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H^2 solutions for a Degasperis–Procesi-type equation

Giuseppe Maria Coclite (*) – Lorenzo di Ruvo (**)

Abstract – The Degasperis–Procesi equation can be regarded as a model for shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm equation. Here we consider a generalization of the Degasperis–Procesi equation and prove the existence and uniqueness of H^2 solutions for the Cauchy problem.

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1. Introduction

In this study, we investigate the existence and uniqueness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \delta\partial_x u - \dfrac{4\gamma}{\beta^2} u\partial_x u + \kappa\partial_x^3 u \\ \qquad -\beta^2\partial_t\partial_x^2 u + \gamma\partial_x u\partial_x^2 u + \gamma u\partial_x^3 u = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with

$$\delta, \gamma, \beta, \kappa \in \mathbb{R}, \quad \gamma, \beta \neq 0. \tag{1.2}$$

(*) *Indirizzo dell'A.*: Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, via E. Orabona 4, 70125 Bari, Italy; giuseppemaria.coclite@poliba.it

(**) *Indirizzo dell'A.*: Dipartimento di Matematica, Università di Bari, via E. Orabona 4, 70125 Bari, Italy; lorenzo.diruvo@uniba.it, lorenzo.diruvo77@gmail.com

On the initial datum, we assume

$$u_0 \in H^2(\mathbb{R}), \quad u_0 \neq 0. \tag{1.3}$$

Equation (1.1) is a particular case of

$$\partial_t u + \delta\partial_x u + au\partial_x u + \kappa\partial_x^3 u - \beta^2\partial_t\partial_x^2 u + \alpha\partial_x u\partial_x^2 u + \gamma u\partial_x^3 u = 0, \tag{1.4}$$

which is analyzed by Degasperis and Procesi [21]. In particular, they found that there are only three equations that satisfy the asymptotic integrability.

The first family is given by taking $\delta = \beta = \alpha = \gamma = 0$ in (1.4), that is,

$$\partial_t u + au\partial_x u + \kappa\partial_x^3 u = 0. \tag{1.5}$$

Equation (1.5), known as the Korteweg–de Vries equation, models weakly nonlinear unidirectional long waves, and arises in various physical contexts. For example, it models surface waves of small amplitude and long wavelength in shallow water. In this context, $u(t,x)$ represents the wave height above a flat bottom, with x being proportional to distance in the propagation direction and t being proportional to the elapsed time. Equation (1.5) is completely integrable and possesses solitary wave solutions that are solitons. The Cauchy problem for (1.5) is studied in [6, 35] and the references cited therein.

The second family is given by taking $\alpha = 2\gamma$ in (1.4), that is,

$$\partial_t u + \delta\partial_x u + au\partial_x u + \kappa\partial_x^3 u - \beta^2\partial_t\partial_x^2 u + 2\gamma\partial_x u\partial_x^2 u + \gamma u\partial_x^3 u = 0. \tag{1.6}$$

Equation (1.6), known as the Camassa–Holm equation, models the unidirectional propagation of shallow water waves over a flat bottom. In this case $u(t,x)$ represents the fluid velocity at time t in the spatial x direction, while δ is a parameter related to the critical shallow water speed [4, 24, 33]. Equation (1.6) is also a model for the propagation of axially symmetric waves in hyperelastic rods [19, 20]. It has a bi-Hamiltonian structure [25, 37] and is completely integrable [4, 15].

From a mathematical point of view, local existence and uniqueness results for (1.6) are proven in [16, 27, 39, 42, 44], while the existence of global solutions for a certain class of initial data and solutions that blow up in finite time for a large class of initial data are proven in [14, 16, 17]. Existence and uniqueness results for global weak solutions for (1.6) are proven in [2, 3, 17, 18, 28–30, 34, 46, 47]. The existence of the traveling wave solutions can be found in [38], and the existence and uniqueness of periodic solutions for the Cauchy problem in [13, 43]. Finally, in [5], using a compensated compactness argument in the L^p setting [36, 45], the convergence of the solution of (1.1) to the unique entropy solution of the Burgers equation is proven, and in [31], using a kinetic approach [32].

The third family is given by taking $a = -4\gamma/\beta^2$ and $\alpha = \gamma$ in (1.4), which is precisely equation (1.1) and it is known as the Degasperis–Procesi equation.

