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Weak Dynkin type and the universality of non-negative Coxeter-regular integral quadratic forms

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Abstract. An integral quadratic form is called Coxeter-regular if its integer coefficients satisfy a divisibility condition equivalent to the fact that the associated Coxeter transformation and Weyl group are integral. Such forms are known to be useful in the study of finite-dimensional associative algebras, Lie algebras and certain singularities. We show that a non-negative (connected) Coxeter-regular form q is universal (that is, q represents all non-negative integers) if and only if q represents the integers $1, 2, 3, 7$ and 14. This may be viewed as a specialization (and, actually, an extension) of the Conway–Schneeberger/Bhargava "15 Theorem". As one of the main tools we provide a complete classification, up to $\mathbb{Z}$-equivalence, of all non-negative Coxeter-regular forms by means of so-called weak Dynkin type, which is a certain equivalence class of a Dynkin (bi)graph. In this way, we obtain a generalization of the known result of Barot-de la Peña for unit forms and simply-laced Dynkin diagrams.

1. Introduction

An integral quadratic form (of dimension $n \geq 1$) is a mapping $q\colon \mathbb{Z}^n \to \mathbb{Z}$ defined by a homogeneous polynomial of the second degree

$$q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j, \tag{1.1}$$

with integer coefficients $q_i, q_{ij} \in \mathbb{Z}$. Such forms appear in many mathematical contexts, especially in number theory, algebra and geometry, and their study has been a guiding light in mathematics for many years. For instance, the following two natural problems related with integral forms inspired the developments of modern algebraic number theory and the geometry of numbers:

(P1) Given an integral form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ and an integer $d \in \mathbb{Z}$, verify whether q *represents* d, that is, whether the Diophantine equation

$$q(x_1, \ldots, x_n) = d, \tag{1.2}$$

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has an integral solution $x_1, \ldots, x_n \in \mathbb{Z}$. Recall that if q represents all integers (or all non-negative integers in case q is non-negative) then q is called *universal*, cf. [6, 10].

(P2) Provide a classification of integral quadratic forms (from a given class) up to the $\mathbb{Z}$-equivalence $\sim$ defined as $q \sim q'$ if and only if $q' = q \circ T$ for some $T \in \mathrm{Gl}_n(\mathbb{Z})$.

Both problems have a long history and in the full generality they are highly non-trivial. Regarding (P2), recall the works of Gauss [15] on the classification of binary forms, or the results of Minkowski [30] on integral quadratic forms of small dimensions (see also [11, Chapter 15]), which later gave rise to the well-known Hasse–Minkowski–Witt theory of rational quadratic forms. Note that integral quadratic forms were considered to be "inherently unclassifiable" in dimension $n \geq 24$, see [11, p. 353].

In the context of problem (P1) we should mention the classical works of, among others, Fermat, Legendre and Lagrange on the representation of an integer as a sum of squares. For example, the known Lagrange "Four-Squares Theorem" states that the (positive) integral form $q(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} x_i^2$ is universal. Note that the representation problem may be viewed as studying the integral points on the affine variety in $\mathbb{C}^n$ defined by the polynomial equation (1.2). More general universality problems were considered later by many authors, for instance, we recall the classification of 4-dimensional (positive) diagonal forms due to Ramanujan [39]. Another remarkable result, which is one of the main inspirations for our study, is the celebrated "15 Theorem" of Bhargava, Conway, Schneeberger [6, 10] stating that a positive integral quadratic form q having an integer Gram matrix G_q (that is, an integral form (1.1) with $\frac{q_{ij}}{2} \in \mathbb{Z}$ for every $i < j$) is universal if and only if it represents the integers 1, 2, 3, 5, 6, 7, 10, 14, and 15. We also refer to the unpublished manuscript "Universal quadratic forms and the 290-theorem" of Bhargava and Hanke containing a more general criterion for positive forms. These results provide in some sense a general solution to the universality problem. However, it is still interesting to study certain more specialized variants of this problem, especially if they embrace quadratic forms which are not positive.

In the present paper we address both problems (P1) and (P2) for *Coxeter-regular forms* (shortly, *Cox-regular forms* [23]), that is, integral forms q as in (1.1) such that $q_i > 0$ for each $i = 1, \ldots, n$ and additionally

$$\frac{q_{ij}}{q_i}, \frac{q_{ij}}{q_j} \in \mathbb{Z}, \tag{1.3}$$

for all $i < j$. This class of integral forms has algebraic origins, cf. [3, 13, 23, 41, 48]. Namely, the condition (1.3) is equivalent to the integrality of the associated Coxeter transformation, Weyl group and Weyl roots (see Proposition 4.4 for the precise statement). This may be viewed as a fundamental "reason" that the following integral quadratic forms associated to algebraic structures are often Cox-regular: Euler or Tits quadratic forms of a finite-dimensional associative algebra [1, 7, 9, 12, 13, 20, 40], or integral form induced by the Killing form of a Lie algebra [3, 18, 21], cf. [5, 37]. Observe that in this context problems (P1) and (P2) have relevant interpretation. Namely, if two algebras of finite global dimension are derived equivalent then their Euler forms are $\mathbb{Z}$-equivalent, see [17, 33], see also Remark 4.9. Moreover, in certain cases positive solutions of the equation (1.2) for small d's (usually $d = 0$ and 1) for the Euler or Tits form q_A of an algebra A are related with classifications of indecomposable A-modules, see [9,12,20,40]. On the other hand, Cox-regular forms are closely related to the so-called quasi-Cartan matrices in the sense of Barot–Geiss–Zelevinsky [3] defined in the context of cluster algebras (see [25] for more details, cf. [27, 37, 38, 48]), and generalized intersection matrices in the sense of Slodowy [50]. The latter matrices were successfully used in the classification of rational singularities, in particular in the celebrated Arnold $\mathbb{A}$-$\mathbb{D}$-$\mathbb{E}$ classification of the simple hypersurface singularities (see Gabrielov [14] for the details, cf. Simson [48]).

Nevertheless, the class of Cox-regular forms is still not well understood and besides a few papers of the Ukrainian school (see, e.g., [35, 41, 53]) there are not many known results concerning general (especially number-theoretic) properties of these forms, see also [13,23,34] and references therein. More effort has been put in understanding the very special case of Cox-regular forms, namely, the *unit forms*, which by the definition are the integral forms (1.1) with $q_1 = \cdots = q_n = 1$. Classification of all non-negative unit forms with respect to $\sim$ was given by Barot-de la Peña in [2] by means of the *Dynkin type*, that is, the unique simply-laced Dynkin diagram $\mathbf{Dyn}(q) \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ associated to a unit form q. We refer also to [46,49] for an alternative classification, and the results on (weakly) positive unit forms of Ovsienko [35], von Höhne [51,52] and Simson [43,44,48].

In this paper we present two main results on Cox-regular forms related to problems (P1) and (P2), respectively. The first main result of the paper may be viewed as a specialized variant of the Conway–Schneeberger/Bhargava "15 Theorem", extended (in a quite natural way) to non-negative forms. We refer to Section 2 for the detailed definitions.

Theorem 1. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Coxeter-regular form of rank $r = \mathbf{rk}(q) := \mathbf{rk}(G_q) \geq 0$. Then the following conditions are equivalent.*

(a) *The form q is universal.*

(b) *The rank r is at least* 4.

(c) *The form q represents the integers* 2, 3, 7 *and* 14.

Note that obviously if a quadratic form q is universal then it is irreducible, cf. Section 2 and Corollary 8.4. We emphasize that Theorem 1 is independent of the "15 Theorem" in the sense that we do not apply the latter in our proof. On the other hand, note that "15 Theorem" does not admit a counterpart of the condition (b) since, e.g., the class of all diagonal positive forms of dimension 4 contains universal as well as non-universal forms, see [39], cf. Remark 8.5.

The proof of Theorem 1 is given in Section 8.1. It applies, among others, the following classification of non-negative Cox-regular forms which we treat as the second main contribution of the paper. It generalizes the main theorem of Barot-de la Peña [2] from unit forms to Cox-regular forms.

Theorem 2. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Coxeter-regular form of rank $r = \mathbf{rk}(q) \geq 1$. Then the following holds.*

(a) *There exists a $\mathbb{Z}$-invertible transformation $T\colon \mathbb{Z}^n \to \mathbb{Z}^n$ such that*

$$q \circ T(x_1, \ldots, x_n) = q_{D_r}(x_1, \ldots, x_r), \tag{1.4}$$

where $q_{D_r}\colon \mathbb{Z}^r \to \mathbb{Z}$ is the integral form associated to a Dynkin bigraph $D_r \in \{\mathbb{A}_r, \mathcal{B}_r, \mathcal{C}_r, \mathbb{D}_r, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathcal{F}_4, \mathcal{G}_2\}$ (see Table 1), which is uniquely determined by q up to $\mathbb{Z}$-equivalence $\sim$.

(b) *There exists a positive connected irreducible restriction $\hat{q}$ of q such that $\hat{q} \sim q_{D_r}$.*

The class $\mathbf{wDyn}(q) := [D_r]_{\sim_d}$ of Dynkin bigraphs $\mathbb{Z}$-equivalent with D_r is called the *weak Dynkin type* of q, see Definition 7.3 and Remark 7.7. Observe that Theorem 2 provides a complete classification of non-negative Cox-regular forms (of arbitrary dimension) up to $\mathbb{Z}$-equivalence, and the right-hand side of (1.4) (see also (7.1)) may be seen as a kind of a canonical "reduced" form of q, cf. Corollaries 7.6 and 7.10. The proof of Theorem 2 is given in Section 7.1. It applies the proven existence of a certain special basis of the radical $\ker q$ of q (see Theorem 6.2) and our earlier results concerning positive and principal Cox-regular forms [25, 27, 34] (see also [28, 35, 53]) and the so-called Gabrielov equivalence $\sim_G$, a stronger (than $\sim$) equivalence of Cox-regular forms having its origins in Lie theory and singularity theory (cf. [5, 14, 37, 50] and Remark 4.9).

The paper contains seven sections following this introduction. Sections 2-5 have a preparatory character. In Section 2 we recall basic definitions and facts used in the paper. Next, in Section 3 we develop techniques related with the so-called omissible vertices and special bases for arbitrary (i.e., not necessarily Cox-regular) non-negative integral quadratic forms. These techniques allow us to perform certain reductions and inductive reasoning in the main proofs later. In Section 4 (resp. Section 5) we survey the main properties of Cox-regular forms and their transformations (resp. the distinguished classes of Cox-regular forms related to Dynkin and Euclidean bigraphs), and we supplement a few new facts. In Section 6 we prove the key technical result on the existence of the above-mentioned special basis of $\ker q$. The last Sections 7 and 8 contain the proofs of Theorems 2 and 1, respectively, together with related facts and some consequences and remarks.

2. Basic notions and facts

By $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$) we denote the ring of integers, the fields of rational and real numbers, respectively. Given $n \geq 1$, by $\underline{n}$ we denote the set $\underline{n} = \{1, 2, \ldots, n\} \subseteq \mathbb{Z}$. For a ring R or the abelian group $\frac{1}{2}\mathbb{Z} = \{\frac{n}{2} : n \in \mathbb{Z}\}$, by $\mathbb{M}_n(R)$ we denote the set of $n \times n$ matrices with coefficients in R. For $M \in \mathbb{M}_n(R)$, by M^{tr} we denote the transpose of M. We often identify automorphisms of the abelian group $\mathbb{Z}^n$ (whose elements are viewed as column vectors) with their matrices in $\mathrm{Gl}_n(\mathbb{Z}) := \{M \in \mathbb{M}_n(\mathbb{Z}) : \det(M) = \pm 1\}$ written in the standard basis $\mathbf{e}_1 = \mathbf{e}_1^{(n)} = [1, 0, \ldots, 0]^{\mathrm{tr}}, \ldots, \mathbf{e}_n = \mathbf{e}_n^{(n)} = [0, \ldots, 0, 1]^{\mathrm{tr}}$ of $\mathbb{Z}^n$. Conversely, each matrix $M \in \mathrm{Gl}_n(\mathbb{Z})$ induces the automorphism of $\mathbb{Z}^n$ given by $v \mapsto M(v) = M \cdot v$ for each $v \in \mathbb{Z}^n$. We apply similar conventions for $\mathrm{Gl}_n(R) := \{M \in \mathbb{M}_n(R) : \det(M) \neq 0\}$ and linear automorphisms in R^n for $R = \mathbb{Q}, \mathbb{R}$. We say that a vector $v = [v_1, \ldots, v_n]^{\mathrm{tr}} \in \mathbb{Z}^n$ is *sincere* (resp. *positive*) if $v_i \neq 0$ for all $i \in \underline{n}$ (resp. if $v \neq 0$ and $v_i \geq 0$ for all $i \in \underline{n}$). Note that $v \in \mathbb{Z}^n$ is positive and sincere iff $[1, 1, \ldots, 1]^{\mathrm{tr}} \leq v$, where $\leq$ denotes the coordinate-wise partial order in $\mathbb{Z}^n$.

An integral quadratic form $q: \mathbb{Z}^n \to \mathbb{Z}$ as in (1.1) is viewed as $q(x) = \sum_{i=1}^n q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j$ for $x = [x_1, \ldots, x_n]^{\mathrm{tr}} \in \mathbb{Z}^n$. We set $q_{ji} := q_{ij}$ for all $i < j$. Recall that if $q_1 = \cdots = q_n = 1$ then q is called *unitary* (or a *unit form*). Moreover, we say that q is *irreducible* if $q = \alpha q'$ implies that $\alpha = \pm 1$, for each $\alpha \in \mathbb{Z}$ and an integral quadratic form $q': \mathbb{Z}^n \to \mathbb{Z}$. We say that q is *positive* (resp. *non-negative*) if $q(v) > 0$ (resp. $q(v) \geq 0$) for all $0 \neq v \in \mathbb{Z}^n$.

The unique symmetric matrix $G_q \in \mathbb{M}_n(\frac{1}{2}\mathbb{Z})$ such that $q(x) = x^{\mathrm{tr}} G_q x$ for each $x \in \mathbb{Z}^n$ is called the (symmetric) *Gram matrix* of q. Note that $G_q \in \mathbb{M}_n(\mathbb{Z})$ iff $\frac{q_{ij}}{2} \in \mathbb{Z}$ for every $i < j$, cf. [10]. By $q(-,-): \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{2}\mathbb{Z}$ we denote the *polarization* of q, that is, the symmetric bilinear form given by $q(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y)) = x^{\mathrm{tr}} G_q y$, for $x, y \in \mathbb{Z}^n$.

