intersection of L different divisors $D_{\delta_i(\Gamma)}^{\mathcal{H}}$. In particular, $D_\Gamma^{\mathcal{H}}$ has codimension L in $\overline{\mathcal{H}}$.*

Fix now an enumeration of two level graphs, i.e., a bijection between $\mathrm{LG}_1(B)$ and $\{1, \dots, |\,\mathrm{LG}_1(B)|\}$, and define

$$D_{i_1,\dots,i_L} \subseteq \bigcap_{j=1}^{L} D_{\Gamma_{i_j}}$$

to be the subset of all D_Λ, with $\Lambda \in \mathrm{LG}_L(B)$, such that $\delta_j(\Lambda) = \Gamma_{i_j}$ for all $j = 1, \dots, L$. The previous proposition allows to show, via the same argument as the proof of [13, Proposition 5.1], the key result in order to argue inductively.

Corollary 3.3. *If $\bigcap_{j=1}^{L} D^{\mathcal{H}}_{\Gamma_{i_j}}$ is not empty, there is a unique ordering $\sigma \in \mathrm{Sym}_L$ on the set of indices $I = \{i_1, \dots, i_L\}$ such that*

$$D_{\sigma(I)} = \bigcap_{j=1}^{L} D^{\mathcal{H}}_{\Gamma_{i_j}}.$$

Moreover, if $i_k = i_{k'}$ for a pair of indices $k \neq k'$, then $D_{i_1,\dots,i_L} = \emptyset$.

The next statement is crucial to inductively apply the formulas in this paper. We now need to refine our analysis by looking at irreducible components of $D^{\mathcal{H}}_{\Gamma}$. Recall that $\mathrm{LG}_1^{+}(\mathcal{H})$ is the set of pairs $\Gamma^{+} = (\Gamma, i)$, where $\Gamma \in \mathrm{LG}_1(\mathcal{H})$ and $i \in \{1, \dots, n_\Gamma(\overline{\mathcal{H}})\}$ is the index set of irreducible components of $D^{\mathcal{H},\circ}_{\Gamma}$. We denote by $D^{\mathcal{H}}_{\Gamma+}$ the irreducible component of $D^{\mathcal{H}}_{\Gamma}$ corresponding to Γ^{+}. Recall that p_Γ and c_Γ are the projection and clutching morphisms of the diagram (2).

Proposition 3.4. *There are generalized linear submanifolds $\Omega\mathcal{H}^{[i]}_{\Gamma+} \to \Omega\mathcal{M}^{\mathfrak{R}_i}_{\mathbf{g}_i,\mathbf{n}_i}(\boldsymbol{\mu}_i)$ of dimension d_i with projectivization $\mathcal{H}^{[i],\circ}_{\Gamma+}$, such that*

$$\sum_{i=-L}^{0} d_i = d_{\mathcal{H}} - L$$

and such that the normalizations $\mathcal{H}^{[i]}_{\Gamma+} \to B^{[i]}_{\Gamma}$ of closures of $\mathcal{H}^{[i],\circ}_{\Gamma+}$ together give a product decomposition

$$\mathcal{H}_{\Gamma+} = \prod_{i=-L}^{0} \mathcal{H}^{[i]}_{\Gamma+}$$

of the normalization of the p_Γ-image of the c_Γ-preimage of $\mathrm{Im}(D^{\mathcal{H}}_{\Gamma+}) \subset \mathbb{P}\,\Xi\,\overline{\mathcal{M}}_{g,n}(\mu)$.

We will call $\mathcal{H}^{[i]}_{\Gamma+} \to B^{[i]}_{\Gamma+}$ the *i-th level linear manifold*. Our ultimate goal here is to show the following decomposition. The terminology is explained along with the definition of coordinates.

Proposition 3.5. *Let Γ be an arbitrary level graph with L levels below zero. In a small neighborhood U of a point in $D_\Gamma^{\mathcal{H}}$ there is a direct sum decomposition*

$$\Omega^1_{\overline{\mathcal{H}}}(\log \partial\mathcal{H})|_U = \bigoplus_{i=-L}^{0} \left(\Omega_i^{\mathrm{hor}}(\log) \oplus \Omega_i^{\mathrm{lev}}(\log) \oplus \Omega_i^{\mathrm{rel}}\right) \tag{6}$$

for certain subsheaves such that the natural restriction map induces surjections

$$\Omega^{\mathrm{hor}}_{i,B}(\log)|_{\overline{\mathcal{H}}} \twoheadrightarrow \Omega_i^{\mathrm{hor}}(\log), \quad \Omega^{\mathrm{lev}}_{i,B}(\log)|_{\overline{\mathcal{H}}} \simeq \Omega_i^{\mathrm{lev}}(\log) \quad \textit{and} \quad \Omega^{\mathrm{rel}}_{i,B}|_{\overline{\mathcal{H}}} \twoheadrightarrow \Omega_i^{\mathrm{rel}}.$$

Moreover, the statements in items (i) *and* (ii) *of Section 3.3 hold verbatim for the linear submanifold with the same ℓ_Γ.*

As a consequence we may use the symbols ℓ_Γ and ℓ_{Γ_i} ambiguously for strata and their linear submanifolds.

We summarize the relevant parts of [7]. Equations of $\mathcal{H}$ are interpreted as homology classes and we say that a *horizontal node is crossed by an equation*, if the corresponding vanishing cycle has non-trivial intersection with the equation. The horizontal nodes are partitioned into *$\mathcal{H}$-cross-equivalence classes* by simultaneous appearance in equations for $\mathcal{H}$. A main observation is that ω-periods of the vanishing cycles in an $\mathcal{H}$-cross-equivalence class are proportional. Similarly, for each equation and for any level passage the intersection numbers of the equation with the nodes crossing that level add up to zero when weighted appropriately with the residue times ℓ_Γ/κ_e ([7, Proposition 3.11]).

Next, in [7] they sort the equations by level and then write them in reduced row echelon from. One may order the periods so that the distinguished $c_1^{[i]}$ (whose period is close to the level parameter t_{-i}) is among the pivots of the echelon form for each i. The second main observation is that each defining equation of $\mathcal{H}$ can be split into a sum of defining equations, denoted by $F_k^{[i]}$, with the following properties. The upper index i indicate the highest level, whose periods are involved in the equation. Moreover, either $F_k^{[i]}$ has non-trivial intersection with some (vanishing cycles of a) horizontal node at level i and then no intersection with a horizontal node at lower level, or else no intersection with a horizontal node at all.

As a result, $\mathcal{H}$ is cut out by two sets of equations, see [7, equations (4.2)–(4.4)]. First, there are the equations $G_k^{[i]}$ that are $t_{\lceil -i\rceil}$-rescalings of linear functions

$$G_k^{[i]} = L_k^{[i]}\left(\widetilde{c}_{2-\delta_{i,0}^{[i]}}, \dots, \widetilde{c}^{[i]}_{N(i)-h(i)}\right) \tag{7}$$

in the periods at level i. (To get this form from the version in [7] absorb the terms from lower level periods into the function $c_j^{[i]}$, where $j = j(k,i)$ is the pivot of the equation $F_k^{[i]}$. This does not affect the truth of (3)).

Second, there are multiplicative monomial equations among the exponentiated periods, that can be written as bi-monomial equations with positive exponents

$$H_k^{[i]} = (\widetilde{\mathbf{q}}^{[i]})^{J_{1,k}} - (\widetilde{\mathbf{q}}^{[i]})^{J_{2,k}}, \tag{8}$$

where $\widetilde{\mathbf{q}}^{[i]}$ is the tuple of the variables $\widetilde{q}_j^{[i]}$ and $J_{1,k}$, $J_{2,k}$ are tuples of non-negative integers. (In the multiplicative part, Benirschke–Dozier–Grushevsky [7] already incorporated the lower level blurring into the pivot variable.)

Proof of Proposition 3.1. This follows directly from the form of the binomial equations (8), see [7, Theorem 1.6]. ■

Proof of Proposition 3.2. Smoothness and normal crossing is contained in [7, Corollary 1.8]. The transversality claimed there contains the dimension drop claimed in the proposition. The more precise statement in [7, Theorem 1.5] says that after each intersection of $\overline{\mathcal{H}}$ with a vertical boundary divisor the result is empty or contained in the open boundary divisor $D_\Gamma^{B,\circ}$. ■

Proof of Proposition 3.4. This is the main result of [6] or the restatement in [7, Proposition 3.3], and this together with Proposition 3.2 implies the dimension statement. ■

Proof of Proposition 3.5. Immediate from (7) and (8), which are equations among the respective set of generators of the decomposition in (5). The additional claim in item (ii) follows from the isomorphism of level parameters and transversality. Item (i) is a consequence of this. ■

3.6. Push-pull comparison for linear submanifolds

For recursive computations, we will transfer classes from $\mathcal{H}_{\Gamma+}^{[i]}$, which were defined via Proposition 3.4, to $D_{\Gamma+}^{\mathcal{H}}$ essentially via $p_{\Gamma+}$-pullback and $c_{\Gamma+}$-pushforward. More precisely, taking the normalizations into account, we have to use the maps $c_{\Gamma+,\mathcal{H}}$ and $p_{\Gamma+,\mathcal{H}}$ defined on the normalization $\mathcal{H}_{\Gamma+}^{s}$ of the $c_{\Gamma+}$-preimage of the image of $D_{\Gamma+}^{\mathcal{H}}$ in $D_{\Gamma+}^{B}$. To compute degrees we use the analog of the inner triangle in (2) and give a concrete description of $\mathcal{H}_{\Gamma+}^{s}$.

Recall from the introduction that $K_{\Gamma+}^{\mathcal{H}}$ is the product of the number of prong-matchings on each edge of Γ that are actually contained in $D_{\Gamma+}^{\mathcal{H}}$:

$$\begin{array}{ccccccccc}
(\Omega\mathcal{H}_{\Gamma+}^{\circ})^{\mathrm{pm}} & & & \longrightarrow & \mathcal{H}_{\Gamma+}^{s,\circ} & & & & \\
\big\downarrow & & & \overset{p_{\Gamma+,\mathcal{H}}}{\swarrow} & \big\downarrow & \overset{c_{\Gamma+,\mathcal{H}}}{\searrow} & & & \\
 & & & & B_{\Gamma,\Gamma}^{s} & & & & \\
 & & & \overset{p_\Gamma^\Gamma}{\swarrow} & & \overset{c_\Gamma^\Gamma}{\searrow} & & & \\
\Omega\mathcal{H}_{\Gamma+}^{\circ} & \longrightarrow & \mathcal{H}_{\Gamma+}^{\circ} & \longrightarrow & B_{\Gamma,\Gamma} & & D_\Gamma^{B,\circ} & \longleftarrow & D_{\Gamma+}^{\mathcal{H},\circ}.
\end{array} \tag{9}$$

Consider $\Omega\mathcal{H}^{\circ}_{\Gamma^+} := \prod \Omega\mathcal{H}^{[i]}_{\Gamma^+}$ as a moduli space of differentials subject to some (linear) conditions imposed on its periods. Consider, moreover, the moduli space

$$(\Omega\mathcal{H}^{\circ}_{\Gamma^+})^{\mathrm{pm}} := \Big(\prod \Omega\mathcal{H}^{[i]}_{\Gamma^+}\Big)^{\mathrm{pm}},$$

where we add the additional datum of one of the $K^{\mathcal{H}}_{\Gamma^+}$ prong-matchings reachable from the interior. The torus $(\mathbb{C}^*)^{L+1}$ acts on $\Omega\mathcal{H}^{\circ}_{\Gamma^+}$ with quotient $\mathcal{H}^{\circ}_{\Gamma^+} = \prod \mathcal{H}^{[i],\circ}_{\Gamma^+}$. On the other hand, if we take the quotient of $(\Omega\mathcal{H}^{\circ}_{\Gamma^+})^{\mathrm{pm}}$ by

$$(\mathbb{C}^*)^{L+1} = (\mathbb{C}^*) \times \big(\mathbb{C}^L/\mathrm{Tw}^s_\Gamma\big),$$

we obtain a space $\mathcal{H}^{s,\circ}_{\Gamma^+}$, which is naturally the normalization of a subspace of $U^s_{\Gamma^+}$, since it covers $D^{\mathcal{H},\circ}_{\Gamma^+}$ with marked (legs and) edges and whose generic isotropy group does not stem from Gh_Γ (it might be non-trivial, e.g., if a level of Γ^+ consists of a hyperelliptic stratum), while the generic isotropy group of $D^{\mathcal{H},\circ}_{\Gamma^+}$ is an extension of Gh_Γ by possibly some group of graph automorphisms and possibly isotropy groups of the level strata.

Lemma 3.6. *The ratio of the degrees the maps in (9) on $\mathcal{H}^s_{\Gamma^+}$ is*

$$\frac{\deg(p_{\Gamma^+,\mathcal{H}})}{\deg(c_{\Gamma^+,\mathcal{H}})} = \frac{K^{\mathcal{H}}_{\Gamma^+}}{|\mathrm{Aut}_{\mathcal{H}}(\Gamma^+)|\ell_{\Gamma^+}},$$

where $\mathrm{Aut}_{\mathcal{H}}(\Gamma^+)$ is the subgroup of $\mathrm{Aut}(\Gamma)$ whose induced action on a neighborhood of $D^{\mathcal{H}}_{\Gamma^+}$ preserves $\overline{\mathcal{H}}$ and $\ell_{\Gamma^+} = \ell_\Gamma$.

Proof. We claim that $\deg(p_{\Gamma^+,\mathcal{H}}) = K^{\mathcal{H}}_{\Gamma^+}/[R_\Gamma : \mathrm{Tw}^s_\Gamma]$, where $R_\Gamma \cong \mathbb{Z}^L \subset \mathbb{C}^L$ is the level rotation group. In fact, this follows since in the left quadrilateral in (9) the left vertical arrow has degree $K^{\mathcal{H}}_{\Gamma^+}$ while the bottom arrow is the quotient by $(\mathbb{C}^*) \times ((\mathbb{C}^*)^L/R_\Gamma)$ and the top arrow is quotient by $(\mathbb{C}^*) \times (\mathbb{C}^*)^L/\mathrm{Tw}^s_\Gamma$.

On the other side under the map c^Γ_Γ of the ambient stratum two points have the same image only if they differ by an automorphism of Γ. However, only the subgroup $\mathrm{Aut}_{\mathcal{H}}(\Gamma^+) \subset \mathrm{Aut}(\Gamma)$ acts on $c_{\Gamma^+,\mathcal{H}}(\mathcal{H}^{s,\circ}_{\Gamma^+})$ and its normalization and contributes to the local isotropy group of the normalization. Thus only this subgroup contributes to the degree of $c_{\Gamma^+,\mathcal{H}}$. The claimed equality now follows because $[R_\Gamma : \mathrm{Tw}^s_\Gamma] = \ell_\Gamma$. ∎

Consider $\Gamma^+ = (\Gamma, j) \in \mathrm{LG}^+_L(\mathcal{H})$ and $\Delta^+ \in \mathrm{LG}^+_1(\mathcal{H}^{[i]}_{\Gamma^+})$ defining an irreducible component of a divisor in $\mathcal{H}^{[i]}_{\Gamma^+}$. We aim to compute its pullback to $D^s_{\Gamma^+}$ and the push forward to D_{Γ^+} and to $\overline{\mathcal{H}}$. For this purpose we need extend the commensurability diagram (9) to include degenerations of the boundary strata. This works by copying verbatim the construction that lead in [13] to the commensurability diagram (2). We will indicate with subscripts $\mathcal{H}$ to the morphisms that we work in this adapted setting.

Recall from this construction that in $B^s_{\Gamma,\Gamma}$ (and hence in $\mathcal{H}^{s,\circ}_{\Gamma+}$) the edges of Γ have been labeled once and for all (we write $\Gamma^\dagger$ for this labeled graph) and that the level strata $\mathcal{H}^{[i]}_{\Gamma+}$ inherit these labels. Consequently, there is a unique irreducible component $D_{\widetilde{\Delta}^\dagger_+}$ associated to a level graph $\widetilde{\Delta}^\dagger$ which is a degeneration of $\Gamma^\dagger$ and such that the products of the levels i and $i-1$ of $D_{\widetilde{\Delta}^\dagger_+}$ equals $\mathcal{H}^\circ_{\Delta+}$. The resulting refined unlabeled graph will simply be denoted by $\widetilde{\Delta}^+$. For a fixed labeled graph $\Gamma^\dagger$ we denote by $J(\Gamma^\dagger, \widetilde{\Delta}^+)$ the set of $\Delta^+ \in \mathrm{LG}^+_1(\mathcal{H}^{[i]}_{\Gamma+})$ such that $\widetilde{\Delta}^+$ is the result of that procedure. Obviously the graphs in $J(\Gamma^\dagger, \widetilde{\Delta}^+)$ differ only by the labeling of their half-edges and the following lemma computes its cardinality.

Lemma 3.7. *The cardinality of $J(\Gamma^\dagger, \widetilde{\Delta}^+)$ is determined by*

$$|J(\Gamma^\dagger, \widetilde{\Delta}^+)| \cdot |\operatorname{Aut}_{\mathcal{H}}(\widetilde{\Delta}^+)| = |\operatorname{Aut}_{\mathcal{H}^{[i]}_{\Gamma+}}(\Delta^+)| \cdot |\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)|.$$

Proof. The proof is analogous to the one of [13, Lemma 4.6], where one considers the kernel and cokernel of the map

$$\varphi\colon \operatorname{Aut}_{\mathcal{H}}(\widetilde{\Delta}^+) \to \operatorname{Aut}_{\mathcal{H}}(\Gamma^+)$$

given by undegeneration. ■

We now determine the multiplicities of the push-pull procedure. Recall from Section 3.3 the definition of $\ell_{\Gamma,j} = \ell_{\delta_j(\Gamma)}$ for $j \in \mathbb{Z}_{\geq 1}$.

Proposition 3.8. *For a fixed $\Delta^+ \in \mathrm{LG}^+_1(\mathcal{H}^{[i]}_\Gamma)$, the divisor classes of $D^{\mathcal{H}}_{\widetilde{\Delta}+}$ and the clutching of $D^{\mathcal{H}}_{\Delta+}$ are related by*

$$\frac{|\operatorname{Aut}_{\mathcal{H}}(\widetilde{\Delta}^+)|}{|\operatorname{Aut}_{\mathcal{H}^{[i]}_{\Gamma+}}(\Delta^+)||\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)|} \cdot c^*_{\Gamma+,\mathcal{H}}[D^{\mathcal{H}}_{\widetilde{\Delta}+}] = \frac{\ell_\Delta}{\ell_{\widetilde{\Delta},-i+1}} \cdot p^{[i],*}_{\Gamma+,\mathcal{H}}[D^{\mathcal{H}}_{\Delta+}] \tag{10}$$

in $\mathrm{CH}^1(\mathcal{H}^s_{\Gamma+})$, and consequently by

$$\frac{|\operatorname{Aut}_{\mathcal{H}}(\widetilde{\Delta}^+)|}{|\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)|} \cdot \ell_{\widetilde{\Delta},-i+1} \cdot [D^{\mathcal{H}}_{\widetilde{\Delta}+}] = \frac{|\operatorname{Aut}_{\mathcal{H}^{[i]}_{\Gamma+}}(\Delta^+)|}{\deg(c_{\Gamma+,\mathcal{H}})} \cdot \ell_\Delta \cdot c_{\Gamma+,\mathcal{H},*}\big(p^{[i],*}_{\Gamma,\mathcal{H}}[D^{\mathcal{H}}_{\Delta+}]\big) \tag{11}$$

in $\mathrm{CH}^1(D_{\Gamma+})$.

Here (10) is used later for the proofs of the main theorems while (11) is implemented in `diffstrata` for the special case of k-differentials to compute the pullback of tautological classes from $D^{\mathcal{H}}_{\Delta+}$ to $D^{\mathcal{H}}_{\widetilde{\Delta}+}$, see also Section 7.

Proof. The proof is similar to the one of [13, Proposition 4.7] and works by comparing the ramification orders of the maps $c^{\widetilde{\Delta}+}_{\Gamma+,\mathcal{H}}$ and $p^{\widetilde{\Delta}+}_{\Gamma+,\mathcal{H}}$. The main difference to the original proof is only that the automorphism factors appearing in the clutching morphisms are the ones fixing irreducible components of $\mathcal{H}$. ■

The final part of this section is to compare various natural vector bundles under pullback along the maps $c_{\Gamma+,\mathcal{H}}$ and $p_{\Gamma+,\mathcal{H}}$. The first bundle we consider is $\mathcal{E}^{\top}_{\Gamma+}$, a vector bundle of rank $N^{\top}_{\Gamma+} - 1$ on $D^{\mathcal{H}}_{\Gamma+}$ that should be thought of as the top level version of the logarithmic cotangent bundle. Formally, let $U \subset D^{\mathcal{H}}_{\Gamma+}$ be an open set centered at a degeneration of the top level of Γ^+ into k level passages. Then we define

$$\mathcal{E}^{\top}_{\Gamma+|U} = \bigoplus_{i=-k}^{0} \Omega^{\mathrm{lev}}_i(\log)_{|U} \oplus \Omega^{\mathrm{hor}}_i(\log)_{|U} \oplus \Omega^{\mathrm{rel}}_i{}_{|U}. \tag{12}$$

Let, moreover, $\xi^{[i]}_{\Gamma+,\mathcal{H}}$ be the first Chern class of the line bundle on $D^{\mathcal{H}}_{\Gamma+}$ generated by the multi-scale component at level i and $\mathcal{L}^{[i]}_{\Gamma+}$ be the line bundle whose divisor is given by the degenerations of the i-th level of Γ^+, as defined more formally in (16) below.

We have the following compatibilities.