After an appropriate rescaling, equation (1.1) can be written in the form

$$\partial_t u + \partial_x u + 3\partial_x u^2 + \partial_x^3 u - \beta^2 \partial_t \partial_x^2 u - \frac{9\beta^2}{2}\partial_x u \partial_x^2 u - \frac{3\beta^2}{2} u \partial_x^3 u = 0. \tag{1.7}$$

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, equation (1.7), can be transformed into the famous form (see [22, 23])

$$\partial_t u - \partial_t \partial_x^2 u + 4u\partial_x u = 3\partial_x u \partial_x^2 u + u \partial_x^3 u. \tag{1.8}$$

From a mathematical point of view, the local and global existence and uniqueness of (1.8) in energy spaces is proven in [26, 48–50] and the references cited therein. In [9], the existence and uniqueness of the entropy solution is proven, while in [11] the well-posedness of the homogeneous initial boundary value problem is studied. In [10] the existence and uniqueness of the periodic solution of (1.8) is analyzed, while in [12] the convergence of some numerical schemes is proven. Possible estimates on the blow-up time T for (1.8) are given in [39–41] and the references cited therein.

We use the following definition of a solution.

Definition 1.1. We say that a function $u\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a weak solution of (1.1) if

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})), \quad \partial_t\partial_x u \in L^\infty(0,T;L^2(\mathbb{R})), \qquad T > 0,$$
$$u(0,\cdot) = u_0 \text{ a.e. in } (0,\infty)\times\mathbb{R}$$

and for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support,

$$\int_0^\infty \int_{\mathbb{R}} \Big(\partial_t u \varphi + \delta \partial_x u \varphi - \frac{4\gamma}{\beta^2} u \partial_x u \varphi - \kappa \partial_x^2 u \partial_x \varphi + \beta^2 \partial_t \partial_x u \partial_x \varphi \gamma u \partial_x^2 u \partial_x \varphi \Big)\, dt\, dx = 0.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume* (1.2) *and* (1.3)*. Given* δ, γ, κ, β, *there exists a unique weak solution* u *of* (1.1) *in the sense of Definition* 1.1*, such that for every* $T > 0$,

$$\begin{aligned} &u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}),\\ &\partial_t \partial_x u \in L^\infty(0,T;L^2(\mathbb{R})). \end{aligned} \tag{1.9}$$

Moreover, if u_1 and u_2 are two weak solutions of (1.1) in correspondence with the initial data $u_{1,0}$ and $u_{2,0}$, we have that, for every $T > 0$,

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|^2_{H^1(\mathbb{R})} \le \frac{\tau_2^2 e^{C(T)t}}{\tau_3^2} \|u_{1,0} - u_{2,0}\|^2_{H^1(\mathbb{R})}, \tag{1.10}$$

for some suitable $C(T) > 0$, and every $t \in [0,T]$, where

$$\tau_1^2 = \min\{1, \beta^2\}, \quad \tau_2^2 = \max\{1, \beta^2\}. \tag{1.11}$$

We note that Theorem 1.1 affirms that equation (1.1) admits a unique global-in-time weak solution. Unfortunately, some of the estimates blow up as $T \to \infty$, and indeed we prove them in every stripe $(0,T) \times \mathbb{R}$, $T > 0$. Finally, we cannot use the arguments of [9] because the L^2 norm is not a conserved quantity of (1.1); we can only prove that it stays bounded in any finite time interval $(0,T)$ (see (2.3) below).

The proof of Theorem 1.1 is given in the next section.

2. Proof of the Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1); see e.g. [1].

Fix a small number $0 < \varepsilon < 1$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following problem [8]:

$$\begin{cases} \partial_t u_\varepsilon + \delta\partial_x u_\varepsilon - \dfrac{4\gamma}{\beta^2} u_\varepsilon \partial_x u_\varepsilon + \kappa\partial_x^3 u_\varepsilon - \beta^2 \partial_t \partial_x^2 u_\varepsilon \\ \qquad + \gamma \partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \gamma u_\varepsilon \partial_x^3 u_\varepsilon = -\varepsilon \partial_x^4 u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0, such that

$$\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \le \|u_0\|_{H^2(\mathbb{R})}, \quad \sqrt{\varepsilon}\|\partial_x^3 u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \le C_0, \quad u_0 \ne 0, \tag{2.2}$$

where C_0 is a positive constant independent of ε.

Let us prove some a priori estimates on u_ε, denoting by C_0 the constants which depend only on the initial data, and by $C(T)$ the constants which depend also on T.

Inspired by the arguments in [7] we prove the following lemma.

Lemma 2.1 (H^2 estimate). *Given δ, γ, κ, β, the following estimates hold:*

$$\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}, \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}, \|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \tag{2.3}$$

$$\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{2.4}$$

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{2.5}$$

$$\varepsilon \int_0^t \|\partial_x^3 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\, ds \le C(T), \tag{2.6}$$

for every $0 \le t \le T$.

