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The relativistic Vlasov–Maxwell system: Local smooth solvability for a weak topology

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Abstract. This article is devoted to the relativistic Vlasov–Maxwell system in space dimension three. We prove the local well-posedness (existence and uniqueness) for initial data $(\mathrm{f}_0, \mathrm{E}_0, \mathrm{B}_0) \in L^\infty \times H^1 \times H^1$, with f_0 compactly supported in momentum. As a byproduct, we obtain the uniqueness of weak solutions to the 3D relativistic Vlasov–Maxwell system. This result is at the interface of the classical solutions in the sense of Glassey–Strauss, and the weak solutions in the sense of DiPerna–Lions. It is the consequence of the local smooth solvability for the weak topology associated with $L^\infty \times H^1 \times H^1$. We derive our result from a representation formula decoding how the momentum spreads and revealing that the domain of influence in momentum is controlled by mild information. We do so by developing a Radon Fourier analysis on the RVM system, leading to the study of a class of singular weighted integrals. In parallel, we implement our method to construct smooth solutions to the RVM system in the regime of dense, hot and strongly magnetized plasmas.

1. Main results

Let $C_c^n(\mathbb{R}^m)$ be the class of compactly supported n times continuously differentiable functions on $\mathbb{R}^m$, endowed with the norm

$$\|\mathrm{U}\|_n := \sup\{\|\partial^\alpha \mathrm{U}\|_{L^\infty(\mathbb{R}^m)} : |\alpha| \leq n\}, \quad \|\cdot\|_0 \equiv \|\cdot\|_{L^\infty}, \quad (n,m) \in \mathbb{N}^2.$$

Consider a Cauchy problem which is locally well-posed in C_c^n with $n \in \mathbb{N}^*$. Given some initial data $\mathrm{U}_0 \in C_c^n$, denote by $T(\mathrm{U}_0) \in \mathbb{R}_+^* \cup \{+\infty\}$ the lifespan of the associated smooth (say C^1) solution. Select a subspace $\mathfrak{N} \subset C_c^n$ which is equipped with a norm $\mathcal{N}$ weaker than $\|\cdot\|_n$, meaning that $\mathcal{N} \lesssim \|\cdot\|_n$ on $\mathfrak{N}$. Now, for all $\mathrm{S}_0 \in \mathbb{R}_+$, we can introduce the lower bound

$$T(\mathfrak{N}, \mathcal{N}; \mathrm{S}_0) := \inf\{T(\mathrm{U}_0) : \mathrm{U}_0 \in \mathfrak{N} \text{ and } \mathcal{N}(\mathrm{U}_0) \leq \mathrm{S}_0\} \in \mathbb{R}_+ \cup \{+\infty\}. \tag{1.1}$$

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Definition 1.1 (Smooth solvability of initial value problems). We say that a Cauchy problem is locally smoothly solvable (LSS) for $(\mathfrak{N}, \mathcal{N})$ when $T(\mathfrak{N}, \mathcal{N}; \mathrm{S}_0)$ is a positive number for all $\mathrm{S}_0 \in \mathbb{R}_+$. It is globally smoothly solvable (GSS) for $(\mathfrak{N}, \mathcal{N})$ when $T(\mathfrak{N}, \mathcal{N}; \mathrm{S}_0) = +\infty$ for all $\mathrm{S}_0 \in \mathbb{R}_+$.

The notion of smooth solvability for $(\mathfrak{N}, \mathcal{N})$ focuses on the persistence of regularity. It must be distinguished from the classical concept of "well-posedness", which is usually understood in the sense of Hadamard (with existence, uniqueness and stability related to the normed space associated with $\mathcal{N}$). It is less demanding. In particular, it says nothing about the continuous dependence with respect to $\mathcal{N}$.

For a large class of first-order quasilinear symmetric hyperbolic systems (say QSH systems) whose prototypes are Burgers' equation and the 3D compressible Euler equations (away from the vacuum), the Cauchy problem is known to be locally smoothly solvable for $(C^n_c, \|\cdot\|_n)$ as long as n is large enough. But, due to the finite-time singularity formation, it is certainly not globally smoothly solvable for such $(C^n_c, \|\cdot\|_n)$.

For another whole range of (three-dimensional) nonlinear equations, including the incompressible Euler equations, the Navier–Stokes equations and the relativistic Vlasov–Maxwell system (RVM system in abbreviated form), we have LSS (for adequate norms $\mathcal{N}$) but GSS is still open. Standard results involving norms $\|\cdot\|_n$ with n large provide with a lower bound for $T(\mathrm{U}_0)$, which is typically of the form $\|\mathrm{U}_0\|_n^{-1}$. Starting from there, the local smooth solvability for weaker norms $\mathcal{N}$ (with $\mathcal{N} \lesssim \|\cdot\|_n$) may not be granted. It is recognized that the challenge behind this issue (and behind GSS) lies in the development of turbulence.

This text is devoted to the three-dimensional RVM system. The local Hadamard well-posedness for $C^1 \times C^2 \times C^2$ is known, see [16, 17, 38]. The same applies to the global weak existence [9, 33] for data in $L^\infty \times L^2 \times L^2$ (with finite energy). Our goal here is to make a progress at the interface and to show the local smooth solvability for the norm $L^\infty \times H^1 \times H^1$.

The present section summarizes our outcomes. In Section 1.1, we specify our results on LSS. In Section 1.2, the focus is on dense, hot, collisionless and strongly magnetized plasmas. As a by-product of our analysis, we are able to construct a framework within which the local uniform existence of smooth solutions may be achieved while a large magnetic field is applied. This furnishes mathematical tools for a better understanding of plasmas during their confinement (like in fusion devices), including the several complicated behaviors that can occur.

1.1. On the Cauchy problem for the RVM system in a strong-weak setting

The unknowns are the distribution function f and the electromagnetic field (E, B) depending on the time $t \in \mathbb{R}$, on the spatial position $x \in \mathbb{R}^3$, and on the momentum $\xi \in \mathbb{R}^3$, as indicated below:

$$\mathrm{f} : \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi \to \mathbb{R}_+, \quad \mathrm{E} : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3 \quad \text{and} \quad \mathrm{B} : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3.$$

The speed of light is normalized to one. We deal with the relativistic velocity

$$v(\xi) := \langle \xi \rangle^{-1} \xi, \quad \langle \xi \rangle := \sqrt{1 + |\xi|^2}, \quad |v(\xi)| < 1.$$

The RVM system is composed of the Vlasov equation

$$\partial_t \mathrm{f} + v(\xi) \cdot \nabla_x \mathrm{f} + \mathrm{F}(t,x,\xi) \cdot \nabla_\xi \mathrm{f} = 0, \quad \mathrm{F}(t,x,\xi) := \mathrm{E}(t,x) + v(\xi) \times \mathrm{B}(t,x), \tag{1.2}$$

coupled with Maxwell's equations

$$\partial_t \mathrm{E} - \nabla_x \times \mathrm{B} = \mathrm{J} := -\int_{\mathbb{R}^3} v(\xi)\,\mathrm{f}(t,x,\xi)\,d\xi, \quad \nabla_x \cdot \mathrm{E} = \int_{\mathbb{R}^3} \mathrm{f}(t,x,\xi)\,d\xi - \rho, \tag{1.3a}$$

$$\partial_t \mathrm{B} + \nabla_x \times \mathrm{E} = 0, \quad \nabla_x \cdot \mathrm{B} = 0. \tag{1.3b}$$

The system (1.2)–(1.3) on $\mathrm{U} := (\mathrm{f}, \mathrm{E}, \mathrm{B})$ is completed with initial data

$$\mathrm{U}_{|t=0} = \mathrm{U}_0 = (\mathrm{f}_0, \mathrm{E}_0, \mathrm{B}_0), \tag{1.4}$$

satisfying, for some given function $\rho(x) \in C_c^1(\mathbb{R}^3; \mathbb{R}_+)$, the compatibility conditions

$$\nabla_x \cdot \mathrm{E}_0 = \int_{\mathbb{R}^3} \mathrm{f}_0(x,\xi)\,d\xi - \rho, \quad \nabla_x \cdot \mathrm{B}_0 = 0. \tag{1.5}$$

Theorem 1.2. *Fix* $\mathrm{P}_0 \in \mathbb{R}_+$. *Denote by* $B(0,\mathrm{P}_0]$ *the closed ball of* $\mathbb{R}^3$ *with radius* P_0. *Select any Sobolev exponent* $\bar{p} \in]3/2, 2]$. *The RVM is locally smoothly solvable for the subspace*

$$\mathfrak{N} := \big\{ \mathrm{U}_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \times C_c^2(\mathbb{R}^3) \times C_c^2(\mathbb{R}^3) : \operatorname{supp} \mathrm{f}_0 \subset \mathbb{R}^3 \times B(0,\mathrm{P}_0] \big\}, \tag{1.6}$$

equipped with the product norm

$$\mathcal{N}(\mathrm{U}_0) := \|\mathrm{f}_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} + \|\mathrm{E}_0\|_{W^{1,\bar{p}}(\mathbb{R}^3)} + \|\mathrm{B}_0\|_{W^{1,\bar{p}}(\mathbb{R}^3)}. \tag{1.7}$$

At the level of (1.6) and (1.7), the topologies related to f_0 and $(\mathrm{E}_0, \mathrm{B}_0)$ are managed separately. The choice of C^1 for f_0 and C^2 for $(\mathrm{E}_0, \mathrm{B}_0)$ inside (1.6) is classical, see [16]. Passing to (1.7), each of these indices go down one level. To put into perspective Theorem 1.2, as a first step, we can work in a simplified context. With this in mind, we can handle initial data which are restricted to

$$\mathfrak{N}_r := \big\{ \mathrm{U}_0 : \mathrm{f}_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3),\ \operatorname{supp} \mathrm{f}_0 \subset \mathbb{R}^3 \times B(0,\mathrm{P}_0],\ \mathrm{E}_0 \equiv 0,\ \mathrm{B}_0 \equiv 0 \big\} \subset \mathfrak{N}.$$

Observe that a non-trivial electromagnetic field (E, B) is generated by interaction. Tested on $\mathfrak{N}_r$, the norm $\mathcal{N}$ reduces to the sup-norm (on the sole component f_0). As a direct consequence of Theorem 1.2, the RVM system is locally solvable for $(\mathfrak{N}_r, \|\cdot\|_0)$. Now, the RVM system shares many commonalities with QSH systems. Thus, it is instructive to compare this information with what is obtained for hyperbolic systems. Starting from a smooth solution leading to blow-up in finite time, a scaling argument indicates that general QSH systems (with constant coefficients) are not locally solvable for $(C_c^n, \|\cdot\|_0)$. Why is there such discrepancy?

This is because there are also notable differences separating the RVM system from common QSH systems. First, the RVM system is not scaling invariant (due to the Lorentz factor). Secondly, the nonlinearity occurs only on a transport part which preserves the sup-norm. There is, however, a coupling (through the integral term J) which produces a self-consistent electromagnetic field. The feedback of this (E, B) on f may destroy the C^1-regularity. But this impact is lessened due to a number of specificities, among which relativistic effects and transfers from time (in t) and space (in x) derivatives to kinetic (in ξ) derivatives, which can be neutralized at the level of the electric current J through arguments from [4, 5, 23].

For data in $\mathfrak{N}_r$, we simply work with

$$U_{|t=0} = U_0 = (f_0, E_0, B_0) = (f_0, 0, 0), \tag{1.8}$$

where f_0 satisfies (for some $P_0 \in \mathbb{R}_+$ and $S_0 \in \mathbb{R}_+$)

$$f_0 \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}), \tag{1.9a}$$

$$\operatorname{supp} f_0 \subset \mathbb{R}^3 \times B(0, P_0], \tag{1.9b}$$

$$\| f_0 \|_0 := \sup_{(x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3} | f_0(x, \xi)| \le S_0, \tag{1.9c}$$

together with the compatibility condition inherited from (1.5) and (1.8), that is,

$$0 = \int_{\mathbb{R}^3} f_0(x, \xi)\, d\xi - \rho. \tag{1.10}$$

The initial data f_0, while remaining smooth, can undergo large fluctuations allowing to trigger filamentation or coherent structures [6, 7], whose counterparts in the fluid description are shearing and cascade of phases. The point is that the regularity persists locally in time, uniformly with respect to the small norm $\mathcal{N}$ (instead of $\| \cdot \|_1$). In other words, despite arbitrarily large derivatives of f_0, a control remains within reach. Basically, this relates to the minimal radius $P(t)$ of the balls containing the momentum support of $f(t, \cdot)$, i.e.,

$$P(t) := \inf\big\{R \in \mathbb{R}_+ : f(t, x, \xi) = 0 \text{ for all } x \in \mathbb{R}^3 \text{ and all } \xi \in \mathbb{R}^3, \text{ with } R \le |\xi|\big\}. \tag{1.11}$$

The role of P is essential in the forthcoming discussion. From the definition (1.6) of $\mathfrak{N}$, we can assert that $P(0) \le P_0$. In what follows, we will exhibit (Proposition 4.1) a positive increasing continuous function $\mathcal{F}$ depending only on S_0, allowing to control P as indicated in (4.6). Now, from Glassey–Strauss' conditional [16], singularities may develop only if $P(t)$ explodes. Looking at (3.10), this may occur only if $\|E(t, \cdot)\|_{L^\infty}$ goes to $+\infty$. This explosion mechanism does not necessarily apply to $\|f(t, \cdot)\|_{W^{1,\infty}}$. Large values of $\|f(t, \cdot)\|_{W^{1,\infty}}$ are not known to be a triggering factor of the rapid breakdown of smooth solutions. In line with this, Theorem 1.2 shows that large Lipschitz norms of f_0 are compatible with a uniform life-span.

Another way to put Theorem 1.2 in context is to derive a priori sup-norm estimates on all fluid quantities (Corollary 4.2), without the need for looking at derivatives of f_0.

In fact, our approach brings into play strong and weak characteristics: *strong* in view of the (technical) regularity assumption (1.9a), *weak* in the sense of the relaxed conditions (1.9b) and (1.9c). Now, by compactness arguments, we can exhibit a notion of local well-posedness (without stability) in the framework of $L^\infty \times H^1 \times H^1$. Such "strong-weak" solutions satisfy (4.6) and they are unique, see Corollary 4.3. Note that the existence results [9, 33] of weak solutions do not provide information about the persistence of regularity or about the property (4.6). On the other hand, to the best of the authors' knowledge, the uniqueness of weak solutions has been a big open problem together with the global regularity, and the conservation of energy for weak solutions. We refer to [19] for an overview and a result (in the absence of coupling). Besides the uniqueness result for the decoupled Maxwell's equations in a vacuum by Jabin–Masmoudi, we believe our result is the first result for the uniqueness of weak solutions to the full RVM system.

Theorem 1.2 is derived from Sections 3 and 4. It is a compilation of two intermediate important stages. First, in Section 3, we bring to the fore a representation formula for the momentum increment (Definition 3.1) with some original proof (resorting to a Radon transform [18] with respect to $x \in \mathbb{R}^3$ and oscillatory integrals). Second, in Section 4, we control the momentum spread. To this end, we have to study a class of singular weighted integrals. A few ideas leading to (4.6) are new in comparison to the preceding approaches (presented in Section 2.3), and they are exposed in Section 2.4.

1.2. Application: the regime of dense, hot and strongly magnetized plasmas

In kinetic theory, the time evolution of a single species of charged particles (typically electrons in a background of ions) is described (in dimensionless units) by the system (1.2)–(1.3). The study of (1.2)–(1.3) is of fundamental interest because it reveals the interactions between the matter (driven by f) and the fields (represented by E and B). In (1.3a), the function $\rho \in C^1_c(\mathbb{R}^3;\mathbb{R}_+)$ stands for the density of charges associated with protons.

Property (4.6), which prevails for strong and (adequate) weak solutions, is likely to furnish a wide range of applications. It is related to precise quantitative information on the lifespan, and therefore it is well suited to the study of the RVM system at specific scales. Particular emphasis is placed here on situations involving (after nondimensionalization) large magnetic fields. As a matter of fact, such models constitute a real challenge in plasma physics (Section 1.2.1) and provide means (Section 1.2.2) to generate solutions undergoing rapid oscillations.

1.2.1. The physical model. In many important applications, such as planetary magnetospheres (in the case of Earth, Van Allen's belt is an example of plasma) or fusion devices (like tokamaks), a large inhomogeneous external magnetic field $\varepsilon^{-1}\mathrm{B}_e(x)$ is applied. Here, the number ε is the inverse of the electron gyrofrequency; it is a small dimensionless parameter which is typically in the order of $\varepsilon \approx 10^{-5}$.

By extension and to highlight the smallness of ε, we will work with $\varepsilon \in]0,1]$. Since the large weight ε^{-1} is in factor of B_e, the plasma is *strongly magnetized* as soon as $\mathrm{B}_e \not\equiv 0$. The function $\mathrm{B}_e(\cdot)$ often takes the form of a smooth bounded solenoidal and irrotational vector field:

$$\mathrm{B}_e \in C^1_b(\mathbb{R}^3;\mathbb{R}^3), \quad \nabla_x \cdot \mathrm{B}_e \equiv 0, \quad \nabla_x \times \mathrm{B}_e \equiv 0. \tag{1.12}$$

In this way, the given external magnetic field B_e does fit in with the second condition inside (1.5), which means that it induces zero current. The variations of B_e are quite important since they account for the spatial inhomogeneities, which are usually issued from the underlying physical geometries (such as toroidal shapes). Physical plasmas are generally comprised of a dominant part which stays at the thermodynamic equilibrium.

In the relativistic framework, the reference model at rest is the Maxwell–Jüttner distribution. We consider here that most of charged particles are in a steady state. This may be represented by a distribution profile for $\mathrm{f}(\cdot)$ of the form $\mathrm{M}(\varepsilon, \langle\xi\rangle)$, which may depend smoothly on $\varepsilon \in [0, 1]$. More precisely,

$$\text{(1.13)} \qquad \mathrm{M} \equiv \mathrm{M}^\varepsilon \equiv \mathrm{M}(\varepsilon, \langle\xi\rangle), \quad \mathrm{M}(\varepsilon, r) \in C^1_c([0, 1] \times [1, +\infty[; \mathbb{R}_+).$$

The plasma is called *dense* when there exists a constant $c_1 \in \mathbb{R}^*_+$ such that

$$\text{(1.14)} \qquad 0 < c_1 \le \|\mathrm{M}(\varepsilon, \cdot)\|_\infty \quad \text{for all } \varepsilon \in [0, 1].$$

It is called *hot* when there exists a constant $c_2 \in \mathbb{R}^*_+$ such that

$$\text{(1.15)} \qquad \operatorname{supp} \mathrm{M}(\varepsilon, \cdot) \cap [1 + c_2, +\infty[\ne \emptyset \quad \text{for all } \varepsilon \in [0, 1].$$

To fit with (1.5), we must assume that $\rho \equiv \rho^\varepsilon$ is a constant adjusted in such a way that

$$\text{(1.16)} \qquad \rho^\varepsilon = \int_{\mathbb{R}^3} \mathrm{M}(\varepsilon, \langle\xi\rangle)\, d\xi \quad \text{for all } \varepsilon \in [0, 1].$$

Then, for all parameters $\varepsilon \in]0, 1]$, the expression

$$\text{(1.17)} \qquad \tilde{\mathrm{U}}^\varepsilon_a(t, x, \xi) := (\mathrm{M}(\varepsilon, \langle\xi\rangle), 0, \varepsilon^{-1}\mathrm{B}_e(x)) \equiv \tilde{\mathrm{U}}^\varepsilon_a(0, x, \xi)$$

is clearly a stationary solution to the RVM system. When $\mathrm{M} \not\equiv 0$ does not depend on ε and $\mathrm{B}_e \not\equiv 0$, it is the prototype of a solution belonging to the regime of dense, hot and strongly magnetized plasmas. It can serve as a working example. Retain however that we deal with a more general class of *approximate solutions* denoted by $\mathrm{U}^\varepsilon_a = (\mathrm{f}^\varepsilon_a, \mathrm{E}^\varepsilon_a, \mathrm{B}^\varepsilon_a)$, adjusted as follows.

Proposition 1.3. *The family $\{\mathrm{U}^\varepsilon_a\}_\varepsilon$ is well-prepared in the sense of Definition 5.1.*

The reader is referred to Section 5.1.1 for a precise description and a discussion about the content of U^ε_a. The expression U^ε_a does not necessarily have to be an exact solution to (1.2)–(1.3). It may produce a non-zero remainder R^ε_a. It may also reveal a full range of plasma distinctive features which are not detected by $\tilde{\mathrm{U}}^\varepsilon_a$, see, for instance, [6, 7] for a preview of what can happen. From now on, we fix a family $\{\mathrm{U}^\varepsilon_a\}_\varepsilon$ of approximate solutions and, at the initial time $t = 0$, we modify $\mathrm{U}^\varepsilon_a(0, \cdot)$ according to

$$\text{(1.18)} \qquad \mathrm{U}^\varepsilon_0 = \mathrm{U}^\varepsilon_a(0, \cdot) + U^\varepsilon_0, \quad U^\varepsilon_0 = (f^\varepsilon_0, 0, 0), \quad \varepsilon \in]0, 1].$$

In (1.18), there is no initial electromagnetic perturbation $(E^\varepsilon_0, B^\varepsilon_0)$ interfering with U^ε_a. This is because, under adequate assumptions that will be investigated in Section 5.1.1, the influence of such $(E^\varepsilon_0, B^\varepsilon_0)$ can be incorporated inside $(\mathrm{E}^\varepsilon_a, \mathrm{B}^\varepsilon_a)(0, \cdot)$.

1.2.2. Perturbation theory. The realistic plasmas always include a larger or smaller part of matter which is out of equilibrium and which may have destabilizing effects (like electron beams). Such aspects can be taken into account by introducing some $f^\varepsilon_0 \not\equiv 0$ inside (1.18).

Assumption 1.4. The family $\{f^\varepsilon_0\}_\varepsilon$ is compatible in the sense of Definition 5.2.

Basically, Assumption 1.4 means that we impose (1.9)–(1.10) uniformly on $\{f_0^\varepsilon\}_\varepsilon$, and moreover that we add the L^2-smallness condition (5.18) which is calibrated by N_0 given in Definition 5.2. Below, we fix the values of $\mathrm{P}_0 \in \mathbb{R}_+^*$, $\mathrm{S}_0 \in \mathbb{R}_+^*$ and $\mathrm{N}_0 \in \mathbb{R}_+^*$ occurring in Definition 5.2, and we select $\{f_0^\varepsilon\}_\varepsilon$ accordingly. Then we seek the solution to (1.2)–(1.3) in the form

$$\mathrm{U}^\varepsilon(t,x,\xi) = \mathrm{U}_a^\varepsilon(t,x,\xi) + U^\varepsilon(t,x,\xi), \quad U^\varepsilon = (f^\varepsilon, E^\varepsilon, B^\varepsilon). \tag{1.19}$$

Theorem 1.5 (Uniform life-span for classical solutions). *Fix a well-prepared family* $\{\mathrm{U}_a^\varepsilon\}_\varepsilon$ *of approximate solutions. Select a compatible family* $\{f_0^\varepsilon\}_\varepsilon$ *of initial data controlled by* P_0*,* S_0 *and* N_0*. Then there exists a positive time* $\mathcal{T}$ *depending only on* $(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0)$*, say* $\mathcal{T} \equiv \mathcal{T}(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0) \in \mathbb{R}_+^*$*, such that for all* $\varepsilon \in\,]0,1]$*, the Cauchy problem* (1.2)–(1.4)*, with* U_0^ε *as in* (1.18)*, has a smooth* C_c^1*-solution on the domain* $[0,\mathcal{T}] \times \mathbb{R}^3 \times \mathbb{R}^3$*. Moreover, there exists a positive continuous function* $\mathcal{F}$*, depending only on* $(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0)$*, say* $\mathcal{F} \equiv \mathcal{F}(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0) : [0,\mathcal{T}] \to \mathbb{R}_+^*$*, such that*

$$\operatorname{supp} f^\varepsilon(t,\cdot) \subset \mathbb{R}^3 \times B(0,\mathcal{F}(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0)(t)) \quad \textit{for all } t \in [0,\mathcal{T}(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0)]. \tag{1.20}$$

Resorting to approximate solutions U_a^ε allows to break (1.8) since $(\mathrm{E}_a^\varepsilon,\mathrm{B}_a^\varepsilon)(0,\cdot)$ is aimed at being non-trivial. As can be guessed by looking at (1.17), Definition 5.1 gives access to large magnetic fields B_a^ε. That is, however, not the only aspect. Implementing U_a^ε or f_0^ε is also a way to generate rapid fluctuations, and then to measure their quantitative impact. In Theorem 1.5, the absorption of large fields and oscillations requires the L^2-smallness conditions (5.6b) on the remainder R_a^ε and (5.18) on the initial data f_0^ε. This is really not demanding in comparison with the common assumptions (like coherence, H_ε^s-estimates and so on) in nonlinear geometric optics [25, 31], in comparison with the already improved contexts of [8] or [30], and in comparison with the strong conditions imposed in [32] and recently in [37].

