Rev. Mat. Iberoam. 41 (2025), no. 2, 509–550
DOI 10.4171/RMI/1507

Supercaloric functions for the porous medium equation in the fast diffusion case

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Abstract. We study a generalized class of supersolutions, so-called supercaloric functions, to the porous medium equation in the fast diffusion case. Supercaloric functions are defined as lower semicontinuous functions obeying a parabolic comparison principle. We prove that bounded supercaloric functions are weak supersolutions. In the supercritical range, we show that unbounded supercaloric functions can be divided into two mutually exclusive classes dictated by the Barenblatt solution and the infinite point-source solution, and give several characterizations for these classes. Furthermore, we study the pointwise behavior of supercaloric functions and obtain connections between supercaloric functions and weak supersolutions.

1. Introduction

In this paper, we study supersolutions to the porous medium equation (PME for short), which can be written as

$$\partial_t u - \Delta(u^m) = 0, \tag{1.1}$$

for $0 < m < \infty$ and nonnegative u. We are concerned with the case $0 < m < 1$, the fast diffusion range, and in particular, with some of the main results in the supercritical fast diffusion range $(n-2)/n < m < 1$. Furthermore, we suppose that the spatial dimension satisfies $n \geq 2$. For the standard theory of the porous medium equation, we refer to the monographs [10, 30, 31].

Our main objective is to investigate a general class of supersolutions, so-called supercaloric functions, for the porous medium equation. They are defined as lower semicontinuous functions, which are finite in a dense set, and satisfy a parabolic comparison principle, see Definition 3.1. Supercaloric functions for (1.1) can be regarded as a counterpart of superharmonic functions in the classical theory, and they arise naturally for example in obstacle problems, Perron's method, and in questions related to boundary regularity, see, e.g., [4, 5, 19]. In contrast to weak supersolutions, supercaloric functions are

Mathematics Subject Classification 2020: 35B51 (primary); 35D30, 35K55, 35K67 (secondary).
Keywords: porous medium equation, supercaloric function, weak supersolution, comparison principle, fast diffusion.

not required to satisfy the equation or any Sobolev space property a priori. In the range $(n-2)/n < m < 1$, functions such as the Barenblatt solution (6.1) and the infinite point-source solution (equation (7.2), see [8]) are examples of supercaloric functions which are not weak supersolutions, since the Sobolev space requirement fails to hold (when $0 < m < 1$, see also (7.1)). Furthermore, in the supercritical fast diffusion case, these functions will be the main examples from two disjoint classes of supercaloric functions, which we will study in this paper.

The theory of supercaloric functions for the parabolic p-Laplace equation in the supercritical case is well developed. In the slow diffusion case, Sobolev space properties of locally bounded supercaloric functions were proven in [17], and the classification theory of unbounded supercaloric functions is summarized in [21]. In [20], the study of bounded supercaloric functions was extended to the supercritical fast diffusion range, and for the classification theory in this case for unbounded supercaloric functions, we refer to [14].

For the porous medium equation, the analogous theory in the slow diffusion case is also well established. Sobolev space properties of supercaloric functions were studied in [18], and for the classification theory in the unbounded case we refer to [16]. The theory in the fast diffusion range, which we address in this paper, is currently open. To our knowledge, many questions in the critical and subcritical cases are still open for both equations, and are left to subjects of future research.

The structure of the porous medium equation poses some well-known challenges. For example, solutions are not closed under addition or multiplication by constants. In our case, the former poses a serious difficulty in obtaining an appropriate Caccioppoli inequality and comparison principles, for example. A critical feature that occurs is that one cannot approximate nonnegative solutions with strictly positive ones by adding constants, and in this way avoid the set $\{u = 0\}$ where the equation becomes singular. In order to overcome this difficulty, we are able to show that in each connected component of the domain, every supercaloric function is either strictly positive or vanishes identically on any given time-slice, see Lemma 4.2. The proof of this property relies on an expansion of positivity result for weak solutions (see [11]), which holds in the whole fast diffusion range $0 < m < 1$. Furthermore, this allows us to express the set where a supercaloric function is strictly positive as a countable union of time intervals in every connected component of the domain. The described phenomenon is strongly tied to the nature of fast diffusion, and it does not occur as such in the slow diffusion case.

In Section 5, we show that the class of locally bounded supercaloric functions is included in the class of weak supersolutions; a result which was shown for the parabolic p-Laplace equation in [14, 17, 20], and for the porous medium equation in the slow diffusion case, in [18]. The proof is roughly divided into two parts. First, the result is shown for strictly positive supercaloric functions in Lemma 5.2, whose proof relies on a suitable obstacle problem stated in Theorem 5.1, which is based on the results in [7, 9, 25, 26, 29]. In the second step, this result is generalized to hold for nonnegative supercaloric functions (Theorem 5.3). The geometry of positivity sets of supercaloric functions established in Section 4 plays an important role in the second part of the proof.

In the supercritical case, we show that supercaloric functions can be divided into two mutually exclusive classes, which we call the Barenblatt class and the complementary class. The former is modeled by the Barenblatt solution (6.1), while the latter is modeled by the infinite point-source solution (7.2), see [8]. Functions in the Barenblatt class have some regularity properties, e.g., in terms of integrability (Theorem 6.8), while functions in the complementary class are not guaranteed to have any (Theorem 7.3). As was noticed already in the case of the parabolic p-Laplace equation ([14, 21]), prominent singularities of functions in the complementary class are qualitatively different in the fast diffusion case than in the slow diffusion case ([16]). Roughly speaking, variables in space and time change their roles in this respect. For Sobolev space properties in the Barenblatt class, we use a Moser type iteration, which is based on the combination of the Sobolev inequality and a suitable Caccioppoli inequality. On the other hand, proofs in the complementary class are based on Harnack type inequalities stated in Section 4.

In the final section, we study the pointwise behavior of supercaloric functions. It is well known that every weak supersolution is lower semicontinuous after possible redefinition in a set of measure zero, see [3, 23]. More precisely, pointwise values can be recovered almost everywhere by the ess lim inf of the function, where only instances of time in the past are relevant. For the parabolic p-Laplace equation, it was shown in [17], and for the porous medium equation in the slow diffusion case in [18] that supercaloric functions enjoy the same property at every point in their domain (for the elliptic case, see also [15]). In Section 8, we show that the same property holds for supercaloric functions for the porous medium equation in the fast diffusion case. We conclude the paper by summarizing the connections between supercaloric functions and weak supersolutions in Corollary 8.5.

2. Weak supersolutions

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $T > 0$, we denote by $\Omega_T := \Omega \times (0, T)$ a space-time cylinder in $\mathbb{R}^{n+1}$. The parabolic boundary of Ω_T is defined as $\partial_p \Omega_T := (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T))$. We call Ω_T a $C^{k,\alpha}$-cylinder if $\Omega \subset \mathbb{R}^n$ is a bounded $C^{k,\alpha}$-domain for $k \in \mathbb{N}$ and $\alpha \in (0, 1]$.

2.1. Notion of weak solutions

We begin by defining the concept of weak (super- and sub)solutions.

Definition 2.1. A measurable function $u\colon \Omega_T \to [0, \infty]$ satisfying

$$u^m \in L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega)) \cap L^{1/m}_{\mathrm{loc}}(\Omega_T)$$

is called a weak solution to the PME (1.1) if and only if u satisfies the integral equality

$$\iint_{\Omega_T} (-u\, \partial_t \varphi + \nabla u^m \cdot \nabla \varphi)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{2.1}$$

for every $\varphi \in C_0^\infty(\Omega_T)$. Further, we say that u is a weak supersolution if the integral above is nonnegative for all nonnegative test functions $\varphi \in C_0^\infty(\Omega_T)$. If the integral is nonpositive for such test functions, we call u a weak subsolution.

Finally, we say that $u\colon \Omega_T \to [0, \infty]$ is a global weak solution to the PME (1.1) if it is a weak solution with the property

$$u^m \in L^2(0, T; H^1(\Omega)) \cap L^{1/m}(\Omega_T).$$

Now we recall a comparison principle for weak super(sub)solutions, see [5, 10, 31].

Lemma 2.2. *Let $0 < m < 1$ and let Ω_T be a $C^{2,\alpha}$-cylinder with $\alpha \in (0,1]$. Suppose that u is a nonnegative weak supersolution and v is a nonnegative weak subsolution to (1.1) in Ω_T, such that $u^m, v^m \in L^2(0,T;H^1(\Omega)) \cap L^{2/m}(\Omega_T)$. If, in addition,*

$$(v^m - u^m)_+(\cdot,t) \in H_0^1(\Omega), \quad \text{for a.e. } t \in (0,T),$$

and

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \int_\Omega (v-u)_+ \,\mathrm{d}x\,\mathrm{d}t = 0$$

holds true, then $0 \le v \le u$ a.e. in Ω_T.

The following maximum principle also holds, see Lemma 2.8 in [26].

Lemma 2.3. *Let $m > 0$. Let u be a nonnegative weak subsolution with the property that $u^m \in L^2(0,T;H^1(\Omega)) \cap L^{1/m}(\Omega_T)$ and $k \in \mathbb{R}_{\ge 0}$. If $(u^m - k^m)_+(\cdot,t) \in H_0^1(\Omega)$ for a.e. $t \in (0,T)$ and*

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \int_\Omega (u-k)_+ \,\mathrm{d}x\,\mathrm{d}t = 0,$$

then

$$u \le k \quad \text{a.e. in } \Omega_T.$$

Even though we cannot add constants to solutions, we can show the following result for weak solutions with perturbed boundary values. For the proof in the case $m > 1$, see Lemma 3.2 in [19].

Lemma 2.4. *Suppose that $0 < m < 1$ and $\Omega \Subset \mathbb{R}^n$. Let g be a nonnegative function satisfying $g^m \in L^2(0,T;H^1(\Omega))$, $g \in C([0,T];L^{m+1}(\Omega)) \cap L^\infty(\Omega_T)$. Denote $g_\varepsilon = (g^m + \varepsilon^m)^{1/m}$, for $\varepsilon \in (0,1]$. Let u and u_ε be global weak solutions in Ω_T (in class $C([0,T];L^{m+1}(\Omega))$), taking boundary values g and g_ε, respectively, in the Sobolev sense on the lateral boundary, and $u(x,0) = g(x,0)$ and $u_\varepsilon(x,0) = g_\varepsilon(x,0)$ for a.e. $x \in \Omega$. Then, there exists $c = c(m, \|g\|_\infty, |\Omega|, T) > 0$ such that*

$$\iint_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m)\,\mathrm{d}x\,\mathrm{d}t \le c\delta(\varepsilon),$$

in which $\delta(\varepsilon) := \max\{\varepsilon^m, \int_\Omega (g_\varepsilon(x,0) - g(x,0))\,\mathrm{d}x\} \to 0$ as $\varepsilon \to 0$.

Proof. We use the Oleinik type test function

$$\eta(x,t) := \begin{cases} \int_t^T (u_\varepsilon^m - u^m - \varepsilon^m)\,\mathrm{d}s, & \text{for } 0 < t < T, \\ 0, & \text{for } t \ge T, \end{cases}$$

in the weak formulation. Observe that this function vanishes on the lateral boundary in Sobolev sense, and

$$\partial_t \eta = -(u_\varepsilon^m - u^m) + \varepsilon^m, \quad \nabla\eta = \int_t^T \nabla(u_\varepsilon^m - u^m)\,\mathrm{d}s \quad \text{on } \Omega_T.$$

By subtracting the weak formulations with the given test function, we obtain

$$\iint_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m - \varepsilon^m) + \nabla(u_\varepsilon^m - u^m) \cdot \int_t^T \nabla(u_\varepsilon^m - u^m)\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_\Omega (g_\varepsilon(x,0) - g(x,0)) \int_0^T (u_\varepsilon^m - u^m - \varepsilon^m)\, \mathrm{d}s\, \mathrm{d}x$$
$$= \int_\Omega (g_\varepsilon(x,0) - g(x,0)) \int_0^T (u_\varepsilon^m - u^m)\, \mathrm{d}s\, \mathrm{d}x - \varepsilon^m T \int_\Omega (g_\varepsilon(x,0) - g(x,0))\, \mathrm{d}x.$$

The divergence part on the left-hand side equals

$$\frac{1}{2} \int_\Omega \Big(\int_0^T (\nabla u_\varepsilon^m - \nabla u^m)\, \mathrm{d}t \Big)^2 \mathrm{d}x \geq 0,$$

so we can estimate it away and obtain the equality above as inequality $\leq$ without that term. Similarly, since $g_\varepsilon \geq g$, the very last term is negative and we can omit that as well. Now by denoting

$$M := \|g\|_\infty,$$

in total we have

$$\iint_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m)\, \mathrm{d}x\, \mathrm{d}t$$
$$\leq \varepsilon^m \iint_{\Omega_T} (u_\varepsilon - u)\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega (g_\varepsilon(x,0) - g(x,0)) \int_0^T (u_\varepsilon^m - u^m)\, \mathrm{d}s\, \mathrm{d}x$$
$$\leq \varepsilon^m C(m,M)\, |\Omega_T| + C(m,M)\, T \int_\Omega (g_\varepsilon(x,0) - g(x,0))\, \mathrm{d}x,$$

since the maximum principle, Lemma 2.3, implies $u \leq M$ and $u_\varepsilon \leq (M^m + 1)^{1/m}$ a.e. in Ω_T. Now we have that

$$g_\varepsilon(x,0) - g(x,0) = (g^m(x,0) + \varepsilon^m)^{1/m} - g(x,0) \xrightarrow{\varepsilon \to 0} 0$$

pointwise a.e. in Ω. Also,

$$0 \leq g_\varepsilon(x,0) - g(x,0) \leq (2^{(1-m)/m} - 1)\, g(x,0) + 2^{(1-m)/m} \in L^1(\Omega),$$

so the dominated convergence theorem implies

$$\lim_{\varepsilon \to 0} \int_\Omega (g_\varepsilon(x,0) - g(x,0))\, \mathrm{d}x = 0.$$

By choosing

$$\delta(\varepsilon) = \max \Big\{ \varepsilon^m, \int_\Omega (g_\varepsilon(x,0) - g(x,0))\, \mathrm{d}x \Big\},$$

the claim follows. ∎

2.2. Continuous weak solutions

As an auxiliary tool, we will also use a local notion of continuous very weak solution, see [1, 2].

Definition 2.5. We say that a nonnegative function $u \in C(\overline{\Omega_T})$ is a continuous very weak solution with boundary values $g \in C(\overline{\partial_p \Omega_T})$ if $u = g$ on $\overline{\partial_p \Omega_T}$ and, for every $0 < t_1 < t_2 \le T$ and smooth $Q \Subset \Omega$,

$$\iint_{Q_{t_1,t_2}} -(u\,\partial_t \eta + u^m \Delta\eta)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{\partial Q} u^m\,\partial_\nu \eta\,\mathrm{d}\sigma\,\mathrm{d}t = \int_Q u(x,t_1)\,\eta(x,t_1)\,\mathrm{d}x - \int_Q u(x,t_2)\,\eta(x,t_2)\,\mathrm{d}x$$

holds true for all $\eta \in C^{2,1}(\overline{Q_{t_1,t_2}})$ vanishing on $\partial Q \times (t_1,t_2]$, where ν is the outward-directed normal vector to Q at points on ∂Q.

We recall existence, comparison and stability results for the notion defined above from [1, 2].

Theorem 2.6. *Let $0 < m < 1$ and let Ω_T be a $C^{1,\alpha}$-cylinder with $\alpha \in (0,1]$. Then, for any nonnegative function $g \in C(\overline{\partial_p \Omega_T})$, there exists a unique locally Hölder continuous very weak solution $u \in C(\overline{\Omega_T})$ in the sense of Definition 2.5 such that*

$$u = g \quad \text{on } \overline{\partial_p \Omega_T}.$$

Furthermore, if u_1 and u_2 are very weak solutions with nonnegative boundary values g_1 and g_2, respectively, satisfying $g_1, g_2 \in C(\overline{\partial_p \Omega_T})$ and $g_1 \le g_2$, then $u_1 \le u_2$.

Theorem 2.7 (Corollary 2.3 in [2]). *Let $0 < m < 1$ and let Ω_T be a $C^{1,\alpha}$-cylinder, with $\alpha \in (0,1]$. Also, let $h_j \in C(\overline{\partial_p \Omega_T})$ be nonnegative, and let $u_j \in C(\overline{\Omega_T})$ be the corresponding very weak solution given by Theorem 2.6, for $j \in \mathbb{N}_0$. If we have*

$$\sup_{\overline{\partial_p \Omega_T}} |h_j - h_0| \to 0 \quad \text{as } j \to \infty,$$

then

$$\lim_{j\to\infty} u_j = u_0 \quad \text{in } \overline{\Omega_T},$$

and the convergence is locally uniform in $\Omega \times (0,T]$ as $j \to \infty$.

The advantage here is that the comparison principle and the stability results hold for solutions according to Definition 2.5, even though such solutions are not required to belong to the (global) parabolic Sobolev space, in contrast to Lemma 2.2. This will be the case, e.g., for weak solutions appearing in the definition of supercaloric functions (Definition 3.1 (iii)/(iii')), which will only belong to the local parabolic Sobolev space in their domain of definition.

Then we are at the stage of stating a useful result concerning existence and comparison of continuous weak solutions.