Lemma 3.9. *The first Chern classes of the tautological bundles on the levels of a boundary divisor are related by*

$$c^*_{\Gamma+,\mathcal{H}}\, \xi^{[i]}_{\Gamma+,\mathcal{H}} = p^{[i],*}_{\Gamma+,\mathcal{H}}\, \xi_{\mathcal{H}^{[i]}_{\Gamma+}} \quad \text{in } \mathrm{CH}^1(\mathcal{H}^s_{\Gamma+}).$$

It is also true that

$$p^{[i]*}_{\Gamma+,\mathcal{H}} \mathcal{L}_{\mathcal{H}^{[i]}_{\Gamma+}} = c^*_{\Gamma+,\mathcal{H}} \mathcal{L}^{[i]}_{\Gamma+}, \quad \text{where } \mathcal{L}_{\mathcal{H}^{[i]}_{\Gamma+}} = \mathcal{O}_{\mathcal{H}^{[i]}_{\Gamma+}}\Bigg(\sum_{\Delta \in \mathrm{LG}_1(\mathcal{H}^{[i]}_{\Gamma+})} \ell_\Delta D_\Delta\Bigg).$$

Similarly for the logarithmic cotangent bundles, we have

$$p^{[0],*}_{\Gamma+,\mathcal{H}} \Omega^1_{\mathcal{H}^{[0]}_{\Gamma+}}\big(\log D_{\mathcal{H}^{[0]}_{\Gamma+}}\big) = c^*_{\Gamma+,\mathcal{H}} \mathcal{E}^{\top}_{\Gamma+,\mathcal{H}}.$$

Proof. The first claim is just the global compatibility of the definitions of the bundles $\mathcal{O}(-1)$ on various spaces, compare [13, Proposition 4.9].

The second claim is a formal consequence of Lemma 3.7 and Proposition 3.8, just as in [13, Lemma 7.4].

The last claim follows as in [13, Lemma 9.6] by considering local generators, which are given in equation (12) and have for linear submanifolds the same shape as for strata. ■

In the final formulas we will use these compatibilities together with the following restatement of Lemma 3.6.

Lemma 3.10. *Suppose that* $\alpha_{\Gamma^+} \in \mathrm{CH}_0(D^{\mathcal{H}}_{\Gamma^+})$ *is a top degree class and that*

$$c^*_{\Gamma^+,\mathcal{H}}\alpha_{\Gamma^+} = \prod_{i=0}^{-L(\Gamma^+)} p^{[i],*}_{\Gamma^+,\mathcal{H}}\alpha_i$$

for some α_i*. Then*

$$\int_{D^{\mathcal{H}}_{\Gamma^+}} \alpha_{\Gamma^+} = \frac{K^{\mathcal{H}}_{\Gamma^+}}{|\operatorname{Aut}_{\mathcal{H}}(\Gamma^+)|\ell_{\Gamma^+}} \prod_{i=0}^{-L(\Gamma^+)} \int_{\mathcal{H}^{[i]}_{\Gamma^+}} \alpha_i .$$

4. Evaluation of tautological classes

This section serves two purposes. First, we briefly sketch a definition of the tautological ring of linear submanifolds and how the results of the previous section can be used to evaluate expressions in the tautological ring, provided the classes of the linear manifold are known. Second, we provide formulas to compute the first Chern class of the normal bundle $\mathcal{N}^{\mathcal{H}}_{\Gamma} = \mathcal{N}_{D^{\mathcal{H}}_{\Gamma}}$ to a boundary divisor $D^{\mathcal{H}}_{\Gamma}$ of a projectivized linear submanifold $\overline{\mathcal{H}}$. This is needed both for the evaluation algorithm and as an ingredient to prove our main theorems.

4.1. Vertical tautological ring

We denote by $\psi_i \in \mathrm{CH}^1(\overline{\mathcal{H}})$ the pullbacks of the classes $\psi_i \in \mathrm{CH}^1(\overline{\mathcal{M}}_{g,n})$ to a linear submanifold $\overline{\mathcal{H}}$. The *clutching maps* are defined as $\mathrm{cl}_{\Gamma^+,\mathcal{H}} = \mathrm{i}_{\Gamma^+,\mathcal{H}} \circ c_{\Gamma^+,\mathcal{H}}$, where $\mathrm{i}_{\Gamma^+,\mathcal{H}}\colon D^{\mathcal{H}}_{\Gamma^+} \to \overline{\mathcal{H}}$ is the inclusion map of an irreducible components of the boundary divisor. We define the refined *(vertical) tautological ring* $R^\bullet_v(\overline{\mathcal{H}})$ *of* $\overline{\mathcal{H}}$ to be the ring with additive generators

$$\mathrm{cl}_{\Gamma^+,\mathcal{H},*}\left(\prod_{i=0}^{-L} p^{[i],*}_{\Gamma^+,\mathcal{H}}\alpha_i\right), \tag{13}$$

where Γ^+ runs over all irreducible components of boundary components associated to level graphs without horizontal edges for all boundary strata of $\mathcal{H}$, including the trivial graph, and where α_i is a monomial in the ψ-classes supported on level i of the graph Γ^+. That this is indeed a ring follows from the excess intersection formula [13, Proposition 8.1] that works exactly the same for linear submanifolds, and the normal bundle formula Proposition 4.4 which allows together with Proposition 4.1 to rewrite products in terms of our standard generators. We do not claim that pushforward $R^\bullet_v(\overline{\mathcal{H}}) \to \mathrm{CH}^\bullet(\overline{\mathcal{M}}_{g,n})$ maps to the tautological ring $R^\bullet(\overline{\mathcal{M}}_{g,n})$, since the fundamental classes of linear submanifolds, e.g., loci of double covers of elliptic curves, may be non-tautological in $\overline{\mathcal{M}}_{g,n}$ (see, e.g., [29]).

If $\alpha \in \mathrm{CH}_0(\overline{\mathcal{H}})$ is a top-degree class which is also an additive generator of the tautological ring, i.e., it is has an expression as in (13), we can apply Lemma 3.10 to obtain

$$\int_{\overline{\mathcal{H}}} \alpha = \int_{\overline{\mathcal{H}}} \mathrm{cl}_{\Gamma^+,\mathcal{H},*}\left(\prod_{i=0}^{-L} p_{\Gamma^+,\mathcal{H}}^{[i],*}\alpha_i\right) = \frac{K_{\Gamma^+}^{\mathcal{H}}}{|\mathrm{Aut}_{\mathcal{H}}(\Gamma^+)|\ell_{\Gamma^+}} \prod_{i=0}^{-L(\Gamma^+)} \int_{\mathcal{H}_{\Gamma^+}^{[i]}} \alpha_i .$$

To evaluate this expression, one needs to determine the fundamental classes of the level linear submanifolds $\mathcal{H}_{\Gamma^+}^{[i]}$ in their corresponding generalized strata, which is in general a non-trivial task.

In the case where $\alpha \in \mathrm{CH}_0(\overline{\mathcal{H}})$ is a special top-degree class supported on a full boundary stratum D_Γ, and not only on one of its components D_{Γ^+}, there is a possibly different way to evaluate it. Indeed, note first that

$$\alpha = \mathrm{cl}_{\Gamma,\mathcal{H},*}\left(\prod_{i=0}^{-L} p_{\Gamma,\mathcal{H}}^{[i],*}\left(\prod_{j=1}^{l(i)} \psi_j^{p_j}\right)\right) = \psi_1^{p_1}\cdots\psi_n^{p_n}\cdot[D_\Gamma^{\mathcal{H}}]$$

since the ψ classes are compatible under clutchings and projections.

If one knows the class $[\overline{\mathcal{H}}] \in \mathrm{CH}_{\dim(\mathcal{H})}(\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu))$ and this class happens to be tautological, one may evaluate

$$\int_{\overline{\mathcal{H}}} \alpha = \int_{\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)} \psi_1^{p_1}\cdots\psi_n^{p_n}\cdot[D_\Gamma]\cdot[\overline{\mathcal{H}}]$$

using the methods described in [13]. This has the advantage of not requiring the computation of the classes of all the level linear submanifolds $\mathcal{H}_\Gamma^{[i]}$.

4.2. Evaluation of $\xi_{\mathcal{H}}$

If we want to evaluate a top-degree class in $\mathrm{CH}_0(\overline{\mathcal{H}})$ that is not just a product of ψ-classes and a boundary stratum, but also involves the $\xi_{\mathcal{H}}$-class, we can reduce to the previous case by applying the following proposition.

Proposition 4.1. *The class $\xi_{\mathcal{H}}$ on the closure of a projectivized linear submanifold $\overline{\mathcal{H}}$ can be expressed as*

$$\xi_{\mathcal{H}} = (m_i+1)\psi_i - \sum_{\Gamma\in_i \mathrm{LG}_1(\mathcal{H})} \ell_\Gamma[D_\Gamma^{\mathcal{H}}],$$

where ${}_i\mathrm{LG}_1(\mathcal{H})$ are two-level graphs with the leg i on the lower level.

Proof. The formula is obtained by pulling-back the formula in [13, Proposition 8.1] to $\overline{\mathcal{H}}$ and thereby using the transversality statement from Proposition 3.2. ■

We remark here that in some cases it is possible to directly evaluate the top $\xi_{\mathcal{H}}$-powers by using that we can represent the powers of the $\xi_{\mathcal{H}}$-class via an explicit closed current.

Let $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ be a *holomorphic stratum*, i.e., a stratum of flat surfaces of finite area or equivalently all the entries of μ are non-negative. Then there is a canonical hermitian metric on the tautological bundle $\mathcal{O}_{\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)}(-1)$ given by the flat area form

$$h(X,\omega,\mathbf{z}) = \mathrm{area}_X(\omega) = \frac{i}{2}\int_X \omega \wedge \overline{\omega},$$

which extends to a singular hermitian metric of the tautological bundle on the space $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. If $\overline{\mathcal{H}} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is the compactification of a linear submanifold of such a holomorphic stratum, then the area metric induces a singular hermitian metric, which we denote again by h, on the pullback $\mathcal{O}_{\overline{\mathcal{H}}}(-1)$ of the tautological bundle to $\overline{\mathcal{H}}$. Recall from Proposition 3.1 (combined with the level-wise decomposition in Proposition 3.4) that the singularities of $\overline{\mathcal{H}}$ are toric. Let $\overline{\mathcal{H}}^{\mathrm{tor}} \to \overline{\mathcal{H}}$ be a resolution of singularities which is locally toric.

Proposition 4.2. *Let $\overline{\mathcal{H}}^{\mathrm{tor}} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ be a resolution of a compactified linear submanifold of a holomorphic stratum. The curvature form $\frac{i}{2\pi}[F_h]$ of the pull metric h to $\overline{\mathcal{H}}^{\mathrm{tor}}$ is a closed current that represents the first Chern class $c_1(\mathcal{O}_{\overline{\mathcal{H}}^{\mathrm{tor}}}(-1))$. More generally, the d-th wedge power of the curvature form represents $c_1(\mathcal{O}_{\overline{\mathcal{H}}^{\mathrm{tor}}}(-1))^d$ for any $d \geq 1$.*

Proof. In [15, Proposition 4.3] it was shown that on the neighborhood U of a boundary point of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ in the interior of the stratum D_Γ the metric h has the form

$$h(X,q) = \sum_{i=0}^{L} |t_{\lceil i\rceil}|^2 \big(h^{\mathrm{tck}}_{(-i)} + h^{\mathrm{ver}}_{(-i)} + h^{\mathrm{hor}}_{(-i)}\big), \tag{14}$$

where $h^{\mathrm{tck}}_{(-i)}$ (coming from the 'thick' part) are smooth positive functions bounded away from zero and

$$h^{\mathrm{ver}}_{(-i)} := -\sum_{p=1}^{i} R^{\mathrm{ver}}_{(-i),p} \log|t_p|, \quad h^{\mathrm{hor}}_{(-i)} := -\sum_{j=1}^{E^h_{(-i)}} R^{\mathrm{hor}}_{(-i),j} \log|q_j^{[i]}|, \tag{15}$$

where $R^{\mathrm{ver}}_{(-i),p}$ is a smooth non-negative function and $R^{\mathrm{hor}}_{(-i),j}$ is a smooth positive function bounded away from zero, both involving only perturbed period coordinates on levels $-i$ and below.

The statement of the proposition in loc. cit. follows by formal computations from the shape of (14) and the properties of its coefficients, see [15, Propositions 4.4 and 4.5]. We thus only need to show that in local coordinates of a point in $\overline{\mathcal{H}}^{\rm tor}$ (mapping to the given stratum D_Γ) the metric has the same shape (14). For this purpose, recall that by Proposition 3.4, the level parameters t_i are among the coordinates. On the other hand, a toric resolution of the toric singularities arising from (8) is given by fan subdivision and thus by a collection of variables $y_j^{[i]}$ for each level i, each of which is a product of integral powers of the $q_j^{[i]}$ at that level i. Conversely, the map $\overline{\mathcal{H}}^{\rm tor} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is given locally by $q_j^{[i]} = \prod_k (y_k^{[i]})^{b_{i,j,k}}$ for some $b_{i,j,k} \in \mathbb{Z}_{\geq 0}$, not all of the $b_{i,j,k} = 0$ for fixed (i, j). Plugging this into (14) and (15) gives an expression of the same shape and with coefficients satisfying the same smoothness and positivity properties. Mimicking the proof in loc. cit. thus implies the claim. ∎

For a linear submanifold $\mathcal{H}$ consider the vector space given in local period coordinates by the intersection of the tangent space of the unprojectivized linear submanifold with the span of relative periods. We call this space the REL space of $\mathcal{H}$ and we denote by $R_{\mathcal{H}}$ its dimension.

Using Proposition 4.2 we can now generalize the result about vanishing of top ξ-powers on non-minimal strata of differentials to linear submanifolds with non-zero REL (see [50, Proposition 3.3] for the holomorphic Abelian strata case).

Corollary 4.3. *Let $\overline{\mathcal{H}} \to \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ be a linear submanifold of a holomorphic stratum. Then*

$$\int_{\overline{\mathcal{H}}} \xi_{\overline{\mathcal{H}}}^i \alpha = 0 \quad \text{for } i \geq d_{\mathcal{H}} - R_{\mathcal{H}} + 1,$$

where $d_{\mathcal{H}}$ is the dimension of $\mathcal{H}$ and $R_{\mathcal{H}}$ is the dimension of the REL space and where α is any class of dimension $d_{\mathcal{H}} - i$.

Proof. Since the area is given by an expression in absolute periods, the pullback of ξ to $\overline{\mathcal{H}}^{\rm tor}$ is represented by Proposition 4.2 by a $(1, 1)$-form involving only absolute periods (see [50, Lemma 2.1] for the explicit expression in the case of strata). Taking a wedge power that exceeds the dimension of the space of absolute periods gives zero. ∎

4.3. Normal bundles

Finally, we state the normal bundle formula, which is necessary to evaluate self-intersections, which is for example needed to evaluate powers of $\xi_{\mathcal{H}}$. More generally, we provide formulas for the normal bundle of an inclusion

$$\mathrm{j}_{\Gamma^+,\Pi^+} \colon D_{\Gamma^+}^{\mathcal{H}} \hookrightarrow D_{\Pi^+}^{\mathcal{H}}$$

between irreducible components of non-horizontal boundary strata of relative codimension one, say defined by the L-level graph Π and one of its $(L+1)$-level graph degenerations Γ. This generalization is needed for recursive evaluations. Such an inclusion is obtained by splitting one of the levels of Π^+, say, for example, the level $i \in \{0, -1, \ldots, -L\}$. Here we use the structure of the equations cutting out the linear manifold in Section 3.4 to observe that j is a regular embedding, in fact with ideal sheaf locally generated by the parameter t_i, to talk about normal bundles (as opposed to merely normal sheaves). In particular, these regular embeddings j and thus also their compositions i come with classes in operational Chow groups (see [25, Section 17] for background and, e.g., [1, Section 2] for the extension to stacks). This is the language that justifies all the intersection theory we need working on the (singular) stack $\overline{\mathcal{H}}$. We do not reflect this in our notation of Chow groups since for the morphisms we consider, all formulas of the classical setting carry over. We define

$$\mathcal{L}_{\Gamma^+}^{[i]} = \mathcal{O}_{D_{\Gamma^+}^{\mathcal{H}}}\Bigg(\sum_{\Gamma^+ \overset{[i]}{\rightsquigarrow} \widetilde{\Delta}^+} \ell_{\widetilde{\Delta}^+,-i+1} D_{\widetilde{\Delta}^+}^{\mathcal{H}}\Bigg) \quad \text{for any } i \in \{0, -1, \ldots, -L\}, \tag{16}$$

where the sum is over all refined graphs $\widetilde{\Delta}^+ \in \mathrm{LG}_{L+2}^+(\mathcal{H})$ that yield divisors in $D_{\Gamma^+}^{\mathcal{H}}$ by splitting the i-th level, which in terms of undegenerations means $\delta_{-i+1}^{\complement}(\widetilde{\Delta}^+) = \Gamma^+$. The following result contains the formula for the normal bundle as the special case where Π is the trivial graph.

Proposition 4.4. *For $\Pi^+ \overset{[i]}{\rightsquigarrow} \Gamma^+$ (or equivalently for $\delta_{-i+1}^{\complement}(\Gamma^+) = \Pi^+$), the Chern class of the normal bundle $\mathcal{N}_{\Gamma^+,\Pi^+}^{\mathcal{H}} := \mathcal{N}_{D_{\Gamma^+}^{\mathcal{H}}/D_{\Pi^+}^{\mathcal{H}}}$ is given by*

$$c_1\big(\mathcal{N}_{\Gamma^+,\Pi^+}^{\mathcal{H}}\big) = \frac{1}{\ell_{\Gamma,(-i+1)}}\big(-\xi_{\Gamma^+,\mathcal{H}}^{[i]} - c_1\big(\mathcal{L}_{\Gamma^+,\mathcal{H}}^{[i]}\big) + \xi_{\Gamma^+,\mathcal{H}}^{[i-1]}\big) \quad \text{in } \mathrm{CH}^1(D_{\Gamma^+}^{\mathcal{H}}).$$

Proof. We use the transversality statement Proposition 3.2 of $\mathcal{H}$ with a boundary stratum $D_{\Gamma^+}^{B}$ in order to have that the transversal parameter is given by t_i. Then the proof is the same as the one in the case of Abelian strata, see [13, Proposition 7.5]. ■

Since in Section 8 we will need to compute the normal bundle to horizontal divisors for strata of k-differentials, we provide here the general formula for the case of smooth horizontal degenerations of linear submanifolds.

Proposition 4.5. *Let $D_h^{\mathcal{H}} \subset D^{\mathcal{H}}$ be a divisor in a boundary component $D^{\mathcal{H}}$ obtained by horizontal degeneration. Suppose that the linear submanifold is smooth along $D_h^{\mathcal{H}}$ and let e be one of the new horizontal edges in the level graph of $D_h^{\mathcal{H}}$. Then the first Chern class of the normal bundle $\mathcal{N}_{D_h}^{\mathcal{H}}$ is given by*

$$c_1(\mathcal{N}_{D_h}^{\mathcal{H}}) = -\psi_{e^+} - \psi_{e^-} \in \mathrm{CH}^1(D^{\mathcal{H}}),$$

where e^+ and e^- are the half-edges associated to the two ends of e.

Proof. Similarly to the proof of [13, Proposition 7.2], consider the divisor D_e in $\overline{\mathcal{M}}_{g,n}$ corresponding to the single edge e and denote by $\mathcal{N}_e$ its normal bundle. The forgetful map $f\colon D_h \to D_e$ induces an isomorphism $\mathcal{N}^{\mathcal{H}}_{D_h} \to f^*\mathcal{N}_{D_e}$ (compare local generators!) and the formula follows from the well-known expression of $\mathcal{N}_{D_e}$ in terms of ψ-classes. ∎

We will need the following result about pullbacks of normal bundles to apply the same arguments as in [13] recursively over inclusions of boundary divisors. The proof is the same as in [13, Corollary 7.7], since it follows from Proposition 4.4 that we can j-pullback properties of ξ and $\mathcal{L}^{[i]}_{\Gamma}$ that hold on the whole stratum and hence on linear submanifolds.

Lemma 4.6. *Let $\Gamma^+ \in \mathrm{LG}^+_L(\mathcal{H})$ and let $\widetilde{\Delta}^+$ be a codimension one degeneration of the $(-i+1)$-th level of Γ^+, i.e., such that $\Gamma^+ = \delta^{\complement}_i(\widetilde{\Delta}^+)$ for some $i \in \{1, \ldots, L+1\}$. Then*

$$\mathrm{j}^*_{\widetilde{\Delta}^+,\Gamma^+}\big(\ell_{\Gamma,j}\, c_1\big(\mathcal{N}^{\mathcal{H}}_{\Gamma^+/\delta^{\complement}_j(\Gamma^+)}\big)\big) = \begin{cases} \ell_{\widetilde{\Delta},j}\, c_1\big(\mathcal{N}^{\mathcal{H}}_{\widetilde{\Delta}^+/\delta^{\complement}_j(\widetilde{\Delta}^+)}\big) & \text{for } j < i, \\ \ell_{\widetilde{\Delta}^+,j+1}\, c_1\big(\mathcal{N}^{\mathcal{H}}_{\widetilde{\Delta}^+/\delta^{\complement}_{(j+1)}(\widetilde{\Delta}^+)}\big) & \text{otherwise.} \end{cases}$$

5. Chern classes of the cotangent bundle via the Euler sequence

The core of the computation of the Chern classes is given by two exact sequences that are the direct counterparts of the corresponding theorems for Abelian strata. The proof should be read in parallel with [13, Section 6 and 9] and we mainly highlight the differences and where the structure theorems of the compactification from Section 3.5 are needed.