The regime of dense, hot and strongly magnetized plasmas is of practical interest. For this reason, it has been intensively studied in physics. Up to now, the mathematical advances [3, 9] are based on weak solutions which do not confer the ability to describe and control the solutions (due to lack of uniqueness). Theorem 1.5 achieves that goal with smooth solutions on a pertinent observation time ($t \sim 1$). It can be interpreted as a kind of stability statement about the regularity of solutions near WKB expansions (such as U_a^ε), when some large C^1-perturbations are applied. Theorem 1.5 is proved in Section 5. For more details on Theorems 1.2 and 1.5, the reader is recommended to look at Section 2.

2. Detailed introduction

The construction of solutions to the RVM system has a long history. This goes back to the pioneering works of Wollman [38], Glassey and Strauss [16, 17], and DiPerna and Lions [9] in the 1980s. This includes the alternative methods proposed by Klainerman and Staffilani [23], and Bouchut, Golse and Pallard [4, 5] in the early 2000s. This also relates to recent developments in connection with plasmas involving large fields [8, 30, 32, 37]. In this section, we explain better how our results and techniques stand in relation to the preceding contributions, and also how they differ from them. In Section 2.1, we recall the historical backdrop. In Section 2.2, we give some insight into localized and oscillating solutions. In Section 2.3, we list the preceding approaches. In Section 2.4, we compare them with our strategy; we also detail the plan and the content of the text, while giving a sketch of ideas and proofs.

2.1. Historical background

We give here a quick overview of previous contributions.

2.1.1. Local and global smooth well-posedness. The existence of local stable classical solutions dates back to [38]. The issue of global existence was then raised in [16,17]. Then much has been done to make progress on the global smooth solvability. The advances concern small initial data [15] (much smaller than in Theorem 1.5), reduced dimensions [12,14] (less than the six dimensions of the actual phase space), symmetry conditions see [36] (broken by the inhomogeneities of the external field $\mathrm{B}_{\boldsymbol{e}}$), stability properties [32] (under stronger topologies), and so on. For further information, we can refer to the survey article [13]. The most elaborate result in relation with our theme is perhaps the latest article [37] of Wei and Yang, which (among other things) can include a large electromagnetic field but which also requires (in compensation) a very small density distribution. In view of Proposition 4.8 of [37], interpreted in our framework, this density should be of size less that $\varepsilon^8 |\ln \varepsilon|^{-11}$. This is far from the regime under consideration in Theorem 1.5. Now, for general data, many continuation criteria have been established, see [24, 28, 29] and the references therein. However, the global existence of classical solutions remains open. This is a longstanding and still active problem. On the other hand, failing to reach global existence, in connection with concrete applications, we can seek for more precise quantitative information on the life-span of smooth solutions. That is the position of the present text.

2.1.2. About nonlinear geometric optics. The construction of oscillating solutions for quasilinear systems of conservation laws is the core subject of nonlinear geometric optics. However, the general results [25, 31] do not furnish optimal information when applied to the RVM system. There are several reasons for this. The transport part (the Vlasov equation) and the integral source term (the electric courant) make things easier. Also, importantly, the variable coefficients with ε^{-1} in factor (generated by $\mathrm{B}_{\boldsymbol{e}}$) can be handled through different arguments than the usual quite restrictive conditions.

With the goal of better exploiting the specificities of (1.2)–(1.3) and also motivated by the applications, there has been some progress to remedy this situation. The *cold* configuration for which $|\xi| \lesssim \varepsilon$, or $\mathrm{M}^\varepsilon \equiv M(|\xi|/\varepsilon)$ with $M \in C_c^1(\mathbb{R}_+;\mathbb{R}_+)$, is examined in [8]. The hot but *dilute* situation, for which $\mathrm{M}^\varepsilon \equiv \varepsilon M(\xi)$ with $M \in C_c^1(\mathbb{R}^3;\mathbb{R}_+)$ and $(f_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)$ is *small in Lipschitz norm* (of size ε), is investigated in [30]. Therein, large amplitude profiles $\mathrm{M}^\varepsilon \approx 1$ as well as weak norms of U_0^ε are excluded for deeper reasons related to the method used.

To our knowledge, Theorem 1.5 is completely new. Implemented in the dense, hot and strongly magnetized framework, that is when $\mathrm{M} \approx 1$, $\mathrm{P}_0 \approx 1$ and $|\mathrm{B}_{\boldsymbol{e}}| \approx 1$, the criteria yielding global existence are clearly not applicable. On the other hand, all preceding approaches seem to furnish a life-span $\mathcal{T}^\varepsilon \in \mathbb{R}_+^*$ that shrinks very rapidly to 0 when ε goes to 0. In space dimension two, global classical solutions to strongly magnetized plasmas were studied by Nguyen, Nguyen and Strauss [26] in a 2D infinite strip with infinite external magnetic potential, and by Jang, Strain and Wong [20] in a 2D bounded annulus with finite external magnetic field.

2.1.3. Global weak existence. As is well known, weak global solutions are available. This has been shown by DiPerna and Lions in the seminal contribution [9], just after [16, 17] in the late 1980s. But such solutions are obtained by compactness methods, and much information has not yet been provided. In the context of (1.2)–(1.3), little can currently be said about the uniqueness, the stability, the regularity, or the (oscillating) form of these solutions. Note, however, that uniqueness and stability can be addressed from the perspective of the renormalization property [10]. We refer to paragraph 1.3 in [19] for recent developments (in a decoupled situation) and a nice presentation of results in this direction. Recall also that weak solutions have been considered in the framework (similar to Section 1.2) of large magnetic fields in order to give partial information, develop asymptotic models or enrich gyrokinetic theory (see, for instance, [3]). In view of Theorem 1.5, some strong control (related to the smoothness) on the solutions does persist. But how is this possible, and why? The first step (Section 2.2) is to identify the underlying difficulties; the second step (Section 2.3) is to complement our presentation by recalling the previous strategies; the third step (Section 2.4) is to present our plan and to explain our approach.

2.2. The impact of localizations and oscillations

Plasmas out of equilibrium can exhibit a wide variety of complicated behaviors, which are manifested by oscillating coherent structures (in subdomains of the phase space [6,7]) or by chaotic motions (shearing due to the sensitivity of characteristics under changes of initial data, especially near separatrices). As a consequence, the Lipschitz norm of f^ε is in practice often very large. Our aim in this subsection is not to inventory all the phenomena that can occur. Instead, we want to indicate why the distribution function f^ε should undergo large rapid variations (when $\mathrm{B}_e \not\equiv 0$).

For the sake of simplicity, we do not (always) mention the dependence on the parameter ε at the level of U, U, f, E and B. In the simplified framework (1.17), emphasis may be placed on the role of f_0^ε, and its impact on f and on the self-consistent electromagnetic field (E, B). The presence of some $f_0^\varepsilon \not\equiv 0$ has many consequences, among which are the following:

(i) The onset of *anisotropic features* (in momentum variable ξ). A spatial localization of f_0^ε necessarily implies that $\nabla_x f \not\equiv 0$. Then the part $v(\xi) \cdot \nabla_x f \not\equiv 0$ inside (1.2) is switched on, and it cannot be only a function of $|\xi|$. The same applies for f.

(ii) The emergence of *fast oscillations*. The Lorentz force F can be decomposed into

$$\text{(2.1)} \qquad \mathrm{F} = F + \varepsilon^{-1} v(\xi) \times \mathrm{B_e}(x), \quad F(t,x,\xi) := E(t,x) + v(\xi) \times B(t,x).$$

Due to the anisotropy of f, when computing the contribution $(\xi \times \mathrm{B_e}) \cdot \nabla_\xi \mathrm{f}$, the above fast rotating term is certainly activated. This means that, as allowed by (1.9c), $\| f \|_{W^{1,\infty}}$ is designed to be not uniformly bounded with respect to $\varepsilon \in]0,1]$. The condition (1.9c) does not preclude the choice of initial data f_0^ε having arbitrarily large Lipschitz norms (as compared to P_0 and S_0).

The above considerations deserve to be highlighted by a concrete example (more elaborate models can be found in [6,7]). Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a global diffeomorphism. Select

$$f_0^\varepsilon(x,\xi) := \mathscr{P}(x,\varepsilon^{-1}\phi(x),\xi), \quad \mathscr{P} \in C_c^1(\mathbb{R}_x^3 \times \mathbb{R}^3 \times \mathbb{R}_\xi^3;\mathbb{R}).$$

The profile $\mathscr{P}$ may be periodic or (partially) compactly supported with respect to its second variable (to represent a localized input of electrons for the initialization of an electron beam). For adequate choices of $\mathscr{P}$ and ϕ, we have clearly access to (5.18), and Theorem 1.5 does apply. Now, we can get a preview of what happens at least when neglecting most of the terms inside (1.2). With this in mind, we can consider the elementary transport equation

$$\partial_t f + \varepsilon^{-1}(v(\xi) \times \mathrm{B}_e(x)) \cdot \nabla_\xi f = 0, \quad \mathrm{B}_e(x) = b_e(x_3)\,{}^t(0,0,1). \tag{2.2}$$

In (2.2), the field B_e has a fixed (vertical) direction and a varying amplitude b_e (depending only on x_3 to satisfy $\nabla \cdot B_e = 0$, $\nabla \times B_e = 0$). In the cylindrical coordinate system for ξ, with

$$\xi = (r\cos\theta, r\sin\theta, \xi_3), \quad r^2 = \xi_1^2 + \xi_2^2,\ \theta \in \mathbb{T},$$
$$\tilde{\mathscr{P}}(\cdot, r, \theta, \xi_3) := \mathscr{P}(\cdot, r\cos\theta, r\sin\theta, \xi_3),$$

the solution is simply given by

$$\tilde{f}(x,r,\theta,\xi_3) := f(t,x,r\cos\theta,r\sin\theta,\xi_3) = \tilde{\mathscr{P}}\Big(x, \frac{\phi(x)}{\varepsilon}, r, \theta + \frac{t b_e(x_3)}{\varepsilon\sqrt{1+r^2}}, \xi_3\Big). \tag{2.3}$$

The transport part $v(\xi)\cdot\nabla_x f$ of (1.2) has been removed at the level of (2.2). This suppresses the first effect (i). But still we are faced with (ii). Indeed, when $\partial_\theta \tilde{\mathscr{P}} \not\equiv 0$, the first order derivatives in almost all directions (time, space and momentum) of the expression $\tilde{f}$ given by (2.3) are of large size ε^{-1}. As a consequence, uniform Lipschitz estimates (with respect to ε) are certainly not available. The same is sure to apply to the solutions provided by Theorem 1.5.

In the dilute situation [30], the expression U_a^ε and the perturbation f_a^ε are of small amplitude ε. This weight ε which can be put in factor of the data can be used to absorb the oscillations at the frequency ε^{-1}. This boils down to a sort of weakly nonlinear regime. Then, uniform Lipschitz estimates may become available and, by this way, oscillating approximate solutions to (1.2) can be constructed and justified by standard arguments. There is nothing like this in the framework of Theorem 1.5.

In fact, the main reason why it is complicated to achieve $\mathcal{T}^\varepsilon \sim 1$ is the following. So far, the methods which have been implemented require, at a moment or another, to compute derivatives of U. But this is proving to be very costly in terms of negative powers of ε because these derivatives are – in all reasonable norms – at least of size ε^{-1}, compromising the existence of a uniform life-span. By contrast, we are able here to avoid this problem.

2.3. Previous approaches for smooth solutions

Estimates on P (or on similar quantities) have been a central part in the study of the RVM system because singularities do not develop as long as the momentum support of $\mathrm{f}(t,\cdot)$

remains bounded. This is Glassey–Strauss' conditional theorem [16], which has inspired many works and which has been revisited in [4]. Such bounds have been achieved by pursuing three noteworthy lines of research.

2.3.1. The historical procedure. As already mentioned, the standard method has been initiated in [15–17]. It is based on a representation formula for (E, B), see Theorem 3 of [16] and [14] for its simplified two-dimensional version. In this line, the stability under (very) small perturbations in Lipschitz norm of smooth solutions (such as U_a^ε) has been investigated in [32].

2.3.2. An alternative path. The second way is to proceed similarly to what has been done in Klainerman–Staffilani [23] or even simpler in Bouchut, Golse and Pallard [4, 5]. These authors have obtained with an economy of means Lipschitz estimates on the field (E, B). To this end, they have implemented three principal arguments:

(a) *Commuting vector field techniques* [23] for the wave equation related to Maxwell's equations.

(b) *Non-resonant smoothing property* [5], stating that $1 \pm v(\xi) \cdot \omega$, with $\omega \in \mathbb{S}^2$, remains away from zero. This property of ellipticity deteriorates when $|\xi|$ grows up, and this is one of the difficulties.

(c) A *division lemma*, see Lemma 3.1 of [4], whose aim is to convert transversal derivatives to the light cone into the derivative $\partial_t + v(\xi) \cdot \nabla_x$ (appearing in the transport part of RVM).

2.3.3. Global existence through sharp decay estimates. The third method is to implement (by way of a fixed-point iteration) sup-norm controls and decay estimates (related to the dispersion) on (E, B), as was achieved by Wei and Yang in [37]. But this requires controls in $C^1 \times C^2 \times C^2$ which are at the origin of the strong smallness condition ($\leq \varepsilon^8 |\ln \varepsilon|^{-11}$) on f_0^ε noted above.

The three approaches of Sections 2.3.1, 2.3.2 and 2.3.3 are based on different assumptions; they resort to distinct techniques; they produce complementary information; and they are good starting points. But, to achieve our goal, they need to be supplemented by novel procedures. We will adapt these methods by following at some points a different logic, yielding other consequences.

2.4. Plan of the text, and main ideas

This article is organized around Sections 3, 4 and 5, whose contents are detailed in below.

2.4.1. Content of Section 3. In Section 3, we deal with smooth compactly supported solutions to the RVM system (1.2)–(1.3). We derive a representation formula (Proposition 3.2) which is reminiscent of [16]. In contrast to [4, 16], we do not look at (E, B). Instead, we focus on the *momentum increment* $\mathrm{D}(t, y, \eta)$, see Definition 3.1. The construction of $\mathrm{D}(t, y, \eta)$ is related to the characteristic $(X, \Xi)(t, y, \eta)$ emanating from a point (y, η) and associated with the Vlasov equation (1.2). Equivalently, $\mathrm{D}(t, y, \eta)$ is the difference $\langle \Xi(t, y, \eta) \rangle - \langle \eta \rangle$ which is built from the solution (X, Ξ) to the ordinary differential equation (3.1). The quantity D depends on t. It is a dynamical quantity which allows to control the size $\mathrm{P}(t)$ of the momentum support. Obviously, when $\mathrm{E} \equiv 0$, we find that D is invariant, that is $D(t) = D(0)$ for all time t. But certainly this becomes not true in the presence of interactions. This raises an interesting question:

Question 1. Can we identify the factors that alter D? And evaluate their quantitative effects?

The determination of D is based on $|\Xi|$, not on Ξ. This remark is important because, in the regime under consideration, the directions of Ξ may be strongly oscillating with respect to the parameter $\varepsilon \in]0,1]$ (see, for instance, (2.3)), and therefore they would be impractical to control. In fact, resorting on D is a way to avoid rapid fluctuations of $\Xi/|\Xi|$, and to dispense with having to implement the (possibly large) momentum speed of propagation. The quantity D is recovered after some integration (with underlying cancellation effects), while still informing about the expanse of the momentum support.

To evaluate D, a first option is to consider (3.10), and to directly extract the sup-norm of E. But this would imply pointwise estimates (like the estimates of von Wahl [35] or the Strichartz inequalities [21, 34]) at the level of a three-dimensional wave equation with source term $\partial_t \mathrm{J}$. This would take us back to the control of derivatives with again important losses in terms of negative powers of ε. This is why we adopt an alternative strategy which is partly inspired by [4, 28]. As in [4], in Section 3.1.1, we interpret the RVM system in terms of the Lienard–Wiechert potential $\mathrm{u}(t,x,\xi)$. As in [28], we aim at substituting for (3.10) integral expressions which are not based on dt but instead which integrate with respect to the time, the space and the velocity. But, to this end, we exploit other arguments. Our main innovation to prove (3.4) is as follows.

First, to perform a Radon transform (with respect to the position variable $x \in \mathbb{R}^3$). The Radon transform (see the book [18] and Section 3.1.2) allows to convert $\mathrm{f}(t,x,\xi)$ into $\mathrm{g}(t,\omega,p,\xi)$, and $\mathrm{u}(t,x,\xi)$ into $\mathrm{v}(t,\omega,p,\xi)$. The equation (3.23) for $\mathrm{g}(\cdot,\omega,\cdot,\xi)$ becomes (with respect to t and p) a one-dimensional transport equation; the equation (3.22) for $\mathrm{v}(\cdot,\omega,\cdot,\xi)$ becomes (with respect to t and p) a one-dimensional wave equation. These one-dimensional features entail many simplifications. Indeed, we recover a simple one-dimensional mean field equation (comparable to the model studied in [11]) with all the three-dimensional geometry encoded in the sphere (with the angle $\omega \in \mathbb{S}^2$ serving as a parameter).

Second, to interpret D as an oscillatory integral. In this way, the geometry of propagation is driven by the phase $\pm|X(s) - x| + s - r$ which reveals very well the *joint* properties of the transport equation (1.2) and of the wave equation inside (1.3).

It follows that

(a) the commutating vector fields considerations of [23] are bypassed,

(b) the non-resonant smoothing property of [5] reduces to the manipulation of a non-zero coefficient (vanishing to 0 when $|\xi|$ tends to $+\infty$) occurring in a non-stationary phase,

(c) the division lemma of [4] is replaced by a straightforward argument (there is no more three spatial directions but instead only one direction p with a parameter $\omega \in \mathbb{S}^2$).

On the other hand, the return through the inverse Radon transform to the original spatial configuration (in x) involves oscillatory integrals which, thanks to Lemmas 3.3 and 3.4, simplify and lead to the explicit formula (3.4), which is easy to use.

The content of (3.4) does not imply any derivative of f, E or B. It does neither require any pointwise estimate on f, E or B. Instead, it relies on integral computations with respect to the volume element $ds\,d\omega\,d\xi$. Modulo the (singular) Jacobian (4.16), which is issued from a pushforward to the original phase space, this amounts to work with the Lebesgue measure $dx\,d\xi$. The great advantage is that this Liouville measure $dx\,d\xi$ is preserved by the flow (3.1). Henceforth, we obtain a connection between the value of $\mathrm{D}(t,y,\eta)$, regarding one particular characteristic issued from (y,η), and the computation of the total energy

$$\mathcal{E}(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle\, \mathrm{f}(t,x,\xi)\,dx\,d\xi + \frac{1}{2}\int_{\mathbb{R}^3} |\mathrm{E}(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\mathrm{B}(t,x)|^2\,dx, \tag{2.4}$$

which is for smooth solutions a conserved quantity [33] while, for weak solutions, this issue has not yet been completely resolved (see [1] for a discussion about Onsager's conjecture). The fact remains that, in the actual context of smooth solutions, we have

$$\mathcal{E}(t) = \mathcal{E}_0 := \mathcal{E}(0), \quad t \geq 0. \tag{2.5}$$

Thus, the search for pointwise estimates on D is driven by the invariant energy estimate (2.5). In that sense, we can again assert that our approach is at the junction between results about weak solutions (which rely on the sole preservation of $\mathcal{E}$ and which do not bring into play derivatives) and results yielding a kind of strong information (namely, sup-norm estimates for adequate quantities) without resorting to the computation of derivatives.

Amongst other things, the formula (3.4) has the benefit of revealing and delineating the role of a series of weights $\mathrm{W}_\star$ defined on $\mathbb{R}_+ \times \mathbb{S}^2_\omega \times \mathbb{R}^3_\xi$. Depending on the respective positions of the directions $\Xi(s)$, ω and ξ, these weights may be larger or smaller. This remark gives access to quantitative estimates which differentiate between the localizations of $(\Xi(s),\omega,\xi)$. This process is precisely what motivates Section 4.

2.4.2. Content of Section 4. Section 4 answers Question 1. To this end, in Section 4.1, we start by a preparatory work: we tie together the controls of D and P (Section 4.1.1); we express the pushforward of the measure $ds\,d\omega\,d\xi$ as a density with respect to $dx\,d\xi$ (Section 4.1.2); and we introduce useful tools concerning the microlocal weights $\mathrm{W}_\star$ (Section 4.1.3); this allows (in Section 4.1.4) to evaluate the sizes of $\mathrm{W}_\star$. This leads (in Section 4.2) to the study and to the control of weighted integrals. In the end (Section 4.3), this provides key inputs for proving Proposition 4.1 (which also investigates large time issues).

2.4.3. Content of Section 5. Section 5 is devoted to Theorem 1.5. In Section 5.1, we specify the underlying framework: we introduce the notion of well-prepared approximate solutions U^ε_a (Section 5.1.1); we write the equations for the perturbation U (Section 5.1.2); and we explain what is meant by compatible initial data f^ε_0 (Section 5.1.3). In Section 5.2, we come back to the question of energy estimates. It turns out that (2.5) is not very useful. Instead, we have to propagate the L^2-norm of $U(t)$ at the level of the linearized equations along U^ε_a. That is the only way to ensure that the smallness L^2-conditions (5.6b) and (5.18) is passed on to $U(t)$. In Section 5.3, we prove Theorem 1.5. This is done by estimating separately the approximate momentum increment D^ε_a, bilinear terms obtained by freezing D_n along U^ε_a, and a full nonlinear contribution D_n. In this process, it is essential to know that $U(t)$ is of size ε in L^2. Indeed, this emerges as an indispensable prerequisite to compensate the presence inside (5.12) of the large factor ε^{-1}. Again, it is noteworthy that the preceding approaches would furnish, if any, much less information than in Theorem 1.5.

3. The proof of a representation formula

In this section, we consider a solution (f, E, B) to the Cauchy problem (1.2)–(1.4), which is assumed to be smooth and compactly supported with respect to ξ, and which is defined on $[0, T[$ for some $T \in \mathbb{R}_+^* \cup \{+\infty\}$. In what follows, the time t is chosen in $[0, T[$.

The Vlasov equation (1.2) is linked to a dynamical system on the phase space $\mathbb{R}_x^3 \times \mathbb{R}_\xi^3$. Recall that F is the Lorentz force given by (1.2), and consider the flow (X, Ξ) obtained by solving the ordinary differential equation:

$$\text{(3.1)} \qquad \begin{cases} \dfrac{dX}{dt}(t, y, \eta) = v(\Xi), & X(0, y, \eta) = y, \\ \dfrac{d\Xi}{dt}(t, y, \eta) = \mathrm{F}(t, X, \Xi), & \Xi(0, y, \eta) = \eta. \end{cases}$$

The solution f may be recovered by integrating along the characteristics, in the sense that

$$\text{(3.2)} \qquad \mathrm{f}(t, x, \xi) = \mathrm{f}_0(X(-t, x, \xi), \Xi(-t, x, \xi)).$$

Definition 3.1 (Momentum increment). The momentum increment at the time t associated with an initial phase point (y, η) is the difference

$$\mathrm{D}(t, y, \eta) := \langle \Xi(t, y, \eta) \rangle - \langle \eta \rangle.$$

In the absence of an electric field, that is when $\mathrm{E} \equiv 0$ as it is the case concerning the stationary solution $\tilde{\mathrm{U}}_a^\varepsilon$ of (1.17), the kinetic energy $|\Xi|^2/2$ is just constant, and the same applies to $\langle \Xi \rangle$ so that $\mathrm{D} \equiv 0$. But in general, we have $\mathrm{D} \not\equiv 0$ for two main reasons:

- The impact of a non-zero initial data $(\mathrm{E}_0, \mathrm{B}_0) \not\equiv 0$, with $\nabla_x \times \mathrm{B}_0 \not\equiv 0$ when $\mathrm{E}_0 \equiv 0$. Then a non-trivial electric field $\mathrm{E} \not\equiv 0$ persists or is created (at least for small times t). It can be approximated by solving the homogeneous version of Maxwell's equations:

$$\text{(3.3)} \qquad \partial_{tt}^2 \mathrm{E}_h - \Delta_x \mathrm{E}_h = 0, \quad \mathrm{E}_{h|t=0} = \mathrm{E}_0, \quad \partial_t \mathrm{E}_{h|t=0} = \nabla_x \times \mathrm{B}_0,$$

  where the subscript h is for homogeneous. Retain that the access to E_h is determined only by $(\mathrm{E}_0, \mathrm{B}_0)$; it is obtained by solving a linear wave equation; and it is completely decoupled from the Vlasov equation.
- The effect of nonlinear interactions (the quadratic terms in the Vlasov equation) together with the coupling (electric current in Maxwell's equations).

  The momentum increment $\mathrm{D}(t)$ can be computed from E_h, f and F as indicated below.