Given $d \in \mathbb{Z}$, elements of the set

$$R_q(d) := q^{-1}(d) = \{x \in \mathbb{Z}^n : q(x) = d\} \tag{2.1}$$

are called *d-roots* of q. The set of 0-roots $\ker q := R_q(0)$ is called the *kernel* of q. The subgroup

$$\operatorname{rad} q := \{x \in \mathbb{Z}^n : \ q(-, x) = 0\} = \{x \in \mathbb{Z}^n : \ G_q x = 0\} \subseteq \mathbb{Z}^n \tag{2.2}$$

is called the *radical* of q. Clearly, $\operatorname{rad} q \subseteq \ker q$. We recall that non-negative forms have the following nice properties.

Lemma 2.3. *Let $q: \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative integral quadratic form. Then:*

(a) $\operatorname{rad} q = \ker q$*; in particular,* $\ker q$ *is a subgroup of* $\mathbb{Z}^n$*,*

(b) $q(x + h) = q(x)$ *for each* $x \in \mathbb{Z}^n$ *and* $h \in \ker q$*,*

(c) $\ker q$ *is a pure subgroup of* $\mathbb{Z}^n$ *(that is,* $\alpha h \in \ker q$ *implies* $h \in \ker q$ *for each* $h \in \mathbb{Z}^n$ *and* $0 \neq \alpha \in \mathbb{Z}$*); in particular,* $\mathbb{Z}^n / \ker q$ *is free and* $\ker q$ *is a direct summand of* $\mathbb{Z}^n$*.*

Proof. For the proof of (a) we refer to [42, Proposition 2.8]. For claim (b) note that

$$q(x + h) = q(x + h, x + h) = q(x, x) + 2q(x, h) + q(h, h) = q(x)$$

by (a) and (2.2).

To see (c) note that $q(\alpha h) = \alpha^2 q(h)$ so clearly $\ker q$ is pure and $\mathbb{Z}^n / \ker q$ is a torsion-free (finitely generated) abelian group, hence it is free. This also implies that $\mathbb{Z}^n = \ker q \oplus V$ for the subgroup $V := \mathbb{Z}v^1 \oplus \cdots \oplus \mathbb{Z}v^t \subseteq \mathbb{Z}^n$ where $\{v^i\}_{i=1}^t \subseteq \mathbb{Z}^n$ are chosen such that $\{v^i + \ker q\}_{i=1}^t$ is a basis of $\mathbb{Z}^n / \ker q$, cf. [4, Proposition 4.1]. ∎

By the *rank* (resp. *corank*) of an integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ we mean the rank $\mathbf{rk}(q) := \mathbf{rk}(G_q)$ of the Gram matrix of q (resp. its corank $\mathbf{crk}(q) := n - \mathbf{rk}(q)$). If q is non-negative, then $\mathbf{crk}(q)$ equals the rank of the subgroup $\ker q$, see Lemma 2.3 (a). In particular, in this case q is positive if and only if $\mathbf{crk}(q) = 0$, equivalently, if $\mathbf{rk}(q) = n$. If $\mathbf{crk}(q) = 1$ then q is called *principal*.

We say that two integral quadratic forms $q, q'\colon \mathbb{Z}^n \to \mathbb{Z}$ are *$\mathbb{Z}$-equivalent* (or *$\mathbb{Z}$-congruent*) if $q' = q \circ T$, i.e., $q'(v) = q(T(v))$ for each $v \in \mathbb{Z}^n$ (equivalently, $q'(v, w) = q(T(v), T(w))$ for all $v, w \in \mathbb{Z}^n$), for some $T \in \mathrm{Gl}_n(\mathbb{Z})$. We write then $q \sim q'$ or $q \sim^T q'$. We say that q and q' are *trivially equivalent* and we write $q \cong q'$ if $q \sim^P q'$, for some permutation automorphism (matrix) $P \in \mathrm{Gl}_n(\mathbb{Z})$, that is, $P(\mathbf{e}_i) = \mathbf{e}_{\rho(i)}$ for all $i \in \underline{n}$ and some permutation $\rho\colon \underline{n} \to \underline{n}$ (in this case we write $P = P^\rho$).

The following fact is straightforward (cf. [32, Proposition 2.5]).

Lemma 2.4. *Given $T \in \mathrm{Gl}_n(\mathbb{Z})$ and two integral quadratic forms $q, q'\colon \mathbb{Z}^n \to \mathbb{Z}$, T induces the $\mathbb{Z}$-equivalence $q \sim^T q'$ if and only if $G_{q'} = T^{\mathrm{tr}} G_q T$. In this case*

(a) *T induces a bijection $T_| : R_{q'}(d) \to R_q(d)$ for every $d \in \mathbb{Z}$,*

(b) *q is non-negative iff so is q', and in this case $\mathbf{crk}(q) = \mathbf{crk}(q')$; in particular, q is positive (resp. principal) iff so is q'.*

Fix $n \geq 1$ and a subset $J = \{j_1 < j_2 < \cdots < j_r\} \subseteq \underline{n}$. Let $\iota_J\colon \mathbb{Z}^r \to \mathbb{Z}^n$ be the inclusion given by $\mathbf{e}_t^{(r)} \mapsto \mathbf{e}_{j_t}^{(n)}$ for all $1 \leq t \leq r$. We also consider the projection $\pi_J\colon \mathbb{Z}^n \to \mathbb{Z}^r$ defined as $\pi_J := \iota_J^{\mathrm{tr}}$, that is, $\pi_J(x_1, \ldots, x_n) = (x_{j_1}, \ldots, x_{j_r})$. Clearly, $\pi_J \circ \iota_J = \mathrm{id}_{\mathbb{Z}^r}$, and $\iota_J \circ \pi_J(h) = h$ only if $\pi_{\underline{n}\setminus J}(h) = 0$ (that is, $h_i = 0$ for all $i \in \underline{n} \setminus J$).

Given an integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ we define the *restriction* $q^J\colon \mathbb{Z}^r \to \mathbb{Z}$ of q to J by setting $q^J(y) := q(\iota_J(y))$ for every $y \in \mathbb{Z}^r$. Given $j \in \underline{n}$ we write $q^{(j)} := q^{\underline{n}\setminus\{j\}}$. For $m \geq 1$ and a quadratic form $q'\colon \mathbb{Z}^m \to \mathbb{Z}$ we define the *direct sum* $q \oplus q'\colon \mathbb{Z}^{n+m} \to \mathbb{Z}$ as the quadratic form given by

$$(q \oplus q')(x_1, \ldots, x_{n+m}) := q(x_1, \ldots, x_n) + q'(x_{n+1}, \ldots, x_{n+m}).$$

Observe that $G_{q\oplus q'} = G_q \oplus G_{q'} := \left[\begin{smallmatrix} G_q & 0 \\ 0 & G_{q'} \end{smallmatrix}\right]$.

We say that q is *connected* if q is not trivially equivalent to a direct sum of integral quadratic forms.

Lemma 2.5. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form. Then*

(a) *if q is non-negative then for every subset $J \subseteq \underline{n}$:*

  (a1) *the restriction q^J is non-negative,*

  (a2) *$\iota_J(\ker q^J) \subseteq \ker q$, so $\mathbf{crk}(q^J) \leq \mathbf{crk}(q)$,*

  (a3) *$\pi_{\underline{n}\setminus J}(h) = 0$ implies $\pi_J(h) \in \ker q^J$, for each $h \in \ker q$,*

  (a4) *$\pi_{\underline{n}\setminus J}(h) = 0$ if and only if $h \in \iota_J(\ker q^J)$, for each $h \in \ker q$;*

(b) *if $q = q' \oplus q''$ then q is non-negative if and only if so are q' and q''; in this case $\mathbf{crk}(q) = \mathbf{crk}(q') + \mathbf{crk}(q'')$.*

Proof. The assertions on non-negativity in (a) and (b) are obvious. Claim (a2) follows from the fact that $q^J = q \circ \iota_J$ and ι_J is a monomorphism. To show (a3) recall that $\pi_{\underline{n}\setminus J}(h) = 0$ implies that $\iota_J(\pi_J(h)) = h$. Thus $q^J(\pi_J(h)) = q(\iota_J(\pi_J(h))) = q(h) = 0$, so $\pi_J(h) \in \ker q^J$. This also means that $h = \iota_J(\pi_J(h))$ belongs to $\iota_J(\ker q^J)$ which proves the implication "$\Rightarrow$" in (a4). The implication "$\Leftarrow$" in (a4) follows directly from the definition of ι_J.

To show $\mathbf{crk}(q) = \mathbf{crk}(q') + \mathbf{crk}(q'')$ in (b) observe that $q^I = q'$ and $q^J = q''$ for $I = \underline{k}$ and $J = \underline{n} \setminus \underline{k}$, where k is the dimension of q'. Thus $\iota_I(\ker q') \oplus \iota_J(\ker q'') \subseteq \ker q$ by (a2). On the other hand, if $h = \iota_I(h^1) + \iota_J(h^2) \in \ker q$ for $h^1 \in \mathbb{Z}^k$ and $h^2 \in \mathbb{Z}^{n-k}$ then $0 = q(h) = q'(h^1) + q''(h^2)$ so $h^1 \in \ker q'$ and $h^2 \in \ker q''$ since q' and q'' are non-negative. Therefore $\ker q = \iota_I(\ker q') \oplus \iota_J(\ker q'') \cong \ker q' \oplus \ker q''$. ∎

Following [2,23], with an integral form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ as in (1.1) we associate the *bigraph* $\Delta = \Delta_q$, i.e., $\Delta = (\Delta_0, \Delta_1)$ is an undirected multigraph with the set of vertices $\Delta_0 := \underline{n} = \{1, \dots, n\}$ and the (multi)set Δ_1 of edges of two kinds: solid edges and dotted edges, defined as follows. For vertices $i \neq j$ the set Δ_1 contains $|q_{ij}|$ edges between i and j. These edges are solid if $q_{ij} < 0$ or dotted if $q_{ij} > 0$. Moreover, for every vertex $i \in \Delta_0$ there are $|q_i - 1|$ solid loops (resp. dotted loops) at i if $q_i \leq 0$ (resp. $q_i > 0$). Given a bigraph $\Delta = (\Delta_0 = \underline{n}, \Delta_1)$, by $q_\Delta\colon \mathbb{Z}^n \to \mathbb{Z}$ we denote the unique integral quadratic form such that $\Delta_{q_\Delta} = \Delta$ (we assume that each pair of vertices in Δ is joined by edges of the same kind, see [23,25] for the details). We often identify Δ with q_Δ, in particular, given two bigraphs Δ, Δ' we often write $\Delta \sim \Delta'$ instead of $q_\Delta \sim q_{\Delta'}$.

Remark 2.6. Basic properties of quadratic forms have the following obvious interpretation in terms of their bigraphs. An integral form q is connected (resp. unitary) if and only if Δ_q is a connected bigraph (resp. Δ_q is loop-free). Moreover, the bigraph $\Delta_{q\oplus q'}$ of a direct sum $q \oplus q'$ is the disjoint union $\Delta_q \sqcup \Delta_{q'}$ (up to suitable renumbering of the vertices). The bigraph Δ_{q^J} of a restriction q^J coincides with the full subbigraph Δ^J of $\Delta = \Delta_q$ induced by the vertices from J. In particular, given $j \in \underline{n}$, $\Delta_{q^{(j)}}$ is the bigraph $\Delta^{(j)}$ obtained from $\Delta = \Delta_q$ by removing the vertex j and all the edges incident with j. Note that trivial equivalences of quadratic forms correspond to bigraph isomorphisms.

The following useful observation states that given a non-negative form q, if a vertex i in Δ_q has a single solid loop then i is an isolated vertex, cf. [19, Lemma 3.2]. By $\xi\colon \mathbb{Z} \to \mathbb{Z}$ we denote the *zero form* of dimension 1 given by $\xi(x_1) = 0$.

Lemma 2.7. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative integral quadratic form as in* (1.1)*. If $q_i = 0$ for some $i \in \underline{n}$ then $q_{ij} = 0$ for all $j \in \underline{n}$, $j \neq i$. In particular, q is disconnected and not positive, and $q \cong q' \oplus \xi$ for some non-negative integral form $q'\colon \mathbb{Z}^{n-1} \to \mathbb{Z}$ of* $\mathbf{crk}(q') = \mathbf{crk}(q) - 1$.

Proof. If $q_i = 0$, then $\mathbf{e}_i \in \ker q$. Therefore, $\mathbf{e}_i \in \operatorname{rad} q$ since q is non-negative, see Lemma 2.3 (a). It means that $0 = q(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i^{\mathrm{tr}} G_q \mathbf{e}_j = \frac{1}{2} q_{ij}$ for all $j \neq i$, cf. (2.2). The remaining assertions are simple consequences of this fact and Lemma 2.5 (b). ∎

3. Omissible vertices and special bases

Similar methods to these presented in this section were discussed for unit forms in [4, Section 3.4] and [16, 47]. One checks that the arguments can be quite easily extended for the general case. For the completeness we present short proofs. We also provide some new examples at the end.

Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative integral quadratic form. Following [2,4], we say that $s \in \{1,\dots,n\} = \Delta_0$ for $\Delta = \Delta_q$ is an *omissible vertex* of q (or of Δ) if there exists a vector $h \in \ker q$ such that $h_s = 1$.

