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Cyclic subgroup transitivity for Abelian groups

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Abstract – In previous work, the first two authors studied the notion of transitivity with respect to cyclic subgroups for separable Abelian p-groups and modules over the ring of p-adic integers. Here we consider briefly how the notion can be used in the context of torsion-free Abelian groups and also look at the situation for non-separable p-groups and direct sums of infinite-rank homocyclic p-groups.

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1. Introduction

In earlier works [6, 7], the following notions of transitivity for Abelian p-groups and p-adic modules were introduced and studied.

Definition 1.1. A p-group G is said to be *transitive with respect to cyclic subgroups* if when X, Y are cyclic subgroups of G with (i) $X \cong Y$ and (ii) $G/X \cong G/Y$, there exists an automorphism ϕ of G with $\phi(X) = Y$.

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We use the abbreviation "G is CS-transitive" for the full statement "G is transitive with respect to cyclic subgroups".

Definition 1.2. A p-group M is said to be *quotient-transitive* if, given any pair of non-zero elements $x, y \in M$, with $M/\langle x\rangle \cong M/\langle y\rangle$, there is an automorphism ϕ of M with $\phi(x) = y$.

Note that a quotient-transitive group is always CS-transitive but the converse is not true, an easy example being furnished by the group $G = A \oplus B$, where A is an elementary p-group of infinite rank and B is cyclic of order p^2 – see [6, Example 2.1]. The notions, of course, coincide for finite Abelian p-groups and finitely generated p-adic modules. Among the principal results obtained in [6, 7] were the seemingly unknown facts that both finite Abelian p-groups and finitely generated p-adic modules are quotient-transitive. The arguments depended heavily on the technical [6, Proposition 3.1], which gives information on the structure of certain quotient groups; this result will be useful in some of our arguments in this work.

Throughout, all groups will be additively written Abelian groups and we will generally omit the adjective Abelian. We will examine further the notion of CS-transitivity in the areas of non-separable p-groups and direct sums of infinite-rank homocyclic p-groups; we will also look briefly at the situation in relation to torsion-free groups. Our main results are that (i) the non-separable generalised Prüfer groups of the form $H_{\omega+n}$, with n finite, are CS-transitive but $H_{\omega+\omega}$ is not (Theorem 3.4); (ii) if $\mathcal{S} = \{n_1, n_2, \ldots\}$ is an infinite set of integers with $n_1 < n_2 < \cdots$ and $G_{\mathcal{S}} = \bigoplus_{i=1}^{\infty} G_i$, where each G_i is an infinite-rank homocyclic p-group of exponent n_i, then $G_{\mathcal{S}}$ is CS-transitive provided that for each $i, n_{i+1} \geq 2n_i + 1$ (Corollary 4.7). For torsion-free groups we show that both transitive and strongly separable torsion-free groups are CS-transitive; in particular homogeneous completely decomposable groups are CS-transitive.

We also provide further evidence of the complexity of characterising CS-transitive groups of the type $G_{\mathcal{S}}$ above.

In the classical theory of transitivity due to Kaplansky [9, 10] using height or Ulm sequences, it turns out that in some not too precisely defined way, "most" groups are transitive. However, the classes of quotient-transitive and CS-transitive groups are "small", but still large enough to be interesting.

Our notation is largely standard and terminology in relation to Abelian group theory may be found in the standard works of Fuchs [4, 5], an exception being that we indicate a jump in a height sequence of an element using $\uparrow$, so that $U(x) = (r_0, r_1, \ldots, \uparrow r_{n_1}, r_{n_1} + 1, \ldots, \uparrow r_{n_2}, r_{n_2} + 1, \ldots, \uparrow r_{n_t} = \infty)$ indicates that there are jumps immediately before $r_{n_1}, \ldots, r_{n_t}$.

2. CS-transitivity in torsion-free groups

In this section we will take a brief look at the situation in torsion-free groups in relation to CS-transitivity. The general problem is quite complex and is deserving of a full treatment in its own right; we will content ourselves by presenting some simple results that require a minimum of technical background. Our first observation is that the two notions of transitivity with respect to cyclic subgroups and quotient transitivity coincide in this situation: this is immediate since any two cyclic subgroups of a torsion-free group are isomorphic, both being isomorphic to the group of integers $\mathbb{Z}$. We will state our results in terms of CS-transitivity.

Our first result is to show that transitive torsion-free groups are CS-transitive.

LEMMA 2.1. *Let G be a torsion-free Abelian group and $x, y \in G$ such that $G/\langle x\rangle \cong G/\langle y\rangle$. Then the height sequences of x and y coincide.*

PROOF. Let $x, y \in G$ be as stated in the lemma. Obviously we have $G/(\langle x\rangle_*) \cong (G/\langle x\rangle)/(\langle x\rangle_*/\langle x\rangle)$ where, as usual, for a subgroup H of a torsion-free group G, H_* denotes the purification of H in G. Since the left-hand side of the equation is torsion-free we conclude that $\langle x\rangle_*/\langle x\rangle$ is the torsion part of $G/\langle x\rangle$. Hence, if $G/\langle x\rangle \cong G/\langle y\rangle$, then $\langle x\rangle_*/\langle x\rangle \cong \langle y\rangle_*/\langle y\rangle$. By [1, Theorem 1.4], it follows that the height sequences of x and y are the same. ∎

COROLLARY 2.2. *Let G be a transitive torsion-free Abelian group. Then G is CS-transitive. In particular, rank 1 torsion-free groups are CS-transitive.*

PROOF. Assume that $x, y \in G$ and that $G/\langle x\rangle \cong G/\langle y\rangle$. By Lemma 2.1 above we conclude that x and y have the same height sequences and thus transitivity implies that there is an automorphism of G mapping x onto y, as required for CS-transitivity. ∎

Our next result shows that non-CS-transitive groups can exist even at rank 2.

EXAMPLE 2.3. Let $H = R_1 e \oplus R_2 f$ be a completely decomposable group of rank 2 such that the types $\mathrm{tp}(R_1)$ and $\mathrm{tp}(R_2)$ are incomparable and satisfy $R_1 \cap R_2 = \mathbb{Z}$. Assume that there is a prime p such that $1/p \notin R_1$ and $1/p \notin R_2$. Then H is not CS-transitive.

PROOF. We need to find elements $x, y \in H$ such that the two quotients $H/\langle x\rangle$ and $H/\langle y\rangle$ are isomorphic, but x and y cannot be mapped onto each other by an automorphism of H. Let p be as in the hypothesis of the lemma and put $x = pe + f$ *and* $y = e + pf$. Obviously, any isomorphism of H that maps x onto y would have to send pe onto e, which is impossible since multiplication by p is not an isomorphism of R_1.

We first claim that x and y are pure elements of H. Assume that $nh \in \langle x \rangle$ for some $n \in \mathbb{N}$ and $h = r_1 e + r_2 f \in H$ where $r_i \in R_i$. Thus there is $m \in \mathbb{N}$ with $(n, m) = 1$ and $n(r_1 e + r_2 f) = nh = mx = m(pe + f)$. Equating coefficients we obtain $nr_1 = mp$ *and* $nr_2 = m$ and hence $\frac{n}{p} r_1 = nr_2$, which implies that $pr_2 = r_1$. Consequently, $r_1 e + r_2 f = pr_2 e + r_2 f = r_2(pe + f)$. Moreover, since $R_1 \cap R_2 = \mathbb{Z}$ and $1/p \notin R_i$ for $i = 1, 2$ we conclude $r_2 \in \mathbb{Z}$ and thus

$$r_1 e + r_2 f = pr_2 e + r_2 f = r_2(pe + f) \in \langle x \rangle.$$

Similar arguments show that also y is pure in H and it remains to prove that $H/\langle x \rangle \cong H/\langle y \rangle$. However, it is easily seen that both quotients are torsion-free of type $R_1 + R_2$ and hence must be isomorphic. ∎

Note that the group in Example 2.3 above is completely decomposable but is not homogeneous.

We now turn our attention to strongly separable torsion-free groups. Recall that a torsion-free group is said to be *strongly separable* if each of its pure rank 1 subgroups is a direct summand; homogeneous completely decomposable groups are, of course, strongly separable.

Proposition 2.4. *Let G be a strongly separable torsion-free group. Then G is CS-transitive.*

Proof. Assume that G is strongly separable and let $x, y \in G$ such that $G/\langle x \rangle \cong G/\langle y \rangle$. By Lemma 2.1 we conclude that x and y have the same height sequences. Moreover, since G is strongly separable it follows that the pure subgroups $\langle x \rangle_*$ and $\langle y \rangle_*$ are direct summands of G, i.e. $G = \langle x \rangle_* \oplus G_x = \langle y \rangle_* \oplus G_y$. Obviously, there is a homomorphism from $\langle x \rangle_*$ to $\langle y \rangle_*$ since x and y are of the same type. By assumption we also have

$$(\langle x \rangle_* / \langle x \rangle) \oplus G_x = G/\langle x \rangle \cong G/\langle y \rangle = (\langle y \rangle_* / \langle y \rangle) \oplus G_y,$$

and since G_x and G_y are torsion-free while $\langle y \rangle_* / \langle y \rangle$ and $\langle x \rangle_* / \langle x \rangle$ are torsion, we conclude that $G_y \cong G_x$. Putting this together with the homomorphism between $\langle x \rangle_*$ and $\langle y \rangle_*$ we obtain an automorphism of G mapping x onto y. ∎

Our final results in this brief introduction to CS-transitivity for torsion-free groups establish conditions which ensure that certain subgroups of torsion-free CS-transitive groups are again CS-transitive.

LEMMA 2.5. *Let $G = G_1 \oplus G_2$ be a CS-transitive torsion-free Abelian group such that $\{G_1, G_2\}$ is a semi-rigid pair, i.e. either G_1 or G_2 is fully invariant in G. Then G_1 and G_2 are both CS-transitive.*

PROOF. Let $x, y \in G_2$ be such that $G_2/\langle x\rangle$ and $G_2/\langle y\rangle$ are isomorphic. Then also

$$G/\langle x\rangle = G_1 \oplus (G_2/\langle x\rangle) \cong G_1 \oplus (G_2/\langle y\rangle) = G/\langle y\rangle,$$

and by the CS-transitivity of G we obtain an isomorphism α of G mapping $(0, x)$ onto $(0, y)$. If G_2 is fully invariant in G, then clearly α induces an isomorphism on G_2 that maps x onto y. So assume that G_1 is fully invariant in G, i.e. $\mathrm{Hom}(G_1, G_2) = 0$. Let

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

By assumption $\alpha_3 = 0$, and hence α is an isomorphism if and only if α_1 and α_4 are isomorphisms. Thus α_4 is an automorphism of G_2 that maps x onto y. An identical argument holds if one starts with $x, y \in G_1$. ■

We have an immediate corollary.

COROLLARY 2.6. *Let $A = \bigoplus_{\tau\in T_{\mathrm{cr}}(A)} A_\tau$ be a completely decomposable group where $T_{\mathrm{cr}}(A)$ is the critical typeset of A. Fix a type $\sigma \in T_{\mathrm{cr}}(A)$ and let $A = A(\sigma) \oplus A^\sigma$ where $A^\sigma = \bigoplus_{\tau\in T_{\mathrm{cr}}(A), \tau\not\geq\sigma} A_\tau$ and $A(\sigma) = \bigoplus_{\tau\in T_{\mathrm{cr}}(A),\tau\geq\sigma}$.*
If A is CS-transitive, then $A(\sigma)$ and A^σ are both CS-transitive.