Theorem 5.1. *There is a vector bundle $\mathcal{K}$ on $\overline{\mathcal{H}}$ that fits into an exact sequence*

$$0 \to \mathcal{K} \xrightarrow{\psi} (\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_{\overline{\mathcal{H}}}(-1) \xrightarrow{\mathrm{ev}} \mathcal{O}_{\overline{\mathcal{H}}} \to 0,$$

where $\overline{\mathcal{H}}^1_{\mathrm{rel}}$ is the Deligne extension of the local subsystem that defines the tangent space to $\Omega\mathcal{H}$ inside the relative cohomology $\overline{\mathcal{H}}^1_{\mathrm{rel},B}|_{\overline{\mathcal{H}}}$, such that the restriction of $\mathcal{K}$ to the interior $\mathcal{H}$ is the cotangent bundle $\Omega^1_{\mathcal{H}}$ and for U as in Proposition 3.5, we have

$$\mathcal{K}|_U = \bigoplus_{i=-L}^{0} t_{\lceil -i \rceil} \cdot \big(\Omega^{\mathrm{hor}}_i(\log) \oplus \Omega^{\mathrm{lev}}_i(\log) \oplus \Omega^{\mathrm{rel}}_i\big).$$

The definition of the evaluation map and the notion of Deligne extension on a stack with toric singularities requires justification given in the proof. For the next result we define the abbreviations

$$\mathcal{E}_{\mathcal{H}} = \Omega^1_{\overline{\mathcal{H}}}(\log \overline{\partial}\mathcal{H}) \quad \text{and} \quad \mathcal{L}_{\mathcal{H}} = \mathcal{O}_{\overline{\mathcal{H}}}\Big(\sum_{\Gamma \in \mathrm{LG}_1(\mathrm{B})} \ell_\Gamma D^{\mathcal{H}}_\Gamma \Big)$$

that are consistent with the level-wise definitions in (12) and (16).

Theorem 5.2. *There is a short exact sequence of quasi-coherent $\mathcal{O}_{\overline{\mathcal{H}}}$-modules*

$$0 \to \mathcal{E}_{\mathcal{H}} \otimes \mathcal{L}^{-1}_{\mathcal{H}} \to \mathcal{K} \to \mathcal{C} \to 0, \tag{17}$$

where $\mathcal{C} = \bigoplus_{\Gamma \in \mathrm{LG}_1(\mathcal{H})} \mathcal{C}_\Gamma$ is a coherent sheaf supported on the non-horizontal boundary divisors, whose precise form is given in Proposition 5.4 below.

Proof of Theorem 5.1. We start with the definition of the maps in the Euler sequence for the ambient stratum, see the middle row in the commutative diagram below. It uses the evaluation map

$$\mathrm{ev}_B \colon (\overline{\mathcal{H}}^1_{\mathrm{rel},B})^\vee \otimes \mathcal{O}_{\overline{B}}(-1) \to \mathcal{O}_{\overline{B}}, \quad \gamma \otimes \omega \mapsto \int_\gamma \omega,$$

restricted to $\overline{\mathcal{H}}$. The first map in the sequence is

$$\mathrm{d}c_i \mapsto \Big(\gamma_i - \frac{c_i}{c_k}\alpha_k\Big) \otimes \omega, \quad i = 1, \dots, \widehat{k}, \dots, N, \tag{18}$$

as usual in the Euler sequence, on a chart of $\mathcal{H}$ where c_k is non-zero. The exactness of the middle row is the content of [13, Theorem 6.1].

We next define the sheaf Eq. In the interior, Eq is the local system of equations cutting out $\Omega\mathcal{H}$, and thus the quotient $(\mathcal{H}^1_{\mathrm{rel}})^\vee = (\mathcal{H}^1_{\mathrm{rel},B})^\vee/\mathrm{Eq}$ is the relative homology local system, by definition of a linear manifold. The proof in [13, Section 6.1] concerning the restriction of the sequence to the interior $\mathcal{H}$ uses that $\mathcal{H}$ has a linear structure with tangent space modeled on the local system $\mathbb{H}^1_{\mathrm{rel}}$. In particular, it gives the claim about $\mathcal{K}|_{\mathcal{H}}$.

As an interlude, we introduce notation for the Deligne extension of $(\mathcal{H}^1_{\mathrm{rel},B})^\vee$. For each $\gamma^{[i]}_j$, we let $\widehat{\gamma}^{[i]}_j$ be it extension, the sum of the original cycles and vanishing cycles times logarithms of the coordinates of the boundary divisors to kill monodromies. The functions

$$\widehat{c}^{[i]}_j = \frac{1}{t_{\lceil -i \rceil}} \int_{\widehat{\gamma}^{[i]}_j} \omega$$

are called *log periods* in [7].

We now *define* Eq at the boundary, say locally near a point $p \in D_\Gamma$, to be the subsheaf of $(\overline{\mathcal{H}}^1_{\mathrm{rel},B})^\vee$ generated by the defining equations $F^{[i]}_k$ constructed in Section 3.5, but with each variable replaced by its Deligne extension. It requires justification that this definition near the boundary agrees with the previous definition in the interior. We can verify this for the distinguished basis consisting of the $F_k^{[i]}$. Equations that do not intersect horizontal nodes agree with their Deligne extension. This cancellation of the compensation terms is [7, Proposition 3.11] (see also the expression for $F_k^{[i]}$ after [7, Proposition 4.1]) which displays the ω-integrals of the terms to be compared. For equations $F_k^{[i]}$ that do intersect horizontal nodes (thus only at level i by construction) the difference $F_k^{[i]}(c_j^{[s]}, \text{all } (j,s)) - F_k^{[i]}(\widehat{c}_j^{[s]}, \text{all } (j,s))$ vanishes thanks to the proportionality of the periods of horizontal nodes in an $\mathcal{H}$-equivalence class and since on $\overline{\mathcal{H}}$ the equation $H_k^{[i]}$ holds.

By the very definition of defining equation its periods evaluate to zero, explaining the right arrow in the top row of the following diagram and showing that ev is well defined on the quotient:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_{\mathrm{Eq}} & \longrightarrow & \mathrm{Eq} \otimes \mathcal{O}_{\overline{\mathcal{H}}}(-1) & \longrightarrow & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{i=0}^{L} t_{\lceil -i \rceil} \cdot \Omega_B^{[i]}|_{\overline{\mathcal{H}}} & \xrightarrow{\psi} & (\overline{\mathcal{H}}^1_{\mathrm{rel},B})^\vee \otimes \mathcal{O}_{\overline{\mathcal{H}}}(-1) & \xrightarrow{\mathrm{ev}_B} & \mathcal{O}_{\overline{\mathcal{H}}} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle q_\Omega} & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{i=0}^{L} t_{\lceil -i \rceil} \cdot \Omega^{[i]} & \longrightarrow & (\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_{\overline{\mathcal{H}}}(-1) & \xrightarrow{\mathrm{ev}} & \mathcal{O}_{\overline{\mathcal{H}}} & \longrightarrow & 0.
\end{array}$$

Here we used the abbreviations

$$\Omega_B^{[i]} = \Omega_{i,B}^{\mathrm{hor}}(\log) \oplus \Omega_{i,B}^{\mathrm{lev}}(\log) \oplus \Omega_{i,B}^{\mathrm{rel}}, \quad \Omega^{[i]} = \Omega_i^{\mathrm{hor}}(\log) \oplus \Omega_i^{\mathrm{lev}}(\log) \oplus \Omega_i^{\mathrm{rel}}.$$

The surjectivity of q_Ω follows from the definition of the summands in (6). It requires justification that the image is not larger, since the derivatives of the local equations of $\mathcal{H}$ do not respect the direct sum decomposition (5). More precisely we claim that $\mathcal{K}_{\mathrm{Eq}}$ is generated by two kinds of equations. Before analyzing them, note that the log periods satisfy by construction an estimate of the form

$$\widetilde{c}_j^{[-i]} - \widehat{c}_j^{[-i]} = \sum_{s>i} \frac{t_{\lceil s \rceil}}{t_{\lceil i \rceil}} \widehat{E}_{j,i}^{[-s]} \tag{19}$$

with some error term $\widehat{E}_{j,i}^{[-k]}$ depending on the variables $c_j^{[-s]}$ on the lower level $-s$ as in (3).

For each of the equations (7) the corresponding linear function $L_k^{[i]}$ in the variables $c_j^{[i]}$ is an element in Eq. We use the comparisons (19) and (3) to compute its ψ-preimage in $\mathcal{K}_{\mathrm{Eq}}$ via (18). It is $t_{\lceil -i \rceil}$ times the corresponding expression in the $\widehat{c}_j^{[i]}$ plus a linear combination of the terms $t_{\lceil -s \rceil} \widehat{E}_{j,i}^{[s]}$. The quotient by such a relation does not yield any quotient class beyond those in $\bigoplus_{i=0}^{i} t_{\lceil -i \rceil} \cdot \Omega^{[i]}$.

We write the other equations (8) as $(\mathbf{q}^{[i]})^{J_{1,k}-J_{2,k}} = 1$ since we are interested in torus-invariant differential forms and can compute on the boundary complement. Consider d log of this equation. Under the first map ψ of the Euler sequence

$$dq_j^{[i]}/q_j^{[i]} = d\log(q_j^{[i]}) = d\left(2\pi I \frac{b_j^{[i]}}{a_j^{[i]}}\right) \mapsto \frac{2\pi I}{a_j^{[i]}}\left(\beta_j^{[i]} - \frac{b_j^{[i]}}{a_j^{[i]}}\alpha_j^{[i]}\right) \otimes \omega.$$

Recall from summary of [7] in Section 3.5 that the functions $a_j^{[i]}$ for all j, where $(v_1,\dots,v_{N(i)-h(i)}) := J_{1,k} - J_{2,k}$ is non-zero, are rational multiples of each other. Note moreover that

$$\beta_j^{[i]} - \frac{b_j^{[i]}}{a_j^{[i]}}\alpha_j^{[i]} = \beta_j^{[i]} - \frac{1}{2\pi I}\log(q_j^{[i]})\alpha_j^{[i]}$$

is the Deligne extension of $\beta_j^{[i]}$ across all the boundary divisors that stem from horizontal nodes at level i. For the full Deligne extension $\widehat{\beta}_j^{[i]}$ the correction terms for the lower level nodes have to be added. Together with (4) we deduce that the ψ-image of

$$\sum_{m=1}^{h(i)} v_m a_m^{[i]} \frac{d\widetilde{q}_m^{[i]}}{\widetilde{q}_m^{[i]}} = \sum_{m=1}^{h(i)} v_j c_{j(m)}^{[i]} \frac{d\widetilde{q}_m^{[i]}}{\widetilde{q}_m^{[i]}}$$

differs from the element in Eq responsible for the equation $H_k^{[i]}$ only by terms from lower level s, which come with a factor $t_{\lceil -s\rceil}$. In this equation, we used that $a_m^{[i]} = c_{j(m)}^{[i]}$ for an appropriate $j(m)$. Since $c_{j(m)}^{[i]}$ is close to $t_{\lceil -i\rceil}\widetilde{c}_{j(m)}^{[i]}$, compare with (3) this element indeed belongs to the kernel of ψ as claimed in the commutative diagram. The quotient by such a relation does not yield any quotient class beyond those above either. Since the (8) and (7) correspond to a basis (in fact, in reduced row echelon form) of Eq, this completes the proof. ■

Proof of Theorem 5.2. Uses that the summands of $\mathcal{K}|_U$ are, up to t-powers, the decomposition of the logarithmic tangent sheaf by Proposition 3.5. ■

Corollary 5.3. *The Chern character and the Chern polynomial of the kernel $\mathcal{K}$ of the Euler sequence are given by*

$$\mathrm{ch}(\mathcal{K}) = Ne^{\xi_{\mathcal{H}}} - 1 \quad \textit{and} \quad \mathrm{c}(\mathcal{K}) = \sum_{i=0}^{N-1}\binom{N}{i}\xi_{\mathcal{H}}^i.$$

Proof. As a Deligne extension of a local system, $(\overline{\mathcal{H}}^1_{\mathrm{rel},B})^\vee|_{\overline{\mathcal{H}}}$ has trivial Chern classes except for c_0. By construction, the pullback of the sheaf Eq to an allowable modification (toric resolution with normal crossing boundary, see the proof of Proposition 2.1) is the Deligne extension of a local system. It follows that all Chern classes but c_0 of this pullback vanish and by push-pull this holds for Eq, too. The Chern class vanishing for $(\mathcal{H}^1_{\mathrm{rel}})^\vee$ and the corollary follows. ■

To start with the computation of $\mathcal{C}$, we will also need an infinitesimal thickening of the boundary divisor $D_\Gamma^{\mathcal{H}}$, that is, we define $D_{\Gamma,\bullet}^{\mathcal{H}}$ to be its ℓ_Γ-th thickening, the non-reduced substack of $\overline{\mathcal{H}}$ defined by the ideal $\mathcal{I}_{D_\Gamma^{\mathcal{H}}}^{\ell_\Gamma}$. We will factor the above inclusion using the notation

$$\mathrm{i}_\Gamma = \mathrm{i}_{\Gamma,\bullet} \circ j_{\Gamma,\bullet} \colon D_\Gamma^{\mathcal{H}} \overset{j_{\Gamma,\bullet}}{\hookrightarrow} D_{\Gamma,\bullet}^{\mathcal{H}} \overset{\mathrm{i}_{\Gamma,\bullet}}{\hookrightarrow} \overline{\mathcal{H}}.$$

We will denote by $\mathcal{L}_{\Gamma,\bullet}^\top = (j_{\Gamma,\bullet})_*(\mathcal{L}_\Gamma^\top)$ and $\mathcal{E}_{\Gamma,\bullet}^\top = (j_{\Gamma,\bullet})_*(\mathcal{E}_\Gamma^\top)$ the push-forward to the thickening of the vector bundles defined in (16) and (12).

Proposition 5.4. *The cokernel of* (17) *is given by*

$$\mathcal{C} = \bigoplus_{\Gamma \in \mathrm{LG}_1(\mathrm{B})} \mathcal{C}_\Gamma, \quad \text{where } \mathcal{C}_\Gamma = (\mathrm{i}_{\Gamma,\bullet})_*\big(\mathcal{E}_{\Gamma,\bullet}^\top \otimes (\mathcal{L}_{\Gamma,\bullet}^\top)^{-1}\big).$$

Moreover, there is an equality of Chern characters

$$\mathrm{ch}\big((\mathrm{i}_{\Gamma,\bullet})_*\big(\mathcal{E}_{\Gamma,\bullet}^\top \otimes (\mathcal{L}_{\Gamma,\bullet}^\top)^{-1}\big)\big) = \mathrm{ch}\Bigg((\mathrm{i}_\Gamma)_*\Bigg(\bigoplus_{j=0}^{\ell_\Gamma - 1} \mathcal{N}_\Gamma^{\otimes -j} \otimes \mathcal{E}_\Gamma^\top \otimes (\mathcal{L}_\Gamma^\top)^{-1}\Bigg)\Bigg).$$

Proof. The second part of the statement is justified by the original argument in [15, Lemma 9.3].

The first part of the statement follows since, from Theorem 5.1, we know that

$$\mathcal{K}|_U = \bigoplus_{i=-L}^{0} \prod_{j=1}^{-i} t_j^{\ell_j} \cdot \big(\Omega_i^{\mathrm{hor}}(\log) \oplus \Omega_i^{\mathrm{lev}}(\log) \oplus \Omega_i^{\mathrm{rel}}\big),$$

and from Proposition 3.5 we also know that

$$(\mathcal{E}_{\mathcal{H}} \otimes \mathcal{L}_{\mathcal{H}}^{-1})|_U = \bigoplus_{i=-L}^{0} \prod_{j=1}^{L} t_j^{\ell_j} \cdot \big(\Omega_i^{\mathrm{hor}}(\log) \oplus \Omega_i^{\mathrm{lev}}(\log) \oplus \Omega_i^{\mathrm{rel}}\big),$$

where Γ is an arbitrary level graph with L levels below zero and U is a small neighborhood of a point in $D_\Gamma^{\mathcal{H},\circ}$. ■

We can finally compute the following result.

Proposition 5.5. *The Chern character of the twisted logarithmic cotangent bundle $\mathcal{E}_{\mathcal{H}} \otimes \mathcal{L}_{\mathcal{H}}^{-1}$ can be expressed in terms of the twisted logarithmic cotangent bundles of the top levels of non-horizontal divisors as*

$$\mathrm{ch}(\mathcal{E}_{\mathcal{H}} \otimes \mathcal{L}_{\mathcal{H}}^{-1}) = Ne^{\xi} - 1 - \sum_{\Gamma \in \mathrm{LG}_1(\mathrm{B})} \mathrm{i}_{\Gamma *}\left(\mathrm{ch}(\mathcal{E}_{\Gamma}^{\top}) \cdot \mathrm{ch}(\mathcal{L}_{\Gamma}^{\top})^{-1} \cdot \frac{(1 - e^{-\ell_{\Gamma} c_1(\mathcal{N}_{\Gamma})})}{c_1(\mathcal{N}_{\Gamma})}\right).$$

Proof. The proof of [15, Proposition 9.5] works in the same way, since the only tool that was used is the Grothendieck–Riemann–Roch theorem applied to the map $f = \mathrm{i}_{\Gamma}$, which is still a regular embedding. ∎

Proof of Theorem 1.1 and Theorem 1.2. The final formulas of the full twisted Chern character, Chern polynomials and Euler characteristic follow from the arguments used for Abelian strata in [15, Section 9], since they were purely formal starting from the previous proposition. Here we need a more refined sum distinguishing irreducible components, but this works since the relevant inputs needed are the compatibility statement of Lemma 3.9, the formula for pulling back normal bundles given in Lemma 4.6 and Corollary 3.3 which work in this more refined setting. ∎

Proof of Theorem 1.3. A formal consequence of Theorem 1.2 and the rewriting in [13, Theorem 9.10] (with the reference to [13, Proposition 4.9] replaced by Lemma 3.9) is

$$\chi(\mathcal{H}) = (-1)^d \sum_{L=0}^{d} \sum_{\Gamma+ \in \mathrm{LG}_L^{+}(\mathcal{H})} N_{\Gamma+}^{\top} \cdot \ell_{\Gamma+} \cdot \int_{D_{\Gamma+}^{\mathcal{H}}} \prod_{i=-L}^{0} \left(\xi_{\Gamma+,\mathcal{H}}^{[i]}\right)^{d_{\Gamma+}^{[i]}}. \tag{20}$$

We now use Lemma 3.10 to convert integrals on a boundary component into the product of integrals of its the level strata. ∎

6. Example: Euler characteristic of the eigenform locus

For a non-square $D \in \mathbb{N}$ with $D \equiv 0$ or $1 \pmod 4$, let

$$\Omega E_D(1,1) \subseteq \Omega\mathcal{M}_{2,2}(1,1) \quad \text{and} \quad \Omega W_D \subseteq \Omega\mathcal{M}_{2,1}(2)$$

be the eigenform loci for real multiplication by $\mathcal{O}_D$ in the given stratum; see [8,42,43] for the first proofs that these loci are linear submanifolds and some background. We define $E_D := \mathbb{P}\Omega E_D(1,1)$ and *Weierstrass curve* $W_D := \mathbb{P}\Omega W_D$ as the projectivized eigenform loci. Associating with the curve its Jacobian, the projectivized eigenform loci map to the *Hilbert modular surface*

$$X_D = \mathbb{H} \times \mathbb{H} / \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^{\vee}).$$

Inside X_D let $P_D \subseteq X_D$ denote the *product locus*, i.e., the curve consisting of those surfaces which are polarized products of elliptic curves. The images of E_D and W_D are contained in the complement $X_D \setminus P_D$.

The goal of this section is to provide a new short proof of Theorem 1.4.

Proof of Theorem 1.4. The Hilbert modular surface X_D is the disjoint union of the symmetrization of the eigenform locus $E_D \subset \Omega\mathcal{M}_{2,1}(1,1)$, the product locus P_D of reducible Jacobians and the Teichmüller curve W_D. This gives

$$\chi(P_D) + \chi(W_D) + \frac{1}{2}\chi(E_D) = \chi(X_D). \tag{21}$$

The numerical input is

$$\chi(X_D) = 2\zeta(-1) \quad \text{and} \quad \chi(P_D) = -\frac{5}{2}\chi(X_D) = -5\zeta(-1), \tag{22}$$

where $\zeta = \zeta_{\mathbb{Q}(\sqrt{D})}$ is the Dedekind zeta function. The first formula is due to Siegel [53] (see also [56, Theorem IV.1.1]), the second is given in [2, Theorem 2.22] viewing P_D as the vanishing locus of the product of odd theta functions.

We are left to compute $\chi(E_D)$, which we will do using the formula for the Euler characteristic provided in Theorem 1.3. We first need to list the vertical boundary strata of the linear submanifold $\overline{E}_D \subset \mathbb{P}\Xi\mathcal{M}_{2,2}(1,1)$. This list consists of two divisorial strata only, given in Figure 1.

Hence, in this situation, the formula of Theorem 1.3 gives

$$\chi(E_D) = 3\int_{E_D} \xi_{E_D}^2 + 2\frac{K_{\Gamma_P}^{E_D}}{|\mathrm{Aut}_{E_D}(\Gamma_P)|}\int_{D_{\Gamma_P}^{\top}} \xi_{D_{\Gamma_P}^{\top}} \int_{D_{\Gamma_P}^{\bot}} 1$$
$$+ 2\frac{K_{\Gamma_W}^{E_D}}{|\mathrm{Aut}_{E_D}(\Gamma_W)|}\int_{D_{\Gamma_W}^{\top}} \xi_{D_{\Gamma_W}^{\top}} \int_{D_{\Gamma_W}^{\bot}} 1.$$

Firstly, note that the top-ξ-integral on E_D vanishes by Corollary 4.3, since E_D is a linear submanifold with REL non-zero.