Proposition 3.2 (Representation formula for D). *We have*

$$\mathrm{D} = \mathrm{D}_0 + \mathrm{D}_h + \mathrm{D}_l + \mathrm{D}_n,$$

with

$$\mathrm{D}_0 := \int_0^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \mathrm{W}_0(s,\omega,\xi) \mathrm{f}_0(X(s)+s\omega,\xi)\, ds\, d\omega\, d\xi, \tag{3.4a}$$

$$\mathrm{D}_h := \int_0^t \nu \circ \Xi(s) \cdot \mathrm{E}_h(s, X(s))\, ds, \tag{3.4b}$$

$$\mathrm{D}_l := \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \mathrm{W}_l(s,\omega,\xi)\, \mathrm{f}(r, X(s)+(s-r)\omega,\xi)\, ds\, d\omega\, d\xi \Big)\, dr, \tag{3.4c}$$

$$\begin{aligned}\mathrm{D}_n := \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} & \mathrm{W}_n(r,s,\omega,\xi) \cdot \mathrm{F}(r, X(s)+(s-r)\omega,\xi) \\ & \times \mathrm{f}(r, X(s)+(s-r)\omega,\xi)\, ds\, d\omega\, d\xi \Big)\, dr,\end{aligned} \tag{3.4d}$$

where the microlocal weights $\mathrm{W}_\star$ *are given by*

$$\mathrm{W}_0(s,\omega,\xi) := -\frac{s}{4\pi}\, \frac{\nu\circ\Xi(s)\cdot(\nu(\xi)+\omega)}{1+\omega\cdot\nu(\xi)}, \tag{3.5a}$$

$$\mathrm{W}_l(s,\omega,\xi) := -\frac{1}{4\pi}\, \frac{1}{\langle\xi\rangle^2}\, \frac{\nu\circ\Xi(s)\cdot\big(\omega+\nu(\xi)\big)}{(1+\omega\cdot\nu(\xi))^2}, \tag{3.5b}$$

$$\mathrm{W}_n(r,s,\omega,\xi) := -\frac{s-r}{4\pi}\, \nabla_\xi \Big(\frac{\nu\circ\Xi(s)\cdot(\nu(\xi)+\omega)}{1+\omega\cdot\nu(\xi)} \Big). \tag{3.5c}$$

In (3.4), the subscripts 0, h, l and n stand, respectively, for $t=0$ (initial time), homogeneous, linear with respect to f, and nonlinear in terms of f and F. We can separate inside D_n the bilinear interactions involving E and B. We have $\mathrm{D}_n = \mathrm{D}_{ne} + \mathrm{D}_{nb}$. The expressions D_{ne} and D_{nb} are defined as D_n with $(\mathrm{W}_n, \mathrm{F})$ replaced, respectively, by $(\mathrm{W}_{ne}, \mathrm{E})$ and $(\mathrm{W}_{nb}, \mathrm{B})$, where

$$\mathrm{W}_{ne}(r,s,\omega,\xi) := \mathrm{W}_n(r,s,\omega,\xi), \qquad \mathrm{W}_{nb}(r,s,\omega,\xi) := \mathrm{W}_n(r,s,\omega,\xi) \times \nu(\xi). \tag{3.6}$$

In the pioneering works of Glassey and Strauss [16, 17], the electromagnetic field (E, B) was represented in terms of E_h, f and F. The pointwise estimates on E and B thus obtained were exploited (in [16, 17] and subsequent works) to extract information on D. By contrast, we focus here directly on D. As a consequence, we will be able to control D without resorting to (costly) sup-norm estimates on E and B but only through integrals involving f, E and B. Seen in this light, our approach is more in line with [4, 23], and especially [27]. It can be interpreted as an alternative to these contributions. Still, it differs from those both in its formulation, conception (some arguments we use to prove Proposition 3.2 are original), and consequences.

This section is devoted to the decomposition of D into (3.4). In Section 3.1, we introduce basic tools. In Section 3.2, we compute two oscillatory integrals. Then, in Section 3.3, we show Proposition 3.2.

For the sake of simplicity, we will prove Proposition 3.2 for smooth solutions which are compactly supported with respect to both variables x and ξ. The finite speed of propagation in x allows ultimately to relax this condition on the spatial support. Note also that Proposition 3.2 should remain true for less regular (weak) solutions under adequate integrability conditions in ξ. But this aspect will not be investigated here.

3.1. Prerequisites

In Section 3.1.1, we start by adopting the intrinsic viewpoint of [4, 5, 27] to express D in terms of the microscopic electromagnetic potential. In Section 3.1.2, we recall basic facts about the Radon transform [18].

3.1.1. Lienard–Wiechert potentials. Choose a vector field $A_i : \mathbb{R}^3_x \to \mathbb{R}^3$, where i stands for initial, such that $\nabla_x \cdot A_i = 0$ and $\nabla_x \times A_i = B_0$. Then solve the wave equation

$$\partial^2_{tt} A_h - \Delta_x A_h = 0, \quad A_{h|t=0} = A_i, \quad \partial_t A_{h|t=0} = -E_0. \tag{3.7}$$

This allows to recover E_h through the relation $E_h = -\partial_t A_h$. As first noted in [5] and exploited, for instance, in [4, 8], the RVM system can be recast as a coupling between a wave equation and a Vlasov equation. To this end, it suffices to define the microscopic electromagnetic potential $u(t, x, \xi)$ which solves the initial value problem

$$(\partial^2_{tt} - \Delta_x) u = f, \quad u_{|t=0} = 0, \quad \partial_t u_{|t=0} = 0. \tag{3.8}$$

Then, the electromagnetic field (E, B) can be computed from u by the two identities

$$E(t, x) = -\partial_t A_h(t, x) - \int_{\mathbb{R}^3} [\nu(\xi)\partial_t u + \nabla_x u](t, x, \xi)\, d\xi, \tag{3.9a}$$

$$B(t, x) = \nabla_x \times A_h(t, x) + \int_{\mathbb{R}^3} \nabla_x \times [u\, \nu(\xi)](t, x, \xi)\, d\xi. \tag{3.9b}$$

System (1.2), (3.8)–(3.9) is self-contained. It is equivalent to (1.2)–(1.3). This is why, in what follows, it will also be referred to as the “RVM system”. From (3.1) and due to the special structure inside (1.2) of F, we have

$$D(t, y, \eta) = \int_0^t \frac{d}{ds} \langle \Xi(s) \rangle\, ds = \int_0^t \nu \circ \Xi(s) \cdot E(s, X(s))\, ds. \tag{3.10}$$

We can plug (3.9a) into (3.10). The part $-\partial_t A_h$ inside (3.9a) leads to D_h. More precisely, we have $D = D_h + \check{D}$, with

$$\begin{aligned} \check{D}(t, y, \eta) = -\int_0^t \int_{\mathbb{R}^3} \{ & \nu \circ \Xi(s) \cdot \nu(\xi) \partial_s u(s, X(s), \xi) \\ & + \nu \circ \Xi(s) \cdot \nabla_x u(s, X(s), \xi) \}\, ds\, d\xi. \end{aligned} \tag{3.11}$$

This explains the origin of the term D_h inside (3.4). Observe that the influence of A_h is not limited to D_h. It does also impact $\check{D}$ (through u). Indeed, equations (1.2) and (3.8) are coupled with A_h appearing inside (1.2) because the Lorentz force $F = E + \nu(\xi) \times B$ must be computed with E and B given by (3.9).

On the one hand, the formula (3.10) seems to indicate that the control of D should require a sup-norm estimate on E. On the other hand, the second identity (3.11) suggests that the access to $\check{D}$ should imply sup-norm estimates on $\partial_s u$ and $\nabla_x u$. As is well known, see for instance [4, 5, 23, 27], this gives false impressions. As will be seen, more can be done.

3.1.2. Reminders on the Radon transform. The Radon transform [18] is the map

$$R : C_c^0(\mathbb{R}^3;\mathbb{C}) \to C_c^0(\mathbb{S}^2\times\mathbb{R};\mathbb{C}), \quad \mathrm{f}(x) \mapsto R\,\mathrm{f}(\omega,p),$$

where

$$\mathrm{g}(\omega,p) := R\,\mathrm{f}(\omega,p) := \int_{H_{\omega,p}} \mathrm{f}(x)\,dm(x) = \mathrm{g}(-\omega,-p), \tag{3.12}$$

and dm is the Euclidean measure on the hyperplane $H_{\omega,p} := \{x\in\mathbb{R}^3 : \omega\cdot x = p\}$. Recall that $R\mathrm{f}$ is linked to the Fourier transform $\hat{\mathrm{f}}$ through the relation

$$\hat{\mathrm{f}}(\rho\omega) = \int_{-\infty}^{+\infty} e^{-i\rho p} R\,\mathrm{f}(\omega,p)\,dp, \quad \rho\in\mathbb{R}_+.$$

It follows that

$$R(\partial_{x_i}\mathrm{f})(\omega,p) = \omega_i\,\partial_p\,\mathrm{g}(\omega,p) \quad \text{and} \quad R(\Delta\mathrm{f})(\omega,p) = \partial^2_{pp}\,\mathrm{g}(\omega,p). \tag{3.13}$$

As explained, for instance, in p. 19 of [18], with $\eta = \rho\omega\in\mathbb{R}^3$, we obtain that

$$\begin{aligned} \mathrm{f}(x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\eta}\,\hat{\mathrm{f}}(\eta)\,d\eta = \frac{1}{(2\pi)^3}\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\rho(\omega\cdot x)}\,\hat{\mathrm{f}}(\rho\omega)\,\rho^2\,d\rho\,d\omega \\ &= \frac{1}{2}\,\frac{1}{(2\pi)^3}\int_{\mathbb{S}^2}\Big\{\int_{-\infty}^{+\infty}\rho^2\,e^{i\rho(\omega\cdot x)}\Big[\int_{-\infty}^{+\infty} e^{-i\rho p}\,R\,\mathrm{f}(\omega,p)\,dp\Big]\,d\rho\Big\}\,d\omega, \end{aligned} \tag{3.14}$$

where the change of variables (ω,ρ) into $(-\omega,-\rho)$ has been exploited to pass from the first to the second line. This yields the (three-dimensional) inversion formula

$$\mathrm{f}(x) = R^{-1}\mathrm{g}(x) := -\frac{1}{2}\,\frac{1}{(2\pi)^2}\int_{\mathbb{S}^2}\partial^2_{pp}\,\mathrm{g}(\omega,\omega\cdot x)\,d\omega. \tag{3.15}$$

Also observe that

$$(i\rho)^n\,\hat{\mathrm{f}}(\rho\omega) = \mathcal{F}_p((\partial_p^n R\,\mathrm{f})(\omega,\cdot))(\rho) \quad \text{for all } n\in\mathbb{N}. \tag{3.16}$$

As a consequence, we have

$$(\partial_p^n R\,\mathrm{f})(\omega,p) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^3} e^{i\rho(p-\omega\cdot x)}\,(i\rho)^n\,\mathrm{f}(x)\,d\rho\,dx \quad \text{for all } n\in\mathbb{N}. \tag{3.17}$$

The relation (3.13) indicates that a derivative in x is on the Radon side a derivative in p. But the Radon transform is clearly associated with a smoothing effect (due to the integration). And thereby, its inverse can consume derivatives of g, as demonstrated on (3.15). This means that gains of derivatives (in p) are hidden behind the integration with respect to ω. This phenomenon is detected in Section 3.2 and exploited in Section 3.3.

3.2. Two oscillatory integrals

As in the case of the Fourier transform, the study of partial differential equations with the Radon transform may reveal the role of oscillatory integrals. In the present context, these integrals are of two types: the first (Section 3.2.1) involves a compact surface (the sphere $\mathbb{S}^2$); the second (Section 3.2.2) implies the whole space $\mathbb{R}\times\mathbb{R}^3$. In both cases, the aim is to take advantage of cancellation properties induced by oscillations.

3.2.1. Oscillatory integrals on the sphere. In view of (3.16), a derivative ∂_p on the Radon side costs a multiplication by ρ. On the other hand, as can be inferred from equation (3.17), this loss may be associated with oscillatory integrals implying phases looking like $\rho(\omega \cdot \mathrm{X} + \tau)$. It is compensated at the level of (3.15) by an integration with respect to ω. Thus, there is some underlying spatial averaging effect (on $\mathbb{S}^2$). This is highlighted below.

Lemma 3.3 (Gain of one derivative). *Fix* $\mathrm{X} \in \mathbb{R}^3 \setminus \{0\}$ *and* $(\rho, \tau) \in \mathbb{R}^2$. *We have*

$$\int_{\mathbb{S}^2} e^{i\rho(\omega\cdot \mathrm{X}+\tau)}(i\rho)\, d\omega = \frac{2\pi}{|\mathrm{X}|} \sum_{\pm} \pm e^{i\rho(\pm|\mathrm{X}|+\tau)} = 4\pi i\, e^{i\rho\tau}\, \frac{\sin(\rho|\mathrm{X}|)}{|\mathrm{X}|}. \tag{3.18}$$

Proof. Let $\mathcal{R}$ be a rotation such that $\mathcal{R}(\mathrm{X}/|\mathrm{X}|) = {}^t(0,0,1)$. We can change ω into $\mathrm{\omega} := \mathcal{R}\omega$ and then work in spherical coordinates, that is, with

$$\mathrm{\omega} = \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{pmatrix}, \qquad \varphi \in [0, 2\pi],\ \vartheta \in [0,\pi],\ d\mathrm{\omega} = \sin\vartheta\, d\vartheta\, d\varphi. \tag{3.19}$$

Since $\mathcal{R}^{-1}\mathrm{\omega} \cdot \mathrm{X} = \mathrm{\omega} \cdot \mathcal{R}\,\mathrm{X}$, this furnishes

$$\begin{aligned}
\int_{\mathbb{S}^2} e^{i\rho(\omega\cdot \mathrm{X}+\tau)}(i\rho)\, d\omega &= \int_{\mathbb{S}^2} e^{i\rho(\mathcal{R}^{-1}\mathrm{\omega}\cdot \mathrm{X}+\tau)}(i\rho)\, d\mathrm{\omega} = \int_{\mathbb{S}^2} e^{i\rho(|X|\,\mathrm{\omega}\cdot\mathcal{R}\,(\mathrm{X}/|\mathrm{X}|)+\tau)}(i\rho)\, d\mathrm{\omega} \\
&= \int_0^{\pi}\int_0^{2\pi} e^{i\rho(|\mathrm{X}|\cos\vartheta+\tau)}(i\rho)\sin\vartheta\, d\vartheta\, d\varphi = -\frac{2\pi}{|\mathrm{X}|}\int_0^{\pi} \partial_\vartheta\{e^{i\rho[|\mathrm{X}|\cos\vartheta+\tau)}\}\, d\vartheta \\
&= \frac{2\pi}{|\mathrm{X}|}\{e^{i\rho[+|\mathrm{X}|+\tau]} - e^{i\rho[-|\mathrm{X}|+\tau]}\},
\end{aligned}$$

which is exactly (3.18). ∎

Let $\nu \in \mathbb{R}^3$. Applying the differential operator $\nu \cdot \nabla_{\mathrm{X}}$ to the identity (3.18), we end up with

$$\int_{\mathbb{S}^2} \nu\cdot\omega\, e^{i\rho(\omega\cdot \mathrm{X}+\tau)}\rho^2\, d\omega = 2\pi\,\frac{\nu\cdot \mathrm{X}}{|\mathrm{X}|^3}\sum_{\pm}(\pm 1 - i\rho|\mathrm{X}|)\, e^{i\rho(\pm|\mathrm{X}|+\tau)}. \tag{3.20}$$

3.2.2. Oscillatory integrals on the whole space. Replace X by $\mathrm{X} - x$. Looking at the right-hand side of (3.18), we see that the integration with respect to $d\omega$ may produce singular weights (like $1/|\mathrm{X}-x|$ near $\mathrm{X} = x$) in factor of oscillations. Now, the integration with respect to $d\rho\, dx$ of such expressions multiplied by f can produce the integral of the trace of f on spheres.

Lemma 3.4 (Passage from singular weights to traces). *Let* $\mathrm{f}\colon \mathbb{R}^3 \to \mathbb{R}$ *be a compactly supported function of class* C^1. *Let* $\mathrm{K}\colon \mathbb{S}^2 \to \mathbb{R}$ *be a bounded function. Fix* $\mathrm{X} \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$ *and* $\tau \in \mathbb{R}$. *Then, for* $\alpha \le 2$, *we have*

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^3} & e^{i\rho[\pm|\mathrm{X}-x|+\tau]}\, \mathrm{K}\Big(\frac{x-\mathrm{X}}{|x-\mathrm{X}|}\Big)\,\frac{\mathrm{f}(x)}{|\mathrm{X}-x|^\alpha}\, d\rho\, dx \\
&= 2\pi \int_{\mathbb{S}^2} |\tau|^{2-\alpha}\, H(\mp\tau)\, \mathrm{K}(\omega)\, \mathrm{f}(X + |\tau|\omega)\, d\omega,
\end{aligned} \tag{3.21}$$

where H is the Heaviside function (in the half-maximum convention, which is important at least when $\alpha = 2$ and $\tau = 0$), namely,

$$H(\tau) := \frac{1}{2}\left[\mathbb{1}_{\mathbb{R}_+}(\tau-) + \mathbb{1}_{\mathbb{R}_+}(\tau+)\right] = \begin{cases} 0 & \text{if } \tau < 0, \\ 1/2 & \text{if } \tau = 0, \\ 1 & \text{if } \tau > 0. \end{cases}$$

Proof. In spherical coordinates $r\omega$ for $x - \mathrm{X}$, we have to deal with

$$\int_{\mathbb{S}^2}\int_{\mathbb{R}} \mathrm{K}(\omega)\Big(\int_{\mathbb{R}_+} r^{2-\alpha}\, \mathrm{f}(\mathrm{X} + r\,\omega)\, e^{-i\rho(\mp r - \tau)}\, dr\Big)\, d\rho\, d\omega$$
$$= \int_{\mathbb{S}^2} \mathrm{K}(\omega)\Big\{\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \psi_{\mp}(\mathrm{r})\, e^{-i\rho\mathrm{r}}\, d\mathrm{r}\Big)\, d\rho\Big\}\, d\omega = \int_{\mathbb{S}^2} \mathrm{K}(\omega)\,\Big\{\int_{\mathbb{R}} \mathcal{F}(\psi_{\mp})(\rho)\, d\rho\Big\}\, d\omega,$$

where we have changed r into $\mathrm{r} := \mp r - \tau$, and where we have introduced the function $\psi_{\mp}\colon \mathbb{R} \to \mathbb{R}$ (depending on ω) given by

$$\psi_{\mp}(\mathrm{r}) := |\mathrm{r} + \tau|^{2-\alpha}\, \mathbb{1}_{\mathbb{R}_+}(\mp(\mathrm{r} + \tau))\, \mathrm{f}(\mathrm{X} + |\mathrm{r} + \tau|\omega).$$

The function $\psi_{\mp}$ is compactly supported. It is bounded and piecewise C^1 as long as $\alpha \leq 2$, with only one possible discontinuity when $\alpha = 2$ (located at $\mathrm{r} = -\tau$). Thus, we can apply the Dirichlet condition for inversion of Fourier integrals, which furnishes

$$\int_{\mathbb{R}} \mathcal{F}(\psi_{\mp})(\rho)\, d\rho = 2\pi(\mathcal{F}^{-1} \circ \mathcal{F})(\psi_{\mp})(0) = \pi(\psi_{\mp}(0+) + \psi_{\mp}(0-)).$$

After substitution, we find (3.21). ■

3.3. Proof of Proposition 3.2

The demonstration is done in three stages. In Section 3.3.1, we expressed $\check{\mathrm{D}}$ in terms of $\partial^2_{pp}\,\mathrm{g}$. In Section 3.3.2, we exploit (3.17) to exhibit adequate oscillatory integrals. Then, in Section 3.3.3, we draw the conclusions.

3.3.1. The Radon side picture. We introduce

$$\mathrm{v}(t,\omega,p,\xi) := R[\mathrm{u}(t,\cdot,\xi)](\omega,p) \quad \text{and} \quad \mathrm{g}(t,\omega,p,\xi) := R[\mathrm{f}(t,\cdot,\xi)](\omega,p).$$

Under the action of the Radon transform, the three-dimensional wave equation is transformed into a one-dimensional wave equation. With $\mathrm{v}_1 \equiv 0$, the Cauchy problem (3.8) becomes

$$(\partial^2_{tt} - \partial^2_{pp})\,\mathrm{v} = \mathrm{g}, \quad \mathrm{v}_{|t=0} = 0, \quad \partial_t \mathrm{v}_{|t=0} = \mathrm{v}_1. \tag{3.22}$$

On the other hand, from the Vlasov equation, we can deduce that

$$\partial_t\,\mathrm{g} + v(\xi)\cdot\omega\,\partial_p\,\mathrm{g} + \mathrm{div}_\xi\, R(\mathrm{f}\,\mathrm{F}) = 0, \quad \mathrm{g}_{|t=0} = \mathrm{g}_0 := R[\mathrm{f}_0(\cdot,\xi)]. \tag{3.23}$$

At the level of (3.11), the integral $\check{\mathrm{D}}$ is built with four types of derivative: $\partial_s\mathrm{u}$ and $\partial_{x_i}\mathrm{u}$ with $i \in \{1,2,3\}$. In paragraph (a), we show that $\partial_s\mathrm{u}$ can be expressed as a function of $\partial^2_{pp}\,\mathrm{g}$. In paragraph (b), we do the same for $\nabla_x\mathrm{u}$. In this way, in paragraph (c), we can extract a reformulation of $\check{\mathrm{D}}$ in terms of one type of derivative, namely, $\partial^2_{pp}\,\mathrm{g}$. Thus, contrary to the division lemma [4, 23], looking at $\check{\mathrm{D}}$ as depending on g instead of u allows to reduce the kind of derivatives which are needed: we can concentrate on ∂_p only.

(a) Computation of ∂_su. The solution to the one-dimensional wave equation (3.22) can be obtained through Duhamel's formula,

$$\mathrm{v}(t,\omega,p,\xi) = \frac{1}{2}\int_{p-t}^{p+t} \mathrm{v}_1(\omega,y,\xi)\,dy + \frac{1}{2}\iint_{\triangle(t,p)} \mathrm{g}(s,\omega,y,\xi)\,ds\,dy, \tag{3.24}$$

where $\triangle(t,p)$ is the triangle

$$\triangle(t,p) := \{(s,y) : s \in [0,t],\ p-(t-s) \le y \le p+(t-s)\}.$$

From (3.22) with $\mathrm{v}_1 \equiv 0$, we find also

$$(\partial^2_{tt} - \partial^2_{pp})(\partial_t \mathrm{v}) = \partial_t \mathrm{g}, \quad \partial_t \mathrm{v}_{|t=0} = 0,$$

as well as

$$\partial_t(\partial_t \mathrm{v})_{|t=0} = \mathrm{g}_{|t=0} + (\partial^2_{pp}\mathrm{v})_{|t=0} = \mathrm{g}_0.$$

Applying (3.24) to the above equation, we get

$$\begin{aligned}
\partial_t \mathrm{v}(t,\omega,p,\xi) &= \frac{1}{2}\int_{p-t}^{p+t} \mathrm{g}_0(y,\xi)\,dy + \frac{1}{2}\iint_{\triangle(t,p)} \partial_s \mathrm{g}(s,\omega,y,\xi)\,ds\,dy \\
&= \frac{1}{2}\int_{p-t}^{p+t} \mathrm{g}_0(y,\xi)\,dy + \frac{1}{2}\Big\{\int_{p-t}^{p}\Big(\int_0^{y-p+t} \partial_s \mathrm{g}(s,\omega,y,\xi)\,ds\Big)\,dy \\
&\quad + \int_p^{p+t}\Big(\int_0^{-y+p+t} \partial_s \mathrm{g}(s,\omega,y,\xi)\,ds\Big)\,dy\Big\} \\
&= \frac{1}{2}\int_0^t [\mathrm{g}(s,\omega,p+t-s,\xi) + \mathrm{g}(s,\omega,p-t+s,\xi)]\,ds.
\end{aligned} \tag{3.25}$$

From (3.25), we can infer that

$$\begin{aligned}
\partial_s \mathrm{u}(s,x,\xi) = R^{-1}(\partial_s \mathrm{v})(s,x,\xi) &= -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbb{S}^2} (\partial^2_{pp}\partial_s \mathrm{v})(s,\omega,\omega\cdot x,\xi)\,d\omega \\
&= -\frac{1}{4}\frac{1}{(2\pi)^2}\sum_{\pm}\int_{\mathbb{S}^2}\Big\{\int_0^s \partial^2_{pp}\mathrm{g}(r,\omega,\omega\cdot x \pm (s-r),\xi)\,dr\Big\}\,d\omega.
\end{aligned} \tag{3.26}$$

Since $\partial^2_{pp}\mathrm{g}(r,\cdot,\xi)$ is in view of (3.12) an even function, we know that

$$\partial^2_{pp}\mathrm{g}(r,-\omega,-\omega\cdot x-(s-r),\xi) = \partial^2_{pp}\mathrm{g}(r,\omega,\omega\cdot x+(s-r),\xi), \tag{3.27}$$

and therefore, changing ω into $-\omega$, we can deduce that

$$\partial_s \mathrm{u}(s,x,\xi) = -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbb{S}^2}\Big\{\int_0^s \partial^2_{pp}\mathrm{g}(r,\omega,\omega\cdot x+(s-r),\xi)\,dr\Big\}\,d\omega. \tag{3.28}$$

(b) Computation of ∇_xu. From (3.22) with $\mathrm{v}_1 \equiv 0$, we can extract

$$(\partial^2_{tt} - \partial^2_{pp})(\partial_p \mathrm{v}) = \partial_p \mathrm{g}, \quad (\partial_p \mathrm{v})_{|t=0} = 0, \quad \partial_t(\partial_p \mathrm{v})_{|t=0} = 0. \tag{3.29}$$

Applying (3.24), this yields

$$\partial_p \mathrm{v}(t,\omega,p,\xi) = \frac{1}{2}\iint_{\Delta(t,p)} \partial_p \mathrm{g}(s,\omega,y,\xi)\,ds\,dy = \frac{1}{2}\int_0^t [\mathrm{g}(s,\omega,p+t-s,\xi) - \mathrm{g}(s,\omega,p-t+s,\xi)]\,ds. \tag{3.30}$$

On the other hand, from (3.13), we have

$$R[\nu\circ\Xi(s)\cdot\nabla_x \mathrm{u}](s,\omega,p,\xi) = \nu\circ\Xi(s)\cdot\omega\,\partial_p \mathrm{v}(s,\omega,p,\xi),$$

and therefore, using (3.27) again and (3.30), we obtain that

$$\begin{aligned}\nu\circ\Xi(s)\cdot\nabla_x \mathrm{u}(s,x,\xi) &= R^{-1}[\nu\circ\Xi(s)\cdot\omega\,\partial_p \mathrm{v}](s,x,\xi)\\ &= -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbb{S}^2} \nu\circ\Xi(s)\cdot\omega\,\partial_{pp}^2(\partial_p \mathrm{v})(s,\omega,\omega\cdot x,\xi)\,d\omega\\ &= -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbb{S}^2}\Big\{\int_0^s \nu\circ\Xi(s)\cdot\omega\,\partial_{pp}^2\mathrm{g}(r,\omega,\omega\cdot x+s-r,\xi)\,dr\Big\}\,d\omega. \end{aligned}\tag{3.31}$$

(c) Summary. It suffices to plug (3.28) and (3.31) inside (3.11) to get a representation formula for $\check{\mathrm{D}}$ on the Radon side:

$$\check{\mathrm{D}} = \frac{1}{8\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\int_0^t\Big\{\int_0^s \nu\circ\Xi(s)\cdot(\omega+\nu(\xi))\partial_{pp}^2\mathrm{g}(r,\omega,\omega\cdot X(s)+s-r,\xi)\,dr\Big\}\,ds\,d\omega\,d\xi.$$

Recall (3.13), which says that $\partial_{pp}^2 = R\Delta$. Thus, the computation of $\check{\mathrm{D}}$ seems to consume two derivatives of f. The aim of the next paragraph is to show that this is not the case. The goal is to remove the presence of ∂_{pp}^2 inside D.