Theorem 2.8. *Let $0 < m < 1$, and let Ω_T be a $C^{1,\alpha}$-cylinder with $\alpha \in (0, 1]$. Suppose that the nonnegative function $g \in C(\overline{\Omega_T})$ satisfies $g^m \in L^2(0, T; H^1(\Omega))$ and $\partial_t g^m \in L^{(m+1)/m}(\Omega_T)$. Then, there exists a unique global weak solution u to (1.1) such that $u \in C(\overline{\Omega_T})$, u is locally Hölder continuous and $u = g$ on $\partial_p \Omega_T$. Moreover, if g' satisfies conditions above, $g \leq g'$ on $\partial_p \Omega_T$ and $h' \in C(\overline{\Omega_T})$ is a global weak solution with boundary values g' on $\partial_p \Omega_T$, then $h \leq h'$ in Ω_T.*

Proof. By Theorem 1.2 in [28], there exists a global weak solution u to (1.1) such that $u \in L^\infty(0, T; L^{m+1}(\Omega))$ and $u^m \in L^2(0, T; H^1(\Omega))$, and u attains the lateral boundary values in the sense $u^m - g^m \in L^2(0, T; H_0^1(\Omega))$, and the initial values $g_o = g(x, 0)$ in L^{m+1}-sense. Observe that since $g \in L^\infty(\Omega_T)$, also $u \in L^\infty(\Omega_T)$ by the maximum principle, Lemma 2.3. Now Theorem 18.1 in Chapter 6 of [11] implies that u is locally Hölder continuous and [26] that $u \in C(\overline{\Omega_T})$. Furthermore, the solution is unique by Theorem 5.3 in [31]. It is a straightforward consequence that u is a very weak solution according to Definition 2.5 with boundary values g.

By Theorem 2.6, there exists a unique locally Hölder continuous very weak solution $\tilde{u} \in C(\overline{\Omega_T})$ according to Definition 2.5 such that $\tilde{u} = g$ on $\overline{\partial_p \Omega_T}$. By uniqueness, u and $\tilde{u}$ coincide. The comparison principle holds by Theorem 2.6. ■

2.3. Some properties of weak supersolutions

Next we state a Caccioppoli inequality for bounded weak supersolutions, see Lemma 2.15 in [18].

Lemma 2.9. *Let $m > 0$. Suppose that $0 \leq u \leq M$ is a weak supersolution in Ω_T. Then, there exists a numerical constant $C > 0$ such that*

$$\int_{t_1}^{t_2} \int_\Omega \xi^2 |\nabla u^m|^2 \, \mathrm{d}x \mathrm{d}t \leq C M^{2m} T \int_\Omega |\nabla \xi|^2 \, \mathrm{d}x + C M^{m+1} \int_\Omega \xi^2 \, \mathrm{d}x$$

for every $\xi = \xi(x) \in C_0^\infty(\Omega)$ with $\xi \geq 0$, and any t_1, t_2 satisfying $0 < t_1 < t_2 < T$.

In the following, for $v \in L^1_{\mathrm{loc}}(\Omega_T)$, $h > 0$ and $\tau_1 > 0$, we use the mollification in time defined as

$$[\![u]\!]_h(x,t) = \frac{1}{h} \int_{\tau_1}^t e^{(s-t)/h} u(x,s) \, \mathrm{d}s, \quad \text{for any } t \in (\tau_1, T). \tag{2.2}$$

For the standard properties of this mollification, see, e.g., Lemma 2.2 in [17].

The proof of the next lemma follows the lines of the proof of Lemma A.1 in [6], see also Lemma 2.7 in [24].

Lemma 2.10. *Let $m > 0$. If u is a nonnegative weak supersolution in Ω_T, then $\min\{u, k\}$ is a weak supersolution in Ω_T for every $k \geq 0$.*

Proof. Let us start with a mollified weak formulation,

$$\iint_{\Omega_T} \partial_t [\![u]\!]_h \varphi + [\![\nabla u^m]\!]_h \cdot \nabla \varphi \, \mathrm{d}x \mathrm{d}t \geq 0,$$

for $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0})$, and use a test function

$$\varphi = \eta \, \frac{(u^m - k^m)_-}{(u^m - k^m)_- + \sigma}, \quad \text{with } \sigma > 0 \text{ and } \eta \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0}).$$

For the divergence part, we have

$$\begin{aligned}
&\lim_{h\to 0} \iint_{\Omega_T} [\![\nabla u^m]\!]_h \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_{\Omega_T} \nabla u^m \cdot \Big(\nabla\eta \, \frac{(u^m - k^m)_-}{(u^m - k^m)_- + \sigma} + \sigma\eta \, \frac{\nabla(u^m - k^m)_-}{[(u^m - k^m)_- + \sigma]^2} \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \iint_{\Omega_T} \nabla u^m \cdot \nabla\eta \, \frac{(u^m - k^m)_-}{(u^m - k^m)_- + \sigma} \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \iint_{\Omega_T} \nabla(\min\{u, k\}^m) \cdot \nabla\eta \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

as $\sigma \to 0$, by the dominated convergence theorem. For the parabolic part, we obtain

$$\begin{aligned}
\iint_{\Omega_T} \partial_t [\![u]\!]_h \varphi \,\mathrm{d}x\,\mathrm{d}t &= \iint_{\Omega_T} \eta\, \partial_t [\![u]\!]_h \, \frac{([\![u]\!]_h^m - k^m)_-}{([\![u]\!]_h^m - k^m)_- + \sigma} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{\Omega_T} \eta\, \partial_t [\![u]\!]_h \Big(\frac{(u^m - k^m)_-}{(u^m - k^m)_- + \sigma} - \frac{([\![u]\!]_h^m - k^m)_-}{([\![u]\!]_h^m - k^m)_- + \sigma} \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \iint_{\Omega_T} \eta\, \partial_t [\![u]\!]_h \, \frac{([\![u]\!]_h^m - k^m)_-}{([\![u]\!]_h^m - k^m)_- + \sigma} \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

since the map

$$s \mapsto \frac{(s^m - k^m)_-}{(s^m - k^m)_- + \sigma}$$

is decreasing and

$$\partial_t [\![u]\!]_h = \frac{1}{h}(u - [\![u]\!]_h).$$

Now we can estimate further

$$\begin{aligned}
&\iint_{\Omega_T} \eta\, \partial_t [\![u]\!]_h \, \frac{([\![u]\!]_h^m - k^m)_-}{([\![u]\!]_h^m - k^m)_- + \sigma} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \iint_{\Omega_T} \eta\, \partial_t \Big[k - \int_{[\![u]\!]_h}^k \frac{(s^m - k^m)_-}{(s^m - k^m)_- + \sigma} \,\mathrm{d}s \Big] \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = -\iint_{\Omega_T} \partial_t \eta \Big[k - \int_{[\![u]\!]_h}^k \frac{(s^m - k^m)_-}{(s^m - k^m)_- + \sigma} \,\mathrm{d}s \Big] \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \xrightarrow{h\to 0} -\iint_{\Omega_T} \partial_t \eta \Big[k - \int_u^k \frac{(s^m - k^m)_-}{(s^m - k^m)_- + \sigma} \,\mathrm{d}s \Big] \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \xrightarrow{\sigma\to 0} -\iint_{\Omega_T} \partial_t \eta \, [k - (u - k)_-] \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $k-(u-k)_-=\min\{u,k\}$, in total we have

$$\iint_{\Omega_T} -\min\{u,k\}\partial_t\eta + \nabla(\min\{u,k\}^m)\cdot\nabla\eta \,\mathrm{d}x\mathrm{d}t \geq 0,$$

which completes the proof. ∎

A result in [23] states that every weak supersolution has a lower semicontinuous representative.

Theorem 2.11. *Let $m>0$ and let u be a nonnegative weak supersolution in Ω_T. Then, there exists a lower semicontinuous function u_* such that $u_*(x,t)=u(x,t)$ for a.e. $(x,t)\in\Omega_T$. Moreover,*

$$u_*(x,t) = \operatorname*{ess\,lim\,inf}_{\substack{(y,s)\to(x,t)\\ s<t}} u(y,s),$$

for every $(x,t)\in\Omega_T$.

3. Notion of supercaloric functions

Up next, we define (quasi-)super- and subcaloric functions.

Definition 3.1. Let $U\subset\mathbb{R}^{n+1}$ be an open set. A function $u\colon U\to[0,\infty]$ is called a supercaloric function if

(i) u is lower semicontinuous,

(ii) u is finite in a dense subset,

(iii) u satisfies the comparison principle in every subcylinder $Q_{t_1,t_2}=Q\times(t_1,t_2)\Subset U$: if $h\in C(\overline{Q}_{t_1,t_2})$ is a weak solution in Q_{t_1,t_2} and if $h\leq u$ on the parabolic boundary of Q_{t_1,t_2}, then $h\leq u$ in Q_{t_1,t_2}.

We call u a quasi-supercaloric function if (i) and (ii) hold, and (iii) is replaced by

(iii') u satisfies the comparison principle in every $C^{2,\alpha}$-subcylinder $Q_{t_1,t_2}=Q\times(t_1,t_2)\Subset U$: if $h\in C(\overline{Q}_{t_1,t_2})$ is a weak solution in Q_{t_1,t_2} and if $h\leq u$ on the parabolic boundary of Q_{t_1,t_2}, then $h\leq u$ in Q_{t_1,t_2}.

A function $u\colon\Omega_T\to[0,\infty)$ is called subcaloric function if the conditions (i), (ii) and (iii) above hold with (i) replaced by upper semicontinuity, and the inequalities in (iii), by $\geq$. The function u is called quasi-subcaloric if (iii') holds instead of (iii) with $\geq$.

The notion of quasi-supercaloric functions is only used as an auxiliary construct for the following proofs. In fact, it turns out that the classes of supercaloric and quasi-supercaloric functions coincide, see Proposition 3.5. However, the proof requires a more detailed analysis of quasi-supercaloric functions and is therefore postponed to the end of this section.

Our next goal is to prove that every lower semicontinuous weak supersolution is a supercaloric function. Observe that a weak supersolution is lower semicontinuous after a possible redefinition in a set of measure zero by Theorem 2.11. However, since the comparison principle from Lemma 2.2 is limited to $C^{2,\alpha}$-cylinders, as a first step we only obtain the following preliminary result.

Lemma 3.2. *Let* $0 < m < 1$. *If* u *is a nonnegative weak supersolution in* Ω_T, *then* u_* *is a quasi-supercaloric function in* Ω_T.

Remark 3.3. At the end of this section, we will improve this result and show that lower semicontinuous weak supersolutions are supercaloric functions, see Lemma 3.6.

Proof. We only need to show the comparison principle (iii') from the definition of quasi-supercaloric functions. Let $Q_{t_1,t_2} \Subset \Omega_T$ be a $C^{2,\alpha}$-cylinder, and let $h \in C(\overline{Q_{t_1,t_2}})$ be a weak solution, which implies $h^m \in L^2_{\mathrm{loc}}(t_1, t_2; H^1_{\mathrm{loc}}(Q))$. We are not able to use the comparison principle between weak subsolutions and supersolutions, Lemma 2.2, directly, since we would need $h^m \in L^2(t_1, t_2; H^1(Q))$. Thus we proceed as follows.

Denote

$$\tilde{u} = \min\Big\{u_*, \max_{\overline{Q}_{t_1,t_2}} h\Big\},$$

which is a lower semicontinuous weak supersolution by Lemma 2.10.

We let $\bar{h}_j : \overline{Q_{t_1,t_2}} \to \mathbb{R}_{\geq 0}$ be Lipschitz functions for $j = 1, 2, \ldots$, such that for $h_j := \bar{h}_j|_{\partial_p Q_{t_1,t_2}}$ we have $0 \leq h_j \leq h^m$ on $\partial_p Q_{t_1,t_2}$ and

$$\sup_{\partial_p Q_{t_1,t_2}} |h_j^{1/m} - h| \xrightarrow{j\to\infty} 0. \tag{3.1}$$

By Theorem 2.8, there exists a unique weak solution $\hat{h}_j \in C(\overline{Q_{t_1,t_2}})$ in Q_{t_1,t_2} taking the boundary values $h_j^{1/m}$ continuously, and $\hat{h}_j^m - \bar{h}_j \in L^2(t_1, t_2; H^1_0(Q))$. By Lemma 2.2, we have that $\hat{h}_j(x,t) \leq \tilde{u}(x,t) \leq u_*(x,t)$ for a.e. $(x,t) \in Q_{t_1,t_2}$. Since $u = u_*$ a.e. by Theorem 2.11, it follows that $(u_*)_* = u_*$ everywhere. Together with continuity of $\hat{h}_j$ it follows that $\hat{h}_j(x,t) \leq u_*(x,t)$ for every $(x,t) \in Q_{t_1,t_2}$.

Furthermore, since the condition (3.1) holds, Theorem 2.7 implies that also in the limit $j \to \infty$, $h(x,t) \leq u_*(x,t)$ holds for every $(x,t) \in Q_{t_1,t_2}$. Thus u_* is a quasi-supercaloric function. ∎

In the next lemma, we show that the comparison principle for super(sub)caloric functions holds in general space-time cylinders. The proof follows the lines of Theorem 3.6 in [5] (see also Theorem 3.3 in [19]), in which the result was proved in case $m \geq 1$. Observe that the result is proved for quasi-super(sub)caloric functions, which implies that the result also holds for super(sub)caloric functions.

Lemma 3.4. *Suppose that* $0 < m < 1$. *Let* $Q_{t_1,t_2} \Subset \mathbb{R}^{n+1}$ *be a cylinder. Suppose that* u *is a nonnegative (quasi-)supercaloric and* v *is a nonnegative (quasi-)subcaloric function in* Q_{t_1,t_2}. *If*

$$\infty \neq \limsup_{Q_{t_1,t_2} \ni (y,s) \to (x,t)} v(y,s) \leq \liminf_{Q_{t_1,t_2} \ni (y,s) \to (x,t)} u(y,s)$$

for every $(x,t) \in \partial_p Q_{t_1,t_2}$, *then* $v \leq u$ *in* Q_{t_1,t_2}.

Proof. Fix $\delta > 0$ and denote $\tau_2 := t_2 - \delta$, $\tilde{\tau}_2 := t_2 - \delta/2$ and $\hat{\tau}_2 := t_2 - \delta/4$. If u is unbounded, we may consider $\tilde{u} = \min\{u, \sup_{Q_{t_1,\hat{\tau}_2}} v\}$ instead of u in the proof, which is a bounded quasi-supercaloric function in $Q_{t_1,\tilde{\tau}_2}$ as a truncation of a quasi-supercaloric function. Then in the end, by proving $v \le \tilde{u}$ in Q_{t_1,τ_2} this implies $v \le u$ in Q_{t_1,τ_2} since $\tilde{u} \le u$ in Q_{t_1,τ_2}. Therefore, from now on we assume that u is bounded. Furthermore, observe that v is locally bounded in Q_{t_1,t_2} by definition, and the assumption implies that v is bounded in $Q_{t_1,\hat{\tau}_2}$.

We extend u up to the parabolic boundary by setting

$$u(x,t) := \liminf_{Q_{t_1,t_2} \ni (y,s) \to (x,t)} u(y,s) \quad \text{for every } (x,t) \in \overline{\partial_p Q_{t_1,\tilde{\tau}_2}}.$$

The function v is extended analogously via lim sup. By standard arguments it follows that u (v) is lower(upper) semicontinuous in $\overline{Q_{t_1,\tilde{\tau}_2}}$.

For $\varepsilon_j = 1/j$, take nested $C^{2,\alpha}$-cylinders $Q^j_{s_j,\tilde{\tau}_2} \Subset Q \times (t_1, \tilde{\tau}_2]$ with

$$\bigcup_{j=1}^{\infty} Q^j = Q, \quad s_j \xrightarrow{j\to\infty} t_1$$

and

$$v^m \le u^m + \frac{1}{2}\varepsilon_j^m \quad \text{in } \overline{Q_{t_1,\tilde{\tau}_2}} \setminus \left(Q^j \times (s_j, \tilde{\tau}_2]\right).$$

We can find a nondecreasing sequence of functions $\bar{h}_j \in C^{0,1}(\overline{Q_{t_1,\tau_2}}, \mathbb{R}_{\ge 0})$ such that

$$\bar{h}_j \xrightarrow{j\to\infty} u^m \quad \text{pointwise in } \overline{Q_{t_1,\tau_2}},$$

satisfying

$$v^m \le \bar{h}_j + \varepsilon_j^m \le u^m + \varepsilon_j^m \quad \text{in } \overline{Q_{t_1,\tau_2}} \setminus \left(Q^j \times (s_j, \tau_2]\right).$$

Observe that by construction,

$$\|\bar{h}_j\|_{L^\infty(Q_{t_1,\tau_2})} \le \|u^m\|_{L^\infty(Q_{t_1,\tau_2})} < \infty \quad \text{for every } j \in \mathbb{N}.$$

In view of Theorem 2.8, we can find continuous global weak solutions h_j and $\hat{h}_j$ in $Q^j_{s_j,\tau_2}$ that take the boundary values $\bar{h}_j^{1/m}$ and $(\bar{h}_j + \varepsilon_j^m)^{1/m}$ continuously and in the Sobolev/trace sense on $\partial_p Q^j_{s_j,\tau_2}$. Since v is quasi-sub- and u quasi-supercaloric, and $Q^j_{s_j,\tau_2} \Subset Q_{t_1,t_2}$ are $C^{2,\alpha}$-cylinders, we have that

$$u \ge h_j \quad \text{and} \quad v \le \hat{h}_j \quad \text{in } Q^j_{s_j,\tau_2}.$$

By extending h_j by $\bar{h}_j^{1/m}$ and $\hat{h}_j$ by $(\bar{h}_j + \varepsilon_j^m)^{1/m}$ to $Q_{t_1,\tau_2} \setminus Q^j_{s_j,\tau_2}$, the inequalities above hold also in this set. Furthermore, we clearly have

$$h_j \le \hat{h}_j \quad \text{in } Q_{t_1,\tau_2} \setminus Q^j_{s_j,\tau_2},$$

and

$$h_j \le \hat{h}_j \quad \text{in } Q^j_{s_j,\tau_2}$$

by the comparison principle for weak solutions, see Lemma 2.2. Furthermore, sequences of functions h_j and $\hat{h}_j$ are uniformly bounded in Q_{t_1,τ_2} since $\bar{h}_j$ is by the maximum principle from Lemma 2.3.