Proof. Consider F a positive constant. Multiplying (2.1) by $2u_\varepsilon - 2F\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+F)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + F\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\big)\\
&\quad = 2\int_{\mathbb{R}} u\partial_t u_\varepsilon\, dx - 2\beta^2\int_{\mathbb{R}} u_\varepsilon \partial_t\partial_x^2 u_\varepsilon\, dx - 2F\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon\, dx\\
&\qquad + 2F\beta^2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon\, dx\\
&\quad = -2\delta\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\, dx + 2F\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx + \frac{8\gamma}{\beta^2}\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\, dx\\
&\qquad - \frac{4F\gamma}{\beta^2}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx - 2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\, dx + 2F\kappa\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon\, dx\\
&\qquad - 2\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx + 2F\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\, dx - 2\gamma\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon\, dx\\
&\qquad + 2F\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon\, dx - 2\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^4 u_\varepsilon\, dx + 2F\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\, dx\\
&\quad = \frac{2F\gamma}{\beta^2}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx + 2\kappa\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx + \gamma\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx\\
&\qquad + F\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\, dx + 4\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx + 2\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^3 u_\varepsilon\, dx\\
&\qquad - 2F\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad = \frac{2F\gamma}{\beta^2}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx - \gamma\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx + F\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\, dx\\
&\qquad - 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2F\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+F)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + F\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\big)\\
&\qquad + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2F\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad = \frac{2F\gamma}{\beta^2}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx - \gamma\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx + F\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\, dx.
\end{aligned} \tag{2.7}$$

Introducing the function

$$
\begin{aligned}
X_\varepsilon(t) := {}& \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2 + F)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&+ F\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{2.8}
$$

equation (2.7) reads

$$
\begin{aligned}
&\frac{dX_\varepsilon(t)}{dt} + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2F\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad = \frac{2F\gamma}{\beta^2}\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3\,dx - \gamma\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3\,dx + F\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\,dx.
\end{aligned} \tag{2.9}
$$

Observe that, thanks to (2.8),

$$
\begin{aligned}
\frac{2F\gamma}{\beta^2}\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3\,dx &\le \frac{2F|\gamma|}{\beta^2}\int_{\mathbb{R}} |\partial_x u_\varepsilon|^3\,dx \\
&\le \frac{2F|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{2(F+\beta^2)|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{2|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad\qquad + (F+\beta^2)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) \\
&\quad + \frac{2|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}F\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&= \frac{2|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}X_\varepsilon(t),
\end{aligned}
$$

$$
\begin{aligned}
-\gamma\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3\,dx &\le |\gamma|\int_{\mathbb{R}} |\partial_x u_\varepsilon|^3\,dx \\
&\le \frac{|\gamma|}{\beta^2}\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{|\gamma|}{\beta^2}(F+\beta^2)\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad\qquad + (F+\beta^2)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) \\
&\quad + \frac{|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}F\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&= \frac{|\gamma|}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}X_\varepsilon(t),
\end{aligned}
$$

$$
\begin{aligned}
F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2\, dx &\le F|\gamma| \int_{\mathbb{R}} |\partial_x u_\varepsilon| (\partial_x^2 u_\varepsilon)^2\, dx \\
&\le \frac{|\gamma|}{\beta^2} F\beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\le \frac{|\gamma|}{\beta^2} (\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + F\beta^2 \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) \\
&\quad + \frac{|\gamma|}{\beta^2} (F+\beta^2) \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&= \frac{|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t).
\end{aligned}
$$

Consequently, by (2.9), we have

$$
\frac{dX_\varepsilon(t)}{dt} \le \frac{4|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t). \tag{2.10}
$$

Observe that, by (2.8),

$$
(\beta^2 + F)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le X_\varepsilon(t), \quad F\beta^2 \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le X_\varepsilon(t). \tag{2.11}
$$

Therefore, thanks to (2.11) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x u_\varepsilon(t,x))^2 &= 2\int_{-\infty}^{x} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon\, dy \le 2\int_{\mathbb{R}} |\partial_x u_\varepsilon|\, |\partial_x^2 u_\varepsilon|\, dx \\
&\le 2\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \\
&= \frac{2}{\sqrt{F\beta^2}} \sqrt{\beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}} \sqrt{F\beta^2 \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}} \\
&\le \frac{2}{\sqrt{F\beta^2}} \sqrt{(F+\beta^2) \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}} \sqrt{F\beta^2 \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}} \\
&\le \frac{2}{\sqrt{F\beta^2}} X_\varepsilon(t).
\end{aligned}
$$