Lemma 3.1. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative integral quadratic form. Then:*

(a) $0 \leq \mathbf{crk}(q) - \mathbf{crk}(q^{(s)}) \leq 1$ *for each vertex $s \in \underline{n}$,*

(b) $\mathbf{crk}(q^{(s)}) = \mathbf{crk}(q) - 1$*, provided s is an omissible vertex of q.*

Proof. (a) The first inequality follows from Lemma 2.5 (a2). To show the second one take the inclusion $\iota_s := \iota_{\underline{n}\setminus\{s\}}\colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ (cf. Section 2), and fix a $\mathbb{Z}$-basis $v^1,\dots,v^c \in \mathbb{Z}^{n-1}$ of $\ker q^{(s)}$, where $c = \mathbf{crk}(q^{(s)})$. Observe that $\iota_s(\ker q^{(s)})$ is a pure subgroup of $\ker q$, hence the basis $w^1 = \iota_s(v^1),\dots,w^c = \iota_s(v^c)$ of $\iota_s(\ker q^{(s)})$ may be completed to a basis $w^1,\dots,w^c,w^{c+1},\dots,w^{c'} \in \mathbb{Z}^n$ of $\ker q$, where $c' = \mathbf{crk}(q)$, see Lemma 2.3 (c), cf. [4, Proposition 4.1]. Assume that $c' > c+1$. Then the coefficients $\alpha := w_s^{c'-1}$ and $\beta := w_s^{c'}$ are non-zero and $(\beta w^{c'-1} - \alpha w^{c'})_s = 0$ hence $\beta w^{c'-1} - \alpha w^{c'} \in \iota_s(\ker q^{(s)})$ by Lemma 2.5 (a4). Thus $\beta w^{c'-1} - \alpha w^{c'} = \sum_{i=1}^{c} \alpha_i w^i$ for some $\alpha_i \in \mathbb{Z}$. This gives a contradiction with the linear independence of $w^1,\dots,w^{c'}$. Therefore $c' - c \leq 1$.

(b) Fix $h \in \ker q$ such that $h_s = 1$ and consider the group homomorphism $\delta_s\colon \mathbb{Z}^n \to \mathbb{Z}^n$ given by $\delta_s(x) = x_s h$. Since $h_s = 1$ then δ_s induces the epimorphism $\delta'_s\colon \ker q \twoheadrightarrow \mathbb{Z}h \subseteq \ker q$ yielding the group direct sum decomposition $\ker q = \mathbb{Z}h \oplus \ker \delta'_s$. Now again using that $h_s = 1$ we verify that the inclusion $\iota_s\colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ induces the isomorphism $\ker q^{(s)} \cong \ker \delta'_s$. Thus $\mathbf{crk}(q^{(s)}) = \mathrm{rk}(\ker \delta'_s) = \mathbf{crk}(q) - 1$. ∎

Let H be a subgroup in $\mathbb{Z}^n$ of rank c. Following [16, 46, 47] we say that a $\mathbb{Z}$-basis $h^1,\dots,h^c \in \mathbb{Z}^n$ of H is a *special basis* if there exists a set of indices $S := \{s_1 < s_2 < \dots < s_c\} \subseteq \underline{n}$ such that $h^i_{s_i} = 1$ and $h^i_{s_j} = 0$ for all $i = 1,\dots,c$ and $j \neq i$. We also say that such basis is an S-special basis of H. Observe that in this case the subgroup H is pure and

$$\mathbb{Z}^n = \Big(\bigoplus_{j\in J} \mathbb{Z}\mathbf{e}_j\Big) \oplus H \tag{3.2}$$

for $J := \underline{n} \setminus S$, cf. Lemma 2.3 (c). It should be emphasized that not every pure subgroup of $\mathbb{Z}^n$ admits a special basis, see Example 3.7 (e). Given an integral form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ and a subset $S \subseteq \underline{n}$, we use the following notation for the restriction of q to $\underline{n} \setminus S$

$$q^{(S)} := q^{\underline{n}\setminus S} = q \circ \iota_{\underline{n}\setminus S}\colon \mathbb{Z}^{n-|S|} \to \mathbb{Z}. \tag{3.3}$$

Note that the bigraph $\Delta_{q'}$ of $q' = q^{(S)}$ is the bigraph $\Delta^{(S)}$ obtained from $\Delta = \Delta_q$ by removing all vertices from $S \subseteq \underline{n} = \Delta_0$ (and all the edges incident with them).

The following fact generalizes [16, Theorem 3.2 (b), (c)] and the claims in [46, pp. 28–29].

Proposition 3.4. *Let* $q\colon \mathbb{Z}^n \to \mathbb{Z}$ *be a non-negative integral quadratic form of corank* $c = \mathbf{crk}(q) \geq 1$. *Assume that the subgroup* $\ker q$ *admits an* S*-special basis for* $S \subseteq \underline{n}$. *Then the following holds.*

(a) $\mathbf{crk}(q^{(\widehat{S})}) = c - |\widehat{S}|$ *and* $\ker q^{(\widehat{S})}$ *admits an* $(S \setminus \widehat{S})$*-special basis (up to suitable renumbering of* $S \setminus \widehat{S}$*), for every subset* $\widehat{S} \subseteq S$,

(b) $q^{(S)}$ *is positive,*

(c) $q \sim q'$ *implies* $q^{(S)} \sim q'^{(S')}$ *for each non-negative integral quadratic form* $q'\colon \mathbb{Z}^n \to \mathbb{Z}$ *whose kernel* $\ker q'$ *admits an* S'*-special basis,*

(d) *if* q *and* q' *are principal forms then* $q \sim q'$ *implies* $q^{(s)} \sim q'^{(s')}$ *for all omissible vertices* s *and* s' *of* q *and* q', *respectively.*

Proof. (a) Let $S = \{s_1 < \cdots < s_c\} \subseteq \underline{n}$ and fix an S-special basis $h^1, \ldots, h^c \subseteq \mathbb{Z}^n$ of $\ker q$. Observe that all the elements of S are omissible vertices of q, by the definition of a special basis. To show the first part we proceed by induction on $t := |\widehat{S}| \geq 1$ for $\widehat{S} \subseteq S$. If $t = 1$ then $\mathbf{crk}(q^{(\widehat{S})}) = c - 1 = c - |\widehat{S}|$ by Lemma 3.1 (b). Assume that $t \geq 2$ and fix $s_k \in \widehat{S}$. Then by the inductive assumption we have $\mathbf{crk}(q^{(S_1)}) = c - |S_1| = c - t + 1$ where $S_1 := \widehat{S} \setminus \{s_k\}$. Using Lemma 2.5 (a3) we check that $\hat{h}^k := \pi_{\underline{n}\setminus S_1}(h_k)$ belongs to $\ker q^{(S_1)}$. Since $\hat{h}^k_{s_k} = 1$ it follows that s_k is omissible in $q^{(S_1)}$. So again applying Lemma 3.1 (b) we get $\mathbf{crk}(q^{(\widehat{S})}) = \mathbf{crk}((q^{(S_1)})^{(s_k)}) = c - t$.

To show the second part we verify directly by applying Lemma 2.5 (a) that the set

$$\big\{\pi_{\underline{n}\setminus\widehat{S}}(h^i) : s_i \in S \setminus \widehat{S}\big\} \subseteq \ker q^{(\widehat{S})}$$

is linearly independent, it generates $\ker q^{(\widehat{S})}$, and it satisfies the definition of an $(S \setminus \widehat{S})$-special basis. (Actually, this can also serve as an alternative argument for the first part of (a)). Assertion (b) follows obviously from (a) applied for $\widehat{S} := S$.

(c) Fix $T \in \mathrm{Gl}_n(\mathbb{Z})$ such that $q \sim^T q'$, that is, $q' = q \circ T$. Note that $T(H') = H$ where $H := \ker q$ and $H' := \ker q'$. Consider the mapping $F = F^S_q\colon \mathbb{Z}^r \to \mathbb{Z}^n/H$ defined as $F(u) := \iota_{(S)}(u) + H$, where $r := n - c$ and $\iota_{(S)} := \iota_{\underline{n}\setminus S}\colon \mathbb{Z}^r \to \mathbb{Z}^n$ is the inclusion as in Section 2. Since $\ker q$ admits an S-special basis then using the decomposition (3.2) we show that the mapping F is a group isomorphism (indeed, note that F induces a bijection on $\mathbb{Z}$-bases $\{\mathbf{e}^{(r)}_t\}_{1\leq t\leq r}$ and $\{\mathbf{e}^{(n)}_j + H\}_{j\in\underline{n}\setminus S}$ of $\mathbb{Z}^r$ and $\mathbb{Z}^n/H$, respectively). Moreover, using Lemma 2.3 (a), (b) we check that the mapping $\bar{q}\colon \mathbb{Z}^n/H \to \mathbb{Z}$ given by $\bar{q}(v + H) := q(v)$ is well defined. Additionally, observe that

$$\bar{q} \circ F(u) = \bar{q}\big(\iota_{(S)}(u) + H\big) = q\big(\iota_{(S)}(u)\big) = q^{(S)}(u) \tag{3.5}$$

for $u \in \mathbb{Z}^r$. Analogously we define the group isomorphism $F' = F^{S'}_{q'}\colon \mathbb{Z}^r \to \mathbb{Z}^n/H'$ and the mapping $\bar{q}'\colon \mathbb{Z}^n/H' \to \mathbb{Z}$.

Let $\overline{T}\colon \mathbb{Z}^n/H' \to \mathbb{Z}^n/H$ be the quotient mapping defined by $\overline{T}(v + H') := T(v) + H$. Taking into account that $T(H') = H$ we check that $\overline{T}$ is a well-defined group isomorphism. Now consider the composition $X := F^{-1} \circ \overline{T} \circ F' \colon \mathbb{Z}^r \to \mathbb{Z}^r$ of the constructed isomorphisms. Using (3.5) for q and q' we check that:

$$q^{(S)} \circ X = \big(q^{(S)} \circ (F_q^S)^{-1}\big) \circ \overline{T} \circ F_{q'}^{S'} = (\bar{q} \circ \overline{T}) \circ F_{q'}^{S'} = \bar{q}' \circ F_{q'}^{S'} = q'^{(S')},$$

thus X defines the equivalence $q^{(S)} \sim^X q'^{(S')}$.

(d) Note that if s is an omissible vertex of a principal form q, then there is a generator h of $\ker q$ with $h_s = 1$ and $\{h\}$ forms an $\{s\}$-special basis. Hence (d) follows from (c). ■

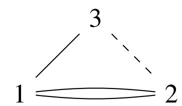
Remark 3.6. Given a non-negative integral quadratic form $q \colon \mathbb{Z}^n \to \mathbb{Z}$ of corank $c \geq 1$, the induced mapping $\bar{q} \colon \mathbb{Z}^n / \ker q \to \mathbb{Z}$ as in the proof of Proposition 3.4 (c) is always well defined and by construction $\bar{q}(\bar{v}) > 0$ for each $\bar{v} \neq 0$. If $\ker q$ admits an S-special basis then $\bar{q} \circ F = q^{(S)}$ as in (3.5). One can say that a special basis yields a "constructive realization" of the mapping $\bar{q}$.

Example 3.7. Consider the following examples of omissible vertices of integral forms and induced restrictions.

(a) Take the unit form $q \colon \mathbb{Z}^3 \to \mathbb{Z}$ given by

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - x_1x_3 + x_2x_3$$

with the associated bigraph $\Delta = \Delta_q$:



Then q is principal with $\ker q = \mathbb{Z}[1, 1, 0]^{\mathrm{tr}}$, hence the vertices 1 and 2 are omissible but 3 is not. We check that $\Delta^{(1)} \sim \mathbb{A}_2$ and $\Delta^{(2)} = \mathbb{A}_2$ are positive and $\Delta^{(3)} = \widetilde{\mathbb{A}}_1$ is principal (cf. Tables 1 and 2). In other words, $\mathbf{crk}(q) = \mathbf{crk}(q^{(3)}) = 1$ and $\mathbf{crk}(q^{(1)}) = \mathbf{crk}(q^{(2)}) = 0$. Note that $\Delta \sim \widetilde{\mathbb{A}}_2$ and $\mathbf{Euc}(q) = \widetilde{\mathbb{A}}_2$, see Definition 5.10.

(b) Take the Euclidean bigraph $\Delta = \widetilde{\mathcal{F}}_{41}$ from Table 2. Then Δ has two omissible vertices 1 and 5, cf. Lemma 5.1 (c). So by Proposition 3.4 (d) we get $\Delta^{(1)} \sim \Delta^{(5)}$. Observe that $\Delta^{(1)} = \mathcal{F}_4$ and $\Delta^{(5)} = \mathcal{C}_4$, cf. Lemma 5.3 (a) and Table 1.

(c) Take now the Euclidean bigraph $\Delta = \widetilde{\mathcal{F}}_{42}$. Then Δ has only one omissible vertex 5. Observe that $\Delta^{(1)} = \mathcal{B}_4$ and $\Delta^{(5)} = \mathcal{F}_4$ but $\mathcal{B}_4 \not\sim \mathcal{F}_4$ by Lemma 5.3 (c). This shows the importance of the assumption of Proposition 3.4 (d) that both vertices are omissible.

(d) [2, Proposition 2.5] shows that if q is a connected non-negative unit form then $\Delta_q^{(s)}$ is connected for any omissible vertex s. It is no longer true for non-unit forms. Take $\Delta = \widetilde{\mathcal{B}}_n$ for a fixed $n \geq 2$. Then every vertex of Δ is omissible but, e.g., $\Delta^{(2)}$ is disconnected.

(e) Not every non-negative integral quadratic form has an omissible vertex. Take $q(x_1, x_2) = 4x_1^2 + 9x_2^2 - 12x_1x_2 = (2x_1 - 3x_2)^2 \geq 0$. The form q is principal with $\ker q = \mathbb{Z}[3,2]^{\mathrm{tr}}$. Later we distinguish a class of integral forms in which every non-negative form has an omissible vertex, see Theorem 6.2.

4. Reflections, Gabrielov transformations and Cox-regular forms

Fix $n \geq 1$, an integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ as in (1.1) and the associated polarization $q(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{2}\mathbb{Z}$. Then q and $q(-,-)$ induce the mappings $q\colon \mathbb{R}^n \to \mathbb{R}$ and $q(-,-)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which we denote by the same symbols. Given $u \in \mathbb{R}^n$ such that $q(u) \neq 0$, the linear isomorphism $\sigma_u = \sigma_u^q\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\sigma_u^q(w) = w - \frac{2q(u,w)}{q(u)} \cdot u, \tag{4.1}$$

for $w \in \mathbb{R}^n$, is called the *reflection* at $u \in \mathbb{Z}^n$ with respect to q, see [18]. Note that $\sigma_u^2 = \mathrm{id}$ and $q \circ \sigma_u = q$. In particular, if $\sigma_u \in \mathbb{M}_n(\mathbb{Z})$ (that is, the matrix of σ_u in the standard basis has integral coefficients) then $q \sim^{\sigma_u} q$.

