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Two further characterizations of orthodiagonal quadrilaterals

Hans Humenberger

Hans Humenberger got his PhD 1993 and 1998 his habilitation in the field of mathematics education, both at the University of Vienna. From 2000 until 2005, he was assistant professor at the University of Dortmund (Germany). Since 2005, he is a full professor for mathematics at the University of Vienna with special emphasis on mathematics education. He is the head of the group Didactics of Mathematics and School Mathematics and responsible for the education of teachers for lower secondary and high schools.

1 A characterization using maltitudes

In summer 2024, we read the interesting article [4] by Josefsson about orthodiagonal quadrilaterals and learned the term *maltitude of a quadrilateral* (the term is older; see e.g. Wells [11, p. 146]). There are several well-known theorems about maltitudes, for instance that the maltitudes of a cyclic quadrilateral are concurrent; see e.g. Wells [11] or De Villiers [9]. *Maltitude* is an abbreviation for *midpoint altitude* and denotes the line emanating from a midpoint of a side perpendicular to the opposite side. A quadrilateral $ABCD$ with side midpoints E, F, G, H and corresponding maltitudes (dotted) is shown in Figure 1. Intersecting maltitudes of adjacent quadrilateral sides yields four points, in Figure 1 labeled by I, J, K, L.

The arising quadrilateral $IJKL$ (we call it *maltitudes quadrilateral*) reminded us immediately of constructing a quadrilateral using the perpendicular bisectors of an initial quadrilateral, a famous and well-known process with many interesting features (see

Es gibt zahlreiche verschiedene Charakterisierungen für Vierecke mit orthogonalen Diagonalen. Im Englischen ist dafür der Fachbegriff *orthodiagonal quadrilateral* weiter verbreitet als im Deutschen *orthodiagonales Viereck*. Die Liste solcher Charakterisierungen wird im folgenden Beitrag um zwei erweitert, die offenbar so noch nicht in der Literatur vorkommen. Die erste der beiden behandelten Charakterisierungen hat zu tun mit im Englischen sogenannten *maltitudes quadrilaterals*, wofür im Deutschen leider ein entsprechender Begriff fehlt. So viel sei aber schon hier verraten: Der englische Begriff *maltitude* ist ein Kofferwort aus *midpoint* (Mittelpunkt) und *altitude* (Höhe). Die zweite Charakterisierung bezieht sich auf Gelenksvierecke.

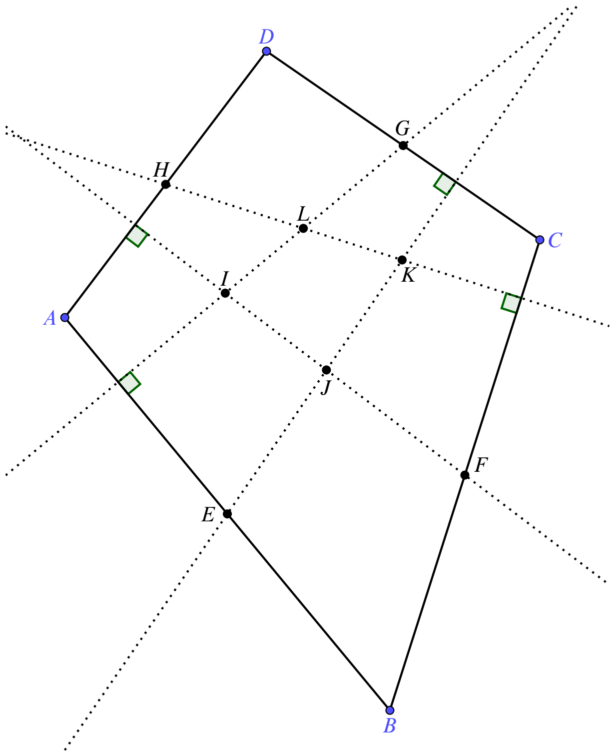

Figure 1. Maltitudes of a quadrilateral

e.g. Radko & Tsukerman [8]). And indeed, using maltitudes, or perpendicular bisectors respectively, for constructing a quadrilateral from a given one are two closely related processes: the maltitudes quadrilateral and the perpendicular bisectors quadrilateral are congruent; even more specific, they are symmetric with respect to the centroid Z of the initial quadrilateral (Figure 2). This is clear because a perpendicular bisector (e.g. EL') is sent to a maltitude (e.g. GL) by a half turn about the centroid Z, and vice versa; see Mammana [5] and Mammana & Micale [6]. So all the theorems about perpendicular bisector quadrilaterals (e.g. concerning their similarity when iterating the process of constructing them: all the quadrilaterals with even index are similar to each other, and all the quadrilaterals with odd index are similar to each other – dividing all the iterated perpendicular bisector quadrilaterals into two classes) also hold for maltitudes quadrilaterals.