By the estimate for the local Hölder continuity (see Theorem 18.1 in Chapter 6 of [11]), we have that the families h_j and $\hat{h}_j$ are locally equicontinuous, which by Arzelà–Ascoli and a diagonal argument shows that there exist subsequences h_j and $\hat{h}_j$ that converge locally uniformly in Q_{t_1,τ_2} to continuous functions h and $\hat{h}$, which satisfy $h \le \hat{h}$, and by earlier inequalities, also

$$u \ge h \quad \text{and} \quad v \le \hat{h} \quad \text{in } Q_{t_1,\tau_2}. \tag{3.2}$$

Let us restrict to a subsequence for which the aforementioned convergences hold. By using Corollary 3.11 in [7] and Lemma 2.4, we have

$$\begin{aligned}
&\iint_{Q_{t_1,\tau_2}} |\hat{h}_j^m - h_j^m|^{(m+1)/m} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \iint_{Q^j_{s_j,\tau_2}} (\hat{h}_j - h_j)(\hat{h}_j^m - h_j^m) \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_{t_1,\tau_2}\setminus Q^j_{s_j,\tau_2}} |\hat{h}_j^m - h_j^m|^{(m+1)/m} \,\mathrm{d}x\,\mathrm{d}t \\
&\le c(m, \|\bar{h}_j\|_\infty, |Q|, t_2 - t_1) \max\left\{\varepsilon_j^m, \int_{Q^j} (\bar{h}_j(x,s_j) + \varepsilon_j^m)^{1/m} - \bar{h}_j(x,s_j)^{1/m} \,\mathrm{d}x\right\} \\
&\quad + \varepsilon_j^{m+1} |Q_{t_1,\tau_2} \setminus Q^j_{s_j,\tau_2}| \\
&\le c(m, \|u\|_\infty, |Q|, t_2 - t_1) \max\left\{\varepsilon_j^m, (\|u\|_\infty^m + \varepsilon_j^m)^{1/m} - \|u\|_\infty\right\} \xrightarrow{j\to\infty} 0,
\end{aligned}$$

where we used the facts that $\|\bar{h}_j\|_\infty \le \|u\|_\infty^m < \infty$ and $s \mapsto (s + \varepsilon_j^m)^{1/m} - s^{1/m}$ is a nondecreasing mapping.

Since the functions h_j and $\hat{h}_j$ are uniformly bounded in Q_{t_1,τ_2}, $h_j \to h$ and $\hat{h}_j \to \hat{h}$ pointwise in Q_{t_1,τ_2}, the estimate above together with the dominated convergence theorem implies

$$\iint_{Q_{t_1,\tau_2}} |\hat{h}^m - h^m|^{(m+1)/m} \,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Thus $\hat{h} = h$ a.e. in $Q \times (t_1, \tau_2)$. By continuity of $\hat{h}$ and h this holds at every point, which together with (3.2) concludes the result in $Q \times (t_1, \tau_2) = Q \times (t_1, t_2 - \delta)$. Since $\delta > 0$ was arbitrary, the result holds in $Q \times (t_1, t_2)$. ∎

For the proof of the following two lemmas in the case $m \ge 1$, see Proposition 3.8 and Theorem 3.5 in [5].

Proposition 3.5. *Let $0 < m < 1$. If u is a nonnegative quasi-supercaloric function, then u is a supercaloric function.*

Proof. Let $Q_{t_1,t_2} \Subset \Omega_T$ and let $h \in C(\overline{Q_{t_1,t_2}})$ be a weak solution in Q_{t_1,t_2} such that $h \le u$ on $\partial_p Q_{t_1,t_2}$. Since h is continuous in $\overline{Q_{t_1,t_2}}$, it is also bounded in $\overline{Q_{t_1,t_2}}$. By an analogous proof as in Lemma 3.2, h is a quasi-subcaloric function. Since u is a quasi-supercaloric function, we may use Lemma 3.4 to conclude that $h \le u$ in Q_{t_1,t_2}, which implies the claim. ∎

Combining the preceding proposition with Lemma 3.2, we obtain the desired improvement of Lemma 3.2.

Lemma 3.6. *Let $0 < m < 1$ and let u be a nonnegative weak supersolution in Ω_T. Then, u_* is a supercaloric function in Ω_T.*

In the next lemma, we show that supercaloric functions can be extended by zero in the past.

Lemma 3.7. *Let $0 < m < 1$ and let $v\colon \Omega_T \to [0, \infty]$ be a supercaloric function in Ω_T. Then*

$$u = \begin{cases} v & \text{in } \Omega \times (0, T), \\ 0 & \text{in } \Omega \times (-\infty, 0], \end{cases}$$

is a supercaloric function in $\Omega \times (-\infty, T)$.

Proof. Clearly u satisfies items (i) and (ii) in Definition 3.1 since v does, and $v \geq 0$. By showing (iii'), the claim holds by Proposition 3.5.

Fix a $C^{2,\alpha}$-cylinder $Q_{t_1,t_2} \Subset \Omega \times (-\infty, T)$, and let $h \in C(\overline{Q_{t_1,t_2}})$ be a weak solution in Q_{t_1,t_2} such that $h \leq u$ on $\partial_p Q_{t_1,t_2}$. Furthermore, suppose that $Q_{t_1,t_2} \cap (\Omega \times \{0\}) \neq \varnothing$ since otherwise the comparison (in (iii') of Definition 3.1) clearly holds.

By definition of u, we have that

$$h \leq v = 0 \quad \text{on } \partial_p[Q \times (t_1, 0)],$$

i.e., $h = 0$ on $\partial_p[Q \times (t_1, 0)]$. This implies that $h = 0$ in $Q \times (t_1, 0)$. Since h is continuous, this implies that $h = 0$ in $Q \times (t_1, 0]$. Now by using also continuity of h we have that

$$\limsup_{Q_{0,t_2} \ni (y,s) \to (x,t)} h(y, s) = h(x, t) \leq \liminf_{Q_{0,t_2} \ni (y,s) \to (x,t)} v(y, s)$$

for all $(x, t) \in \partial_p Q_{0,t_2}$. Since v is supercaloric and h is subcaloric in Q_{0,t_2}, it follows that $h \leq v$ in Q_{0,t_2} by Lemma 3.4 completing the proof. ∎

Then we recall a parabolic comparison principle for super(sub)caloric functions in noncylindrical bounded sets from Theorem 5.1 in [5].

Lemma 3.8. *Let $m > 0$ and let $U \subset \mathbb{R}^{n+1}$ be a bounded open set. Suppose that u is a nonnegative supercaloric and v is a nonnegative subcaloric function in U. Let $T \in \mathbb{R}$ and assume that*

$$\limsup_{U \ni (y,s) \to (x,t)} v(y, s) < \liminf_{U \ni (y,s) \to (x,t)} u(y, s)$$

for all $(x, t) \in \{(x, t) \in \partial U : t < T\}$. Then

$$v \leq u \quad \text{in } \{(x, t) \in U : t < T\}.$$

The following result shows that the class of supercaloric functions is closed under increasing limits, provided that the limit function is finite in a dense set, see Proposition 4.6 in [5].

Lemma 3.9. *Let $m > 0$ and let u_k be a nondecreasing sequence of nonnegative supercaloric functions in Ω_T. If $u := \lim_{k\to\infty} u_k$ is finite in a dense subset of Ω_T, then u is a supercaloric function in Ω_T.*

4. Positivity sets of supercaloric functions

First we recall the following result on expansion of positivity for weak solutions.

Theorem 4.1 (Proposition 7.2 in Chapter 4 of [11]). *Let* $0 < m < 1$. *Assume that* u *is a locally bounded, nonnegative weak solution to* (1.1) *in the class* $C_{\mathrm{loc}}(0, T; L^{m+1}_{\mathrm{loc}}(\Omega))$. *Suppose that for some* $(x_o, t_o) \in \Omega_T$ *and* $r > 0$,

$$|\{u(\cdot, t_o) \geq M\} \cap B(x_o, r)| \geq \alpha |B(x_o, r)|$$

holds true for some $M > 0$ *and* $\alpha \in (0, 1)$. *Then there exist constants* $\varepsilon, \delta, \eta \in (0, 1)$, *depending only on* n, m *and* α, *such that*

$$u(\cdot, t) \geq \eta M \quad \text{in } B(x_o, 2r), \text{ for all } t \in [t_o + (1-\varepsilon)\delta M^{1-m} r^2, t_o + \delta M^{1-m} r^2],$$

provided that $B(x_o, 16r) \times (t_o, t_o + \delta M^{1-m} r^2) \Subset \Omega_T$.

We use the expansion of positivity for the following characterization of the positivity set of supercaloric functions in the fast diffusion case.

Lemma 4.2. *Let* $0 < m < 1$ *and assume that* u *is a nonnegative supercaloric function in* Ω_T, *where* $\Omega \subset \mathbb{R}^n$ *is open and connected. Then, for any time* $t \in (0, T)$, *either* u *is positive on the whole time slice* $\Omega \times \{t\}$, *or* u *vanishes on the whole time slice.*

Proof. As a first step, we prove the claim for a continuous, nonnegative, bounded weak solution to (1.1). Let us fix a time $t \in (0, T)$. We claim that $u_o := u(x_o, t) > 0$ for some $x_o \in \Omega$ implies

$$u(\cdot, t) > 0 \quad \text{in } B(x_o, r), \tag{4.1}$$

for any $r > 0$ with $B(x_o, 16r) \Subset \Omega$. First, we note that the continuity of u implies

$$u \geq \frac{1}{2} u_o \quad \text{in } B(x_o, \varrho) \times [t - \varrho^2, t] \subset \Omega_T \tag{4.2}$$

for some $\varrho > 0$. If $r \leq \varrho$, this already implies claim (4.1). Otherwise, we apply Theorem 4.1 with the parameter $\alpha := (\varrho/r)^n \in (0, 1)$. Let $\delta = \delta(n, m, \alpha) \in (0, 1)$ be the number determined by this theorem. We choose $M \in (0, \frac{1}{2} u_o]$ so small that

$$\delta M^{1-m} r^2 \leq \varrho^2,$$

and let $t_o := t - \delta M^{1-m} r^2 \in [t - \varrho^2, t]$. Because of (4.2) and $M \leq \frac{1}{2} u_o$, we have

$$|\{u(\cdot, t_o) \geq M\} \cap B(x_o, r)| \geq |B(x_o, \varrho)| = \alpha |B(x_o, r)|.$$

Therefore, Theorem 4.1 implies

$$u(\cdot, t) \geq \eta M \quad \text{in } B(x_o, 2r)$$

for some $\eta > 0$, which implies claim (4.1). Next, we observe that this yields the implication

$$u(x_o, t) > 0 \text{ in some point } x_o \in \Omega \implies u(x_1, t) > 0 \text{ in any point } x_1 \in \Omega. \tag{4.3}$$

For the derivation of this claim, we recall that Ω is connected and consider a curve $\Gamma \subset \Omega$ that connects x_o and x_1. Then we cover Γ by finitely many balls $B(x_i, r)$, $i = 1, \dots, L$, with $x_{i+1} \in B(x_i, r)$ for any $i = 0, \dots, L-1$. Since Γ is compactly contained in Ω, we can choose the radius $r > 0$ small enough to ensure $B(x_i, 16r) \Subset \Omega$ for each $i = 1, \dots, L$. Repeated applications of the positivity result (4.1) imply that u is positive on each of the balls $B(x_i, r) \times \{t\}$, and in particular, $u(x_1, t) > 0$.

This proves claim (4.3). The contraposition of this implication ensures that $u(x_1, t) = 0$ for some $x_1 \in \Omega$ implies $u(x_o, t) = 0$ in any point $x_o \in \Omega$. We conclude that either u is positive or zero on the whole time slice $\Omega \times \{t\}$. This proves the claim for a continuous, bounded weak solution.

Now, we consider a supercaloric function $u\colon \Omega_T \to [0, \infty]$. Let us assume for contradiction that there is a time $t \in (0, T)$ for which $\Omega_+ := \{x \in \Omega\colon u(x,t) > 0\}$ satisfies $\varnothing \neq \Omega_+ \subsetneq \Omega$. By lower semicontinuity of u, the set Ω_+ is open. Because Ω is connected, its subset Ω_+ cannot be relatively closed. Therefore, there exists a point $x_o \in \partial\Omega_+ \cap \Omega$, in which we have $u(x_o, t) = 0$. We choose a neighborhood $B(x_o, r) \subset \Omega$. Because of $x_o \in \partial\Omega_+$, there exists a point $x_+ \in B(x_o, r)$ with $u(x_+, t) > 0$. If $u(x_+, t) < \infty$, let $a := u(x_+, t)$. If $u(x_+, t) = \infty$, let $a \in \mathbb{R}_{>0}$. By lower semicontinuity of u, there exists a $\delta > 0$ such that

$$u > \frac{1}{2} a \quad \text{on } B(x_+, 2\delta) \times [t-\delta, t+\delta] \Subset \Omega_T.$$

We choose a function $\eta \in C_0^\infty(B(x_+, 2\delta), [0,1])$ with $\eta \equiv 1$ in $B(x_+, \delta)$, and abbreviate $\varrho := |x_o - x_+|$. Then we consider the weak solution to the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t v - \Delta v^m = 0 & \text{in } B(x_o, \varrho) \times (t-\delta, t+\delta), \\ v = \frac{1}{2} a \eta^{1/m} & \text{on } \partial_p[B(x_o, \varrho) \times (t-\delta, t+\delta)]. \end{cases}$$

Theorem 2.8 implies that v is nonnegative, bounded and continuous up to the boundary. Therefore, the first part of the proof implies that for every time $s \in (t-\delta, t+\delta)$, the function v is either positive on the whole time slice $B(x_o, \varrho) \times \{s\}$, or it vanishes on the whole time slice. However, since $(x_+, s) \in \partial_p[B(x_o, \varrho) \times (t-\delta, t+\delta)]$ for every $s \in (t-\delta, t+\delta)$, and in this point we have $v(x_+, s) = \frac{1}{2}a > 0$, we can exclude the second alternative. This proves $v > 0$ on the whole domain $B(x_o, \varrho) \times (t-\delta, t+\delta)$.

Moreover, by construction we have $u \geq v$ on $\partial_p[B(x_o, \varrho) \times (t-\delta, t+\delta)]$. Therefore, by definition of the supercaloric function u, we have $u \geq v$ on $B(x_o, \varrho) \times (t-\delta, t+\delta)$, and in particular,

$$u(x_o, t) \geq v(x_o, t) > 0.$$

Since $u(x_o, t) = 0$ by construction, this yields the desired contradiction. Therefore, we have established the claim also in the case of a supercaloric function. ■

Corollary 4.3. *Let $0 < m < 1$ and assume that u is a nonnegative supercaloric function in Ω_T, where $\Omega \subset \mathbb{R}^n$ is open and connected. Then, the set*

$$\Lambda_+ := \{t \in (0,T)\colon u \text{ is positive on } \Omega \times \{t\}\} \tag{4.4}$$

can be written as a countable union $\Lambda_+ = \bigcup_i \Lambda_i$, where Λ_i is an open subinterval of $(0,T)$ for every i.

Proof. In view of Lemma 4.2 and since u is lower semicontinuous, the set

$$\Lambda_+ := \{t \in (0,T) \colon u \text{ is positive on } \Omega \times \{t\}\}$$

is an open subset of $(0,T)$. We decompose Λ_+ in its connected components $\Lambda_i = (t_{i,1}, t_{i,2})$, $i \in I$, i.e., $\Lambda_+ = \bigcup_{i \in I} \Lambda_i$, with disjoint open intervals Λ_i. Since Λ_+ is an open subset of the real line, there can be at most countably many connected components, i.e., we can choose the index set either as $I = \mathbb{N}$ or of the form $I = \{1, \dots, L\}$. ∎

We state Harnack type estimates for weak solutions that will be used later on. In the following, we denote $\lambda := n(m-1) + 2$.

Lemma 4.4 (Theorem 17.1 in Chapter 6 of [11]). *Let $(n-2)/n < m < 1$. Suppose that u is a nonnegative weak solution in the class $C_{\mathrm{loc}}(0,T; L^{m+1}_{\mathrm{loc}}(\Omega))$. Then there exists $\gamma = \gamma(n,m)$ such that*

$$\sup_{B(y,r)\times[s,t]} u \le \frac{\gamma}{(t-s)^{n/\lambda}} \Big(\inf_{2s-t<\tau<t} \int_{B(y,2r)} u(x,\tau)\,\mathrm{d}x \Big)^{2/\lambda} + \gamma \Big(\frac{t-s}{r^2} \Big)^{1/(1-m)}$$

for all cylinders $B(y,2r) \times [s-(t-s), s+(t-s)] \Subset \Omega_T$.

Lemma 4.5 (Proposition B.1.1 in [11]). *Let $0 < m < 1$. Suppose that u is a continuous nonnegative weak solution in Ω_T. Then there exists $\gamma = \gamma(n,m) \ge 1$ such that*

$$\sup_{s<\tau<t} \int_{B(y,r)} u(x,\tau)\,\mathrm{d}x \le \gamma \inf_{s<\tau<t} \int_{B(y,2r)} u(x,\tau)\,\mathrm{d}x + \gamma \Big(\frac{t-s}{r^\lambda} \Big)^{1/(1-m)}$$

for all cylinders $B(y,2r) \times [s,t] \Subset \Omega_T$.

Up next, we prove a weak Harnack inequality for supercaloric functions. The proof follows the approach in Proposition 3.1 of [13].

Lemma 4.6. *Let $(n-2)/n < m < 1$ and u be a nonnegative supercaloric function in Ω_T. Then, there exist constants $c_1, c_2, \alpha \in (0,1)$, depending only on n and m, such that the following holds. Assume that for some $s \in (0,T)$, we have*

$$\theta := c_2 \Big(\fint_{B(x_o,2r)} u(x,s)\,\mathrm{d}x \Big)^{1-m} > 0,$$

and $B(x_o, 64r) \times (s, s+\theta r^2) \Subset \Omega_T$. Then the estimate

$$\inf_{B(x_o,2r)} u(\cdot,t) \ge c_1 \fint_{B(x_o,2r)} u(x,s)\,\mathrm{d}x$$

holds for any $t \in [s+\alpha\theta r^2, s+\theta r^2]$.