Given $s \in \underline{n}$, if $q_s = q(\mathbf{e}_s) \neq 0$ then $\sigma_s := \sigma_{\mathbf{e}_s}$ is called the *simple reflection* (at s). If $q_s \neq 0$ for all $s \in \underline{n}$ then by $\mathbb{W}_q$ we denote the subgroup of $\mathrm{Gl}_n(\mathbb{R})$ generated by all simple reflections. $\mathbb{W}_q$ is called the *Weyl group* of q, cf. [18, 41]. A composition of all simple reflections $\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n \in \mathbb{W}_q$ (in this particular order, cf. [34, p. 221] and [13, p. 8]) is called the *Coxeter transformation* associated with q.

Lemma 4.2 ([34, Proposition 4.5]). *For an integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ with $q_s \neq 0$ for all $s \in \underline{n}$, the Coxeter transformation is given by the* Coxeter matrix $\Phi_q := -\check{G}_q^{-1}\check{G}_q^{\mathrm{tr}} \in \mathrm{Gl}_n(\mathbb{Q})$*, where $\check{G}_q \in \mathbb{M}_n(\mathbb{Z})$ is the unique upper-triangular matrix such that $G_q = \frac{1}{2}(\check{G}_q^{\mathrm{tr}} + \check{G}_q)$.*

A vector $v \in \mathbb{R}^n$ is called a *Weyl root* if $v = \sigma(\mathbf{e}_k)$ for some $\sigma \in \mathbb{W}_q$ and $k \in \underline{n}$. Denote by $R_q^\bullet \subseteq \mathbb{R}^n$ the set of all Weyl roots of q. Note that $v \in R_q^\bullet$ iff $v = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_t}(\mathbf{e}_k)$ for some $t \geq 0$ and $i_1, \dots, i_t, k \in \underline{n}$.

Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ with $n \geq 1$, be an integral quadratic form given by $q(x) = \sum_{i=1}^n q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j$ as in (1.1). Recall that q is called a *Coxeter-regular form* (shortly, *Cox-regular form*) if $q_i > 0$ for each $i \in \underline{n}$, and $\frac{q_{ij}}{q_i}, \frac{q_{ij}}{q_j} \in \mathbb{Z}$, for all $i \neq j$, cf. [23, 24, 41].

Remark 4.3. The following properties follow easily from definitions.

(a) If an integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is Cox-regular then so is its multiple αq, for each integer $\alpha \geq 1$. A Cox-regular form q is irreducible if and only if $\gcd(q_1, \dots, q_n) = 1$. Each Cox-regular form q is an integer multiple of a unique irreducible Cox-regular $\check{q}$.

(b) Each unit form is irreducible and Cox-regular.

(c) Each restriction q^J of a Cox-regular form q is also Cox-regular. However, for an irreducible q, its restriction q^J is not necessarily irreducible.

We note that Roiter in [41] calls Cox-regular forms simply "integral quadratic forms", see also [53]. For this reason sometimes Cox-regular forms are called Roiter's integral quadratic forms, cf. [34].

The following characterization of Cox-regular forms is one of our main motivations for their study.

Proposition 4.4. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form with $q_i > 0$ for each $i \in \underline{n}$. Then the following conditions are equivalent.*

(a) *The form q is Cox-regular.*

(b) *The Weyl group $\mathbb{W}_q$ of q is a subgroup of $\mathrm{Gl}_n(\mathbb{Z})$.*

(c) *The set of Weyl roots $R_q^\bullet$ is contained in $\mathbb{Z}^n$.*

(d) *The Coxeter matrix Φ_q of q has integer coefficients, i.e., $\Phi_q \in \mathrm{Gl}_n(\mathbb{Z})$.*

Proof. Observe that $\sigma_i(\mathbf{e}_j) = \mathbf{e}_j - \frac{q_{ij}}{q_i}\mathbf{e}_i$ for any $i \neq j$. Thus the equivalences (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) hold easily by the definitions, cf. [34, Remark 4.4]. The implication (b)$\Rightarrow$(d) follows from Lemma 4.2. Whereas (d)$\Rightarrow$(a) is shown in [23, Theorem 3.8], cf. [34, Corollary 4.7]. ∎

We refer to [34] for a more detailed study of Weyl roots of Cox-regular forms.

Fix a Cox-regular form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ and $i, j \in \underline{n}$, $i \neq j$. Let $T_{ij} = T_{ij}^q\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the linear transformation defined as follows:

$$T_{ij}(\mathbf{e}_k) = \begin{cases} \mathbf{e}_k, & \text{if } k \neq j, \\ \sigma_i(\mathbf{e}_j) = \mathbf{e}_j - \frac{q_{ij}}{q_i}\mathbf{e}_i, & \text{if } k = j. \end{cases} \tag{4.5}$$

T_{ij}^q is called the *Gabrielov transformation* (of q) at (i, j), cf. [40, 53]. If $q_{ij} > 0$ (resp. $q_{ij} < 0$) then T_{ij}^q is called an *inflation* (resp. *deflation*), cf. [2, 22, 44].

The following properties of Gabrielov transformations are known and easy to check from the definition, cf. [53, p. 3631], [34, Lemma 4.9] and [28, (2.2)].

Lemma 4.6. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Cox-regular form, and let $T = T_{ij}^q$ be a Gabrielov transformation of q for fixed $i \neq j$. The composite mapping $q' := q \circ T_{ij}^q\colon \mathbb{Z}^n \to \mathbb{Z}$ has the following properties:*

(a) *q' is a Cox-regular form,*

(b) *q' is connected (resp. irreducible) if and only if so is q,*

(c) *$(T_{ij}^q)^{-1} = T_{ij}^{q'}$ is a Gabrielov transformation of q'; moreover, T_{ij}^q is an inflation (resp. deflation) iff $T_{ij}^{q'}$ is a deflation (resp. inflation),*

(d) *$q'_{ij} = -q_{ij}$ and $q'_s = q_s$ for each $s \in \underline{n}$; moreover,*

$$q'_{k\ell} = \begin{cases} q_{k\ell}, & \text{if } k \neq \ell \text{ and } k \neq j \neq \ell, \\ q_{kj} - \frac{q_{ki}q_{ij}}{q_i}, & \text{if } i \neq k \neq \ell = j. \end{cases} \tag{4.7}$$

Given $n \geq 1$ and $s \in \underline{n}$, we consider the *sign inversion* at s, i.e., the automorphism $T_s \in \mathrm{Gl}_n(\mathbb{Z})$ defined by $T_s(\mathbf{e}_s) = -\mathbf{e}_s$ and $T_s(\mathbf{e}_i) = \mathbf{e}_i$ for $i \neq s$. If $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is a (irreducible, connected) Cox-regular form then clearly so is $q' := q \circ T_s$.

As in [5, 37, 53] we consider the following equivalence relation on the set of Cox-regular forms, induced by Gabrielov transformations, sign inversions and trivial equivalences.

Definition 4.8. Given $n \geq 1$, we say that two Cox-regular forms $q, q' \colon \mathbb{Z}^n \to \mathbb{Z}$ are *Gabrielov equivalent* or shortly, *G-equivalent*, if there exist $t \geq 1$ and a sequence of Cox-regular forms

$$q = q^0, q^1, \dots, q^t = q'$$

and $\mathbb{Z}$-automorphisms $T^1, \dots, T^t \in \mathrm{Gl}_n(\mathbb{Z})$ such that $q^s = q^{s-1} \circ T^s$ and the automorphism T^s is either a Gabrielov transformation $T^s = T_{i_s j_s}^{q^{s-1}}$ of q^{s-1}, a sign-inversion $T^s = T_{i_s}$, or a permutation matrix $T^s = P$, for each $s = 1, \dots, t$. In particular, $q' = q \circ T$ (i.e., $q \sim^T q'$) for $T := T^1 \cdot \dots \cdot T^t$. We write $q \sim_G q'$ or $q \sim_G^T q'$.

Remark 4.9. G-equivalence $\sim_G$ has its origins and relevant interpretation in Lie theory and singularity theory. For instance, we recall from [5,37,50] that for unit forms as well as positive Cox-regular forms G-equivalence corresponds to isomorphism of the associated Lie algebras (by means of Serre-type relations induced by quasi-Cartan and intersection matrices related with quadratic forms). Moreover, G-equivalences preserve Weyl roots of (not necessarily non-negative) Cox-regular forms, see [34, Proposition 5.1]. We refer also to [3, 14, 38, 48, 50] for related results from singularity theory and the theory of cluster algebras.

Corollary 4.10. *If two Cox-regular forms $q, q' \colon \mathbb{Z}^n \to \mathbb{Z}$ are G-equivalent then there exists a permutation $\rho \colon \underline{n} \to \underline{n}$ such that $q'_i = q_{\rho(i)}$ for each $i \in \underline{n}$. In particular, q is a unit form if and only if so is q'.*

Proof. Follows from Lemma 4.6 (d) and obvious properties of sign inversions and trivial equivalences. ∎

Remark 4.11. It is known that in the class of connected, non-negative unit forms the relations $\sim$ and $\sim_G$ coincide, see [5, Proposition 1.2], cf. [2]. This is no longer true for non-unitary Cox-regular forms, see Lemmata 5.3 and 5.8.

We finish this section with the following useful fact.

Proposition 4.12 ([26, Theorem 5.3 (b)]). *Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative irreducible Cox-regular form. Then there exists at least one diagonal coefficient q_i of q with $q_i = 1$.*

We refer to [26] for a more extensive discussion on the coefficients of non-negative irreducible Cox-regular forms and related quasi-Cartan matrices.

5. Positive and principal Cox-regular forms

In this section we focus on the properties of non-negative Cox-regular forms of corank 0 and 1. It is known that they are classified by means of Dynkin and Euclidean bigraphs, respectively, presented in Tables 1 and 2, see Theorem 5.9 and Corollary 5.11. These bigraphs correspond to the well-known Dynkin and Euclidean diagrams, see [8, 13, 18, 21, 31], see also [34, Remark 2.6 (c)] and [25] for details of this correspondence.

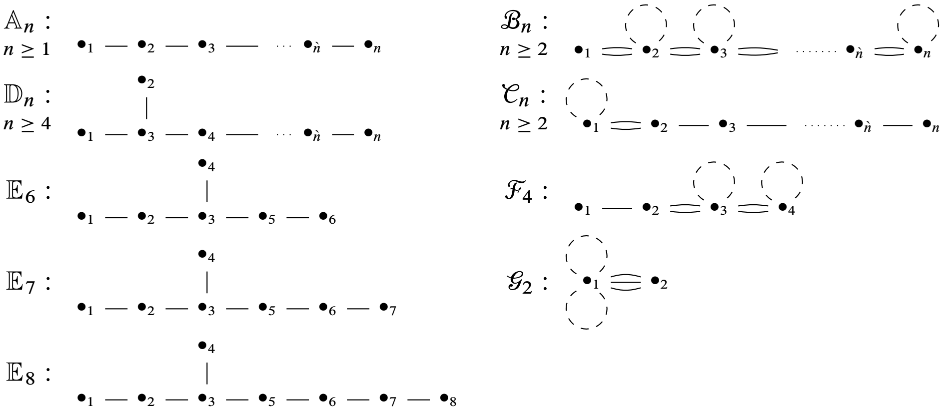


Table 1. *Dynkin bigraphs.* The indices denote a chosen numbering of the vertices fixed in the paper. We set $\grave{n} := n - 1$. Note that $\mathcal{B}_2 \cong \mathcal{C}_2$.

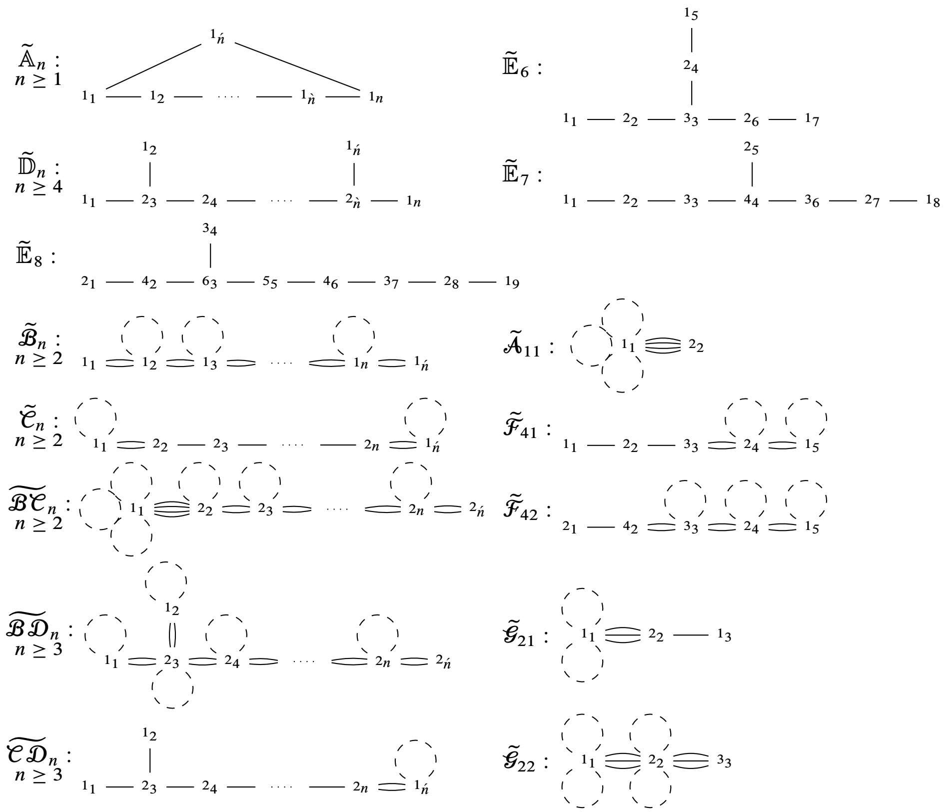


Table 2. *Euclidean bigraphs.* Vertices contain the coordinates of the positive generator of $\ker q_D$ for a given Euclidean bigraph D (see Lemma 5.1 (c)). The indices denote the chosen numbering of the vertices fixed in the paper. We set $\grave{n} := n - 1$ and $\acute{n} := n + 1$.