Hence,

$$
\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \le \frac{2}{\sqrt{F\beta^2}} X_\varepsilon(t). \tag{2.12}
$$

It follows from (2.10) and (2.12) that

$$
\frac{dX_\varepsilon(t)}{dt} \le \frac{4\sqrt{2}|\gamma|}{\sqrt[4]{F}|\beta|^3} X_\varepsilon^{\frac{3}{2}}(t),
$$

which gives

$$
X_\varepsilon^{-\frac{3}{2}}(t) \frac{dX_\varepsilon(t)}{dt} \le \frac{4\sqrt{2}|\gamma|}{\sqrt[4]{F}|\beta|^3}.
$$

Integrating on $(0, t)$, we obtain

$$\frac{1}{\sqrt{X_\varepsilon(t)}} \geq \frac{1}{\sqrt{X_\varepsilon(0)}} - \frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3}. \tag{2.13}$$

Using (2.8) in (2.13), we have

$$\begin{aligned}&\frac{1}{\sqrt{A_\varepsilon(t) + (\beta^2 + F)B_\varepsilon(t) + F\beta^2 E_\varepsilon(t)}} + \frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3}\\ &\geq \frac{1}{\sqrt{A_\varepsilon(0) + (\beta^2 + F)B_\varepsilon(0) + F\beta^2 E_\varepsilon(0)}}\end{aligned} \tag{2.14}$$

for every $0 \leq t \leq T$, where F is an arbitrary positive constant, and

$$\begin{aligned}&A_\varepsilon(t) := \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}, && B_\varepsilon(t) := \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ &E_\varepsilon(t) := \|\partial_x^2 u(t,\cdot)\|^2_{L^2(\mathbb{R})}, && A_\varepsilon(0) := \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})},\\ &B_\varepsilon(0) := \|\partial_x u_{\varepsilon,0}\|_{L^2(\mathbb{R})}, && E_\varepsilon(0) := \|\partial_x^2 u_{\varepsilon,0}\|_{L^2(\mathbb{R})}.\end{aligned} \tag{2.15}$$

We prove (2.3). Observe that, by (2.2),

$$\begin{aligned}(F + \beta^2)\|\partial_x u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} &\leq (F + \beta^2)\|\partial_x u_0\|^2_{L^2(\mathbb{R})},\\ F\beta^2\|\partial_x^2 u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} &\leq F\beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{2.16}$$

Consequently, (2.2), (2.15), (2.9), and (2.16) give

$$\begin{aligned}X_\varepsilon(0) &= A_\varepsilon(0) + (F + \beta^2)B_\varepsilon(0) + F\beta^2 E_\varepsilon(0)\\ &\leq Y_0 + (F + \beta^2)Y_1 + F\beta^2 Y_2,\end{aligned}$$

where

$$Y_0 := \|u_0\|^2_{L^2(\mathbb{R})}, \quad Y_1 := \|\partial_x u_0\|^2_{L^2(\mathbb{R})}, \quad Y_2 := \|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})}.$$

Therefore, we have

$$\begin{aligned}&\frac{1}{\sqrt{A_\varepsilon(0) + (F + \beta^2)B_\varepsilon(0) + F\beta^2 E_\varepsilon(0)}}\\ &\geq \frac{1}{\sqrt{Y_0 + (F + \beta^2)Y_1 + F\beta^2 Y_2}}.\end{aligned} \tag{2.17}$$

Moreover,

$$\frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3} \leq \frac{8\sqrt{2}|\gamma|T}{\sqrt[4]{F}|\beta|^3} \quad \Rightarrow \quad -\frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3} \geq -\frac{8\sqrt{2}|\gamma|T}{\sqrt[4]{F}|\beta|^3}. \tag{2.18}$$

It follows from (2.14), (2.15), (2.8), (2.13), (2.17), and (2.18) that

$$\begin{aligned}\frac{1}{\sqrt{X_\varepsilon(t)}} &\geq \frac{1}{\sqrt{Y_0+(F+\beta^2)Y_1+F\beta^2Y_2}} - \frac{8\sqrt{2}|\gamma|T}{\sqrt[4]{F}|\beta|^3}\\ &= \frac{\sqrt[4]{F}|\beta|^3 - 8\sqrt{2}|\gamma|T\sqrt{Y_0+(F+\beta^2)Y_1+F\beta^2Y_2}}{\sqrt[4]{F}|\beta|^3\sqrt{Y_0+(F+\beta^2)Y_1+F\beta^2Y_2}}.\end{aligned} \tag{2.19}$$

