Interfaces Free Bound. 27 (2025), 235–283
DOI 10.4171/IFB/533

A thresholding algorithm for Willmore-type flows via fourth-order linear parabolic equation

Katsuyuki Ishii, Yoshihito Kohsaka, Nobuhito Miyake, and Koya Sakakibara

Abstract. We propose a thresholding algorithm for Willmore-type flows in $\mathbb{R}^N$. This algorithm is constructed based on the asymptotic expansion of the solution to the initial value problem for a fourth-order linear parabolic partial differential equation whose initial data is the indicator function on the compact set Ω_0. The main results of this paper demonstrate that the boundary $\partial\Omega(t)$ of the new set $\Omega(t)$, generated by our algorithm, is included in $O(t)$-neighborhood of $\partial\Omega_0$ for small $t > 0$ and that the normal velocity from $\partial\Omega_0$ to $\partial\Omega(t)$ is nearly equal to the L^2-gradient of Willmore-type energy for small $t > 0$. Finally, numerical examples of planar curves governed by the Willmore flow are provided by using our thresholding algorithm.

1. Introduction

In this paper, we propose a thresholding algorithm for the L^2-gradient flow of Willmore-type energy in $\mathbb{R}^N$. Let Γ be a hypersurface in $\mathbb{R}^N$. The Willmore-type energy $\mathcal{E}_\lambda^N(\Gamma)$ for Γ is defined as

$$\mathcal{E}_\lambda^N(\Gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Gamma \kappa^2 ds + \lambda \int_\Gamma ds & \text{if } N = 2, \\ \dfrac{1}{2}\displaystyle\int_\Gamma H^2 dS_\Gamma - \dfrac{2}{3}\int_\Gamma \sum_{\substack{i,j\in\Lambda \\ i<j}} \kappa_i\kappa_j dS_\Gamma + \lambda \int_\Gamma dS_\Gamma & \text{if } N \geq 3, \end{cases}$$

where $\lambda \in \mathbb{R}$. For $N = 2$, κ is the curvature of a planar curve Γ and s is the arc-length parameter. For $N \geq 3$, κ_i ($i \in \Lambda := \{1, 2, \dots, N-1\}$) are the principal curvatures of Γ and H is the (($N-1$)-times) mean curvature of Γ. This can be regarded as a generalization of the Willmore energy. Note that the energy $\mathcal{E}_0^N(\Gamma)$ with $N \geq 3$ appears in the asymptotic expansion of the heat content (see Angiuli–Massari–Miranda [1]).

Let $\{\Gamma(t)\}_{t\geq 0}$ be a family of hypersurfaces in $\mathbb{R}^N$ and assume that the motion of $\Gamma(t)$ is governed by the L^2-gradient flow

$$V = -\nabla_{L^2}\mathcal{E}_\lambda^N(\Gamma(t)), \tag{1.1}$$

Mathematics Subject Classification 2020: 53E40 (primary); 35K30, 35R37 (secondary).
Keywords: thresholding algorithm, Willmore-type energy, gradient flow.

where V is the normal velocity of $\Gamma(t)$ and $\nabla_{L^2}\mathcal{E}_\lambda^N(\Gamma)$ is the L^2-gradient of $\mathcal{E}_\lambda^N(\Gamma)$ given by

$$\nabla_{L^2}\mathcal{E}_\lambda^N(\Gamma) = \begin{cases} \partial_s^2\kappa + \dfrac{1}{2}\kappa^3 - \lambda\kappa & \text{if } N = 2, \\ \Delta_g H + H|A|^2 - \dfrac{1}{2}H^3 - \lambda H & \text{if } N = 3, \\ \Delta_g H + H|A|^2 - \dfrac{1}{2}H^3 + 2\displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j<k}} \kappa_i\kappa_j\kappa_k - \lambda H & \text{if } N \geq 4. \end{cases} \tag{1.2}$$