Secondly, the full automorphism groups of the graphs Γ_P and Γ_W are trivial and all three prong-matchings for Γ_W are reachable since they belong to one orbit of the prong rotation group. Hence, we obtain

$$\frac{K_{\Gamma_P}^{E_D}}{|\mathrm{Aut}_{E_D}(\Gamma_P)|} = 1, \qquad \frac{K_{\Gamma_W}^{E_D}}{|\mathrm{Aut}_{E_D}(\Gamma_W)|} = 3.$$

Thirdly, we can identify the top levels $D_{\Gamma_P}^{\top}$ and $D_{\Gamma_W}^{\top}$ with P_D and W_D respectively. Hence, again by applying Theorem 1.3 to the top level strata, we get

$$2\int_{D_{\Gamma_P}^{\top}} \xi_{D_{\Gamma_P}^{\top}} = -\chi(P_D), \quad 2\int_{D_{\Gamma_W}^{\top}} \xi_{D_{\Gamma_W}^{\top}} = -\chi(W_D).$$

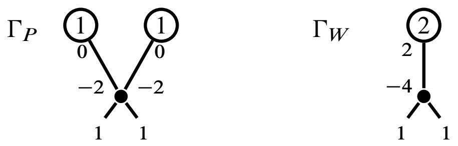

Figure 1. The boundary divisors of the eigenform locus E.

Finally, is it clear that

$$\int_{D^{\perp}_{\Gamma_P}} 1 = 1 \quad \text{and} \quad \int_{D^{\perp}_{\Gamma_W}} 1 = 1,$$

since there is unique differential up to scale of type $(1, 1, -2, -2)$ on $\mathbb{P}^1$ with vanishing residues and $D^{\perp}_{\Gamma_W} \cong \mathcal{M}_{0,3}$.

From the previous computations, we hence obtain that

$$\chi(E_D) = -\chi(P_D) - 3\chi(W_D).$$

This, together with (21) and the numerical inputs (22), yields the desired result

$$\begin{aligned} \chi(W_D) &= -2\chi(X_D) + \chi(P_D) \\ &= -9\zeta(-1). \end{aligned}$$

■

7. Strata of k-differentials

Our goal here is to prove Corollary 1.5 that gives a formula for the Euler characteristic of strata $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ of k-differentials. Those strata can be viewed as linear submanifolds of strata of Abelian differentials $\mathbb{P}\Omega\mathcal{M}_{\widehat{g},\widehat{n}}(\widehat{\mu})$ via the canonical covering construction and thus Theorem 1.3 applies. This is however of little practical use as we do not know the classes of k-differential strata in $\mathbb{P}\Omega\mathcal{M}_{\widehat{g},\widehat{n}}(\widehat{\mu})$. However, we do know their classes in $\overline{\mathcal{M}}_{g,n}$ via Pixton's formulas for the DR-cycle ([1, 34]). As a consequence the formula in Corollary 1.5 can be implemented, and the `diffstrata` package does provide such an implementation. In this section we thus recall the basic definitions of the compactification and collect all the statements to perform evaluation of expressions in the tautological rings on strata of k-differentials.

7.1. Compactification of strata of k-differentials

We want to work on the multi-scale compactification $\overline{\mathcal{Q}} := \overline{\mathcal{Q}}_k := \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ of the space of k-differentials. As topological space this compactification was given in [15], reviewing the plumbing construction from [5], but without giving the stack structure. Here we consider a priori the compactification of Section 3. We give some details, describing auxiliary stacks usually by giving $\mathbb{C}$-valued points and morphisms, from which the reader can easily deduce the notion of families following the procedure in [5]. From this description it should become clear that the two compactifications, the one of Section 3 and [15], agree up to explicit isotropy groups (see Lemma 7.2). In particular, the compactification $\overline{\mathcal{Q}}_k$ is smooth. This follows also directly from the definition of Section 3, since the only potential singularities are at the horizontal nodes. There however the local equations (8) simply compare monomials (with exponent one), the various q-parameters of the k preimages of a horizontal node.

We start by recalling notation for the canonical k-cover *in the primitive case*. Let X be a Riemann surface of genus g and let q be a *primitive* meromorphic k-differential of type $\mu = (m_1, \dots, m_n)$, i.e., not the d-th power of a k/d-differential for any $d > 1$. This datum defines (see, e.g., [4, Section 2.1]) a connected k-fold cover $\pi\colon \widehat{X} \to X$ such that $\pi^* q = \omega^k$ is the k-power of an Abelian differential. This differential ω is of type

$$\widehat{\mu} := \Big(\underbrace{\widehat{m}_1, \dots, \widehat{m}_1}_{g_1 := \gcd(k, m_1)}, \underbrace{\widehat{m}_2, \dots, \widehat{m}_2}_{g_2 := \gcd(k, m_2)}, \dots, \underbrace{\widehat{m}_n, \dots, \widehat{m}_n}_{g_n := \gcd(k, m_n)}\Big),$$

where $\widehat{m}_i := (k + m_i)/\gcd(k, m_i) - 1$. (Here and throughout marked points of order zero may occur.) We let $\widehat{g} = g(\widehat{X})$ and $\widehat{n} = \sum_i \gcd(k, m_i)$. The type of the covering determines a natural subgroup $S_{\widehat{\mu}} \subset S_{\widehat{n}}$ of the symmetric group that allows only the permutations of each the $\gcd(k, m_i)$ points corresponding to a preimage of the i-th point. In the group $S_{\widehat{\mu}}$, we fix the element

$$\tau_0 = (1\ 2\ \cdots\ g_1)(g_1 + 1\ g_1 + 2\ \cdots\ g_1 + g_2) \cdots \Big(1 + \sum_{i=1}^{n-1} g_i \cdots \sum_{i=1}^{n} g_n\Big),$$

i.e., the product of cycles shifting the g_i points in the π-preimage of each point in $\mathbf{z}$. We fix a primitive k-th root of unity ζ_k throughout.

We consider the stack $\Omega\mathcal{H}_k := \Omega\mathcal{H}_k(\widehat{\mu})$ whose points are

$$\big\{(\widehat{X}, \widehat{\mathbf{z}}, \omega, \tau) : \tau \in \operatorname{Aut}(\widehat{X}),\ \operatorname{ord}(\tau) = k,\ \tau^*\omega = \zeta_k \omega,\ \tau|_{\widehat{\mathbf{z}}} = \tau_0\big\}. \tag{23}$$

Families are defined in the obvious way. Morphisms are morphisms of the underlying pointed curves that commute with τ. Since the marked points determine the differential up to scale, the differentials are identified by the pullback of morphisms up to scale. Commuting with τ guarantees that morphisms descend to the quotient curves by $\langle \tau \rangle$ (for a morphism f to descend, a priori $f \tau f^{-1} = \tau^a$ for some a would be sufficient, but the action on ω implies that in fact $a = 1$). It will be convenient to label the tuple of points $\hat{\mathbf{z}}$ by tuples (i, j) with $i = 1, \dots, n$ and $j = 1, \dots, \gcd(k, m_i)$. There is a natural forgetful map $\Omega\mathcal{H}_k \to \Omega\mathcal{M}_{\hat{g},\hat{n}}$ and period coordinates (say, after providing both sides locally with a Teichmüller marking) show that this map is the normalization of its image and the image is cut out by linear equations, i.e., that $\Omega\mathcal{H}_k$ is a linear submanifold as defined in Section 3.1.

The subgroup

$$G = \left\langle (1\,2\,\cdots\,g_1), \left(g_1+1\; g_1+2\,\cdots\, g_1+g_2\right), \dots, \left(1+\sum_{i=1}^{n-1} g_i \cdots \sum_{i=1}^{n} g_n\right)\right\rangle \subset S_{\hat{\mu}} \tag{24}$$

generated by the cycles that τ_0 is made from acts on $\Omega\mathcal{H}_k$ and on the projectivization $\mathcal{H}_k$. We denote the quotient of the latter by $\mathcal{H}_k^{\mathrm{mp}} := \mathcal{H}_k/G$, where the upper index is an abbreviation of *marked (only) partially*.

Since τ has ω as eigendifferential, its k-th power naturally descends to (projectivized) k-differential $[q]$ on the quotient $X = \widehat{X}/\langle\tau\rangle$, which is decorated by the marked points $\mathbf{z}$, the images of $\hat{\mathbf{z}}$.

We denote by $\mathcal{Q}$ the stack given by the rigidification of $\mathcal{H}_k^{\mathrm{mp}}$ by the action of $\langle\tau\rangle$, i.e., the stack with the same underlying set as $\mathcal{H}_k^{\mathrm{mp}}$, but where morphisms are given by the morphisms of $(X/\langle\tau\rangle, \mathbf{z}, [q])$ in $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$. Written out on curves, a morphism in $\mathcal{Q}$ is a map $f\colon \widehat{X}/\langle\tau\rangle \to \widehat{X}'/\langle\tau'\rangle$, such that there exists a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \overset{\tilde{f}}{\dashrightarrow} & \widehat{X}' \\ \downarrow & & \downarrow \\ X = \widehat{X}/\langle\tau\rangle & \overset{f}{\longrightarrow} & X' = \widehat{X}'/\langle\tau'\rangle. \end{array}$$

If two such maps $\tilde{f}$ exist, they differ by pre- or postcomposition with an automorphism of $\widehat{X}$ resp. $\widehat{X}'$. Via the canonical cover construction, the stack $\mathcal{Q}$ is isomorphic to $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$. The non-uniqueness of $\tilde{f}$ exhibits $\mathcal{H}_k^{\mathrm{mp}} = \mathcal{Q}/\langle\tau\rangle$ as the quotient stack by a group of order k, acting trivially.

As in Section 3, we denote by

$$\overline{\Omega\mathcal{H}}_k := \overline{\Omega\mathcal{H}}_k(\mu)$$

the normalization of the closure of $\Omega\mathcal{H}_k$ in $\Xi\overline{\mathcal{M}}_{\hat{g},\hat{n}}(\mu)$, and let $\overline{\mathcal{H}}_k := \overline{\mathcal{H}}_k(\mu)$ be the corresponding projectivizations. We next describe the boundary strata of $\overline{\mathcal{H}}_k$. These are indexed by enhanced level graphs $\widehat{\Gamma}$ together with an $\langle\tau\rangle$-action on them. We will leave the group action implicit in our notation. The following lemma describes the objects parametrized by the boundary components $D_{\widehat{\Gamma}}^{\mathcal{H}_k}$ (using the notation from Section 3) of the compactification $\overline{\mathcal{H}}_k$.

Lemma 7.1. *A point in the interior of the boundary stratum $D_{\widehat{\Gamma}}^{\mathcal{H}_k}$ is given by a tuple*

$$\big\{(\widehat{X},\widehat{\Gamma},\widehat{\mathbf{z}},[\boldsymbol{\omega}],\boldsymbol{\sigma},\tau) : \tau \in \mathrm{Aut}(\widehat{X}),\ \mathrm{ord}(\tau)=k,\ \tau^*\boldsymbol{\omega}=\zeta_k\boldsymbol{\omega},\ \tau|_{\widehat{\mathbf{z}}}=\tau_0\big\},$$

where $(\widehat{X},\widehat{\Gamma},\widehat{\mathbf{z}},[\boldsymbol{\omega}],\boldsymbol{\sigma}) \in \mathbb{P}\Xi\overline{\mathcal{M}}_{\widehat{g},\widehat{n}}(\widehat{\mu})$ is a multi-scale differential, and where moreover the prong-matching $\boldsymbol{\sigma}$ is equivariant with respect to the action of $\langle\tau\rangle$.

The equivariance of prong-matching requires an explanation: Suppose x_i and y_i are standard coordinates near the node corresponding to an edge e of Γ, so that the prong-matching at e is given by

$$\sigma_e = \frac{\partial}{\partial x_i} \otimes -\frac{\partial}{\partial y_i}$$

(compare [5, Section 5] for the relevant definitions). Then $\tau^* x_i$ and $\tau^* y_i$ are standard coordinates near $\tau(e)$. We say that a global prong-matching $\boldsymbol{\sigma} = \{\sigma_e\}_{e\in E(\widehat{\Gamma})}$ is *equivariant* if

$$\sigma_{\tau(e)} = \frac{\partial}{\partial \tau^* x_i} \otimes -\frac{\partial}{\partial \tau^* y_i}$$

for each edge e.

Proof. The necessity of the conditions on the boundary points is obvious from the definition in (23), except for the prong-matching equivariance. This follows from the construction of the induced prong-matching in a degenerating family in [5, Proposition 8.4] and applying τ to it.

Conversely, given $(\widehat{X},\widehat{\Gamma},\widehat{\mathbf{z}},[\boldsymbol{\omega}],\boldsymbol{\sigma},\langle\tau\rangle)$ as above with equivariant prong-matchings, we need to show that it is in the boundary of $\mathcal{H}_k$. This is achieved precisely by the equivariant plumbing construction given in [4]. ■

The group G still acts on the compactification $\overline{\Omega\mathcal{H}_k}$ and on its projectivization $\overline{\mathcal{H}}_k$. As above we denote the quotient by $\overline{\mathcal{H}}_k^{\mathrm{mp}} = \overline{\mathcal{H}}_k/G$ to indicate that the points $\widehat{\mathbf{z}}$ are now marked only partially. By Lemma 7.1 we may construct $\overline{\mathcal{Q}}$ just as in the uncompactified case.

The map $\overline{\mathcal{H}}_k^{\mathrm{mp}} \to \overline{\mathcal{Q}}$ is in general non-representable due to the existence of additional automorphisms of objects in $\overline{\mathcal{H}}_k^{\mathrm{mp}}$. This resembles the situation common for Hurwitz spaces, where the target map is in general non-representable, too. We denote by $s\colon \overline{\mathcal{H}}_k \to \overline{\mathcal{H}}_k^{\mathrm{mp}} \to \overline{\mathcal{Q}}$ the composition of the maps.

7.2. Generalized strata of k-differentials

Our notion of generalized strata is designed for recursion purposes so that the extraction of levels of a boundary stratum of $\overline{\mathcal{Q}}$ is an instance of a generalized stratum (of

k-differentials). This involves incorporating disconnected strata, differentials that are non-primitive on some components, and residue conditions. Moreover, we aim for a definition of a space of k-fold covers on which the group G acts, to match with the previous setup. The key is to record which of the marked points is adjacent to which component of the canonical cover, an information that is obviously trivial in the case of primitive k-differentials.

A map $\mathcal{A}\colon \widehat{\mathbf{z}} \to \pi_0(\widehat{X})$ that records which marked point is adjacent to which component of $\widehat{X}$ is called an *adjacency datum*. Such an adjacency datum is equivalent to specifying a one-level graph of a generalized stratum, which is indeed the information we get when we extract level strata. Note in particular that from the adjacency datum it is possible to reconstruct the unique u such that the Abelian differentials on $(\widehat{X}, \widehat{\mathbf{z}})$ are u-th power of primitive k/u differentials.

More abstractly, an adjacency datum is given by a set π_0 with a transitive action of $\mathbb{Z}/k\mathbb{Z}$ together with a map $\mathcal{A}\colon \widehat{\mathbf{z}} \to \pi_0$ that is equivariant with respect to the action of $\mathbb{Z}/k\mathbb{Z}$. We say that $(\widehat{X}, \widehat{\mathbf{z}})$ has adjacency $\mathcal{A}$ if there is a $\mathbb{Z}/k\mathbb{Z}$-equivariant bijection $\pi_0 \cong \pi_0(\widehat{X})$ such that $\mathcal{A}$ records the adjacency of the markings $\widehat{\mathbf{z}}$ in the components of $\widehat{X}$. The subgroup G from (24) acts on the triples $(\widehat{X}, \widehat{\mathbf{z}}, \mathcal{A})$ of pointed stable curves with adjacency map by acting simultaneously on $\widehat{\mathbf{z}}$ and on $\mathcal{A}$ by precomposition. For a fixed adjacency datum $\mathcal{A}$ we consider the stack $\Omega\widetilde{\mathcal{H}}_k(\widehat{\mu}, \mathcal{A})$ whose points are

$$\big\{(\widehat{X}, \widehat{\mathbf{z}}, \omega, \tau) : (\widehat{X}, \widehat{\mathbf{z}}) \text{ have adjacency } \mathcal{A}, \\ \tau \in \mathrm{Aut}(\widehat{X}), \mathrm{ord}(\tau) = k, \tau^*\omega = \zeta_k\omega, \tau|_{\widehat{\mathbf{z}}} = \tau_0\big\}.$$

We denote by $\Omega\mathcal{H}_k(\widehat{\mu}, [\mathcal{A}]) := G \cdot \Omega\widetilde{\mathcal{H}}_k(\widehat{\mu}, \mathcal{A})$ the G-orbit of this space.

A *residue condition* is given by a τ-invariant partition $\lambda_{\mathfrak{R}}$ of a subset of the set $H_p \subseteq \{1, \ldots, \widehat{n}\}$ of marked points such that $\widehat{m}_i < -1$. We often also call the associated linear subspace

$$\mathfrak{R} := \left\{(r_i)_{i \in H_p} \in \mathbb{C}^{H_p} : \sum_{i \in \lambda} r_i = 0 \text{ for all } \lambda \in \lambda_{\mathfrak{R}}\right\}$$

the residue condition. The linear subspace $\mathfrak{R}$ is obviously not G-invariant in general.

We denote by $\Omega\mathcal{H}_k^{\mathfrak{R}}(\widehat{\mu}, \mathcal{A}) \subseteq \Omega\mathcal{H}_k(\widehat{\mu}, \mathcal{A})$ the subset where for each $R \in \mathfrak{R}$ the residues of $\widehat{\omega}$ at all the points $z_i \in R$ add up to zero. If $(\widehat{X}, \widehat{\mathbf{z}}, \omega, \tau)$ is contained in $\Omega\mathcal{H}_k^{\mathfrak{R}}(\widehat{\mu}, \mathcal{A})$, then $g \cdot (\widehat{X}, \widehat{\mathbf{z}}, \omega, \tau)$ is contained in $\Omega\mathcal{H}_k^{g\cdot\mathfrak{R}}(\widehat{\mu}, g \cdot \mathcal{A})$ for any $g \in G$. That is, the G-action simultaneously changes the residue condition and the adjacency datum. We denote by $[\mathfrak{R}, \mathcal{A}]$ the G-orbit of this pair and use the abbreviation

$$\Omega\mathcal{H}_k^{[\mathfrak{R},\mathcal{A}]} := G \cdot \Omega\mathcal{H}_k^{\mathfrak{R}}(\widehat{\mu}, \mathcal{A})$$

for the G-orbit of the spaces, $\widehat{\mu}$ being tacitly fixed throughout.

As above, we denote the projectivization of $\Omega\mathcal{H}_k^{[\mathfrak{R},\mathcal{A}]}$ by $\mathcal{H}_k^{[\mathfrak{R},\mathcal{A}]}$ and the G-quotient by $\mathcal{H}_k^{\mathfrak{R},\mathrm{mp}} := \mathcal{H}_k^{[\mathfrak{R},\mathcal{A}]}/G$, dropping the information about adjacency and the connected components to ease notation. Finally, we denote by $\mathcal{Q}^{\mathfrak{R}}$ the stack with the same underlying set as $\mathcal{H}_k^{\mathfrak{R},\mathrm{mp}}$ and with morphisms defined in the same way as above for $\mathcal{Q}$. Recall that the curves in $\mathcal{Q}^{\mathfrak{R}}$ may be disconnected. We call such a stratum with possibly disconnected curves and residue conditions a *generalized stratum of k-differentials*. Since $\mathcal{H}_k^{[\mathfrak{R},\mathcal{A}]}$ is a linear submanifold, we can still compactify them as before and a version of Lemma 7.1 with adjacency data still holds.

We will now compute the degree of the map s from the linear submanifolds to the strata of k-differential. Our definition of generalized strata of k-differentials makes the degree of this map the same in the usual and in the generalized case.

Lemma 7.2. *The map $s\colon \overline{\mathcal{H}}_k^{[\mathfrak{R},\mathcal{A}]} \to \overline{\mathcal{Q}}^{\mathfrak{R}}$ is proper, quasi-finite, unramified and of degree*

$$\deg(s) = \frac{1}{k}\prod_{m_i\in\mu}\gcd(m_i,k).$$

Proof. The composition of the map s with the quotient map $\overline{\mathcal{Q}}^{\mathfrak{R}} \to \overline{\mathcal{H}}_k^{[\mathfrak{R},\mathcal{A}]}$ by the trivial action of $\mathbb{Z}/k\mathbb{Z}$ is the quotient by a group of order $|G| = \prod_{m_i\in\mu}\gcd(m_i,k)$. Since degrees are multiplicative under compositions, the claimed formula for $\deg(s)$ follows.

The map is unramified as both quotient maps are unramified. ∎

7.3. Decomposing boundary strata

Having constructed strata of k-differentials, we now want to decompose their boundary strata again as a product of generalized strata of k-differentials and argue recursively. In fact, the initial stratum should be a generalized stratum $\overline{\mathcal{Q}}^{\mathfrak{R}}$, thus coming with its own residue condition, but we suppress this in our notation, focusing on the new residue condition that arise when decomposing boundary strata. Here 'decomposition' of the boundary strata should be read as a construction of a space finitely covering both of them, as given by the following diagram:

$$\begin{array}{ccccc} & & D_\pi^{\circ,\mathcal{H}_k,s} & & \\ & & {\scriptstyle p_\pi}\swarrow \quad \searrow{\scriptstyle c_\pi} & & \\ \mathcal{H}_k(\pi) := \prod_{i=0}^{-L}\mathcal{H}_k(\pi_{[i]}) & \xleftarrow{\supseteq} & \mathrm{Im}(p_\pi) & & D_\pi^{\circ,\mathcal{H}_k} \\ {\scriptstyle s_\pi}\downarrow & & & & \downarrow{\scriptstyle d_\pi} \\ \mathcal{Q}(\pi) := \prod_{i=0}^{-L}\mathcal{Q}(\pi_{[i]}) & & & & D_\pi^{\circ,\mathcal{Q}}, \end{array} \tag{25}$$

whose notation we now start to explain. Note that the diagram is for the open boundary strata throughout, since we mainly need the degree all these maps as in Lemma 3.6 (the existence of a similar diagram over the completions follows as at the beginning of Section 3.2).