Proof. Let us assume $(x_0, s) = (0,0)$ and $Q_S := B_{64r} \times (0,S) \Subset \Omega_T$, for some $S < T$. Let u be a supercaloric function in Ω_T, and let $u_k := \min\{u,k\}$ be its truncation of level $k = 1, 2, \dots$ We want to solve a Dirichlet problem in Q_S with $u_k(x,0)\chi_{B(0,2r)}$ on the initial boundary and zero on the lateral boundary. However, in order to guarantee existence of a (unique and continuous) solution, we solve a regularized problem instead. To this end, we rely on the lower semicontinuity of $u_k^m(x,0)\chi_{B(0,2r)}$ to approximate it pointwise from below by Lipschitz functions $\psi_{k,i}^m$, such that $0 \le \psi_{k,i} \le \psi_{k,i+1} \le u_k(x,0)\chi_{B(0,2r)}$ in $\Omega \times \{0\}$ with $\psi_{k,i}(x) \to u_k(x,0)\chi_{B(0,2r)}$ pointwise in Ω as $i \to \infty$. That is, we consider the problem

$$\begin{cases} \partial_t h_{k,i} - \Delta(h_{k,i}^m) = 0 & \text{in } Q_S, \\ h_{k,i}(x,t) = 0 & \text{on } \partial B_{64r} \times (0,S), \\ h_{k,i}(x,0) = \psi_{k,i}(x) & \text{on } \overline{B}(0,64r) \times \{0\}. \end{cases}$$

By Theorem 2.8, a unique global weak solution $h_{k,i} \in C(\overline{Q}_S)$ exists such that $h_{k,i} = 0$ on the lateral boundary and $h_{k,i} = \psi_{k,i}$ on the initial boundary. Since $0 \le \psi_{k,i} \le u_k \le u$ on the parabolic boundary, from the comparison principle in the definition of supercaloric functions, it follows that $0 \le h_{k,i} \le u_k \le u$ in Q_S. From Theorem 2.8, it also follows that $0 \le h_{k,i} \le h_{k,i+1}$ in Q_S for every i, so that $h_{k,i}$ forms a nondecreasing sequence with respect to $i \in \mathbb{N}$. We set

$$\tilde{\theta} = \Big(⨍_{B_{8r}} h_{k,i}(x,0)\,\mathrm{d}x \Big)^{1-m} = 4^{-n(1-m)} \Big(⨍_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x \Big)^{1-m}$$

and

$$\tilde{\delta} = \Big(\frac{|B_1|}{\gamma} \Big)^{1-m} \tilde{\theta} r^2,$$

where γ is the constant from Lemma 4.5. By Lemma 4.4, we have

$$\sup_{B_{4r}\times(\tilde{\delta}/2,\tilde{\delta})} h_{k,i} \le \gamma_1 ⨍_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x,$$

for $\gamma_1 = \gamma_1(n,m) > 0$. From Lemma 4.5, it follows that

$$\inf_{0<\tau<\tilde{\delta}} \int_{B_{4r}} h_{k,i}(x,\tau)\,\mathrm{d}x \ge \frac{1}{2\gamma} \int_{B_{2r}} h_{k,i}(x,0)\,\mathrm{d}x.$$

By using the previous two estimates, we obtain

$$\begin{aligned} \frac{1}{2^{n+1}\gamma} ⨍_{B_{2r}} h_{k,i}(x,0)\,\mathrm{d}x &\le ⨍_{B_{4r}} h_{k,i}(x,\tau)\,\mathrm{d}x \\ &= \frac{1}{|B_{4r}|} \int_{\{h_{k,i}(\cdot,\tau)>c_o\}\cap B_{4r}} h_{k,i}(x,\tau)\,\mathrm{d}x + \frac{1}{|B_{4r}|} \int_{\{h_{k,i}(\cdot,\tau)\le c_o\}\cap B_{4r}} h_{k,i}(x,\tau)\,\mathrm{d}x \\ &\le \frac{|\{h_{k,i}(\cdot,\tau) > c_o\} \cap B_{4r}|}{|B_{4r}|} \gamma_1 ⨍_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x + c_o, \end{aligned}$$

for any $\tau \in (\tilde{\delta}/2, \tilde{\delta})$ and an arbitrary constant $c_o > 0$. By choosing

$$c_o = \frac{1}{2^{n+2}\gamma} ⨍_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x,$$

the estimate above gives

$$|\{h_{k,i}(\cdot,\tau) > c_o\} \cap B_{4r}| \geq \frac{1}{2^{n+2}\gamma\gamma_1}|B_{4r}|$$

for any $\tau \in (\tilde{\delta}/2, \tilde{\delta})$. At this point we can apply the expansion of positivity, Theorem 4.1. This gives that there exist constants $\varepsilon, \sigma, \eta \in (0,1)$, depending only on n and m, such that

$$h_{k,i}(\cdot,t) \geq \frac{\eta}{2^{n+2}\gamma} \fint_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x \quad \text{in } B_{8r}$$

for all $t \in [\tau + (1-\varepsilon)\sigma c_o^{1-m} r^2, \tau + \sigma c_o^{1-m} r^2]$. Observe that this holds for any $\tau \in (\tilde{\delta}/2, \tilde{\delta})$. Now if we choose the constant $c > 0$ such that

$$c := \Big(\frac{|B_1|}{4^n\gamma}\Big)^{1-m} + \sigma\Big(\frac{1}{2^{n+2}\gamma}\Big)^{1-m}$$

and

$$\theta_{k,i} := c\Big(\fint_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x\Big)^{1-m}, \quad \delta_{k,i} := \theta_{k,i} r^2,$$

we have that

$$\inf_{B_{8r}\times(\alpha\delta_{k,i},\delta_{k,i})} h_{k,i} \geq \frac{\eta}{2^{n+2}\gamma} \fint_{B_{2r}} \psi_{k,i}(x)\,\mathrm{d}x, \tag{4.5}$$

where $\alpha \in (0,1)$ depends only on n and m. Moreover, if we first let $i \to \infty$ and then $k \to \infty$, by monotone convergence we have

$$\delta_{k,i} \to \delta, \quad \text{where} \quad \delta := c\Big(\fint_{B_{2r}} u(x,0)\,\mathrm{d}x\Big)^{1-m} r^2.$$

The left-hand side of (4.5) can be estimated from above by using comparison as

$$\inf_{B_{8r}\times(\alpha\delta_{k,i},\delta_{k,i})} h_{k,i} \leq \inf_{B_{8r}\times(\alpha\delta_{k,i},\delta_{k,i})} u_k \leq \inf_{B_{8r}\times(\alpha\delta_{k,i},\delta_{k,i})} u.$$

By passing to the limit in (4.5), first in $i \to \infty$ and then in $k \to \infty$, we obtain

$$\inf_{B_{8r}\times(\alpha\delta,\delta)} u \geq \frac{\eta}{2^{n+2}\gamma} \fint_{B_{2r}} u(x,0)\,\mathrm{d}x$$

by using the monotone convergence theorem on the right-hand side. The claim follows. ■

If Ω is connected, as a consequence of Lemma 4.2, the positivity set of a supercaloric function in Ω_T has the form $\Omega \times \Lambda_+$, where the set $\Lambda_+ \subset (0,T)$ is a countable union of open time intervals. The next lemma guarantees that the supercaloric function vanishes at the endpoint t_o of each of these time intervals, provided $t_o < T$.

Lemma 4.7. *Let $0 < m < 1$, and suppose that $u\colon \Omega_T \to [0,\infty]$ is a supercaloric function in Ω_T such that for some $t_o \in (0,T)$, we have $u(x,t_o) = 0$ for all $x \in \Omega$. Then,*

$$\lim_{t \uparrow t_o} \int_K u(x,t)\,\mathrm{d}x = 0 \quad \textit{for every } K \Subset \Omega.$$

Proof. Since an arbitrary compact set $K \Subset \Omega_T$ can be covered by finitely many balls B_r with $B_{4r} \Subset \Omega$, it suffices to prove the claim for the case $K = B_r$ with $B_{4r} \Subset \Omega$.

Let $B(y, 4r) \times [s, t_o] \Subset \Omega_T$. Consider the regularized Dirichlet problem as in the proof of Lemma 4.6 in $\mathcal{Q} = B(y, 4r) \times (s, t_o + \delta) \Subset \Omega_T$. By using Lemma 4.5 together with the comparison principle $h_{k,i} \le u_k \le u$ in $\mathcal{Q}$, it follows that

$$\begin{aligned}\int_{B(y,r)} \psi_{k,i}(x)\,\mathrm{d}x &\le \gamma \inf_{s<\tau<t_o} \int_{B(y,2r)} h_{k,i}(x,\tau)\,\mathrm{d}x + \gamma \left(\frac{t_o - s}{r^\lambda}\right)^{1/(1-m)} \\ &= \gamma \left(\frac{t_o - s}{r^\lambda}\right)^{1/(1-m)},\end{aligned}$$

since $u(\cdot, t_o) \equiv 0$. By using the monotone convergence theorem, we can pass to the limit $i \to \infty$ and $k \to \infty$ to obtain

$$\int_{B(y,r)} u(x,s)\,\mathrm{d}x \le \gamma \left(\frac{t_o - s}{r^\lambda}\right)^{1/(1-m)}.$$

Since $s < t_o$ was arbitrary, provided that $B(y, 4r) \times [s, t_o] \Subset \Omega_T$ holds, we may pass to the limit $s \to t_o$ in the estimate above, from which the claim follows. ∎

We prove a variant of Lemma 2.9 when the supersolution vanishes at the final instant of time. The result will be important in the following section.

Lemma 4.8. *Let $0 < m < 1$. Let $u\colon \Omega_T \to [0, \infty]$ be a supercaloric function in Ω_T such that u is a weak supersolution in $\Omega \times (t_1, t_2)$ for some interval $(t_1, t_2) \Subset (0, T)$. Furthermore, suppose that $u(x, t_2) = 0$ for every $x \in \Omega$. Then,*

$$\int_{t_1}^{t_2} \int_\Omega \eta^2 |\nabla u^m|^2\,\mathrm{d}x\mathrm{d}t \le 4M^{2m}(t_2 - t_1) \int_\Omega |\nabla \eta|^2\,\mathrm{d}x$$

for any nonnegative $\eta \in C_0^\infty(\Omega)$ and $M = \|u\|_{L^\infty(\mathrm{spt}(\eta)\times(t_1,t_2))}$. If u does not vanish at t_2, then we have

$$\int_{t_1}^{t_2} \int_\Omega \eta^2 |\nabla u^m|^2\,\mathrm{d}x\mathrm{d}t \le 4M^{2m}(t_2 - t_1) \int_\Omega |\nabla \eta|^2\,\mathrm{d}x + 2M^{m+1} \int_\Omega \eta^2\,\mathrm{d}x.$$

Proof. We start with a mollified weak formulation for u, which can be written as

$$\int_{\tau_1}^{\tau_2} \int_\Omega \partial_t [\![u]\!]_h \varphi + [\![\nabla u^m]\!]_h \cdot \nabla \varphi\,\mathrm{d}x\mathrm{d}t \ge \frac{1}{h} \int_\Omega u(x,\tau_1) \int_{\tau_1}^{\tau_2} e^{(\tau_1 - s)/h} \varphi(x,s)\,\mathrm{d}s\mathrm{d}x \ge 0$$

for a.e. $\tau_2 \in (t_1, t_2)$ and a.e. $\tau_1 \in (t_1, \tau_2)$. The time mollification $[\![\cdot]\!]_h$ is defined as in (2.2). Up next, we use a test function

$$\varphi = (M^m - u^m)\,\alpha_\varepsilon \eta^2,$$

where $\eta \in C_0^\infty(\Omega, \mathbb{R}_{\ge 0})$ and α_ε is a piecewise affine approximation of $\chi_{\tau_1,\tau_2}(t)$.

For the parabolic part, we have

$$\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_\Omega \partial_t [\![u]\!]_h\, \varphi \,\mathrm{d}x\mathrm{d}t &= \int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon \eta^2 M^m\, \partial_t [\![u]\!]_h \,\mathrm{d}x\mathrm{d}t - \int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon \eta^2 u^m\, \partial_t [\![u]\!]_h \,\mathrm{d}x\mathrm{d}t \\
&\le \int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon \eta^2 M^m\, \partial_t [\![u]\!]_h \,\mathrm{d}x\mathrm{d}t - \int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon \eta^2\, [\![u]\!]_h^m\, \partial_t [\![u]\!]_h \,\mathrm{d}x\mathrm{d}t \\
&= -\int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon' \eta^2 M^m\, [\![u]\!]_h \,\mathrm{d}x\mathrm{d}t + \frac{1}{m+1}\int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon' \eta^2\, [\![u]\!]_h^{m+1} \,\mathrm{d}x\mathrm{d}t \\
&\xrightarrow{h\to 0} -\int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon' \eta^2 M^m u \,\mathrm{d}x\mathrm{d}t + \frac{1}{m+1}\int_{\tau_1}^{\tau_2}\int_\Omega \alpha_\varepsilon' \eta^2 u^{m+1} \,\mathrm{d}x\mathrm{d}t \\
&\xrightarrow{\varepsilon\to 0} -\int_\Omega \eta^2 M^m u(\tau_1)\,\mathrm{d}x + \frac{1}{m+1}\int_\Omega \eta^2 u^{m+1}(\tau_1)\,\mathrm{d}x \\
&\quad + \int_\Omega \eta^2 M^m u(\tau_2)\,\mathrm{d}x - \frac{1}{m+1}\int_\Omega \eta^2 u^{m+1}(\tau_2)\,\mathrm{d}x
\end{aligned}$$

for a.e. $\tau_2 \in (t_1, t_2)$ and a.e. $\tau_1 \in (t_1, \tau_2)$. Since $\frac{1}{m+1}u^{m+1} \le M^m u$, the sum of the first two terms on the right-hand side is nonpositive, and we can discard it. After passing to the limit $h \to 0$, for the integrand of the divergence part we have

$$\nabla u^m \cdot \nabla\varphi = -\alpha_\varepsilon \eta^2 |\nabla u^m|^2 + 2\alpha_\varepsilon \eta (M^m - u^m)\nabla\eta\cdot\nabla u^m.$$

For the latter term, we use Young's inequality and obtain

$$2\alpha_\varepsilon \eta (M^m - u^m)\nabla\eta\cdot\nabla u^m \le 2\alpha_\varepsilon \eta M^m |\nabla\eta|\,|\nabla u^m| \le \frac{1}{2}\alpha_\varepsilon \eta^2 |\nabla u^m|^2 + 2\alpha_\varepsilon M^{2m}|\nabla\eta|^2.$$

By passing to the limit $\varepsilon \to 0$ and combining the estimates, we have

$$\begin{aligned}
\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_\Omega \eta^2 |\nabla u^m|^2 \,\mathrm{d}x\mathrm{d}t &\le 2M^{2m}(\tau_2 - \tau_1)\int_\Omega |\nabla\eta|^2\,\mathrm{d}x \\
&\quad + \int_\Omega \eta^2 M^m u(\tau_2)\,\mathrm{d}x - \frac{1}{m+1}\int_\Omega \eta^2 u^{m+1}(\tau_2)\,\mathrm{d}x.
\end{aligned}$$

By multiplying this inequality by 2 and letting $\tau_2 \to t_2$ and $\tau_1 \to t_1$, the first claim follows by using Lemma 4.7, while the second one follows by using $0 \le u(\tau_2) \le M$. ■

5. Bounded supercaloric functions

First we state a result concerning the obstacle problem that will have significant importance in further results of this paper. The existence and regularity results stated in the following theorem can be extracted from [7, 9, 26, 29] (see also [25]). The proof of properties (i) and (iv) can be found in [27].

Theorem 5.1. *Let $0 < m < 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let ψ satisfy $\psi^m \in C^1(\overline{\Omega_T}, \mathbb{R}_{\ge 0})$. Then, there exists a function $u \in C(\overline{\Omega_T})$ with the following properties*:

(i) *u is a weak supersolution in Ω_T,*

(ii) *$u \ge \psi$ everywhere in Ω_T,*

(iii) *$u = \psi$ on $\partial_p \Omega_T$,*

(iv) *u is a weak solution in the set $\{u > \psi\}$.*

We start by proving that supercaloric functions are weak supersolutions on their positivity set.

Lemma 5.2. *Let $0 < m < 1$. Let $u > 0$ be a locally bounded supercaloric function in Ω_T, where $\Omega \subset \mathbb{R}^n$ is an open set. Then u is a weak supersolution in Ω_T.*

Proof. Consider a compactly contained cylinder $\mathcal{Q} = Q_{t_1,t_2} := B(x_o, r) \times (t_1, t_2) \Subset \Omega_T$ and choose a larger cylinder $\widetilde{\mathcal{Q}}$ with $\mathcal{Q} \Subset \widetilde{\mathcal{Q}} \Subset \Omega_T$. Observe that by lower semicontinuity of u and $u > 0$ in Ω_T, we have that $u \geq \delta > 0$ in $\widetilde{\mathcal{Q}}$, for some $\delta > 0$. Furthermore, there exists a sequence (ψ_k) with the properties $\psi_k \in C^\infty(\Omega_T)$ for each $k = 1, 2, \ldots$,

$$0 < \psi_1 < \psi_2 < \cdots < u \quad \text{and} \quad \lim_{k\to\infty} \psi_k = u \quad \text{in } \widetilde{\mathcal{Q}}.$$

Next we consider the obstacle problem in Theorem 5.1, with obstacle ψ_k. By Theorem 5.1, there exists a solution $v_k \in C(\overline{Q_{t_1,t_2}})$ to the obstacle problem, with $v_k = \psi_k$ on $\partial_p Q_{t_1,t_2}$. In the set

$$U_k := \{(x,t) \in Q_{t_1,t_2} : v_k(x,t) > \psi_k(x,t)\},$$

v_k is a weak solution. Since $v_k = \psi_k$ on $\partial_p Q_{t_1,t_2}$, it follows that $v_k = \psi_k$ on ∂U_k, except possibly when $t = t_2$. That is,

$$v_k = \psi_k < u \quad \text{on } \partial U_k \cap \{t < t_2\}.$$

We want to use now Lemma 3.8 to conclude that

$$v_k \leq u \quad \text{in } U_k \cap \{t < t_2\}. \tag{5.1}$$

Since v_k is continuous in $\overline{Q_{t_1,t_2}}$, it follows that v_k is continuous in $\overline{U_k} \cap \{t < t_2\}$. From here it follows that

$$\limsup_{U_k \ni (y,s)\to(x,t)} v_k(y,s) = \psi_k(x,t) < u(x,t) \leq \liminf_{U_k \ni (y,s)\to(x,t)} u(y,s)$$

for each $(x,t) \in \{(x,t) \in \partial U_k : t < t_2\}$ by using also lower semicontinuity of u. Now we can use Lemma 3.8 to conclude (5.1).