Here, ∂_s^2 is the second-order differential operator with respect to s, Δ_g is the Laplace–Beltrami operator on Γ by the induced metric $g = (g_{ij})$, and $|A|^2$ denotes the norm of the second fundamental form $A = (h_{ij})$, which is defined as $|A|^2 := \sum_{i,j,k,\ell\in\Lambda} g^{ij}g^{k\ell}h_{ik}h_{j\ell}$ $(= \sum_{i\in\Lambda}\kappa_i^2)$, where $g^{-1} = (g^{ij})$ is the inverse matrix of $g = (g_{ij})$. If $\Gamma(t)$ is embedded and encloses a domain $D(t)$, we choose the orientation induced by the outer unit normal so that V is positive if $D(t)$ grows and H is negative if $\Gamma(t)$ is a spherical surface. Note that the term $\sum_{i,j,k\in\Lambda}\kappa_i\kappa_j\kappa_k$ for $N \geq 4$ in (1.2) is derived as the first variation of the integral of $(-1/3)\sum_{i,j\in\Lambda}\kappa_i\kappa_j$ on Γ (see, e.g., Reilly [36]). Also, note that if $N = 3$ and the topology of Γ is fixed, the integral of $\kappa_1\kappa_2$ on Γ is constant by virtue of the Gauss–Bonnet theorem so that its first variation is zero.

Equation (1.1) with $N = 2, 3$ and $\lambda = 0$ is the Willmore flow (WF for short). For the results of the existence and the asymptotic behavior of the WF and related flows, see, e.g., Simonett [40], Kuwert–Schätzle [23, 24], Dziuk–Kuwert–Schätzle [11], Okabe–Wheeler [34], and Rupp [37] and references therein. As for the approximation schemes and the methods of numerical computations of the flow by (1.1), there are many results taking account of various applications. Mayer–Simonett's work [32] is one of the first numerical approaches for the WF in $\mathbb{R}^3$. They used a finite difference scheme to the WF and numerically observed that the WF can develop singularities in finite time. Rusu [38] presented an algorithm for the WF in $\mathbb{R}^3$ based on the variational method and studied a semi- and a fully discrete scheme in space and a semi-implicit method in time. Dziuk [10] introduced a parametric finite-element method to the WF in general space dimensions. In [3, 4], etc. Barrett, Garcke, and Nürnberg studied parametric finite-element methods for fourth-order geometric evolution problems, such as surface diffusion flow and the WF. Furthermore, it is well known that the WF can be approximated by the fourth-order phase-field equations or equivalent systems of PDE's, which are derived from approximations of the Willmore functional. Loreti–March [30] obtained the formal asymptotic expansions of solutions of the fourth-order phase-field equations (or equivalent systems) and derived (1.1) for $N = 3$. Bretin–Masnou–Oudet [6] gave similar results to some modified versions of the fourth-order phase-field equations and the related energies. They also presented in [6] some numerical simulations of the flows for $N = 2, 3$, based on their formal asymptotic expansions. Colli–Laurençot [7, 8] studied the well-posedness of a phase-field approximation to the WF with volume and area constraints in $\mathbb{R}^N$ $(1 \leq N \leq 3)$.

Rätz–Röger [35] introduced a new diffuse-interface approximation of the WF, avoiding intersections of phase boundaries that do not correspond to the intended sharp interface evolution. They also justified the approximation property by a Gamma convergence for the energies and a matched asymptotic expansion for the flow. Fei–Liu [16] rigorously proved the convergence of the zero-level set of the solutions of the phase-field system to the WF for $N = 2, 3$ if the smooth WF exists.

The purpose of this paper is to introduce a thresholding algorithm by using the following Cauchy problem for the fourth-order linear parabolic equation:

$$\begin{cases} u_t = -\Delta^2 u + \lambda \Delta u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\boldsymbol{x},0) = \chi_{\Omega_0}(\boldsymbol{x}) := \begin{cases} 1 & \text{in } \Omega_0, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega_0. \end{cases} \end{cases} \tag{1.3}$$

Here, $N \geq 2$, $\lambda \in \mathbb{R}$, and $\Omega_0 \subset \mathbb{R}^N$ is compact set with smooth boundary. By the derivation of a threshold function from the solution to the above problem, we obtain a thresholding algorithm to the motion of $\Gamma(t)$ by (1.1) at least formally. The outline of our algorithm is as follows: set $h > 0$ as a time step. For a given compact set Ω_0 in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega_0$, we solve the initial value problem (1.3). Next, let u^0 be the solution to (1.3) and set $u_a^0(\boldsymbol{x},t) := u^0(\boldsymbol{x}, a^4 t)$ for $a > 0$. Then, define a threshold function U^0 as

$$U^0(\boldsymbol{x},t) := u_{3a}^0(\boldsymbol{x},t) - 3u_{2a}^0(\boldsymbol{x},t) + 3u_a^0(\boldsymbol{x},t)$$

and give a new set Ω_1 by

$$\Omega_1 := \left\{ \boldsymbol{x} \in \mathbb{R}^N \mid U^0(\boldsymbol{x},h) \geq \frac{1}{2} \right\}.$$