We denote by $\widehat{\Gamma}$ the level graphs indexing the boundary strata of $\mathbb{P}\Xi\overline{\mathcal{M}}_{\widehat{g},\widehat{n}}(\widehat{\mu})$, and thus of $\overline{\mathcal{H}}_k$. Following our general convention for strata their legs are labeled, but not the edges. In $\overline{\mathcal{H}}_k^{\mathrm{mp}}$ the leg-marking is only well defined up to the action of G. A graph with such a marking is said to be *marked (only) partially* and denoted by $\widehat{\Gamma}_{\mathrm{mp}}$. Even though curves in $\overline{\mathcal{H}}_k$ are marked (and not only marked up to the action of G), the boundary strata of $\overline{\mathcal{H}}_k$ are naturally indexed by partially marked graphs as well: If $\widehat{\Gamma}$ is the dual graph of one stable curve in the boundary of $\overline{\mathcal{H}}_k$, then for all $g \in G$ the graph $g \cdot \widehat{\Gamma}$ is the dual graph of another stable curve in the boundary of $\overline{\mathcal{H}}_k$. The existence of τ implies that level graphs $\widehat{\Gamma}$ at the boundary of $\overline{\mathcal{H}}_k$ come with the quotient map by this action. To each boundary stratum of $\overline{\mathcal{Q}}$ we may thus associate a k-cyclic covering of graphs $\pi\colon \widehat{\Gamma}_{\mathrm{mp}} \to \Gamma$ (see [15, Section 2] for the definitions of such covers). We denote the corresponding (open) boundary strata by $D_\pi^{\circ,\mathcal{Q}} \subset \overline{\mathcal{Q}}$ and the (open) boundary strata corresponding to such a G-orbit of graphs by $D_\pi^{\circ,\mathcal{H}_k} \subset \overline{\mathcal{H}}_k$. The map $d_\pi\colon D_\pi^{\circ,\mathcal{H}_k} \to D_\pi^{\circ,\mathcal{Q}}$ is the restriction of the map $s\colon \overline{\mathcal{H}}_k \to \overline{\mathcal{Q}}$.

Next we construct the commensurability roof just as in (9), though for each $\widehat{\Gamma}$ in the G-orbit separately, so that $D_\pi^{\circ,\mathcal{H}_k,s}$ is the disjoint union of a G-orbit of the roofs in (9).

Next we define the spaces $\mathcal{H}_k(\pi_{[i]})$. Consider the linear submanifolds of generalized strata of k-differentials with signature and adjacency datum given by the i-th level of one marked representative $\widehat{\Gamma}$ of $\widehat{\Gamma}_{\mathrm{mp}}$ (the resulting strata are independent of the choice of a representative). Their product defines the image $\mathrm{Im}(p_\pi)$. For every level i, consider the orbit under $G(\mathcal{H}_k(\pi_{[i]}))$, where $G(\mathcal{H}_k(\pi_{[i]}))$ is the group as in (24) for the i-th level, of the linear submanifolds we extracted from the levels. We define $\mathcal{H}_k(\pi_{[i]})$ to be these orbits, which in particular are then linear submanifolds associated to generalized strata of k-differentials as we defined them above. We can hence consider, for every level, the morphism given by the quotient by $G(\mathcal{H}_k(\pi_{[i]}))$ composed with the rigidification by the action of $\langle\tau\rangle$ at each level and denote by $\mathcal{Q}(\pi_{[i]})$ its image, which is called the generalized stratum of k-differentials at level i. The map $\mathbf{s}_\pi$ in diagram (25) is just a product of maps like the map s above, thus Lemma 7.2 immediately implies the following.

Lemma 7.3. *The degree of the map* $\mathbf{s}_\pi$ *in the above diagram* (25) *is*

$$\deg(\mathbf{s}_\pi) = \frac{1}{k^{L+1}} \prod_{i=1}^{n} \gcd(m_i, k) \prod_{e \in E(\Gamma)} \gcd(\kappa_e, k)^2,$$

where κ_e *is the* k*-enhancement of the edge* e.

We recall Lemma 3.6 and compute explicitly the coefficients appearing in our setting here. Note that the factor $|\operatorname{Aut}_{\mathcal{H}}(\Gamma)|$ there should be called $|\operatorname{Aut}_{\mathcal{H}_k}(\widehat{\Gamma})|$ in the notation used in this section.

Lemma 7.4. *The ratio of the degrees of the topmost maps in (25) is*

$$\frac{\deg(p_\pi)}{\deg(c_\pi)} = \frac{K_{\widehat{\Gamma}}^{\mathcal{H}_k}}{|\operatorname{Aut}_{\mathcal{H}_k}(\widehat{\Gamma})| \cdot \ell_{\widehat{\Gamma}}},$$

where the number of reachable prong-matchings is given by

$$K_{\widehat{\Gamma}}^{\mathcal{H}_k} = \prod_{e \in E(\Gamma)} \frac{\kappa_e}{\gcd(\kappa_e, k)}$$

and $\operatorname{Aut}_{\mathcal{H}_k}(\widehat{\Gamma})$ *is the subgroup of automorphisms of* $\widehat{\Gamma}$ *commuting with* τ*.*

We remark that the quantity $\ell_{\widehat{\Gamma}}$ is intrinsic to Γ, for a two-level graph it is given by $\ell_{\widehat{\Gamma}} = \operatorname{lcm}(\kappa_e / \gcd(\kappa_e, k)$ for $e \in E(\Gamma))$.

Proof. The first statement is exactly the one of Lemma 3.6 since the topmost maps in (25) are given by a disjoint union of the topmost maps in (9).

For the second statement, consider an edge $e \in E(\Gamma)$. The edge e has $\gcd(\kappa_e, k)$ preimages, each with an enhancement $\kappa_e / \gcd(\kappa_e, k)$. The prong-matching at one of the preimages determines the prong-matching at the other preimages by Lemma 7.1, as they are related by the action of the automorphism.

For the third statement, we need to prove that the subgroup of $\operatorname{Aut}(\widehat{\Gamma})$ fixing setwise the linear subvariety $\overline{\mathcal{H}}_k$ is precisely the subgroup commuting with τ. If $\rho \in \operatorname{Aut}(\widehat{\Gamma})$ commutes with τ, then it descends to a graph automorphism of Γ and gives an automorphism of families of admissible covers of stable curves, thus preserving $\overline{\mathcal{H}}_k$. Conversely, if ρ fixes $\overline{\mathcal{H}}_k$, it induces an automorphism of families of admissible covers of stable curves, thus of coverings of graphs. A priori this implies only that ρ normalizes the subgroup generated by τ. Note however that on $\overline{\mathcal{H}}_k$ the automorphism τ acts by a fixed root of unity ζ_k. If $\rho\tau\rho^{-1}$ is a non-trivial power of τ, this leads to another (though isomorphic) linear subvariety. We conclude that ρ indeed commutes with τ. ∎

The aim of the following paragraphs is to rewrite the evaluation Lemma 3.10 in our context in order to find the shape of the formula in Corollary 1.5. We elaborate on basic definitions to distinguish notions of isomorphisms and automorphisms. The underlying graph of an enhanced (k-)level graph can be written as a tuple

$$\Gamma = \big(V, H, L,\ a\colon H \cup L \to V,\ i\colon H \to H\big),$$

where V, H and L are the sets of vertices, half-edges and legs, a is the attachment map and i is the fixpoint free involution that specifies the edges. An isomorphism of graphs $\sigma\colon \Gamma \to \Gamma'$ is a pair of bijections

$$\sigma = \big(\sigma_V\colon V \to V',\ \sigma_H\colon H \to H'\big)$$

that preserve the attachment of the half-edges and legs and the identification of the half-edges to edges, i.e., the diagrams

$$\begin{array}{ccc} H \cup L & \xrightarrow{a} & V \\ \big\downarrow{\scriptstyle \sigma_H \cup \mathrm{id}_L} & & \big\downarrow{\scriptstyle \sigma_V} \\ H' \cup L & \xrightarrow{a'} & V', \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{i} & H \\ \big\downarrow{\scriptstyle \sigma_H} & & \big\downarrow{\scriptstyle \sigma_H} \\ H' & \xrightarrow{i'} & H' \end{array} \tag{26}$$

commute. If the graph is an enhanced level graph, we additionally ask that σ preserves the enhancements and level structure. In the presence of a deck transformation τ, we moreover ask that σ commutes with τ.

In the sequel we will encounter isomorphisms of graphs with the same underlying sets of vertices and half-edges. We emphasize that in this case an isomorphism σ is an *automorphism* if and only if it preserves the maps a and i, i.e., if

$$\sigma_V^{-1} \circ a \circ (\sigma_H \cup \mathrm{id}_L) = a \quad \text{and} \quad \sigma_H^{-1} \circ i \circ \sigma_H = i. \tag{27}$$

We now define the group of level-wise half-edge permutations compatible with the cycles of τ, i.e., we let

$$\mathbf{G} := \mathbf{G}_\pi = \prod_{i=0}^{-L} G\big(\mathcal{H}_k(\pi_{[i]})\big),$$

where $G(\mathcal{H}_k(\pi_{[i]}))$ is the group G from (24) applied to the i-th level stratum. An element of the group $\mathbf{G}$ is a permutation $g\colon H \cup L \to H \cup L$ and acts on a graph $\widehat{\Gamma}$ via $g \cdot \widehat{\Gamma} = (V, H, L, a \circ g, i)$.

There is a natural action of the group $\mathbf{G}$ on the set of all (possibly disconnected) graphs with the same set of underlying vertices as $\widehat{\Gamma}_{\mathrm{mp}}$. We denote by

$$\mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma}) := \big\{g \in \mathbf{G} : g\widehat{\Gamma} \cong \widehat{\Gamma}\big\} \tag{28}$$

the stabilizer. Note that this is in general not a group, as it is not the stabilizer of an element but of an isomorphism class. We also denote by $\mathrm{Stab}_{\mathbf{G}}(\mathcal{H}(\pi))$ the set of elements of $\mathbf{G}$ which fix the adjacency data (or equivalently the 1-level graphs) of the level-wise linear manifolds $\mathcal{H}(\pi_{[i]})$, i.e., elements which permute vertices with the same signature and permute legs of the same order on the same vertex.

Lemma 7.5. *We have*

$$|\operatorname{Aut}_{\mathcal{H}_k}(\widehat{\Gamma})| \cdot |\operatorname{Stab}_{\mathbf{G}}(\widehat{\Gamma})| = |\operatorname{Aut}(\Gamma)| \prod_{e \in E(\Gamma)} \gcd(\kappa_e, k) \cdot |\operatorname{Stab}_{\mathbf{G}}(\mathcal{H}(\pi))|.$$

Proof. Fix a cover $\widehat{\Gamma} \to \Gamma$. We may assume that the vertices of Γ are $\{1, \dots, v_\Gamma\}$, the legs are $\{1, \dots, n\}$ and the half-edges are $\{1^\pm, \dots, h_\Gamma^\pm\}$ with the convention that

$$i(h^\pm) = h^\mp.$$

For $\widehat{\Gamma}$, we may assume that the preimages of vertex v are $(v, 1), \dots, (v, p_v)$ such that $\tau((v, q)) = (v, q+1)$, where equality in the second entry is to be read mod p_v. Similarly, we index the legs of $\widehat{\Gamma}$ by tuples $(m, 1), \dots, (m, p_m)$ for $m = 1, \dots, n$, and the half-edges by tuples $(h^\pm, 1), \dots, (h^\pm, p_{h^\pm})$ for $h^\pm = 1, \dots, h_\Gamma^\pm$, again such that (h^+, q) and (h^-, q) form an edge.

We consider the group $\mathcal{P}$ of pairs of permutations $\sigma = (\sigma_V, \sigma_H)$ of the vertices and half-edges of $\widehat{\Gamma}$ that are of the following form: There exists $\gamma = (\gamma_V, \gamma_H) \in \operatorname{Aut}(\Gamma)$, integers $\lambda_v \in \mathbb{Z}/p_v\mathbb{Z}$ for any $v \in V(\Gamma)$, and integers $\mu_{h^\pm} \in \mathbb{Z}/p_{h^\pm}\mathbb{Z}$ for any $h^\pm \in E(\Gamma)$ such that

$$\sigma_V = \big\{(v, q) \mapsto \big(\gamma_V(v), q + \lambda_v\big)\big\} \quad \text{and} \quad \sigma_H = \big\{(h^\pm, q) \mapsto \big(\gamma_H(h^\pm), q + \mu_{h^\pm}\big)\big\}.$$

We let this group act on $\widehat{\Gamma}$ via $\sigma \cdot \widehat{\Gamma} = (V, H, L, \sigma_V^{-1} \circ a \circ (\sigma_H \cup \operatorname{id}_L), i)$. An element $\sigma \in \mathcal{P}$ acts always as an isomorphism since the diagrams (26) commute. If we denote by e the edge given by $h^\pm$, we have $p_{h^\pm} = \gcd(\kappa_e, k)$. Hence, the group $\mathcal{P}$ has cardinality

$$|\mathcal{P}| = |\operatorname{Aut}(\Gamma)| \cdot \prod_{e \in E(\Gamma)} \gcd(\kappa_e, k) \cdot \prod_{v \in V(\Gamma)} p_v.$$

Recall that the group $\mathbf{G}$ is a product cyclic groups and thus Abelian. The stabilizer $\operatorname{Stab}_{\mathbf{G}}(\mathcal{H}_k(\pi))$ has a subgroup Stab^f where only half-edges and legs attached to the same vertex are permuted (the superscript f is for *fixed*), i.e., the elements $g \in \operatorname{Stab}^f$ are exactly those for which $a \circ g = a$. The quotient $\operatorname{Stab}^p := \operatorname{Stab}_{\mathbf{G}}(\mathcal{H}_k(\pi)) / \operatorname{Stab}^f$ can be identified with those elements of $\mathbf{G}$ that permute legs and half-edges in such a way that whenever a leg or half-edge attached to a vertex v_1 is moved to another vertex v_2, then all the legs and half-edges attached to v_1 are moved to v_2. So we may alternatively identify Stab^p with τ-invariant permutations of the vertices of $\widehat{\Gamma}$ (hence the superscript p for *permutation*). This yields $|\operatorname{Stab}^p| = \prod_{v \in V(\Gamma)} p_v$.

The group $\mathcal{P}$ comes with a commutative triangle

$$\begin{array}{ccc} \operatorname{Aut}_{\mathcal{H}}(\widehat{\Gamma}) & \hookrightarrow & \mathcal{P} \\ & \searrow & \big\downarrow \\ & & \operatorname{Aut}(\Gamma), \end{array}$$

where the vertical map is the forgetful map, the diagonal map is the quotient by G-map and the horizontal map is natural injection. Since we computed above $|\mathcal{P}|$, we know that the kernel of the surjective map $\mathcal{P} \to \mathrm{Aut}(\Gamma)$ has cardinality

$$\prod_{e \in E(\Gamma)} \gcd(\kappa_e, k) \cdot \prod_{v \in V(\Gamma)} p_v.$$

We note now that the group Stab^f acts on the set $\mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma})$ and we denote by $\mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma})/\mathrm{Stab}^f$ the space of orbits. We are done if we can identify elements of $\mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma})/\mathrm{Stab}^f$ with elements of the cosets in $\mathcal{P}/\mathrm{Aut}_{\mathcal{H}}(\widehat{\Gamma})$.

For this identification, first consider $g \in \mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma})$. By definition, there exists an isomorphism $\sigma(g)\colon g \cdot \widehat{\Gamma} \to \widehat{\Gamma}$ such that $g \cdot \widehat{\Gamma} = \sigma(g)(\widehat{\Gamma})$. This induces a map $\sigma\colon \mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma}) \to \mathcal{P}$. Note that Stab^f is a subgroup of $\mathrm{Aut}_{\mathcal{H}}(\widehat{\Gamma})$. If we had chosen a different representative g' in the orbit $g \cdot \mathrm{Stab}^f$, the resulting element $\sigma(g') \in \mathcal{P}$ would differ by an element of $\mathrm{Aut}_{\mathcal{H}}(\widehat{\Gamma})$. Hence, σ induces a well-defined map

$$\mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma})/\mathrm{Stab}^f \to \mathcal{P}/\mathrm{Aut}_{\mathcal{H}}(\widehat{\Gamma}).$$

We now construct an inverse map for σ. For any $\rho \in \mathcal{P}$, we need to find an element $g \in \mathbf{G}$ such that $\sigma(g) = \rho$, i.e., such that $g \cdot \widehat{\Gamma} = \rho(\widehat{\Gamma})$. This implies that g must satisfy the equation

$$a \circ g = \rho_V^{-1} \circ a \circ (\rho_H \cup \mathrm{id}_L),$$

which determines the element g up to the action of Stab^f. The resulting g does not depend on the choice of a representative of the coset $\rho/\mathrm{Aut}_{\mathcal{H}}(\widehat{\Gamma})$ because of (27). ∎

We let now

$$S(\pi) = \frac{|G|}{|\mathbf{G}|} \cdot \frac{|\mathrm{Stab}_{\mathbf{G}}(\widehat{\Gamma})|}{|\mathrm{Stab}_G(\widehat{\Gamma})|} = \frac{|\mathrm{Stab}_{\mathbf{G}/G}(\widehat{\Gamma})|}{\prod_e \gcd(\kappa_e, k)^2}, \tag{29}$$

where the stabilizers are defined in a way analogous to (28).

Remark 7.6. We have the ratio $S(\pi) = 1$ for many coverings of graphs $\pi\colon \widehat{\Gamma} \to \Gamma$, e.g., when all vertices of Γ have exactly one preimage in $\widehat{\Gamma}$. In this case $\mathbf{G}/G$ only permutes half-edges adjacent to one vertex, and this always stabilizes the graph. Thus $S(\pi) = 1$, as $|\mathbf{G}/G| = \prod_e \gcd(\kappa_e, k)^2$. More generally, $S(\pi) = 1$ if each edge of Γ is adjacent to at least one vertex which has exactly one preimage in $\widehat{\Gamma}$. In this case it is straightforward to verify that the obvious generators of $\mathbf{G}/G$ are stabilizing the graph.

If there are vertices of Γ with more than one preimage in $\widehat{\Gamma}$, then $S(\pi)$ is in general non-trivial. Consider for example the covering of graphs π depicted in Figure 2, for which $S(\pi) = 1/2$.

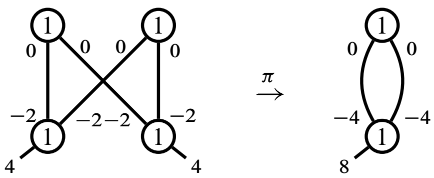

Figure 2. A covering of graphs $\pi\colon \widehat{\Gamma} \to \Gamma$ in $\Xi^2\bar{\mathcal{M}}_{3,1}(8)$ with non-trivial $S(\pi)$.

As a consequence of the degree computation in Lemma 7.4 and Lemma 7.5, we can write an evaluation lemma for k-differentials analogous to Lemma 3.10. We give two versions, for $\mathcal{H}_k$ and $\mathcal{Q}$ respectively.

Lemma 7.7. *Let* $(\pi\colon \widehat{\Gamma}_{\mathrm{mp}} \to \Gamma) \in \mathrm{LG}_L(\mathcal{H}_k^{\mathrm{mp}})$ *and* $\widehat{\Gamma}$ *a marked version of* $\widehat{\Gamma}_{\mathrm{mp}}$. *Suppose that* $\alpha_\pi \in \mathrm{CH}_0(D_\pi^{\mathcal{H}_k})$ *and* $\beta_\pi \in \mathrm{CH}_0(D_\pi^{\mathcal{Q}})$ *are top degree classes and that*

$$c_\pi^* \alpha_\pi = p_\pi^* \prod_{i=0}^{-L} \alpha_i \quad \text{and} \quad c_\pi^* d_\pi^* \beta_\pi = p_\pi^* \mathbf{s}_\pi^* \prod_{i=0}^{-L} \beta_i$$

for some α_i *and* β_i. *Then*

$$\int_{D_\pi^{\mathcal{H}_k}} \alpha_\pi = S(\pi) \cdot \frac{\prod_{e\in E(\Gamma)} \kappa_e}{|\mathrm{Aut}(\Gamma)| \cdot \prod_{e\in E(\Gamma)} \gcd(\kappa_e, k)^2 \cdot \ell_{\widehat{\Gamma}}} \cdot \prod_{i=0}^{-L} \int_{\mathcal{H}_k(\pi_{[i]})} \alpha_i$$

and

$$\int_{D_\pi^{\mathcal{Q}}} \beta_\pi = S(\pi) \cdot \frac{\prod_{e\in E(\Gamma)} \kappa_e}{k^L \cdot |\mathrm{Aut}(\Gamma)| \cdot \ell_{\widehat{\Gamma}}} \cdot \prod_{i=0}^{-L} \int_{\mathcal{Q}(\pi_{[i]})} \beta_i .$$

Proof. In order to show the first statement, we first apply Lemma 7.4 and note that the map p_π is not surjective in general. It is now enough to check that the number of adjacency data appearing in $\mathcal{H}_k(\pi)$ is $|\mathbf{G}|/|\operatorname{Stab}_{\mathbf{G}}(\mathcal{H}_k(\pi))|$, while the one appearing in the image of p_π is $|G|/|\operatorname{Stab}_G \widehat{\Gamma}|$. We finally use Lemma 7.5 to rewrite the prefactor. For the second statement, we additionally apply Lemma 7.2 and Lemma 7.3. ∎

We are finally ready to prove Corollary 1.5.