Consequently, we have that

$$\psi_k \leq v_k \leq u \quad \text{in } Q_{t_1,t_2},$$

which implies that $v_k \to u$ as $k \to \infty$ pointwise in Q_{t_1,t_2}. By Lemma 2.9, $|\nabla v_k^m|$ is uniformly bounded in $L^2(V \times (t_1,t_2))$ for every subdomain $V \Subset B(x_o, r)$. This together with pointwise convergence implies that ∇v_k^m converges weakly to ∇u^m in $L^2(V \times (t_1,t_2), \mathbb{R}^n)$. This implies that u is a weak supersolution in any $Q_{t_1,t_2} \Subset \Omega_T$. Since being a weak supersolution is a local property, it follows that u is a weak supersolution in Ω_T. That is,

$$\iint_{\Omega_T} \left(-u\, \partial_t \varphi + \nabla u^m \cdot \nabla \varphi\right) \mathrm{d}x\,\mathrm{d}t \geq 0$$

for any nonnegative $\varphi \in C_0^\infty(\Omega_T)$. ∎

Next, we generalize the preceding result to nonnegative supercaloric functions.

Theorem 5.3. *Suppose $0 < m < 1$. Let $u \geq 0$ be a locally bounded supercaloric function in Ω_T. Then u is a weak supersolution in Ω_T.*

Proof. Write Ω as a union of its connected components, i.e., $\Omega = \bigcup_{j\in\mathbb{N}} \Omega^j$, in which each Ω^j is open and connected. By Corollary 4.3, we may decompose the positivity set

$$\Lambda_+^j := \{t \in (0,T) \colon u \text{ is positive on } \Omega^j \times \{t\}\}$$

into at most countably many disjoint open intervals $\Lambda_+^j = \bigcup_{i\in I_j} \Lambda_i^j$, where $\Lambda_i^j = (t_{i,1}^j, t_{i,2}^j)$.

On each of the sets $\Omega^j \times \Lambda_i^j$, Lemma 5.2 implies that u is a weak supersolution to (1.1), i.e., $u^m \in L^2_{\mathrm{loc}}(\Lambda_i^j; H^1_{\mathrm{loc}}(\Omega^j))$ and

$$\int_{\Lambda_i^j}\int_{\Omega^j} (-u\partial_t\varphi + \nabla u^m\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \tag{5.2}$$

for all nonnegative test functions $\varphi \in C_0^\infty(\Omega^j \times \Lambda_i^j)$.

First we show that $u^m \in L^2_{\mathrm{loc}}(0,T; H^1_{\mathrm{loc}}(\Omega))$. To this end, let $K \subset \Omega$ be compact and $(s_1,s_2) \Subset (0,T)$. Choose an open set K' such that $K \subset K' \Subset \Omega$ and a cutoff function $\eta \in C_0^\infty(K')$ such that $\eta \equiv 1$ in K and $|\nabla\eta| \leq c\,\operatorname{dist}(K,\partial K')^{-1}$ with a numerical constant $c > 0$. Denote $K^j := K \cap \Omega^j$, which is compact since $K_j = K \setminus (\bigcup_{i\neq j}\Omega^i)$ is closed.

For each $\Lambda_i^j \Subset (0,T)$, Lemma 4.8 implies that $u^m \in L^2(\Lambda_i^j; H^1_{\mathrm{loc}}(\Omega^j))$. Denote

$$I_j' := \{i \in I_j : \Lambda_i^j \cap (s_1,s_2) \neq \varnothing\}.$$

Observe that for every $t \in (0,T)\setminus\Lambda_+^j$ we have $u(\cdot,t) \equiv 0$ and $\nabla u^m(\cdot,t) \equiv 0$ on Ω^j. By applying Lemma 4.8 on the sets $\Omega^j \times (\Lambda_i^j \cap (s_1,s_2))$, we obtain

$$\begin{aligned}
\int_{s_1}^{s_2}\int_{K^j} |\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t \leq \int_{s_1}^{s_2}\int_{\Omega^j} \eta^2|\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t &= \sum_{i\in I_j'}\int_{\Lambda_i^j\cap(s_1,s_2)}\int_{\Omega^j}\eta^2|\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leq 4M^{2m}\int_{\Omega^j}|\nabla\eta|^2\,\mathrm{d}x\sum_{i\in I_j'}(t_{i,2}^j - t_{i,1}^j) + 2M^{m+1}\int_{\Omega^j}\eta^2\,\mathrm{d}x\\
&\leq 4TM^{2m}\int_{\Omega^j}|\nabla\eta|^2\,\mathrm{d}x + 2M^{m+1}\int_{\Omega^j}\eta^2\,\mathrm{d}x < \infty
\end{aligned}$$

for $M = \|u\|_{L^\infty(K'\times(s_1,s_2))}$, where the last integral can be omitted in the case $s_2 \notin \Lambda_+^j$. Since Ω^j and K^j are disjoint, and $\Omega = \bigcup_{j\in\mathbb{N}}\Omega^j$ and $K = \bigcup_{j\in\mathbb{N}} K^j$, we can sum over $j \in \mathbb{N}$ and obtain

$$\int_{s_1}^{s_2}\int_K |\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t \leq 4TM^{2m}\int_\Omega|\nabla\eta|^2\,\mathrm{d}x + 2M^{m+1}\int_\Omega\eta^2\,\mathrm{d}x < \infty.$$

Since K, s_1 and s_2 were arbitrary, this finally implies that $u^m \in L^2_{\mathrm{loc}}(0,T; H^1_{\mathrm{loc}}(\Omega))$.

Then we show that the integral inequality (5.2) holds in Ω_T for all test functions $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0})$. Observe that this implies $\varphi \in C_0^\infty(\Omega^j \times (0,T), \mathbb{R}_{\geq 0})$ for every $j \in \mathbb{N}$. Fix $i \in I_j$. For such a test function a standard cut-off argument yields

$$\int_{\tau_1}^{\tau_2} \int_{\Omega^j} (-u\partial_t \varphi + \nabla u^m \cdot \nabla \varphi)\,\mathrm{d}x\,\mathrm{d}t \geq -\int_{\Omega^j \times \{\tau_2\}} u\varphi\,\mathrm{d}x$$

for every $\tau_1 \in \Lambda_i^j$ and a.e. $\tau_2 \in \Lambda_i^j$ with $\tau_2 > \tau_1$. In the case $t_{i,2}^j < T$, the last term vanishes in the limit $\tau_2 \uparrow t_{i,2}^j$ due to Lemma 4.7. If $t_{i,2} = T$, we only consider test functions that vanish in a neighborhood of $\Omega^j \times \{T\}$, so that we can omit the last integral also in this case. Since φ vanishes also in a neighborhood of $\Omega^j \times \{0\}$, we may pass to the limit $\tau_1 \to t_{i,1}^j$ as well. Thus we get

$$\int_{t_{i,1}^j}^{t_{i,2}^j} \int_{\Omega^j} (-u\partial_t \varphi + \nabla u^m \cdot \nabla \varphi)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

By recalling that $u(\cdot,t) \equiv 0$ and $\nabla u^m(\cdot,t) \equiv 0$ for every $t \in (0,T) \setminus \Lambda_+^j$, we obtain

$$\iint_{\Omega^j \times (0,T)} (-u\partial_t \varphi + \nabla u^m \cdot \nabla \varphi)\,\mathrm{d}x\,\mathrm{d}t = \sum_{i \in I_j} \int_{\Lambda_i^j} \int_{\Omega^j} (-u\partial_t \varphi + \nabla u^m \cdot \nabla \varphi)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

By summing up over $j \in \mathbb{N}$ and using the fact that the Ω^j are disjoint, we conclude the proof. ∎

We show that a supercaloric function is a weak supersolution also if it belongs to the appropriate energy space.

Lemma 5.4. *Let $0 < m < 1$. Let $u: \Omega_T \to [0,\infty]$ be a supercaloric function in Ω_T such that $u^m \in L^2_{\mathrm{loc}}(0,T; H^1_{\mathrm{loc}}(\Omega)) \cap L^{1/m}_{\mathrm{loc}}(\Omega_T)$. Then u is a weak supersolution.*

Proof. By Theorem 5.3, the truncation $u_k = \min\{u,k\}$ is a weak supersolution for every $k \in \mathbb{N}$, $u_k(x,t) \leq u_{k+1}(x,t)$ and $\lim_{k\to\infty} u_k(x,t) = u(x,t)$ for every $(x,t) \in \Omega_T$. This implies that

$$\lim_{k\to\infty} -\iint_{\Omega_T} \partial_t \varphi\, u_k\,\mathrm{d}x\,\mathrm{d}t = -\iint_{\Omega_T} \partial_t \varphi\, u\,\mathrm{d}x\,\mathrm{d}t$$

for every $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0})$, by the dominated convergence theorem and the fact that $u \in L^1_{\mathrm{loc}}(\Omega_T)$.

There also holds

$$\lim_{k\to\infty} \nabla u_k^m(x,t) = \nabla u^m(x,t) \quad \text{for a.e. } (x,t) \in \Omega_T,$$
$$|\nabla u_k^m| \leq |\nabla u^m| \quad \text{for every } k = 1,2,\ldots, \quad \text{and} \quad |\nabla u^m| \in L^2_{\mathrm{loc}}(\Omega_T).$$

Again, by dominated convergence theorem, we can conclude that

$$\lim_{k\to\infty} \iint_{\Omega_T} \nabla u_k^m \cdot \nabla \varphi\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T} \nabla u^m \cdot \nabla \varphi\,\mathrm{d}x\,\mathrm{d}t$$

for every $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0})$, which concludes the proof. ∎

6. Barenblatt solutions

In the case $(n-2)/n < m < 1$, the Barenblatt solution can be written as

$$\mathcal{B}(x,t) = (Ct)^{1/(1-m)}\,(At^{2/\lambda} + |x|^2)^{-1/(1-m)} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty),$$

in which $\lambda = n(m-1)+2$, $C = 2m\lambda/(1-m)$ and $A > 0$. The Barenblatt solution is a continuous weak solution in $\mathbb{R}^n \times (0,\infty)$. However, we may define a function u in the whole space as

$$u(x,t) = \begin{cases} \mathcal{B}(x,t), & t > 0, \\ 0, & t \le 0, \end{cases} \tag{6.1}$$

which is not even a weak supersolution in $\mathbb{R}^n \times \mathbb{R}$. That is because the integrability assumption for the gradient fails in any neighborhood of the origin, i.e., because $|\nabla u^m| \notin L^2_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$. However, u is a supercaloric function in the whole space $\mathbb{R}^n \times \mathbb{R}$. This is due to Lemma 3.7, since $\mathcal{B}$ is a supercaloric function as a continuous weak solution in the upper half-space by Lemma 3.6.

The Barenblatt solution is the leading example of a supercaloric function in Barenblatt class that, on the other hand, is not a weak supersolution. The Barenblatt solution defined in (6.1) satisfies

$$\partial_t u - \Delta u^m = M\delta \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

in the weak sense, where δ is Dirac's delta at the origin and $M > 0$ represents the mass at the origin (A is a decreasing function of M). Furthermore,

$$\int_{t_1}^{t_2}\int_{B(0,r)} u^{m+2/n}\,\mathrm{d}x\,\mathrm{d}t = \infty \quad \text{and} \quad \int_{t_1}^{t_2}\int_{B(0,r)} |\nabla u^m|^{1+1/(1+mn)}\,\mathrm{d}x\,\mathrm{d}t = \infty,$$

for every $t_1 \le 0$, $t_2 > 0$ and $r > 0$. Later on, this will show that the integrability exponents obtained in Lemmas 6.5 and 6.6 are sharp. We interpret

$$\nabla u^m = \lim_{k\to\infty} \nabla \min\{u,k\}^m \tag{6.2}$$

for a supercaloric function u. The weak gradient of the truncation is well defined for each $k \in \mathbb{N}$, since $\min\{u,k\}^m \in L^2_{\mathrm{loc}}(0,T;H^1_{\mathrm{loc}}(\Omega))$ by Theorem 5.3. If the gradient defined in (6.2) is a locally integrable function (together with u^m), then it is the weak gradient of u^m in the standard sense. Observe that $\nabla u^m = 0$ a.e. in $\{u = \infty\}$, since $\nabla \min\{u,k\}^m = 0$ a.e. in $\{u=\infty\}$ for every $k \in \mathbb{N}$.

We will make use of the following Caccioppoli inequality. For the case $m > 1$, see also Lemma 2.4 in [22].

Lemma 6.1. *Let $0 < m < 1$. Suppose that $u \ge 0$ is a supercaloric function in Ω_T and let $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\ge 0})$. Then there exist numerical constants $c_1, c_2 > 0$ such that*

$$\iint_{\Omega_T} u^{-m-\varepsilon}|\nabla u^m|^2\varphi^2\,\mathrm{d}x\,\mathrm{d}t + \operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega u^{1-\varepsilon}\varphi^2\,\mathrm{d}x \le \frac{c_1}{\varepsilon^2}\iint_{\Omega_T} u^{m-\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{c_2}{\varepsilon(1-\varepsilon)}\iint_{\Omega_T} u^{1-\varepsilon}|\partial_t(\varphi^2)|\,\mathrm{d}x\,\mathrm{d}t$$

holds for every $\varepsilon \in (0,m)$.

Remark 6.2. In points with $u = 0$, we interpret the first integrand on the left-hand side as zero. This is reasonable since formally for $\varepsilon \in (0, m)$, we have

$$u^{-m-\varepsilon}|\nabla u^m|^2 = \frac{4m^2}{(m-\varepsilon)^2}|\nabla u^{(m-\varepsilon)/2}|^2 \quad \text{and} \quad \frac{m-\varepsilon}{2} > 0.$$

Remark 6.3. The result in Lemma 6.1 holds also if u is a weak supersolution by Theorem 2.11 and Lemma 3.6.

Proof of Lemma 6.1. We again notice that $\Omega = \bigcup_{j\in\mathbb{N}} \Omega^j$, where each Ω^j is open and connected. First, we consider an arbitrary connected component Ω^j, but denote it by Ω for simplicity. By Corollary 4.3, we may decompose the positivity set

$$\Lambda_+ := \{t \in (0,T) \colon u \text{ is positive on } \Omega \times \{t\}\}$$

into at most countably many disjoint open intervals $\Lambda_+ = \bigcup_{i\in I} \Lambda_i$.

Let $\tau_1, \tau_2 \in \Lambda_i =: (t_{i,1}, t_{i,2})$ for some $i \in I$. We consider truncations $u_k = \min\{u, k\}$, $k = 1, 2, \ldots$, which are supercaloric functions with the same positivity set as u. For simplicity we denote u_k by u. By Lemma 5.2, u satisfies the mollified weak formulation

$$\int_{\tau_1}^{\tau_2}\int_\Omega \varphi\,\partial_t[u]_h + [\nabla u^m]_h \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \geq 0$$

for any nonnegative $\varphi \in C_0^\infty(\Omega \times (\tau_1, \tau_2))$. By a standard approximation argument, the same holds more generally for test functions $\varphi \in C_0^\infty(\Omega_T)$. Here $[\cdot]_h$ denotes the standard mollification in time, and we consider $h < \frac{1}{2}\mathrm{dist}(\partial\Lambda_i, (\tau_1, \tau_2))$. Observe that in $(\Omega \times (\tau_1 - h, \tau_2 + h)) \cap \mathrm{spt}(\varphi)$, we have $0 < \delta \leq u \leq k < \infty$ for some $\delta > 0$. We test the mollified formulation with $[u]_h^{-\varepsilon}\varphi^2 \in L^2(\tau_1, \tau_2; H_0^1(\Omega))$. From the parabolic part, we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_\Omega \varphi^2 [u]_h^{-\varepsilon}\,\partial_t[u]_h \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\frac{1}{1-\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega [u]_h^{1-\varepsilon}\,\partial_t(\varphi^2)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{1-\varepsilon}\int_\Omega ([u]_h^{1-\varepsilon}\varphi^2)(\cdot,\tau_2)\,\mathrm{d}x \\
&\qquad - \frac{1}{1-\varepsilon}\int_\Omega ([u]_h^{1-\varepsilon}\varphi^2)(\cdot,\tau_1)\,\mathrm{d}x \\
&\quad \to -\frac{1}{1-\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega u^{1-\varepsilon}\,\partial_t(\varphi^2)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{1-\varepsilon}\int_\Omega (u^{1-\varepsilon}\varphi^2)(\cdot,\tau_2)\,\mathrm{d}x \\
&\qquad - \frac{1}{1-\varepsilon}\int_\Omega (u^{1-\varepsilon}\varphi^2)(\cdot,\tau_1)\,\mathrm{d}x
\end{aligned}$$

as $h \to 0$, for a.e. $\tau_1 < \tau_2$ in Λ_i. Observe also that the second term on the right-hand side converges to 0 when $\tau_2 \to t_{i,2}$.