3.3.2. Analysis through oscillatory integrals. To better understand the content of $\check{\mathrm{D}}$, we can apply (3.17) with $n=2$ to exhibit the following oscillatory integral:

$$\begin{aligned}\check{\mathrm{D}} = -\frac{1}{16\pi^3}\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\int_{\mathbb{S}^2}\nu\circ\Xi(s)\cdot(\omega+\nu(\xi))\,e^{i\rho[\omega\cdot X(s)-\omega\cdot x+s-r]}\rho^2\,d\omega\Big)\\ \times\mathrm{f}(r,x,\xi)\,ds\,dr\,d\rho\,dx\,d\xi.\end{aligned}$$

We can apply Lemma 3.3 and (3.20) with $\mathrm{X} = X(s)-x$ and $\tau = s-r$ to get

$$\begin{aligned}\check{\mathrm{D}} = &\sum_{\pm}\mp\frac{1}{8\pi^2}\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nu\circ\Xi(s)\cdot(X(s)-x)}{|X(s)-x|^3}\\ &\times\mathrm{f}(r,x,\xi)\cdot e^{i\rho[\pm|X(s)-x|+s-r]}\,ds\,dr\,d\rho\,dx\,d\xi\\ &+\sum_{\pm}\pm\frac{1}{8\pi^2}\int_0^t\int_0^s\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\frac{\nu\circ\Xi(s)\cdot\nu(\xi)}{|X(s)-x|}\pm\frac{\nu\circ\Xi(s)\cdot(X(s)-x)}{|X(s)-x|^2}\Big)\\ &\times\mathrm{f}(r,x,\xi)(i\rho)\,e^{i\rho[\pm|X(s)-x|+s-r]}\,ds\,dr\,d\rho\,dx\,d\xi.\end{aligned}$$

In the second sum, there is still ρ in factor (which corresponds to the lost of one derivative). The next idea is to eliminate this weight ρ by exploiting the underlying presence of oscillations. This is the principle of non-stationary phase. To this end, the strategy is to perform an integration by parts with respect to r and x. As usual, this operation costs time and spatial derivatives of the symbol f. It must be done without introducing unmanageable derivatives of f. In practice, taking into account the Vlasov equation, we can convert the derivative $\partial_r + \nu(\xi) \cdot \nabla_x$ into derivatives with respect to ξ. With this in mind, we look at

$$(3.32) \qquad (\partial_r + \nu(\xi) \cdot \nabla_x)\{e^{i\rho[\pm|X(s)-x|+s-r]}\} = \Big(-1 \pm \frac{x - X(s)}{|x - X(s)|} \cdot \nu(\xi)\Big)(i\rho) \cdot e^{i\rho\,[\pm|X(s)-x|+s-r]}.$$

Since $|\nu(\xi)| < 1$, the multiplicative factor in the right-hand side is negative, and therefore it can be inverted. This argument exploits a microlocal ellipticity property. As mentioned in Section 2.3.2 (b), it is referred to as a *non-resonant smoothing* property [5]. The identity (3.32) connects $\partial_r + \nu(\xi) \cdot \nabla_x$ to the multiplication by ρ. In view of (3.16), this amounts to apply the derivative ∂_p. Now, recall that

$$R(\partial_{x_i} \mathrm{f}) = \omega_i \partial_p (R\mathrm{f}),$$

and thereby $\omega_i \partial_p$ may be viewed on the Radon side as a condensed version of the spatial derivatives ∂_{x_i}. Thus, a link is established between $\partial_r + \nu(\xi) \cdot \nabla_x$ and ∂_{x_i}. Historically, see [4,23], this was done by converting the derivative $\partial_{x_i} \mathrm{u}$ inside (3.11) into $\partial_r \mathrm{f} + \nu(\xi) \cdot \nabla_x \mathrm{f}$. Here, there is no need of such division lemma, see Section 2.3.2 (c). We just observe that the light cones (related to the wave equation for the electromagnetic fields), which may be viewed as the level surfaces in $\mathbb{R} \times \mathbb{R}^3$ of the phase functions $\pm|X(s) - x| + s - r$, are transversal to the derivative $\partial_r + \nu(\xi) \cdot \nabla_x$ (which is related to the time-spatial transport in the Vlasov equation).

With the help of (3.32), in the second sum defining $\check{\mathrm{D}}$, we can interpret the coefficient which is in factor of f according to

$$\Big(\frac{\nu \circ \Xi(s) \cdot \nu(\xi)}{|X(s) - x|} \pm \frac{\nu \circ \Xi(s) \cdot (X(s) - x)}{|X(s) - x|^2}\Big)(i\rho)e^{i\rho\,[\pm|X(s)-x|+s-r]} = -k_{\mp}(s, x, \xi)(\partial_r + \nu(\xi) \cdot \nabla_x)\{e^{i\rho\,[\pm|X(s)-x|+s-r]}\},$$

with

$$k_{\mp}(s, x, \xi) := \frac{1}{|X(s) - x|} K_{\mp}\Big(s, \frac{x - X(s)}{|x - X(s)|}, \xi\Big)$$

and

$$K_{\mp}(s, \omega, \xi) := \frac{\nu \circ \Xi(s) \cdot (\nu(\xi) \mp \omega)}{1 \mp \omega \cdot \nu(\xi)}.$$

Let T_k be the multiplicative operator by the function k, that is,

$$(3.33) \qquad T_k(\mathrm{f})(r, s, x, \xi) := k(s, x, \xi)\, \mathrm{f}(r, x, \xi).$$

Now, we can integrate by parts with respect to r and x to obtain

$$\begin{aligned}
\check{\mathrm{D}} = & \sum_{\pm} \mp \frac{1}{8\pi^2} \int_0^t \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nu \circ \Xi(s) \cdot (X(s) - x)}{|X(s) - x|^3} \\
& \times \mathrm{f}(r, x, \xi)\, e^{i\rho[\pm|X(s)-x|+s-r]}\, ds\, dr\, d\rho\, dx\, d\xi \\
& + \sum_{\pm} \pm \frac{1}{8\pi^2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \Big\{ \int_0^s \int_{\mathbb{R}^3} (\partial_r + \nu(\xi) \cdot \nabla_x)(T_{k_\mp}(\mathrm{f})) \\
& \times e^{i\rho[\pm|X(s)-x|+s-r]}\, dx\, dr \Big\}\, d\xi\, d\rho\, ds \\
& - \sum_{\pm} \pm \frac{1}{8\pi^2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \Big[\int_{\mathbb{R}^3} T_{k_\mp}(\mathrm{f}) e^{i\rho[\pm|X(s)-x|+s-r]}\, dx \Big]_0^s\, d\xi\, d\rho\, ds.
\end{aligned}$$

This yields two boundary terms (D_{b0} and D_{bs} issued, respectively, for $r = 0$ and $r = s$), a term D_l including the integrands which have f in factor, as well as a contribution D_n which implies derivatives of f. More precisely, we have

$$\check{\mathrm{D}} = \mathrm{D}_{b0} + \mathrm{D}_{bs} + \mathrm{D}_l + \mathrm{D}_n,$$

with

$$\mathrm{D}_{b0} = \sum_{\pm} \pm \mathrm{D}_{b0}^{\pm}, \quad \mathrm{D}_{bs} = \sum_{\pm} \mp \mathrm{D}_{bs}^{\pm}, \quad \mathrm{D}_l = \sum_{\pm} \pm \mathrm{D}_l^{\pm}, \quad \mathrm{D}_n = \sum_{\pm} \pm \mathrm{D}_n^{\pm},$$

and where

$$\begin{aligned}
\mathrm{D}_{b0}^{\pm} &:= \frac{1}{8\pi^2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T_{k_\mp}(\mathrm{f})(0, s, x, \xi)\, e^{i\rho[\pm|X(s)-x|+s]}\, ds\, d\rho\, dx\, d\xi, \\
\mathrm{D}_{bs}^{\pm} &:= \frac{1}{8\pi^2} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T_{k_\mp}(\mathrm{f})(s, s, x, \xi)\, e^{\pm i\rho|X(s)-x|}\, ds\, d\rho\, dx\, d\xi, \\
\mathrm{D}_l^{\pm} &:= \frac{1}{8\pi^2} \int_0^t \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T_{\tilde{k}_\mp}(\mathrm{f})\, e^{i\rho[\pm|X(s)-x|+s-r]}\, ds\, dr\, d\rho\, dx\, d\xi, \\
\mathrm{D}_n^{\pm} &:= \frac{1}{8\pi^2} \int_0^t \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T_{k_\mp}(\partial_r \mathrm{f} + \nu(\xi) \cdot \nabla_x \mathrm{f}) \\
&\qquad \times e^{i\rho[\pm|X(s)-x|+s-r]}\, ds\, dr\, d\rho\, dx\, d\xi.
\end{aligned}$$

By construction, we find

$$\tilde{k}_\mp := -\frac{\nu \circ \Xi(s) \cdot (X(s) - x)}{|X(s) - x|^3} + \nu(\xi) \cdot \nabla_x k_\mp = \frac{1}{|X(s) - x|^2} \tilde{K}_\mp\Big(s, \frac{x - X(s)}{|x - X(s)|}, \xi\Big),$$

with

$$\tilde{K}_\mp(s, \omega, \xi) := \frac{1}{\langle \xi \rangle^2} \frac{\nu \circ \Xi(s) \cdot (\omega \mp \nu(\xi))}{(1 \mp \omega \cdot \nu(\xi))^2}.$$

Now, the time spatial derivative $\partial_r \mathrm{f} + \nu(\xi) \cdot \nabla_x \mathrm{f}$ can be exchanged with velocity derivatives. Indeed, using the Vlasov equation, it can be converted into derivatives with respect to ξ (which are harmless because the coefficients are smooth in ξ and because the phase does not depend on ξ). We find that

$$\mathrm{D}_n^{\pm} = \frac{1}{8\pi^2} \int_0^t \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_\xi k_{\mp} \cdot \mathrm{F f} e^{i\rho[\pm|X(s)-x|+s-r]} \, ds\, dr\, d\rho\, dx\, d\xi.$$

On the other hand, in $\mathrm{D}_l^{\pm}$ and $\mathrm{D}_n^{\pm}$, we can switch the order of integrations according to

$$\int_0^t \Big(\int_0^s \cdots dr \Big) ds = \int_0^t \Big(\int_r^t \cdots ds \Big) dr.$$

3.3.3. Epilog. In general, the Fourier analysis as well as the Radon analysis of nonlinear partial differential equations lead to complex formulas. And indeed, the above oscillatory integrals seem complicated. Surprisingly, they can be significantly simplified by applying Lemma 3.4. As a matter of fact, all the computations can be made explicit:

- *Study of* D_{b0}. We take $\mathrm{K} = K_{\mp}$, $\mathrm{X} = X(s)$, $\alpha = 1$ and $\tau = s \geq 0$. We find that $\mathrm{D}_{b0}^{+} = 0$ and $\mathrm{D}_{b0} = -\mathrm{D}_{b0}^{-}$. With $\mathrm{W}_0 = -s\, K_{+}/(4\pi)$, we can recognize D_0 as in (3.4a).
- *Study of* D_{bs}. We take $\mathrm{K} = K_{\mp}$, $\mathrm{X} = X(s)$, $\alpha = 1$ and $\tau = 0$ to see that $\mathrm{D}_{bs}^{\pm} = 0$.
- *Study of* D_l. We take $\mathrm{K} = \tilde{K}_{\mp}$, $\mathrm{X} = X(s)$, $\alpha = 2$ and $\tau = s - r \geq 0$ to obtain $\mathcal{D}_l^{+} = 0$ so that $\mathrm{D}_l = -\mathrm{D}_l^{-}$. This is coherent with (3.4c), where $\mathrm{W}_l = -\tilde{K}_{+}/(4\pi)$ as in (3.5b).
- *Study of* D_n. We deal with a vector valued version of (3.21), where f is replaced by Ff. We take $\mathrm{K} = \nabla_\xi K_{\mp}$, $\mathrm{X} = X(s)$, $\alpha = 1$ and $\tau = s - r \geq 0$ to see that $\mathcal{D}_n^{+} = 0$ and therefore that $\mathrm{D}_n = -\mathrm{D}_n^{-}$. In this way, with $\mathrm{W}_n = -(s-r)\, \nabla_\xi K_{+}/(4\pi)$, we find that D_n is given by (3.4d).

  The proof of Proposition 3.2 is now complete.

4. The control of the momentum spread

In this section, we consider a solution U to the Cauchy problem (1.2)–(1.4), assumed to be smooth, compactly supported (in ξ), and defined on $[0, T[$, where $T \equiv T(\mathrm{U}_0) \in \mathbb{R}_+^* \cup \{+\infty\}$ is the maximum life-span of this smooth solution. From (3.2), we know that the support of $\mathrm{f}(t,\cdot)$ is the image by the map $(X, \Xi)(t,\cdot)$ of the support of f_0. Given $t \in [0, T[$, we can define the maximal size $\mathrm{Q}(t)$ of the spatial support of $\mathrm{f}(t,\cdot)$, which is

(4.1) $$\mathrm{Q}(t) := \inf\big\{R \in \mathbb{R}_+ : \mathrm{f}(t,x,\xi) = 0 \text{ for all } \xi \in \mathbb{R}^3 \text{ and all } x \in \mathbb{R}^3 \text{ with } R \leq |x|\big\},$$

as well as the maximal size $\mathrm{P}(t)$ of the momentum support of $\mathrm{f}(t,\cdot)$, which is

(4.2) $$\mathrm{P}(t) := \inf\big\{R \in \mathbb{R}_+ : \mathrm{f}(t,x,\xi) = 0 \text{ for all } x \in \mathbb{R}^3 \text{ and all } \xi \in \mathbb{R}^3 \text{ with } R \leq |\xi|\big\}.$$

By construction, we have

$$\operatorname{supp} \mathrm{f} \subset \big\{(t,x,\xi) \in [0,T[\times \mathbb{R}^3 \times \mathbb{R}^3 : |x| \leq \mathrm{Q}(t),\ |\xi| \leq \mathrm{P}(t)\big\}.$$

The quantity Q yields a control on the size of the *spatial domain of influence*. In view of the first equation of (3.1), the spatial speed of propagation is bounded by one, so that

(4.3) $$\mathrm{Q}(t) \leq \mathrm{Q}_0 + t, \quad \mathrm{Q}_0 := \mathrm{Q}(0).$$

Both Vlasov and Maxwell's equations have a finite spatial speed of propagation (bounded by 1). Thus, exploiting the notion of region of influence, to prove the local smooth solvability, it suffices to work with solutions that are compactly supported with respect to the space variable x. To simplify, we can localize the spatial and momentum support in the same ball (say of size P_0). With this in mind, we replace the condition inside (1.6) on $\operatorname{supp} \mathrm{f}_0$ by

$$\operatorname{supp} \mathrm{f}_0 \subset B(0, \mathrm{P}_0] \times B(0, \mathrm{P}_0]. \tag{4.4}$$

On the other hand, the quantity P gives a bound on the extent of the *momentum domain of influence*. It provides insight into the *momentum spread*. Without a control involving the sup-norm of F (that is equivalently of E and B), the second equation of (3.1) does not provide with a bound for the momentum speed of propagation. This is usually resolved by looking at Lipschitz bounds on $\mathrm{U} = (\mathrm{f}, \mathrm{E}, \mathrm{B})$. From the pioneering works [15–17, 38] on smooth solutions, we know that there exist a time $\mathcal{T}_s \in \mathbb{R}_+^*$ and a continuous function $\mathcal{F}_s \colon [0, \mathcal{T}_s] \to \mathbb{R}_+$ (the subscript s stands for smooth), depending both on the Lipschitz norm of U_0, such that

$$\mathrm{P}(t) \le \mathcal{F}_s(t) \quad \text{for all } t \in [0, \mathcal{T}_s],\ 0 < \mathcal{T}_s \le T. \tag{4.5}$$

A key issue is whether (4.5) remains true under less restrictive criteria on the initial data U_0. Our purpose here is to remove the Lipschitz condition on f_0 and to relax the cost of two derivatives concerning $(\mathrm{E}_0, \mathrm{B}_0)$.

Proposition 4.1 (Control on the momentum spread by mild information). *Fix $\mathrm{P}_0 \in \mathbb{R}_+$, and consider the corresponding subspace $\mathfrak{N}$ adjusted as in (1.6). Given any $\mathrm{S}_0 \in \mathbb{R}_+^*$, select initial data $\mathrm{U}_0 \in \mathfrak{N}$ satisfying $\mathcal{N}(\mathrm{U}_0) \le \mathrm{S}_0$, with $\mathcal{N}$ as in (1.7). Then there exist a time $\mathcal{T} \in \mathbb{R}_+^*$ and a continuous increasing function $\mathcal{F} \colon [0, \mathcal{T}] \to \mathbb{R}_+$, both depending only on S_0, so that $\mathcal{T} \equiv \mathcal{T}(\mathrm{S}_0)$ and $\mathcal{F} \equiv \mathcal{F}(\mathrm{S}_0; \cdot)$, such that*

$$\mathrm{P}(t) \le \mathcal{F}(\mathrm{S}_0; t) \quad \textit{for all } t \in [0, \min(T(\mathrm{U}_0); \mathcal{T}(\mathrm{S}_0))[. \tag{4.6}$$

Moreover, under (4.4), the behavior of $\mathcal{T}$ for small values of S_0 is bounded below according to

$$\textit{there exists } c \in \mathbb{R}_+^* \textit{ such that } c\,\mathrm{S}_0^{-1} \le \mathcal{T}(\mathrm{S}_0) \quad \textit{for all } \mathrm{S}_0 \in \mathbb{R}_+^*. \tag{4.7}$$

From the Glassey–Strauss continuation criterion [16], assuming that $T(\mathrm{U}_0) \in \mathbb{R}_+^*$, the size of $\mathrm{P}(t)$ must explode when $t \to T(\mathrm{U}_0)^-$. Looking at (4.6), since $\mathcal{F}$ is continuous on $[0, \mathcal{T}(\mathrm{S}_0)]$ and therefore bounded, there is a contradiction if $T(\mathrm{U}_0) < \mathcal{T}(\mathrm{S}_0)$. Expressed in terms of (1.1), this means that

$$0 < \mathcal{T}(\mathrm{S}_0) \le T(\mathfrak{N}, \mathcal{N}; \mathrm{S}_0) \le T(\mathrm{U}_0). \tag{4.8}$$

Theorem 1.2 is a direct consequence of (4.8). Given P_0, the line (4.7) specifies how fast $T(\mathfrak{N}, \mathcal{N}; \mathrm{S}_0)$ tends to $+\infty$ when S_0 goes to zero. Without (4.4), a version of (4.7) is still available under adaptations. To this end, the spatial support of f_0 must be truncated on a larger ball of size Ct with $C > 1$, and the smallness parameters must be revisited.

Now, let $G\colon \mathbb{R}^3 \to \mathbb{R}^q$, with $q \in \mathbb{N}^*$. The usual fluid description of plasmas (MHD) involves macroscopic quantities like

$$\mathrm{M_G}(t,x) := \int_{\mathbb{R}^3} \mathrm{G}(\xi)\,\mathrm{f}(t,x,\xi)\,d\xi.$$

For $\mathrm{G} \equiv 1$, we deal with the number density. For $\mathrm{G} \equiv \nu(\xi)$, we recover the current density. For $\mathrm{G}_n(\xi) = \xi \otimes \xi \otimes \cdots \otimes \xi$, where ξ is multiplied n times with $n \in \mathbb{N}^*$, we find the n-th moment, with is, in particular, the momentum density (for $n = 1$).

Corollary 4.2 (Sup-norm controls on all fluid quantities under mild information). *In the context of Proposition 4.1, for all* $\mathrm{G} \in L^\infty(\mathbb{R}^3;\mathbb{R}^q)$*, we have*

$$(4.9)\quad \|\mathrm{M_G}(t,x)\| \le 4\pi \mathrm{S}_0 \sup_{|\xi|\le \mathcal{F}(\mathrm{S}_0;t)} \|\mathrm{G}(\xi)\|^3 < +\infty \quad \textit{for all } (t,x) \in [0,\mathcal{T}(\mathrm{S}_0)] \times \mathbb{R}^3.$$

This furnishes a range of *a priori sup-norm estimates* which, contrary to (4.5), do not require any regularity on f_0, and which implement (relatively) weak estimates on $(\mathrm{E}_0,\mathrm{B}_0)$. This may seem surprising in the quasilinear context (1.2)–(1.3) under study. Recall, however, that such bounds are basically inherited from the transport part (the Vlasov equation) after its (complicated) interaction with Maxwell's equations.

Our construction is also a gateway to a notion of solutions which is at the interface between the strong and weak versions of, respectively, [16, 17] and [9].

Corollary 4.3 (Strong-weak solutions). *Fix* $(\mathrm{P}_0,\mathrm{S}_0) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. *Select* $\mathrm{f}_0 \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ *satisfying* (1.9b), (1.9c) *and* (1.10). *Choose* $\mathrm{E}_0 \in H^1(\mathbb{R}^3)$ *and* $\mathrm{B}_0 \in H^1(\mathbb{R}^3)$. *Assume that* $\mathcal{N}(\mathrm{U}_0) \le \mathrm{S}_0$*, where* $\mathcal{N}$ *is as in* (1.7) *with* $\bar p = 2$. *Then we can find a time* $\mathcal{T} \equiv \mathcal{T}(\mathrm{S}_0) \in \mathbb{R}_+^*$*, depending only on* S_0*, such that the Cauchy problem* (1.2)–(1.4) *has a unique solution satisfying*

$$(4.10)\qquad \operatorname{supp} \mathrm{f}(t,\cdot) \subset \mathbb{R}^3 \times B(0,\mathcal{F}(\mathrm{S}_0;t))] \quad \textit{for all } t \in [0,\mathcal{T}],$$

as well as $\mathrm{E} \in H^1([0,\mathcal{T}]\times\mathbb{R}^3)$ *and* $\mathrm{B} \in H^1([0,\mathcal{T}]\times\mathbb{R}^3)$.

Proof. Any bounded function f_0 satisfying (1.9b), (1.9c) and (1.10) can be approximated by a sequence $(\mathrm{f}_0^n)_n$ subject to (1.9)–(1.10) uniformly with respect to $n \in \mathbb{N}$. Similarly, E_0 and B_0 can be approximated by a sequence $(\mathrm{E}_0^n)_n$ and $(\mathrm{B}_0^n)_n$, with $\mathrm{E}_0^n \in C_c^2$ and $\mathrm{B}_0^n \in C_c^2$. Moreover, it can be ensured that $\mathrm{U}_0^n := (\mathrm{f}_0^n,\mathrm{E}_0^n,\mathrm{B}_0^n)$ is such that $\mathcal{N}(\mathrm{U}_0^n) \le \mathrm{S}_0$ for all $n \in \mathbb{N}$. Theorem 1.2 gives access to solutions U^n on $[0,\mathcal{T}]$ associated with the initial data U_0^n, and satisfying (4.6). By compactness arguments (based on averaging lemmas [9, 33]), passing to the limit ($n \to +\infty$), we can extract a corresponding weak solution $\mathrm{U} = (\mathrm{f},\mathrm{E},\mathrm{B}) \in L^\infty \times L^2 \times L^2$, which still satisfies (4.6). But from Theorem 1 of [5] (or alternatively [2]) together with (4.6), we can deduce that ξ-averages of u (without the need of momentum cutoff) are in $H^2([0,\mathcal{T}]\times\mathbb{R}^3)$,[1] and therefore that $\mathrm{E} \in H^1$ and $\mathrm{B} \in H^1$. From there, applying [10], we can recover the uniqueness. ∎

[1]This argument has been reported to us by Nicolas Besse. Observe that the information (4.6) is crucial to recover the H^1-regularity. It is missing in the case of the weak solutions provided by DiPerna–Lions [9].