Now let us come to a new (at least for us; if a reader happens to know this phenomenon somewhere described in the literature or in the WWW, with or without proof, please inform us and send us the corresponding references; this applies also for the phenomenon described in Section 2) connection between maltitudes quadrilaterals and orthodiagonal quadrilaterals. In [4], ten characterizations (i.e. necessary and sufficient conditions) of orthodiagonal quadrilaterals are described and proved, and here we present another one. The idea for it was triggered by looking at [4, Figure 2].

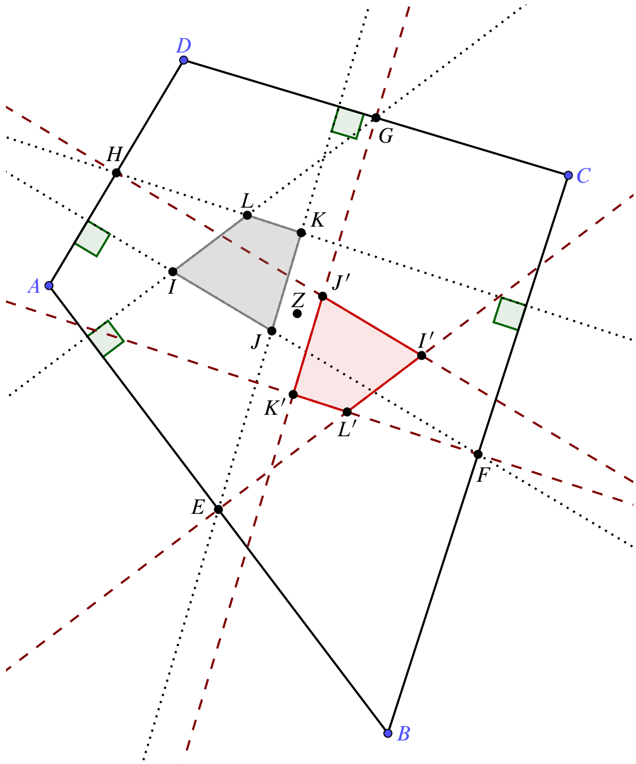

Figure 2. Maltitudes quadrilateral $IJKL$ and perpendicular bisector quadrilateral $I'J'K'L'$

Theorem 1. *The vertices of the maltitudes quadrilateral lie on the diagonals of a quadrilateral $ABCD$ if and only if $ABCD$ is orthodiagonal (in case that the quadrilateral $ABCD$ is cyclic, the maltitudes quadrilateral degenerates to a single point, the so-called anticenter; for an orthodiagonal cyclic quadrilateral, its anticenter coincides with the intersection point of the diagonals).*

Proof. Let $ABCD$ be an orthodiagonal quadrilateral and E, F, G, H the midpoints of the sides (Figure 3), and T the diagonals' point of intersection. The two maltitudes emanating from G and H intersect at L. We must show that L lies on diagonal BD (analogous with the other vertices of the maltitudes quadrilateral).

We draw the parallels to the maltitudes emanating from G and H through the points C and A, intersecting at Q. Then Q must be the orthocenter of $\triangle ABC$ and thus lie on its third altitude BT and hence on BD (note that $BT \perp AC$). Since Q must also lie on straight line DL (homothety centered at D, factor 2), we can conclude that L lies on the diagonal BD.