For the gradient, we have

$$\nabla([u]_h^{-\varepsilon}\varphi^2) = 2\varphi\,[u]_h^{-\varepsilon}\,\nabla\varphi - \frac{\varepsilon}{m}\varphi^2[u]_h^{-\varepsilon-1}\,[u^{1-m}\,\nabla u^m]_h.$$

Observe that since $0 < \delta \le u \le k$, we also have $\delta \le [u]_h \le k$. Now each mollified term above converges pointwise a.e. when $h \to 0$. In particular, the last term is majorized by

$$\varphi^2 [u]_h^{-\varepsilon-1} |[u^{1-m} \nabla u^m]_h| |[\nabla u^m]_h| \le \delta^{-\varepsilon-1} k^{1-m} \chi_{\mathrm{spt}(\varphi)} \|\varphi\|_\infty^2 \big[|\nabla u^m|\big]_h^2,$$

and for the integral of the majorant, we have the convergence

$$\begin{aligned}\lim_{h\to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \delta^{-\varepsilon-1} k^{1-m} \chi_{\mathrm{spt}(\varphi)} \|\varphi\|_\infty^2 \big[|\nabla u^m|\big]_h^2 \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\tau_1}^{\tau_2} \int_\Omega \delta^{-\varepsilon-1} k^{1-m} \chi_{\mathrm{spt}(\varphi)} \|\varphi\|_\infty^2 |\nabla u^m|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty,\end{aligned}$$

since $[\nabla u^m]_h \to \nabla u^m$ in $L^2_{\mathrm{loc}}(\Omega_T)$ when $h \to 0$. Thus, we can use a variant of the dominated convergence theorem (see Theorem 4 in Chapter 1.3 of [12]) to conclude

$$\lim_{h\to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \varphi^2 [u]_h^{-\varepsilon-1} [\nabla u^m]_h \cdot [u^{1-m} \nabla u^m]_h \,\mathrm{d}x\,\mathrm{d}t = \int_{\tau_1}^{\tau_2} \int_\Omega \varphi^2 u^{-m-\varepsilon} |\nabla u^m|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

We can argue similarly with the other term in the divergence part, which implies

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} \int_\Omega [\nabla u^m]_h \cdot \nabla([u]_h^{-\varepsilon} \varphi^2) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \to 2 \int_{\tau_1}^{\tau_2} \int_\Omega \varphi u^{-\varepsilon} \nabla u^m \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \frac{\varepsilon}{m} \int_{\tau_1}^{\tau_2} \int_\Omega \varphi^2 u^{-m-\varepsilon} |\nabla u^m|^2 \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

when $h \to 0$. By Young's inequality, we have

$$2\varphi u^{-\varepsilon} \nabla u^m \cdot \nabla\varphi \le \frac{\varepsilon}{2m} \varphi^2 u^{-m-\varepsilon} |\nabla u^m|^2 + \frac{2m}{\varepsilon} u^{m-\varepsilon} |\nabla\varphi|^2.$$

By combining the results, we obtain

$$\begin{aligned}&\frac{\varepsilon}{2m} \int_{\tau_1}^{\tau_2} \int_\Omega \varphi^2 u^{-m-\varepsilon} |\nabla u^m|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{1-\varepsilon} \int_\Omega (u^{1-\varepsilon} \varphi^2)(\cdot, \tau_1) \,\mathrm{d}x \\ &\quad \le \frac{2m}{\varepsilon} \int_{\tau_1}^{\tau_2} \int_\Omega u^{m-\varepsilon} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{1-\varepsilon} \int_{\tau_1}^{\tau_2} \int_\Omega u^{1-\varepsilon} |\partial_t(\varphi^2)| \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \frac{1}{1-\varepsilon} \int_\Omega (u^{1-\varepsilon} \varphi^2)(\cdot, \tau_2) \,\mathrm{d}x.\end{aligned}$$

Now we can pass to the limit $\tau_2 \to t_{i,2}$ so that the last term vanishes due to Lemma 4.7 if $t_{i,2} < T$ and also in the case $t = T$, since φ vanishes in a neighborhood of $\Omega \times \{T\}$. On the right-hand side, we may integrate over $\Omega \times \Lambda_i$. At this point, we also pass to the limit $k \to \infty$ in the truncations. Using Fatou's lemma for the first term on the left-hand side and the monotone convergence theorem for the remaining terms, we obtain the inequality above for the original function u. Observe that if the right-hand side tends to infinity, the estimate clearly holds. Thus we may assume that the right-hand side is finite. By considering separately the terms on the left-hand side, in the first term we can pass to the limit $\tau_1 \to t_{i,1}$. In the second term on the left-hand side, we take the supremum over $\tau_1 \in \Lambda_i$. In this way, we arrive at the bound

$$\begin{aligned}\iint_{\Omega\times\Lambda_i} \varphi^2 u^{-m-\varepsilon} |\nabla u^m|^2 \,\mathrm{d}x\,\mathrm{d}t + \operatorname*{ess\,sup}_{t\in\Lambda_i} \int_\Omega (u^{1-\varepsilon}\varphi^2)(\cdot,t)\,\mathrm{d}x \\ \le \frac{4m^2+2m\varepsilon(1-\varepsilon)}{\varepsilon^2} \iint_{\Omega\times\Lambda_i} u^{m-\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{2m+\varepsilon-\varepsilon^2}{\varepsilon(1-\varepsilon)} \iint_{\Omega\times\Lambda_i} u^{1-\varepsilon}|\partial_t(\varphi^2)|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Observe that in $\Omega_T \setminus (\bigcup_{i\in I} \Omega\times\Lambda_i)$, both sides are zero since in this set $u \equiv 0$ and $\nabla u^m \equiv 0$, see also Remark 6.2. By summing up over $i \in I$, we have

$$\begin{aligned}\iint_{\Omega_T} u^{-m-\varepsilon}|\nabla u^m|^2\varphi^2\,\mathrm{d}x\,\mathrm{d}t + \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega u(x,t)^{1-\varepsilon}\varphi(x,t)^2\,\mathrm{d}x \\ \le \frac{c_1}{\varepsilon^2}\iint_{\Omega_T} u^{m-\varepsilon}|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{c_2}{\varepsilon(1-\varepsilon)}\iint_{\Omega_T} u^{1-\varepsilon}|\partial_t(\varphi^2)|\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

for numerical constants $c_1, c_2 > 0$. In the end, we may sum up over all connected components of Ω, which concludes the proof. ■

We recall Sobolev's inequality, see [11,21].

Lemma 6.4. *Assume that $w \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ and $\varphi \in C_0^\infty(\Omega_T)$, and $r > 0$. There exists a constant $c = c(n,p,r)$ such that the inequality*

$$\iint_{\Omega_T} |\varphi w|^q\,\mathrm{d}x\,\mathrm{d}t \le c^q \iint_{\Omega_T} |\nabla(\varphi w)|^p\,\mathrm{d}x\,\mathrm{d}t \left(\operatorname*{ess\,sup}_{0<t<T}\int_\Omega |\varphi w|^r\,\mathrm{d}x\right)^{p/n}, \tag{6.3}$$

is valid for $q = p + pr/n$.

Up next, we prove a local integrability result for supercaloric functions by exploiting a Moser type iteration.

Lemma 6.5. *Let $(n-2)/n < m < 1$ and let Ω be an open set in $\mathbb{R}^n$. Suppose that u is a nonnegative supercaloric function in Ω_T. If $u \in L^s_{\mathrm{loc}}(\Omega_T)$ for some $s > \frac{n}{2}(1-m)$, then $u \in L^q_{\mathrm{loc}}(\Omega_T)$ whenever $q < m + 2/n$.*

Proof. By Theorem 5.3, the truncations $u_k := \min\{u,k\}$ are weak supersolutions for any $k > 0$ and satisfy the Caccioppoli estimate in Lemma 6.1. Up next, we combine the Sobolev inequality, Lemma 6.4 and Caccioppoli inequality, Lemma 6.1.

Let $\varphi \in C_0^\infty(\Omega_T)$, $0 \le \varphi \le 1$ and $\varphi = 1$ in a compact subset of Ω. Since $m > (n-2)/n$, it follows that $\frac{n}{2}(1-m) < 1$. Therefore, there exists $\varepsilon \in (0,m)$ with $s = 1-\varepsilon > \frac{n}{2}(1-m)$. We choose

$$w = u_k^{(s-(1-m))/2} = u_k^{(m-\varepsilon)/2}, \quad p = 2 \quad \text{and} \quad r = \frac{2s}{s-(1-m)} > 2$$

in Sobolev's inequality, and start to estimate the right-hand side. For the first term, we have

$$
\begin{aligned}
&\iint_{\Omega_T} \left|\nabla\left(\varphi u_k^{(m-1+s)/2}\right)\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le 2 \iint_{\Omega_T} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + c \iint_{\Omega_T} u_k^{-m-1+s} |\nabla(u_k^m)|^2 \varphi^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le c \iint_{\Omega_T} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + c \iint_{\Omega_T} u_k^{s} |\partial_t(\varphi^2)| \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

in which $c = c(m,\varepsilon) > 0$. In the last step we applied the Caccioppoli inequality from Lemma 6.1 with $\varepsilon = 1 - s$. With the aforementioned lemma we can also estimate the second term from the Sobolev inequality (6.3). Since $r > 2$, the function $\varphi^{r/2} \in C_0^1(\Omega_T)$ is an admissible test function in the Caccioppoli inequality, which gives

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega} u_k^s (\varphi^{r/2})^2 \,\mathrm{d}x \\
&\qquad \le c \iint_{\Omega_T} u_k^{m-1+s} \varphi^{r-2} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + c \iint_{\Omega_T} u_k^{s} \varphi^{r-2} |\partial_t(\varphi^2)| \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le c \iint_{\Omega_T} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + c \iint_{\Omega_T} u_k^{s} |\partial_t(\varphi^2)| \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where $c = c(m,\varepsilon) > 0$. By using the Sobolev inequality, Lemma 6.4, and the two inequalities above, we obtain

$$
\begin{aligned}
&\iint_{\Omega_T} \varphi^q u_k^{s(1+\frac{2}{n})-(1-m)} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \left(c \iint_{\Omega_T} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + c \iint_{\Omega_T} u_k^{s} |\partial_t(\varphi^2)| \,\mathrm{d}x\,\mathrm{d}t \right)^{1+2/n},
\end{aligned}
$$

with a constant $c = c(n,m,\varepsilon) > 0$. We can estimate

$$
\begin{aligned}
&\iint_{\Omega_T} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \iint_{\Omega_T} \chi_{\{u_k>1\}} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} \chi_{\{u_k\le 1\}} u_k^{m-1+s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \iint_{\Omega_T} u_k^{s} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

and further

$$
\begin{aligned}
&\iint_{\Omega_T} \varphi^q u_k^{s(1+\frac{2}{n})-(1-m)} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le c(n,m,\varepsilon) \left(\iint_{\Omega_T} u_k^{s} \left(|\nabla\varphi|^2 + |\partial_t(\varphi^2)|\right) \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1+2/n} \\
&\qquad \le c(n,m,\varepsilon) \left(\iint_{\Omega_T} u^{s} \left(|\nabla\varphi|^2 + |\partial_t(\varphi^2)|\right) \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{1+2/n}.
\end{aligned}
$$

Now we can pass to the limit $k \to \infty$ and use monotone convergence theorem on the left-hand side, which implies

$$u \in L^{s(1+\frac{2}{n})-(1-m)}_{\mathrm{loc}}(\Omega_T).$$

We can repeat this procedure as long as $\varepsilon > 0$, i.e., the integrability exponent is strictly less than 1. By iteration we obtain a sequence of integrability exponents

$$s_i = s_{i-1}\Big(1 + \frac{2}{n}\Big) - (1-m),$$

provided $s_{i-1} < 1$. The exponents can be written in terms of the integrability exponent $s_0 = 1 - \varepsilon > \frac{n}{2}(1-m)$ as

$$s_i = \Big(1 + \frac{2}{n}\Big)^i \Big(s_0 - \frac{n}{2}(1-m)\Big) + \frac{n}{2}(1-m).$$

In a finite number of iteration steps we obtain the integrability $u \in L^1_{\mathrm{loc}}(\Omega_T)$. Then, we let $\sigma \in (0, m)$ and $s = 1 - \frac{\sigma}{1+2/n}$. Combining the Sobolev and Caccioppoli inequalities once more we obtain

$$u \in L^{m+2/n-\sigma}_{\mathrm{loc}}(\Omega_T).$$

Since $\sigma \in (0, m)$ is arbitrary, the claim follows. ■

Next we prove a local integrability result for the gradient ∇u^m.

Lemma 6.6. *Let $(n-2)/n < m < 1$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that u is a nonnegative supercaloric function with $u \in L^s_{\mathrm{loc}}(\Omega_T)$ for some $s > \frac{n}{2}(1-m)$. Then, the weak gradient ∇u^m exists and $|\nabla u^m| \in L^q_{\mathrm{loc}}(\Omega_T)$ for any $q < 1 + 1/(1+mn)$.*

Proof. By Lemma 6.5, it already follows that $u \in L^r_{\mathrm{loc}}(\Omega_T)$ whenever $r < m + 2/n$. In particular, $u \in L^1_{\mathrm{loc}}(\Omega_T)$. First we start with truncations $u_k = \min\{u, k\}$. Let $\Omega' \Subset \Omega$, $0 < t_1 < t_2 < T$ and $\varepsilon \in (0, m)$. By Theorem 5.3, the truncation u_k is a weak supersolution for every $k \in \mathbb{N}$. Now, for $q < 1 + 1/(1+mn)$ and $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi = 1$ in $\Omega' \times (t_1, t_2)$ and $\varphi \geq 0$, we have

$$\begin{aligned}
\int_{t_1}^{t_2}\int_{\Omega'} |\nabla u_k^m|^q \,\mathrm{d}x\,\mathrm{d}t &= \int_{\Lambda_+\cap(t_1,t_2)}\int_{\Omega'} |\nabla u_k^m|^q \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Lambda_+\cap(t_1,t_2)}\int_{\Omega'} \big(u_k^{-(m+\varepsilon)/2}\,|\nabla u_k^m|\big)^q\, u_k^{q\,(m+\varepsilon)/2} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \Big(\int_{t_1}^{t_2}\int_{\Omega'} u_k^{-m-\varepsilon}\,|\nabla u_k^m|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{q/2}\Big(\int_{t_1}^{t_2}\int_{\Omega'} u_k^{\frac{q}{2-q}(m+\varepsilon)} \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/2} \\
&\leq \Big(c\iint_{\Omega_T} \big(u_k^{m-\varepsilon}\,|\nabla\varphi|^2 + u_k^{1-\varepsilon}\,|\partial_t(\varphi^2)|\big) \,\mathrm{d}x\,\mathrm{d}t\Big)^{q/2}\Big(\int_{t_1}^{t_2}\int_{\Omega'} u_k^{\frac{q}{2-q}(m+\varepsilon)} \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/2} \\
&\leq \Big(c\iint_{\Omega_T} \big(u^{m-\varepsilon}\,|\nabla\varphi|^2 + u^{1-\varepsilon}\,|\partial_t(\varphi^2)|\big) \,\mathrm{d}x\,\mathrm{d}t\Big)^{q/2}\Big(\int_{t_1}^{t_2}\int_{\Omega'} u^{\frac{q}{2-q}(m+\varepsilon)} \,\mathrm{d}x\,\mathrm{d}t\Big)^{1-q/2}
\end{aligned}$$

for $c = c(\varepsilon) > 0$ by using Hölder's inequality and the Caccioppoli inequality, Lemma 6.1. The first integral on the right-hand side is clearly bounded since $u \in L^1_{\mathrm{loc}}(\Omega_T)$, and the second is as well whenever $\frac{q}{2-q}(m+\varepsilon) < m + 2/n$ by Lemma 6.5. Since $\varepsilon > 0$ can be chosen arbitrarily small, the second integral is finite whenever $q < 1 + 1/(1+mn)$, which completes the proof. ∎

Remark 6.7. Observe that in the case $0 < m \le (n-2)/n$ (and in particular, when $m = (n-2)/n$), the proof of the preceding lemma also implies that if $u \in L^1_{\mathrm{loc}}(\Omega_T)$ is a supercaloric function in Ω_T, then $|\nabla u^m| \in L^q_{\mathrm{loc}}(\Omega_T)$ for every $q < 2/(m+1)$. Indeed, in that case, $2/(m+1) > 1$ and ∇u^m is a weak gradient of u^m.

Finally, we state characterizations for Barenblatt type supercaloric functions.

Theorem 6.8. *Let $(n-2)/n < m < 1$ and let Ω be an open set in $\mathbb{R}^n$. Suppose that u is a nonnegative supercaloric function in Ω_T. Then the following statements are equivalent:*

(i) $u \in L^q_{\mathrm{loc}}(\Omega_T)$ *for some* $q > \frac{n}{2}(1-m)$.

(ii) $u \in L^{\frac{n}{2}(1-m)}_{\mathrm{loc}}(\Omega_T)$.

(iii') *There exists* $\alpha \in (\frac{n}{2}(1-m), 1)$ *such that*

$$\sup_{\delta < t < T-\delta} \int_{\Omega'} u(x,t)^\alpha \,\mathrm{d}x < \infty,$$

whenever $\Omega' \Subset \Omega$ *and* $\delta \in (0, T/2)$,

(iii) *There holds*

$$\sup_{\delta < t < T-\delta} \int_{\Omega'} u(x,t) \,\mathrm{d}x < \infty,$$

whenever $\Omega' \Subset \Omega$ *and* $\delta \in (0, T/2)$.

Proof. (i) ⇒ (ii): Hölder's inequality.

(iii') ⇒ (i): Elementary.

(i) ⇒ (iii'): This is a direct consequence of the Caccioppoli inequality, Lemma 6.1.

(iii) ⇒ (iii'): Hölder's inequality.

(ii) ⇒ (i): Follows from proving contraposition ¬(i) ⇒ ¬(ii) in Theorem 7.3.