Repeating this procedure inductively, we obtain a sequence $\{\Omega_k\}_{k=0}^{\infty}$ of compact subsets of $\mathbb{R}^N$. Set

$$\Omega^h(t) := \Omega_k \quad \text{for } kh \leq t < (k+1)h,\ k = 0, 1, 2, \ldots.$$

Then, letting $h \to 0$, we observe at least formally that $\Omega^h(t)$ converges to a compact set $\Omega(t)(\subset \mathbb{R}^N)$ and that $\partial\Omega(t)$ moves by (1.1) if we choose a suitable constant a. In order to justify the thresholding algorithm explained above, we derive the asymptotic expansion of the solution to (1.3) near $\partial\Omega_0$. For the details of the justification, see the argument in Section 4.1.

Thresholding algorithms for the geometric evolution equations were first introduced by Bence–Merriman–Osher [5] to numerically compute the mean curvature flows. Based on the level set approach for geometric evolution equations, the convergence and generalizations of their algorithm were studied by Mascarenhas [31], Evans [15], Barles–Georgelin [2], Ishii [21], Ishii–Pires–Souganidis [22], Vivier [41], Leoni [29], and so on. Recently, another approach was suggested by Esedoḡlu–Otto [13], which was considered the thresholding algorithm for the multi-phase mean curvature flow. They gave the interpretation such that the thresholding algorithm can be regarded as a minimizing movement scheme. For the result of the convergence and the further development on this approach, see Laux–Otto [26–28], Laux–Lelmi [25], and Fuchs–Laux [18].

On the thresholding algorithm for the WF, there are results by Grzhibovskis–Heintz [20] in $\mathbb{R}^3$ and Esedoḡlu–Ruuth–Tsai [14] in $\mathbb{R}^2$. In [14, 20], the asymptotic expansion of the convolution $(t^{-N/4}\rho(|\cdot|/t^{1/4}) * \chi_{\Omega_0})(\boldsymbol{x})$ is used to define a threshold function. Here, $t^{-N/4}\rho(|\cdot|/t^{1/4})$ is a modified Gauss kernel or some similar ones. Note that in [14] the L^2-gradient flow of the Helfrich functional in $\mathbb{R}^2$ was also considered. On the details of the difference between their thresholding algorithm and ours, see Remark 4.1. Metivet–Sengers–Ismaïl–Maitre [33] treated the diffusion-resistance scheme in $\mathbb{R}^2$ or $\mathbb{R}^3$, which is a variant of the algorithm by [14].

Referring to [14, 20], the space-time scale $|\boldsymbol{x}|/t^{1/4}$ plays a key role in obtaining the WF from the formal asymptotic expansions of their convolutions. Indeed, in [14, 20], they used a modified Gauss kernel whose space-time scale is $|\boldsymbol{x}|/t^{1/4}$ instead of a natural space-time scale $|\boldsymbol{x}|/t^{1/2}$ of the usual Gauss kernel (see Remark 4.1 on the details of the calculation). Based on this fact, we arrive at the idea of using the fundamental solution to the fourth-order linear parabolic equation in (1.3) to construct a thresholding algorithm for the flow by (1.1) since a natural space-time scale of its solution is $|\boldsymbol{x}|/t^{1/4}$. Another reason to consider the equation in (1.3) is that (1.1) can be written by the signed distance function d as follows (cf. [6, Section 3.3] and [16, Lemma A.2]):

$$
\begin{aligned}
d_t = {} & -\Delta^2 d + \langle D(\langle D(\Delta d), Dd\rangle), Dd\rangle + 2\langle D(\Delta d), Dd\rangle \Delta d \\
& + \frac{1}{2}(\Delta d)^3 + \lambda \Delta d \quad \text{on } \Gamma(t).
\end{aligned}
$$

Dropping all of the nonlinear terms, we have the fourth-order linear parabolic equation in (1.3), and hence, it is regarded as the “rough” approximation of this equation. Furthermore, as the benefit of using the fundamental solution to (1.3), the part related to the area constraint is naturally derived from the term of Laplacian of the equation in (1.3). Such a derivation considering the structure is difficult if we use a modified Gauss kernel. We remark that for a thresholding algorithm for the WF, at present, there are no results on the convergence to some suitable solution to the WF. Since it seems that our approach is more natural compared with that in [14, 20], it is expected that the construction of some suitable solution and the convergence to it are shown based on our results. Indeed, in order to prove the convergence, the ideas based on the gradient flow as in [18, 25–28] may be useful for the WF. We think that our algorithm has a strong possibility for connecting to their approach.