For the converse, let L lie on diagonal BD. We must show that $AC \perp BD$. Again, we draw the parallels to the maltitudes through A and C, intersecting at Q. This point Q must lie on the straight line DL (homothety centered at D, factor 2), and like above, Q must be

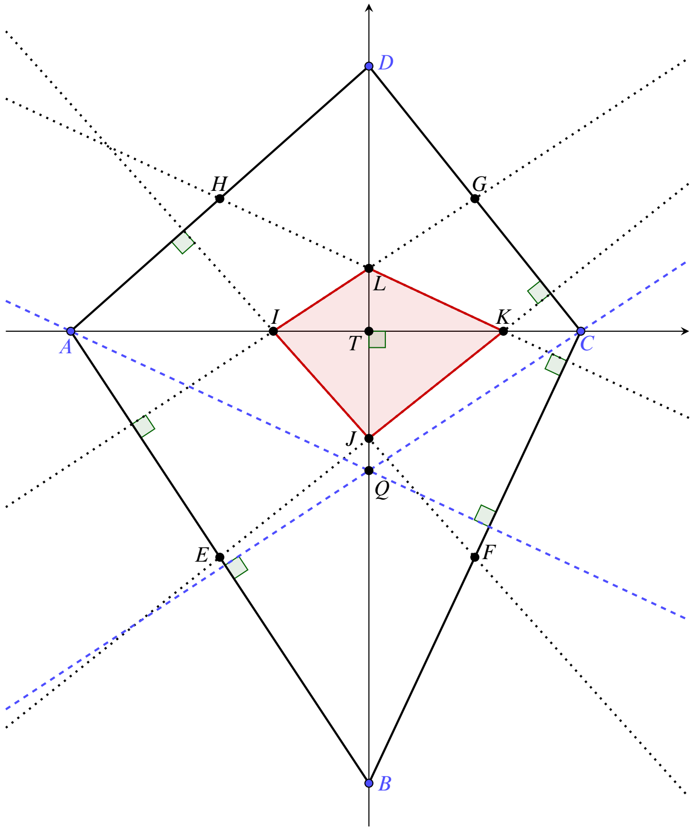

Figure 3. Orthodiagonal quadrilateral $ABCD$ with I, J, K, L on its diagonals

the orthocenter of $\triangle ABC$. This means that Q must also lie on its third altitude through B, and finally that $AC \perp BD$, as claimed. ■

A very well-known and famous characterization of orthodiagonal quadrilaterals with sides a, b, c, d is

$$a^2 + c^2 = b^2 + d^2 \quad \text{or equivalently} \quad b^2 + d^2 - a^2 - c^2 = 0;$$

this we will need also for the next section. The proof is, in one direction, straightforward with Pythagoras' Theorem, since both $a^2 + c^2$ and $b^2 + d^2$ equal the sum of the four squared distances from the orthodiagonal quadrilateral's vertices to the diagonals' point of intersection. Nevertheless, it also holds in the other direction

$$a^2 + c^2 = b^2 + d^2 \implies \text{orthodiagonal;}$$

see e.g. [4].

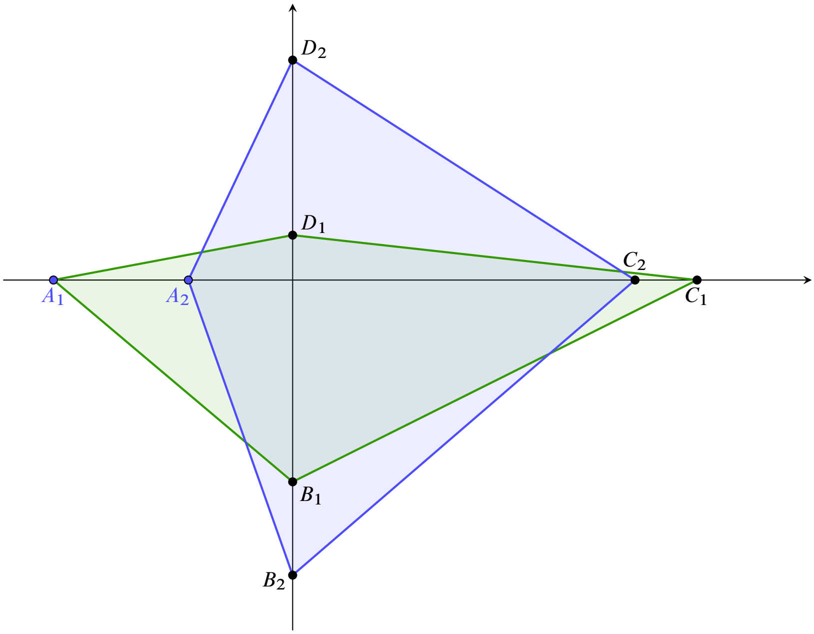

Figure 4. Two possible shapes of an orthodiagonal four-bar linkage $ABCD$, the coordinate axes are fixed "tracks" on which the points can move