PROOF. Since $A(\sigma)$ is fully invariant in A and $\mathrm{Hom}(A(\sigma), A^\sigma) = 0$, the claim follows from Lemma 2.5 above. ■

3. CS-transitivity for non-separable p-groups

In the earlier works [6, 7], the notions of quotient transitivity and CS-transitivity were investigated in the context primarily of separable p-groups. In this section we turn our attention to the situation for non-separable p-groups. Recall that in [6, Theorem 3.2, Corollary 3.4] it was established that finite p-groups and reduced semi-standard separable p-groups are both quotient transitive and thus CS-transitive.

PROPOSITION 3.1. *Suppose that G is a reduced semi-standard p-group and*

(i) *$p^\omega G$ is cyclic of order p^n;*

(ii) *$G/p^{\omega+n-1}G$ is CS-transitive.*

Then G is CS-transitive.

Proof. Suppose x, y are arbitrary in G such that $o(x) = o(y)$ and $G/\langle x\rangle \cong G/\langle y\rangle$. There are three outcomes to consider:

(a) $\langle x\rangle \cap p^\omega G = 0 = \langle y\rangle \cap p^\omega G$;

(b) $\langle x\rangle \cap p^\omega G \neq 0 \neq \langle y\rangle \cap p^\omega G$;

(c) without loss of generality $\langle x\rangle \cap p^\omega G = 0$ but $\langle y\rangle \cap p^\omega G \neq 0$.

Case (a) is easily handled by [6, Theorem 4.2 and Lemma 4.3].

Case (c). We next show that case (c) cannot occur. Since $\langle x\rangle$, $\langle y\rangle$ are both finite, they are nice in the sense of Hill, therefore, $p^\omega(\frac{G}{\langle x\rangle}) = \frac{p^\omega G+\langle x\rangle}{\langle x\rangle} \cong \frac{p^\omega G}{\langle x\rangle\cap p^\omega G}$, and similarly for $\langle y\rangle$. Since $p^\omega(\frac{G}{\langle x\rangle}) \cong p^\omega(\frac{G}{\langle y\rangle})$, so $\frac{p^\omega G}{\langle x\rangle\cap p^\omega G} \cong \frac{p^\omega G}{\langle y\rangle\cap p^\omega G}$. Thus case (c) cannot occur; note that we have used only the finiteness of $p^\omega G$ to show that this case cannot occur.

Case (b). Let $p^\omega G = \langle a\rangle$ and set $H = p^{\omega+n-1}G \cong \mathbb{Z}(p)$. Now $\langle x\rangle \cap p^\omega G = p^k(p^\omega G)$ and $\langle y\rangle \cap p^\omega G = p^l(p^\omega G)$, where $0 \leq k, l < n$. Since $G/\langle x\rangle \cong G/\langle y\rangle$ we conclude $k = l$.

Furthermore, $p^k a = p^r ux$ for some r relatively prime to p. Similarly, $p^k a = p^s vy$ for some s relatively prime to p. Since $o(x) = o(y)$ we must have $r = s$. Now consider the elements $\bar{x}$, $\bar{y}$ of $\overline{G} = G/H$, where $\bar{x} = x + H$, $\bar{y} = y + H$; note that $o(\bar{x}) = o(\bar{y}) = p^{r+n-k-1}$. We also have that $\langle x\rangle + H = \langle x\rangle$ and similarly for y, so

$$\frac{G}{\langle x\rangle} \cong \frac{G/H}{\langle x\rangle/H} \cong \frac{\overline{G}}{\langle \bar{x}\rangle} \cong \frac{G}{\langle y\rangle} \cong \frac{\overline{G}}{\langle \bar{y}\rangle}.$$

Since we are assuming $\overline{G}$ is CS-transitive, there is an automorphism of $\overline{G}$ mapping $\bar{x} \mapsto \bar{y}$; in particular, $U_{\overline{G}}(\bar{x}) = U_{\overline{G}}(\bar{y})$:

$$\begin{aligned} U_G(x) &= (\mathrm{ht}_G(x), \ldots, \mathrm{ht}_G(p^{n-k+r-1}x), \infty) \\ &= (\mathrm{ht}_G(x), \ldots, \mathrm{ht}_G(p^{n-k+r-2}x), \omega + n - 1, \infty). \end{aligned}$$

We have a similar result for the Ulm sequence of y. Now consider the canonical projection $\eta \colon G \to G/H = \overline{G}$. The $\operatorname{Ker}\eta$ consists of elements with generalised p-heights $\geq \sigma = \omega + n - 1$. By [4, Lemma 37.1], it follows that $\mathrm{ht}_{\overline{G}}(\eta(g)) = \mathrm{ht}_G(g)$ provided $\mathrm{ht}_G(g) < \sigma$. In particular,

$$\mathrm{ht}_G(x) = \mathrm{ht}_{\overline{G}}(\bar{x}), \ldots, \mathrm{ht}_G(p^{n-k+r-2}x) = \mathrm{ht}_{\overline{G}}\left(\overline{p^{n-k+r-2}\bar{x}}\right),$$

and similarly for y and $\bar{y}$.

So we conclude that

$$U_G(x) = \big(\mathrm{ht}_{\bar{G}}(\bar{x}), \ldots, \mathrm{ht}_{\bar{G}}(\overline{p^{n-k+r-2}\bar{x}}), \omega + n - 1, \infty\big)$$
$$= \big(\mathrm{ht}_{\bar{G}}(\bar{y}), \ldots, \mathrm{ht}_{\bar{G}}(\overline{p^{n-k+r-2}\bar{y}}), \omega + n - 1, \infty\big) = U_G(y).$$

Now $p^\omega G$ is cyclic and hence G is transitive in the sense of Kaplansky (see, for example, [2, Lemma 2]), so there is an automorphism φ of G with $\varphi(x) = y$. Since x, y were arbitrary subject to $o(x) = o(y)$ and $G/\langle x\rangle \cong G/\langle y\rangle$, we thus have that G is CS-transitive. ∎

Corollary 3.2. *If G is a semi-standard reduced p-group with $p^\omega G$ cyclic of order p, then G is CS-transitive.*

Proof. Take $n = 1$ in Proposition 3.1 and note that $G/p^\omega G$ is semi-standard since G is semi-standard. Since $G/p^\omega G$ is necessarily separable, it follows from [6, Corollary 4.4] that $G/p^\omega G$ is CS-transitive, and thus G is CS-transitive. ∎

We want to apply Proposition 3.1 to certain of the so-called Prüfer groups H_σ. We recall some well-known properties of these groups; a detailed discussion of them may be found in, for example, [4, Section 81]. The groups are constructed inductively starting from $H_0 = 0$ to satisfy the following four conditions:

(i) H_σ is of length σ;

(ii) $p^\sigma H_{\sigma+1}$ is cyclic of order p and $H_{\sigma+1}/p^\sigma H_{\sigma+1} \cong H_\sigma$;

(iii) for a limit ordinal σ, $H_\sigma = \bigoplus_{\rho<\sigma} H_\rho$;

(iv) every Ulm invariant of H_σ is at most $|\sigma|$.

We state two properties of Prüfer groups which are well known and easy to prove by induction using the defining properties above:

- For each $1 \le n \le \omega$, $p^\omega H_{\omega+n} \cong \mathbb{Z}(p^n)$.
- For each $1 \le n \le \omega$, $H_{\omega+n}$ is semi-standard; in fact, the Ulm invariant $f_k(H_{\omega+n}) = 1$ for all $k < \omega$.

Lemma 3.3. *For each $1 \le n < \omega$, $H_{\omega+n}$ is CS-transitive.*

Proof. If $n = 1$, $p^\omega H_{\omega+1} \cong \mathbb{Z}(p)$ and $H_{\omega+1}/p^\omega H_{\omega+1} \cong B$, a standard p-group and hence CS-transitive by [6, Corollary 4.4]. Since $H_{\omega+1}$ is also semi-standard, we have by Corollary 3.2 that $H_{\omega+1}$ is CS-transitive.

Proceed by induction: Assume $H_{\omega+n}$ is CS-transitive and consider $H_{\omega+n+1}$. Then from the bullet points above, $H_{\omega+n+1}$ is semi-standard and $p^\omega H_{\omega+n+1} \cong \mathbb{Z}(p^{n+1})$.

Furthermore, $H_{\omega+n+1}/p^{\omega+n}H_{\omega+n+1} \cong H_{\omega+n}$ is CS-transitive by our induction hypothesis and so, applying Proposition 3.1, gives the desired result that $H_{\omega+n+1}$ is CS-transitive. ∎

Thus, we have established the core part of the following:

THEOREM 3.4. *For each $1 \le n < \omega$, $H_{\omega+n}$ is CS-transitive, but $H_{\omega+\omega}$ is not CS-transitive.*

PROOF. It remains only to show that $H_{\omega+\omega}$ is not CS-transitive. Since $H_{\omega+\omega} = H_0 \oplus H_1 \oplus \cdots \oplus H_\omega \oplus H_{\omega+1} \oplus \cdots$, utilising the bullet points above we see that

$$\begin{aligned} f_0(H_{\omega+\omega}) &= f_0(H_0) + f_0(H_1) + \cdots + f_0(H_{\omega+1}) + \cdots \\ &\ge f_0(H_{\omega+1}) + \cdots \\ &\ge 1 + 1 + \cdots \ge \aleph_0; \end{aligned}$$

similarly, $f_1(H_{\omega+\omega}) \ge \aleph_0$. But then it follows from [6, Proposition 5.9] that $H_{\omega+\omega}$ is not CS-transitive. ∎

The proof of our next result has many similarities with that of Proposition 3.1 but the result is somewhat more powerful.

THEOREM 3.5. *Suppose G is a semi-standard reduced p-group which is transitive (in the sense of Kaplansky) and $p^\omega G$ is finite. Then G is CS-transitive.*

PROOF. Let x, y be arbitrary in G such that $o(x) = o(y)$ and $G/\langle x\rangle \cong G/\langle y\rangle$. Then $p^\omega(\frac{G}{\langle x\rangle}) = \frac{p^\omega G + \langle x\rangle}{\langle x\rangle} \cong \frac{p^\omega G}{\langle x\rangle \cap p^\omega G}$ and similarly for $p^\omega(\frac{G}{\langle y\rangle})$. As in the proof of Proposition 3.1, the same three cases need to be considered. Case (a) follows exactly as in that proposition, while case (c) cannot happen due to the finiteness of $p^\omega G$, as observed previously.

So we restrict our consideration to case (b) where $\langle x\rangle \cap p^\omega G \ne 0 \ne \langle y\rangle \cap p^\omega G$.

Since $p^\omega(G/\langle x\rangle) \cong p^\omega(G/\langle y\rangle)$ we see that $|\langle x\rangle \cap p^\omega G| = |\langle y\rangle \cap p^\omega G|$ and since both groups are cyclic we may write $\langle x\rangle \cap p^\omega G = \langle a\rangle$ and $\langle y\rangle \cap p^\omega G = \langle b\rangle$ for some $a, b \in G$. Thus $|\langle a\rangle| = |\langle b\rangle|$ and as $o(x) = o(y)$, we can write $a = p^r x$, $b = p^r y$ for some integer r. Hence $p^\omega(G/\langle a\rangle) \cong p^\omega(G/\langle b\rangle)$ and $o(a) = o(b)$. Now, since $p^\omega G$ is finite, it is CS-transitive by [6, Theorem 4.2]. Thus, there is an automorphism of $p^\omega G$ mapping a onto b. Hence $U_{p^\omega G}(p^r x) = U_{p^\omega G}(p^r y)$. Note that $U_{p^\omega G}(p^r x)$ is of the form $(n_1, n_2, \ldots, n_k, \infty)$, where the n_i are integers and so it follows that $U_G(p^r x) = (\omega + n_1, \ldots, \omega + n_k, \infty) = U_G(p^r y)$.

