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A note on Huppert's theorem and Chen's theorem

Jiangtao Shi (*) – Mengjiao Shan (**) – Fanjie Xu (***)

Abstract – Let G be a finite group. We prove that every maximal subgroup of G has prime index if and only if every maximal subgroup of G that contains the normalizer of some Sylow subgroup has prime index, which implies that the hypothesis in Huppert's theorem and the hypothesis in Chen's theorem are actually equivalent. Moreover, we prove that the hypothesis in a theorem of Shao and Beltrán and the hypothesis in a theorem of Li et al. are also equivalent.

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1. Introduction

In this paper all groups are assumed to be finite. It is known that Huppert's theorem [2, Chapter VI, Theorem 9.2] shows that if every maximal subgroup of a group G has prime index then G is supersolvable. As a generalization of Huppert's theorem, Chen [1, Theorem 7.25] obtained the following theorem:

Theorem 1.1 ([1, Theorem 7.25]). *Let G be a group. If every maximal subgroup of G that contains the normalizer of some Sylow subgroup has prime index, then G is supersolvable.*

(*) *Indirizzo dell'A.*: School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China; shijt2005@163.com, shijt2005@pku.org.cn

(**) *Indirizzo dell'A.*: School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China; 2423436767@qq.com, shanmj2023@s.ytu.edu.cn

(***) *Indirizzo dell'A.*: School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China; 19818728116@163.com, xufj2023@s.ytu.edu.cn

In this paper, we will indicate that the hypothesis that every maximal subgroup has prime index in Huppert's theorem and the hypothesis that every maximal subgroup that contains the normalizer of some Sylow subgroup has prime index in Chen's theorem are actually equivalent. Our main result is as follows, the proof of which is given in Section 3.

Theorem 1.2. *Let G be a group. Then every maximal subgroup of G has prime index if and only if every maximal subgroup of G that contains the normalizer of some Sylow subgroup has prime index.*

As another generalization of Huppert's theorem, Shao and Beltrán [4, Theorem B] had the following result.

Theorem 1.3 ([4, Theorem B]). *Let G and A be groups of coprime orders and assume that A acts on G by automorphisms. If the index of every maximal A-invariant subgroup of G is prime, then G is supersolvable.*

And as a generalization of Chen's theorem, Li et al. [3, Theorem 1.2] gave the following result.

Theorem 1.4 ([3, Theorem 1.2]). *Suppose that A acts on G via automorphisms and that $(|A|, |G|) = 1$. If every maximal A-invariant subgroup of G that contains the normalizer of some Sylow subgroup has prime index, then G is supersolvable.*

In Section 4 of this paper, we will prove that the hypothesis in [4, Theorem B] and the hypothesis in [3, Theorem 1.2] are also equivalent.

Theorem 1.5. *Let A and G be groups of coprime orders and assume that A acts on G by automorphisms. Then every maximal A-invariant subgroup of G has prime index if and only if every maximal A-invariant subgroup of G that contains the normalizer of some Sylow subgroup has prime index.*

2. A simple lemma

Lemma 2.1. *Let p be the largest prime divisor of the order of a group G and $P \in \mathrm{Syl}_p(G)$. Then either $P \trianglelefteq G$ or any maximal subgroup of G that contains $N_G(P)$ has composite index.*

Proof. Suppose that P is not normal in G. Let M be maximal in G satisfying $N_G(P) \leq M$ and $|G : M| = m$. In the following we will show that m is a composite number.

Otherwise, if m is a prime, then $m < p$ by the hypothesis. Since $|G : M| = m$, one has that G/M_G is isomorphic to a subgroup of the symmetric group S_m, where $M_G = \bigcap_{g \in G} M^g$ is the largest normal subgroup of G that is contained in M. It is clear that $p \nmid |S_m|$, which implies that $p \nmid |G/M_G|$. It follows that $P \in \mathrm{Syl}_p(M_G)$. By the Frattini argument, one has $G = M_G N_G(P) \leq M_G M = M$, a contradiction. Therefore, m is a composite number. ■

3. Proof of Theorem 1.2

We only need to prove the sufficiency part.

Let G be a counterexample of minimal order. It is easy to see that the hypothesis of the theorem also holds for any quotient group of G.

Suppose that p is the largest prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$. By the hypothesis and Lemma 2.1, one has $P \trianglelefteq G$. Let P_0 be a minimal normal subgroup of G satisfying $P_0 \leq P$. Then P_0 is an elementary abelian group. By the minimality of G, every maximal subgroup of the quotient group G/P_0 has prime index.

Suppose that M is any maximal subgroup of G that has composite index. Then $P_0 \not\leq M$. It follows that $G = P_0 M$. In particular, one has $G = P_0 \rtimes M$ by the minimality of P_0.

For any maximal subgroup M_1 of M, it is clear that $P_0 \rtimes M_1$ is maximal in G and $P_0 \rtimes M_1 > P_0$. By the above argument, one has $|M : M_1| = |G : P_0 \rtimes M_1| = |G/P_0 : (P_0 \rtimes M_1)/P_0|$ is a prime.

Let q be the largest prime divisor of $|M|$ and $Q \in \mathrm{Syl}_q(M)$. If $Q \ntrianglelefteq M$, then $N_M(Q) < M$. Suppose that M_2 is a maximal subgroup of M such that $N_M(Q) \leq M_2$. One has that $|M : M_2|$ is a composite number by Lemma 2.1. This contradicts that every maximal subgroup of M has prime index. Therefore, $Q \trianglelefteq M$.