The information (4.6) is also adapted to *concrete applications* and to further stability results which, as in [4], may be inherited from (4.6). Since the study of strongly magnetized plasmas were our point of entry [8, 30], this axis of research is prioritized in the present article. For the moment, we can only say that since S_0 can be taken small, we can assume $T \geq \tau$. And we can already guess that T could remain fixed for a large magnetic field whose size is adequately compensated by the smallness of S_0. This will be confirmed in Section 5.

In Section 4.1, we lay the background for a phase space analysis. These preliminaries lead in Section 4.2 to the study of weighted integrals. This results, in Section 4.3, in the proof of Proposition 4.1. From now on, we will work implicitly with $t < T$.

4.1. Preparatory work

In Section 4.1.1, we make the connection between the size P of the momentum spread and the quantities $\mathrm{D}^{\mathrm{a}}_\star$ (derived from the $\mathrm{D}_\star$). In Section 4.1.2, we take the pushforward of the measure $ds\,d\omega\,d\xi$ to recover (modulo a Jacobian) the Liouville measure $dx\,d\xi$. In Section 4.1.3, we show three lemmas that are helpful in Section 4.1.4 to estimate the weight functions $|\mathrm{W}_\star|$.

4.1.1. Control of the momentum spread through the representation formula. By construction, for all $(s,x) \in [0,t] \times \mathbb{R}^3$, the momentum support of $\mathrm{f}(s,x,\cdot)$ is contained in the ball of radius

$$\langle \mathrm{P} \rangle_\infty(t) := \sup \big\{ (1+\mathrm{P}(s)^2)^{1/2} : s \in [0,t] \big\}. \tag{4.11}$$

By compactness arguments using the continuity of f, we can find some (t_0,x_0,ξ_0) such that

$$(t_0,x_0,\xi_0) \in [0,t] \times \operatorname{supp} \mathrm{f}(t_0,\cdot), \quad \langle \xi_0 \rangle = \langle \mathrm{P} \rangle(t_0) = \langle \mathrm{P} \rangle_\infty(t).$$

In view of (3.2), the position (x_0,ξ_0) is necessarily the image by $(X,\Xi)(t_0,\cdot)$ of some (y_0,η_0) in the support of f_0. In other words, we have $\xi_0 = \Xi(t_0,y_0,\eta_0)$ for some η_0 satisfying $|\eta_0| \leq \mathrm{P}_0$. It follows that

$$\langle \mathrm{P} \rangle_\infty(t) = \langle \Xi(t_0,y_0,\eta_0) \rangle = \langle \eta_0 \rangle + \mathrm{D}(t_0,y_0,\eta_0) \leq 1 + \mathrm{P}_0 + |\mathrm{D}(t_0,y_0,\eta_0)|. \tag{4.12}$$

In this way, the control of $\langle \mathrm{P} \rangle_\infty(t)$ boils down to the study of $|\mathrm{D}(t_0,y_0,\eta_0)|$. Moreover, for $s \neq t_0$, we find that $(X,\Xi)(s,y_0,\eta_0)$ must be in the support of $\mathrm{f}(s,\cdot)$ so that

$$\langle \Xi(s,y_0,\eta_0) \rangle \leq \langle \mathrm{P} \rangle_\infty(s) \leq \langle \mathrm{P} \rangle_\infty(t) \quad \text{for all } s \in [0,t]. \tag{4.13}$$

Unless necessary, the selection of (y_0,η_0) will not be mentioned again.

For $\star \in \{0,h,l,ne,nb\}$, we denote by $\mathrm{D}^{\mathrm{a}}_\star$ (with superscript "a" for absolute value) the expression $\mathrm{D}_\star$, where f, $\mathrm{W}_\star$, E and B are replaced by $|\mathrm{f}|$ $(=\mathrm{f})$, $|\mathrm{W}_\star|$, $|\mathrm{E}|$ and $|\mathrm{B}|$, respectively, and where the scalar products (like $\mathrm{W}_n \cdot \mathrm{E}$) are just substituted for products of norms (like $|\mathrm{W}_n|\,|\mathrm{E}|$). By definition, the $\mathrm{D}^{\mathrm{a}}_\star$ are increasing functions of the time t, and therefore

$$\mathrm{D}_\star(t_0) \leq \mathrm{D}^{\mathrm{a}}_\star(t_0) \leq \mathrm{D}^{\mathrm{a}}_\star(t).$$

From Proposition 3.2, together with (4.12), we can assert that

$$0 \leq \langle \mathrm{P} \rangle_\infty(t) \leq 1 + \mathrm{P}_0 + \mathrm{D}^{\mathrm{a}}_0(t) + \mathrm{D}^{\mathrm{a}}_h(t) + \mathrm{D}^{\mathrm{a}}_l(t) + \mathrm{D}^{\mathrm{a}}_{ne}(t) + \mathrm{D}^{\mathrm{a}}_{nb}(t). \tag{4.14}$$

Retain also that, from (1.9c) and (3.2), we get

$$0 \leq \mathrm{f}(r, x, \xi) \leq \mathrm{S}_0 \quad \text{for all } (r, x, \xi) \in [0, T[\times \mathbb{R}^3 \times \mathbb{R}^3. \tag{4.15}$$

Looking at (4.14), we can already assert that $\langle \mathrm{P} \rangle_\infty$ is qualitatively controlled by zero-order information on U, with no derivatives of f, E_h, E or B. In fact, the situation is even better since the influence of f, E and B is expressed through integrals with respect to $ds\, d\omega\, d\xi$. This aspect is examined in the next paragraph.

4.1.2. Comparison of measures, microlocal pictures and related difficulties. All the integrals $\mathrm{D}^{\mathrm{a}}_\star$ (except for $\star = h$) involve the differential element $ds\, d\omega$. It is worth noting that $ds\, d\omega$ appears in our analysis after special procedures, by mixing different aspects:

- The part ds comes from (3.10). We do not look at the speed of propagation (in momentum), which in our context may be large. Instead, we consider some integral version of it. When dealing with (3.1), we exploit Duhamel's principle.
- The part $d\omega$ is issued from (3.15). The Radon transform involves integrals on families of (two-dimensional) hyperplanes. Its inverse implies integrals on (two-dimensional) spheres with respect to $d\omega$.

By assembling ds and $d\omega$, we find $ds\, d\omega$. This combination of the variables s and ω emerges also at the level of (3.4). Indeed, in the $\mathrm{D}^{\mathrm{a}}_\star$, all expressions, like $\mathrm{f}(r, \cdot, \xi)$, are evaluated at the specific position $X(s) + (s - r)\,\omega$ (note that $r = 0$ in case of $\mathrm{D}^{\mathrm{a}}_0$). This furnishes a link to the Lebesgue measure dx. To this end, we can (for instance) exploit Lemma 2.2 of [28], which is recalled below for the sake of completeness.

Lemma 4.4 (Pallard, [28]). *Let $r \in \mathbb{R}$. The map*

$$\mathcal{X} : [r, t] \times \mathbb{S}^2 \to \mathbb{R}^3, \quad (s, \omega) \mapsto \mathcal{X}(s, \omega) := X(s) + (s - r)\,\omega,$$

is a C^1-diffeomorphism onto some region of $\mathbb{R}^3$. Its Jacobian J is given by

$$J(r, s, \omega) := (s - r)^2\, (1 + \omega \cdot \nu \circ \Xi(s)), \qquad J\, ds\, d\omega = dx. \tag{4.16}$$

Thus, the pushforward (by $\mathcal{X}$) of the measure $ds\, d\omega$ is absolutely continuous with respect to the Lebesgue measure: it is just $J^{-1}\, dx$. Now, we can extend this argument at a microlocal level. The Radon transform is an integral (hence non-local) operator and, as such, it is not suitable for transferring localizations. However, what counts inside the $\mathrm{D}^{\mathrm{a}}_\star$ is $\mathcal{X}$, which can effectively translates (microlocal) localizations in terms of (s, ω, ξ) into (microlocal) localizations in terms of (x, ξ) – and conversely. In doing so, the landscape is changing. The advantage is that working with (s, ω, ξ) instead of (x, ξ) is much easier.

This transfer of microlocal localizations is achieved by the map $\mathcal{X}$ which is built (through X) on a complete nonlinear evolution (on a special solution to the RVM system) and on the choice of a specific characteristic (namely, X for a special selection of y_0 and η_0). Passing to the phase space, we have to deal with the pushforward by $\mathcal{X} \otimes \mathrm{Id}_\xi$ of the measure $ds\,d\omega\,d\xi$. Modulo the Jacobian J, this enables a connection with the Liouville measure $dx\,d\xi$.

One of the main interests of Proposition 3.2 associated with (4.16) is to produce explicit weights which indicate where f, E and B contribute most to the $\mathrm{D}^{\mathrm{a}}_{\star}$. Depending upon the relative positions of $\Xi(s)$, ω and ξ, these weights can be larger or smaller. To perform estimates by taking care of the respective localizations of $\Xi(s)$, ω and ξ is what we call here *Radon Fourier analysis*. We use the expression "*Radon*" because the angle ω (respectively, the measure $d\omega$) has appeared after a Radon transform (respectively, through its inverse R^{-1}).

In the end, we are faced with three types of singularities, related to the content of the weight functions $|\mathrm{W}_\star|$ or to negative powers of J thus introduced:

(a) Complication at $\omega = -\xi/|\xi|$ coming from negative powers of $1+\omega\cdot\nu(\xi)$ inside the $|\mathrm{W}_\star|$. This leads to unbounded coefficients in the perspective of a global analysis ($\xi \in \mathbb{R}^3$).

(b) Problem at $s = r$ due to the introduction of J^{-1} (or $J^{-1/p}$ in case of Hölder estimates). This induces a singularity in the time variable.

(c) Difficulty at $\omega = -\Xi(s)/|\Xi(s)|$, issued from negative powers of $1+\omega\cdot\nu\circ\Xi(s)$ which are provided by J^{-1} (or $J^{-1/p}$). This reflects some interplay between the phase space and the momentum component $\Xi(s)$ of the characteristics.

4.1.3. Three useful tools. The origin of negative powers of $1+\omega\cdot\nu(\eta)$ inside (3.4) is a manifold. For $\eta = \xi$, they go with the gain of one derivative (in Lemma 3.3). For $\eta = \xi$ again, they are also due to the rate at which the transversality condition between the derivative $\partial_t + \nu(\xi)\cdot\nabla_x$ and the light cones can degenerate for large values of $|\xi|$. Or for $\eta = \Xi(s)$, they could be issued from the inverse $J^{-1/p}$ of the Jacobian. From a quantitative perspective, they may furnish large weights. We examine below what happens in diverse situations (Cases A, B and C).

Case A. This case deals with sup-norm estimates on the coefficients. It is particularly relevant for the study of $\mathrm{D}^{\mathrm{a}}_{ne}$ and $\mathrm{D}^{\mathrm{a}}_{nb}$ because only L^2-estimates are available when dealing with E and B.

Lemma 4.5 (Maximal loss due to the proximity of $\nu(\eta)$ to the light cone). *We have*

$$0 \le (1+\omega\cdot\nu(\eta))^{-1} \le 2\langle\eta\rangle^2 \quad \textit{for all } (\omega,\eta)\in\mathbb{S}^2\times\mathbb{R}^3. \tag{4.17}$$

Proof. It suffices to note that

$$(1+\omega\cdot\nu(\eta))^{-1} \le (1-(|\eta|/\langle\eta\rangle))^{-1} = \langle\eta\rangle(\langle\eta\rangle+|\eta|). \qquad \blacksquare$$

The bound (4.17) is (almost) optimal when $|\eta|$ is large, and ω becomes close to $-\eta/|\eta|$. However, for other values of ω, the upper bound (4.17) furnishes only a rough control. The question is therefore to evaluate the impact of this singular factor in terms of the measure $d\omega$.

Case B. Large weights may have a limited impact when they focus on a domain of small measure. This is helpful at the level of $\mathrm{D}^{\mathrm{a}}_0$ and $\mathrm{D}^{\mathrm{a}}_l$ because we know that f_0 and f are bounded functions. Below, this effect appears after integration with respect to ω.

Lemma 4.6 (Gain after averaging along the sphere). *For all* $(\delta,\eta)\in\mathbb{R}_+\times\mathbb{R}^3$, *we have*

$$0\le\int_{\mathbb{S}^2}(1+\omega\cdot\nu(\eta))^{-\delta}\,d\omega\lesssim\begin{cases}\langle\eta\rangle^{2(\delta-1)} & \textit{when } 1<\delta,\\ 1+\ln\langle\eta\rangle & \textit{when } \delta=1,\\ 1 & \textit{when } \delta<1.\end{cases}\tag{4.18}$$

Proof. By performing a rotation in ω, we can always assume $\nu(\eta)=(|\eta|/\langle\eta\rangle)\,{}^t(0,0,1)$. Then we can work in spherical coordinates as in (3.19) to see that

$$\int_{\mathbb{S}^2}\frac{d\omega}{(1+\omega\cdot\nu(\eta))^{\delta}}=\int_0^{2\pi}\int_0^{\pi}\frac{\sin\vartheta\,d\vartheta\,d\varphi}{(1+|\eta|\cos\vartheta/\langle\eta\rangle)^{\delta}}=(2\pi)\frac{\langle\eta\rangle}{|\eta|}\int_{-|\eta|/\langle\eta\rangle}^{+|\eta|/\langle\eta\rangle}\frac{ds}{(1-s)^{\delta}}$$
$$=(2\pi)\frac{\langle\eta\rangle}{|\eta|}\times\begin{cases}\frac{1}{1-\delta}\big[-(1-|\eta|/\langle\eta\rangle)^{1-\delta}+(1+|\eta|/\langle\eta\rangle)^{1-\delta}\big] & \text{when } \delta\neq1,\\ -\ln(1-|\eta|/\langle\eta\rangle)+\ln(1+|\eta|/\langle\eta\rangle) & \text{when } \delta=1.\end{cases}$$

In this way, we can easily deduce (4.18). ∎

Case C. On different occasions, we will have to evaluate the distance between $\nu(\eta)$ and ω. This will serve, for instance, to control the vector valued functions W_{ne} and W_{nb}.

Lemma 4.7 (Comparison between $\nu(\eta)$ and ω). *We have*

$$|\nu(\eta)+\omega|\le\sqrt{2}\,(1+\omega\cdot\nu(\eta))^{1/2}\qquad \textit{for all } (\omega,\eta)\in\mathbb{S}^2\times\mathbb{R}^3.\tag{4.19}$$

Proof. This is just because

$$|\nu(\eta)+\omega|^2=|\nu(\eta)|^2+2\omega\cdot\nu(\eta)+1\le2(1+\omega\cdot\nu(\eta)).$$ ∎

4.1.4. Bounds for the weight functions. The purpose of this paragraph is to evaluate carefully the amplitudes of $|\mathrm{W}_0|$, $|\mathrm{W}_l|$, $|\mathrm{W}_{ne}|$ and $|\mathrm{W}_{nb}|$.

We begin by studying $|\mathrm{W}_0|$. From (3.5a), note that

$$\mathrm{W}_0(s,\omega,\xi):=\frac{s}{4\pi}\Big(1-\frac{(\nu\circ\Xi(s)+\omega)\cdot(\nu(\xi)+\omega)}{1+\omega\cdot\nu(\xi)}\Big).$$

With (4.19), it is obvious that

$$|\mathrm{W}_0(s,\omega,\xi)|\le\frac{s}{4\pi}\Big(1+2\frac{(1+\omega\cdot\nu\circ\Xi(s))^{1/2}}{(1+\omega\cdot\nu(\xi))^{1/2}}\Big).\tag{4.20}$$

Next we study $|\mathrm{W}_l|$. From (3.5b), observe that

$$\mathrm{W}_l(s,\omega,\xi):=\frac{1}{4\pi}\frac{1}{\langle\xi\rangle^2}\Big(\frac{1}{(1+\omega\cdot\nu(\xi))}-\frac{(\nu\circ\Xi(s)+\omega)\cdot(\omega+\nu(\xi))}{(1+\omega\cdot\nu(\xi))^2}\Big).$$

Then, as a corollary of Lemmas 4.5 and 4.7, we can assert that

$$|\mathrm{W}_l(s,\omega,\xi)|\le\frac{1}{\pi}\Big(1+\frac{(1+\omega\cdot\nu\circ\Xi(s))^{1/2}}{(1+\omega\cdot\nu(\xi))^{1/2}}\Big).\tag{4.21}$$

Finally, we deal with $|\mathrm{W}_{ne}|$ and $|\mathrm{W}_{nb}|$. We start by computing

$$\nabla_\xi \Big\{ \frac{\nu \circ \Xi(s) \cdot (\nu(\xi) + \omega)}{1 + \omega \cdot \nu(\xi)} \Big\} = \frac{1}{\langle \xi \rangle (1 + \omega \cdot \nu(\xi))} [\nu \circ \Xi(s) - (\nu(\xi) \cdot \nu \circ \Xi(s)) \nu(\xi)] - \frac{\nu \circ \Xi(s) \cdot \big(\nu(\xi) + \omega\big)}{\langle \xi \rangle (1 + \omega \cdot \nu(\xi))^2} [\omega - (\nu(\xi) \cdot \omega) \nu(\xi)].$$

This is a vector valued function which can be decomposed with respect to the moving "frame" made of the three directions ω, $\nu \circ \Xi(s) + \omega$ and $\nu(\xi) + \omega$. This gives rise to

$$\begin{aligned} &\nabla_\xi \Big\{ \frac{\nu \circ \Xi(s) \cdot (\nu(\xi) + \omega)}{1 + \omega \cdot \nu(\xi)} \Big\} \\ &\quad = - \frac{1}{\langle \xi \rangle} \frac{1 + \omega \cdot \nu \circ \Xi(s)}{1 + \omega \cdot \nu(\xi)} \omega + \frac{1}{\langle \xi \rangle} \frac{1}{1 + \omega \cdot \nu(\xi)} (\nu \circ \Xi(s) + \omega) \\ &\qquad + \frac{1}{\langle \xi \rangle} \Big(\frac{1 + \omega \cdot \nu \circ \Xi(s)}{1 + \omega \cdot \nu(\xi)} - \frac{(\nu(\xi) + \omega) \cdot (\omega + \nu \circ \Xi(s))}{(1 + \omega \cdot \nu(\xi))^2} \Big) (\nu(\xi) + \omega). \end{aligned}$$

The three vectors ω, $\nu \circ \Xi(s) + \omega$ and $\nu(\xi) + \omega$ are clearly uniformly bounded (by 2) as functions of (s, ω, ξ). When doing the above decomposition, we can observe that the coefficient in factor of ω is small (at least smaller than what appears at first sight) due to various cancellations that are revealed during its decomposition. On the other hand, the sizes of $\nu \circ \Xi(s) + \omega$ and $\nu(\xi) + \omega$ can be estimated through Lemma 4.7. Briefly, from (3.6), we can deduce that

$$|\mathrm{W}_{n\star}(r, s, \omega, \xi)| \leq \frac{3\sqrt{2}}{2\pi} \frac{(s - r)}{\langle \xi \rangle} \frac{(1 + \omega \cdot \nu \circ \Xi(s))^{1/2}}{1 + \omega \cdot \nu(\xi)} \quad \text{for all } \star \in \{e, b\}. \tag{4.22}$$

4.2. Weighted integrals

The goal of this subsection is to estimate the contributions provided by the $\mathrm{D}^{\mathrm{a}}_\star$ with $\star \in \{0, h, l, ne, nb\}$. This will be done in separate paragraphs, one for each $\mathrm{D}^{\mathrm{a}}_\star$. Before starting, we would like to accurately define the scope of our discussion. Indeed, different courses of action are possible when studying the $\mathrm{D}^{\mathrm{a}}_\star$.

In the perspective of continuation criteria, one might attempt to minimize the powers of $\langle \mathrm{P} \rangle_\infty$ needed to control the $\mathrm{D}^{\mathrm{a}}_\star$. As in [24, 28, 29], it seems that additional conditions (to be identified) are needed to recover the global existence. This interesting option is not pursued here.

On the other hand, to fit in with Proposition 4.1, one can insist on the role of S_0 when looking at the momentum increment. This is the path that we follow below. However, when doing this, a cautionary note is in order. This is because the functional $\mathcal{N}$ (and thereby S_0) involves different types of norms: first, the sup-norm concerning f_0; and secondly, the Sobolev norm $W^{1,\bar{p}}$ in the case of $(\mathrm{E}_0, \mathrm{B}_0)$. Let us explain the origin of this distinction:

- The handling of the $\mathrm{D}^{\mathrm{a}}_\star$ with $\star \in \{0, l, ne, nb\}$ does not prove to be demanding in terms of regularity. It will only require the use of $\| f_0 \|_0$ and $\mathcal{E}_0$.
- The manipulation of $\mathrm{D}^{\mathrm{a}}_h$ could be based just on a sup-norm estimate concerning E_h. It is the transcription of such uniform bound in terms of the initial data $(\mathrm{E}_0, \mathrm{B}_0)$ that

generates the implementation of $W^{1,\bar p}$. By the way, note that the use of $\bar p \in]3/2,3[$ instead of $\bar p = +\infty$ (or even higher levels of regularity) is a subtle refinement that will be clarified in Section 4.2.2.

In other words, the focus is on the *minimal regularity* required on the complete *initial data* U_0 in order to control the quantities $\mathrm{D}^{\mathrm{a}}_\star$. In doing so, for the sake of simplicity, we have highlighted in Proposition 4.1 the role of the sole parameter S_0, which serves in fact to cover different aspects. For instance, from (1.9), we get that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle\, \mathrm{f}_0(x,\xi)\,dx\,d\xi \lesssim \langle \mathrm{P}\rangle_0^7\,\mathrm{S}_0, \quad \langle \mathrm{P}\rangle_0 := (1+\mathrm{P}_0^2)^{1/2}, \tag{4.23}$$

where the symbol '$\lesssim$' is for '$\le C$', with some (universal) constant C not depending on P_0 or S_0. On the other hand, the Sobolev embedding theorem (which holds since $\bar p < 3$) ensures that $\mathrm{E}_0 \in L^{\tilde p}$ for some $\tilde p > 3$. By interpolation, we get that $W^{1,\bar p} \hookrightarrow L^2$. Now, since $\mathcal{N}(\mathrm{U}_0) \le \mathrm{S}_0$, we have

$$\int_{\mathbb{R}^3} |\mathrm{E}_0(x)|^2\,dx + \int_{\mathbb{R}^3} |\mathrm{B}_0(x)|^2\,dx \lesssim \mathrm{S}_0^2. \tag{4.24}$$

Coming back to (2.4) and using (4.4), it follows that (for fixed P_0 and small S_0)

$$\mathcal{E}_0 \lesssim \langle \mathrm{P}\rangle_0^7\,\mathrm{S}_0 + \mathrm{S}_0^2 \lesssim \mathrm{S}_0. \tag{4.25}$$

It is clear that the S_0 of (4.23) comes from the sup-norm of f_0, while the S_0 of (4.24) is issued from the $W^{1,\bar p}$-norm of $(\mathrm{E}_0,\mathrm{B}_0)$. These contributions are mixed (with different powers) at the level of (4.25). To avoid having to introduce too much material, our decision is to not make the various origins of S_0 apparent in the final statement. However, the interested reader can easily trace S_0 in the forthcoming analysis.

4.2.1. Study of $\mathbf{D}^{\mathrm{a}}_0$. By definition, $\mathrm{D}^{\mathrm{a}}_0$ is built on U_0. It is therefore a known quantity. Still, it is interesting to estimate $\mathrm{D}^{\mathrm{a}}_0$ to see how this works and to have access to its time behavior.