Now consider the elements $\bar{x}$, $\bar{y}$ of $\overline{G} = G/p^{\omega}G$, where $\bar{x} = x + p^{\omega}G$ and $\bar{y} = y + p^{\omega}G$. Similar to the proof of Proposition 3.1, we have $\frac{\overline{G}}{\langle \bar{x} \rangle} \cong \frac{\overline{G}}{\langle \bar{y} \rangle}$ and $o(\bar{x}) = o(\bar{y}) = p^r$. Since G is semi-standard, so too is $\overline{G} = G/p^{\omega}G$ and the latter is separable, so it follows that $\overline{G} = G/p^{\omega}G$ is CS-transitive by [6, Corollary 4.4]. Hence there is an automorphism of $\overline{G}$ mapping $\bar{x}$ onto $\bar{y}$, and $U_{\overline{G}}(\bar{x}) = U_{\overline{G}}(\bar{y})$. By [4, Lemma 37.1], $\text{ht}_G(x) = \text{ht}_{\overline{G}}(x + p^{\omega}G) = \text{ht}_{\overline{G}}(\bar{x})$ since $\text{ht}_G(G) < \omega$; similarly, $\text{ht}_G(px) = \text{ht}_{\overline{G}}(p\bar{x}), \ldots, \text{ht}_G(p^{r-1}x) = \text{ht}_{\overline{G}}(p^{r-1}\bar{x})$, and similarly for y. Piecing together this information with that pertaining to $U_G(p^r x)$, we conclude that $U_G(x) = U_G(y)$. Now, since G is transitive (in the sense of Kaplansky) by hypothesis, there is an automorphism of G mapping x onto y, and so G is CS-transitive. ∎

The additional strength of Theorem 3.5 allows us to extend our results on Prüfer groups to certain direct sums of these groups.

Corollary 3.6. *If $G = \bigoplus_{j=1}^{N} H_{\omega+n_j}$, where $H_{\omega+n_j}$ is a Prüfer group and for each j, $1 \leq n_j < \omega$. Then G is CS-transitive and so too is any direct summand of G.*

Proof. Note that G is totally projective by [4, Theorem 82.3] and hence by a well-known result of Hill [8], it is transitive. Furthermore, it follows easily from the bullet points above that G is also semi-standard. The remaining hypothesis in Theorem 3.5 is immediate since $p^{\omega}G$ is a direct sum of finitely many cyclic p-groups and hence G is CS-transitive.

If H is any direct summand of G, then H is also totally projective and hence transitive. Since G is CS-transitive, so too is its transitive summand H by [6, Proposition 4.4]. ∎

4. Non-semi-standard groups

Before embarking on our study of non-semi-standard groups in relation to CS-transitivity, we look at the difference between quotient-transitivity and CS-transitivity in a simple case. In [6, Theorem 4.8], we saw that a group G of the form $G = F \oplus \bigoplus_{\lambda} \mathbb{Z}(p^n)$, with F finite and λ infinite, is quotient-transitive if, and only if the exponent of F is strictly less than n. The situation is somewhat different for CS-transitivity.

Proposition 4.1. *Suppose that $B = \bigoplus_{\lambda} \mathbb{Z}(p^n)$, where λ is an infinite cardinal, is an infinite homocyclic group and F is a finite p-group. Then $G = B \oplus F$ is CS-transitive.*

Proof. Let $x, y \in G$ be such that $o(x) = o(y)$ and $G/\langle x \rangle \cong G/\langle y \rangle$. Similar to the proof of [6, Theorem 4.2], using Baer's lemma [4, Lemma 65.4] we may find two finite summands C, D of G with $x \in C$, $y \in D$, such that for every integer α, the αth Ulm–Kaplansky invariants of C, D are either 0 or 1. Write $G = C \oplus H = D \oplus K$. Set

$$\begin{aligned} \mathcal{S} &= \{(\alpha, f_\alpha(C)) \mid \alpha \neq n-1,\ f_\alpha(C) \neq 0\}, \\ \mathcal{T} &= \{(\beta, f_\beta(D)) \mid \beta \neq n-1,\ f_\beta(D) \neq 0\}, \\ \mathcal{S}' &= \{(\gamma, f_\gamma(C/\langle x\rangle)) \mid \gamma \neq n-1,\ f_\gamma(C/\langle x\rangle) \neq 0\}, \\ \mathcal{T}' &= \{(\delta, f_\delta(D/\langle y\rangle)) \mid \delta \neq n-1,\ f_\delta(D/\langle y\rangle) \neq 0\}. \end{aligned}$$

Since $G/\langle x\rangle \cong C/\langle x\rangle \oplus H$ and $G/\langle y\rangle \cong D/\langle y\rangle \oplus K$, then for every α, $f_\alpha(G/\langle x\rangle) = f_\alpha(C/\langle x\rangle) + f_\alpha(H)$ and $f_\alpha(G/\langle y\rangle) = f_\alpha(D/\langle y\rangle) + f_\alpha(K)$. In particular, if $\alpha \neq n-1$, then all Ulm–Kaplansky invariants f_α of G, C, D, H, K are finite. Thus, $f_\alpha(G) = f_\alpha(C) + f_\alpha(H)$ and $f_\alpha(G) = f_\alpha(D) + f_\alpha(K)$. By substituting, we get

$$\begin{aligned} f_\alpha(G/\langle x\rangle) &= f_\alpha(C/\langle x\rangle) + f_\alpha(G) - f_\alpha(C), \\ f_\alpha(G/\langle y\rangle) &= f_\alpha(D/\langle y\rangle) + f_\alpha(G) - f_\alpha(D). \end{aligned}$$

Furthermore, by hypothesis, $G/\langle x\rangle \cong G/\langle y\rangle$ and since $f_\alpha(G)$ is finite we thus have

$$(\star) \qquad f_\alpha(C/\langle x\rangle) + f_\alpha(D) = f_\alpha(D/\langle y\rangle) + f_\alpha(C).$$

Now let $\alpha \in \mathcal{S}$. Then it follows from ($\star$) that $f_\alpha(C/\langle x\rangle) + f_\alpha(D) = f_\alpha(D/\langle y\rangle) + f_\alpha(C)$, and note that as $f_\alpha(C) \neq 0$, we know from [6, Proposition 4.1] that $f_\alpha(C/\langle x\rangle) = 0$, thus, $f_\alpha(C/\langle x\rangle) + f_\alpha(D) = f_\alpha(D) = f_\alpha(D/\langle y\rangle) + f_\alpha(C) \geq f_\alpha(D/\langle y\rangle) + 1$, which implies that $\alpha \in \mathcal{T}$. Since α is arbitrarily chosen from $\mathcal{S}$, we conclude that $\mathcal{S} \subset \mathcal{T}$. Reversing the roles of $\mathcal{S}$, $\mathcal{T}$ in the above proof, we also get $\mathcal{S} \subset \mathcal{T}$, therefore, we have $\mathcal{S} = \mathcal{T}$

Note that for every integer α, the αth Ulm–Kaplansky invariants of C, D are either 0 or 1, so if $\alpha \neq n-1$, then $f_\alpha(C) = f_\alpha(D)$.

So there are now three possibilities to handle:

(i) $f_{n-1}(C) = f_{n-1}(D) = 1$;

(ii) $f_{n-1}(C) = f_{n-1}(D) = 0$;

(iii) one of $f_{n-1}(C)$, $f_{n-1}(D)$ is equal to 1 and the other is 0.

Before analysing the three possibilities, we first note a further consequence of the equality ($\star$). If $\gamma \in \mathcal{S}'$, then $f_\gamma(C/\langle x\rangle) \neq 0$ and $f_\gamma(C/\langle x\rangle) + f_\gamma(D) = f_\gamma(D/\langle y\rangle) + f_\gamma(C)$. Since $\gamma \neq n-1$, then $f_\alpha(C) = f_\alpha(D)$, so $f_\gamma(C/\langle x\rangle) = f_\gamma(D/\langle y\rangle) \neq 0$. So $\gamma \in \mathcal{S}'$, and as γ is arbitrary from $\mathcal{S}'$, thus $\mathcal{S}' \subset \mathcal{T}'$. Reversing the roles of $\mathcal{S}'$ and $\mathcal{T}'$, we conclude $\mathcal{S}' = \mathcal{T}'$.

Case (iii) does not occur. Assume, for a contradiction, that $f_{n-1}(C) = 1$ and $f_{n-1}(D) = 0$, so $C \cong D \oplus \mathbb{Z}(p^n)$. Note that since $f_{n-1}(C) = 1$, by [6, Proposition 4.1], $f_{n-1}(C/\langle x\rangle) = 0$. Observe also that $f_{n-1}(D/\langle y\rangle) = 0$; if not, then $D/\langle y\rangle \geq C/\langle x\rangle \oplus \mathbb{Z}(p^n)$. So we have $\frac{|D|}{o(y)} \geq \frac{|C|}{o(x)} \cdot p^n = \frac{|D|\cdot p^n}{o(x)} \cdot p^n$ – impossible as $o(x) = o(y)$. Thus, $f_{n-1}(C/\langle x\rangle) = f_{n-1}(D/\langle y\rangle) = 0$ and so $C/\langle x\rangle \cong D/\langle y\rangle$ as they have the same set of Ulm–Kaplansky invariants. Note that $o(x) = o(y)$, so $C/\langle x\rangle \cong D/\langle y\rangle$ implies $|C| = |D|$, which is contrary to $C \cong D \oplus \mathbb{Z}(p^n)$. Therefore, case (iii) cannot happen.

Now consider the remaining cases (i) and (ii).

In case (i), $f_{n-1}(C) = f_{n-1}(D) = 1$ implies $C \cong D$. Furthermore, by [6, Proposition 4.1], $f_{n-1}(C) \neq 0$ implies $f_{n-1}(C/\langle x\rangle) = 0$; similarly for D and $D/\langle y\rangle$. Thus, in this case, $C/\langle x\rangle \cong D/\langle y\rangle$. So by [6, Theorem 4.2] and by cancellation theory (see, for example, [3]), the existence of an automorphism of G mapping x to y follows.

Similar arguments in case (ii) give $C \cong D$ and since $o(x) = o(y)$, we must have $|C/\langle x\rangle| = |D/\langle y\rangle|$. Since $f_\alpha(C/\langle x\rangle) = f_\alpha(D/\langle y\rangle)$ for all $\alpha \neq n-1$, this forces $f_{n-1}(C/\langle x\rangle) = f_{n-1}(D/\langle y\rangle)$ and hence $C/\langle x\rangle \cong D/\langle y\rangle$. Therefore, as in case (i), there is an automorphism of G mapping x to y. ■

In the final section of [6] the situation pertaining to groups which were not semi-standard was briefly considered and it was quite easy to find some arithmetic conditions which ensure that a group is not CS-transitive. We want to investigate this further and in a somewhat more systematic way. So throughout this section we will consider a set $\mathcal{S}$, usually finite, of strictly increasing integers, $\mathcal{S} = \{n_1, \ldots, n_t\}$, and the associated group $G_\mathcal{S} = G_1 \oplus G_2 \oplus \cdots \oplus G_t$, where each G_i is a homocyclic group of infinite rank and exponent n_i. The rank of the homocyclic components will not be of interest other than the fact that each is infinite. It is easy to show that $G_\mathcal{S}$ is not CS-transitive if any of the following hold – see [6, Proposition 4.9, Example 4.11]:

- $n_j = 2n_i$; $n_i + n_k = 2n_j$ for some $i < j < k$.