Proof of Corollary 1.5. The orbifold Euler characteristics of $\mathcal{Q} = \mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ and $\mathcal{H}_k$ are related by

$$\chi\big(\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)\big) = \frac{1}{\deg(s)} \cdot \chi(\mathcal{H}_k).$$

We apply the general Euler characteristic formula in the form (20) to $\mathcal{H}_k$ and group the level graphs $\widehat{\Gamma} \in \mathrm{LG}_L(\mathcal{H}_k)$ by those with the same graph $\widehat{\Gamma}_{\mathrm{mp}}$ that is marked partially. Since the integrals do not depend on the marking, we obtain

$$\chi(\mathcal{Q}) = \frac{k}{|G|}(-1)^d \sum_{L=0}^{d} \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_L(\mathcal{H}_k^{\mathrm{mp}})} N_\pi^\top \cdot \ell_{\widehat{\Gamma}} \cdot \int_{D_\pi^{\mathcal{H}_k}} \prod_{i=-L}^{0} (\xi_{\widehat{\Gamma},\mathcal{H}_k}^{[i]})^{d_\Gamma^{[i]}},$$

where we used the notation that $\widehat{\Gamma}$ is a fully marked representative of $\widehat{\Gamma}_{\mathrm{mp}}$. Thanks to Lemma 3.9, we can apply Lemma 7.7 and convert the integral over $D_\pi^{\mathcal{H}_k}$ into a ξ-integral over the product of $\mathcal{H}_k(\pi_{[i]})$. We hence obtain

$$\begin{aligned}
&\chi\big(\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)\big)\\
&= \frac{k}{|G|}\cdot(-1)^d \sum_{L=0}^{d} \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_L(\mathcal{H}_k^{\mathrm{mp}})} S(\pi)\frac{\prod_{e\in E(\Gamma)}\kappa_e \cdot N_\pi^\top}{|\mathrm{Aut}(\Gamma)|\cdot\prod_e \gcd(\kappa_e,k)^2}\\
&\quad \cdot \prod_{i=0}^{-L}\int_{\mathcal{H}_k(\pi_{[i]})} \xi^{d_\pi^{[i]}}\\
&= \Big(\frac{-1}{k}\Big)^d \sum_{L=0}^{d} \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_L(\mathcal{Q})} S(\pi)\cdot\frac{\prod_{e\in E(\Gamma)}\kappa_e \cdot N_\pi^\top}{|\mathrm{Aut}(\Gamma)|}\cdot\prod_{i=0}^{-L}\int_{\mathcal{Q}(\pi_{[i]})}\zeta^{d_\pi^{[i]}}.
\end{aligned}$$

For the second equality, we used that

$$s^*\zeta = k\xi, \quad \text{and hence } d_*\xi = \frac{\deg(s)}{k}\zeta \tag{30}$$

for any level stratum, together with the dimension statement of Proposition 3.4. The final result is what we claimed in Corollary 1.5. ∎

7.4. Evaluating tautological classes

In this section we explain how to evaluate any top degree class of the form

$$\beta := \zeta^{p_0}\psi_1^{p_1}\cdots\psi_n^{p_n}\cdots[D_{\pi_1}^{\mathcal{Q}}]\cdots[D_{\pi_w}^{\mathcal{Q}}] \in \mathrm{CH}_0(\overline{\mathcal{Q}}) \tag{31}$$

for any generalized stratum $\overline{\mathcal{Q}}$ of k-differentials. First, we show how to transform the previous class into the form

$$\beta = \sum_i \psi_1^{q_{i,1}}\cdots\psi_1^{q_{i,n}}[D_{\sigma_i}^{\mathcal{Q}}].$$

Then by Lemma 7.7, we can write every summand of β as a product of ψ-classes evaluated on generalized strata of k-differentials. We finally will explain how to evaluate such classes.

Let us start with the first task. The relations in the Chow ring of a general linear submanifold we obtained in Section 4 immediately apply to the covering $\overline{\mathcal{H}}_k$ and we want to restate them in the Chow ring of the generalized stratum $\overline{\mathcal{Q}}$ of k-differentials. Let i be the index of a marked point in $\overline{\mathcal{Q}}$ and (i, j) be the index of a preimage of this point in $\overline{\mathcal{H}}_k$. Moreover, let m_i denote the order of the k-differential at the i-th marked point, and let $\widehat{m}_{i,j}$ denote the order of the Abelian covering at the (i, j)-th marked point. Then the relation

$$\psi_{i,j} = \frac{\gcd(m_i, k)}{k} \cdot d^*\psi_i \tag{32}$$

holds; see, for example, [52, Lemma 3.9]. Using the relation

$$\widehat{m}_{i,j} + 1 = (m_i + k)/\gcd(m_i, k)$$

and applying push-pull, we obtain

$$(\widehat{m}_{i,j} + 1)d_*\psi_{i,j} = \frac{\deg(d)}{k}(m_i + k)\psi_i. \tag{33}$$

We are now in a position to write the analog of Proposition 4.1 for the first Chern class $\zeta \in \mathrm{CH}^1(\overline{Q})$ of the tautological line bundle on the stratum of k-differentials.

Corollary 7.8. *The class ζ can be expressed as*

$$\begin{aligned}\zeta &= (m_i + k)\psi_i - \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in_i \mathrm{LG}_1(\overline{\mathcal{Q}})} k\ell_{\widehat{\Gamma}_{\mathrm{mp}}}[D_\pi^{\mathcal{Q}}] \\ &= (m_i + k)\psi_i - \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in_i \mathrm{LG}_1(\overline{\mathcal{Q}})} S(\pi)\frac{\prod_{e\in E(\Gamma)}\kappa_e}{|\mathrm{Aut}(\Gamma)|}\, \mathrm{cl}_{\pi,*}\, p_\pi^*\mathbf{s}_\pi^*[\mathcal{Q}(\pi)],\end{aligned}$$

where ${}_i\mathrm{LG}_1(\overline{\mathcal{Q}})$ are covers of two-level graphs with the leg i on the lower level and $\mathrm{cl}_\pi = \mathrm{i}_\pi \circ d_\pi \circ c_\pi$ is the clutching morphism analogous to (13).

Proof. The first equation is obtained by pushing forward the equation in Proposition 4.1 along d and using the relations (30) and (33). The second equation is obtained from the first by Lemma 7.7. ■

Remark 7.9. The expression given by the second line of Corollary 7.8 reproves the formula of [51, Theorem 3.12] and computes explicitly the coefficients appearing in loc. cit., which were computed only for special two-level graphs.

To state the formula for the normal bundle, let

$$\mathcal{L}_\pi^\top = \mathcal{O}_{D_\pi^{\mathcal{Q}}}\Bigg(\sum_{\substack{(\sigma:\widehat{\Delta}_{\mathrm{mp}}\to\Delta)\in\mathrm{LG}_2(\overline{\mathcal{Q}})\\ \delta_2(\sigma)=\pi}} \ell_{\widehat{\Delta},1} D_\sigma^{\mathcal{H}}\Bigg)$$

denote the top level correction bundle.

Corollary 7.10. *Suppose that D_π is a divisor in $\overline{\mathcal{Q}}$ corresponding to a covering of graphs $(\pi\colon \widehat{\Gamma}_{\mathrm{mp}} \to \Gamma) \in \mathrm{LG}_1(\overline{\mathcal{Q}})$. Then the first Chern class of the normal bundle is given by*

$$c_1(\mathcal{N}_\pi) = \frac{1}{\ell_{\widehat{\Gamma}}}\Big(-\frac{1}{k}\zeta_\pi^\top - c_1(\mathcal{L}_\pi^\top) + \frac{1}{k}\zeta_\pi^\perp\Big) \in \mathrm{CH}^1(D_\pi^{\mathcal{Q}}),$$

where $\zeta_\pi^\top$, resp. $\zeta_\pi^\perp$, is the first Chern class of the line bundle generated by the top, resp. bottom, level multi-scale component.

Proof. We can pull back the right- and left-hand sides of the relation via d. Using the expression (30), we see that the pulled-back relation holds since it agrees with the one of Proposition 4.4. Since d is a quasi-finite proper unramified map, we are done. The same argument, together with Proposition 4.5, works for the second statement about horizontal divisors. ∎

Using the same arguments as [13, Proposition 8.1], it is possible to show an excess intersection formula in this context of k-differentials. We will not explicitly do this here since the methods and the result are exactly parallel to the original ones for Abelian differentials. Using the previous ingredients we can then reduce the computation of the class β in (31) to the computation of a top-degree product of ψ-classes

$$\alpha := \psi_1^{p_1} \cdots \psi_n^{p_n} \in \mathrm{CH}_0(\overline{\mathcal{Q}})$$

on a generalized stratum. If we can describe the class of a generalized stratum in its corresponding moduli space of pointed curves, then we are done since it is possible to compute top-degree tautological classes on the moduli space of curves, e.g., with the SageMath package `admcycles`, see [16].

One of the advantages in comparison to the situation with general linear submanifolds (as explained in Section 4) is that the fundamental classes of strata of primitive k-differentials $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ are known in $\overline{\mathcal{M}}_{g,n}$, see [1].

More generally, if $\mathcal{Q}$ parametrizes k-differentials on a curve with connected τ-quotient, which are d-th powers of primitive $k' := k/d$-differentials, then we can compare ψ-classes on $\overline{\mathcal{Q}}$ to ψ-classes on the stratum of primitive k' differentials $\mathbb{P}\Xi^{k'}\overline{\mathcal{M}}_{g,n}(\mu/d)$ via the diagram

$$\begin{array}{ccc} \mathcal{H}_k^{\mathrm{mp}}(\mu) & \xrightarrow{\ \phi\ } & \mathcal{H}_{k'}^{\mathrm{mp}}(\mu/d) \\ \big\downarrow{\scriptstyle d_1} & & \big\downarrow{\scriptstyle d_2} \\ \mathcal{Q} & & \mathbb{P}\Xi^{k'}\overline{\mathcal{M}}_{g,n}(\mu/d), \end{array}$$

where the map ϕ sends the disconnected curve

$$\Big(\bigcup_{i=1}^{d} \widehat{X}_i, \bigcup_{i=1}^{d} \widehat{\mathbf{z}}_i, \bigcup_{i=1}^{d} \omega_i, \tau\Big)$$

to $(\widehat{X}_1, \mathbf{z}_1, \omega_1, \tau^d|_{\widehat{X}_1})$. The map ϕ has degree $\deg(\phi) = d^{n-1}$, since up to the action of τ there are many such ways to distribute the marked points $\widehat{\mathbf{z}}$ onto the connected components of $\widehat{X}$. Using $\deg(d_1) = 1/k$ and $\deg(d_2) = 1/k'$, we can evaluate α as

$$\int_{\mathcal{Q}} \alpha = d^n \int_{\mathbb{P}\Xi^{k'}\bar{\mathcal{M}}_{g,n}(\mu/d)} \psi_1^{p_1} \cdots \psi_n^{p_n}.$$

If $\mathcal{Q}$ parametrizes primitive differentials on disconnected curves, then $\int_{\mathcal{Q}} \alpha = 0$, since we go down in dimension by looking at the image of the projection to the moduli spaces of curves.

It remains to explain how to evaluate intersection numbers in the presence of residue conditions. In addition to the space $\mathfrak{R}$ defined starting from a τ-invariant partition $\lambda_{\mathfrak{R}}$, we consider the linear subspace

$$R := \Big\{ (r_i)_{i\in H_p} \in \mathbb{C}^{H_p} : \sum_{i\in\mathcal{A}^{-1}(\widehat{X}')} r_i = 0 \text{ for all } \widehat{X}' \in \pi_0(\widehat{X}), \\ r_i = \zeta_k^{-1} r_{\tau(i)} \text{ for all } i \in H_p \Big\}$$

cut out by the residue theorem on each component and the deck transformation. Recall that $\lambda_{\mathfrak{R}}$ is τ-invariant. Let $\lambda_{\mathfrak{R}_0}$ denote a subset of $\lambda_{\mathfrak{R}}$ obtained by removing one element, and let $\mathfrak{R}_0$ denote the new set of residue conditions. For ease of notation, let for now

$$H_k^{\mathfrak{R}} := \mathbb{P}\Omega\mathcal{H}_k^{[\mathfrak{R},\mathcal{A}]} \quad \text{and} \quad H_k^{\mathfrak{R}_0} := \mathbb{P}\Omega\mathcal{H}_k^{[\mathfrak{R}_0,\mathcal{A}]}.$$

If $R \cap \mathfrak{R} = R \cap \mathfrak{R}_0$, then $\mathcal{H}_k^{\mathfrak{R}} = \mathcal{H}_k^{\mathfrak{R}_0}$. So assume that $R \cap \mathfrak{R} \neq R \cap \mathfrak{R}_0$, in which case $\mathcal{H}_k^{\mathfrak{R}} \subsetneq \mathcal{H}_k^{\mathfrak{R}_0}$ is a divisor since removing one element from $\lambda_{\mathfrak{R}}$ forces to remove its τ-orbit. For a divisor $D_\pi^{\mathcal{H}_k^{\mathfrak{R}}} \subseteq \bar{\mathcal{H}}_k^{\mathfrak{R}}$, we denote by $\mathfrak{R}^\top$ the residue conditions induced by $\mathfrak{R}$ on the top-level stratum $\mathcal{H}_k(\pi_{[0]})$. It can be simply computed by discarding from the parts of $\lambda_{\mathfrak{R}}$ all indices of legs that go to lower level in $D_\pi^{\mathcal{H}_k^{\mathfrak{R}}}$. Moreover, we denote be $R^\top$ the linear subspace belonging to the top-level stratum of π that is cut out by the residue theorem and the deck transformation.

Proposition 7.11. *The class of $\bar{\mathcal{H}}_k^{\mathfrak{R}}$ compares inside the Chow ring of $\bar{\mathcal{H}}_k^{\mathfrak{R}_0}$ to the class ξ by the formula*

$$[\bar{\mathcal{H}}_k^{\mathfrak{R}}] = -\xi - \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_1^{\mathfrak{R}}(\bar{\mathcal{H}}_k^{\mathfrak{R}_0})} \ell_{\widehat{\Gamma}}[D_\pi^{\mathcal{H}_k^{\mathfrak{R}_0}}] - \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_{1,\mathfrak{R}}(\bar{\mathcal{H}}_k^{\mathfrak{R}_0})} \ell_{\widehat{\Gamma}}[D_\pi^{\mathcal{H}_k^{\mathfrak{R}_0}}],$$

where $\mathrm{LG}_1^{\mathfrak{R}}(\bar{\mathcal{H}}_k^{\mathfrak{R}_0})$ are the two-level graphs with $R^\top \cap \mathfrak{R}^\top = R^\top \cap \mathfrak{R}_0^\top$, i.e., where the GRC on top level induced by $\mathfrak{R}$ does no longer introduce an extra condition, and where $\mathrm{LG}_{1,\mathfrak{R}}(\bar{\mathcal{H}}_k^{\mathfrak{R}_0})$ are the two-level graphs where all the legs involved in the condition forming $\mathfrak{R} \setminus \mathfrak{R}_0$ go to the lower level.

Proof. The formula is obtained by intersecting the formula in [13, Proposition 8.3] with $\overline{\mathcal{H}}_k^{\mathfrak{R}_0}$ and thereby using the transversality statement from Proposition 3.2. ∎

By pushing down this relation along d and applying relation (30), we obtain a similar relation for a generalized stratum of k-differentials $\mathcal{Q}^{\mathfrak{R}}$ with residue conditions $\mathfrak{R}$.

Corollary 7.12. *The class of $\overline{\mathcal{Q}}^{\mathfrak{R}}$ compares inside the Chow ring of $\overline{\mathcal{Q}}^{\mathfrak{R}_0}$ to the class ζ by the formula*

$$[\overline{\mathcal{Q}}^{\mathfrak{R}}] = -\frac{1}{k}\zeta - \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_1^{\mathfrak{R}}(\overline{\mathcal{Q}}^{\mathfrak{R}_0})} \ell_{\widehat{\Gamma}}[D_\pi^{\mathcal{Q}^{\mathfrak{R}_0}}] - \sum_{(\pi:\widehat{\Gamma}_{\mathrm{mp}}\to\Gamma)\in \mathrm{LG}_{1,\mathfrak{R}}(\overline{\mathcal{Q}}^{\mathfrak{R}_0})} \ell_{\widehat{\Gamma}}[D_\pi^{\mathcal{Q}^{\mathfrak{R}_0}}],$$

where $\mathrm{LG}_1^{\mathfrak{R}}(\overline{\mathcal{Q}}^{\mathfrak{R}_0})$ are the two-level graphs with $R^\top \cap \mathfrak{R}^\top = R^\top \cap \mathfrak{R}_0^\top$, i.e., where the GRC on top level induced by $\mathfrak{R}$ does no longer introduce an extra condition and where $\mathrm{LG}_{1,\mathfrak{R}}(\overline{\mathcal{Q}}^{\mathfrak{R}_0})$ are the two-level graphs where all the legs involved in the condition forming $\mathfrak{R} \setminus \mathfrak{R}_0$ go to the lower level.

The last expression allows us, in the presence of residue conditions, to reduce to the previous situations without residue conditions when we want to evaluate α.

7.5. Values and cross-checks

In this section, we provide in Table 2 and Table 3 some Euler characteristics for strata of k-differentials. We abbreviate

$$\chi_k(\mu) := \chi\big(\mathbb{P}\Omega^k_{\mathrm{pr}}\mathcal{M}_{g,n}(\mu)\big)$$

for the orbifold Euler characteristic of strata of primitive k-differentials. Moreover, we provide several cross-checks for our values.

The second power of the projectivized Hodge bundle over $\mathcal{M}_2$ is the union of the strata of quadratic differentials of type (4), (2, 2), $(2, 1^2)$ and (1^4), if all of them are taken with unmarked zeros. (Note that there are no quadratic differentials of type (3, 1).) All quadratic differentials of type (4) are second powers of Abelian differentials of type (2). The stratum (2, 2) contains both primitive quadratic differentials and second powers of Abelian differentials of type (1, 1). From Table 2 and [13, Table 1], we read off that

$$\chi_1(2) + \frac{1}{2}\chi_2(2,2) + \frac{1}{2}\chi_1(1,1) + \frac{1}{2}\chi_2(2,1^2) + \frac{1}{4!}\chi_2(1^4) = -\frac{1}{80} = \chi(\mathbb{P}^2)\chi(\mathcal{M}_2).$$

Similarly, one checks for the third power of the projectivized Hodge bundle over $\mathcal{M}_2$ that the numbers in provided in Table 3 add up to $-1/48 = \chi(\mathbb{P}^4)\chi(\mathcal{M}_2)$. In the above checks we have used that $\chi(\mathcal{M}_2) = -1/240$ by [32].

μ	$(2,2)$	$(2,1^2)$	(1^4)	$(5,-1)$	$(4,1,-1)$
$\chi_2(\mu)$	$-\frac{1}{8}$	$\frac{1}{5}$	-1	$-\frac{7}{15}$	$\frac{6}{5}$
μ	$(3,2,-1)$	$(3,1^2,-1)$	$(2^2,1,-1)$	$(2,1^3,-1)$	$(1^5,-1)$
$\chi_2(\mu)$	$\frac{5}{3}$	-5	-6	26	-147

Table 2. Euler characteristics of the strata of primitive quadratic differentials in genus 2 with at most one simple pole.

Now consider the second power of the projectivized Hodge bundle twisted by the universal section over $\mathcal{M}_{2,1}$. It decomposes into the unordered strata (4), $(5,-1)$, $(4,1,-1)$, $(3,2,-1)$, $(2,1^2)$, $(3,1^2,-1)$, $(2^2,1,-1)$, $(2,1^3,-1)$, $(1^5,-1)$, $(4,0)$, $(2^2,0)$, $(2,1^2,0)$, $(1^4,0)$, the ordered stratum (2^2), $(2,1^2)$ (since the zero at the unique marked point is distinguished) and the partially ordered stratum (1^4). The stratum $(2,1^2)$ appears two times in the list: the first time the unique marked point is the zero of order 2, the second time it is one of the simple zeros. On the stratum (1^4) one of the simple zeros is distinguished, while the others may be interchanged. Note that

$$\chi_k(m_1,\ldots,m_n,0) = (2-2g-n)\chi_k(m_1,\ldots,m_n).$$

The contributions in Table 2 and [13, Table 1] add up to $1/30 = \chi(\mathbb{P}^3)\chi(\mathcal{M}_{2,1})$, where we have used that $\chi(\mathcal{M}_{2,1}) = 1/120$ by [32].

We present some further cross-checks suggested by the referee.

The stratum $\mathbb{P}\Omega^2_{\mathrm{pr}}\mathcal{M}_{2,3}(2,1,1)$ is isomorphic to the space of 3-marked curves where the markings are at a Weierstrass point and at two hyperelliptic conjugate points (see [40, Theorem 1.2]). The latter space is isomorphic to $\mathcal{M}_{0,7}/S_5$, where the symmetric group S_5 permutes the first five markings, while the last two markings correspond to the three marked points of the genus two curve under the hyperelliptic map. Then indeed we have $\chi_2(2,1,1) = \chi(\mathcal{M}_{0,7}/S_5) = 1/5$.