(i) ⇒ (iii): Follows from proving contraposition ¬(iii) ⇒ ¬(i) in Theorem 7.3. ∎

Observe that every supercaloric function u in the Barenblatt class satisfies

$$0 \le \lim_{k\to\infty} \iint_{\Omega_T} -u_k \partial_t \varphi + \nabla u_k^m \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T} -u\, \partial_t \varphi + \nabla u^m \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t,$$

for every nonnegative $\varphi \in C_0^\infty(\Omega_T)$ by Theorem 5.3 and Lemmas 6.5, 6.6. Together with Riesz' representation theorem, this implies that for every supercaloric function u in the Barenblatt class, there exists a nonnegative Radon measure μ in Ω_T such that

$$\iint_{\Omega_T} -u\, \partial_t \varphi + \nabla u^m \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \varphi \,\mathrm{d}\mu \quad \text{for every } \varphi \in C_0^\infty(\Omega_T).$$

7. Infinite point-source solutions

In this section, we consider supercaloric functions that do not fall into the class described by Theorem 6.8. As a starting point, we recall that a function

$$u(x,t) = |x|^{-(n-2)/m}, \quad \text{for } n \geq 3,\ 0 < m < 1, \tag{7.1}$$

based on the fundamental solution to the elliptic (Laplace) equation is a supercaloric function to the porous medium equation in the whole space $\mathbb{R}^{n+1}$. In the supercritical case, the singularity of the function in (7.1) is mild enough to guarantee that it belongs to the Barenblatt class. However, $|\nabla u^m| \notin L^2_{\mathrm{loc}}(\mathbb{R}^{n+1})$, which implies that u is not a weak supersolution.

For the rest of this section, we focus only on the supercritical range $(n-2)/n < m < 1$. In the complementary class, the leading example is the infinite point-source solution, which possesses a slightly similar behavior as (7.1). The infinite point-source solution (see [8]) can be written as

$$\mathcal{U}(x,t) = \left(\frac{Ct}{|x|^2}\right)^{1/(1-m)}, \quad \text{where} \quad C = \frac{2m}{1-m}(2 - n(1-m)) > 0, \tag{7.2}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

This function is a continuous weak solution to (1.1) in $(\mathbb{R}^n \setminus \{0\}) \times (0,\infty)$. However, $u \notin L^{\frac{n}{2}(1-m)}_{\mathrm{loc}}(\mathbb{R}^n \times (0,\infty))$, which implies that u is not even an integrable function in $\mathbb{R}^n \times (0,\infty)$. However, $\mathcal{U}$ is a supercaloric function in $\mathbb{R}^n \times (0,\infty)$, which we show in the next lemma.

Lemma 7.1. *The infinite point-source solution $\mathcal{U}$ defined in (7.2) is a supercaloric function in $\mathbb{R}^n \times (0,\infty)$.*

Proof. Denote $\mathcal{U}_k = \min\{\mathcal{U}, k\}$. Now $\mathcal{U}_k$ is clearly continuous in $\mathbb{R}^n \times (0,\infty)$, and it is a supercaloric function in $(\mathbb{R}^n \setminus \{0\}) \times (0,\infty)$ as a truncation of a continuous weak solution. Let $Q_{t_1,t_2} = Q \times (t_1,t_2) \Subset \mathbb{R}^n \times (0,\infty)$ be a $C^{2,\alpha}$-cylinder such that $0 \in Q$, and let $h \in C(\overline{Q_{t_1,t_2}})$ be a weak solution in Q_{t_1,t_2}, with $h \leq \mathcal{U}_k$ on $\partial_p Q_{t_1,t_2}$. This immediately implies that $h \leq k$ in Q_{t_1,t_2} and, in particular, that $h \leq \mathcal{U}_k = k$ on $\{0\} \times [t_1,t_2)$. Then, since h is subcaloric, we can use Lemma 3.4 to conclude that we also have $h \leq \mathcal{U}_k$ in $(Q \setminus \{0\}) \times (t_1,t_2)$.

If $0 \in \partial Q$, we can use the fact that $\mathcal{U}_k = k$ in $\overline{B_r(0)} \times (t_1,t_2)$ with $r = (\frac{Ct_1}{k^{1-m}})^{1/2}$. Since $h \leq k$ in Q_{t_1,t_2}, it follows that $h \leq \mathcal{U}_k$ in $(\overline{B_r(0)} \cap Q) \times (t_1,t_2)$ with the previously defined r. In the set $(Q \setminus \overline{B_r(0)}) \times (t_1,t_2)$, we can use Lemma 3.4 to conclude that $h \leq \mathcal{U}_k$ in $(Q \setminus \overline{B_r(0)}) \times (t_1,t_2)$, and therefore in the whole cylinder Q_{t_1,t_2}. Thus $\mathcal{U}_k$ is supercaloric in $\mathbb{R}^n \times (0,\infty)$.

By Lemma 3.9, also the pointwise limit $\lim_{k\to\infty} \mathcal{U}_k = \mathcal{U}$ turns out to be supercaloric in $\mathbb{R}^n \times (0,\infty)$. ∎

Again, zero extension of $\mathcal{U}$ to nonpositive times $t \leq 0$, say u, is supercaloric in $\mathbb{R}^n \times \mathbb{R}$ by Lemma 3.7. However, $u \notin L^{\frac{n}{2}(1-m)}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$.

We can modify the example above to obtain supercaloric functions with as bad singularity as we please. We can define

$$\mathcal{U}(x,t) = \Big(\frac{Ct}{|x|^q}\Big)^{1/(1-m)},$$

in which $q \geq 2$ can be as large as we wish and

$$C = qm\Big(2 + \frac{qm}{1-m} - n\Big).$$

This is still a supercaloric function in $B(0,1) \times (0,\infty)$. However, for given $\varepsilon > 0$, $\mathcal{U} \notin L^{\varepsilon}_{\mathrm{loc}}$ if $q \geq \frac{n}{\varepsilon}(1-m)$.

Before stating characterizations in the complementary class, we state and prove an auxiliary result, which is analogous to Lemma 4.5 in [14].

Lemma 7.2. *Let $(n-2)/n < m < 1$. Let u be a nonnegative supercaloric function in Ω_T. Suppose that there exist a point $x_o \in \Omega$ and a sequence (t_j) in $(0,T)$ with $t_j \to t_o \in (0,T)$ as $j \to \infty$, such that*

$$\lim_{j\to\infty} \int_{B(x_o,r)} u(x,t_j)\,\mathrm{d}x = \infty$$

whenever $r > 0$ and $B(x_o,r) \Subset \Omega$. Then,

$$\liminf_{(x,s)\to(x_o,t)} u(x,s)\,|x-x_o|^{2/(1-m)} > 0$$

for every $t > t_o$.

Proof. Fix $r > 0$ with $B(x_o,64r) \Subset \Omega$ and let $t \in (t_o,T)$. Then, for large enough j we have that

$$\fint_{B(x_o,r)} u(x,t_j)\,\mathrm{d}x \geq 4c\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)},$$

where $c = c(n,m)$ is the constant from Lemma 4.5 with integral averages. There exist truncations $u_{k_j} := \min\{u,k_j\}$ such that

$$\fint_{B(x_o,r)} u_{k_j}(x,t_j)\,\mathrm{d}x = 2c\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)}. \tag{7.3}$$

By lower semicontinuity of u_{k_j}, there exists a sequence of Lipschitz functions $(\psi_{k_j,i})_{i\in\mathbb{N}}$, such that

$$0 \leq \psi_{k_j,i} \leq \psi_{k_j,i+1} \leq u_{k_j}^m \quad \text{and} \quad \psi_{k_j,i} \to u_{k_j}^m \quad \text{pointwise in } \Omega_T \text{ as } i \to \infty.$$

By Theorem 2.8, there exists a unique continuous solution $h_{k_j,i} \in C(\overline{B(x_o,2r) \times (t_j,T)})$, such that $h_{k_j,i} = \psi_{k_j,i}^{1/m}$ on the parabolic boundary of $B(x_o,2r) \times (t_j,T)$. By the comparison principle from the definition of supercaloric functions, it then follows that $h_{k_j,i} \leq u_{k_j}$ for every $i \in \mathbb{N}$.

By taking $s = t_j$ and $t < T$ in Lemma 4.5, we have that

$$\sup_{t_j<\tau<t} \fint_{B(x_o,r)} h_{k_j,i}(x,\tau)\,dx \le c \inf_{t_j<\tau<t} \fint_{B(x_o,2r)} h_{k_j,i}(x,\tau)\,dx + c\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)}$$
$$\le c \inf_{t_j<\tau<t} \fint_{B(x_o,2r)} u_{k_j}(x,\tau)\,dx + c\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)},$$

where comparison was used in the second inequality. We can further bound the left-hand side from below as

$$\sup_{t_j<\tau<t} \fint_{B(x_o,r)} h_{k_j,i}(x,\tau)\,dx \ge \fint_{B(x_o,r)} h_{k_j,i}(x,t_j)\,dx = \fint_{B(x_o,r)} \psi_{k_j,i}^{1/m}(x,t_j)\,dx.$$

By combining the inequalities above and passing to the limit $i \to \infty$ and using (7.3), we obtain

$$2c\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)} = \fint_{B(x_o,r)} u_{k_j}(x,t_j)\,dx = \lim_{i\to\infty} \fint_{B(x_o,r)} \psi_{k_j,i}^{1/m}(x,t_j)\,dx$$
$$\le c \fint_{B(x_o,2r)} u(x,\tau)\,dx + c\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)}$$

for any $\tau \in (t_j, t)$ and large enough j. From here, it follows that

$$\Big(\frac{t-t_j}{r^2}\Big)^{1/(1-m)} \le \fint_{B(x_o,2r)} u(x,\tau)\,dx.$$

By passing to the limit $j \to \infty$, this implies

$$r^2\Big(\fint_{B(x_o,2r)} u(x,\tau)\,dx\Big)^{1-m} \ge t - t_o,$$

for any $\tau \in (t_o, t)$. Observe that $r > 0$ was arbitrary. By taking any sequence (r_j) with $0 < r_j \to 0$ as $j \to \infty$, we have

$$\liminf_{j\to\infty} r_j^2\Big(\fint_{B(x_o,2r_j)} u(x,\tau)\,dx\Big)^{1-m} \ge t - t_o > 0$$

for any $\tau \in (t_o, t)$.

For the constant $c_2 = c_2(n,m)$ from Lemma 4.6, we fix $\varepsilon \in (0, \min\{c_2(t-t_o), T-t\})$, $\tau \in (t_o, t)$ and choose truncation levels k_j such that

$$c_2 r_j^2\Big(\fint_{B(x_o,2r_j)} u_{k_j}(x,\tau)\,dx\Big)^{1-m} = \varepsilon$$

holds for all large enough j. Now we can apply Lemma 4.6 and obtain

$$\inf_{B(x_o,2r_j)} u(\cdot,s) \ge \inf_{B(x_o,2r_j)} u_{k_j}(\cdot,s) \ge c(n,m) \fint_{B(x_o,2r_j)} u_{k_j}(x,\tau)\,dx$$
$$= c(n,m)\varepsilon^{1/(1-m)} r_j^{-2/(1-m)}$$

for any $s \in [\tau + \alpha\varepsilon, \tau + \varepsilon]$, where $\alpha = \alpha(n,m) \in (0,1)$ is the constant from Lemma 4.6. Since the sequence (r_j) and numbers $\tau \in (t_o, t)$ and $\varepsilon \in (0, \min\{c_2(t-t_o), T-t\})$ could be chosen freely, the claim follows. ∎

Next we state characterizations for the complementary class.

Theorem 7.3. *Let $(n-2)/n < m < 1$ and let Ω be an open set in $\mathbb{R}^n$. Assume that u is a nonnegative supercaloric function in Ω_T. Then the following statements are equivalent*:

(i) $u \notin L^q_{\mathrm{loc}}(\Omega_T)$ *for any* $q > \frac{n}{2}(1-m)$.

(ii) $u \notin L^{\frac{n}{2}(1-m)}_{\mathrm{loc}}(\Omega_T)$.

(iii') *For every* $\alpha \in (\frac{n}{2}(1-m), 1)$ *there exist* $\Omega' \Subset \Omega$ *and* $\delta \in (0, T/2)$ *such that*

$$\sup_{\delta<t<T-\delta} \int_{\Omega'} u(x,t)^\alpha \,\mathrm{d}x = \infty.$$

(iii) *There exist* $\Omega' \Subset \Omega$ *and* $\delta \in (0, T/2)$ *such that*

$$\sup_{\delta<t<T-\delta} \int_{\Omega'} u(x,t) \,\mathrm{d}x = \infty.$$

(iv) *There exists* $(x_o, t_o) \in \Omega_T$ *such that*

$$\liminf_{(x,s)\to(x_o,t)} u(x,s)|x-x_o|^{2/(1-m)} > 0 \quad \textit{for every } t > t_o.$$

Proof. (ii) $\Rightarrow$ (i): Hölder's inequality.

(iii') $\Rightarrow$ (iii): Hölder's inequality.

(i) $\Rightarrow$ (iii'): Elementary.

(iv) $\Rightarrow$ (iii'): Fix $t > t_o$. Then, for some $r > 0$ there exists $\varepsilon > 0$ such that

$$u(x,s)\,|x-x_o|^{2/(1-m)} \ge \varepsilon$$

whenever $(x,s) \in (B(x_o,r) \setminus \{x_o\}) \times ((t-r, t+r) \setminus \{t_o\})$. This implies that

$$\int_{B(x_o,r)} u(x,t)^\alpha \,\mathrm{d}x = \infty$$

for every $\alpha \ge \frac{n}{2}(1-m)$ and $t \in ((t-r, t+r) \setminus \{t_o\})$. This implies (iii').

(iv) $\Rightarrow$ (ii): Same argument as above.

(iii) $\Rightarrow$ (iv): By (iii), there exist an instant of time $t_o \in (0,T)$ and a sequence (t_j) in $(0,T)$ with $t_j \to t_o$, such that

$$\lim_{j\to\infty} \int_{\Omega'} u(x,t_j) \,\mathrm{d}x = \infty$$

for some $\Omega' \Subset \Omega$.

Let us fix a small $r_o > 0$. We claim that there exists a point $x_o \in \overline{\Omega'}$ such that

$$\lim_{j\to\infty} \int_{B(x_o,r)} u(x,t_j) \,\mathrm{d}x = \infty$$

for every $r \in (0, r_o)$. This can be shown by contradiction. Assume that for any $y \in \overline{\Omega'}$ there exists a radius $r_y \in (0, r_o)$ such that

$$\limsup_{j\to\infty} \int_{B(y,r_y)} u(x,t_j) \,\mathrm{d}x < \infty.$$

Take an open cover $\{B(y, r_y) : y \in \overline{\Omega'}\}$ of $\overline{\Omega'}$. By compactness of $\overline{\Omega'}$, this has a finite subcover, say $\{B(y_k, r_k) : k = 1, 2, \dots, M\}$, which implies

$$\int_{\Omega'} u(x, t_j)\,\mathrm{d}x \le \sum_{k=1}^{M} \int_{B(y_k, r_k)} u(x, t_j)\,\mathrm{d}x,$$

for any $j \in \mathbb{N}$. When $j \to \infty$, the left-hand side tends to infinity, while the right-hand side stays bounded, implying the desired contradiction. Thus we have established that there exists a point $x_o \in \Omega$ such that

$$\lim_{j\to\infty} \int_{B(x_o, r)} u(x, t_j)\,\mathrm{d}x = \infty$$

for arbitrarily small $r > 0$. Now we can use Lemma 7.2 to conclude the proof. ■

8. Pointwise behavior of supercaloric functions

In this section, we show that every supercaloric function coincides with its ess lim inf-regularization, cf. Theorem 2.11 for the case of weak supersolutions. Proofs are partly based on [17, 18].

Theorem 8.1. *Let $0 < m < 1$, $\Omega \subset \mathbb{R}^n$ be an open set, and $u : \Omega_T \to [0, \infty]$ a supercaloric function in Ω_T. Then,*

$$u(x, t) = \operatorname*{ess\,lim\,inf}_{\substack{(y,s)\to(x,t)\\ s<t}} u(y, s) \quad \textit{for every } (x, t) \in \Omega_T.$$

First, we prove existence and some properties of a Poisson modification we will use in the proof.

Proposition 8.2. *Let $0 < m < 1$. Let (h_k) be a nondecreasing sequence of nonnegative continuous weak solutions in Ω_T, i.e., $h_k^m \in L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega))$ for each $k \in \mathbb{N}$, and suppose that the pointwise limit $\lim_{k\to\infty} h_k = h$ is bounded in Ω_T. Then, h is a locally Hölder continuous weak solution in Ω_T with $h^m \in L^2(0, T; H^1_{\mathrm{loc}}(\Omega))$, and $\nabla h_k^m \rightharpoonup \nabla h^m$ weakly in $L^2_{\mathrm{loc}}(\Omega_T)$.*

Proof. First observe that the sequence (h_k) is bounded, since $h_k \le h$ for every $k \in \mathbb{N}$. By Theorem 18.1 in Chapter 6 of [11], it follows that the family (h_k) is locally equicontinuous. Then the Arzelá–Ascoli theorem implies that there exists a subsequence h_{k_i} that converges uniformly to some function g, which is locally continuous in Ω_T by the uniform limit theorem. Furthermore, since $\lim_{k\to\infty} h_k = h$ pointwise, it follows that $g = h$. Lemma 2.9 implies that $\nabla h_k^m \rightharpoonup \nabla h^m$ weakly in $L^2_{\mathrm{loc}}(\Omega_T)$, which further implies that h is a weak solution in Ω_T and that $h^m \in L^2(0, T; H^1_{\mathrm{loc}}(\Omega))$. As a bounded weak solution, the function h is locally Hölder continuous by Theorem 18.1 in Chapter 6 of [11]. ■

Proposition 8.3. *Let $0 < m < 1$ and let $Q_{t_1,t_2} \Subset \Omega_T$ be a $C^{2,\alpha}$-cylinder. Let (v_k) be a nondecreasing sequence of nonnegative continuous weak supersolutions in Ω_T such that $\lim_{k\to\infty} v_k = v$, in which v is a bounded supercaloric function in Ω_T. Then, there exists a Poisson modification defined as*

$$u_P^k = \begin{cases} h_k & \textit{in } Q \times (t_1, t_2], \\ v_k & \textit{otherwise}, \end{cases}$$

where $h_k \in C(\overline{Q_{t_1,t_2}})$ is a weak solution in Q_{t_1,t_2} with $h_k^m \in L^2(t_1, t_2; H^1(Q))$ such that $h_k = v_k$ on $\partial_p Q_{t_1,t_2}$ and $h_k^m - v_k^m \in L^2(t_1, t_2; H_0^1(\Omega))$.