There are two further elementary conditions which lead to the failure of CS-transitivity:

- $n_i + n_j = n_k$; $n_i + n_\ell = n_j + n_k$ for some $n_i < n_j < n_k < n_\ell$.

We give the short proof of the final claim leaving the simpler proof of the first to the reader. Let e_i be a generator of a cyclic summand of G_i and set $x = p^{n_k} e_\ell$, so that $e(x) = n_\ell - n_k$ and $G/\langle x\rangle \cong G \oplus \mathbb{Z}(p^{n_k}) \cong G$. However, if $y = p^{n_i} e_j$ then $e(y) = n_j - n_i = n_\ell - n_k = e(x)$ and $G/\langle x\rangle \cong G \oplus \mathbb{Z}(p^{n_i}) \cong G \cong G/\langle y\rangle$. Clearly, no automorphism of G can map $x \mapsto y$, so G is not CS-transitive.

Our first two results illustrate situations in which the exclusion of some of the above relations also gives sufficient conditions for CS-transitivity.

Proposition 4.2. *Let $G = A \oplus B$, where $A = \bigoplus_\lambda \mathbb{Z}(p^r)$, $B = \bigoplus_\mu \mathbb{Z}(p^s)$, λ, μ are infinite, and $r < s$. Then G is CS-transitive if and only if $2r \neq s$.*

Proof. The necessity follows from the bullet points above. For the sufficiency, assume $2r \neq s$. Suppose that $x, y \in G$. Since G is certainly transitive (in the sense of Kaplansky), it suffices to show that if $G/\langle x\rangle \cong G/\langle y\rangle$ and $e(x) = e(y)$, then $U(x) = U(y)$. Suppose that the Ulm (height) sequence of x is $U(x) = (r_0, r_1, \ldots, \uparrow r_{n_1}, r_{n_1} + 1, \ldots, \uparrow r_{n_2}, r_{n_2} + 1, \ldots, \uparrow r_{n_t} = \infty)$, and $U(y) = (s_0, s_1, \ldots, \uparrow s_{m_1}, s_{m_1} + 1, \ldots, \uparrow s_{m_2}, r_{m_2} + 1, \ldots, \uparrow r_{m_u} = \infty)$. Note that in this case, the exponent $e(x)$ of x is n_t, and also $e(y) = m_u$. The proof is divided into three cases:

(i) both Ulm sequences have two gaps;

(ii) one Ulm sequence has two gaps, and the other one has one gap;

(iii) both Ulm sequences have one gap.

In case (i), $U(x) = (r_0, r_1, \ldots, \uparrow r_{n_1}, r_{n_1} + 1, \ldots, \uparrow r_{n_2} = \infty)$ and $U(y) = (s_0, s_1, \ldots, \uparrow s_{m_1}, s_{m_1} + 1, \ldots, \uparrow s_{m_2} = \infty)$. By [4, Lemma 65.4], there are integers n_1, n_2, k_1, k_2 with $0 < n_1 < n_2$, $0 \leq k_1 < k_2$, and $c_1, c_2 \in G$ with the following properties:

(a) $C = \langle c_1\rangle \oplus \langle c_2\rangle$ is a summand of G with $e(c_1) = n_1 + k_1$, $e(c_2) = n_2 + k_2$, and $r_0 = k_1$, $r_{n_1} = n_1 + k_2$;

(b) x can be written as $x = p^{k_1}c_1 + p^{k_2}c_2$.

A similar result holds for y: there are integers m_1, m_2, l_1, l_2 with $0 < m_1 < m_2$, $0 \leq l_1 < l_2$, and $d_1, d_2 \in G$ with the following properties:

(a′) $D = \langle d_1\rangle \oplus \langle d_2\rangle$ is a summand of G with $e(d_1) = m_1 + l_1$, $e(d_2) = m_2 + l_2$, and $s_0 = l_1$, $s_{m_1} = m_1 + l_2$;

(b′) y can be written as $y = p^{l_1}d_1 + p^{l_2}d_2$.

By [6, Proposition 4.1], $G/\langle x\rangle$ is isomorphic to $G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1})$ and $G/\langle y\rangle \cong G \oplus \mathbb{Z}(p^{l_1}) \oplus \mathbb{Z}(p^{l_2+m_1})$.

In this case, note that $n_1 + k_1 < n_2 + k_2$, and both belong to the set $\{r, s\}$. So $n_1 + k_1 = r$, $n_2 + k_2 = s$. Similarly, $m_1 + l_1 = r$, $m_2 + l_2 = s$. Since $e(x) = e(y)$ by hypothesis, $n_2 = m_2$. Thus $k_2 = l_2$.

Furthermore, $G/\langle x\rangle \cong G/\langle y\rangle$, so $G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \cong G \oplus \mathbb{Z}(p^{l_1}) \oplus \mathbb{Z}(p^{l_2+m_1})$. Observe that $r < k_2 + n_1, l_2 + m_1 < s$, $f_G(k_2 + n_1 - 1) = f_G(l_2 + m_1 - 1) = 0$, that is to say, G has no cyclic summand isomorphic to $\mathbb{Z}(p^{k_2+n_1})$ or to $\mathbb{Z}(p^{l_2+m_1})$. Hence we see $k_2 + n_1 = l_2 + m_1$ and so $n_1 = m_1$. By the same reasoning $k_1 = l_1$, since $0 \leq k_1, l_1 < r$.

We conclude in this case that $n_i = m_i$, $k_i = l_i$, $i = 1, 2$. Thus, $U(x) = U(y)$ and there is an automorphism of G taking x onto y.

For case (ii), without loss of generality, assume x is the same as in case (i), $U(y) = (s_0, s_1, \ldots, \uparrow s_{m_1} = \infty)$. Then there is a summand $D = \langle d_1 \rangle$ of G with $e(d_1) = m_1 + l_1$, $s_0 = l_1$ and y can be written as $y = p^{l_1} d_1$. Thus $G/\langle y \rangle \cong G \oplus \mathbb{Z}(p^{l_1})$. First note that $e(x) = e(y)$ implies $n_2 = m_1$.

By assumption, $G/\langle x \rangle \cong G/\langle y \rangle$, that is, $G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \cong G \oplus \mathbb{Z}(p^{l_1})$. As in case (i), we have $k_2 + n_1 = l_1$; so $k_2 + n_1 = l_1 > r$ and $l_1 + m_1 \in \{r, s\}$, therefore, $l_1 + m_1 = s$. But then $s = l_1 + m_1 = k_2 + n_1 + m_1 = k_2 + n_1 + n_2 = s + n_1$ – a contradiction since $n_1 > 0$.

In case (iii), since $U(x) = (r_0, r_1, \ldots, \uparrow r_{n_1} = \infty)$, we have by a similar argument to that in case (i) that there is a summand $C = \langle c_1 \rangle$ of G with $e(c_1) = n_1 + k_1$, $r_0 = k_1$ and x can be written as $x = p^{k_1} c_1$ and $G/\langle x \rangle \cong G \oplus \mathbb{Z}(p^{k_1})$. Similarly, since $U(y) = (s_0, s_1, \ldots, \uparrow s_{m_1} = \infty)$, as in case (i) there is a summand $D = \langle d_1 \rangle$ of G with $e(d_1) = m_1 + l_1$, $s_0 = l_1$ and y can be written as $y = p^{l_1} d_1$ and that $G/\langle y \rangle \cong G \oplus \mathbb{Z}(p^{l_1})$. First notice that $e(x) = e(y)$ implies that $n_1 = m_1$.

There are two possibilities for k_1, l_1: (1) $k_1 = l_1$, (2) $k_1 \neq l_1$, with $k_1, l_1 \in \{0, r, s\}$. In possibility (1), then $n_1 = m_1$, $k_1 = l_1$, so $U(x) = U(y)$ and there is an automorphism of G from x onto y. For possibility (2), since $s \geq k_1 + n_1 > k_1$, we have $s \geq l_1 + m_1 > l_1$, so actually, $k_1, l_1 \in \{0, r\}$. Then without loss of generality, assume $k_1 = 0$, $l_1 = r$. Thus, $k_1 + n_1 = n_1 < l_1 + m_1 = r + n_1$. As $k_1 + n_1$, $l_1 + m_1 \in \{r, s\}$, we have $k_1 + n_1 = n_1 = r$, $l_1 + m_1 = r + n_1 = s$, so $2r = s$ – contrary to our assumption that $2r \neq s$. ∎

The idea in the proof of the next result is similar to that in Proposition 4.2 above, but more cases need to be handled. The necessity is justified by the bullet points above.

Proposition 4.3. *Let $G = A_1 \oplus A_2 \oplus A_3$, where $A_1 = \bigoplus_\lambda \mathbb{Z}(p^r)$, $A_2 = \bigoplus_\mu \mathbb{Z}(p^s)$, $A_3 = \bigoplus_\nu \mathbb{Z}(p^t)$, λ, μ, ν are infinite, and $r < s < t$. Then G is CS-transitive if and only if $2r \neq s$, $2r \neq t$, $2s \neq t$, $r + s \neq t$, and $r + t \neq 2s$.*

Proof. For the sufficiency, assume $2r \neq s$, $2r \neq t$, $2s \neq t$, $r + s \neq t$, and $r + t \neq 2s$. Suppose that $x, y \in G$. Since G is transitive in the sense of Kaplansky, it suffices to show that if $G/\langle x \rangle \cong G/\langle y \rangle$ and $e(x) = e(y)$, then $U(x) = U(y)$. There are three cases:

(i) one Ulm sequence of x, y has three gaps;

(ii) one Ulm sequence has two gaps, the other has at most two gaps;

(iii) both Ulm sequences have one gap.

Case (i). Without loss of generality, assume $U(x) = (r_0, r_1, \ldots, \uparrow r_{n_1}, r_{n_1} + 1, \ldots, \uparrow r_{n_2}, r_{n_2} + 1, \ldots, \uparrow r_{n_3} = \infty)$. As in the proof of Proposition 4.2, by [4, Lemma 65.4], we have integers $n_1, n_2, n_3, k_1, k_2, k_3$ with $0 < n_1 < n_2 < n_3$, $0 \leq k_1 < k_2 < k_3$, and $c_1, c_2, c_3 \in G$ with the following properties:

(a) $C = \langle c_1 \rangle \oplus \langle c_2 \rangle \oplus \langle c_3 \rangle$ is a summand of G with $e(c_1) = n_1 + k_1$, $e(c_2) = n_2 + k_2$, $e(c_3) = n_3 + k_3$, and $r_0 = k_1$, $r_{n_1} = n_1 + k_2$, $r_{n_2} = n_2 + k_3$;

(b) x can be written as $x = p^{k_1}c_1 + p^{k_2}c_2 + p^{k_3}c_3$.