We search for F such that

$$\sqrt[4]{F}|\beta|^3 - 8\sqrt{2}|\gamma|T\sqrt{Y_0+(F+\beta^2)Y_1+F\beta^2Y_2} > 0. \tag{2.20}$$

Taking $|\beta|$ as

$$|\beta| = F^n, \quad n > \frac{1}{8}, \tag{2.21}$$

(2.20) reads

$$F^{\frac{12n+1}{4}} - 8\sqrt{2}|\gamma|T\sqrt{Y_0+(F+F^{2n})Y_1+F^{2n+1}Y_2} > 0,$$

which gives

$$F^{\frac{12n+1}{2}} > 128\gamma^2T^2(Y_0+(F+F^{2n})Y_1+F^{2n+1}Y_2). \tag{2.22}$$

Consequently,

$$\begin{aligned}&F^{\frac{12n+1}{2}} - 128\gamma^2T^2Y_2F^{2n+1} - 128\gamma^2T^2Y_1F^{2n}\\ &\quad - 128\gamma^2T^2Y_1F - 128\gamma^2T^2Y_0 > 0.\end{aligned} \tag{2.23}$$

Taking n as in (2.21), we have

$$\begin{aligned}\lim_{F\to\infty} &F^{\frac{12n+1}{2}} - 128\gamma^2T^2Y_2F^{2n+1} - 128\gamma^2T^2Y_1F^{2n}\\ &- 128\gamma^2T^2Y_1F - 128\gamma^2T^2Y_0 = \infty.\end{aligned} \tag{2.24}$$

In fact, we have (2.24) if only if

$$\frac{12n+1}{2} > 2n+1,$$

that is

$$n > \frac{1}{8}.$$

Therefore, thanks to (2.24), we have that (2.20) (which is equivalent to (2.22)) holds. Taking F very large and up to rescaling, we can have $|\beta| = F^n$, with n defined in (2.21).

Consequently, by (2.19), (2.20), (2.22), (2.23), and (2.24),

$$\frac{1}{\sqrt{X_\varepsilon(t)}} \geq C(T). \tag{2.25}$$

Thanks to (2.8) and (2.25), we get

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2 + F)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + F\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T),$$

which gives (2.3).

We prove (2.4). By (2.8) and (2.12) with $F = 1$, we have

$$\begin{aligned}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq \frac{2}{\beta^2}(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}).\end{aligned} \tag{2.26}$$

Inequality (2.4) follows from (2.3) and (2.26).

We prove (2.5). Thanks to (2.3) and the Hölder inequality,

$$\begin{aligned}u_\varepsilon^2(t,x) &= 2\int_{-\infty}^x u_\varepsilon \partial_x u_\varepsilon \, dy \leq 2\int_{\mathbb{R}} |u_\varepsilon|\,|\partial_x u_\varepsilon|\, dx\\ &\leq 2\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(T).\end{aligned}$$

Hence,

$$\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \leq C(T),$$

which gives (2.5).

Finally, we prove (2.6). Observe that (2.7) with $F = 1$ is written

$$\begin{aligned}&\frac{d}{dt}\big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\big)\\ &\quad + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &= \frac{\gamma(2-\beta^2)}{\beta^2}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3\, dx + \gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\, dx.\end{aligned} \tag{2.27}$$

Thanks to (2.3) and (2.4),

$$\begin{aligned}&\frac{|\gamma(2-\beta^2)|}{\beta^2}\int_{\mathbb{R}}|\partial_x u_\varepsilon|^3\, dx\\ &\qquad\leq \frac{|\gamma(2-\beta^2)|}{\beta^2}\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T),\\ &|\gamma|\int_{\mathbb{R}}|\partial_x u_\varepsilon|(\partial_x^2 u_\varepsilon)^2\, dx\\ &\qquad\leq |\gamma|\,\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T).\end{aligned} \tag{2.28}$$

It follows from (2.27) and (2.28) that

$$\begin{aligned}\frac{d}{dt}\big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\big)\\ + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T).\end{aligned}$$

Integrating on $(0,t)$, by (2.2) we get

$$\begin{aligned}&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\qquad + 2\varepsilon\int_0^t \|\partial_x^2 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds + 2\varepsilon\int_0^t \|\partial_x^3 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds\\ &\quad \le C_0 + C(T)t \le C(T),\end{aligned}$$