By a direct check, one obtains the following fact, see [23], [29, Lemma 2.1], cf. [13, 21]. Recall that an integral quadratic form q is called *classic* if $q_{ij} \leq 0$ for all $i \neq j$, cf. [41, 53].

Lemma 5.1. *Let $q = q_D\colon \mathbb{Z}^m \to \mathbb{Z}$ be the integral quadratic form associated to a Dynkin or Euclidean bigraph D from Tables 1 and 2. Then the following holds.*

(a) *The quadratic form q_D is an irreducible, connected, classic Cox-regular form.*

(b) *If D is a Dynkin bigraph then q_D is positive.*

(c) *If D is a Euclidean bigraph then q_D is principal and the kernel $\ker q_D = \mathbb{Z}h$ is generated by the vector $h = h_D \in \mathbb{Z}^m$ whose coefficients are depicted in Table 2 in the corresponding vertices.*

Recall that the quadratic forms associated to Dynkin and Euclidean bigraphs are actually the only non-negative classic Cox-regular forms, i.e., the following holds (see [13, Proposition 1.2], cf. [29, Proposition 2.8]).

Proposition 5.2. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected, irreducible and non-negative Cox-regular form. Assume that q is classic. Then*

(a) *q is positive if and only if q is (up to trivial equivalence) the quadratic form associated with one of the Dynkin bigraphs,*

(b) *if q is not positive then q is principal,*

(c) *q is principal if and only if q is (up to trivial equivalence) the quadratic form associated with one of the Euclidean bigraphs.*

Note that assertion (b) of the proposition is not true for non-classic Cox-regular forms. In particular, there exist non-negative Cox-regular forms of arbitrary corank, see [25, 26].

The following equivalences between Dynkin bigraphs were observed in [24].

Lemma 5.3. *The following relations between quadratic forms associated to Dynkin bigraphs hold:*

(a) *$\mathcal{G}_2 \sim \mathbb{A}_2$, $\mathcal{C}_3 \sim \mathbb{A}_3$ and $\mathcal{C}_4 \sim \mathcal{F}_4$,*

(b) *$\mathcal{C}_n \sim \mathbb{D}_n$ for all $n \geq 4$ (in particular, $\mathcal{C}_4 \sim \mathcal{F}_4 \sim \mathbb{D}_4$),*

(c) *there are no non-trivial equivalences between Dynkin bigraphs other than these from* (a)–(b).

Proof. We refer to [24, Proposition 2.4] for the explicit shapes of all the equivalences from (a) and (b). Alternatively, one can prove the existence of the desired equivalences by applying Proposition 3.4 (d) to Euclidean bigraphs $\widetilde{\mathcal{G}}_{21}$, $\widetilde{\mathcal{F}}_{41}$ and $\widetilde{\mathcal{CD}}_n$ for $n \geq 3$, as in Example 3.7 (b).

To show (c) note that if $q \sim^B q'$ for $B \in \mathrm{Gl}_n(\mathbb{Z})$ then $\det(G_{q'}) = \det(B^{\mathrm{tr}} G_q B) = \det(G_q)$. Now (c) follows from the inspection of the known determinants of the Gram matrices of Dynkin bigraphs, see [34, (3.6)], cf. [18, p. 63]. ∎

It is useful to have the following analogous fact for the Euclidean bigraphs.

Lemma 5.4. *The following relations between quadratic forms associated to Euclidean bigraphs hold:*

(a) $\widetilde{\mathbb{A}}_1 \sim \widetilde{\mathcal{A}}_{11}$, $\widetilde{\mathbb{A}}_2 \sim \widetilde{\mathcal{G}}_{21} \sim \widetilde{\mathcal{G}}_{22}$ *and* $\widetilde{\mathcal{B}}_2 \sim \widetilde{\mathcal{C}}_2 \sim \widetilde{\mathcal{BC}}_2$,

(b) $\widetilde{\mathbb{A}}_3 \sim \widetilde{\mathcal{C}}_3 \sim \widetilde{\mathcal{CD}}_3$ *and* $\widetilde{\mathcal{C}}_4 \sim \widetilde{\mathcal{F}}_{41} \sim \widetilde{\mathcal{F}}_{42}$,

(c) $\widetilde{\mathcal{B}}_n \sim \widetilde{\mathcal{BC}}_n \sim \widetilde{\mathcal{BD}}_n$ *for all* $n \geq 3$,

(d) $\widetilde{\mathcal{C}}_n \sim \widetilde{\mathbb{D}}_n \sim \widetilde{\mathcal{CD}}_n$ *for all* $n \geq 4$ *(in particular,* $\widetilde{\mathcal{C}}_4 \sim \widetilde{\mathcal{F}}_{41} \sim \widetilde{\mathcal{F}}_{42} \sim \widetilde{\mathbb{D}}_4 \sim \widetilde{\mathcal{CD}}_4$*),*

(e) *there are no non-trivial equivalences between Euclidean bigraphs other than these from* (a)–(d).

Proof. In the proof we take Euclidean bigraphs with the vertices ordered as in Table 2.

(a)+(b) Consider the following $\mathbb{Z}$-invertible matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & \hat{1} & 0 & 0 \\ 0 & 0 & \hat{1} & 1 \\ 1 & 0 & \hat{1} & 0 \\ 0 & 0 & 0 & \hat{1} \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \hat{1} \\ 0 & 0 & 1 & 0 & \hat{1} \\ 0 & 0 & 0 & 1 & \hat{1} \end{bmatrix}, \tag{5.5}$$

where $\hat{1} := -1$. We check directly that $A^{\mathrm{tr}} G_q A = G_{q'}$ for $q = q_{\widetilde{\mathcal{A}}_{11}}$ and $q' = q_{\widetilde{\mathbb{A}}_1}$ that is, A defines the equivalence $\widetilde{\mathcal{A}}_{11} \sim^A \widetilde{\mathbb{A}}_1$. Similarly, we check that $\widetilde{\mathcal{G}}_{21} \sim^B \widetilde{\mathbb{A}}_2$, $\widetilde{\mathcal{G}}_{22} \sim^C \widetilde{\mathcal{G}}_{21}$, $\widetilde{\mathcal{C}}_2 \sim^B \widetilde{\mathcal{B}}_2 \sim^{X_2} \widetilde{\mathcal{BC}}_2$, $\widetilde{\mathbb{A}}_3 \sim^D \widetilde{\mathcal{C}}_3 \sim^{T_3} \widetilde{\mathcal{CD}}_3$ and $\widetilde{\mathcal{F}}_{41} \sim^E \widetilde{\mathcal{F}}_{42}$, where $X_2 \in \mathrm{Gl}_3(\mathbb{Z})$ and $T_3 \in \mathrm{Gl}_4(\mathbb{Z})$ are the matrices given in (5.6) below. For $\widetilde{\mathcal{C}}_4 \sim \widetilde{\mathcal{F}}_{41}$ see [34, Example 3.4].

(c)+(d) Consider the following four families of $\mathbb{Z}$-invertible matrices:

$$X_l = \begin{bmatrix} \hat{1} & 0 & 0 & \cdots & 0 & 0 \\ \hat{1} & 0 & 0 & \ddots & 1 & \hat{1} \\ \vdots & 0 & 0 & \ddots & 0 & \hat{1} \\ \hat{1} & 0 & 1 & \cdots & 0 & \hat{1} \\ \hat{1} & 1 & 0 & \cdots & 0 & \hat{1} \\ 1 & 0 & 0 & \cdots & 0 & \hat{1} \end{bmatrix}, \quad Y_l = \begin{bmatrix} \hat{1} & 0 & 0 & \cdots & 0 & 0 \\ \hat{1} & 0 & 0 & \ddots & 1 & \hat{1} \\ \vdots & 0 & 0 & \ddots & 0 & \hat{1} \\ \hat{1} & 0 & 1 & \cdots & 0 & \hat{1} \\ 0 & 1 & 0 & \cdots & 0 & \hat{1} \\ 1 & 0 & 0 & \cdots & 0 & \hat{1} \end{bmatrix},$$
$$Z_l = \mathrm{I}_{l+1} + \mathrm{E}_{l+1}^{2,1} + \mathrm{E}_{l+1}^{l,l+1}, \quad T_l = \mathrm{I}_{l+1} + \mathrm{E}_{l+1}^{2,1} \tag{5.6}$$

of size $l + 1 \geq 3$, where $\mathrm{I}_t, \mathrm{E}_t^{i,j} \in \mathbb{M}_t(\mathbb{Z})$ denote the identity matrix and the matrix with 1 at (i, j)-th entry and zeros elsewhere, respectively. We verify that $\widetilde{\mathcal{BD}}_n \sim^{Y_n^{-1}} \widetilde{\mathcal{B}}_n \sim^{X_n} \widetilde{\mathcal{BC}}_n$ for all $n \geq 3$, and $\widetilde{\mathcal{CD}}_n \sim^{T_n^{-1}} \widetilde{\mathcal{C}}_n \sim^{Z_n} \widetilde{\mathbb{D}}_n$ for all $n \geq 4$.

(e) For each Euclidean bigraph D we chose one omissible vertex $s = s_D \in D_0$. We take $s_{\widetilde{\mathbb{E}}_8} = 9$, $s_{\widetilde{\mathcal{F}}_{42}} = 5$, $s_{\widetilde{\mathcal{G}}_{21}} = 3$ and $s_D = 1$ for the remaining bigraphs D, see Table 2. Then we check that $D^{(s)}$ are as follows:

$$\begin{array}{c||c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} D & \widetilde{\mathbb{A}}_n & \widetilde{\mathbb{D}}_n & \widetilde{\mathbb{E}}_6 & \widetilde{\mathbb{E}}_7 & \widetilde{\mathbb{E}}_8 & \widetilde{\mathcal{A}}_{11} & \widetilde{\mathcal{B}}_n & \widetilde{\mathcal{C}}_n & \widetilde{\mathcal{BC}}_n & \widetilde{\mathcal{BD}}_n & \widetilde{\mathcal{CD}}_n & \widetilde{\mathcal{F}}_{41} & \widetilde{\mathcal{F}}_{42} & \widetilde{\mathcal{G}}_{21} & \widetilde{\mathcal{G}}_{22} \\ \hline D^{(s)} & \mathbb{A}_n & \mathbb{D}_n & \mathbb{E}_6 & \mathbb{E}_7 & \mathbb{E}_8 & \mathbb{A}_1 & \mathcal{B}_n & \mathcal{C}_n & \mathcal{B}_n & \mathcal{B}_n & \mathcal{C}_n & \mathcal{F}_4 & \mathcal{F}_4 & \mathcal{G}_2 & \mathcal{G}_2 \end{array} \tag{5.7}$$

By inspection of table (5.7) and Lemma 5.3 we observe that every pair of (non-isomorphic) Euclidean bigraphs D, D' satisfying $D^{(s_D)} \sim D'^{(s_{D'})}$ is one of the congruent pairs described in (a)–(d). Therefore, by applying Proposition 3.4 (d) we get claim (e). ∎

It appears that the G-equivalence $\sim_G$ (in contrast to $\sim$) separates distinct Dynkin (resp. Euclidean) bigraphs. The explicit proof of the following fact is given in [34, Corollary 4.17].

Lemma 5.8. *Given two Dynkin or Euclidean bigraphs D and D', the G-equivalence $q_D \sim_G q_{D'}$ holds if and only if $D = D'$ (up to isomorphism).*

We recall the following known $\sim_G$-classification of positive and principal Cox-regular forms.

Theorem 5.9. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected, irreducible and non-negative Cox-regular form.*

(a) *q is positive if and only if $q \sim_G q_D$ for a Dynkin bigraph D.*

(b) *q is principal if and only if $q \sim_G q_D$ for a Euclidean bigraph D.*

Proof. Assertion (a) can be derived from the results of Ovsienko [35]. Explicit proof can be found in [36, Theorem 2.1] or in [29, Theorem 1.12 (c)]. Assertion (b) follows from [28, Theorem 1.4] or [25, Theorem 5.4]. It can also be deduced from [53]. Note that all these proofs provide a concrete sequence of Gabrielov transformations (in fact, it is enough to apply inflations only) and sign inversions that induce the desired G-equivalences. We refer also to [25, 27] where new and more efficient algorithms for these tasks are presented. ∎

Lemma 5.8 guarantees that the Dynkin (resp. Euclidean) bigraph D in (a) (resp. (b)) in the above theorem is unique for q (up to bigraph isomorphism). Hence the following notions are well defined, cf. [28, 29, 34].

Definition 5.10. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected, irreducible and non-negative Cox-regular form.

(a) If q is positive, then the unique Dynkin bigraph D from Table 1 such that $q \sim_G q_D$ is called the *Dynkin type* of q and it is denoted by $D = \mathbf{Dyn}(q)$.

(b) If q is principal, then the unique Euclidean bigraph D from Table 2 such that $q \sim_G q_D$ is called the *Euclidean type* of q and it is denoted by $D = \mathbf{Euc}(q)$.

Corollary 5.11. *Let $q, q'\colon \mathbb{Z}^n \to \mathbb{Z}$ be two connected, irreducible and non-negative Cox-regular forms. Then*

(a) $q \sim_G q'$ *if and only if* $\mathbf{Dyn}(q) = \mathbf{Dyn}(q')$*, in case q and q' are positive,*

(b) $q \sim_G q'$ *if and only if* $\mathbf{Euc}(q) = \mathbf{Euc}(q')$*, in case q and q' are principal.*

6. Existence of a special basis

Existence of an omissible vertex as well as a special basis of the kernel of a non-negative integral quadratic form is a non-obvious problem, cf. Example 3.7 (e). Even more, for our purposes we need to have a special basis with the following additional properties (we use notation of Section 2 and (3.3)).

Definition 6.1. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible integral quadratic form.

(a) An omissible vertex $s \in \underline{n}$ of q is called *strictly omissible* if $q^{(s)}$ is irreducible and connected.

(b) An S-special basis of $\ker q$ is called *strictly special* if $q^{(S)}$ is irreducible and connected.