Similarly, if the Ulm sequence of y has three gaps, say $U(y) = (s_0, s_1, \ldots, \uparrow s_{m_1}, s_{m_1} + 1, \ldots, \uparrow s_{m_2}, s_{m_2} + 1, \ldots, \uparrow s_{m_3} = \infty)$, then by [4, Lemma 65.4] we have integers $m_1, m_2, m_3, l_1, l_2, l_3$ with $0 < m_1 < m_2 < m_3$, $0 \leq l_1 < l_2 < l_3$, and $d_1, d_2, d_3 \in G$ with the following properties:

(a′) $D = \langle d_1 \rangle \oplus \langle d_2 \rangle \oplus \langle d_3 \rangle$ is a summand of G with $e(d_1) = m_1 + l_1$, $e(d_2) = m_2 + l_2$, $e(d_3) = m_3 + l_3$, and $r_0 = l_1$, $r_{m_1} = m_1 + l_2$, $r_{m_2} = m_2 + l_3$;

(b′) y can be written as $y = p^{l_1}d_1 + p^{l_2}d_2 + p^{l_3}d_3$.

As $e(x) = e(y)$ we have $n_3 = m_3$, and by [6, Proposition 4.1], $G/\langle x \rangle \cong G/\langle y \rangle$ implies

$$\begin{aligned} &G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \oplus \mathbb{Z}(p^{k_3+n_2}) \\ &\quad \cong G \oplus \mathbb{Z}(p^{l_1}) \oplus \mathbb{Z}(p^{l_2+m_1}) \oplus \mathbb{Z}(p^{l_3+m_2}). \end{aligned}$$

Hence $e(c_1) = n_1 + k_1 = r$, $e(c_2) = n_2 + k_2 = s$, $e(c_3) = n_3 + k_3 = t$, and in the same way, $e(d_1) = m_1 + l_1 = r$, $e(d_2) = m_2 + l_2 = s$, $e(d_3) = m_3 + l_3 = t$. Since $s < k_3 + n_2$, $l_3 + m_2 < t$, we have $k_3 + n_2 = l_3 + m_2$, hence, $n_2 = m_2$; in a similar way, we have $k_2 = l_2$, and $n_1 = m_1$, $k_1 = l_1$. Therefore, $U(x) = U(y)$.

If the Ulm sequence of y has two gaps, say $U(y) = (s_0, s_1, \ldots, \uparrow s_{m_1}, s_{m_1} + 1, \ldots, \uparrow s_{m_2} = \infty)$, appealing again to [4, Lemma 65.4], we have $y = p^{l_1}d_1 + p^{l_2}d_2$. Then $e(x) = e(y)$ implies that $n_3 = m_2$; in addition, by [6, Proposition 4.1], $G/\langle x \rangle \cong G/\langle y \rangle$ means

$$(*) \qquad G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \oplus \mathbb{Z}(p^{k_3+n_2}) \cong G \oplus \mathbb{Z}(p^{l_1}) \oplus \mathbb{Z}(p^{l_2+m_1}).$$

If $k_1 \neq 0$, then $k_1 < r$ and we also have $r < k_2 + n_1 < s < k_3 + n_2 < t$, so that the left-hand side of $(*)$ has three non-zero summands with exponents not in the set $\{r, s, t\}$. Since the right-hand side of $(*)$ has at most two such summands, this is impossible and so we have $k_1 = 0$. Furthermore, we must then have $n_1 = r$, $l_1 = k_2 + n_1$, $l_2 + m_1 = k_3 + n_2$; recall also that $n_3 = m_2$. Thus $l_1 > r$ and so it follows that $l_1 + m_1 = s = n + 2 + k_2$. So $k_2 + n_1 + m_1 = n_2 + k_2$, which shows that $n_2 = n_1 + m_1$ and as $l_2 + m_1 = k_3 + n_1 + m_1$ we deduce that $l_2 = k_3 + n_1$. However, since $l_1 + m_1 = s$, we must have $l_2 + m_2 = t = n_3 + k_3$, which gives $l_2 = k_3$, which contradicts our previous conclusion that $l_2 = k_3 + n_1$. Thus this case cannot occur.

If the Ulm sequence of y has one gap, that is to say, $U(y) = (s_0, s_1, \dots, \uparrow s_{m_1} = \infty)$, then as in the previous case, by [4, Lemma 65.4], write $y = p^{l_1} d_1$. Note that $e(x) = e(y)$ implies $n_3 = m_1$. By [6, Proposition 4.1], we have $G/\langle x\rangle \cong G/\langle y\rangle$, which means $G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \oplus \mathbb{Z}(p^{k_3+n_2}) \cong G \oplus \mathbb{Z}(p^{l_1})$. This is impossible since neither $k_2 + n_1$ nor $k_3 + n_2$ belongs to the set $\{0, r, s, t\}$, and $G \oplus \mathbb{Z}(p^{l_1})$ does not have two cyclic summands isomorphic to $\mathbb{Z}(p^{k_2+n_1})$ and $\mathbb{Z}(p^{k_3+n_2})$ respectively.

Case (ii). Without loss of generality assume $U(x) = (r_0, r_1, \dots, \uparrow r_{n_1}, r_{n_1} + 1, \dots, \uparrow r_{n_2} = \infty)$. Similarly to case (i), by [4, Lemma 65.4] we can write $x = p^{k_1} c_1 + p^{k_2} c_2$. If the Ulm sequence of y has two gaps, by [4, Lemma 65.4], we may write $x = p^{l_1} d_1 + p^{l_2} d_2$. Note that $e(x) = e(y)$ gives $n_2 = m_2$. Using [6, Proposition 4.1], we have $G/\langle x\rangle \cong G/\langle y\rangle$, which means

$$(**) \qquad G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \cong G \oplus \mathbb{Z}(p^{l_1}) \oplus \mathbb{Z}(p^{l_2+m_1}).$$

Subcase (a) $k_2 + n_1 = s$. Note that we then have $k_1 + n_1 = r$, $k_2 + n_2 = t$, and $0 \le k_1 < r$. As $r < l_2 + m_1$ and $(**)$ gives us that $\mathbb{Z}(p^{l_2+m_1})$ is isomorphic to a summand of $G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1})$, we must have $l_2 + m_1 \in \{0, r, s, t\}$ and clearly the only possibility is $l_2 + m_1 = s$. Therefore, $k_1 + n_1 = l_1 + m_1 = r$, $k_2 + n_2 = l_2 + m_2 = t$. Straightforward calculation then gives us that $n_i = m_i$, $k_1 = l_i$, $i = 1, 2$. So $U(x) = U(y)$ and this ensures the existence of an automorphism of G mapping $x \mapsto y$ since G is transitive in the sense of Kaplansky.

Subcase (b) $k_2 + n_1 \ne s$. Note firstly that we may assume $l_2 + m_1 \ne s$ since otherwise we are again in subcase (a). Furthermore, $k_2 + n_1 \notin \{r, t\}$ and similarly for $l_2 + m_1$. It then follows that $k_2 + n_1 = l_2 + m_1$ since these are the exponents of the summands of maximal size not in the set $\{r, s, t\}$ occurring in $(**)$.

If $k_1 + n_1 = l_1 + m_1 = z \in \{r, s\}$ then substituting for n_1, l_1 in the equation $k_2 + n_1 = l_2 + m_2$ we see that $k_2 - k_1 = l_2 - l_1$. Consider the various possibilities:

- If $z = r$ then $k_1, l_1 < r$ and it follows from $(**)$ that $k_1 = l_1$, so that $k_2 = l_2$ and $n_1 = m_1$. The equality of the exponents of x, y gives $n_2 = m_2$ and so we have $U(x) = U(y)$.
- If $z = s$, then $k_2 + n_1 > s$ so that $k_2 + n_2 = t$ and similarly $l_2 + m_2 = t$. As $n_2 = m_2$ this yields $k_2 = l_2$ whence $k_1 = l_1$ and so $n_1 = m_1$. Again we have $U(x) = U(y)$.

So in both of the above situations there will be an automorphism of G mapping $x \mapsto y$.

The remaining possibility is that $k_1 + n_1$ and $l_1 + m_1$ are different and, without loss, we may assume $k_1 + n_1 = r$, $l_1 + m_1 = s$; it follows that $l_2 + m_2 = t$. Also, as $k_1 < r$ it follows from (∗∗) that $k_1 = l_1$. We claim $k_2 + n_2 = s$ since otherwise $k_2 + n_2 = t = l_2 + m_2$, which gives us that $k_2 = l_2$, forcing $n_1 = m_1$. This is clearly impossible as it means $k_1 + n_1 = r = l_1 + m_1 = s$. So we have $k_2 + n_2 = s$, but this is also impossible: $k_2 + n_2 = s$ implies $k_2 + n_1 < s$ while $l_2 + m_1 > l_1 + m_1 = s$, contradicting $l_2 + m_1 = k_2 + n_1$.

In case (ii), if the Ulm sequence of y has one gap, then using similar notation to that in the previous cases, we may write $y = p^{l_1} d_1$. Since $e(x) = e(y)$, then $n_2 = m_1$. By [6, Proposition 4.1], we have $G/\langle x\rangle \cong G/\langle y\rangle$ which means

$$(***) \qquad G \oplus \mathbb{Z}(p^{k_1}) \oplus \mathbb{Z}(p^{k_2+n_1}) \cong G \oplus \mathbb{Z}(p^{l_1}).$$

There are two subcases to consider:

(a) $k_1 + n_1 = r$;

(b) $k_1 + n_1 = s$.

Subcase (a) $k_1 + n_1 = r$. Observe firstly that if $k_1 = 0$ then $n_1 = r$ and it follows from (∗∗∗) that $l_1 = k_2 + n_1$; furthermore, $k_2 + n_1$ must also belong to the set $\{r, s, t\}$ and hence it is immediate that $k_2 + n_1 = s$. However, $l_1 + m_1 = k_2 + n_1 + n_2 \in \{r, s, t\}$ and it follows that $k_2 + n_1 + n_2$ must be equal to t, so that $k_2 + n_2 = s$. This is impossible since then we would have $t = k_2 + n_1 + n_2 = s + r$. So $k_1 \neq 0$ and as $k_1 < r$, it follows from (∗∗∗) that we must have $l_1 = k_1$. Now $l_1 + m_1 = k_1 + n_2 \in \{r, s, t\}$ and $k_1 + n_2 \notin \{r, t\}$, so that $k_1 + n_2 = s$. This is impossible: $r + t = n_1 + k_1 + n_2 + k_2 = (k_2 + n_1) + (k_1 + n_2) = s + s = 2s$. Thus subcase (a) cannot occur.

Subcase (b) $k_1 + n_1 = s$. Note firstly that we then have $k_2 + n_2 = t$ and $n_2 = m_1$. If $k_1 = 0$, then $n_1 = s$ and it follows from (∗∗∗) that $l_1 = k_2 + n_1$. This leads to the contradiction that $l_1 + m_1 = k_2 + n_1 + n_2 = t + n_1 > t$, so $k_1 \neq 0$. However, we cannot have $k_1 = r$, since this would again force $l_1 = n_1 + k_2$ and thus $l_1 + m_1 = n_1 + k_2 + n_2 > t$. Since $k_1 \neq 0, r$ it follows from (∗∗∗) that $l_1 = k_1$ and $k_2 + n_1$ must belong to the set $\{r, s, t\}$. Since $k_2 + n_1 > k_1 + n_1 = s$, we must have $k_2 + n_1 = t = k_2 + n_2$, giving the contradiction that $n_1 = n_2$. So subcase (b), and hence case (ii), cannot occur.