Lemma 4.8 (Control of $\mathrm{D}^{\mathrm{a}}_0$). *For all $p \in]2,+\infty]$, we have*

$$\begin{aligned}\mathrm{D}^{\mathrm{a}}_0(t) &= \int_0^t\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |\mathrm{W}_0(s,\omega,\xi)|\,\mathrm{f}_0(X(s)+s\omega,\xi)\,ds\,d\omega\,d\xi \\ &\lesssim \mathrm{S}_0^{1-1/p}\,\mathcal{E}_0^{1/p}\,t^{2-3/p}(1+\mathrm{P}_0^2)^{3/2-2/p}.\end{aligned} \tag{4.26}$$

Proof. Let q be the conjugate index of p. Note that

$$\begin{aligned}\mathrm{D}^{\mathrm{a}}_0(t) = \int_0^t\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} & \frac{s^{-2/p}|\mathrm{W}_0(s,\omega,\xi)|}{\langle\xi\rangle^{1/p}(1+\omega\cdot\nu\circ\Xi(s))^{1/p}} \\ &\times J(0,s,\omega)^{1/p}\,\langle\xi\rangle^{1/p}\,\mathrm{f}_0(X(s)+s\omega,\xi)\,ds\,d\omega\,d\xi.\end{aligned}$$

As explained in Section 4.1.2, see (4.16), we can assert that

$$\begin{aligned}&\| J^{1/p}\langle\xi\rangle^{1/p}\,\mathrm{f}_0(X(s)+s\omega,\xi)\|_{L^p([0,t]\times\mathbb{S}^2\times\mathbb{R}^3)} \\ &\qquad \le \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle\xi\rangle\,\mathrm{f}_0(x,\xi)^p\,dx\,d\xi\Big)^{1/p} \le \mathrm{S}_0^{1-1/p}\,\mathcal{E}_0^{1/p}.\end{aligned}$$

By Hölder's inequality, exploiting (4.20) and the condition $p > 2$ for the second term in the right-hand side of (4.20), we find that

$$\mathrm{D}_0^{\mathrm{a}}(t) \lesssim \mathrm{S}_0^{1-1/p}\, \mathcal{E}_0^{1/p} \Big(\int_0^t \int_{\mathbb{S}^2} \int_{|\xi| \leq \mathrm{P}_0} \frac{s^{q-2q/p} \langle \xi \rangle^{-q/p}}{(1+\omega \cdot \nu \circ \Xi(s))^{q/p}}\, ds\, d\omega\, d\xi \Big)^{1/q}$$
$$+ \mathrm{S}_0^{1-1/p}\, \mathcal{E}_0^{1/p} \Big(\int_0^t \int_{\mathbb{S}^2} \int_{|\xi| \leq \mathrm{P}_0} \frac{s^{q-2q/p} \langle \xi \rangle^{-q/p}}{(1+\omega \cdot \nu(\xi))^{q/p}}\, ds\, d\omega\, d\xi \Big)^{1/q}.$$

We first integrate with respect to ω. Since $q/p < 1$ (since again $p > 2$), from Lemma 4.6, we have

$$\mathrm{D}_0^{\mathrm{a}}(t) \lesssim \mathrm{S}_0^{1-1/p} \mathcal{E}_0^{1/p} \Big(\int_0^t s^{q-2q/p}\, ds \Big)^{1/q} \Big(\int_{|\xi| \leq \mathrm{P}_0} \langle \xi \rangle^{-q/p}\, d\xi \Big)^{1/q}.$$

Since $q - 2q/p > 0$, we end up with (4.26). ∎

4.2.2. Study of $\mathrm{D}_h^{\mathrm{a}}$. Note that

$$0 \leq \mathrm{D}_h^{\mathrm{a}} := \int_0^t |\nu \circ \Xi(s)|\, |\mathrm{E}_h(s, X(s))|\, ds \leq \int_0^t |\mathrm{E}_h(s, X(s))|\, ds, \quad t \leq T.$$

As already noted, the field $(\mathrm{E}_0, \mathrm{B}_0)$ has an impact on all the $\mathrm{D}_\star$ as well as X because the Lorentz force F is built with (3.9), where A_h (and therefore E_0 and B_0) is activated. It is particularly interesting to further examine its influence on $\mathrm{D}_h^{\mathrm{a}}$. There are two ways of thinking.

The linear viewpoint. That is concentrating on the only role of E_h. This method could be based on the following observations:

(a) In the local (in time) version of Proposition 4.1, it turns out that the restriction on $(\mathrm{E}_0, \mathrm{B}_0)$ may be exchanged with the mild assumption

$$\mathrm{E}_h \in L^1_{\mathrm{loc}}(\mathbb{R}_+; L^\infty(\mathbb{R}^3)). \tag{4.27}$$

(b) The field E_h allows to absorb the main contribution brought by $(\mathrm{E}_0, \mathrm{B}_0)$. The wave equation inside (3.3) is *linear* and completely *decoupled* from (1.2)–(1.3). Moreover, the information (4.27) is available for a whole range of bounded initial data $(\mathrm{E}_0, \mathrm{B}_0)$. The $W^{1,\bar{p}}$-condition is not necessarily (and also not sufficient) for that.

It would be enough to deal with (4.27), but this would be disappointing in terms of the Cauchy problem for the RVM system. Moreover, that would ignore a subtle nuance arising between the time integration of $\mathrm{E}_h(s, X(s))$ and the one of $\mathrm{E}_h(s, x)$. First, recall that (4.27) is not easy to find [34]. In particular, the endpoint Strichartz estimate

$$\|\mathrm{E}_h\|_{L^2_t L^\infty_x} \lesssim \|\mathrm{E}_0\|_{\dot{H}^1} + \|\nabla_x \times \mathrm{B}_0\|_{L^2} \tag{4.28}$$

is known to be false [21]. Furthermore, for data E_0 and B_0 in $W^{1,\bar{p}}(\mathbb{R}^3)$, it is not clear that the integral of E_h along *any* space-time curve makes sense. At this low level of regularity, the meaning of D_h cannot be based solely on the properties of E_h. We have to change the perspective.

The nonlinear viewpoint. This means to look at the expression $\mathrm{D}_h^{\mathrm{a}}$ as a nonlinear functional, pursuant to the influence of X. This approach has a clear advantage. The counter-examples to the inequality (4.28) are exhibited by concentrating solutions along the light cone. But the special curve $\{(s, X(s)) : s \in [0,t]\}$ intersects the light cone transversally, and therefore the time integration of E_h when computing $\mathrm{D}_h^{\mathrm{a}}$ reduces this alignment effects. This (relativistic) feature is a key ingredient because it allows to exploit (as in [4, 16, 19, 22]) the different speeds of propagation between the slow particles and the fields (which propagate at the speed of light). It is crucial here to make sense of D_h in the context of (1.7).

Lemma 4.9 (Control of $\mathrm{D}_h^{\mathrm{a}}$). *We have*

$$\mathrm{D}_h^{\mathrm{a}} \lesssim \mathrm{S}_0(t+1)\langle \mathrm{P}\rangle_\infty(t)^{2/3}. \tag{4.29}$$

Proof. Let us turn to another interpretation of D_h. The field E_h is given by Kirchhoff's formula

$$\begin{aligned} \mathrm{E}_h(t,x) &= t\mathcal{M}_t(\nabla_x \times \mathrm{B}_0) + \partial_t(t\mathcal{M}_t(\mathrm{E}_0)) \\ &= t\mathcal{M}_t(\nabla_x \times \mathrm{B}_0) + t\mathcal{M}_t(\omega\cdot\nabla_x \mathrm{E}_0) + \mathcal{M}_t(\mathrm{E}_0), \end{aligned}$$

where, for $t \in \mathbb{R}_+$, we have introduced the mean operator $\mathcal{M}_t : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ defined by

$$\mathcal{M}_t(k)(x) := \frac{1}{4\pi}\int_{\mathbb{S}^2} k(x+t\omega)\,d\omega, \qquad |||\mathcal{M}_t|||_{\mathcal{L}(L^\infty)} \le 1.$$

After substitution, this means that

$$\begin{aligned} \mathrm{D}_h^{\mathrm{a}} \lesssim \sum_{i=1}^{3}\int_0^t\int_{\mathbb{S}^2} s(|\partial_{x_i}\mathrm{E}_0| + |\partial_{x_i}\mathrm{B}_0|)(s, X(s)+s\omega)\,ds\,d\omega \\ + \int_0^t\int_{\mathbb{S}^2} |\mathrm{E}_0|(s, X(s)+s\omega)\,ds\,d\omega. \end{aligned} \tag{4.30}$$

First, consider the contribution coming from $\partial_{x_i}\mathrm{E}_0$ (do the same with $\partial_{x_i}\mathrm{B}_0$). Since E_0 is selected in $W^{1,\bar p}$, we know that $\partial_{x_i}\mathrm{E}_0 \in L^{\bar p}$. Let $\bar q := \bar p/(\bar p - 1)$ be the Hölder conjugate of $\bar p$. By Hölder's inequality, we can assert that

$$\begin{aligned} &\int_0^t\int_{\mathbb{S}^2} s\,|\partial_{x_i}\mathrm{E}_0|(s, X(s)+s\omega)\,ds\,d\omega \\ &\quad\lesssim \Big(\int_0^t\int_{\mathbb{S}^2} J\,|\partial_{x_i}\mathrm{E}_0|^{\bar p}(s, X(s)+s\omega)\,ds\,d\omega\Big)^{1/\bar p}\Big(\int_0^t\int_{\mathbb{S}^2}\frac{s^{\bar q}}{J^{\bar q/\bar p}}\,ds\,d\omega\Big)^{1/\bar q} \\ &\quad\lesssim \mathrm{S}_0\,\|\mathrm{E}_0\|_{W^{1,\bar p}}\Big(\int_0^t\int_{\mathbb{S}^2}\frac{s^{\bar q - 2\bar q/\bar p}}{(1+\omega\cdot\nu\circ\Xi(s))^{\bar q/\bar p}}\,ds\,d\omega\Big)^{1/\bar q}. \end{aligned}$$

To estimate the right-hand side, we start by applying Lemma 4.6 with $1 \le \delta = \bar q/\bar p \le 2$; then, we use (4.13) and finally, to obtain the time integrability near $s = 0$, it suffices to note that $\bar q - 2\bar q/\bar p > -1$ (since $3/2 < \bar p$). This furnishes

$$\Big(\int_0^t\int_{\mathbb{S}^2}\frac{s^{\bar q - 2\bar q/\bar p}}{(1+\omega\cdot\nu\circ\Xi(s))^{\bar q/\bar p}}\,ds\,d\omega\Big)^{1/\bar q} \lesssim t^{2-3/\bar p}\langle \mathrm{P}\rangle_\infty(t)^{4/\bar p - 2}.$$

Since $2-3/\bar{p}<1$ and $4/\bar{p}-2\le 2/3$ when $3/2<\bar{p}\le 2$, we find the right-hand side of (4.29). Secondly, we look at the contribution brought inside (4.30) by E_0. The difficulty is that "s" is no more in factor. But this lost may be compensated by extra integrability concerning E_0.

Indeed, the Sobolev embedding theorem gives $\mathrm{E}_0\in L^{\tilde{p}}$ for some $\tilde{p}>3$. Let $\tilde{q}:=\tilde{p}/(\tilde{p}-1)$ be the Hölder conjugate of $\tilde{q}$ so that $2\tilde{q}/\tilde{p}<1$. By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{S}^2}|\mathrm{E}_0|(s,X(s)+s\omega)\,ds\,d\omega\\
&\qquad\lesssim\Big(\int_0^t\int_{\mathbb{S}^2}J\,|\mathrm{E}_0|^{\tilde{p}}(s,X(s)+s\omega)\,ds\,d\omega\Big)^{1/\tilde{p}}\Big(\int_0^t\int_{\mathbb{S}^2}J^{-\tilde{q}/\tilde{p}}\,ds\,d\omega\Big)^{1/\tilde{q}}\\
&\qquad\lesssim \mathrm{S}_0\|\mathrm{E}_0\|_{L^{\tilde{p}}}\Big(\int_0^t\int_{\mathbb{S}^2}\frac{s^{-2\tilde{q}/\tilde{p}}}{(1+\omega\cdot\nu\circ\Xi(s))^{\tilde{q}/\tilde{p}}}\,ds\,d\omega\Big)^{1/\tilde{q}}\\
&\qquad\lesssim \mathrm{S}_0\|\mathrm{E}_0\|_{W^{1,\bar{p}}}\,t^{-2/\tilde{p}+1/\tilde{q}}\qquad\lesssim \mathrm{S}_0\,(t+1).
\end{aligned}$$

Again, this is consistent with (4.29). ■

4.2.3. Study of $\mathbf{D}^a_l$. Observe that $\mathrm{W}_l(s,\cdot)$ is an odd function. Thus, there is no benefit from the sign condition on f when studying D^a_l.

Lemma 4.10 (Control of $\mathrm{D}^{\mathrm{a}}_l$). *For all $p\in\,]3,+\infty]$, we have*

$$\begin{aligned}
(4.31)\qquad \mathrm{D}^a_l(t)&=\int_0^t\Big(\int_r^t\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\mathrm{W}_l(s,\omega,\xi)\,\mathrm{f}(r,X(s)+(s-r)\omega,\xi)\,ds\,d\omega\,d\xi\Big)\,dr\\
&\lesssim \mathrm{S}_0^{1-1/p}\mathcal{E}_0^{1/p}\,t^{1-3/p}\int_0^t\langle\mathrm{P}\rangle_\infty(r)^{3-4/p}\,dr.
\end{aligned}$$

Proof. The inequalities (4.20) and (4.21) are similar, except that $(s-r)$ does not appear in factor in the right-hand side of (4.21). Due to this additional difficulty, there are some nuances in comparison to what has been done in Section 4.2.1. From (4.21), as soon as $p>2$, we have

$$\begin{aligned}
\mathrm{D}^a_l(t)\lesssim&\int_0^t\Big(\int_r^t\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\frac{(J^{1/p}\langle\xi\rangle^{1/p}\,\mathrm{f})(r,X(s)+(s-r)\omega,\xi)}{(s-r)^{2/p}(1+\omega\cdot\nu\circ\Xi(s))^{1/p}\langle\xi\rangle^{1/p}}\,ds\,d\omega\,d\xi\Big)\,dr\\
&+\int_0^t\Big(\int_r^t\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\frac{(J^{1/p}\langle\xi\rangle^{1/p}\,\mathrm{f})(r,X(s)+(s-r)\omega,\xi)}{(s-r)^{2/p}(1+\omega\cdot\nu(\xi))^{1/2}\langle\xi\rangle^{1/p}}\,ds\,d\omega\,d\xi\Big)\,dr.
\end{aligned}$$

We apply Hölder's inequality to obtain

$$\begin{aligned}
\mathrm{D}^a_l(t)\lesssim\ &\mathrm{S}_0^{1-1/p}\mathcal{E}_0^{1/p}\int_0^t\Big(\int_r^t\int_{\mathbb{S}^2}\int_{|\xi|\le\mathrm{P}(r)}\frac{ds\,d\omega\,d\xi}{(s-r)^{2q/p}(1+\omega\cdot\nu\circ\Xi(s))^{q/p}\langle\xi\rangle^{q/p}}\Big)^{1/q}dr\\
&+\mathrm{S}_0^{1-1/p}\mathcal{E}_0^{1/p}\int_0^t\Big(\int_r^t\int_{\mathbb{S}^2}\int_{|\xi|\le\mathrm{P}(r)}\frac{ds\,d\omega\,d\xi}{(s-r)^{2q/p}(1+\omega\cdot\nu(\xi))^{q/2}\langle\xi\rangle^{q/p}}\Big)^{1/q}dr.
\end{aligned}$$

Knowing that $q/p<1$ and $q/2<1$, we can integrate with respect to ω through Lemma 4.6. Then we have to take $p>3$ (so that $2q/p<1$) in order to be sure that the integral with respect to ds is convergent, giving rise to (4.31). ■

4.2.4. Study of $\mathrm{D}^{\mathrm{a}}_{ne}$ and $\mathrm{D}^{\mathrm{a}}_{nb}$. The expressions $\mathrm{D}^{\mathrm{a}}_{ne}$ and $\mathrm{D}^{\mathrm{a}}_{nb}$, with $\star \in \{e, b\}$ can be viewed as bilinear forms. Indeed, with $\mathrm{W}^{\mathrm{a}}_{n\star} := |\mathrm{W}_{n\star}|$, where $\mathrm{W}_{n\star}$ is as in (3.6), together with the conventions

$$\mathrm{G}_e := |\mathrm{E}| \quad \text{and} \quad \mathrm{G}_b := |\mathrm{B}|,$$

we have to deal with

$$\mathrm{D}^{\mathrm{a}}_{n\star}(t) = \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \langle \mathrm{P} \rangle_\infty(r)} [(J^{-1/2}\, \mathrm{W}^{\mathrm{a}}_{n\star}\, \mathrm{f})(J^{1/2}\, \mathrm{G}_\star)](r, X(s) + (s-r)\,\omega, \xi)\, ds\, d\omega\, d\xi \Big)\, dr.$$

Lemma 4.11 (Control of $\mathrm{D}^{\mathrm{a}}_{n\star}$). *For all $p \in\,]3, +\infty]$, we have*

$$\mathrm{D}^{\mathrm{a}}_{n\star}(t) \lesssim \mathrm{S}_0^{1-1/(2p)}\, \mathcal{E}_0^{1/2+1/(2p)}\, t^{1/2-3/(2p)} \int_0^t \langle \mathrm{P} \rangle_\infty(r)^{4-3/p}\, dr. \tag{4.32}$$

Proof. By the Cauchy–Schwarz inequality, since $\mathrm{G}_\star(r, \cdot) \in L^2(\mathbb{R}^3)$ with a bound which can be viewed as coming from (2.5), we have

$$\begin{aligned} \mathrm{D}^{\mathrm{a}}_{n\star}(t) \lesssim\ & \mathrm{S}_0^{1/2} \mathcal{E}_0^{1/2} \int_0^t \langle \mathrm{P} \rangle_\infty(r)^{3/2} \\ & \times \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \langle \mathrm{P} \rangle_\infty(r)} (J^{-1}\, \mathrm{W}^2_{n\star}\, \mathrm{f})(r, X(s) + (s-r)\,\omega, \xi)\, ds\, d\omega\, d\xi \Big)^{1/2} dr. \end{aligned}$$

From (4.22), for $\star \in \{e, b\}$, we obtain that

$$J^{-1}\, \mathrm{W}^2_{n\star} \lesssim \langle \xi \rangle^{-2} (1 + \omega \cdot \nu(\xi))^{-2}.$$

Again, we select some $p > 3$. By Hölder's inequality, we get

$$\begin{aligned} & \int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \langle \mathrm{P} \rangle_\infty(r)} (J^{-1}\, \mathrm{W}^2_{n\star}\, \mathrm{f})(r, X(s) + (s-r)\,\omega, \xi)\, ds\, d\omega\, d\xi \\ & \lesssim \int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \langle \mathrm{P} \rangle_\infty(r)} \frac{J^{1/p}\, \langle \xi \rangle^{1/p}\, \mathrm{f}(r, X(s) + (s-r)\,\omega, \xi)}{(s-r)^{2/p} \langle \xi \rangle^{2+1/p} (1+\omega \cdot \nu(\xi))^2 (1+\omega \cdot \nu \circ \Xi(s))^{1/p}}\, ds\, d\omega\, d\xi \\ & \lesssim \mathrm{S}_0^{1-1/p} \mathcal{E}_0^{1/p} \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \langle \mathrm{P} \rangle_\infty(r)} \frac{\langle \xi \rangle^{-2q-q/p}}{(s-r)^{2q/p}}\, \frac{(1+\omega \cdot \nu \circ \Xi(s))^{-q/p}}{(1+\omega \cdot \nu(\xi))^{2q}}\, ds\, d\omega\, d\xi \Big)^{1/q}. \end{aligned}$$

On the one hand, from Lemma 4.5, we have

$$\langle \xi \rangle^{-2q-q/p} (1 + \omega \cdot \nu(\xi))^{-2q} \lesssim \langle \xi \rangle^{2q-(3q/p)} (1 + \omega \cdot \nu(\xi))^{-q/p}.$$

On the other hand, from Lemma 4.6 together with the condition $2q/p < 1$, we can assert that

$$\begin{aligned} & \int_{\mathbb{S}^2} (1 + \omega \cdot \nu(\xi))^{-q/p} (1 + \omega \cdot \nu \circ \Xi(s))^{-q/p}\, d\omega \\ & \qquad \lesssim \int_{\mathbb{S}^2} (1 + \omega \cdot \nu(\xi))^{-2q/p}\, d\omega + \int_{\mathbb{S}^2} (1 + \omega \cdot \nu \circ \Xi(s))^{-2q/p}\, d\omega \lesssim 1. \end{aligned}$$

This implies that

$$\Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi|\leq \langle \mathrm{P}\rangle_\infty(r)} \frac{\langle \xi\rangle^{-2q-q/p}}{(s-r)^{2q/p}} \, \frac{(1+\omega\cdot\nu\circ\Xi(s))^{-q/p}}{(1+\omega\cdot\nu(\xi))^{2q}}\, ds\, d\omega\, d\xi\Big)^{1/q} \lesssim t^{1-3/p} \langle \mathrm{P}\rangle_\infty(r)^{5-6/p}.$$

From there, it is easy to deduce (4.32). ∎

4.3. Proof of Proposition 4.1

In this section, we work with bounded values of $\mathrm{S}_0 \in \mathbb{R}_+^*$, which may go to 0. Let $\iota \in]0,1[$. We perform the change of time variable

$$\tau := \mathrm{S}_0^{1-\iota} t, \quad \langle \tilde{\mathrm{P}}\rangle_\infty(\tau) := \langle \mathrm{P}\rangle_\infty(\mathrm{S}_0^{\iota-1}\tau).$$

Let $p>3$. For small values of S_0, we can exploit (4.25) to replace $\mathcal{E}_0$ by S_0. With this in mind, we combine (4.14) with (4.26), (4.29), (4.31) and (4.32) to see that

$$\begin{aligned}\langle \tilde{\mathrm{P}}\rangle_\infty(\tau) \lesssim\ & 1+\mathrm{P}_0+\mathrm{S}_0^{3/p-1+\iota(2-3/p)}\tau^{2-3/p}(1+\mathrm{P}_0^2)^{3/2-2/p}\\ &+(\mathrm{S}_0^\iota\tau+\mathrm{S}_0)\langle \tilde{\mathrm{P}}\rangle_\infty(\tau)^{2/3}\\ &+\mathrm{S}_0^{3/p-1+\iota(2-3/p)}\tau^{(p-3)/p}\int_0^\tau \langle \tilde{\mathrm{P}}\rangle_\infty(r)^{3-4/p}\,dr\\ &+\mathrm{S}_0^{3/(2p)+\iota(p-3)/(2p)}\tau^{(p-3)/(2p)}\int_0^\tau \langle \tilde{\mathrm{P}}\rangle_\infty(r)^{4-3/p}\,dr.\end{aligned}$$

We can adjust $p>3$ sufficiently close to 3 to ensure that

$$0<3/p-1+\iota(2-3/p).$$

We select $\tau\in[0,\tau_0]$ with arbitrary $\tau_0>0$, and using Young's inequality, one can absorb the power $2/3$ of $\langle \tilde{\mathrm{P}}\rangle_\infty(\tau)$. Hence,

$$\langle \tilde{\mathrm{P}}\rangle_\infty(\tau) \lesssim 1+\int_0^\tau \langle \tilde{\mathrm{P}}\rangle_\infty(r)^4\,dr.$$

This estimate is compatible with the Bihari–LaSalle inequality.[2] It furnishes a time $\tilde{\tau}\leq\tau_0$ and a continuous increasing function $\tilde{\mathcal{F}}:[0,\tilde{\tau}]\to\mathbb{R}_+$ (not depending on ι) such that

$$\langle \tilde{\mathrm{P}}\rangle_\infty(\tau)\leq \tilde{\mathcal{F}}(\tau) \quad \text{for all } \tau\in[0,\tilde{\tau}],$$

or equivalently

$$\mathrm{P}(t)\leq \langle \mathrm{P}\rangle_\infty(t)\leq \tilde{\mathcal{F}}(\mathrm{S}_0^{1-\iota}t) \quad \text{for all } t\in[0,T[\,\cap\,[0,\mathrm{S}_0^{\iota-1}\tilde{\tau}]. \tag{4.33}$$

Passing to the limit ($\iota\to 0+$), with $c=\tilde{\tau}$ and $\mathcal{F}(\mathrm{S}_0;t):=\tilde{\mathcal{F}}(\mathrm{S}_0\,t)$, we recover (4.6).

[2] See https://en.wikipedia.org/wiki/Bihari-LaSalle_inequality.

5. Application

This section is still devoted to the study of (1.2)–(1.4). But from now on, we assume that the initial data $\mathrm{U}_0 \equiv \mathrm{U}_0^\varepsilon$ takes the form of (1.18) where $\mathrm{U}_a^\varepsilon(0,\cdot)$ is issued from a well-prepared approximate solution U_a^ε (in the sense of Definition 5.1 below). Thus, we consider a family of Cauchy problems indexed by $\varepsilon \in]0,1]$. From [38], for all $\varepsilon \in]0,1]$, there exists locally in time, say on the interval $[0, T^\varepsilon]$ with $T^\varepsilon \in \mathbb{R}_+^*$, a unique smooth solution $\mathrm{U} \equiv \mathrm{U}^\varepsilon$. The main goal is to show that we can extract a lower bound $\mathcal{T} \in \mathbb{R}_+^*$ such that $\mathcal{T} \le T^\varepsilon$ for all $\varepsilon \in]0,1]$. To this end, Proposition 4.1 is of no use. Indeed, as soon as $\mathrm{B}_a^\varepsilon \sim \varepsilon^{-1}$, the condition $\mathcal{N}(\mathrm{U}_0^\varepsilon) \lesssim 1$ is not uniformly satisfied when $\varepsilon \to 0$. However, the proof of Proposition 4.1 does not exploit a number of specificities that can be detected by working in the vicinity of U_a^ε. Far beyond [8, 30, 37], this will allow us to incorporate the dense, hot and strongly magnetized framework.

In Section 5.1, we furnish some preparatory material. In Section 5.2, we discuss the issue of energy estimates. This is an opportunity to identify the challenges posed by the introduction of U_a^ε. In Section 5.3, we prove Theorem 1.5.