Similarly the stratum $\mathbb{P}\Omega^2_{\mathrm{pr}}\mathcal{M}_{2,2}(2,2)$ is isomorphic to the space of 2-marked curves where the markings are at the Weierstrass points. This space is a $(\mathbb{Z}/2\mathbb{Z})$-gerbe over $\mathcal{M}_{0,6}/S_4$, where the $\mathbb{Z}/2\mathbb{Z}$ comes from the hyperelliptic involution. Also in this case we get the correct number $\chi_2(2,2) = \chi(\mathcal{M}_{0,6}/S_4)/2 = -1/8$.

Finally, the stratum $\mathbb{P}\Omega^2_{\mathrm{pr}}\mathcal{M}_{2,4}(1,1,1,1)$ decomposes as the disjoint union of three copies of the space of curves with two marked pairs of Weierstrass points (the three possibilities arise as the way of grouping the four markings into two pairs). Each of these copies is a double cover of $\mathcal{M}_{0,8}/S_6$, leading to the correct number $\chi_2(1,1,1,1) = 6\cdot\chi(\mathcal{M}_0,8)/(6!) = -1$.

μ	(6)	$(5,1)$	$(4,2)$	$(3,3)$	$(4,1^2)$	$(3,2,1)$
$\chi_3(\mu)$	$\frac{1}{3}$	$-\frac{4}{5}$	$-\frac{9}{8}$	$-\frac{4}{3}$	$\frac{16}{5}$	4
μ	(2^3)	$(3,1^3)$	$(2^2,1^2)$	$(2,1^4)$	(1^6)	
$\chi_3(\mu)$	$\frac{41}{10}$	-16	$-\frac{52}{3}$	90	-567	

Table 3. Euler characteristics of the strata of primitive holomorphic 3-differentials in genus 2.

8. Ball quotients

The goal of this section is to prove Theorem 1.7, which gives an independent proof of the Deligne–Mostow–Thurston construction ([17, 54]) of ball quotients via cyclic coverings. For this proof of concept we consider the special case of surfaces, i.e., lattices in $\mathrm{PU}(1,2)$.

We first prove a criterion for showing that a two-dimensional smooth Deligne–Mumford stack is a ball quotient via the Bogomolov–Miyaoka–Yau equality. Such a criterion exists in many contexts, typically for pairs of a variety and a $\mathbb{Q}$-divisor with various hypothesis on the singularities a priori allowed, see for example [30, 31]. We anyway found no criterion for stacks in the literature. Only the inequality was proven in [12] and only in the compact case.

We then investigate the special two-dimensional strata of k-differentials of genus zero considered in Deligne–Mostow–Thurston, compute all the relevant intersection numbers and construct, via a contraction of some specific divisor, the smooth surface stack which we finally show to be a ball quotient.

8.1. Ball quotient criterion

We provide a version of the Bogomolov–Miyaoka–Yau inequality for stacks in the surface case, based on [37]. Singularity terminology and basics about the minimal model program can be found, e.g., in [38].

Proposition 8.1. *Suppose that $\overline{\mathfrak{B}}$ is a smooth Deligne–Mumford stack of dimension 2 with trivial isotropy group at the generic point and let $\mathcal{D}_1$ be a normal crossing divisor. Moreover, suppose that $K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1)^2 > 0$ and that $K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1)$ intersects positively any curve not contained in $\mathcal{D}_1$. Then the Miyaoka–Yau inequality*

$$c_1^2\big(K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1)\big) \le 3c_2\big(K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1)\big) \tag{34}$$

holds, with equality if and only if $\mathfrak{B} = \overline{\mathfrak{B}} \setminus \mathcal{D}_1$ is a ball quotient, i.e., there is a cofinite lattice $\Gamma \in (1, n)$ such that $\mathfrak{B} = [\mathbb{B}^2/\Gamma]$ as quotient stack, where

$$\mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

is the 2-ball.

Proof. Let $\mathcal{D}$ be the divisor defined as $\mathcal{D}_1$ together with the sum $\mathcal{D}_2$ of the divisors $\mathcal{D}_2^i$ with non-trivial isotropy groups of order b_i. Let $\pi : \overline{\mathfrak{B}} \to \overline{B}$ be the map to the coarse space and let $D_1 = \pi(\mathcal{D}_1)$, $D_2 = \sum(1 - 1/b_i)\pi(\mathcal{D}_2^i)$ and $D = D_1 + D_2$.

We start by assuming that the pair $(\overline{B}, D)$ is log-canonical and the pair $(\overline{B}, D_2)$ is log-terminal. We will show that this assumptions holds in our situation at the end of the proof.

Let $\overline{B}'$ be a log-minimal model given by contracting all the log-exceptional curves in D_1, i.e., contracting all irreducible curves $C \subseteq D_1$ with the properties $C^2 < 0$ and $(c_1(K_{\overline{B}}) + [D_1] + [D_2]) \cdot C \leq 0$, and let D_i' be the image of D_i for $i = 1, 2$. Then

$$K_{\overline{B}}(\log D_1) + D_2 = \pi^*\big(K_{\overline{B}'}(\log D_1') + D_2'\big).$$

Moreover, the log-canonical bundle satisfies

$$K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1) = \pi^*\big(K_{\overline{B}}(\log D_1) + D_2\big). \tag{35}$$

The fact that the support of the log-exceptional curves is in $\mathcal{D}_1$, together with (35), implies that $K_{\overline{B}'} + D_1' + D_2'$ is numerically ample. By the assumption above on the singularities we know that $(\overline{B}, D)$ is log-canonical. Hence, we are in the situation of applying [37, Theorem 12].

As a consequence of (35) we know that $c_1^2(K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1))$ coincides with the left-hand side of the Miyaoka–Yau inequality of [37, Theorem 12] applied to $\overline{B}'$ with boundary divisor $D_1' + D_2'$.

Moreover, by the Gauss–Bonnet theorem for DM-stacks (see, e.g., [13, Proposition 2.1]), we can also identify $c_2(K_{\overline{\mathfrak{B}}}(\log \mathcal{D}_1))$ with the right-hand side of the inequality of [37, Theorem 12] applied to $\overline{B}'$ with boundary divisor $D_1' + D_2'$, up to non-log-terminal singularities (similarly to how it was done in [12, Section 3.2]). By the assumption above, the pair $(\overline{B}, D_2)$ is log-terminal and so the previous identification of the right-hand side of [37, Theorem 12] with $c_2(K_{\overline{\mathfrak{B}}}(\log \mathcal{D}))$ is true without corrections.

This shows inequality (34) and that in the case of equality $\overline{B}' \setminus D_1' \cong \overline{B} \setminus D_1$ is a ball quotient, i.e., $\overline{B} \setminus D_1 \cong \mathbb{B}^2/\Gamma$. Moreover, in this case, the divisors D_2^i are the branch loci of π with branch indices b_i.

Since $\overline{B} \setminus D_1$ is the coarse space associated both to $\overline{\mathfrak{B}} \setminus \mathcal{D}_1$ and to $[\mathbb{B}^2/\Gamma]$, this implies that these two DM stacks have to differ by a composition of root constructions along divisors (see, e.g., [12, Section 3.1]). But since the branch indices of D_2^i can be identified with the isotropy groups of the corresponding divisors in $[\mathbb{B}^2/\Gamma]$, and since they coincide with the isotropy groups of the corresponding divisor $\overline{B} \setminus D_1$, we can identify $\overline{B} \setminus D_1$ with $[\mathbb{B}^2/\Gamma]$, as non-trivial root constructions would have changed the size of such isotropy groups.

We are finally left to show the assumption on the singularities. First, there exists a resolution $\widetilde{\mathfrak{B}}$ of $\overline{\mathfrak{B}}$ where the proper transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is a normal crossing divisor and the exceptional divisors $\mathcal{E}_i$ are log-exceptional, i.e.,

$$\mathcal{E}_i^2 < 0 \quad \text{and} \quad \big(c_1(K_{\widetilde{\mathfrak{B}}}) + [\widetilde{\mathcal{D}}_1]\big) \cdot \mathcal{E}_i \leq 0.$$

Indeed, such a resolution can be obtained by blowing-up smooth points of the DM stack, where the numerical conditions can be checked on an étale chart just as for the usual blow-up of a smooth point of a variety.

In this situation the corresponding exceptional divisors E_i for the coarse space resolution $\widetilde{B}$ of $\overline{B}$ are also log-exceptional, i.e.,

$$\big(c_1(K_{\widetilde{B}}) + [\widetilde{D}_1] + [\widetilde{D}_2]\big) \cdot E_i \leq 0 \quad \text{and} \quad E_i^2 \leq 0.$$

Since contracting log-exceptional divisors does not change the singularity type, this implies that to show that $(\overline{B}, D_1 + D_2)$ is log-canonical and $(\overline{B}, D_2)$ is log-terminal, it is enough to show that $(\widetilde{B}, \widetilde{D}_1 + \widetilde{D}_2)$ is log-canonical and $(\widetilde{B}, \widetilde{D}_2)$ is log-terminal.

In order to do this, we observe that in general since $(\widetilde{\mathfrak{B}}, \widetilde{\mathcal{D}})$ is a smooth DM stack with normal crossing divisor, then $(\widetilde{B}, \widetilde{D}_1 + \sum_i \widetilde{D}_2^i)$ is log-canonical. Details are given in [9, Theorem 5.1], using [33, Proposition A.13] . Then we can use that $\widetilde{B}$ has at worst klt singularities (since it is a surface with quotient singularities and by [38, Proposition 4.18]). It is easy to show that this implies that $(\widetilde{B}, \widetilde{D}_1 + \sum_i t_i \widetilde{D}_2^i)$ has log-canonical singularities and $(\widetilde{B}, \sum_i t_i \widetilde{D}_2^i)$ has log-terminal singularities, for any $0 \leq t_i < 1$. The desired statement follows then by setting $t_i = 1 - 1/b_i$. ∎

8.2. Strata of genus zero satisfying (INT)

Let $(a_1, \ldots, a_5)$ be positive integers such that $\gcd(a_1, \ldots, a_5, k) = 1$ with

$$\sum_{i=1}^{5} a_i = 2k,$$

and for all $i \neq j$,

$$\Big(1 - \frac{a_i}{k} - \frac{a_j}{k}\Big)^{-1} \in \mathbb{Z} \quad \text{if } a_i + a_j < k.$$

The first condition states that $\mu = (-a_1, \ldots, -a_5)$ is a type of a stratum of k-differentials on 5-pointed rational lines and that the intersection form on eigenspace

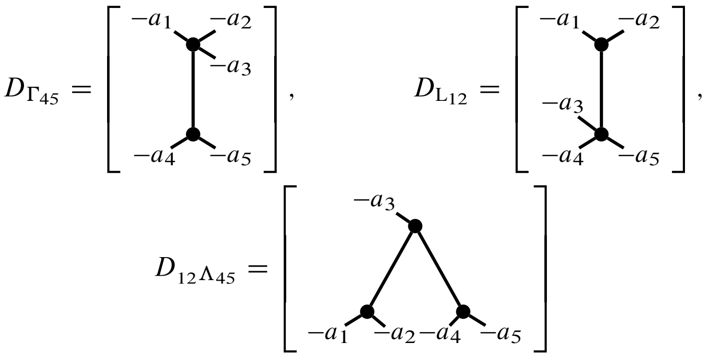

Figure 3. Level graphs of boundary divisors for strata $\Omega\mathcal{M}_{0,5}(a_1,\ldots,a_5)$.

giving period coordinates has the desired signature $(1,2)$. Imposing the gcd-condition lets us work without loss of generality with primitive k-differentials. The last condition is the condition (INT) of [17]. For Deligne–Mostow, this condition is key to ensure that the period map extends as an étale map over all boundary divisors. Thurston [54] uses this condition to show that his cone manifolds are indeed orbifolds. Mostow completed in [48] the $g=0$ picture by showing that up to the variant ΣINT from [47] these are the only ball quotient surfaces uniformized by the VHS of a cyclic cover of 5-punctured projective line. We recall from [17, Section 14] that there are exactly 27 five-tuples satisfying INT, and all of them satisfy in fact the integrality condition INT for all $i \neq j$ with $a_i + a_k \neq k$.

For us the condition INT has the most important consequence that the enhancements $\widehat{\kappa}_e$ of the Abelian covers of the level graphs are all one. This implies that ghost groups of all strata in this section are trivial. However, the condition INT also enters at other places of the following computations of automorphism groups and intersection numbers.

In the sequel we will use the notation $\mathcal{Q} = \Omega^k\mathcal{M}_{0,5}(a_1,\ldots,a_5)$. We now list the boundary divisors without horizontal edges. A short case inspection shows that the only possibilities are the level graphs $\Gamma = \Gamma_{ij}$ (see Figure 3, top left) and $\mathrm{L} = \mathrm{L}_{ij}$ (see Figure 3, top right), that yield the 'dumbbell' divisors with two or three legs on bottom level under the condition that the a_i's on lower level add up to less than k, and the level graphs $\Lambda = {}_{i,j}\Lambda_{p,q}$ that yield 'cherry' divisors (see Figure 3, bottom); V-shaped graphs are ruled out by $\sum a_i = 2k$. We define $\kappa_{i,j} := k - (a_i + a_j)$, which is both the k-enhancement of the single edge of $\Gamma_{i,j}$ and the negative of the k-enhancement of the single edge of $\mathrm{L}_{i,j}$.

Lemma 8.2. *Each of the graphs $\Gamma_{i,j}$, $L_{i,j}$ and ${}_{i,j}\Lambda_{p,q}$ is the codomain of a unique covering of graphs $\pi \in \mathrm{LG}_1(\overline{\mathcal{Q}})$ and for each such covering $S(\pi) = 1$.*

Proof. We will give the argument for $\Gamma_{1,2}$, the argument for the other graphs is similar. The number of preimages of the vertices of $\Gamma_{1,2}$ is $\gcd(k, a_1, a_2)$ for the bottom level and $\gcd(k, a_3, a_4, a_5)$ for the top level, while the edge has $\kappa_{1,2}$ preimages.

We claim that for any cover of graphs $\pi\colon \widehat{\Gamma}_{\mathrm{mp}} \to \Gamma_{1,2}$, the domain is connected. In fact, suppose there are k' components. This subdivides both the top level and the bottom level into subsets of equal size. This implies that $k' \mid \gcd(k, a_1, a_2)$ and $k' \mid \gcd(k, a_3, a_4, a_5)$, and hence $k' = 1$ because $\gcd(k, a_1, \ldots, a_5) = 1$.

To construct such a cover of graphs it suffices to prescribe one edge of $\widehat{\Gamma}_{\mathrm{mp}}$, the other edges are then forced, since τ-acts transitively on edges. Since the vertices on top and bottom level are indistinguishable (forming each one orbit τ-orbit) the resulting graph is independent of the choice of the first edge. In particular, $\widehat{\Gamma}_{\mathrm{mp}}$ is unique and $S(\pi) = 1$. ■

Next we compute (self)-intersection numbers of boundary divisors.

Lemma 8.3. *The self-intersection numbers of the boundary divisors of $\overline{\mathcal{Q}}$ are*

$$[D_\Gamma^{\mathcal{Q}}]^2 = -\frac{\kappa_{i,j}^2}{k^2} - \sum_{\substack{p<q,\, a_p+a_q<k \\ p,q\notin\{i,j\}}} \frac{\kappa_{i,j}\kappa_{p,q}}{k^2},$$

$$[D_L^{\mathcal{Q}}]^2 = -\frac{\kappa_{i,j}^2}{k^2} \quad \textit{and} \quad [D_\Lambda^{\mathcal{Q}}]^2 = -\frac{\kappa_{i,j}\kappa_{p,q}}{k^2}.$$

The mutual intersection numbers are

$$[D_\Gamma^{\mathcal{Q}}]\cdot[D_{\mathrm{L}}^{\mathcal{Q}}] = \begin{cases} |\kappa_{i,j}\kappa_{p,q}|/k^2 & \textit{if } \Gamma\cap\mathrm{L}\neq\emptyset, \\ 0 & \textit{otherwise}, \end{cases}$$

$$[D_\Gamma^{\mathcal{Q}}]\cdot[D_\Lambda^{\mathcal{Q}}] = \begin{cases} \kappa_{i,j}\kappa_{p,q}/k^2 & \textit{if } \Gamma\cap\Lambda\neq\emptyset, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. For the self-intersection numbers consider the formula in Corollary 7.10. As remarked above, the condition (INT) implies that all enhancements of the Abelian coverings are 1 and hence the same is true for the $\widehat{\ell}$-factor in the corollary. Let $\Delta_{i,j}^{p,q}$ denote the slanted cherry with points i, j on bottom level and points p, q on middle level. Together with Corollary 7.8 and Corollary 7.10, we obtain

$$[D_{\Gamma_{i,j}}^{\mathcal{Q}}]^2 = \frac{-1}{k}\zeta^\top - c_1(\mathcal{L}^\top) = -\frac{\kappa_{i,j}^2}{k^2}\int_{\bar{\mathcal{M}}_{0,4}} \psi_1 - \sum_{\substack{p<q,\, a_p+a_q<k \\ p,q\notin\{i,j\}}} [D_{\Delta_{i,j}^{p,q}}^{\mathcal{Q}}].$$

The degree of the slanted cherry is

$$\int_{\overline{\mathcal{Q}}}[D^{\mathcal{Q}}_{\Delta^{p,q}_{i,j}}] = \frac{\kappa_{i,j}\kappa_{p,q}}{k^2}$$

by applying the second formula in Lemma 7.7 and Lemma 8.2. The other numbers are obtained similarly. ∎

8.3. The contracted spaces

We want to construct the compactified ball quotient candidate $\overline{\mathfrak{B}}$ from $\overline{\mathcal{Q}}$ by contracting the all the divisors $D^{\mathcal{Q}}_{\mathrm{L}}$ and $D^{\mathcal{Q}}_{\Lambda}$. This is in fact possible.

Lemma 8.4. *The divisors $D^{\mathcal{Q}}_{\mathrm{L}}$ and $D^{\mathcal{Q}}_{\Lambda}$ of $\overline{\mathcal{Q}}$ are contractible. The DM-stack $\overline{\mathfrak{B}}$ obtained from $\overline{\mathcal{Q}}$ by contracting those divisors is smooth. If $D^{\mathfrak{B}}_{\widetilde{\mathrm{L}}}$ and $D^{\mathfrak{B}}_{\widetilde{\Lambda}}$ denote the points in $\mathfrak{B}$ obtained by contracting the corresponding divisors in $\mathcal{Q}$, then*

$$\int_{\overline{\mathfrak{B}}}[D^{\mathfrak{B}}_{\widetilde{\mathrm{L}}}] = \frac{\kappa_{i,j}^2}{k^2} \quad \textit{and} \quad \int_{\overline{\mathfrak{B}}}[D^{\mathfrak{B}}_{\widetilde{\Lambda}}] = \frac{\kappa_{i,j}\kappa_{p,q}}{k^2}.$$

Proof. For each of the two types of boundary divisors $D^{\mathcal{Q}}_{\mathrm{L}}$ and $D^{\mathcal{Q}}_{\Lambda}$, we will write a neighborhood U as quotient stack $[\widetilde{U}/G]$ with $\widetilde{U}$ smooth, and show that the preimage of the boundary divisor in $\widetilde{U}$ is a $\mathbb{P}^1$ with self-intersection number -1. Castelnuovo's criterion then implies that this curve is smoothly contractible. The order of G will be $k^2/\kappa_{i,j}^2$ for $D^{\mathcal{Q}}_{\mathrm{L}}$ and $k^2/\kappa_{i,j}\kappa_{p,q}$ for $D^{\mathcal{Q}}_{\Lambda}$. After contracting the covering $\mathbb{P}^1$, the quotient is a point with isotropy group G and the claim on the degrees follows.

We first consider a cherry divisor $D^{\mathcal{Q}}_{\Lambda}$. We let $D^{\mathcal{H}_k^{\mathrm{mp}}}_{\Lambda}$ denote its preimage in $\mathcal{H}_k^{\mathrm{mp}}$. Since all of the Abelian enhancements of the cover of ${}_{i,j}\Lambda_{p,q}$ are one, then the divisor $D^{\mathcal{H}_k^{\mathrm{mp}}}_{\Lambda}$ is irreducible, in fact isomorphic to $\mathbb{P}^1$ with coordinates the scales of the differential forms on the cherries.

We compute the order of the automorphism group of any point $(\widehat{X},\widehat{\omega})$ in $D^{\mathcal{H}_k^{\mathrm{mp}}}_{\Lambda}$. Suppose first that $(\widehat{X},\widehat{\omega})$ is generic. The irreducible components of $\widehat{X}$ group into three τ-orbits: The components $\widehat{X}^{\top}$ corresponding to the top-level vertex of ${}_{i,j}\Lambda_{p,q}$, the components $\widehat{X}^{\perp}_{i,j}$ corresponding to the vertex with marked points i, j, and the components $\widehat{X}^{\perp}_{p,q}$ corresponding to the vertex with marked points p, q. Observe that there are $\kappa_{i,j}$ edges between $\widehat{X}^{\top}$ and $\widehat{X}^{\perp}_{i,j}$ and $\kappa_{p,q}$ edges between $\widehat{X}^{\top}$ and $\widehat{X}^{\perp}_{p,q}$. The restriction of τ to each of the three (not necessarily connected) curves $\widehat{X}^{\top}$, $\widehat{X}^{\perp}_{i,j}$, and $\widehat{X}^{\perp}_{p,q}$ has order k. Given an automorphism of the complete curve $\widehat{X}$, its restrictions to $\widehat{X}^{\top}$ and $\widehat{X}^{\perp}_{i,j}$ need to agree on the $\kappa_{i,j}$ nodes, and the analog argument applies to $\widehat{X}^{\perp}_{p,q}$. Hence, after fixing the automorphism on the top-level curve $\widehat{X}^{\top}$, there are $k^2/\kappa_{i,j}\kappa_{p,q}$ possible choices for the automorphism on the two bottom-level curves left. Together with the k choices for the top-level automorphism, we obtain

$$|\operatorname{Aut}(\widehat{X},\widehat{\omega})| = \frac{k^3}{\kappa_{i,j}\kappa_{p,q}}.$$

As the non-representable map $\mathcal{H}_k^{\mathrm{mp}} \to \mathcal{Q}$ has degree $1/k$, this yields that the generic point of $D_\Lambda^{\mathcal{Q}}$ has an isotropy group of size $r := k^2/\kappa_{i,j}\kappa_{p,q}$. Exactly the same argument also applies to the two boundary points of $D_\Lambda^{\mathcal{Q}}$ corresponding to the slanted cherries.