Case (iii). Both Ulm sequences have only one gap. By [4, Lemma 65.4], write $x = p^{k_1} c_1$ and $y = p^{l_1} d_1$ and $e(x) = e(y)$ gives $n_1 = m_1$. By [6, Proposition 4.1], $G/\langle x\rangle \cong G \oplus \mathbb{Z}(p^{k_1})$ and $G/\langle y\rangle \cong G \oplus \mathbb{Z}(p^{l_1})$. By hypothesis, $G \oplus \mathbb{Z}(p^{k_1}) \cong G \oplus \mathbb{Z}(p^{l_1})$. If $k_1 = l_1$, then $U(x) = U(y)$. If $k_1 \neq l_1$, then both $k_1, l_1 \in \{0, r, s\}$. Also notice that $k_1 + n_1, l_1 + m_1 = l_1 + n_1 \in \{r, s, t\}$. Without loss of generality, assume $k_1 < l_1$.

If (1), $k_1 = 0$, $l_1 = r$, then $n_1 \in \{r, s, t\}$, but $l_1 + n_1 = r + n_1 \in \{s, t\}$, and so the only possibilities are $2r = s$, $2r = t$, $r + s = t$, $2s = t$, all of which contradict the assumption.

If (2), $k_1 = 0$, $l_1 = s$, then $n_1 \in \{r, s, t\}$, but $l_1 + n_1 = s + n_1 \in \{t\}$, and so the only possibilities are $r + s = t$, $2s = t$, which are all impossible by the hypothesis.

If (3), $k_1 = r$, $l_1 = s$, then $n_1 \in \{s - r, t - r\}$, now $l_1 + n_1 = s + n_1 \in \{t\}$, and so the only possibility is $r + t = 2s$ – which contradicts the assumption. ■

It is tempting to think that the exclusion of relations as given in the bullet points above will lead to sufficient conditions for CS-transitivity as in Propositions 4.2, 4.3 above. Unfortunately, the situation is much more complex. We illustrate this initially with an example.

Example 4.4. Let $G = G_1 \oplus G_2 \oplus G_3 \oplus G_4$, where the groups G_i are homocyclic of infinite rank and exponents 1, 3, 7, 12 respectively. Then G is not CS-transitive but none of the additive relations above hold.

Proof. Clearly no relation $2n_i = n_j$ holds and the sum of any two elements of the set $\{1, 3, 7, 12\}$ is not a member of the set. Furthermore, the sum of any two elements of this set cannot be equal to the sum of the remaining two elements.

However, if a, b, c, d are generators of cyclic summands of the groups G_1, G_2, G_3, G_4 respectively and we set $x = a + p^2c$, $y = p^7d$, then $e(x) = 5 = e(y)$. Clearly, $G/\langle y\rangle \cong G \oplus \mathbb{Z}(p^7) \cong G$. A simple calculation, either directly or using [6, Proposition 3.1], gives $G/\langle x\rangle \cong G \oplus \mathbb{Z}(p^3) \cong G$. Since x, y have different heights in G, no automorphism of G can map $x \mapsto y$ and hence G is not CS-transitive. ■

Finding sufficient conditions to ensure CS-transitivity is non-trivial and a complete characterisation of CS-transitivity, even for direct sums of infinite-rank homocyclic groups, seems doomed to end in a complicated arithmetic discussion. We can, however, offer a reasonably general sufficient condition which will, in at least one case, be necessary also.

For this we need to introduce a simple concept.

Given a finite set $\mathcal{S}$ of strictly increasing integers, $\mathcal{S} = \{n_1, \ldots, n_t\}$, we say that $\mathcal{S}$ is *well spaced of breadth* t if, given any pair of consecutive terms n_i, n_{i+1} in $\mathcal{S}$, the inequality $n_{i+1} \geq 2n_i + 1$ holds. As before, with a well-spaced set $\mathcal{S}$ of breadth t we associate a group $G_{\mathcal{S}}$ by setting $G_{\mathcal{S}} = G_1 \oplus G_2 \oplus \cdots \oplus G_t$, where each G_i is a homocyclic group of infinite rank and exponent n_i. Our objective is to establish the following theorem:

Theorem 4.5. *If $\mathcal{S}$ is a well-spaced set of arbitrary finite breadth, then the group $G_{\mathcal{S}}$ is CS-transitive.*

The proof requires quite delicate calculations and bears some similarity in general nature to the proof that a finitely generated p-adic module is CS-transitive; see [7, Theorem 3.1].

Proof of Theorem 4.5. The proof will be by induction on the breadth of $\mathcal{S}$; if $\mathcal{S}$ is a singleton, then $G_{\mathcal{S}}$ is just a homocyclic group of infinite rank and this is easily seen to be CS-transitive – see, for example, [6, Corollary 2.3]. Suppose then that the theorem holds for all well-spaced sets of breadth $\leq N$ and let $\mathcal{S} = \{n_1, \ldots, n_N, n_{N+1}\}$ be a well-spaced set of breadth $N+1$ with $G_{\mathcal{S}} = \bigoplus_{i=1}^{N+1} G_i$, with G_i homocyclic of infinite rank and exponent n_i; for convenience of notation we will write $G_{N+1} = D = \bigoplus_{i \in I} \langle d_i \rangle$, where I is infinite and $e(d_i) = n_{N+1} \geq 2n_N + 1$. We will also find it notationally convenient to write $G' = \bigoplus_{i=1}^{N} G_i$ and $G'' = \bigoplus_{i=1}^{N-1} G_i$ and if there is no possibility of ambiguity we write G in place of $G_{\mathcal{S}}$.

So suppose that $x, y \in G$ satisfy $e(x) = e(y)$, $G/\langle x \rangle \cong G/\langle y \rangle$. We show by a case-by-case argument that there is an automorphism ϕ of G with $\phi(x) = y$, thereby ensuring that G is CS-transitive. We split the argument into three main cases, although we will require a number of additional subcases:

(I) $x, y \in G'$;

(II) $x \in G'$, $y \notin G'$;

(III) $x \notin G'$, $y \notin G'$.

Case (I) is easily handled: $G/\langle x \rangle \cong G'/\langle x \rangle \oplus D$ and similarly for y, but then as D is homocyclic of exponent strictly bigger than that of the direct sums of cyclic groups $G'/\langle x \rangle, G'/\langle y \rangle$, we may deduce that $G'/\langle x \rangle \cong G'/\langle y \rangle$. By induction, we can find an automorphism ψ of G' with $\psi(x) = y$. Now set $\phi = \psi \oplus 1_D$ and ϕ is then an automorphism of G mapping $x \mapsto y$.

Case (II). Here $x \in G'$ but $y \notin G'$, so that $y = c + d'$ for some $c \in G'$, $0 \neq d' \in D$. Since D is homocyclic we may choose a decomposition $D = \langle d \rangle \oplus D_0$ with $d' = p^{\alpha} d$ for some $0 \leq \alpha \leq n_{N+1} - 1$; note that $n_{N+1} - 1 \geq 2n_N$. Now $G/\langle x \rangle = G'/\langle x \rangle \oplus \langle d \rangle \oplus D_0$ is a direct sum of cyclic groups where the direct summands either have exponent $\leq n_N$ or have exponent $n_{N+1} \geq 2n_N + 1$; observe that $e(c)$ cannot exceed $e(x)$ since $e(x) = e(y)$.

We consider two subcases:

(a) $e(c) = e(x)$;

(b) $e(c) < e(x)$.

Subcase (a). Suppose $e(c) = e(x) = t$, where $t \leq n_N$. Then $p^t x = 0 = p^t y = p^{\alpha+t} d$, so that $\alpha + t \geq n_{N+1}$, whence $\alpha \geq n_{N+1} - t \geq 2n_N + 1 - t \geq n_N + 1$. Now consider the Ulm sequence $U_G(y)$: since G' and D are direct summands of G, $U_G(y) = U_{G'}(c) \cap U_D(p^\alpha d)$. However, if $U_{G'}(c) = (\gamma_0, \gamma_1, \ldots, \gamma_{t-1}, \infty, \ldots)$, then $\gamma_i < n_N$ for all $0 \leq i \leq t-1$, while $U_D(p^\alpha d) \geq (\alpha, \alpha+1, \ldots, \alpha+t-1, \infty, \ldots)$ and for each i $(0 \leq i \leq t-1)$, $\alpha + i \geq n_N + 1 + i > n_N$. Hence $U_G(y) = U_{G'}(c) = U_G(c)$. Now the group G is, of course, transitive in the sense of Kaplansky, so there is an automorphism θ of G with $\theta(y) = c$. But then $G/\langle y\rangle \cong G/\langle \theta(y)\rangle = G/\langle c\rangle$. However, it then follows that $G/\langle x\rangle \cong G'/\langle x\rangle \oplus D$ and $G/\langle c\rangle \cong G'/\langle c\rangle \oplus D$. Since the exponent of G' is strictly less than that of D, we must have, as observed in case (I), that $G'/\langle x\rangle \cong G'/\langle c\rangle$. By our assumption in subcase (a) we also have $e(x) = e(c)$ and our inductive hypothesis now provides an automorphism ψ of G' mapping $c \mapsto x$; extend this to an automorphism $\phi = \psi \oplus 1_D$ of G and observe that the composition $\phi\theta$ is then an automorphism of G mapping $y \mapsto x$.

Subcase (b). Here we have $e(c) < e(x)$, so suppose $e(x) = t$ and $e(c) < t$; we claim that in this situation it is impossible to have $G/\langle x\rangle \cong G/\langle y\rangle$, so that this subcase cannot arise.

As in subcase (a), we see immediately that $\alpha + t \geq n_{N+1} \geq 2n_N + 1$ and again $\alpha \geq n_N + 1$. On the other hand, $p^{t-1}x \neq 0$, so $p^{t-1}y \neq 0$ while $p^{t-1}c = 0$ by the hypothesis of subcase (b). So $\alpha + t - 1 \leq n_{N+1} - 1 \geq 2n_N$, whence $\alpha + t \geq 2n_N + 1$. Now consider the quotient $H = (G' \oplus \langle d\rangle)/\langle c + p^\alpha d\rangle$, which is generated by elements of the form $c' + \langle c + p^\alpha d\rangle$ with $c' \in G'$ and by $d + \langle c + p^\alpha d\rangle$. The former set of generators all have exponent $\leq n_N$, while the exponent of $d + \langle c + p^\alpha d\rangle$ is easily seen to be $\alpha + e(c)$. It follows that $n_N + 1 \leq \alpha < \alpha + e(c) < \alpha + t = 2n_N + 1$. Hence H is a group with exponent $e(H)$ satisfying $n_N + 1 < e(H) < 2n_N + 1$. Furthermore, since $e(d + \langle c + p^\alpha d\rangle) = e(H)$, H has a direct summand, H' say, with $n_N + 1 < e(H') < 2n_N + 1$. However, $G/\langle y\rangle = H \oplus D_0$ then has a summand of exponent strictly between $n_N + 1$ and $2n_N + 1$ – this is impossible if $G/\langle y\rangle \cong G/\langle x\rangle$ since the latter has summands with exponents either $\leq n_N$ or equal to $n_{N+1} \geq 2n_N + 1$. Thus subcase (b) does not arise and we have completed the proof of case (II).

Case (III). Here we are in the situation where $x = c_1 + p^\alpha d$, $y = c_2 + p^\beta d$ – note that we may assume that we are dealing with the same element $d \in D$ for each of x, y since all the generators of D lie in the same orbit under the action of $\mathrm{Aut}(D)$ and hence of $\mathrm{Aut}(G)$ – where $c_1, c_2 \in G'$, $e(x) = e(y)$, and $G/\langle x\rangle \cong G/\langle y\rangle$. In this situation we need to examine three possible subcases:

(a) $e(x) = e(c_1), e(y) = e(c_2)$;

(b) $e(x) > e(c_1), e(y) > e(c_2)$;

(c) without loss of generality $e(x) > e(c_1), e(y) = e(c_2)$.