Recall that not every omissible vertex of a non-negative Cox-regular form is strictly omissible, see Example 3.7 (d). But there always exists at least one such a vertex, as the following theorem shows. This is a crucial ingredient for our main results in the next sections.

Theorem 6.2. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Cox-regular form of corank $c = \mathbf{crk}(q) \geq 1$. Then*

(a) *q has a strictly omissible vertex,*

(b) $\ker q$ *admits a strictly special basis.*

Proof. Note that $n \geq 2$ since $c \geq 1$. We divide the proof of (a) into three steps.

Step 1^o. Assume that $c = 1$ and q is classic. Then $q \cong q_D$ for a Euclidean bigraph D by Proposition 5.2 (c). Then as in the proof of Lemma 5.4 (e) we observe that each D has an omissible vertex $s = s_D \in D_0$ with $D^{(s)}$ as in table (5.7). Moreover, since every $D^{(s)}$ in the bottom row of (5.7) is irreducible and connected it follows that in all cases the chosen vertex s is strictly omissible. Actually, every omissible vertex s' of a Euclidean bigraph D is strictly omissible except the case $D = \widetilde{\mathcal{B}}_{n-1}$ and $s' \in \{2, \dots, n-1\}$, see Table 2.

Step 2^o. Assume that $c = 1$ and q is not classic. Consider two cases.

Case 2.1^o. The vector $h \in \ker q$, such that $\ker q = \mathbb{Z}h$, is sincere. Applying sign inversions $T_{i_1}, \dots, T_{i_k}$ for all $i_1, \dots, i_k$ such that $h_{i_1} < 0, \dots, h_{i_k} < 0$ we obtain $[1, \dots, 1]^{\mathrm{tr}} \leq h' := T_{i_1} \circ \cdots \circ T_{i_k}(h) \in \ker q'$ with $q' := q \circ T_{i_1} \circ \cdots \circ T_{i_k} \sim q$. Note that $\ker q' = \mathbb{Z}h'$. If q'

is classic, then by Step 1°, q' has a strictly omissible vertex s. Since the sign inversions do not change the underlying multigraph of Δ_q, we conclude that s is a strictly omissible vertex of q.

In the case when q' is not classic, there is a sequence of inflations $T_{a_1b_1}, \dots, T_{a_mb_m}$, for some $m \geq 1$ and $a_i, b_i \in \underline{n}$, $a_i \neq b_i$ for $1 \leq i \leq m$, such that $q'' := q' \circ T_{a_1b_1} \circ \dots \circ T_{a_mb_m}$ is classic, i.e., $q'' \cong q_D$ for a Euclidean bigraph D, see [25, Theorem 5.4 (a), (b)], cf. Proposition 5.2 (c) and [28, Theorem 1.4 (a)]. Since $T_{a_sb_s}$'s are inflations, it follows that

$$h'' := T_{a_mb_m}^{-1} \circ \dots \circ T_{a_1b_1}^{-1}(h') > h' \geq [1, \dots, 1]^{\mathrm{tr}},$$

cf. (4.5). Therefore, since $\ker q'' = \mathbb{Z}h''$, by inspection of Table 2 we see that D is neither isomorphic to $\widetilde{\mathbb{A}}_{n-1}$ nor to $\widetilde{\mathcal{B}}_{n-1}$. Moreover, there are vertices $j_1, \dots, j_\ell$, for $\ell \geq 1$ such that $h''_{j_1} = \dots = h''_{j_\ell} = 1$ and by the arguments in Step 1° all of them are strictly omissible for q''. Observe that $h'_{j_1} = h''_{j_1} = 1, \dots, h'_{j_\ell} = h''_{j_\ell} = 1$, and $\{a_1, \dots, a_m\} \cap \{j_1, \dots, j_\ell\} = \emptyset$ since $h''_{a_1} > 1, \dots, h''_{a_m} > 1$. In particular, each of $j_1, \dots, j_\ell$ is an omissible vertex for q'. We show that they are strictly omissible for q'. To the contrary, assume first that $q'^{(j_s)}$ is disconnected for some $1 \leq s \leq \ell$. Let $\Delta'' := \Delta_{q''}$ and $\Delta' := \Delta_{q'}$. Then $\Delta''^{(j_s)}$ is connected and $\Delta'^{(j_s)}$ is a disjoint union $\Delta'^{(j_s)} = \dot{\Delta} \sqcup \ddot{\Delta}$, so there exist $j' \in \dot{\Delta}_0$, $j'' \in \ddot{\Delta}_0$ such that $q'_{j'j''} = 0$ but $q''_{j'j''} \neq 0$. By the analysis of the formula (4.7) we infer that the only possibility of creating a new edge connecting j' and j'' in Δ'' by applying the inflations $T_{a_1b_1}, \dots, T_{a_mb_m}$ to Δ' is when $j_s \in \{a_1, \dots, a_m\}$. Hence, we get the contradiction. This shows that $\Delta'^{(j_s)}$ is connected, and similarly so are $\Delta'^{(j_t)}$ for the remaining $1 \leq t \leq \ell$. Since the inflations do not affect $q'_i, i \in \underline{n}$, and $q''^{(j_1)}, \dots, q''^{(j_\ell)}$ are irreducible, so are $q'^{(j_1)}, \dots, q'^{(j_\ell)}$ (see Lemma 4.6 (d) and Remark 4.3 (a)). Therefore, any $s \in \{j_1, \dots, j_\ell\}$ is a strictly omissible vertex for q', and obviously, for q.

Case 2.2°. The vector $0 \neq h \in \ker q$ generating $\ker q$ is not sincere. Let $q^J = q \circ \iota_J : \mathbb{Z}^p \to \mathbb{Z}$ be the restriction of q to the set $J := \{i \in \underline{n} : h_i \neq 0\}$ with $p := |J| \geq 1$. Then the non-zero vector $h^J := \pi_J(h) \in \ker q^J$ is sincere and q^J is principal with $\ker q^J = \mathbb{Z}h^J$, see Lemma 2.5 (a3). Observe that q^J is connected. Indeed, otherwise, $q^J \circ P = q' \oplus q''$ for some permutation matrix P and integral forms q' and q''. Thus $(q' \oplus q'')(P^{-1}h^J) = 0$ for sincere $P^{-1}h^J$, which implies that $\mathbf{crk}(q'), \mathbf{crk}(q'') \geq 1$ by the non-negativity of q' and q''. So by Lemma 2.5 (b) we get that $1 = \mathbf{crk}(q^J) = \mathbf{crk}(q') + \mathbf{crk}(q'') \geq 2$, a contradiction.

Since q is Cox-regular, then so is q^J. Hence, there exist an integer $\alpha \geq 1$ and a connected principal irreducible Cox-regular form $\check{q}$ such that $q^J = \alpha\check{q}$ (if q^J is irreducible then $\alpha = 1$ and $\check{q} = q^J$), cf. Remark 4.3 (a). Moreover, $h^J \in \ker \check{q}$ is sincere and $\ker \check{q} = \mathbb{Z}h^J$, so by Case 2.1° and Step 1°, $\check{q}$ has a strictly omissible vertex s. In particular, $\check{q}^{(s)}$ and $(q^J)^{(s)}$ are connected. Now we show that s is a strictly omissible vertex of q (here for simplicity we identify the vertices $\{1, \dots, p\}$ of q^J with the set $J \subset \underline{n}$ via the inclusion $\iota_J : \mathbb{Z}^p \to \mathbb{Z}^n$). To show that $q^{(s)}$ is connected, assume to the contrary that there exist $\dot{\Delta}, \ddot{\Delta}$ such that $\Delta_q^{(s)} = \dot{\Delta} \sqcup \ddot{\Delta}$ and $\Delta_{q^J}^{(s)} \subseteq \dot{\Delta}$. Since q and q^J are connected there exist $k \in J \setminus \{s\} \subseteq \dot{\Delta}_0$ and $k' \in \ddot{\Delta}_0$ such that $q_{ks} \neq 0$ and $q_{k's} \neq 0$. Then by Lemma 2.3 (b) and Lemma 2.5 we get

$$\begin{aligned} q(\mathbf{e}_k+\mathbf{e}_{k'}+\mathbf{e}_s) &= q(\mathbf{e}_k+\mathbf{e}_{k'}+\mathbf{e}_s-h_s\cdot h) = q^J(\mathbf{e}_k+\mathbf{e}_s-h_s\cdot h^J)+q^{\ddot{\Delta}_0}(\mathbf{e}_{k'}) \\ &= q^J(\mathbf{e}_k+\mathbf{e}_s)+q_{k'} = q_k+q_s+q_{ks}+q_{k'}. \end{aligned} \tag{6.3}$$

On the other hand, $q(\mathbf{e}_k+\mathbf{e}_{k'}+\mathbf{e}_s) = q_k+q_{k'}+q_s+q_{ks}+q_{k's}$, which implies that $q_{k's}=0$, a contradiction. To show that $q^{(s)}$ is irreducible, we recall that $q^J=\alpha\check{q}$, where $\alpha\geq 1$ and $\check{q}$ is an irreducible connected Cox-regular form. If q^J is irreducible, then $\alpha=1$ and $(q^J)^{(s)}=\check{q}^{(s)}$ is irreducible and so is $q^{(s)}$. In the case when $\alpha\geq 2$, by Proposition 4.12, there is at least one $i\in \underline{n}\setminus J$ such that $q_i=1$, that is, $q^{(s)}$ is irreducible (since $s\in J$, cf. Remark 4.3 (a)). Therefore, s is a strictly omissible vertex of q.

Step 3^o. Assume that $c\geq 2$. First observe that given arbitrary connected (bi)graph Δ, there exist at least two distinct vertices $s,t\in\Delta_0$ such that both $\Delta^{(s)}$ and $\Delta^{(t)}$ are connected. Indeed, take for instance two distinct leaves of a spanning tree of Δ. Using this observation we can find an order $\{t_1,t_2,\ldots,t_n\}=\underline{n}$ of the vertices $\Delta_0=\underline{n}$ of the bigraph $\Delta=\Delta_q$ of q such that all the subbigraphs $\Delta^{(t_1)}\supseteq\Delta^{(t_1,t_2)}\supseteq\Delta^{(t_1,t_2,t_3)}\supseteq\cdots\supseteq\Delta^{(t_1,t_2,\ldots,t_{n-1})}$ obtained from Δ by removing consecutive vertices $t_1,t_2,\ldots,t_{n-1}$ are connected, and $t_n\in\Delta_0$ is the vertex such that $q_{t_n}=q(\mathbf{e}_{t_n})=1$, see Proposition 4.12. Then by Lemma 3.1 (a) there exists $1\leq\ell\leq n-2$ such that $\Delta':=\Delta^{(t_1,t_2,\ldots,t_\ell)}$ has corank 1, cf. Lemma 2.5 (a1). Moreover, $q':=q^{(t_1,t_2,\ldots,t_\ell)}$ is irreducible since Δ' contains the loop-free vertex t_n, cf. Remark 4.3 (a). Thus q' has a strictly omissible vertex $s\in\Delta'_0$ by Steps 1^o and 2^o. Fix $h'\in\ker q'$ with $h'_s=1$. Then $h:=\iota_J(h')\in\ker q$, for $J=\underline{n}\setminus\{t_1,t_2,\ldots,t_\ell\}$ by Lemma 2.5 (a2). In particular, s (treated as an element of J via ι_J) is omissible for q. We show that s is strictly omissible in q.

First observe that since $q'^{(s)}$ is irreducible then so is $q^{(s)}$, cf. Remark 4.3 (a) (note that $|\Delta'_0|\geq 2$). We show that $q^{(s)}$ is connected by applying analogous arguments as in Case 2.2^o. That is, assume to the contrary that $q^{(s)}$ is disconnected, i.e., there exist $\dot{\Delta},\ddot{\Delta}$ such that $\Delta_q^{(s)}=\dot{\Delta}\sqcup\ddot{\Delta}$ and $\Delta'^{(s)}\subseteq\dot{\Delta}$. Since Δ and Δ' are connected there exist $k\in\Delta_0'^{(s)}$ and $k'\in\ddot{\Delta}_0$ such that $q_{ks}\neq 0$ and $q_{k's}\neq 0$. Since $h_s=1$, analogously to (6.3) we get

$$\begin{aligned} q(\mathbf{e}_k+\mathbf{e}_{k'}+\mathbf{e}_s) &= q(\mathbf{e}_k+\mathbf{e}_{k'}+\mathbf{e}_s-h) = q'(\mathbf{e}_k+\mathbf{e}_s-h')+q_{k'} \\ &= q'(\mathbf{e}_k+\mathbf{e}_s)+q_{k'} = q_k+q_s+q_{ks}+q_{k'}. \end{aligned} \tag{6.4}$$

But $q(\mathbf{e}_k+\mathbf{e}_{k'}+\mathbf{e}_s)=q_k+q_{k'}+q_s+q_{ks}+q_{k's}$ which implies that $q_{k's}=0$, a contradiction. This shows that $q^{(s)}$ is connected so s is strictly omissible for q.

To prove (b) we proceed by induction on $c=\mathbf{crk}(q)\geq 1$. Assume first that $c=1$. By (a) q has a strictly omissible vertex s. In the principal case it means that there is a generator h of $\ker q$ with $h_s=1$ and it is clear that $\{h\}$ forms an $\{s\}$-special basis, thus we get the claim.

Now assume that $c\geq 2$. By (a) q has a strictly omissible vertex $s\in\underline{n}$. Take $h\in\ker q$ with $h_s=1$. It follows that $\hat{q}:=q^{(s)}$ is irreducible, connected and $\hat{q}$ has corank $c-1$ by Lemma 3.1 (b). So by the inductive assumption there exists a strictly special

$\widehat{S}$-basis $\hat{h}^1, \dots, \hat{h}^{c-1}$ of $\ker \hat{q}$ for a subset $\widehat{S} = \{s_1, \dots, s_{c-1}\} \subseteq \underline{n} \setminus \{s\}$ (we identify the set $\underline{n} \setminus \{s\}$ with $\{1, \dots, n-1\}$). Then the vectors $h^i := \iota_s(\hat{h}^i)$ belong to $\ker q$ for each $i = 1, \dots, c-1$, where $\iota_s = \iota_{\underline{n}\setminus\{s\}}\colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n$ is the corresponding inclusion, see Lemma 2.5 (a2). Then clearly the set $\mathcal{H} := \{h^1, \dots, h^{c-1}, h^c := h\}$ is linearly independent. To show that $\mathcal{H}$ generates $\ker q$ take $y \in \ker q$. Then $\hat{y} := \pi_s(y - y_s h)$ belongs to $\ker \hat{q}$ by Lemma 2.5 (a3), so $\hat{y} = \alpha_1 \hat{h}^1 + \dots + \alpha_{c-1}\hat{h}^{c-1}$ for some $\alpha_i \in \mathbb{Z}$. Thus applying the identity $\iota_s \pi_s(y - y_s h) = y - y_s h$ we infer that $y = \alpha_1 h^1 + \dots + \alpha_{c-1} h^{c-1} + y_s h$.