2 A characterization using four-bar linkages

In this section, we consider quadrilaterals $ABCD$ with sides a, b, c, d (fixed) as four-bar linkages. Then the following is easy to see: if $ABCD$ is orthodiagonal, then all other possible shapes of this quadrilateral, thought of as four-bar linkage, are orthodiagonal, so to speak: orthodiagonality in one moment guarantees orthodiagonality "forever" in four-bar linkages! This is because a, b, c, d do not change, and the above-mentioned characterization for orthogonality $a^2 + c^2 = b^2 + d^2$ is fulfilled "forever" (meaning, in all possible positions). One can imagine that A and C are fixed to the x axis, and B and D to the y axis, and $ABCD$ is a four-bar linkage. Then the points can move somehow on the axes (Figure 4); the four-bar linkage is not rigid although the points are fixed to the axes!

A dynamic and kinematic version of it can be seen at https://walser-h-m.ch/hans/Miniaturen/O/Orthodiagonales_Gelenkviereck/Orthodiagonales_Gelenkviereck.html (latest access Nov. 11, 2024, text in German). The orthodiagonal quadrilaterals will prove to be the only type of non-crossed quadrilaterals with the above property: thinking of the quadrilateral as four-bar linkage, the angle between the diagonals does not change when deforming it. In other words, when the vertices of the four-bar linkage are thought to be attached to two fixed "tracks" (diagonals, like in Figure 4 the coordinate axes), the vertices can move somehow on these "tracks". Having shown that this is impossible for non-orthodiagonal quadrilaterals, we will have the next characterization for orthodiagonal quadrilaterals, formulated in Theorem 2.

Theorem 2. *Let $ABCD$ be a non-crossed four-bar linkage (with fixed sides a, b, c, d; for crossed four-bar linkages, in case of an* antiparallelogram*, the angle between the two diagonals stays constant, $0°$ or $180°$, during deformation). Then the angle between the diagonals does not change when deforming the four-bar linkage if and only if $ABCD$ is orthodiagonal.*

Remark. The formulation *the angle between the diagonals does not change when deforming the four-bar linkage* is equivalent to *when the vertices are thought to be attached to fixed "tracks"* (along the diagonals, making an angle of $0° < \varphi := \angle ATB < 180°$; note that φ is not defined as any of the two angles formed by the diagonals but as the specific angle $\angle ATB$; for orthodiagonal quadrilaterals, φ would be $90°$); *then the vertices of the four-bar linkage can move somehow on these "tracks"; the quadrilateral is not rigid.*

For the proof, we need, as preparation, some other well-known results. The first ones are two versions of Bretschneider's formula for the area of a general quadrilateral (convex or concave, not crossed): the area K of a quadrilateral with sides a, b, c, d and diagonals e, f is given by

$$K = \frac{1}{4}\sqrt{4e^2 f^2 - (b^2 + d^2 - a^2 - c^2)^2}, \tag{1}$$

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd[1 + \cos(\alpha + \gamma)]}, \tag{2}$$

where s is the semi-perimeter and α, γ are any two opposite angles (Figure 5; note that $\cos(\alpha + \gamma) = \cos(\beta + \delta)$ because of $\alpha + \beta + \gamma + \delta = 360°$).

Next, an old and short formula for $b^2 + d^2 - a^2 - c^2$ in quadrilaterals can be derived; see e.g. Dostor [1] or Walser [10]. Let $ABCD$ be an arbitrary quadrilateral; in Figure 6, we show a convex case.