5.1. Preliminary background

We start in Section 5.1.1 by generalizing the choice (1.17). In Section 5.1.2, we write the equations which are satisfied by the perturbation $U = (f, E, B)$. In Section 5.1.3, we specify the exact meaning of “compatible initial data”.

5.1.1. Well-prepared approximate solutions. The system built with (1.2) and the two equations on the left of (1.3) is denoted by $\mathcal{L}(\mathrm{U},\partial)\mathrm{U} = 0$. This quasilinear system (with integral source term J) is obtained by ignoring the compatibility conditions.

Definition 5.1 (Well-prepared approximate solutions). We say that the family $\{\mathrm{U}_a^\varepsilon\}_\varepsilon$ is a well-prepared approximate solution to (1.2)–(1.3) if the following hold:

- There exists a time $\mathcal{T}_a \in \mathbb{R}_+^*$ such that, for all $\varepsilon \in]0,1]$, we have

$$\mathrm{U}_a^\varepsilon = (\mathrm{f}_a^\varepsilon, \mathrm{E}_a^\varepsilon, \mathrm{B}_a^\varepsilon) \in C^1([0,\mathcal{T}_a] \times \mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^7). \tag{5.1}$$

- There exists a constant $\mathrm{P}_a \in \mathbb{R}_+^*$ such that, for all $\varepsilon \in]0,1]$, we have

$$\operatorname{supp} \mathrm{f}_a^\varepsilon \subset [0,\mathcal{T}_a] \times \mathbb{R}^3 \times B(0,\mathrm{P}_a). \tag{5.2}$$

- There exist positive constants S_a and H_a such that (in the sup-norm with respect to the domain $[0,\mathcal{T}_a] \times \mathbb{R}^3 \times \mathbb{R}^3$), for all $\varepsilon \in]0,1]$, we have the sup-norm estimates

$$\|\mathrm{f}_a^\varepsilon\|_\infty \le \mathrm{S}_a, \tag{5.3a}$$
$$\|\mathrm{E}_a^\varepsilon\|_\infty \le \mathrm{S}_a, \tag{5.3b}$$
$$\|\varepsilon \mathrm{B}_a^\varepsilon\|_\infty \le \mathrm{H}_a. \tag{5.3c}$$

- There exists a constant L_a such that, for all $\varepsilon \in]0,1]$, we have the Lipschitz estimate

$$\|\nabla_{t,x,\xi} \mathrm{f}_a^\varepsilon\|_\infty \le \mathrm{L}_a. \tag{5.4}$$

- Let E_h^ε be the solution to the linear wave equation (3.3) with initial data
$$\mathrm{E}^\varepsilon_{h|t=0} = \mathrm{E}_a^\varepsilon(0,\cdot), \quad \partial_t \mathrm{E}^\varepsilon_{h|t=0} = \nabla_x \times \mathrm{B}_a^\varepsilon(0,\cdot).$$
There exists a positive constant $\mathrm{S}_h \in \mathbb{R}_+^*$ such that, for all $\varepsilon \in]0,1]$, we have
$$\|\mathrm{E}_h^\varepsilon\|_{L^1([0,\mathcal{T}_a];L_x^\infty)} \le \mathrm{S}_h. \tag{5.5}$$
- We introduce the remainder $\mathrm{R}_a^\varepsilon := \mathcal{L}(\mathrm{U}_a^\varepsilon, \partial)\mathrm{U}_a^\varepsilon$. There exist positive constants S_r and N_r such that, for all $\varepsilon \in]0,1]$, we have
$$\|\mathrm{R}_a^\varepsilon\|_\infty \le \mathrm{S}_r, \tag{5.6a}$$
$$\|\mathrm{R}_a^\varepsilon\|_{L^2} \le \mathrm{N}_r\, \varepsilon, \tag{5.6b}$$
where $\|\mathrm{R}_a^\varepsilon\|_{L^2}$ stands for the following L^2-norm:
$$\|\mathrm{R}_a^\varepsilon\|_{L^2} := \Big(\|\mathrm{R}_{\mathrm{f}a}^\varepsilon\|^2_{L^2([0,\mathcal{T}_a]\times\mathbb{R}^3\times\mathbb{R}^3)} + \|\mathrm{R}_{\mathrm{E}a}^\varepsilon\|^2_{L^2([0,\mathcal{T}_a]\times\mathbb{R}^3)} + \|\mathrm{R}_{\mathrm{B}a}^\varepsilon\|^2_{L^2([0,\mathcal{T}_a]\times\mathbb{R}^3)} \Big)^{1/2}.$$
- For all $(\varepsilon, x) \in]0,1] \times \mathbb{R}^3$, at the time $t = 0$, we impose the compatibility conditions
$$\nabla_x \cdot \mathrm{E}_a^\varepsilon(0,x) = \int_{\mathbb{R}^3} \mathrm{f}_a^\varepsilon(0,x,\xi)\, d\xi - \rho^\varepsilon(x), \tag{5.7a}$$
$$\nabla_x \cdot \mathrm{B}_a^\varepsilon(0,x) = 0. \tag{5.7b}$$

Let us come back to the situation (1.17), where B_e and M^ε are adjusted as in (1.12) and (1.13). This means to deal with $\tilde{\mathrm{U}}_a^\varepsilon$, which gives rise to $\tilde{\mathrm{E}}_h^\varepsilon$ and $\tilde{\mathrm{R}}_a^\varepsilon$. The four conditions (5.1)–(5.4) are clearly satisfied for all $\mathcal{T}_a \in \mathbb{R}_+^*$. Due to the last condition in (1.12), we simply find that $\tilde{\mathrm{E}}_h^\varepsilon \equiv 0$, so that (5.5) is evident. In the same vein, we have
$$\tilde{\mathrm{R}}_a^\varepsilon := \mathcal{L}(\tilde{\mathrm{U}}_a^\varepsilon, \partial)\tilde{\mathrm{U}}_a^\varepsilon \equiv 0,$$
so that (5.6) is achieved. The condition (5.7a) for $\tilde{\mathrm{E}}_a^\varepsilon \equiv 0$ can be guaranteed by adjusting ρ^ε as indicated in (1.16), while the condition (5.7b) for $\tilde{\mathrm{B}}_a^\varepsilon = \varepsilon^{-1}\,\mathrm{B}_e$ is a consequence of the second condition in (1.12). In brief, *the family* $\{\tilde{\mathrm{U}}_a^\varepsilon\}_\varepsilon$ *is a well-prepared approximate solution.*

For a better understanding, Definition 5.1 must be supplemented with a number of remarks. Indeed, (5.1), (5.2) and (5.3a) are just extensions of (1.9). But the other constraints serve other purposes which must be clarified. We discuss the rest of (5.3b), (5.3c) and (5.4)–(5.7) in separate paragraphs.

Link between the notion of well-prepared data and the Lipschitz estimate on f_a^ε. By definition, we have the decomposition $\mathrm{R}_a^\varepsilon = (\mathrm{R}_{\mathrm{f}a}^\varepsilon, \mathrm{R}_{\mathrm{E}a}^\varepsilon, \mathrm{R}_{\mathrm{B}a}^\varepsilon) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$, with, in particular,
$$\mathrm{R}_{\mathrm{f}a}^\varepsilon := \partial_t \mathrm{f}_a^\varepsilon + v(\xi)\cdot\nabla_x \mathrm{f}_a^\varepsilon + \mathrm{E}_a^\varepsilon \cdot \nabla_\xi \mathrm{f}_a^\varepsilon + (v(\xi)\times\mathrm{B}_a^\varepsilon)\cdot\nabla_\xi \mathrm{f}_a^\varepsilon. \tag{5.8}$$
Then, from (5.2), it is easy to see that
$$\operatorname{supp} \mathrm{R}_{\mathrm{f}a}^\varepsilon \subset [0,\mathcal{T}_a] \times \mathbb{R}^3 \times B(0,\mathrm{P}_a). \tag{5.9}$$

Then, from (5.3b), (5.4) and (5.6a), we can also infer that

$$\|(\nu(\xi) \times \mathrm{B}_a^\varepsilon) \cdot \nabla_\xi \mathrm{f}_a^\varepsilon\|_\infty \leq \mathrm{S}_r + (2 + \mathrm{S}_a)\mathrm{L}_a. \tag{5.10}$$

This is not a consequence of (5.3c) and (5.4). This is possible only if B_a^ε and f_a^ε are adjusted accordingly. In particular, this should hold true at the time $t = 0$. This is the common notion of *well-prepared initial data*, see, for instance, [8, 30] or the books [25, 31]. In coherence with (5.4), where the L inside L_a is for Lipschitz, this prevents the emergence of rapid variations inside f_a^ε. Instead, oscillations can be (or are intended to be) introduced through f_0^ε.

Large magnetic fields. Strongly magnetized plasmas are available due to the weight ε which is put in factor of B_a^ε inside (5.3c), where the letter H inside H_a is for High. Again, the prototype of such behavior is $\varepsilon^{-1}\mathrm{B}_e$ inside $\tilde{\mathrm{U}}_a^\varepsilon$. From there, the construction of more elaborate approximate solutions is a subject in its own right. This is a way of revealing various physical phenomena. In the absence of coupling, if we concentrate only on the Vlasov part, this involves a WKB analysis which ties in with the recent advances [6, 7].

The impact of initial electromagnetic fields. The condition (1.18) may seem restrictive since we impose $(E_0^\varepsilon, B_0^\varepsilon) \equiv 0$. This is to forget that the presence at the time $t = 0$ of a non-zero electromagnetic component is considered as an integral part of the construction of U_a^ε. Indeed, the access to U_a^ε may be achieved in two steps. The first is to solve the Cauchy problem (3.3) with initial data $(\mathrm{E}_0, \mathrm{B}_0) \equiv (\mathrm{E}_a^\varepsilon, \mathrm{B}_a^\varepsilon)(0, \cdot)$ in order to get $(\mathrm{E}_h^\varepsilon, \mathrm{B}_h^\varepsilon)$; the second is to seek approximate solutions in the form $\mathrm{U}_a^\varepsilon = (\mathrm{f}_a^\varepsilon, \mathrm{E}_h^\varepsilon + \check{\mathrm{E}}_a^\varepsilon, \mathrm{B}_h^\varepsilon + \check{\mathrm{B}}_a^\varepsilon)$, with $(\check{\mathrm{E}}_a^\varepsilon, \check{\mathrm{B}}_a^\varepsilon)(0, \cdot) \equiv 0$. When doing this, there is not total freedom concerning the choice of $(\mathrm{E}_a^\varepsilon, \mathrm{B}_a^\varepsilon)(0, \cdot)$. Indeed, it is necessary to ensure the property (5.5). Now, for reasons explained in Section 4.2.2, this cannot be taken for granted under only L^2 or even H^1-controls on $(\mathrm{E}_a^\varepsilon, \mathrm{B}_a^\varepsilon)(0, \cdot)$.

About the remainder. The condition (5.6) gives meaning to the word "approximate". The smallness of the remainder is measured by the relatively mild constraint (5.6b).

About the compatibility conditions. In view of (1.5), the restrictions (5.7a) and (5.7b) seem unavoidable. But it is difficult and pointless to preserve the compatibility conditions when constructing approximate solutions U_a^ε. It is preferable to rather leave some freedom to the remainder R_a^ε and therefore to the choice of U_a^ε while, as will be seen, the exact solution U^ε is sure to propagate the initial compatibility conditions.

Changing T^ε if necessary, we can always assume that we work with $T^\varepsilon \leq \mathcal{T}_a$.

5.1.2. The equations for the perturbation. Recall the decomposition (1.19), where U has been put in the form $\mathrm{U} = \mathrm{U}_a^\varepsilon + U$ with $U = (f, E, B)$. The Lorentz force F acts on the charged particles through a contribution coming from U_a^ε and a part issued from the self-consistent electromagnetic field (E, B). We have $\mathrm{F} = \mathrm{F}_a^\varepsilon + F$ with

$$\mathrm{F}_a^\varepsilon := \mathrm{E}_a^\varepsilon(t, x) + \nu(\xi) \times \mathrm{B}_a^\varepsilon(t, x), \qquad F^\varepsilon \equiv F := E(t, x) + \nu(\xi) \times B(t, x). \tag{5.11}$$

After substituting U as in (1.19) inside (1.2)–(1.3), the Vlasov equation becomes

$$\partial_t f + \nu(\xi) \cdot \nabla_x f + (\mathrm{F}_a^\varepsilon + F) \cdot \nabla_\xi f + F \cdot \nabla_\xi f_a^\varepsilon = \mathrm{R}_{\mathrm{f}a}^\varepsilon, \tag{5.12}$$

and the Maxwell's equations reduce to

$$\partial_t E - \nabla_x \times B + \int_{\mathbb{R}^3} v(\xi) f(t,x,\xi)\, d\xi = \mathrm{R}^\varepsilon_{\mathrm{E}a}, \tag{5.13a}$$

$$\partial_t B + \nabla_x \times E = \mathrm{R}^\varepsilon_{\mathrm{B}a}. \tag{5.13b}$$

The system (5.12)–(5.13) is denoted by $\mathcal{L}^\varepsilon_a(U,\partial)U = 0$.

5.1.3. Compatible initial data. In view of (1.18), the initial data are adjusted as follows:

$$f_{|t=0} = f^\varepsilon_0, \quad E_{|t=0} = 0, \quad B_{|t=0} = 0. \tag{5.14}$$

Constraints inherited from (1.9), (1.5)–(5.7) and (5.3c) must be imposed on f^ε_0.

Definition 5.2 (Compatible family of initial data). We say that the family $\{f^\varepsilon_0\}_\varepsilon$ is compatible when the following five conditions are satisfied.

(a) *Regularity*:

$$f^\varepsilon_0 \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}) \quad \text{for all } \varepsilon \in\,]0,1]. \tag{5.15}$$

(b) *Uniform control on the size of the momentum support*:

$$\operatorname{supp} f^\varepsilon_0(\cdot) \subset \mathbb{R}^3 \times B(0,\mathrm{P}_0] \quad \text{for all } \varepsilon \in\,]0,1]. \tag{5.16}$$

(c) *Uniform sup-norm estimate*:

$$\| f^\varepsilon_0 \|_\infty := \sup_{(x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3} |f^\varepsilon_0|(x,\xi) \le \mathrm{S}_0 \quad \text{for all } \varepsilon \in\,]0,1]. \tag{5.17}$$

(d) *L^2-smallness*:

$$\| f^\varepsilon_0 \|_{L^2} \lesssim \mathrm{N}_0\, \varepsilon \quad \text{for all } \varepsilon \in\,]0,1]. \tag{5.18}$$

(e) *Compatibility conditions* (issued from U^ε_a):

$$\int_{\mathbb{R}^3} \mathrm{f}^\varepsilon_0(0,x,\xi)\, d\xi = 0 \quad \text{for all } (x,\varepsilon) \in \mathbb{R}^3 \times\,]0,1]. \tag{5.19}$$

In view of (1.18), (5.7) and (5.19), the compatibility conditions (1.5) are satisfied at time $t = 0$. They are propagated so that

$$\nabla_x \cdot \mathrm{E}^\varepsilon_a(t,x) + \nabla_x \cdot E(t,x) = \int_{\mathbb{R}^3} \mathrm{f}^\varepsilon_a(t,x,\xi)\, d\xi + \int_{\mathbb{R}^3} f(t,x,\xi)\, d\xi - \rho^\varepsilon(x), \tag{5.20a}$$

$$\nabla_x \cdot \mathrm{B}^\varepsilon_a(t,x) + \nabla_x \cdot B(t,x) = 0. \tag{5.20b}$$

We can test Definition 5.2 with $f^\varepsilon_0 \equiv \varepsilon f_0$, where $f_0 \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ does not depend on ε and is as indicated in (1.9) and (1.10) with $\rho \equiv 0$. We have obviously (a), (b) and (c), as well as (e). Moreover, we have (d) with $\mathrm{N}_0 = \| f_0 \|_{L^2}$. Thus, under these assumptions on f_0, we can assert that

The family $\{\varepsilon f_0\}_\varepsilon$ is compatible.

Now, we can apply Theorem 1.5 with $\mathrm{U}^\varepsilon_a \equiv 0$ and $f^\varepsilon_0 = \varepsilon f_0$. Then, just fix $\varepsilon = 1$ to recover the conclusions of Theorem 1.2. Theorem 1.5 can therefore be viewed as an extension of Theorem 1.2. Its proof implies a few subtleties in comparison to what has been done in Section 4.

5.2. Estimating the pertubation

In the perturbative context (5.12)–(5.13), it becomes more complicated to control f and (E, B). In Section 5.2.1, we comment on the structure of the characteristic flow, and then we deduce sup-norm estimates on $f(t,\cdot)$. In Section 5.2.2, we explain why $\mathcal{E}$ is of no use any more, and we follow an alternative path in order to manage the L^2-norm of $(E, B)(t,\cdot)$.

5.2.1. The oscillating characteristic flow. In the context of (1.19), the differential system (3.1) breaks down into

$$
\begin{cases} \dfrac{dX}{dt}(t,y,\eta) = v(\Xi), & X(0,y,\eta) = y, \\ \dfrac{d\Xi}{dt}(t,y,\eta) = \mathrm{F}_a^\varepsilon(t,X,\Xi) + F(t,X,\Xi), & \Xi(0,y,\eta) = \eta. \end{cases} \tag{5.21}
$$

In a first attempt, the impact of the self-consistent electromagnetic field (E, B) may be neglected. In particular, in the case of strongly magnetized plasmas (when $\mathrm{B}_a^\varepsilon \sim \varepsilon^{-1}$), the dominant part inside (5.21) may be thought as

$$
\begin{cases} \dfrac{dX_a^\varepsilon}{dt}(t,y,\eta) = v(\Xi_a^\varepsilon), & X_a^\varepsilon(0,y,\eta) = y, \\ \dfrac{d\Xi_a^\varepsilon}{dt}(t,y,\eta) = \mathrm{F}_a^\varepsilon(t,X_a^\varepsilon,\Xi_a^\varepsilon), & \Xi_a^\varepsilon(0,y,\eta) = \eta. \end{cases} \tag{5.22}
$$

The study of (5.22) when, for instance, $\mathrm{E}_a^\varepsilon = \mathrm{E}_a$ and $\mathrm{B}_a^\varepsilon = \mathrm{B}_a + \varepsilon^{-1}\mathrm{B}_e$ with a given field $(\mathrm{E}_a, \mathrm{B}_a)$ not depending on ε, is very informative. This provides WKB expansions for $(X_a^\varepsilon, \Xi_a^\varepsilon)$ which reveal the high complexity of the underlying motions [6,7]. The flow associated with (5.22) is strongly oscillating in both time, space and momentum; the directions $\Xi_a^\varepsilon/|\Xi_a^\varepsilon|$ are rapidly oscillating, but not $\langle\Xi_a^\varepsilon\rangle$. Let us consider the frozen version $\mathrm{FD}_a^\varepsilon$ of D, which is

$$
\mathrm{FD}_a^\varepsilon(t,y,\eta) := \int_0^t v \circ \Xi_a^\varepsilon(s,y,\eta) \cdot \mathrm{E}_a^\varepsilon(s, X_a^\varepsilon(s,y,\eta))\, ds.
$$

From (5.3b), we can see that

$$
|\mathrm{FD}_a^\varepsilon| \le t\mathrm{S}_a.
$$

This is already an indication that D should remain under control only where the impact of U can be taken into account. To this end, by analogy with (4.2) and (4.11), we define

$$
P(t) := \inf\bigl\{R \in \mathbb{R}_+ : f(t,x,\xi) = 0 \text{ for all } x \in \mathbb{R}^3 \text{ and all } \xi \in \mathbb{R}^3, \text{ with } R \le |\xi|\bigr\},
$$

as well as

$$
\langle P\rangle_\infty(t) := \sup_{s\in[0,t]} (1 + P(s)^2)^{1/2}. \tag{5.23}
$$

The solutions to the Vlasov equation (1.2) are constant along the characteristics, so that

$$
\begin{aligned} 0 &\le \mathrm{f}_a^\varepsilon(t,x,\xi) + f^\varepsilon(t,x,\xi) \\ &= \mathrm{f}_a^\varepsilon(0, X(-t,x,\xi), \Xi(-t,x,\xi)) + f_0^\varepsilon(X(-t,x,\xi), \Xi(-t,x,\xi)). \end{aligned}
$$

As a consequence, we can assert that

$$\|f^\varepsilon(t,\cdot)\|_\infty \lesssim \mathrm{S}_a + \mathrm{S}_0 \quad \text{for all } (t,\varepsilon) \in [0,T^\varepsilon] \times]0,1]. \tag{5.24}$$

5.2.2. Propagation of the L^2-norm. The system (1.2)–(1.3) is endowed with the conserved total energy $\mathcal{E}(t)$, see (2.4)–(2.5). In the strongly magnetized case, we find that $\mathcal{E}(t) = \mathcal{E}(0) \sim \varepsilon^{-1}$, and the use of $\mathcal{E}(t)$ does not help in the perspective of uniform estimates. Instead, we could consider the *relative energy* $\mathcal{E}(t)$ defined by

$$\mathcal{E}(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle f(t,x,\xi)\,dx\,d\xi + \frac{1}{2}\int_{\mathbb{R}^3} (|E|^2 + |B|^2)(t,x)\,dx.$$

Let us first examine what happens in the case of $\tilde{\mathrm{U}}_a^\varepsilon$, with $\tilde{\mathrm{U}}_a^\varepsilon$ as in (1.17).

Lemma 5.3. *Assume that* $\mathrm{U}_a^\varepsilon \equiv \tilde{\mathrm{U}}_a^\varepsilon$. *Then, for all* $t \in [0,T^\varepsilon]$, *we have* $\mathcal{E}(t) = \mathcal{E}(0)$.

Proof. The proof of this conservation of $\mathcal{E}$ may be achieved through direct computations based on (5.12)–(5.13). But it is more instructive to deduce it from the well-known conservation [33] of the total energy $\mathcal{E}(t)$. When $\mathrm{U}_a^\varepsilon \equiv \tilde{\mathrm{U}}_a^\varepsilon$, this gives rise to

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} &\langle\xi\rangle(\mathrm{M}+f)(t,x,\xi)\,dx\,d\xi + \frac{1}{2}\int_{\mathbb{R}^3} |E(t,x)|^2\,dx\\ &+ \frac{1}{2}\int_{\mathbb{R}^3} |\varepsilon^{-1}\mathrm{B}_e(x) + B(t,x)|^2\,dx\\ &\qquad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle(\mathrm{M}+f_0)(x,\xi)\,dx\,d\xi + \frac{1}{2}\int_{\mathbb{R}^3}|\varepsilon^{-1}\mathrm{B}_e(x)|^2\,dx.\end{aligned}$$

This is the same as

$$\mathcal{E}(t) + \varepsilon^{-1}\int_{\mathbb{R}^3} \mathrm{B}_e(x)\cdot B(t,x) = \mathcal{E}(0).$$

Since B_e is an irrotational field, it can be written as the gradient of a magnetic potential A_e. After integration by parts, this yields

$$\int_{\mathbb{R}^3} \mathrm{B}_e(x)\cdot B(t,x)\,dx = -\int_{\mathbb{R}^3} \mathrm{A}_e(x)\nabla_x\cdot B(t,x)\,dx. \tag{5.25}$$

But, for $\mathrm{B}_a^\varepsilon \equiv \varepsilon^{-1}\mathrm{B}_e$ with B_e as in (1.12), the Gauss law (5.20b) reduces to $\nabla_x\cdot B \equiv 0$, which leads to the expected result. ∎

To prove that $0 < \mathcal{T} \le T^\varepsilon$, it seems essential to compensate the large factor ε^{-1} which may occur inside (5.12) in front of $\nabla_\xi f$. To this end, the idea is to obtain some smallness information on U. As already mentioned, this cannot be achieved by passing through $\mathcal{E}$ (as in Section 4). This can also not be obtained by using $\mathcal{E}$. Let us briefly explain why. In view of (5.14), assuming instead of (5.18) that $\|f_0^\varepsilon\|_{L^1} \lesssim \varepsilon$, Lemma 5.3 would imply that $\mathcal{E}(t) \lesssim \varepsilon$. However, resorting to $\mathcal{E}$ is inadequate. There are two reasons for this:

- The quantity $\mathcal{E}$ is not conserved when dealing in a wider context than $\tilde{\mathrm{U}}_a^\varepsilon$, with U_a^ε as in Definition 5.1. This is due (in particular) to the influence of the remainder R_a^ε. In fact, the situation is even more problematic: under the relaxed condition (5.20b), the error term may be of size ε^{-1} because (5.25) does not hold (with B_e replaced by B_a^ε).

- The expression $\mathcal{E}$ is not exploitable because f represents a perturbation and, as such (in contrast to f), it is not necessarily sign definite.