(1) Suppose $q = p$. Then $P_0 \rtimes Q \in \mathrm{Syl}_p(G)$. Since $M \leq N_G(Q)$ and $N_{P_0 \rtimes Q}(Q) > Q$, it follows that $N_G(Q) = G$ by the maximality of M. Then $Q \trianglelefteq G$. Let Q_0 be a minimal normal subgroup of G satisfying $Q_0 \leq Q$. One has that every maximal subgroup of G/Q_0 has prime index by the minimality of G. Note that $Q_0 \leq M$. Then $|G : M| = |G/Q_0 : M/Q_0|$ is a prime, which contradicts the choice of M.

(2) Suppose $q \neq p$. Then $Q \in \mathrm{Syl}_q(G)$. Arguing as in (1), one has $Q \ntrianglelefteq G$. It follows that $M = N_G(Q)$. By the hypothesis, $|G : M|$ is a prime, which is also a contradiction.

Hence the counterexample of minimal order does not exist and so every maximal subgroup of G has prime index. ■

4. Proof of Theorem 1.5

We also only need to prove the sufficiency part.

Let G be a counterexample of minimal order. Assume that N is any A-invariant normal subgroup of G. It is clear that the hypothesis of the theorem also holds for the quotient group G/N.

Suppose that p is the largest prime divisor of $|G|$. Assume all Sylow p-subgroups of G are not normal. Since A acts on G coprimely via automorphisms, we can take P as an A-invariant Sylow p-subgroup of G. Then $N_G(P)$ is a proper A-invariant subgroup of G. Let K be a maximal A-invariant subgroup of G such that $N_G(P) \leq K$. Then $|G:K|$ is a prime by the hypothesis. It follows that K is a maximal subgroup of G that contains $N_G(P)$. However, $|G:K|$ should be a composite number by Lemma 2.1, a contradiction. Therefore, $P \trianglelefteq G$.

We claim $\Phi(G) = 1$. If $\Phi(G) \neq 1$, then since $\Phi(G)$ is an A-invariant normal subgroup of G, one has that every maximal A-invariant subgroup of $G/\Phi(G)$ has prime index by the minimality of G. It follows that every maximal A-invariant subgroup of G has prime index since every maximal A-invariant subgroup of G contains $\Phi(G)$, a contradiction. Therefore, $\Phi(G) = 1$.

Since $P \trianglelefteq G$, one has $\Phi(P) \leq \Phi(G)$. It follows that $\Phi(P) = 1$ and then P is an elementary abelian group. Let P_0 be a minimal A-invariant normal subgroup of G satisfying $P_0 \leq P$. Then P_0 is an elementary abelian group. By the minimality of G, every maximal A-invariant subgroup of the quotient group G/P_0 has prime index.

Suppose that M is any maximal A-invariant subgroup of G that has composite index. Then $P_0 \nleq M$. It follows that $G = P_0M$. Moreover, one has $G = P_0 \rtimes M$ by the minimality of P_0.

For any maximal A-invariant subgroup M_1 of M, it is clear that $P_0 \rtimes M_1$ is a maximal A-invariant subgroup of G and $P_0 \rtimes M_1 > P_0$. Then $|M : M_1| = |G : P_0 \rtimes M_1| = |G/P_0 : (P_0 \rtimes M_1)/P_0|$ is a prime.

Let q be the largest prime divisor of $|M|$.

(1) Suppose $q = p$. Then $P \cap M$ is an A-invariant normal subgroup of G since $G = PM$ and $P \cap M \trianglelefteq PM$. By the minimality of G, every maximal A-invariant subgroup of $G/(P \cap M)$ has prime index, which implies that M has prime index, a contradiction.

(2) Suppose $q \neq p$. Let Q be an A-invariant Sylow q-subgroup of M. Then Q is also an A-invariant Sylow q-subgroup of G. If $Q \trianglelefteq G$, arguing as in (1), we can get a contradiction. Thus $Q \ntrianglelefteq G$. It follows that $N_G(Q) < G$ and then there exists a maximal A-invariant subgroup L of G such that $N_G(Q) \leq L$.

(i) For the case when $P_0 \not\leq L$, then $G = P_0 L = P_0 \rtimes L$. By the hypothesis, $|G : L|$ is a prime, which implies that $|G : M| = |P_0|$ is a prime, a contradiction.

(ii) For another case when $P_0 \leq L$, then $G = P_0 M = LM$. It follows that $|M : M \cap L| = |LM : L| = |G : L|$ is a prime. Note that $M \cap L \geq M \cap N_G(Q) = N_M(Q)$. Then $M \cap L$ is a maximal subgroup of M that contains $N_M(Q)$ and $M \cap L$ has a prime index. By Lemma 2.1, one has $Q \trianglelefteq M$. It follows that $M \leq N_G(Q)$. Since M is a maximal A-invariant subgroup of G and $N_G(Q)$ is a proper A-invariant subgroup of G, one has $M = N_G(Q)$. By the hypothesis, $|G : M|$ is a prime, a contradiction.

So the counterexample of minimal order does not exist and then every maximal A-invariant subgroup of G has prime index. ∎

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