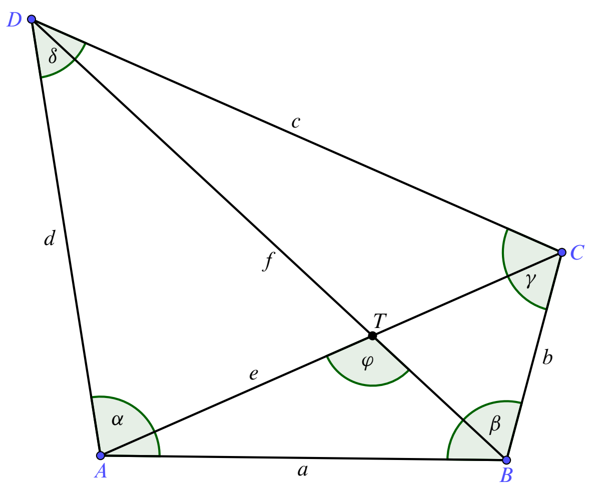

Figure 5. General quadrilateral, here convex

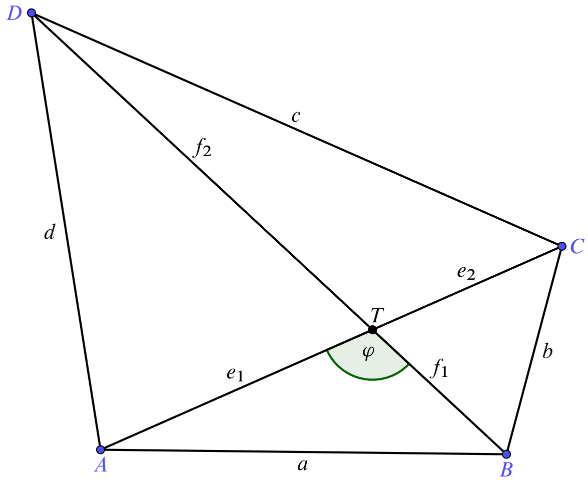

Figure 6. Convex quadrilateral, $e_1 + e_2 = e$, $f_1 + f_2 = f$

Then, according to the law of cosines, we have

$$\begin{aligned}
\text{I:} \quad & a^2 = e_1^2 + f_1^2 - 2e_1 f_1 \cos\varphi, \\
\text{II:} \quad & b^2 = e_2^2 + f_1^2 + 2e_2 f_1 \cos\varphi, \\
\text{III:} \quad & c^2 = e_2^2 + f_2^2 - 2e_2 f_2 \cos\varphi, \\
\text{IV:} \quad & d^2 = e_1^2 + f_2^2 + 2e_1 f_2 \cos\varphi.
\end{aligned}$$

This gives, by adding lines II and IV and subtracting lines I and III,

$$b^2 + d^2 - a^2 - c^2 = 2ef \cos\varphi, \tag{3}$$

since

$$\begin{aligned}
b^2 + d^2 - a^2 - c^2 &= 2\cos\varphi(e_1 f_1 + e_2 f_1 + e_1 f_2 + e_2 f_2) \\
&= 2\cos\varphi \underbrace{(e_1 + e_2)}_{e} \underbrace{(f_1 + f_2)}_{f} = 2ef \cos\varphi.
\end{aligned}$$

Remarks. (i) Formula (3) also follows from (1) using the well-known formula for the area of a quadrilateral $K = \frac{ef}{2} \sin\varphi$ (whose structure is remarkably similar to the trigonometric area formula of a triangle $K = \frac{ab}{2} \sin\gamma$, where γ is the angle between the sides a and b).

(ii) Defining the angle φ and e_1, e_2, f_1, f_2 appropriately, one can see that (3) also holds for concave and even for crossed quadrilaterals.

(iii) For $\varphi \neq 90°$, we can divide by $2\cos\varphi$, yielding a formula for the product of the diagonals of a not orthodiagonal quadrilateral: $ef = \frac{b^2+d^2-a^2-c^2}{2\cos\varphi}$.

(iv) Using formula (3), the above-mentioned characterization of quadrilaterals

$$a^2 + c^2 = b^2 + d^2 \iff \text{orthodiagonal}$$

is an immediate consequence in both directions.

Now we are prepared for proof of Theorem 2.

Proof. The if part is easy; see above. Now for the only if direction, let $ABCD$ be a non-crossed four-bar linkage with fixed sides a, b, c, d and $\varphi \neq 90°$ (not orthodiagonal). Then, according to (3), we see (note that the left side of (3) is a constant) that, when deforming a four-bar linkage, φ is constant if and only if ef is constant. This, in turn, is – due to (1) – equivalent to area K being constant, and according to (2), this is equivalent to $\alpha + \gamma$ being constant. But this is impossible since, when deforming a four-bar linkage, either both angles α, γ increase (in case that diagonal f increases) or decrease (in case that f decreases; f cannot stay constant when changing the shape of the four-bar linkage because, with given values of a, b, c, d, and f, the quadrilateral is uniquely determined). Thus, a non-orthodiagonal four-bar linkage cannot have constant angle φ between the diagonals, which proves the missing direction of Theorem 2. ∎