The automorphism group is thus generated by multiplying the transversal t-parameter (compare Section 3.4) by an r-th root of unity in local charts covering all of ${}_{i,j}\Lambda_{p,q}$. We may thus take for U any tubular neighborhood of $D_\Lambda^{\mathcal{Q}}$ and take a global cover $\widetilde{U}$ of degree $k^2/\kappa_{i,j}\kappa_{p,q}$. Comparing with the degree of the normal bundle in Lemma 8.3 shows that preimage of $D_\Lambda^{\mathcal{Q}}$ in $\widetilde{U}$ is a (-1)-curve.

We now consider a dumbbell divisor $D_{\mathrm{L}}^{\mathcal{Q}}$. As above one checks that the isotropy group at the generic point of $D_{\mathrm{L}}^{\mathcal{Q}}$ is of order $k/|\kappa_{i,j}|$ and that the isotropy groups of the boundary points of the divisor have a quotient group of that order. Consider a tubular neighborhood of $D_{\mathrm{L}}^{\mathcal{Q}}$ and a degree $k/|\kappa_{i,j}|$ cover that trivializes the isotropy group at the generic point. Let $\widetilde{D}_{\mathrm{L}}^{\mathcal{Q}}$ be the preimage of the boundary divisor in this cover.

Let p, q, r denote the three marked points on the bottom level of a point in $\mathrm{L}_{i,j}$. By applying the above line of arguments again, the three boundary points of $\widetilde{D}_{\mathrm{L}}^{\mathcal{Q}}$ have cyclic isotropy groups of sizes $k/\kappa_{p,q}$, $k/\kappa_{p,r}$, and $k/\kappa_{q,r}$ respectively. The triangle group

$$T = T\Big(\frac{k}{\kappa_{p,q}}, \frac{k}{\kappa_{p,r}}, \frac{k}{\kappa_{q,r}}\Big)$$

is always spherical, because $a_i + a_j > k$ implies $a_p + a_q + a_r < k$, and hence

$$2 - \Big(1 - \frac{\kappa_{p,q}}{k}\Big) - \Big(1 - \frac{\kappa_{p,r}}{k}\Big) - \Big(1 - \frac{\kappa_{q,r}}{k}\Big) = 2 - 2\frac{a_p + a_q + a_r}{k} > 0.$$

This implies that the T-cover of $\widetilde{D}_{\mathrm{L}}^{\mathcal{Q}}$ ramified to order $k/\kappa_{p,q}$ along the divisor where $\{p, q\}$ have come together etc., trivializes the isotropy groups on the boundary divisor $\widetilde{D}_{\mathrm{L}}^{\mathcal{Q}}$ and the preimage of $\widetilde{D}_{\mathrm{L}}^{\mathcal{Q}}$ is a $\mathbb{P}^1$. More precisely, the isotropy groups of order $k/\kappa_{p,q}$ do not fix isolated points on the boundary divisor, but have one-dimensional stabilizer, the boundary divisors intersecting $\widetilde{D}_{\mathrm{L}}^{\mathcal{Q}}$. This implies that the above T-cover actually provides a chart of a full tubular neighborhood.

It remains to show that $|T| = k/|\kappa_{i,j}|$ in order to conclude with the normal bundle degree from Lemma 8.3 that this $\mathbb{P}^1$ is a (-1)-curve. To show this, recall that as T is spherical, there are only the cases

$$\Big(\frac{k}{\kappa_{p,q}}, \frac{k}{\kappa_{p,r}}, \frac{k}{\kappa_{q,r}}\Big) = (2,2,n) \quad \text{for } n \in \mathbb{N}_{\geq 2},$$
$$\Big(\frac{k}{\kappa_{p,q}}, \frac{k}{\kappa_{p,r}}, \frac{k}{\kappa_{q,r}}\Big) = (2,3,n) \quad \text{for } n \in \{3,4,5\}$$

to consider. In the first case, the order of $T(2,2,n)$ is $2n$, and assuming that

$$\frac{k}{\kappa_{p,q}} = \frac{k}{\kappa_{p,r}} = 2,$$

one easily checks that

$$2\frac{k}{\kappa_{q,r}} = \frac{k}{|\kappa_{i,j}|}$$

by using $\sum_i a_i = 2k$. In the second case, the order of $T(2,3,n)$ is $2\,\mathrm{lcm}(6,n)$, and the claimed equality follows with a similar argument. ∎

We will now compute the Chern classes of $\overline{\mathfrak{B}}$. Let $c\colon \overline{\mathcal{Q}} \to \overline{\mathfrak{B}}$ denote the contraction map. Let

$$\mathbf{\Gamma} := \big\{(i,j) : i < j,\ a_i + a_j < k\big\},$$
$$\mathbf{L} := \big\{(i,j) : i < j,\ a_i + a_j > k\big\}$$

be the pairs of integers appearing as indices of the $\Gamma_{i,j}$ and $L_{i,j}$. Let $\mathrm{I} = \mathrm{I}_{ij}^{pq}$ denote the common degeneration of Γ_{ij} and L_{pq}, i.e., the three-level graph with points p,q on bottom level, i,j on top level and the remaining point on the middle level. Accordingly, we write

$$\mathbf{\Lambda} := \big\{(i,j,p,q) : i < j,\ i < p < q,\ j \notin \{p,q\},\ a_i + a_j < k,\ a_p + a_q < k\big\},$$
$$\mathbf{I} := \big\{(i,j,p,q) : i < j,\ i < p < q,\ j \notin \{p,q\},\ a_i + a_j > k,\ a_p + a_q < k\big\}$$

for the quadruples of possible indices. Recall that D_{hor} is the union of all boundary divisors $D_{H_{ij}}$ whose level graph has a horizontal edge, i.e., corresponding to pairs (i,j) with $a_i + a_j = k$. We write

$$\mathbf{H} := \big\{(i,j) : i < j,\ a_i + a_j = k\big\}.$$

We summarize the intersections of the boundary divisors: The cherry $D^{\mathcal{Q}}_{i,j}{}_{\Lambda_{p,q}}$ intersects precisely $D^{\mathcal{Q}}_{\Gamma_{ij}}$ and $\Gamma^{\mathcal{Q}}_{pq}$. The divisor $D_{L_{ij}}$ intersects precisely the three divisors $D^{\mathcal{Q}}_{\Gamma_{ab}}$ for any pair (a,b) disjoint from $\{i,j\}$. For the divisor $D^{\mathcal{Q}}_{\Gamma_{ij}}$ consider any pair (p,q) of the three remaining points as $\{p,q,r\}$. This gives an intersection with a cherry if $a_p + a_q < k$, with a horizontal divisor if $a_p + a_q = k$ and with an L-divisor if $a_p + a_q > k$. Consequently, the divisor $D^{\mathcal{Q}}_{H_{ij}}$ intersects precisely the three divisors $D^{\mathcal{Q}}_{\Gamma_{ab}}$ for any pair (a,b) disjoint from $\{i,j\}$.

Lemma 8.5. *The self-intersection numbers of the boundary divisors of* $\overline{\mathfrak{B}}$ *are*

$$[D^{\mathfrak{B}}_{\Gamma_{i,j}}]^2 = -\frac{\kappa_{i,j}^2}{k^2} + \sum_{\substack{p<q,\, a_p+a_q>k \\ p,q\notin\{i,j\}}} \frac{\kappa_{i,j}^2}{k^2} \quad \textit{and} \quad [D^{\mathfrak{B}}_{H_{i,j}}]^2 = -1.$$

The mutual intersection numbers are for $\{i,j\}\cap\{p,q\}=\emptyset$ *given by*

$$[D^{\mathfrak{B}}_{\Gamma_{i,j}}]\cdot[D^{\mathfrak{B}}_{\Gamma_{p,q}}] = \frac{\kappa_{i,j}\kappa_{p,q}}{k^2} \quad \textit{and} \quad [D^{\mathfrak{B}}_{\Gamma_{i,j}}]\cdot[D^{\mathfrak{B}}_{H_{p,q}}] = \frac{\kappa_{i,j}}{k}$$

and for $|\{i,j,p\}|=3$ *by*

$$[D^{\mathfrak{B}}_{\Gamma_{i,j}}]\cdot[D^{\mathfrak{B}}_{\Gamma_{i,p}}] = \begin{cases} \kappa_{i,j}\kappa_{i,p}/k^2 & \textit{if } a_i+a_j+a_p<k, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We claim that the pullback of $[D^{\mathfrak{B}}_{\Gamma_{i,j}}]$ is given by

$$c^*[D^{\mathfrak{B}}_{\Gamma_{i,j}}] = [D^{\mathfrak{Q}}_{\Gamma_{i,j}}] + \sum_{\substack{p<q,\, a_p+a_q>k \\ p,q\notin\{i,j\}}} \frac{\kappa_{i,j}}{|\kappa_{p,q}|}[D^{\mathfrak{Q}}_{L_{p,q}}] + \sum_{\substack{p<q,\, a_p+a_q<k \\ p,q\notin\{i,j\}}} [D^{\mathfrak{Q}}_{i,j\,\Lambda_{p,q}}].$$

To determine the coefficients in the above expression, one may intersect the equation

$$c^*[D^{\mathfrak{B}}_{\Gamma_{i,j}}] = [D^{\mathfrak{Q}}_{\Gamma_{i,j}}] + \sum_{p,q} l_{p,q}[D^{\mathfrak{Q}}_{L_{p,q}}] + \sum_{p,q} \lambda_{p,q}[D^{\mathfrak{Q}}_{i,j\,\Lambda_{p,q}}]$$

with unknown coefficients with each of the divisors $[D^{\mathfrak{Q}}_{L_{p,q}}]$ and $[D^{\mathfrak{Q}}_{i,j\,\Lambda_{p,q}}]$ in turn. The left-hand side vanishes by push-pull, and the intersection numbers on the right-hand side are given by Lemma 8.3. The claimed intersection numbers involving only Γ-divisors follow again by Lemma 8.3.

The pullback of the horizontal divisor is given by $c^*[D^{\mathfrak{B}}_{H_{i,j}}] = [D^{\mathfrak{Q}}_{H_{i,j}}]$. The intersection number

$$[D^{\mathfrak{B}}_{\Gamma_{i,j}}]\cdot[D^{\mathfrak{B}}_{H_{p,q}}] = [D^{\mathfrak{Q}}_{\Gamma_{i,j}}]\cdot[D^{\mathfrak{Q}}_{H_{p,q}}]$$

follows from Lemma 7.7 and Lemma 8.2. Finally, by Proposition 4.5 and (32), the normal bundle of $[D^{\mathfrak{Q}}_{H_{i,j}}]$ is given by $-\psi_e$ in $\mathrm{CH}(D^{\mathfrak{Q}}_{H_{i,j}})$, where ψ_e is the ψ-class supported on the half edge of $H_{i,j}$ that is adjacent to the vertex with three adjacent marked points. ∎

Proposition 8.6. *The log canonical bundle on* $\overline{\mathfrak{B}}$ *has first Chern class*

$$c_1\big(\Omega^1_{\overline{\mathfrak{B}}}(\log D_{\mathrm{hor}})\big) = \sum_{i,j\in\boldsymbol{\Gamma}} \Big(\frac{k}{2\kappa_{i,j}} - 1\Big)[D^{\mathfrak{B}}_{\Gamma_{i,j}}] + \frac{1}{2}[D^{\mathfrak{B}}_{\mathrm{hor}}] \quad \textit{in } \mathrm{CH}_1(\mathfrak{B}). \tag{36}$$

Its square and the second Chern class are given by

$$c_1\big(\Omega^1_{\mathfrak{B}}(\log D_{\mathrm{hor}})\big)^2 = 6 - 3\sum_{i,j\in\mathbf{\Gamma}}\frac{\kappa_{i,j}}{k} + 3\sum_{i,j\in\mathbf{L}}\frac{\kappa_{i,j}^2}{k^2} + 3\sum_{i,j,p,q\in\mathbf{\Lambda}}\frac{\kappa_{i,j}\kappa_{p,q}}{k^2} \tag{37}$$

and

$$c_2\big(\Omega^1_{\mathfrak{B}}(\log D_{\mathrm{hor}})\big) = 2 - \sum_{i,j\in\mathbf{\Gamma}}\frac{\kappa_{i,j}}{k} + \sum_{i,j\in\mathbf{L}}\frac{\kappa_{i,j}^2}{k^2} + \sum_{i,j,p,q\in\mathbf{\Lambda}}\frac{\kappa_{i,j}\kappa_{p,q}}{k^2},$$

respectively.

Proof. To derive (36) from Theorem 1.1, we insert into

$$c_1\big(\Omega^1_{\mathcal{Q}}(\log D_{\mathrm{hor}})\big) = \frac{3}{k}\cdot\zeta + \sum_{\mathrm{L}}[D^{\mathcal{Q}}_{\mathrm{L}}] + \sum_{\Lambda}[D^{\mathcal{Q}}_{\Lambda}]$$

that $5\xi - \sum(m_i+k)\psi_i$ is a sum of boundary terms by the relation (7.8). Consider Keel's relation

$$\psi_i = \frac{1}{6}\sum_{\substack{c<d,\\ i\notin\{c,d\}}}\Delta_{cd} + \frac{1}{2}\sum_{a\neq i}\Delta_{ia},$$

where Δ_{ij} is the boundary divisor in $\bar{\mathcal{M}}_{0,5}$ where the points (i,j) have come together. We pull back this relation via the forgetful map $\pi\colon \mathbb{P}\Xi^k\bar{\mathcal{M}}_{0,5}(\mu)\to\bar{\mathcal{M}}_{0,5}$. Since this map is a root-stack construction and the isotropy groups of the divisors were computed in the proof of Lemma 8.4, we obtain

$$\pi^*\Delta_{ab} = \begin{cases} \frac{1}{|\kappa_{ab}|}[D^{\mathcal{Q}}_{\mathrm{L}_{ab}}] & \text{if } a+b<-k,\\ [D_{\mathrm{H}_{ab}}] & \text{if } a+b=-k,\\ \frac{1}{\kappa_{ab}}[D^{\mathcal{Q}}_{\Gamma_{ab}}] + \sum_{\substack{i<j,\,a_i+a_j<k\\ i,j\notin\{a,b\}}}\frac{1}{\kappa_{ab}}[D^{\mathcal{Q}}_{{}_{i,j}\Lambda_{a,b}}] & \text{if } a+b>-k.\end{cases}$$

Putting everything together, we find in $\mathrm{CH}_1(\mathcal{Q})$ that

$$\begin{aligned} c_1\big(\Omega^1_{\mathcal{Q}}(\log D_{\mathrm{hor}})\big) = &\sum_{i,j\in\mathbf{\Gamma}}\Big(\frac{k}{2\kappa_{i,j}}-1\Big)[D^{\mathcal{Q}}_{\Gamma_{i,j}}] + \sum_{i,j\in\mathbf{L}}\Big(\frac{k}{2|\kappa_{i,j}|}-1\Big)[D^{\mathcal{Q}}_{\mathrm{L}_{i,j}}]\\ &+\sum_{i,j,p,q\in\mathbf{\Lambda}}\Big(\frac{k}{2\kappa_{i,j}}+\frac{k}{2\kappa_{p,q}}-1\Big)[D^{\mathcal{Q}}_{{}_{i,j}\Lambda_{p,q}}] + \frac{1}{2}[D^{\mathcal{Q}}_{\mathrm{hor}}],\end{aligned}$$

and since the divisors $D^{\mathcal{Q}}_{\mathrm{L}_{i,j}}$ and $D^{\mathcal{Q}}_{{}_{i,j}\Lambda_{p,q}}$ are smoothly contractible, we deduce (36).

To derive (37), we first note that $-\frac{1}{4}|\mathbf{\Gamma}| + \frac{1}{2}|\mathbf{\Lambda}| + \frac{5}{4}|\mathbf{H}| + \frac{5}{4}|\mathbf{L}| = 5$ and that for $(i, j) \in \mathbf{L}$ the relation

$$1 + \sum_{\substack{p \in \{1,\dots,5\} \setminus \{i,j\} \\ \{q,r\} = \{1,\dots,5\} \setminus \{i,j,p\}}} \left(-\frac{\kappa_{p,q} + \kappa_{p,r}}{k} + 2\frac{\kappa_{p,q}\kappa_{p,r}}{k^2} + \frac{\kappa_{q,r}^2}{k^2} \right) = 4\frac{\kappa_{i,j}^2}{k^2}$$

holds because $\sum_i a_i = 2k$. Using those relations and the intersection numbers in Lemma 8.5, squaring (36) yields

$$c_1\big(\Omega^1_{\overline{\mathfrak{B}}}(\log D_{\mathrm{hor}})\big)^2 = 5 - \sum_{i,j \in \mathbf{\Gamma}} \left(2\frac{\kappa_{i,j}}{k} + \frac{\kappa_{i,j}^2}{k^2} \right) + 2 \sum_{i,j,p,q \in \mathbf{\Lambda}} \frac{\kappa_{i,j}\kappa_{p,q}}{k^2} + 4 \sum_{i,j \in \mathbf{L}} \frac{\kappa_{i,j}^2}{k^2}$$

and (37) follows because $\sum_i a_i = 2k$ implies

$$1 + \sum_{i,j \in \mathbf{\Gamma}} \left(-\frac{\kappa_{i,j}}{k} + \frac{\kappa_{i,j}^2}{k^2} \right) + \sum_{i,j,p,q \in \mathbf{\Lambda}} \frac{\kappa_{i,j}\kappa_{p,q}}{k^2} - \sum_{i,j \in \mathbf{L}} \frac{\kappa_{i,j}^2}{k^2} = 0.$$

The second Chern class can be computed as

$$\begin{aligned} c_2\big(\Omega^1_{\overline{\mathfrak{B}}}(\log D_{\mathrm{hor}})\big) = \chi(\mathcal{M}_{0,5}) &+ \sum_{i,j \in \mathbf{\Gamma}} \chi\big(D^{\mathfrak{B},\circ}_{\Gamma_{i,j}}\big) \\ &+ \sum_{i,j \in \mathbf{L}} \chi\big(D^{\mathfrak{B}}_{\widetilde{L}_{i,j}}\big) + \sum_{i,j,p,q \in \mathbf{\Lambda}} \chi\big(D^{\mathfrak{B}}_{i,j}{}_{\widetilde{\Lambda}_{p,q}}\big), \end{aligned}$$

where $\chi(D^{\mathfrak{B},\circ}_{\Gamma_{i,j}}) = \chi(D^{\mathcal{Q},\circ}_{\Gamma_{i,j}}) = \kappa_{i,j}/k$ by Lemma 7.7 and Lemma 8.2 and the Euler characteristics of the points are given in Lemma 8.4. ∎

8.4. The ball quotient certificate

We can finally put together the previous intersection numbers and use our ball quotient criterion to show that the contracted spaces are ball quotients.

Proof of Theorem 1.7. We apply Proposition 8.1 and check that first that the only log-exceptional curves for $c_1(\Omega^1_{\overline{\mathfrak{B}}}(\log D_{\mathrm{hor}}))$ are the components of D_{hor}. In fact, since the expression (36) is an effective divisor and since $\overline{\mathfrak{B}} \setminus \mathcal{D} \cong \mathcal{M}_{0,5}$ is affine, we only have to check positivity of c_1^2 and the intersection with $D_{H_{ab}}$ and $D^{\mathfrak{B}}_{\Gamma_{i,j}}$. For the $D^{\mathfrak{B}}_{\Gamma_{i,j}}$-intersections this follows from the intersection numbers in Lemma 8.5. In fact, the self-intersection number of $D^{\mathfrak{B}}_{\Gamma_{i,j}}$ is negative only if $a_p + a_q \le k$ for any pair $\{p, q\}$ disjoint from $\{i, j\}$. Using Lemma 8.3, we compute in this case that

$$[D^{\mathfrak{B}}_{\Gamma_{i,j}}] \cdot c_1\big(\Omega^1_{\overline{\mathfrak{B}}}(\log D_{\mathrm{hor}})\big) = \frac{\kappa_{ij}}{k} \left(\frac{2a_p + 2a_q + 2a_r - a_i - a_j}{k} - 1 \right),$$

where $\{a_1, a_2, a_3, a_4, a_5\} = \{a_i, a_j, a_p, a_q, a_q\}$. Since $a_i + a_j < k$, this expression is positive. Moreover, one directly computes

$$[D_{H_{a,b}}] \cdot c_1\big(\Omega^1_{\mathfrak{B}}(\log D_{\text{hor}})\big) = 0.$$

That $c_1(\Omega^1_{\mathfrak{B}}(\log D_{\text{hor}}))^2 > 0$ is a consequence of the above, since $c_1(\Omega^1_{\mathfrak{B}}(\log D_{\text{hor}}))$ is a linear combination of the divisors $D^{\mathfrak{B}}_{\Gamma_{i,j}}$ and $D^{\mathfrak{B}}_{\text{hor}}$ with positive coefficients, by equation (36). ∎

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Matteo Costantini
Institut für Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany; matteo.costantini@uni-due.de

Martin Möller
Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany; moeller@math.uni-frankfurt.de

Johannes Schwab
Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany; schwab@math.uni-frankfurt.de