Instead of looking at $\mathcal{E}$, we perform usual energy estimates at the level of the system (5.12)–(5.13). With this in mind, we introduce the square of the L^2-energy of $U(t)$, that is,

$$\mathbb{E}(t) := \|U(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,\xi)^2\,dx\,d\xi + \int_{\mathbb{R}^3} |E(t,x)|^2\,dx + \int_{\mathbb{R}^3} |B(t,x)|^2\,dx.$$

Lemma 5.4 (L^2-estimate on the perturbation U). *For all $t \le T^\varepsilon$, we have*

$$\mathbb{E}(t) \le \varepsilon^2(\mathrm{N}_0^2 + \mathrm{N}_r^2)\exp\Big(t + 4(\mathrm{L}_a + 1)\int_0^t \langle P\rangle_\infty(s)^{3/2}\,ds\Big). \tag{5.26}$$

Proof. Multiply (5.12) by $2f$ and integrate on the phase space; multiply (5.13a) and (5.13b) by, respectively, $2E$ and $2B$ and integrate with respect to dx. In this way, we find that

$$\partial_t\mathbb{E}(t) = -2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(f\,F\cdot\nabla_\xi \mathrm{f}^\varepsilon{}_a + E\cdot v(\xi)\,f)\,dx\,d\xi + 2\,\langle U, \mathrm{R}_a^\varepsilon\rangle_{L^2\times L^2}.$$

From (5.4) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(f\,F\cdot\nabla_\xi \mathrm{f}_a^\varepsilon + E\cdot v(\xi)\,f)\,dx\,d\xi\Big| &\le (\mathrm{L}_a+1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(|E|+|B|)(s,x)\,|f(s,x,\xi)|\,dx\,d\xi\\ &\le \sqrt{2}\,(\mathrm{L}_a+1)\Big(\int_{|\xi|\le P(s)} d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}^3}(|E|^2+|B|^2)(s,x)\,dx\Big)^{1/2}\mathbb{E}(s)^{1/2}\\ &\le 4(\mathrm{L}_a+1)P(s)^{3/2}\,\mathbb{E}(s).\end{aligned}$$

After integration in time,

$$\mathbb{E}(t) \le \mathbb{E}(0) + \int_0^t \|\mathrm{R}_a^\varepsilon(s,\cdot)\|_{L^2}^2 ds + \int_0^t (1 + 4\,(\mathrm{L}_a+1)\,\langle P\rangle_\infty(s)^{3/2})\,\mathbb{E}(s)\,ds.$$

Now, it suffices to apply (5.6b) and (5.18) together with Grönwall's inequality. ∎

Starting with f_0 as in (5.16), we have

$$\langle P\rangle_\infty(0) \le 1 + \mathrm{P}_0 \le \mathrm{I}_0, \quad \mathrm{I}_0 := 1 + \max(\mathrm{P}_0, \mathrm{P}_a) + \mathrm{S}_h. \tag{5.27}$$

Recall that $T^\varepsilon \le \mathcal{T}_a$. For T^ε small enough, by continuity, we can always assume that

$$\langle P\rangle_\infty(t) \le 4\mathrm{I}_0 \quad \text{for all } t \in [0, T^\varepsilon]. \tag{5.28}$$

Then, applying (5.26) with

$$\mathrm{N} := (\mathrm{N}_0^2 + \mathrm{N}_r^2)^{1/2},$$

restricting $T^\varepsilon \leq \mathcal{T}_a$ again if necessary, we find

$$\mathbb{E}(t) \lesssim \varepsilon^2 \mathrm{N}^2 \exp(1 + 32(\mathrm{L}_a + 1)\mathrm{I}_0^{3/2})\, t \quad \text{for all } t \in [0, T^\varepsilon].$$

Adjust $\tilde{\mathcal{T}} \in]0, 1]$ with $0 < \tilde{\mathcal{T}} \leq \mathcal{T}_a$ small enough to be sure that

$$\exp(1 + 32(\mathrm{L}_a + 1)\mathrm{I}_0^{3/2})\, \tilde{\mathcal{T}} \leq 2.$$

We define $T^\varepsilon \in]0, 1]$ as the maximal time less than $\tilde{\mathcal{T}}$ leading to (5.28). By construction,

$$\mathbb{E}(t) \lesssim 2\varepsilon^2 \mathrm{N}^2 \quad \text{for all } t \in [0, T^\varepsilon],\ T^\varepsilon \leq \tilde{\mathcal{T}}. \tag{5.29}$$

The quantity $\mathbb{E}$ remains under control (and small) as long as $\langle P \rangle_\infty$ is bounded. Remark that the proof of Lemma 5.4 is simple and that its conclusion (5.29) is not surprising. What is remarkable is the fact that (5.29) is sufficient. Indeed, we do not need to involve costly H^s_ε-estimates with $s > 0$ large, as is typical in nonlinear geometric optics (see, for instance, Chapter 4 in [25]).

5.3. Proof of Theorem 1.5

We start by interpreting the representation formula in terms of the two parts U^ε_a and U of U, see (1.19). We still have $\mathrm{D} = \mathrm{D}_0 + \mathrm{D}_h + \mathrm{D}_l + \mathrm{D}_n$, together with (3.4) where $\mathrm{f}_0 = \mathrm{f}^\varepsilon_a(0,\cdot) + f_0{}^\varepsilon$, where the electric field E_h is given by (3.3) with E_0 and B_0 as in Definition 5.1, whereas $\mathrm{f} = \mathrm{f}^\varepsilon_a + f$ and $\mathrm{F} = \mathrm{F}^\varepsilon_a + F$. Now, we can split the $\mathrm{D}_\star$ into $\mathrm{D}^\varepsilon_{\star a}$ (with subscript "a" for *approximate*) coming from U^ε_a, and $D_\star$ issued from U^ε, as well as bilinear terms. With this in mind, we look at D_n as a bilinear product $\mathcal{B}(\mathrm{F}, \mathrm{f})$ in terms of F (or E and B) and f. We denote by:

- $\mathrm{D}^\varepsilon_{0a}$ and D_0 the expression D_0, where f_0 is replaced, respectively, by $\mathrm{f}^\varepsilon_a(0,\cdot)$ and f^ε_0;
- $\mathrm{D}^\varepsilon_{ha}$ the expression D_h, where E_h stands for E^ε_h as in Definition 5.1;
- $\mathrm{D}^\varepsilon_{la}$ and D_l the expression D_l, where f_0 is replaced, respectively, by f^ε_a and f^ε;
- $\mathrm{D}^\varepsilon_{na} := \mathcal{B}(\mathrm{F}^\varepsilon_a, \mathrm{f}^\varepsilon_a)$ and $D_n := \mathcal{B}(F^\varepsilon, f^\varepsilon)$ the expression D_n, where the couple (F, f) is replaced, respectively, by $(\mathrm{F}^\varepsilon_a, \mathrm{f}^\varepsilon_a)$ and $(F^\varepsilon, f^\varepsilon)$.

With the above conventions, the contributions to the momentum increment brought by the approximate solution $\mathrm{U}^\varepsilon_{0a}$ and the perturbation U^ε are

$$\mathrm{D}^\varepsilon_a := \mathrm{D}^\varepsilon_{0a} + \mathrm{D}^\varepsilon_{ha} + \mathrm{D}^\varepsilon_{la} + \mathrm{D}^\varepsilon_{na}, \qquad D := D_0 + D_l + D_n. \tag{5.30}$$

But we have also to take into account the effect of cross terms, so that

$$\mathrm{D} = \mathrm{D}^\varepsilon_a + \mathcal{B}(F, \mathrm{f}^\varepsilon_a) + \mathcal{B}(\mathrm{F}^\varepsilon_a, f) + D. \tag{5.31}$$

We will estimate separately D^ε_a (Section 5.3.1), $\mathcal{B}(F, \mathrm{f}^\varepsilon_a)$ and $\mathcal{B}(\mathrm{F}^\varepsilon_a, f)$ (Section 5.3.2), as well as D (Section 5.3.3). To this end, we exploit the tools of Section 4. New difficulties arise due to the presence of additional terms (especially those that have ε^{-1} in factor), the absence of sign condition on f, and the need to deal with the energy $\mathbb{E}$ (instead of $\mathcal{E}$ or $\mathfrak{E}$). The estimates below are not meant to be optimal (in terms of powers of ε or P), except for the crucial (singular) contribution $\mathcal{B}(\nu \times \mathrm{B}^\varepsilon_a, f)$, where a compensation between the (possible) large size of B^ε_a and the L^2-smallness of f must be implemented. The proof of Theorem 1.5 is completed last in Section 5.3.4. Recall that $0 \leq t \leq T^\varepsilon \leq \tilde{\mathcal{T}} \leq \min(1, \mathcal{T}_a)$.

5.3.1. Control of the approximate momentum increment D^ε_a. For illustrative purposes, we first examine the case of $\tilde{\mathrm{D}}^\varepsilon_a$, which is D^ε_a, with $\mathrm{U}^\varepsilon_a \equiv \tilde{\mathrm{U}}^\varepsilon_a$. Then we turn to the general situation.

Study of $\tilde{\mathrm{D}}^\varepsilon_a$. When $\mathrm{U}^\varepsilon_a \equiv \tilde{\mathrm{U}}^\varepsilon_a$ with $\tilde{\mathrm{U}}^\varepsilon_a$ as in (1.17), as already seen, we have $\mathrm{E}^\varepsilon_h \equiv 0$ and therefore $\tilde{\mathrm{D}}^\varepsilon_{ha} \equiv 0$. Coming back to (3.4), we have to deal with

$$\begin{aligned}
\tilde{\mathrm{D}}^\varepsilon_{0a} &:= \int_0^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \mathrm{W}_0(s,\omega,\xi)\,\mathrm{M}(\varepsilon,\langle\xi\rangle)\,ds\,d\omega\,d\xi,\\
\tilde{\mathrm{D}}^\varepsilon_{la} &:= \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \mathrm{W}_l(s,\omega,\xi)\,\mathrm{M}(\varepsilon,\langle\xi\rangle)\,ds\,d\omega\,d\xi \Big)\,dr,\\
\tilde{\mathrm{D}}^\varepsilon_{na} &:= \frac{1}{\varepsilon} \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \mathrm{W}_n(r,s,\omega,\xi)\cdot(\nu(\xi)\times \mathrm{B}_e(x))\,\mathrm{M}(\varepsilon,\langle\xi\rangle)\,ds\,d\omega\,d\xi \Big)\,dr.
\end{aligned}$$

In view of (3.5), the functions $\mathrm{W}_0(s,\cdot)$ and $\mathrm{W}_l(s,\cdot)$ are odd (with respect to both variables ω and ξ). It follows that $\tilde{\mathrm{D}}^\varepsilon_{0a} = 0$ and $\tilde{\mathrm{D}}^\varepsilon_{la} = 0$. On the other hand, we can replace W_n as indicated in (3.5c), and then integrate by parts with respect to ξ. Since

$$\nabla_\xi \cdot [(\nu(\xi)\times \mathrm{B}_\mathrm{e}(x))\,\mathrm{M}(\varepsilon,\langle\xi\rangle)] = (\nu(\xi)\times \mathrm{B}_\mathrm{e}(x))\cdot \nabla_\xi[\mathrm{M}(\varepsilon,\langle\xi\rangle)] = 0, \tag{5.32}$$

we can assert that $\tilde{\mathrm{D}}^\varepsilon_{na} = 0$. Briefly, the stationary solution $\tilde{\mathrm{U}}^\varepsilon_a = 0$ does not contribute to the momentum increment. We find that $\tilde{\mathrm{D}}^\varepsilon_a = 0$.

Study of D^ε_a. Let U^ε_a be an approximate solution in the sense of Definition 5.1. From (4.20) and (4.21), together with Lemma 4.6, using (5.2) and (5.3a), we can assert that

$$\begin{aligned}
|\mathrm{D}^\varepsilon_{0a}| &\lesssim \int_0^t \int_{|\xi|\le \mathrm{P}_a} s\,\mathrm{S}_a\,ds\,d\xi \lesssim \mathrm{P}_a^3\,\mathrm{S}_a\,t^2,\\
|\mathrm{D}^\varepsilon_{la}| &\lesssim \int_0^t \Big(\int_r^t \int_{|\xi|\le \mathrm{P}_a} \mathrm{S}_a\,ds\,d\xi \Big)\,dr \lesssim \mathrm{P}_a^3\,\mathrm{S}_a\,t^2.
\end{aligned}$$

From (5.5), we easily get that

$$|\mathrm{D}^\varepsilon_{ha}| \le \mathrm{S}_h.$$

The most problematic term is $\mathrm{D}^\varepsilon_{na}$. Decompose F^ε_a as in (5.11). We exploit again the specific gradient form of W_n inside (3.5c) to perform an integration by parts with respect to the variable ξ in order to get

$$\begin{aligned}
\mathrm{D}^\varepsilon_{na} = &\int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \frac{s-r}{4\pi}\, \frac{\nu\circ \Xi(s)\cdot(\nu(\xi)+\omega)}{1+\omega\cdot\nu(\xi)}\\
&\times (\mathrm{E}^\varepsilon_a + \nu(\xi)\times \mathrm{B}^\varepsilon_a)\cdot \nabla_\xi \mathrm{f}^\varepsilon_a(r, X(s)+(s-r)\omega,\xi)\,ds\,d\omega\,d\xi \Big)\,dr.
\end{aligned}$$

From Lemmas 4.6 and 4.7 together with (5.2), (5.3b), (5.4) and (5.10), we can assert that

$$\begin{aligned}
|\mathrm{D}^\varepsilon_{na}| &\lesssim \int_0^t \Big(\int_r^t \int_{|\xi|\le \mathrm{P}_a} (s-r)\,(\mathrm{S}_r + 2\,(1+\mathrm{S}_a)\,\mathrm{L}_a)\,ds\,d\xi \Big)\,dr\\
&\lesssim \mathrm{P}_a^3\,(\mathrm{S}_r + 2\,(1+\mathrm{S}_a)\,\mathrm{L}_a)\,t^3.
\end{aligned}$$

In the end,

$$|\mathrm{D}^\varepsilon_a| \lesssim \mathrm{S}_h + \mathrm{P}_a^3\,\mathrm{S}_a\,t^2 + \mathrm{P}_a^3\,(\mathrm{S}_r + 2\,(1+\mathrm{S}_a)\,\mathrm{L}_a)\,t^3. \tag{5.33}$$

5.3.2. Influence of the frozen bilinear terms $\mathscr{B}(F, \mathrm{f}_a^\varepsilon)$ and $\mathscr{B}(\mathrm{F}_a^\varepsilon, f)$. We assume here that $t \le T^\varepsilon$.

Control of $\mathscr{B}(F, \mathrm{f}_a^\varepsilon)$. We can use (5.11) to get

$$\mathscr{B}(F, \mathrm{f}_a^\varepsilon) = \mathscr{B}(E, \mathrm{f}_a^\varepsilon) + \mathscr{B}(\nu \times B, \mathrm{f}_a^\varepsilon).$$

From (5.2) and (5.3a), it is clear that

$$\begin{aligned} |\mathscr{B}(E, \mathrm{f}_a^\varepsilon)| \le & \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \mathrm{P}_a} \mathrm{S}_a \, J^{-1/2} \, |\mathrm{W}_{ne}| \\ & \times J^{1/2} |E|(r, X(s) + (s-r)\omega) \, ds \, d\omega \, d\xi \Big) \, dr \\ & \lesssim \mathrm{S}_a \, \mathrm{P}_a^{3/2} \int_0^t \mathbb{E}(r)^{1/2} \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \mathrm{P}_a} J^{-1} |\mathrm{W}_{ne}|^2 \, ds \, d\omega \, d\xi \Big)^{1/2} dr. \end{aligned}$$

From (4.16) and (4.22) together with Lemma 4.6, since $s - r$ appears in factor inside $\mathrm{W}_n \equiv \mathrm{W}_{ne}$, we have

$$\begin{aligned} & \int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le \mathrm{P}_a} J^{-1} \, |\mathrm{W}_{ne}|^2 \, ds \, d\omega \, d\xi \\ & \le \int_r^t \int_{|\xi| \le \mathrm{P}_a} \Big(\int_{\mathbb{S}^2} \frac{d\omega}{\langle \xi \rangle^2 (1 + \omega \cdot \nu(\xi))^2} \Big) \, ds \, d\xi \lesssim \int_r^t \int_{|\xi| \le \mathrm{P}_a} ds \, d\xi \lesssim t \mathrm{P}_a^3. \end{aligned} \tag{5.34}$$

Thus, coming back to (5.29), we recover

$$|\mathscr{B}(E, \mathrm{f}_a^\varepsilon)| \lesssim \mathrm{S}_a \, \mathrm{P}_a^3 \, t^{3/2} \, \varepsilon.$$

Since $|\nu \times B| \le |B|$, the same argument applies to $\mathscr{B}(\nu \times B, \mathrm{f}_a^\varepsilon)$. Thus, we can retain that

$$|\mathscr{B}(F, \mathrm{f}_a^\varepsilon)| \lesssim \mathrm{S}_a \, \mathrm{P}_a^3 \mathrm{N} t^{3/2} \varepsilon \lesssim \mathrm{S}_a \, \mathrm{P}_a^3 \mathrm{N} t. \tag{5.35}$$

Control of $\mathscr{B}(\mathrm{F}_a^\varepsilon, f)$. From (5.3b) and (5.3c), we can infer that

$$\begin{aligned} & |\mathscr{B}(\mathrm{F}_a^\varepsilon, f)| \le (\mathrm{S}_a + \varepsilon^{-1} \, \mathrm{H}_a) \\ & \quad \times \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} |\mathrm{W}_{ne}| \, |f|(r, X(s) + (s-r)\omega, \xi) \, ds \, d\omega \, d\xi \Big) \, dr \\ & \quad \lesssim (\mathrm{S}_a + \varepsilon^{-1} \, \mathrm{H}_a) \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi| \le P(r)} J^{-1} \, |\mathrm{W}_{ne}|^2 \, ds \, d\omega \, d\xi \Big)^{1/2} \\ & \quad \times \Big(\int_r^t \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} J f^2(r, X(s) + (s-r)\omega, \xi) \, ds \, d\omega \, d\xi \Big)^{1/2} dr. \end{aligned}$$

With P_a replaced by $P(r)$, we proceed as in (5.34) to find that

$$|\mathscr{B}(\mathrm{F}_a^\varepsilon, f)| \lesssim (\mathrm{S}_a + \varepsilon^{-1} \mathrm{H}_a) \int_0^t t^{1/2} \, P(r)^{3/2} \, \mathbb{E}(r)^{1/2} \, dr.$$

From (5.29), we obtain that

$$\begin{aligned} |\mathscr{B}(\mathrm{F}_a^\varepsilon, f)| & \lesssim (\varepsilon \, \mathrm{S}_a + \mathrm{H}_a) \mathrm{N} t^{1/2} \int_0^t P(r)^{3/2} \, dr \\ & \lesssim (\mathrm{S}_a + \mathrm{H}_a) \mathrm{N} \Big(t + \int_0^t P(r)^3 \, dr \Big). \end{aligned} \tag{5.36}$$

5.3.3. Impact of the perturbed momentum increment D. We assume again that $t \le T^\varepsilon$. We consider successively D_0, D_l and D_n.

Control of D_0. From (5.16), (5.17), (4.20) and Lemma 4.6, we have

$$|D_0| \lesssim \int_0^t \int_{|\xi|\le \mathrm{P}_0} s\, \mathrm{S}_0\, ds\, d\xi \lesssim \mathrm{P}_0^3\, \mathrm{S}_0\, t^2.$$

Control of D_l. Using (5.24) and then (4.21) together with Lemma 4.6, we have

$$|D_l| \lesssim \int_0^t \Big(\int_r^t \int_{|\xi|\le P(r)} (\mathrm{S}_0 + \mathrm{S}_a)\, ds\, d\xi \Big)\, dr \lesssim (\mathrm{S}_0 + \mathrm{S}_a)\, t \int_0^t P(r)^3\, dr.$$

Control of D_n. Recall that

$$D_n = \mathcal{B}(E, f) + \mathcal{B}(v \times B, f).$$

We handle below $\mathcal{B}(E, f)$, the case of $\mathcal{B}(v \times B, f)$ being completely similar. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |D_n| \le & \int_0^t \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi|\le P(r)} J\, |E(r, X(s) + (s-r)\omega)|^2\, ds\, d\omega\, d\xi \Big)^{1/2} \\ & \times \Big(\int_r^t \int_{\mathbb{S}^2} \int_{|\xi|\le P(r)} J^{-1}\, |\mathrm{W}_{ne}|^2\, f^2\, ds\, d\omega\, d\xi \Big)^{1/2} dr \\ \lesssim & \int_0^t \mathbb{E}(r)^{1/2}\, P(r)^{3/2}\, (\mathrm{S}_0 + \mathrm{S}_a) \\ & \times \Big(\int_r^t \int_{|\xi|\le P(r)} \Big(\int_{\mathbb{S}^2} J^{-1}\, |\mathrm{W}_{ne}|^2\, d\omega \Big)\, ds\, d\xi \Big)^{1/2} dr. \end{aligned}$$

With P_a replaced by $P(r)$, we carry on with (5.34). Then, from (5.29), we get

$$|D_n| \lesssim \varepsilon\, \mathrm{N} (\mathrm{S}_0 + \mathrm{S}_a)\, t^{1/2} \int_0^t P(r)^3\, dr.$$

In brief, we can retain that

$$|D| \lesssim \mathrm{P}_0^3\, \mathrm{S}_0\, t^2 + (\mathrm{S}_0 + \mathrm{S}_a)(1+t) \int_0^t P(r)^3\, dr. \tag{5.37}$$

5.3.4. Compilation of the preceding estimates. We still have (4.12) for some (y_0, η_0) in the support of f_0, that is, for some η_0 satisfying $|\eta_0| \le \max(\mathrm{P}_a, \mathrm{P}_0)$. In view of (5.31), this gives rise to

$$\langle \mathrm{P} \rangle_\infty(t) \le 1 + \max(\mathrm{P}_a, P_0) + |\mathrm{D}_a^\varepsilon| + |\mathcal{B}(F, \mathrm{f}_a^\varepsilon)| + |\mathcal{B}(\mathrm{F}_a^\varepsilon, f)| + |D|.$$

By construction, we have

$$\mathrm{P}(t) \le \max(\mathrm{P}_a, P(t)) \quad \text{and} \quad P(t) \le \max(\mathrm{P}_a, \mathrm{P}(t)), \tag{5.38}$$

and similar inequalities concerning $\langle \mathrm{P}\rangle_\infty(t)$ and $\langle P\rangle_\infty(t)$. In particular,

$$\begin{aligned}\langle P\rangle_\infty(t) &\le \max(\mathrm{P}_a, \langle \mathrm{P}\rangle_\infty(t)) \\ &\le 1+\max(\mathrm{P}_a, P_0)+|\mathrm{D}_a^\varepsilon|+|\mathcal{B}(F,\mathrm{f}_a^\varepsilon)|+|\mathcal{B}(\mathrm{F}_a^\varepsilon,f)|+|D|.\end{aligned}$$

From (5.33) and (5.35)–(5.37), together with the definition of I_0 inside (5.27), we can easily infer that

$$\langle P\rangle_\infty(t) \le \mathrm{I}_0 + \tilde{\mathrm{C}}\,t + \check{\mathrm{C}}\int_0^t \langle P\rangle_\infty(r)^3\,dr,$$

where the constants $\tilde{\mathrm{C}}$ and $\check{\mathrm{C}}$ depend only on L_a, H_a, $\mathrm{S}_\star$, $\mathrm{P}_\star$ and $\mathrm{N}_\star$. We introduce

$$\beta := 16\check{\mathrm{C}}\,\mathrm{I}_0^2,\quad \mathcal{T} := \min(\tilde{\mathcal{T}}, \mathrm{I}_0/\tilde{\mathrm{C}}, (\ln 3-\ln 2)/\beta)\quad\text{and}\quad \tilde{T}^\varepsilon := \min(T^\varepsilon,\mathcal{T}).$$

Taking into account (5.28), we can assert that

$$\langle P\rangle_\infty(t) \le 2\mathrm{I}_0 + \beta\int_0^t \langle P\rangle_\infty(r)\,dr \quad \text{for all } t\in[0,\tilde{T}^\varepsilon].$$

Then, by Grönwall's inequality, the quantity $\langle P\rangle_\infty(r)$ remains controlled according to

$$\langle P\rangle_\infty(t) \le \mathcal{F}(t) := 2\mathrm{I}_0\,e^{\beta t} \le 3\,\mathrm{I}_0 \quad \text{for all } t\in[0,\tilde{T}^\varepsilon]. \tag{5.39}$$

If $T^\varepsilon < \mathcal{T} \le \tilde{\mathcal{T}}$ so that $\tilde{T}^\varepsilon = T^\varepsilon$, due to the definition of T^ε just before (5.29), we must have $\langle P\rangle_\infty(T^\varepsilon) = 4\,\mathrm{I}_0$. This is clearly a contradiction with (5.39) for $t=\tilde{T}^\varepsilon = T^\varepsilon$. Necessarily, we must have $\mathcal{T} \le T^\varepsilon$, and (5.39) holds true on $[0,\mathcal{T}]$ as required in Theorem 1.5. Of course, the function $\mathcal{F}$ inside (1.20) depends also on the various parameters $\dagger_a$, S_h and $\dagger_r$ occurring in Definition 5.1. However, since the family $\{\mathrm{U}_a^\varepsilon\}_\varepsilon$ is viewed as being fixed, this influence has not been reported. On the contrary, we can choose any perturbation $\{f_0^\varepsilon\}_\varepsilon$ as long as it is controlled by P_0, S_0 and N_0 as indicated in Definition 5.2. This is why we have highlighted inside (1.20) the impact of $(\mathrm{P}_0,\mathrm{S}_0,\mathrm{N}_0)$.

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