Hence we have constructed a basis $\mathcal{H}$ of $\ker q$. We modify it to obtain a special basis. Set $\mathcal{H}' := (\mathcal{H} \setminus \{h\}) \cup \{h'\}$ where $h' := h - h_{s_1}h^1 - h_{s_2}h^2 - \dots - h_{s_{c-1}}h^{c-1}$. Note that $\mathcal{H}'$ is also a basis of $\ker q$. Moreover, by construction and the shapes of vectors $h^i = \iota_s(\hat{h}^i)$ it follows that $h'_s = 1$ and $h'_{s_i} = 0$ for $i = 1, \dots, c-1$. This shows that $\mathcal{H}'$ is an S-special basis of $\ker q$ for $S := \widehat{S} \cup \{s\}$. In fact, $\mathcal{H}$ is strictly special since $q^{(S)} = (q^{(s)})^{(\widehat{S})}$ is irreducible and connected. ∎

7. Weak Dynkin type

Having the results of previous sections we are ready to prove Theorem 2 formulated in Introduction. Observe that given $T \in \mathrm{Gl}_n(\mathbb{Z})$, the equality

$$q \circ T(x_1, \dots, x_n) = q_{D_r}(x_1, \dots, x_r)$$

(see (1.4)) is equivalent with

$$q \sim^T q_{D_r} \oplus \xi^c \tag{7.1}$$

for the zero form $\xi^c\colon \mathbb{Z}^c \to \mathbb{Z}$ with $c = n - r$.

7.1. Proof of Theorem 2

Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Cox-regular form of rank $r = \mathbf{rk}(q) \geq 1$ and corank $c = \mathbf{crk}(q) \geq 0$, that is, $n = r + c$. If $c = 0$ then q is positive and both assertions (a) and (b) follow trivially by Theorem 5.9 (a). Assume then that $c \geq 1$.

By Theorem 6.2 (b), there exist a subset

$$S = \{s_1 < s_2 < \dots < s_c\} \subseteq \underline{n}$$

and a strictly S-special basis $h^1, \dots, h^c$ of $\ker q$. Let $\hat{q} := q^{(S)}\colon \mathbb{Z}^r \to \mathbb{Z}$ be the induced positive irreducible connected restriction of q, cf. Proposition 3.4 (b). Take $D = D_r := \mathbf{Dyn}(\hat{q})$ and fix $B \in \mathrm{Gl}_r(\mathbb{Z})$ such that $\hat{q} \sim_G^B q_D$, see Definition 5.10 (a) and Theorem 5.9 (a). Next, let $P = P^\rho \in \mathrm{Gl}_n(\mathbb{Z})$ be the matrix of the permutation $\rho\colon \underline{n} \to \underline{n}$ given by $\rho(s_i) = r + i$ for $i = 1, \dots, c$, and $\rho(j_i) = i$ for $i = 1, \dots, r$, where $\underline{n} \setminus S = \{j_1 < \dots < j_r\}$. Then clearly $q' := q \circ P^{\mathrm{tr}}$ is a non-negative connected irreducible Cox-regular form of corank c. Moreover, it is easy to check that $h'^1 := Ph^1, \dots, h'^c := Ph^c$ is a strictly S'-special basis of $\ker q'$, where $S' = \{r+1, \dots, n-1, n\}$. Let $\hat{q}' := q'^{(S')}\colon \mathbb{Z}^r \to \mathbb{Z}$ be the induced irreducible connected restriction. Then $\hat{q}' \sim^{B'} \hat{q}$ for some $B' \in \mathrm{Gl}_r(\mathbb{Z})$ by Proposition 3.4 (c).

Consider the $\mathbb{Z}$-invertible matrix $H = [\mathbf{e}_1|\cdots|\mathbf{e}_r|h'^1|\cdots|h'^c] \in \mathrm{Gl}_n(\mathbb{Z})$, cf. (3.2). By Lemma 2.3 (a) and (2.2) we have that $q'(y,h') = q'(h',y) = h'^{\mathrm{tr}} G_{q'} y = 0$ for $h' \in \ker q'$ and $y \in \mathbb{Z}^n$. Hence, we check that

$$H^{\mathrm{tr}} G_{q'} H = \left[\begin{array}{c|c} \mathbf{e}_i^{\mathrm{tr}} G_{q'} \mathbf{e}_j & \mathbf{e}_i^{\mathrm{tr}} G_{q'} h'^k \\ \hline {h'^k}^{\mathrm{tr}} G_{q'} \mathbf{e}_j & {h'^k}^{\mathrm{tr}} G_{q'} h'^l \end{array}\right] = \left[\begin{array}{c|c} G_{\hat{q}'} & 0 \\ \hline 0 & 0 \end{array}\right],$$

for $1 \le i,j \le r$ and $1 \le k,l \le c$. This means that $q' \circ H = \hat{q}' \oplus \xi^c$. Summarizing, we obtain the following sequence of $\mathbb{Z}$-equivalences

$$q \sim^{P^{\mathrm{tr}}} q' \sim^{H} \hat{q}' \oplus \xi^c \sim^{B' \oplus \mathrm{I}_c} \hat{q} \oplus \xi^c \sim^{B \oplus \mathrm{I}_c} q_D \oplus \xi^c, \tag{7.2}$$

thus $T := P^{\mathrm{tr}} H (B'B \oplus \mathrm{I}_c) \in \mathrm{Gl}_n(\mathbb{Z})$ is the desired $\mathbb{Z}$-equivalence $q \sim^T q_D \oplus \xi^c$, cf. (7.1).

To show the uniqueness (up to $\sim$) of $D = D_r$ assume that there exists other Dynkin bigraph $D' = D'_r$ such that $q_D \oplus \xi^c \sim q_{D'} \oplus \xi^c$. It is clear that the canonical vectors $\mathbf{e}_{r+1}, \dots, \mathbf{e}_{n-1}, \mathbf{e}_n$ form an S''-special basis of $\ker q_D \oplus \xi^c$ as well as of $\ker q_{D'} \oplus \xi^c$, for $S'' = \{r+1, \dots, n-1, n\}$. So again applying Proposition 3.4 (c) we get that

$$q_D = (q_D \oplus \xi^c)^{(S'')} \sim (q_{D'} \oplus \xi^c)^{(S'')} = q_{D'}.$$

To show the remaining assertion (b) note that the restriction $\hat{q} = q^{(S)}$ of q defined in the proof of (a) satisfies the required properties. This finishes the proof of Theorem 2. ■

Theorem 2 (and its proof) justifies the introduction of the following notion.

Definition 7.3. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Cox-regular form. Fix a strictly S-special basis of $\ker q$ for some $S \subseteq \underline{n}$ and let $q^{(S)}$ be the induced positive irreducible connected restriction of q (if q is positive we take $q^{(S)} := q$). Then the $\sim_d$-equivalence class

$$\mathbf{wDyn}(q) := \big[\mathbf{Dyn}(q^{(S)})\big]_{\sim_d} \tag{7.4}$$

is called the *weak Dynkin type* of q, where $\sim_d$ denotes the relation $\sim$ restricted to connected irreducible classic positive Cox-regular forms, equivalently, to Dynkin bigraphs (cf. Proposition 5.2 (a)) and $\mathbf{Dyn}(q^{(S)})$ is the Dynkin bigraph associated to $q^{(S)}$ as in Definition 5.10 (a).

Lemma 7.5. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Cox-regular form with $r = \mathbf{rk}(q) \ge 1$ and $c = \mathbf{crk}(q) \ge 0$.*

(a) $\mathbf{wDyn}(q)$ *is well defined, in particular, it does not depend on the choice of a strictly special basis of* $\ker q$.

(b) $\mathbf{wDyn}(q) = [D_r]_{\sim_d}$ *if and only if* $q \sim q_{D_r} \oplus \xi^c$, *for any Dynkin bigraph D_r with r vertices.*

Proof. Fix a subset $S \subseteq \underline{n}$ such that there exists a strictly S-special basis of $\ker q$, see Theorem 6.2 (b). Then $q^{(S)}$ is a positive (connected, irreducible) Cox-regular form by Proposition 3.4 (b). Therefore the associated Dynkin bigraph $\mathbf{Dyn}(q^{(S)})$ is defined and unique, see Theorem 5.9 (a) and Lemma 5.8. Now, if there exists a strictly S'-basis of $\ker q$ for another subset $S' \subseteq \underline{n}$, then $q^{(S)} \sim q^{(S')}$ by Proposition 3.4 (c), thus $\mathbf{Dyn}(q^{(S)}) \sim q^{(S)} \sim q^{(S')} \sim \mathbf{Dyn}(q^{(S')})$. This means that $\mathbf{Dyn}(q^{(S)}) \sim_d \mathbf{Dyn}(q^{(S')})$ so (a) holds.

To show (b) recall that $\mathbf{wDyn}(q) = [D'_r]_{\sim_d}$, where $D'_r = \mathbf{Dyn}(q^{(S)})$, see (7.4). On the other hand, $q \sim q^{(S)} \oplus \xi^c$ by the arguments in the proof of Theorem 2, see (7.2). In particular, $q \sim q_{D'_r} \oplus \xi^c$. So the claim holds by the uniqueness (up to $\sim$) of a Dynkin bigraph in Theorem 2 (a). ∎

Corollary 7.6. *Let $q, q' \colon \mathbb{Z}^n \to \mathbb{Z}$ be two non-negative connected irreducible Cox-regular forms. Then $q \sim q'$ if and only if $\mathbf{wDyn}(q) = \mathbf{wDyn}(q')$.*

Proof. Set $r := \mathbf{rk}(q)$, $r' := \mathbf{rk}(q')$, $c := \mathbf{crk}(q)$ and $c' := \mathbf{crk}(q')$. Then $r + c = n = r' + c'$ and by Theorem 2 we get the equivalences $q \sim q_{D_r} \oplus \xi^c$ and $q' \sim q_{D'_{r'}} \oplus \xi^{c'}$ for Dynkin bigraphs D_r and $D'_{r'}$. Hence $\mathbf{wDyn}(q) = [D_r]_{\sim_d}$ and $\mathbf{wDyn}(q') = [D'_{r'}]_{\sim_d}$ by Lemma 7.5 (b).

If $q \sim q'$ then $r = r'$ and $c = c'$, so by the uniqueness (up to $\sim$) of a Dynkin bigraph in Theorem 2 (a) we get that $q_{D_r} \sim q_{D'_r}$. This means that $\mathbf{wDyn}(q) = \mathbf{wDyn}(q')$. Conversely, if $[D_r]_{\sim_d} = [D'_{r'}]_{\sim_d}$ then $r = r'$ so also $c = c'$. Therefore,

$$q \sim q_{D_r} \oplus \xi^c \sim q_{D'_r} \oplus \xi^c \sim q'. \qquad \blacksquare$$

Remark 7.7. By Lemma 5.3 it follows that given a non-negative connected irreducible Cox-regular form q, its weak Dynkin type $\mathbf{wDyn}(q)$ is one of the following (up to trivial equivalences):

$$\begin{aligned}
&[\mathbb{A}_n]_{\sim_d} = \{\mathbb{A}_n\}, \quad \text{for } n = 1 \text{ or } n \geq 4;\\
&[\mathbb{A}_2]_{\sim_d} = \{\mathbb{A}_2, \mathcal{G}_2\}; \quad [\mathbb{A}_3]_{\sim_d} = \{\mathbb{A}_3, \mathcal{C}_3\};\\
&[\mathbb{D}_4]_{\sim_d} = \{\mathbb{D}_4, \mathcal{C}_4, \mathcal{F}_4\};\\
&[\mathbb{D}_n]_{\sim_d} = \{\mathbb{D}_n, \mathcal{C}_n\}, \quad \text{for } n \geq 5;\\
&[\mathbb{E}_6]_{\sim_d} = \{\mathbb{E}_6\}; \quad [\mathbb{E}_7]_{\sim} = \{\mathbb{E}_7\}; \quad [\mathbb{E}_8]_{\sim_d} = \{\mathbb{E}_8\};\\
&[\mathcal{B}_n]_{\sim_d} = \{\mathcal{B}_n\}, \quad \text{for } n \geq 2.
\end{aligned}$$

Thus, speaking a bit informally, there is only one family more (namely, $[\mathcal{B}_n]_{\sim_d}$) of possible weak Dynkin types of Cox-regular forms in comparison with the (classical) Dynkin types $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ of unit forms, cf. [2,46]. On the other hand, we see that $\mathbf{wDyn}(q)$ identifies some of the distinct (more precisely, not G-equivalent, cf. Lemma 5.8) Dynkin types in the positive case. This is the reason we call it "weak" Dynkin type. It is an interesting more specialized open problem to classify of all non-negative (connected, irreducible) Cox-regular forms with respect to the G-equivalence. Our results in this section may be viewed as a useful step towards such classification.

The following fact provides a more general version of claim (b) of Theorem 2, compare also with the analogous [2, Corollary 2.5] for unit forms.