Furthermore, u_P^k is nondecreasing, and the limit $u_P = \lim_{k\to\infty} u_P^k$ can be written as

$$u_P = \begin{cases} h & \textit{in } Q \times (t_1, t_2], \\ v & \textit{otherwise}, \end{cases}$$

in which $h \in C(Q_{t_1,t_2})$ is a weak solution in Q_{t_1,t_2} with $h^m \in L^2(t_1, t_2; H^1(Q))$.

Moreover, u_P is a bounded supercaloric function in Ω_T and $\nabla (u_P^k)^m \rightharpoonup \nabla u_P^m$ weakly in $L^2_{\mathrm{loc}}(\Omega_T)$. In particular,

$$\nabla h_k^m \rightharpoonup \nabla h^m \quad \textit{weakly in } L^2(Q_{t_1,t_2}).$$

Proof. Since v_k is continuous, there exist functions $\psi_k^i \in C^{0,1}(\Omega_T)$ such that

$$\begin{aligned} &0 \le \psi_k^i \le \psi_k^{i+1} \le v_k^m \quad \text{everywhere in } \Omega_T \text{ for every } i \in \mathbb{N}, \\ &\lim_{i\to\infty} \psi_k^i = v_k^m \quad \text{everywhere in } \Omega_T, \text{ and} \\ &\sup_{\partial_p Q_{t_1,t_2}} |(\psi_k^i)^{1/m} - v_k| \xrightarrow{i\to\infty} 0. \end{aligned}$$

Let h_k^i be a weak solution in Q_{t_1,t_2} taking the boundary values $(\psi_k^i)^{1/m}$ on $\overline{\partial_p Q_{t_1,t_2}}$ both continuously and in Sobolev sense (Theorem 2.8). Denote

$$u_P^{k,i} = \begin{cases} h_k^i & \text{in } Q \times (t_1, t_2], \\ v_k & \text{otherwise}. \end{cases}$$

The sequence h_k^i is increasing with respect to i in Q_{t_1,t_2} by the aforementioned theorem, and $h_k^i \in C(\overline{Q_{t_1,t_2}})$ for each $i \in \mathbb{N}$. By Theorem 2.7, $\lim_{i\to\infty} h_k^i = h_k$ pointwise everywhere in Q_{t_1,t_2}, where $h_k \in C(\overline{Q_{t_1,t_2}})$ is a (unique) very weak solution in Q_{t_1,t_2} such that $h_k = v_k$ on $\overline{\partial_p Q_{t_1,t_2}}$. Since the sequence (h_k^i) with respect to i satisfies the assumptions in Proposition 8.2 in Q_{t_1,t_2}, we have that h_k is a locally Hölder continuous weak solution with $h_k^m \in L^2(t_1, t_2; H^1_{\mathrm{loc}}(Q))$. Then Theorem 2.7 implies $u_P^k \in C(\Omega_T \setminus (Q \times \{t_2\}))$, $u_P^{k,i}$ is increasing with respect to i, and $\lim_{i\to\infty} u_P^{k,i} = u_P^k$ pointwise in Ω_T. Since v_k is supercaloric in Ω_T by Theorem 3.6, we have $u_P^{k,i} \le v_k \le v$ everywhere in Ω_T. This implies that also $u_P^k \le v_k \le v$.

Next we show that u_P^k is a (bounded) supercaloric function. Since u_P^k is lower semicontinuous and bounded, properties (i) and (ii) in Definition 3.1 are clear. For (iii'), let $V_{s_1,s_2} \Subset \Omega_T$ be a $C^{2,\alpha}$-cylinder and $g \in C(\overline{V_{s_1,s_2}})$ be a weak solution in V_{s_1,s_2} with $g \le u_P^k$ on $\partial_p V_{s_1,s_2}$. Suppose that V_{s_1,s_2} intersects both Q_{t_1,t_2} and its complement, since otherwise the claim is clear. Now since v_k is supercaloric, we immediately have $g \le v_k$ in Ω_T, which implies $g \le u_P^k$ in $V_{s_1,s_2} \setminus (Q \times (t_1,t_2])$. Since h_k is supercaloric and g is subcaloric in Q_{t_1,t_2}, we can use Theorem 3.4 to conclude that $g \le h_k$ in $V_{s_1,s_2} \cap Q_{t_1,t_2}$. Finally, we have $g \le h_k$ on the slice $(V \cap Q) \times \{t_2\}$ by continuity of g and h_k. This implies that $g \le u_P^k$ in V_{s_1,s_2}, which shows that u_P^k is supercaloric.

Now, since u_P^k is a bounded supercaloric function, Theorem 5.3 implies that we have $u_P^k \in L^2_{\mathrm{loc}}(0,T;H^1_{\mathrm{loc}}(\Omega))$. Furthermore, $h_k^m \in L^2(t_1,t_2;H^1(Q))$, since $u_P^k = h_k$ in Q_{t_1,t_2}. Lemma 2.2 implies that $h_k \le h_{k+1} \le \sup_{\partial_p Q_{t_1,t_2}} v$ for every $k \in \mathbb{N}$, since the sequence (v_k) is increasing. As the sequence (h_k) satisfies the assumptions in Proposition 8.2 in Q_{t_1,t_2}, we have that $h = \lim_{k\to\infty} h_k$ is a locally Hölder continuous weak solution in Q_{t_1,t_2} with $h^m \in L^2(t_1,t_2;H^1_{\mathrm{loc}}(Q))$.

As (u_P^k) is an increasing and uniformly bounded sequence, Lemma 3.9 implies that the limit u_P is a bounded supercaloric function. Furthermore, Theorem 5.3 implies that $u_P^m \in L^2_{\mathrm{loc}}(0,T;H^1_{\mathrm{loc}}(\Omega))$. This further implies $h^m \in L^2(t_1,t_2;H^1(Q))$, since $u_P = h$ in Q_{t_1,t_2}.

Since (u_P^k) is an increasing, uniformly bounded sequence of weak supersolutions in Ω_T converging to u_P, Lemma 2.9 implies that $\nabla(u_P^k)^m \rightharpoonup \nabla u_P^m$ weakly in $L^2_{\mathrm{loc}}(\Omega_T)$. Finally, this implies that $\nabla h_k^m \rightharpoonup \nabla h^m$ weakly in $L^2(Q_{t_1,t_2})$, since $u_P^k = h_k$ and $u_P = h$ in Q_{t_1,t_2}. ■

Before a proof of Theorem 8.1, we state and prove another auxiliary result.

Lemma 8.4. *Let $0 < m < 1$ and let $\Omega \subset \mathbb{R}^n$ be a connected open set. Suppose that $v: \Omega_T \to [0,\infty]$ is a supercaloric function in Ω_T and let $Q_{t_1,t_2} \Subset \Omega_T$ be such that $[t_1,t_2]$ is contained in the positivity set Λ_+ defined in (4.4). Assume that*

$$v = \gamma \quad \text{a.e. in } Q_{t_1,t_2}$$

for some $\gamma \in (0,\infty)$. Then,

$$v(x,t) = \gamma \quad \text{for every } (x,t) \in Q \times (t_1,t_2].$$

Proof. By lower semicontinuity of v, it follows that $v \le \gamma$ everywhere in $\overline{Q_{t_1,t_2}}$. Thus, without loss of generality, we may assume that v is bounded in Ω_T. Since $Q_{t_1,t_2} \Subset \Omega \times \Lambda_+$, it follows that there exists $\delta > 0$ such that $v > 0$ everywhere in $Q_{t_1,t_2+\delta}$. Let $\psi_k \in C^\infty(Q_{t_1,t_2+\delta})$ be such that

$$\psi_1 < \psi_2 < \cdots < v \quad \text{and} \quad \lim_{k\to\infty} \psi_k = v \quad \text{everywhere in } Q_{t_1,t_2+\delta}.$$

Now by applying Theorem 5.1 in a similar fashion as in Lemma 5.2, we can find a sequence of continuous weak supersolutions v_k in $Q_{t_1,t_2+\delta}$ such that $v_1 \le v_2 \le \cdots \le v$ with $\psi_k \le v_k \le v$ everywhere in $Q_{t_1,t_2+\delta}$, which implies $v_k(x,t) \to v(x,t)$ for every $(x,t) \in Q_{t_1,t_2+\delta}$. Observe that we further have $\nabla v_k^m \rightharpoonup \nabla v^m$ weakly in $L^2_{\mathrm{loc}}(Q_{t_1,t_2+\delta})$ by Lemma 2.9.

Fix $t' \in (t_1, t_2)$ such that $v(x, t') = \gamma$ for a.e. $x \in Q$. Observe that this holds for a.e. $t' \in (t_1, t_2)$. Furthermore, fix a $C^{2,\alpha}$-cylinder $Q' \Subset Q$ and define Poisson modifications of v_k and v in $Q'_{t',t_2+\delta}$ as in Proposition 8.3.

Since h_k is a weak solution in $Q'_{t',t_2+\delta}$, it follows that

$$\iint_{Q'_{t',t_2}} -h_k \partial_t \varphi + \nabla h_k^m \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_{Q'} v_k(x, t') \, \varphi(x, t') \, \mathrm{d}x$$

for all $\varphi \in C^\infty(Q'_{t',t_2})$ vanishing on the boundary of Q'_{t',t_2} except possibly on $Q' \times \{t'\}$. By Proposition 8.3, we have that $\nabla h_k^m \rightharpoonup \nabla h^m$ weakly in $L^2(Q'_{t',t_2+\delta})$ when $k \to \infty$. Also $v_k(x, t') \to v(x, t')$ as $k \to \infty$ for every $x \in Q'$. Thus, by passing to the limit $k \to \infty$, we obtain

$$\iint_{Q'_{t',t_2}} -h \, \partial_t \varphi + \nabla h^m \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_{Q'} \gamma \, \varphi(x, t') \, \mathrm{d}x \tag{8.1}$$

since $v(x, t') = \gamma$ for a.e. $x \in Q'$. Since $\gamma > 0$ is a weak solution as a constant, we also have

$$\iint_{Q'_{t',t_2}} -\gamma \, \partial_t \varphi + \nabla \gamma^m \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_{Q'} \gamma \, \varphi(x, t') \, \mathrm{d}x. \tag{8.2}$$

By approximation, we may use test functions satisfying $\varphi \in L^2(t', t_2; H_0^1(Q'))$, with $\partial_t \varphi \in L^2(Q'_{t',t_2})$ and $\varphi(t_2) = 0$. Observe that the Oleinik type test function

$$\varphi(x, t) := \begin{cases} \int_t^{t_2} (v_k^m(x, s) - h_k^m(x, s)) \, \mathrm{d}s, & \text{for } t' < t < t_2, \\ 0, & \text{for } t \geq t_2, \end{cases}$$

is admissible. By using this test function and subtracting (8.1) from (8.2), we obtain

$$\begin{aligned} \mathrm{I}_k &:= \iint_{Q'_{t',t_2}} (\gamma - h)(v_k^m - h_k^m) \, \mathrm{d}x \, \mathrm{d}t \\ &= -\iint_{Q'_{t',t_2}} \nabla(\gamma^m - h^m(x, t)) \cdot \int_t^{t_2} \nabla(v_k^m(x, s) - h_k^m(x, s)) \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}t =: \mathrm{II}_k. \end{aligned}$$

Observe that since $\nabla v_k^m \rightharpoonup \nabla v^m$ and $\nabla h_k^m \rightharpoonup \nabla h^m$ weakly in $L^2(Q'_{t',t_2})$ when $k \to \infty$, and $v = \gamma$ a.e. in Q'_{t',t_2}, we obtain

$$\mathrm{II}_k \xrightarrow{k \to \infty} -\frac{1}{2} \int_{Q'} \left| \int_{t'}^{t_2} \nabla(\gamma^m - h^m(x, t)) \, \mathrm{d}t \right|^2 \mathrm{d}x \leq 0.$$

Thus, by Corollary 3.11 in [7] and using the facts above, we conclude

$$\begin{aligned} \iint_{Q'_{t',t_2}} |\gamma^m - h^m|^{(m+1)/m} \, \mathrm{d}x \, \mathrm{d}t &\leq \iint_{Q'_{t',t_2}} (\gamma - h)(\gamma^m - h^m) \, \mathrm{d}x \, \mathrm{d}t \\ &= \lim_{k \to \infty} \mathrm{I}_k = \lim_{k \to \infty} \mathrm{II}_k \leq 0, \end{aligned}$$

which implies that $h = \gamma$ a.e. in Q'_{t',t_2}. Since $h \in C(Q'_{t',t_2+\delta})$, it follows that $h = \gamma$ everywhere in $Q' \times (t', t_2]$.

Since $h \le v$ everywhere in $Q'_{t',t_2+\delta}$ and $v \le \gamma$ everywhere in $\overline{Q_{t_1,t_2}}$, it follows that $\gamma = h \le v \le \gamma$ everywhere in $Q' \times (t', t_2]$, i.e., $v = \gamma$ everywhere in $Q' \times (t', t_2]$. Since this holds for arbitrary $Q' \Subset Q$ and a.e. $t' \in (t_1, t_2)$, the claim follows. ∎

Proof of Theorem 8.1. Fix $(x_o, t_o) \in \Omega_T$ and denote

$$\lambda = \operatorname*{ess\,lim\,inf}_{\substack{(y,s)\to(x_o,t_o)\\ s<t_o}} u(y,s).$$

Without loss of generality, we may assume that Ω is connected. By lower semicontinuity of u we have that $\lambda \ge u(x_o, t_o)$. Thus, if $\lambda = 0$ there is nothing to prove. Let us suppose that $\lambda > 0$.

Suppose that also $u(x_o, t_o) > 0$. Then, it follows that $t_o \in \Lambda_i$ for some $i \in I$, which further implies that there exists $r_o > 0$ such that $B_r(x_o) \times (t_o - r^2, t_o) \Subset \Omega \times \Lambda_i$ for every $r < r_o$. This implies that $\lambda > 0$. Furthermore, for any $\gamma \in (0, \lambda)$, there exists $r < r_o$ such that $u \ge \gamma$ a.e. in $B_r(x_o) \times (t_o - r^2, t_o)$. Now $v = \min\{u, \gamma\}$ is a supercaloric function satisfying $v = \gamma$ a.e. in $B_r(x_o) \times (t_o - r^2, t_o)$. By Lemma 8.4, it follows that $v = \gamma$ everywhere in $B_r(x_o) \times (t_o - r^2, t_o]$, i.e., $u \ge \gamma$ everywhere in $B_r(x_o) \times (t_o - r^2, t_o]$. In particular, $u(x_o, t_o) \ge \gamma$. Since $\gamma \le u(x_o, t_o) \le \lambda$ and $\gamma \in (0, \lambda)$ was arbitrary, we have $\lambda = u(x_o, t_o)$.

Then suppose that $u(x_o, t_o) = 0$ and $\lambda > 0$. From the latter, it follows that there exist $\varepsilon > 0$ and $r > 0$ such that

$$\operatorname*{ess\,inf}_{B_r(x_o)\times(t_o-r^2,t_o)} u \ge \varepsilon.$$

Thus,

$$\operatorname*{ess\,inf}_{B_r(x_o)} u(\cdot, t) \ge \varepsilon$$

for a.e. $t \in (t_o - r^2, t_o)$. Let (t_i) be a sequence in $(t_o - r^2, t_o)$ for which above holds for every $i \in \mathbb{N}$, and $t_i \to t_o$ as $i \to \infty$. Since $u(x_o, t_o) = 0$ implies that $u(x, t_o) = 0$ for all $x \in \Omega$, we may use Lemma 4.7 to conclude

$$0 < \varepsilon \le \operatorname*{ess\,inf}_{B_r(x_o)} u(\cdot, t_i) \le \fint_{B_r(x_o)} u(x, t_i)\,\mathrm{d}x \xrightarrow{i\to\infty} 0,$$

which is a contradiction. Thus $\lambda = 0$, which completes the proof. ∎

In order to summarize our results on the connections between nonnegative supercaloric functions and weak supersolutions, we consider the classes

$$\begin{aligned}
\mathcal{W} &= \{u_* : u \text{ is a weak supersolution in } \Omega_T\},\\
\mathcal{S} &= \{u : u \text{ is a supercaloric function in } \Omega_T\},\\
\mathcal{S}_E &= \big\{u : u \in \mathcal{S},\ u^m \in L^2_{\mathrm{loc}}(0,T; H^1_{\mathrm{loc}}(\Omega)) \cap L^{1/m}_{\mathrm{loc}}(\Omega_T)\big\},\\
\mathcal{W}_b &= \{u_* : u \in \mathcal{W}, u \text{ is locally essentially bounded in } \Omega_T\},\\
\mathcal{S}_b &= \{u : u \in \mathcal{S}, u \text{ is locally bounded in } \Omega_T\},
\end{aligned}$$

where $(\cdot)_*$ denotes the ess lim inf-regularization defined in Theorem 2.11.

As a direct consequence of Lemmas 3.6, 5.4 (or Theorem 5.3) and Theorem 8.1, together with the examples presented in Sections 6 and 7, we can conclude the following connections of nonnegative supercaloric functions and weak supersolutions.

Corollary 8.5. *Let* $0 < m < 1$. *Then* $\mathcal{W} \subsetneq \mathcal{S}$, $\mathcal{W} = \mathcal{S}_E$ *and* $\mathcal{W}_b = \mathcal{S}_b$.

Funding. K. Moring has been supported by the Magnus Ehrnrooth Foundation.

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Received August 31, 2023; revised July 10, 2024.

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