Corollary 1. *Given the values of a, b, c, d, and $\varphi \neq 90°$ (angle between the diagonals), how many (non-congruent, non-crossed) quadrilaterals are there with these values? The number of solutions is, depending on the given values, either 0, or 1, or 2. The case of one solution occurs if and only if the values come from a cyclic quadrilateral (maximum area, maximum product ef of the diagonals, $\alpha + \gamma = 180°$).*

Proof. According to (3), the value of φ uniquely determines the product ef of the diagonals; in turn, due to (1), this uniquely determines the area K, and via (2), we can see that there are at most two corresponding values of $0° < \alpha + \gamma < 360°$. The value of $\alpha + \gamma$ uniquely defines the shape of the quadrilateral, and the case $\alpha + \gamma = 180°$ yields uniqueness of the quadrilateral. ∎

Corollary 2. *If there are two different (non-crossed) quadrilaterals with the same values of a, b, c, d, and $\varphi \neq 90°$, then they have the same product of their diagonals, the same area, and the values of $\alpha + \gamma$ are symmetric about 180° (same cosine).*

Open Problem. How can these quadrilaterals, if they exist, be constructed?

Corollary 3. *Given a convex four-bar linkage with fixed values of a, b, c, d, and $\varphi \neq 90°$ (this angle changes when deforming the four-bar linkage!), the angle $\varphi \neq 90°$ takes its maximum/minimum if and only if the quadrilateral is cyclic. The case of maximum occurs for $\varphi < 90°$; the case of minimum occurs for $\varphi > 90°$ (this can be best seen experimentally by using dynamic geometry software).*

Proof. From (3), we can conclude that the angle φ cannot change from acute to 90° or to obtuse or vice versa, since the sign of $\cos\varphi$ cannot change for fixed a, b, c, d. (Above, we formulated for four-bar linkages that orthodiagonality in one moment guarantees orthodiagonality "forever". Now we can also say for convex four-bar linkages that $\varphi < 90°$ in one moment guarantees $\varphi < 90°$ forever; $\varphi > 90°$ in one moment guarantees $\varphi > 90°$ "forever"). First, we deal with the case of an acute angle $\varphi < 90°$. Then we have $\cos\varphi > 0$, according to (3), $b^2 + d^2 - a^2 - c^2 > 0$, and the following equivalences:

$$\begin{aligned} \text{quadrilateral is cyclic} &\iff \text{quadrilateral has maximum area} \\ &\iff ef \to \max \end{aligned}$$

$$\iff \underbrace{\frac{b^2+d^2-a^2-c^2}{2ef}}_{>0} \to \min$$

$$\overset{(3)}{\iff} \cos\varphi \to \min,$$

and for acute angles φ, this is equivalent to $\varphi \to \max$. Now, analogously, for the case of an obtuse angle $\varphi > 90°$, we have $\cos\varphi < 0$, according to (3), $b^2 + d^2 - a^2 - c^2 < 0$, and the following equivalences:

$$\begin{aligned} \text{quadrilateral is cyclic} &\iff \text{quadrilateral has maximum area} \\ &\iff ef \to \max \\ &\iff \underbrace{\frac{b^2+d^2-a^2-c^2}{2ef}}_{<0} \to \max \\ &\overset{(3)}{\iff} \cos\varphi \to \max, \end{aligned}$$

and for obtuse angles φ, this is equivalent to $\varphi \to \min$. ■

After having finished this paper, more specifically, during the time the paper was reviewed, we came across two related references. Firstly, Mitchell [7], where it is shown that the area of a quadrilateral is uniquely defined by a, b, c, d, and φ if and only if $\varphi \neq 90°$. Secondly, Johnson [3], where also crossed four-bar linkages are dealt with, especially antiparallelograms (contra-parallelograms, crossed parallelograms); they have the surprising property that the product of the diagonals is constant.

Note added in proof. The open construction problem mentioned on p. 8 could be solved recently by Halbeisen/Hungerbühler/Läuchli. The solution will be published in this journal; see [2].

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Hans Humenberger
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1
1090 Vienna, Austria
hans.humenberger@univie.ac.at