which gives (2.6). ∎

Lemma 2.2. *Given δ, γ, κ, β, we have*

$$\begin{aligned}\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ + \varepsilon^2\int_0^t \|\partial_x^4 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le C(T),\end{aligned} \tag{2.29}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (2.1) by $2\varepsilon\partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}&\frac{d}{dt}(\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})})\\ &\quad = 2\varepsilon\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon\,dx - 2\beta^2\varepsilon\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t\partial_x^2 u_\varepsilon\,dx\\ &\quad = -2\delta\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon\,dx + \frac{8\gamma\varepsilon}{\beta^2}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^4 u_\varepsilon\,dx\\ &\qquad - 2\kappa\varepsilon\int_{\mathbb{R}} \partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon\,dx - 2\gamma\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\,dx\\ &\qquad - 2\gamma\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon\,dx - 2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad = 2\delta\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon\,dx + \frac{8\gamma\varepsilon}{\beta^2}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^4 u_\varepsilon\,dx\\ &\qquad + 3\gamma\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\,dx - 2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad = \frac{8\gamma\varepsilon}{\beta^2}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^4 u_\varepsilon\,dx + 3\gamma\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\,dx\\ &\qquad - 2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}(\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) + 2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$= \frac{8\gamma\varepsilon}{\beta^2}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^4 u_\varepsilon\, dx + 3\gamma\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\, dx. \tag{2.30}$$

Due to (2.3), (2.4), (2.5), and the Young inequality,

$$\begin{aligned}
\frac{8|\gamma|\varepsilon}{\beta^2}\int_{\mathbb{R}}|u_\varepsilon|\,|\partial_x u_\varepsilon|\,|\partial_x^4 u_\varepsilon|\,dx &\leq \frac{8|\gamma|\varepsilon}{\beta^2}\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_\varepsilon|\,|\partial_x^4 u_\varepsilon|\,dx\\
&\leq 2C(T)\varepsilon\int_{\mathbb{R}}|\partial_x u_\varepsilon|\,|\partial_x^4 u_\varepsilon|\,dx\\
&= 2\int_{\mathbb{R}}|C(T)\partial_x u_\varepsilon|\,|\varepsilon\partial_x^4 u_\varepsilon|\,dx\\
&\leq C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
3|\gamma|\varepsilon\int_{\mathbb{R}}|\partial_x u_\varepsilon|(\partial_x^3 u_\varepsilon)^2\,dx &\leq 3|\gamma|\varepsilon\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\leq C(T)\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (2.30) that

$$\begin{aligned}
&\frac{d}{dt}(\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad\leq C(T)\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T).
\end{aligned}$$

Integrating on $(0,t)$, by (2.2) and (2.6), we get

$$\begin{aligned}
&\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^2\int_0^t\|\partial_x^4 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds\\
&\qquad\leq C_0 + C(T)\varepsilon\int_0^t\|\partial_x^3 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds + C(T)t \leq C(T),
\end{aligned}$$

which gives (2.29). ∎

LEMMA 2.3. *Given δ, γ, κ, β, the following estimates hold:*

$$2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T), \tag{2.31}$$

$$\|\partial_t u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T), \tag{2.32}$$

for every $0 \leq t \leq T$.

PROOF. Let $0 \le t \le T$. Multiplying (2.1) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
&2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad = 2\int_{\mathbb{R}} (\partial_t u_\varepsilon)^2\,dx - 2\beta^2\int_{\mathbb{R}} \partial_t u_\varepsilon \partial_t \partial_x^2 u_\varepsilon\,dx\\
&\quad = -2\delta\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_t u_\varepsilon\,dx + \frac{8\gamma}{\beta^2}\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon\,dx\\
&\qquad - 2\kappa\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t u_\varepsilon\,dx - 2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_t u_\varepsilon\,dx\\
&\qquad - 2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_t u_\varepsilon\,dx - 2\varepsilon\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon\,dx\\
&\quad = -2\delta\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_t u_\varepsilon\,dx + \frac{8\gamma}{\beta^2}\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon\,dx\\
&\qquad + 2\kappa\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon\,dx + 2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon\,dx\\
&\qquad + 2\varepsilon\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon\,dx.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad = -2\delta\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_t u_\varepsilon\,dx + \frac{8\gamma}{\beta^2}\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon\,dx\\
&\qquad + 2\kappa\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon\,dx + 2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon\,dx\\
&\qquad + 2\varepsilon\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon\,dx.
\end{aligned}
\tag{2.33}
$$