Subcase (a). Suppose $e(x) = e(c_1) = t$ say; note that we then have $e(y) = e(c_2) = t$ also since $e(x) = e(y)$. Since $e(x) = t$ we have $0 = p^t x = p^t c_1 + p^{\alpha+t} d$, which yields $\alpha + t \geq n_{N+1}$. Now as $t = e(c_1)$, we also have that $t \leq n_N$, so $n_{N+1} \leq \alpha + t \leq \alpha + n_N$ and hence $\alpha \geq n_{N+1} - n_N \geq n_N + 1$; a similar result holds for y and consequently we also have $\beta \geq n_{N+1} - n_N \geq n_N + 1$.

Now arguing exactly as in case (i), subcase (a), we get that $U_G(x) = U_G(c_1) \cap U_G(p^\alpha d) = U_G(c_1)$ and similarly $U_G(y) = U_G(c_2) \cap U_G(p^\beta d) = U_G(c_2)$. As G is transitive in the sense of Kaplansky, there are automorphisms ϕ, θ of G with $\phi(x) = c_1, \theta(c_2) = y$. It then follows that $G/\langle x\rangle \cong G/\langle \phi(x)\rangle = G/\langle c_1\rangle$ and $G/\langle c_2\rangle \cong G/\langle \theta(c_2)\rangle = G/\langle y\rangle$; we immediately deduce that $G/\langle c_1\rangle \cong G/\langle c_2\rangle$. Now $c_1, c_2 \in G'$ so that $G/\langle c_1\rangle = G'/\langle c_1\rangle \oplus D \cong G/\langle c_2\rangle = G'/\langle c_2\rangle \oplus D$. As every summand of $G'/\langle c_1\rangle$, $G'\langle c_2\rangle$ has order at most $n_N < n_{N+1} = e(D)$, we conclude that $G'/\langle c_1\rangle \cong G'/\langle c_2\rangle$. We also have from our assumption that we are in subcase (a) that $e(c_1) = e(c_2)$; by our inductive hypothesis we know that G' is CS-transitive, so there is an automorphism ψ of G' mapping $c_1 \mapsto c_2$. Extend ψ to an automorphism of G which we still call ψ and observe that the composition $\theta\psi\phi$ is an automorphism of G mapping $x \mapsto y$.

Subcase (c). Here we are in the situation where $e(x) > e(c_1)$ and $e(y) = e(c_2)$. Note that exactly as in subcase (a) above we will have $U_G(y) = U_G(c_2)$, which again gives us that $G/\langle y\rangle \cong G/\langle c_2\rangle \cong G'/\langle c_2\rangle \oplus D$.

However, arguing exactly as in case (II), subcase (b) we see that $G/\langle x\rangle$ has a direct summand of exponent strictly greater than n_N and strictly less than n_{N+1}. This is in contradiction with our hypothesis that $G/\langle x\rangle \cong G/\langle y\rangle$ since the latter only has summands of exponent at most n_N or exactly equal to n_{N+1}. Thus this subcase cannot arise.

Before proceeding to the final subcase we need the following general lemma.

Lemma 4.6. *Suppose that $H = C \oplus \langle d\rangle$ is a bounded group and d is an element of exponent $\geq 2e(C) + 1$. Let $z = c + p^\alpha d$, where $c \in C$ and $e(c) < e(z)$. Then if $\alpha + e(c) \geq e(C)$, the quotient $H/\langle z\rangle \cong \mathbb{Z}(p^k) \oplus C/\langle c\rangle$, where $k = \alpha + e(c) \leq e(d) - 1$.*

Proof. The group $H/\langle z\rangle$ is generated by elements of the form $c' + Z$, $d + Z$, where $c' \in C$ and $Z = \langle z\rangle = \langle c + p^\alpha d\rangle$. The first set of generators all have exponents at most $e(C)$, while $e(d + Z)$ is precisely $\alpha + e(c)$. Thus H/Z is bounded of exponent $\max\{\alpha + e(c), e(C)\} = \alpha + e(c)$ by hypothesis. Since $e(d + Z) = e(H/Z)$, the cyclic subgroup generated by $d + Z$ is a direct summand of H/Z and we have

$$H/\langle d + Z\rangle = \langle d + Z\rangle/Z \oplus Y/Z$$

for some $Y \leq H$. Observe that $\langle d + Z\rangle/Z = (\langle c\rangle \oplus \langle d\rangle)/Z$ and so $Y/Z \cong H/(\langle c\rangle \oplus \langle d\rangle) \cong C/\langle c\rangle$. Hence $H/Z \cong C/\langle c\rangle \oplus \mathbb{Z}(p^k)$, where $k = \alpha + e(c)$. Finally, note that $p^{e(c)}z = p^{e(c)+\alpha}d \neq 0$, since by assumption $e(c) < e(z)$. So $k = e(c) + \alpha \leq e(d) - 1$. ∎

We are left with just subcase (b) to consider. Thus we have the following situation: $x = c_1 + p^\alpha d$, $y = c_2 + p^\beta d$, $e(x) = e(y)$, $e(x) > e(c_1)$, $e(y) > e(c_2)$, and the quotients $G/\langle x\rangle, G/\langle y\rangle$ are isomorphic. Observe first of all that in this situation $\alpha = \beta$: a straightforward calculation gives that $e(x) = n_{N+1} - \alpha$, while $e(y) = n_{N+1} - \beta$.

Unfortunately, it seems necessary to split our argument into two further subcases. So let us assume firstly that in addition we have $\alpha + e(c_1) > n_N$, $\alpha + e(c_2) > n_N$. Apply Lemma 4.6 with G' in place of C to get

$$(G' \oplus \langle d\rangle)/\langle c_1 + p^\alpha d\rangle \cong G'/\langle c_1\rangle \oplus \mathbb{Z}(p^{k_1}),$$
$$(G' \oplus \langle d\rangle)/\langle c_2 + p^\alpha d\rangle \cong G'/\langle c_2\rangle \oplus \mathbb{Z}(p^{k_2}),$$

where $k_1 = \alpha + e(c_1) > n_N$ and $k_2 = \alpha + e(c_2) > n_N$. Thus $k_1, k_2 \leq n_{N+1} - 1$. It then follows that

$$G'/\langle c_1\rangle \oplus \mathbb{Z}(p^{k_1}) \oplus D_0 \cong G'/\langle c_2\rangle \oplus \mathbb{Z}(p^{k_2}) \oplus D_0$$

and $n_N < k_1, k_2 \leq n_{N+1} - 1$. Since $G'/\langle c_1\rangle$ and $G'/\langle c_2\rangle$ are of exponent at most n_N, while D_0 has exponent exactly n_{N+1}, we must have $k_1 = k_2$ and $G'/\langle c_1\rangle \cong G'/\langle c_2\rangle$. It now follows that $e(c_1) = e(c_2)$. By our inductive hypothesis, G' is CS-transitive and so there is an automorphism of G' sending $c_1 \mapsto c_2$. Extend θ to $\psi = \theta \oplus 1_D$, an automorphism of G which maps $x \mapsto y$. So in this first subcase we have the desired automorphism of G mapping $x \mapsto y$.

For our final subcase we are in the situation where $x = c_1 + p^\alpha d$, $y = c_2 + p^\alpha d$, $e(c_1) < e(x)$, $e(c_2) < e(y)$ and, without loss of generality we have $\alpha + e(c_1) \leq n_N$.

Note firstly that we cannot have $\alpha + e(c_2) > n_N$ in this situation. If the latter inequality holds, we may apply Lemma 4.6 to get that $G/\langle y\rangle \cong G'/\langle c_2\rangle \oplus \mathbb{Z}(p^{k_2}) \oplus D_0$ with $n_N < k_2 = \alpha + e(c_2) \leq n_{N+1} - 1$. However, as $\alpha + e(c_1) \leq n_N$, we have $e(d + \langle c_1 + p^\alpha d\rangle) = \alpha + e(c_1) \leq n_N$. This means that the summands of $G/\langle x\rangle$ have exponents at most n_N or exactly n_{N+1} and this would violate our hypothesis that $G/\langle x\rangle$ is isomorphic to $G/\langle y\rangle$. So for our final subcase we must have $\alpha + e(c_1)$, $\alpha + e(c_2) \leq n_N$. For our next simplification we write $X = \langle x\rangle = \langle c_1 + p^\alpha d\rangle$.

- If $e(c_1) + \alpha \leq n_N$, then $G' \cap X = 0$.

To see this assume $g \in G' \cap X$ so that $g = p^t(c_1 + p^\alpha d)$ for some t. If $t < e(c_1)$ then $g - p^t c_1 = p^{\alpha+t} d$, which implies that $g - p^t c_1 = 0$ and $\alpha + t \geq n_{N+1}$. But then $e(c_1) + \alpha \geq t + \alpha \geq n_{N+1}$ – a contradiction. So $t \geq e(c_1)$ and hence $g = p^{\alpha+t} d \in G' \cap \langle d \rangle = 0$, as required.

Our next observation is that the quotient $G' \oplus \langle d \rangle / X$ is an extension of G' by $\mathbb{Z}(p^\alpha)$. To see this consider the exact sequence

$$0 \to (G' + X)/X \to (G' \oplus \langle d \rangle)/X \to (G' \oplus \langle d \rangle)/(G' + X) \to 0.$$

The first term is just isomorphic to G' since $G' \cap X = 0$, and as $G' + X = G' \oplus \langle p^\alpha d \rangle$ the final term reduces to $(G' \oplus \langle d \rangle)/(G' \oplus \langle p^\alpha d \rangle) \cong \mathbb{Z}(p^\alpha)$; a similar result holds with X replaced by $Y = \langle y \rangle$.

In fact, the exact sequence above actually splits to yield that $(G' \oplus \langle d \rangle)/X \cong G' \oplus \mathbb{Z}(p^\alpha)$. Since all the groups under consideration are bounded, it suffices to show that $(G' + X)/X$ is pure in $(G' \oplus \langle d \rangle)/X$.

- The group $(G' + X)/X$ is pure in $(G' \oplus \langle d \rangle)/X$.

To see the above claim suppose that $g' + X = p^r(g + p^\gamma d) + X$ for $g', g \in G'$ and some $\gamma < n_{N+1}$. Then $g' - p^r g - p^{\gamma+r} d = p^t(c_1 + p^\alpha d)$ for some suitable integer t. Consequently, $g' - p^r g - p^t c_1 = p^{\gamma+r} d + p^{\alpha+t} d = 0$. Then the right-hand side of the last equation is of the form $p^{\min(\gamma+r,\alpha+t)} \mu d$ with $(\mu, p) = 1$. Thus $\min(\gamma + r, \alpha + t) \geq n_{N+1}$, whence each of $\gamma + r, \alpha + t \geq n_{N+1}$. Now $g' - p^r g - p^t c_1 = 0$ so that $g' + X = p^r(g + X) + p^t c_1 + X$. However, $p^t c_1 + X = [p^t(c_1 = p^\alpha d) - p^{\alpha+t} d] + X = X$. Hence $g' + X = p^r(g + X)$ and as $g \in G'$, purity follows.

Note that a similar result holds for $(G' + Y)/Y$.