Proposition 7.8. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Cox-regular form of rank $r = \mathbf{rk}(q) \geq 1$ and corank $c = \mathbf{crk}(q) \geq 1$. Then for every $0 \leq \hat{c} \leq c$ there exists a connected irreducible restriction $\hat{q}$ of q with $\mathbf{crk}(\hat{q}) = \hat{c}$ and $\mathbf{wDyn}(\hat{q}) = \mathbf{wDyn}(q)$.*

Proof. Let $S \subseteq \underline{n}$ be a subset such that there exists a strictly S-special basis of $\ker q$, cf. Theorem 6.2 (b). Since q and $q^{(S)}$ are connected, we can choose a subset $\widehat{S} \subseteq S$ of size $|\widehat{S}| = c - \hat{c}$ such that $\hat{q} := q^{(\widehat{S})}$ is also connected. Note that $\hat{q}$ is irreducible since so is $q^{(S)}$, cf. Remark 4.3. Moreover, by Proposition 3.4 (a) we have $\mathbf{crk}(\hat{q}) = \hat{c}$ and $\ker \hat{q}$ admits an $(S \setminus \widehat{S})$-special basis. This basis is strictly special since $\hat{q}^{(S\setminus\widehat{S})} = q^{(S)}$ is connected and irreducible. Thus by definition $\mathbf{wDyn}(\hat{q}) = [\mathbf{Dyn}(\hat{q}^{(S\setminus\widehat{S})})]_{\sim_d} = [\mathbf{Dyn}(q^{(S)})]_{\sim_d} = \mathbf{wDyn}(q)$. ■

To complete the picture we note that the weak Dynkin types of principal Cox-regular forms are related with their Euclidean types as follows.

Lemma 7.9. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a principal connected irreducible Cox-regular form and $D = \mathbf{Euc}(q)$ its Euclidean type. Then $\mathbf{wDyn}(q) = [D^{(s)}]_{\sim_d}$ for the Dynkin (sub)bigraph $D^{(s)}$ of D as indicated in table (5.7).*

Proof. Recall that $q \sim q_D$ by Theorem 5.9 (b), see Definition 5.10 (b). In particular,

$$\mathbf{wDyn}(q) = \mathbf{wDyn}(q_D)$$

by Corollary 7.6. Now observe that since every bigraph $D^{(s)}$ in the bottom row of (5.7) is irreducible and connected it follows that in all cases the chosen omissible vertices $s = s_D \in D_0$ in the proof of Lemma 5.4 (e) are strictly omissible. It means that $D^{(s)} \in \mathbf{wDyn}(q_D)$ by Definition 7.3, since in the principal case a generator h of $\ker q$ with $h_s = 1$ forms an $\{s\}$-special basis. ■

Finally, observe that given a non-negative connected irreducible Cox-regular form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ with $c = \mathbf{crk}(q) \geq 1$, its canonical form $q_D \oplus \xi^c$ in Theorem 2 (a) is neither Cox-regular nor connected, since it fails to satisfy $q_i > 0$ for all $i \in \underline{n}$, cf. Lemma 2.7 and Section 4. However, one may consider a slightly more general class of integral quadratic forms q such that $q_i \geq 0$ for all $i \in \underline{n}$, and $q_{ij}/q_i \in \mathbb{Z}$ (resp. $q_{ij}/q_j \in \mathbb{Z}$) for all $i < j$ with $q_i \neq 0$ (resp. $q_j \neq 0$). We call such forms *semi-Cox-regular* (as an analog of semi-unit forms in [2]). Obviously, $q_D \oplus \xi^c$ as above is semi-Cox-regular. Observe that if a non-negative semi-Cox-regular form q is connected, then q is Cox-regular, cf. Lemma 2.7. Moreover, we can extend Theorem 2 (a) to semi-Cox-regular forms as follows.

Corollary 7.10. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative semi-Cox-regular form (not necessarily connected or irreducible) of corank $c = \mathbf{crk}(q) \geq 0$. Then there exist $k \geq 0$ and collections of integers $\alpha_1, \ldots, \alpha_k \geq 1$ and Dynkin bigraphs $D^1_{r_1}, \ldots, D^k_{r_k}$ of sizes $r_1, \ldots, r_k \geq 1$ such that*

$$q \sim \alpha_1 q_{D^1_{r_1}} \oplus \cdots \oplus \alpha_1 q_{D^k_{r_k}} \oplus \xi^c. \tag{7.11}$$

Proof. If $r := \mathbf{rk}(q) = 0$ then q is a zero form hence $q = \xi^n$ and the claim holds trivially. Assume that $r \geq 1$. Then by Lemma 2.7, the form q is trivially equivalent to $q' \oplus \xi^t$ for some $0 \leq t < n$ and a non-negative Cox-regular form $q'\colon \mathbb{Z}^{n-t} \to \mathbb{Z}$. Moreover, there exist irreducible connected Cox-regular forms $q^1, \dots, q^k$ and integers $\alpha_1, \dots, \alpha_k \in \mathbb{Z}$ such that $q' \cong \alpha_1 q^1 \oplus \dots \oplus \alpha_k q^k$, see Remark 4.3 (a). Now applying Theorem 2 (a) to all q^i's we get that $q^i \sim q_{D^i_{r_i}} \oplus \xi^{c_i}$ where $D^i_{r_i}$ is a Dynkin bigraph of size $r_i = \mathbf{rk}(q^i)$, and $c_i = \mathbf{crk}(q^i)$, for each $i = 1, \dots, k$. This gives us decomposition (7.11) since $c_1 + \dots + c_k + t = c$ by Lemma 2.5 (b). ∎

8. Universality

Recall that a non-negative integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is universal if it represents all non-negative integers, that is, the set $R_q(d)$ is non-empty for each $d \geq 1$, cf. (2.1). For one of the crucial arguments for Theorem 1 we need to prove the universality of the following few small Cox-regular forms. Recall that the universality of $q_{\mathbb{A}_4}$ and $q_{\mathbb{D}_4}$ was discussed in [43,45]. However, the proof for $\mathbb{A}_4$ contains a small gap, see [45, p. 359]. For the sake of completeness we give a detailed (slightly different) proof also for this case.

Lemma 8.1. *The quadratic form $q_D\colon \mathbb{Z}^4 \to \mathbb{Z}$ associated to each of the Dynkin bigraphs $D \in \{\mathbb{A}_4, \mathcal{B}_4, \mathcal{C}_4, \mathbb{D}_4, \mathcal{F}_4\}$ with 4 vertices is universal.*

Proof. Consider two integer 4×4 matrices

$$B := \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C := \begin{bmatrix} -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

We check directly that $q_{\mathcal{B}_4} \circ B = q_{\mathrm{Lag}} = q_{\mathcal{C}_4} \circ C$, where

$$q_{\mathrm{Lag}}(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

is the Lagrange "four squares" form. Note that $\det(B) = -1$ so $q_{\mathcal{B}_4} \sim^B q_{\mathrm{Lag}}$, but C does not define a $\mathbb{Z}$-equivalence since $\det(C) = 2$. However, since q_{Lag} is universal by Lagrange theorem, we get that for arbitrary integer $d > 0$ there exists $x \in \mathbb{Z}^4$ such that $q_{\mathrm{Lag}}(x) = d$, therefore $q_{\mathcal{B}_4}(Bx) = d = q_{\mathcal{C}_4}(Cx)$ with $Bx, Cx \in \mathbb{Z}^4$. This shows that $q_{\mathcal{B}_4}$ and $q_{\mathcal{C}_4}$ are universal. Moreover, since $q_{\mathcal{C}_4} \sim q_{\mathbb{D}_4} \sim q_{\mathcal{F}_4}$ by Lemma 5.3, we get that $q_{\mathbb{D}_4}$ and $q_{\mathcal{F}_4}$ are also universal (cf. Lemma 2.4 (a)).

The remaining case $D = \mathbb{A}_4$ needs a bit different approach. Take the rational matrix

$$A := \begin{bmatrix} \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

We verify directly that $2(q_{\mathbb{A}_4} \circ A) = q_{\mathrm{Ram}}$, where $q_{\mathrm{Ram}} \colon \mathbb{Z}^4 \to \mathbb{Z}$, given by

$$q_{\mathrm{Ram}}(y_1, y_2, y_3, y_4) = y_1^2 + 2y_2^2 + 2y_3^2 + 5y_4^2,$$

is one of 54 universal diagonal integral quadratic forms described by Ramanujan in [39]. Now take an arbitrary integer $d > 0$. By the universality of q_{Ram} there exists an integer vector $y = [y_1, y_2, y_3, y_4]^{\mathrm{tr}}$ such that $q_{\mathrm{Ram}}(y) = 2d$. This means that $q_{\mathbb{A}_4}(Ay) = d$. So it remains to show that Ay is an integer vector. To see this, observe that $Ay = [x_1, x_2, x_3, x_4]^{\mathrm{tr}}$, where

$$\begin{aligned} x_1 &= \frac{1}{2}(y_1 + 2y_2 + y_4), \\ x_2 &= y_1 + y_4, \\ x_3 &= \frac{1}{2}(y_1 + 2y_3 + 3y_4), \\ x_4 &= 2y_4. \end{aligned} \tag{8.2}$$

Therefore, to finish the proof, it remains to show that x_1 and x_3 in (8.2) are integers. Since

$$2d = y_1^2 + 2y_2^2 + 2y_3^2 + 5y_4^2 = 2(y_2^2 + y_3^2 + 2y_4^2) + (y_1^2 + y_4^2),$$

the integer $y_1^2 + y_4^2$ is even. Now, we show that $y_1 + y_4$ is even. Assume to the contrary that there exists $k \in \mathbb{Z}$ such that $y_1 + y_4 = 2k + 1$. This assumption yields the contradiction

$$y_1^2 + y_4^2 = (y_1 + y_4)^2 - 2y_1 y_4 = 4k^2 + 4k + 1 - 2y_1 y_4 = 2(2k^2 + 2k - y_1 y_4) + 1.$$

Hence, there exists $k' \in \mathbb{Z}$ such that $y_1 + y_4 = 2k'$. Therefore, $x_1 = \frac{1}{2}(y_1 + 2y_2 + y_4) = k' + y_2 \in \mathbb{Z}$ and $x_3 = \frac{1}{2}(y_1 + 2y_3 + 3y_4) = k' + y_3 + y_4 \in \mathbb{Z}$, and the proof is finished. ∎

8.1. Proof of Theorem 1

Let $q \colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected irreducible Cox-regular form of rank $r = \mathbf{rk}(q) \geq 0$. If $r = 0$ then the form q is the zero form so the hypothesis of Theorem 1 holds trivially. Assume then that $r \geq 1$. By Theorem 2 (a) there exists a Dynkin bigraph D_r with r vertices such that $q \sim q_{D_r} \oplus \xi^c$ for the zero form $\xi^c \colon \mathbb{Z}^c \to \mathbb{Z}$ with $c = n - r$, cf. (7.1). So clearly, q represents precisely the same integers as q_{D_r} (cf. Lemma 2.4 (a)).

We start with the implication "(b)⇒(a)". Assume that $r \geq 4$. Then D_r has at least 4 vertices, so we easily see by inspection of Table 1 that there exists a Dynkin bigraph $D_4 \in \{\mathbb{A}_4, \mathcal{B}_4, \mathcal{C}_4, \mathbb{D}_4, \mathcal{F}_4\}$ with 4 vertices which is a full (induced) subbigraph of D_r. This means that q_{D_4} is a restriction of q_{D_r}, that is, $q_{D_4} = q_{D_r} \circ \iota_J$ for some $J \subseteq \underline{n}$ with $|J| = 4$, cf. Section 2. Now take an arbitrary $d > 0$. By Lemma 8.1 there exists $x \in \mathbb{Z}^4$ such that $q_{D_4}(x) = d$. Thus $d = q_{D_4}(x) = q_{D_r}(\iota_J(x))$. This shows that q_{D_r} is universal, and so is q.

The implication "(a)⇒(c)" is trivial. Therefore, to finish the proof, it remains to show that "(c)⇒(b)". To prove this implication, assume to the contrary that $r < 4$. Then D_r is one of the Dynkin bigraphs $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathcal{B}_2, \mathcal{B}_3, \mathcal{C}_2, \mathcal{C}_3, \mathcal{G}_2$. Case by case verification shows that the truants (i.e., the smallest not represented positive integers) for integral forms associated with these bigraphs are as follows:

D	truant(q_D)
$\mathbb{A}_1$	2
$\mathbb{A}_2, \mathcal{G}_2$	2
$\mathcal{B}_2, \mathcal{C}_2$	3
$\mathcal{B}_3$	7
$\mathbb{A}_3, \mathcal{C}_3$	14

(8.3)

One verifies (8.3) easily by hand, cf. Lemma 5.3. One can also use a simple computer search applying the general known fact stating that: every solution $x \in \mathbb{Z}^n$ of the equation $q(x) = d$ for a positive integral form $q\colon \mathbb{Z}^r \to \mathbb{Z}$ satisfies $\|x\| \leq \sqrt{d/m}$, where $\|x\|$ denotes the usual Euclidean norm and $m := \inf(q(\mathbb{S}^{r-1})) > 0$, for a unit sphere $\mathbb{S}^{r-1} := \{z \in \mathbb{R}^r : \|z\| = 1\}$, see [40, pp. 3–4], cf. [43, Proposition 4.1, Algorithm 4.2].

In particular, (8.3) shows that q_{D_r} (equivalently, q) does not represent at least one of the numbers $2, 3, 7$ or 14. This finishes the proof of the remaining implication "(c)⇒(b)". ■

To conclude the proof of Theorem 1 we note that its main statement remains valid under a slightly weaker assumptions provided we add the number 1 to the list of represented integers. That is, the following fact holds.

Corollary 8.4. *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected (not necessarily irreducible) Coxeter-regular form. Then q is universal if and only if q represents the integers* $1, 2, 3, 7$ *and* 14.

Proof. Note that if q represents 1 then it has to be irreducible. Now apply "(a)⇔(c)" of Theorem 1. ■

Remark 8.5. For arbitrary $n \geq 1$, the unit form $q = q_{\mathbb{A}_n}\colon \mathbb{Z}^n \to \mathbb{Z}$ associated to the Dynkin graph $\mathbb{A}_n$ satisfies all the assumptions of Theorem 1, that is, $q_{\mathbb{A}_n}$ is a positive (hence non-negative), connected, irreducible Cox-regular form. In particular, q is universal if and only if $n \geq 4$. However, for $n \geq 2$ the Gram matrix $G_q \in \mathbb{M}_n(\frac{1}{2}\mathbb{Z})$ is not integral (since $\frac{q_{12}}{2} = \frac{-1}{2} \notin \mathbb{Z}$) hence q is not embraced by "15 Theorem" [6, 10], cf. Section 1.

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