Since $0 < \varepsilon < 1$, thanks to (2.3), (2.5), (2.29), and the Young inequality,

$$
\begin{aligned}
2|\delta|\int_{\mathbb{R}} |\partial_x u_\varepsilon|\,|\partial_t u_\varepsilon|\,dx &= \int_{\mathbb{R}} |2\delta\partial_x u_\varepsilon|\,|\partial_t u_\varepsilon|\,dx\\
&\le 2\delta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
\frac{8|\gamma|}{\beta^2}\int_{\mathbb{R}} |u_\varepsilon|\,|\partial_x u_\varepsilon|\,|\partial_t u_\varepsilon|\,dx &\le \frac{8|\gamma|}{\beta^2}\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x u_\varepsilon|\,|\partial_t u_\varepsilon|\,dx\\
&\le C(T)\int_{\mathbb{R}} |\partial_x u_\varepsilon|\,|\partial_t u_\varepsilon|\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\kappa|\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon|\,|\partial_t\partial_x u_\varepsilon|\,dx &= \int_{\mathbb{R}}\left|\frac{2\kappa\partial_x^2 u_\varepsilon}{\beta}\right|\,|\beta\partial_t\partial_x u_\varepsilon|\,dx\\
&\le \frac{2\kappa^2}{\beta^2}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\gamma|\int_{\mathbb{R}}|u_\varepsilon|\,|\partial_x^2 u_\varepsilon|\,|\partial_t\partial_x u_\varepsilon|\,dx &\le 2|\gamma|\,\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon|\,|\partial_t\partial_x u_\varepsilon|\,dx\\
&\le C(T)\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon|\,|\partial_t\partial_x u_\varepsilon|\,dx\\
&\le \int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u_\varepsilon}{\beta}\right|\,|\beta\partial_t\partial_x u_\varepsilon|\,dx\\
&\le C(T)\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2\varepsilon\int_{\mathbb{R}}|\partial_x^3 u_\varepsilon|\,|\partial_t\partial_x u_\varepsilon|\,dx &= \int_{\mathbb{R}}\left|\frac{2\varepsilon\partial_x^3 u_\varepsilon}{\beta}\right|\,|\beta\partial_t\partial_x u_\varepsilon|\,dx\\
&\le \frac{2\varepsilon^2}{\beta^2}\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \frac{2\varepsilon}{\beta^2}\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.33) that

$$
\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T),
$$

which gives (2.31).

Finally, we prove (2.32). Due to (2.31) and the Hölder inequality,

$$
\begin{aligned}
(\partial_t u_\varepsilon(t,x))^2 &= 2\int_{-\infty}^{x}\partial_t u_\varepsilon\partial_t\partial_x u_\varepsilon\,dy \le 2\int_{\mathbb{R}}|\partial_t u_\varepsilon|\,|\partial_t\partial_x u_\varepsilon|\,dx\\
&\le 2\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T).
\end{aligned}
$$

Hence,

$$\|\partial_t u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C(T),$$

which gives (2.32). ∎

Using the Sobolev immersion theorem, we prove the following result:

LEMMA 2.4. *Assume* (1.2) *and* (1.3). *There exist a subsequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a limit function* u *which satisfies* (1.9) *such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}), \quad 1 \le p < \infty. \tag{2.34}$$

Moreover, u *is a solution of* (1.1).

PROOF. Thanks to Lemmas 2.1 and 2.3,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}), \quad T \ge 0,$$

which gives (2.34).

Observe that, thanks to Lemma 2.1,

$$u \in L^\infty(0,T;H^2(\mathbb{R})), \quad T \ge 0,$$

while, by Lemma 2.3,

$$u \in W^{1,\infty}((0,T)\times\mathbb{R}), \quad T \ge 0.$$

Moreover, by Lemma 2.3, we have

$$\partial_t\partial_x u \in L^2((0,T)\times\mathbb{R}), \quad T \ge 0.$$

Therefore, (1.9) holds and u is a solution of (1.1). ∎

We are finally ready for the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Lemma 2.4 gives the existence of a solution u of (1.1) such that (1.9) holds.

We prove (1.10). Let u_1, u_2 be two solutions of (1.1), which satisfy (1.9), that is,

$$\begin{cases} \partial_t u_i + \delta\partial_x u_i - \dfrac{2\gamma}{\beta^2}\partial_x u_i^2 + \kappa\partial_x^3 u_i \\ \qquad -\beta^2\partial_t\partial_x^2 u_i + \gamma\partial_x(u_i\partial_x^2 u_i) = 0, & t > 0,\ x \in \mathbb{R}, \\ u_i(0,x) = u_{i,0}(x), & x \in \mathbb{R}, \end{cases}$$

for $i = 1, 2$. Then the function

$$\omega = u_1 - u_2 \tag{2.35}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega + \delta \partial_x \omega - \dfrac{2\gamma}{\beta^2} \partial_x (u_i^2 - u_2^2) + \kappa \partial_x^3 \omega \\ \qquad -\beta^2 \partial_t \partial_x^2 \omega + \gamma \partial_x (u_i \partial_x^2 u_i - u_2 \partial_x^2 u_2) = 0, & t > 0,\ x \in \mathbb{R}, \\ \omega_0(x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{2.36}$$