Returning now to the situation where $x = c_1 + p^\alpha d$, $y = c_2 + p^\alpha d$, $e(c_1) < e(x)$, $e(c_2) < e(y)$, and $\alpha + e(c_1), \alpha + e(c_2) \leq n_N$, we see that

$$\begin{aligned} G/X &\cong G' \oplus \mathbb{Z}(p^\alpha) \oplus D_0 \\ &\cong (G' \oplus \langle d \rangle)/X \oplus D_0 \\ &\cong G/Y \cong G' \oplus \mathbb{Z}(p^\alpha) \oplus D_0. \end{aligned}$$

Since by hypothesis $\alpha \leq n_N$, we conclude that $G' \oplus \mathbb{Z}(p^\alpha) \cong (G' \oplus \langle d \rangle)/\langle x \rangle \cong (G' \oplus \langle d \rangle)/\langle y \rangle$.

Now consider the situation where $c_1 = g_1 + p^k w$ with $g_1 \in G''$, $w \in G_N$ and w has height zero in G_N (and hence also in G). Then $0 = p^{e(c_1)} c_1 = p^{e(c_1)} g_1 + p^{k+e(c_1)} w$ and so $p^{k+e(c_1)} w = 0$. This implies that $k + e(c_1) \geq e(w) = n_N$, so $k \geq n_N - e(c_1) \geq \alpha$. A similar argument shows that if $c_2 = g_2 + p^r z$ with $g_2 \in G''$, $z \in G_N$ and z of height zero, then $r \geq \alpha$. Now set, for $i = 1, 2$, $\phi_i\colon \langle d\rangle \to G'$ with $\phi_1(d) = -p^{k-\alpha}w$, $\phi_2(d) = -p^{r-\alpha}z$ and observe that the matrices

$$\Delta_i = \begin{pmatrix} 1_{G'} & \phi_i \\ 0 & 1_{\langle d\rangle} \end{pmatrix}$$

are automorphisms of $G' \oplus \langle d\rangle$ (and hence they extend to automorphisms of G) with $\Delta_1(x) = g_1 + p^\alpha d$, $\Delta_2(y) = g_2 + p^\alpha d$. Hence, there exists an automorphism of G mapping $x \mapsto y$ if, and only if, there exists an automorphism mapping $g_1 + p^\alpha d \mapsto g_2 + p^\alpha d$; note that $g_1, g_2 \in G''$.

The final step in our proof will be to show that there is, indeed, such an automorphism mapping $g_1 + p^\alpha d \mapsto g_2 + p^\alpha d$.

Now $(G' \oplus \langle d\rangle)/\langle x\rangle \cong (G' \oplus \langle d\rangle)/\Delta_1(x) = (G' \oplus \langle d\rangle)/\langle g_1 + p^\alpha d\rangle \cong (G'' \oplus \langle d\rangle)/\langle g_1 + p^\alpha d\rangle \oplus G_N$; a similar result holds for y. Note that $e(g_1) \leq e(c_1) < e(x)$, $e(g_2) \leq e(c_2) < e(y)$, so by our previous argument in this subcase, we will have $(G'' \oplus \langle d\rangle)/\langle g_1 + p^\alpha d\rangle \cong G'' \oplus \mathbb{Z}(p^\alpha)$ and similarly for $g_2 + p^\alpha d$. Hence $G'' \oplus \mathbb{Z}(p^\alpha) \oplus G_N \cong (G'' \oplus \langle d\rangle)/\langle g_1 + p^\alpha d\rangle \oplus G_N$ and a similar result holds for g_2.

Now if $\alpha < n_N = e(G_N)$, a standard argument with Ulm invariants yields that $(G'' \oplus \langle d\rangle)/\langle g_1 + p^\alpha d\rangle \cong G'' \oplus \mathbb{Z}(P^\alpha) \cong (G'' \oplus \langle d\rangle)/\langle g_2 + p^\alpha d\rangle$. Since $e(g_1) \leq e(c_1)$, we have $e(g_1) + \alpha \leq e(c_1) + \alpha \leq \nu_N$ and similarly $e(g_2) + \alpha \leq n_N$. But it then follows that $e(g_1 + p^\alpha d) = n_{N+1} - \alpha = e(g_2 + p^\alpha d)$. Now, by our inductive hypothesis, there is then an automorphism of $G'' \oplus \langle d\rangle$, which extends to an automorphism of G and maps $g_1 + p^\alpha d \mapsto g_2 + p^\alpha d$. Hence, as noted above, there is then an automorphism mapping $x \mapsto y$.

The only remaining thing to check is what happens when $\alpha = n_N$. However, this case is straightforward: we must then have $e(c_1) = e(c_2) = 0$ so that $c_1 = c_2 = 0$ and the identity maps $x \mapsto y$.

So we have established the existence of an automorphism mapping $x \mapsto y$ in all cases and hence the proof is complete. ∎

Corollary 4.7. *Suppose that $\mathcal{S} = \{n_1, n_2, \ldots\}$ is well spaced and of infinite breadth. If $G_\mathcal{S} = \bigoplus_{i=1}^\infty G_i$, where G_i is homocyclic of infinite rank and exponent n_i, then $G_\mathcal{S}$ is CS-transitive.*

Proof. Suppose that $x, y \in G_\mathcal{S} = G$, with $e(x) = e(y)$ and $G/\langle x\rangle \cong G/\langle y\rangle$. Then there is a finite initial segment, $\mathcal{S}'$ of $\mathcal{S}$, such that $x, y \in \mathcal{S}'$; let $\mathcal{S}''$ denote the complement of $\mathcal{S}'$ in $\mathcal{S}$. Denote the largest element of $\mathcal{S}'$ by n_t. Then

$$G/\langle x\rangle = \left(\bigoplus_{i=1}^{t} G_i/\langle x\rangle\right) \oplus \bigoplus_{i\in\mathcal{S}''} G_i \cong \left(\bigoplus_{i=1}^{t} G_i/\langle y\rangle\right) \oplus \bigoplus_{i\in\mathcal{S}''} G_i.$$

Since the exponent of every summand of $\bigoplus_{i=1}^{t} G_i/\langle x\rangle$ (and similarly for y) is at most n_t, while every summand of the complement $\bigoplus_{i\in\mathcal{S}''}$ has exponent strictly greater than n_t, we deduce that $\bigoplus_{i=1}^{t} G_i/\langle x\rangle \cong \bigoplus_{i=1}^{t} G_i/\langle y\rangle$. Clearly the subset $\mathcal{S}'$ is well spaced and of finite breadth, so by the previous theorem there is an automorphism ϕ of $\bigoplus_{i=1}^{t} G_i$ which maps $x \mapsto y$. However, ϕ clearly extends to an automorphism ψ of G with $\psi = \phi \oplus 1$, where 1 denotes the identity map on $\bigoplus_{i\in\mathcal{S}''} G_i$ and $\psi(x) = y$. Since x, y were arbitrary, the result follows. ∎

So from Theorem 4.5 we have sufficient conditions for a direct sum of infinite-rank homocyclic groups to be CS-transitive. However, these conditions are not in general necessary.

Example 4.8. If $G = G_1 \oplus G_2 \oplus G_3$ with G_1, G_2, G_3 homocyclic of infinite rank and exponents 2, 5, 9 respectively, then G is CS-transitive but the set $\{2, 5, 9\}$ is not a well-spaced set.

Proof. This follows immediately from Proposition 4.3. ∎

The situation is, however, radically different if we consider a minimal well-spaced set containing 1, i.e. a set where successive elements n_i, n_{i+1} satisfy $n_{i+1} = 2n_i + 1$. It is possible to give a very general result in this case but we feel the ideas are more simply explained by examples.

Starting from $G_1 \oplus G_2 \oplus G_3 \oplus G_4$, where the G_i are homocyclic of infinite rank and exponents 1, 3, 7, 15 respectively, the smallest possible choice of an exponent for a homocyclic group G_5 of infinite rank making the sum $G_1 \oplus G_2 \oplus G_3 \oplus G_4 \oplus G_5$ CS-transitive is 31. The possibilities $\{16, 17, 18, 19, 21, 22, 23, 27, 29, 30\}$ are easily eliminated due to relations such as $19 + 3 = 22$; details are left to the reader. The remaining possibilities $\{20, 24, 25, 26, 28\}$ all follow the same pattern: let a, b, c, d, e be generators of cyclic summands of $G_1, \ldots, G_5$ respectively and set $x = p^{15}e$, so that $G/\langle x\rangle \cong G$. Then for the various possible values of the exponent of e, one has that x has exponent 5, 9, 10, 11, 13 respectively. Now choose a corresponding element y as $a + p^2c, a + p^6d, pb + p^5d, b + p^4d, a + p^2d$ respectively. A straightforward calculation, using for example [6, Proposition 3.1] or directly, gives us that $G/\langle y\rangle$ is isomorphic to $G \oplus \mathbb{Z}(p^3), G \oplus \mathbb{Z}(p^7), G \oplus \mathbb{Z}(p) \oplus \mathbb{Z}(p^7), G \oplus \mathbb{Z}(p^7), G \oplus \mathbb{Z}(p^3)$ respectively. Since all of these are, in fact, isomorphic to G and in no choice is $\mathrm{ht}_G(x) = \mathrm{ht}_G(y)$, we see that G is not CS-transitive for any of the five possible choices. We summarise the situation as follows:

Proposition 4.9. *If $H = G_1 \oplus G_2 \oplus G_3 \oplus G_4$, where the G_i $(i = 1, 2, 3, 4)$ are homocyclic of infinite rank and exponent* 1, 3, 7, 15 *respectively, and G_5 is* *homocyclic of infinite rank and exponent m, then* $H \oplus G_5$ *is CS-transitive if, and only if,* $\{1, 3, 7, 15, m\}$ *is a well-spaced set.*

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REFERENCES

[1] D. M. ARNOLD, *Finite rank torsion free abelian groups and rings*. Lecture Notes in Math. 931, Springer, Berlin-New York, 1982. Zbl 0493.20034 MR 665251

[2] A. L. S. CORNER, The independence of Kaplansky's notions of transitivity and full transitivity. *Quart. J. Math. Oxford Ser. (2)* **27** (1976), no. 105, 15–20. Zbl 0326.20046 MR 393280

[3] P. CRAWLEY, The cancellation of torsion abelian groups in direct sums. *J. Algebra* **2** (1965), 432–442. Zbl 0135.06004 MR 188293

[4] L. FUCHS, *Infinite abelian groups. Vol. I and Vol. II*. Pure Appl. Math. 36, Academic Press, New York-London, 1970. Zbl 0209.05503 Zbl 0257.20035 MR 255673 MR 349869

[5] L. FUCHS, *Abelian groups*. Springer Monogr. Math., Springer, Cham, 2015. Zbl 1416.20001 MR 3467030

[6] B. GOLDSMITH – K. GONG, Quotient-transitivity and cyclic subgroup-transitivity. *J. Group Theory* **24** (2021), no. 2, 305–319. Zbl 1467.20044 MR 4223848

[7] B. GOLDSMITH – K. GONG, Quotient-transitivity and cyclic submodule-transitivity for p-adic modules. *Colloq. Math.* **166** (2021), no. 2, 187–197. Zbl 1484.13021 MR 4331913

[8] P. HILL, On transitive and fully transitive primary groups. *Proc. Amer. Math. Soc.* **22** (1969), 414–417. Zbl 0192.35601 MR 269735

[9] I. KAPLANSKY, Some results on abelian groups. *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 538–540. Zbl 0047.25804 MR 49189

[10] I. KAPLANSKY, *Infinite abelian groups*. University of Michigan Press, Ann Arbor, MI, 1954. Zbl 0057.01901 MR 65561

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