Observe that, thanks to (2.35),

$$\begin{aligned} \partial_x (u_1^2 - u_2^2) &= \partial_x ((u_1 + u_2)(u_1 - u_2)) = \partial_x ((u_1 + u_2)\omega), \\ \partial_x (u_1 \partial_x^2 u_1 - u_2 \partial_x^2 u_2) &= \partial_x (u_1 \partial_x^2 u_1 - u_1 \partial_x^2 u_2 + u_1 \partial_x^2 u_2 - u_2 \partial_x^2 u_2) \\ &= \partial_x (u_1 \partial_x^2 \omega - \partial_x^2 u_2 \omega) + \partial_x (\partial_x^2 u_2 \omega). \end{aligned} \tag{2.37}$$

Consequently, by (2.37), equation (2.36) reads

$$\begin{aligned} &\partial_t \omega + \delta \partial_x \omega - \beta^2 \partial_t \partial_x^2 \omega - \frac{2\gamma}{\beta^2} \partial_x ((u_1 + u_2)\omega) \\ &\qquad + \kappa \partial_x^3 \omega + \gamma \partial_x (u_1 \partial_x^2 \omega - \partial_x^2 u_2 \omega) + \gamma \partial_x (\partial_x^2 u_2 \omega) = 0. \end{aligned} \tag{2.38}$$

Assume $T \geq 0$ is given. Since $u_1, u_2 \in L^\infty(0, T; H^2(\mathbb{R}))$,

$$\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T). \tag{2.39}$$

Moreover, by (2.39), we have

$$|\partial_x u_1 + \partial_x u_2| \leq |\partial_x u_1| + |\partial_x u_2| \leq C(T). \tag{2.40}$$

Multiplying (2.38) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} &\frac{d}{dt} (\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}) \\ &\quad = 2 \int_{\mathbb{R}} \omega \partial_t \omega - 2\beta^2 \int_{\mathbb{R}} \omega \partial_t \partial_x^2 \omega \, dx \\ &\quad = \frac{4\gamma}{\beta^2} \int_{\mathbb{R}} \omega \partial_x ((u_1 + u_2)\omega)\, dx - 2\delta \int_{\mathbb{R}} \omega \partial_x \omega \, dx - 2\kappa \int_{\mathbb{R}} \omega \partial_x^3 \omega \, dx \\ &\qquad - 2\gamma \int_{\mathbb{R}} \omega \partial_x (u_1 \partial_x^2 \omega - \partial_x^2 u_2 \omega) - 2\gamma \int_{\mathbb{R}} \omega \partial_x (\partial_x^2 u_2 \omega)\, dx \\ &\quad = -\frac{4\gamma}{\beta^2} \int_{\mathbb{R}} (u_1 + u_2) \omega \partial_x \omega \, dx + 2\kappa \int_{\mathbb{R}} \partial_x \omega \partial_x^2 \omega \, dx + 2\gamma \int_{\mathbb{R}} u_1 \partial_x \omega \partial_x^2 \omega \, dx \\ &\quad = \frac{2\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_1 + \partial_x u_2) \omega^2 \, dx - \gamma \int_{\mathbb{R}} \partial_x u_1 (\partial_x \omega)^2 \, dx. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}) \\
&\qquad = \frac{2\gamma}{\beta^2}\int_{\mathbb{R}}(\partial_x u_1 + \partial_x u_2)\omega^2\,dx - \gamma\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2\,dx.
\end{aligned}
\tag{2.41}
$$

Thanks to (2.39) and (2.40),

$$
\begin{aligned}
\frac{2|\gamma|}{\beta^2}\int_{\mathbb{R}}|\partial_x u_1 + \partial_x u_2|\omega^2\,dx &\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
|\gamma|\int_{\mathbb{R}}|\partial_x u_1|(\partial_x\omega)^2\,dx &\le C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.41) that

$$
\begin{aligned}
&\frac{d}{dt}(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}) \\
&\qquad \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad \le C(T)(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}).
\end{aligned}
$$

The Grönwall lemma and (2.36) give

$$
\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le e^{C(T)t}(\|\omega_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega_0\|^2_{L^2(\mathbb{R})}).
$$

By (1.11), we have

$$
\tau_1^2\|\omega\|^2_{H^1(\mathbb{R})} \le \tau_2^2 e^{C(T)t}\|\omega_0\|^2_{H^1(\mathbb{R})}.
\tag{2.42}
$$

Therefore, (1.10) follows from (2.36) and (2.42). ■

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