This paper is organized in the following way. In Section 2, we derive some formulae and pointwise estimates of the fundamental solution $G_{N,\lambda}$ to the operator $\partial_t + (-\Delta)^2 + \lambda(-\Delta)$ and of its derivatives. In Section 3, we discuss the formal asymptotic expansion of the solution to (1.3), which is stated in Theorem 3.4. Section 4 is devoted to the justification of our algorithm. In Section 4.1, we recall the algorithm and the fact to be justified and prove this fact in Section 4.2. Section 5 presents the results of numerical experiments.

We consider the equation in (1.3) with $\lambda = 0$ and $N = 2$, i.e., the case corresponding to the WF in $\mathbb{R}^2$, in a periodic square domain. The last two sections are appendices.

2. Preliminaries

In this section, we derive some properties and estimates of the fundamental solution $G_{N,\lambda}$ to the operator $\partial_t + (-\Delta)^2 + \lambda(-\Delta)$ on $\mathbb{R}^N \times (0,\infty)$, where Δ is the Laplace operator on $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$.

Define the Fourier transform as

$$\mathcal{F}_N[\psi](\boldsymbol{\xi}) := \int_{\mathbb{R}^N} \psi(\boldsymbol{x}) e^{-\mathbf{i}\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle_N} d\boldsymbol{x}, \quad \mathcal{F}_N^{-1}[\psi](\boldsymbol{x}) := c_N \int_{\mathbb{R}^N} \psi(\boldsymbol{\xi}) e^{\mathbf{i}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle_N} d\boldsymbol{\xi}$$

for $\psi \in L^1(\mathbb{R}^N)$, where $\mathbf{i} := \sqrt{-1}$, $c_N := (2\pi)^{-N}$ and $\langle \cdot, \cdot \rangle_N$ is the inner product on $\mathbb{R}^N$. Then, $G_{N,\lambda}$ is given by

$$G_{N,\lambda}(\boldsymbol{x}, t) = \mathcal{F}_N^{-1}[e^{-(|\cdot|^4 + \lambda|\cdot|^2)t}](\boldsymbol{x}) = c_N \int_{\mathbb{R}^N} e^{-(|\boldsymbol{\xi}|^4 + \lambda|\boldsymbol{\xi}|^2)t + \mathbf{i}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle_N} d\boldsymbol{\xi},$$

and we readily see that

$$\int_{\mathbb{R}^N} G_{N,\lambda}(\boldsymbol{x}, t) d\boldsymbol{x} = \mathcal{F}_N[\mathcal{F}_N^{-1}[e^{-(|\cdot|^4 + \lambda|\cdot|^2)t}]](\mathbf{0}) = e^{-(|\boldsymbol{\xi}|^4 + \lambda|\boldsymbol{\xi}|^2)t}\Big|_{\boldsymbol{\xi}=\mathbf{0}} = 1. \quad (2.1)$$

Set $g_N(\boldsymbol{x}) := G_{N,0}(\boldsymbol{x}, 1)$. That is,

$$g_N(\boldsymbol{x}) = \mathcal{F}_N^{-1}[e^{-|\cdot|^4}](\boldsymbol{x}) = c_N \int_{\mathbb{R}^N} e^{-|\boldsymbol{\xi}|^4 + \mathbf{i}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle_N} d\boldsymbol{\xi}. \quad (2.2)$$

We derive the expansion of $G_{N,\lambda}$ by use of g_N and its derivatives.

Proposition 2.1. *$G_{N,\lambda}$ is represented as*

$$G_{N,\lambda}(\boldsymbol{x}, t) = \frac{1}{t^{N/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} (-\Delta_{\boldsymbol{z}})^m g_N\Big(\frac{\boldsymbol{x}}{t^{1/4}}\Big),$$

where $\Delta_{\boldsymbol{z}} g_N$ is the Laplacian of $g_N(\boldsymbol{z})$ with respect to $\boldsymbol{z} \in \mathbb{R}^N$.

Proof. Applying the change of variable $\boldsymbol{\xi} = \boldsymbol{\eta}/t^{1/4}$, we have

$$\begin{aligned} G_{N,\lambda}(\boldsymbol{x}, t) &= \frac{c_N}{t^{N/4}} \int_{\mathbb{R}^N} e^{-|\boldsymbol{\eta}|^4 + \mathbf{i}\langle \frac{\boldsymbol{x}}{t^{1/4}}, \boldsymbol{\eta}\rangle_N} e^{-\lambda|\boldsymbol{\eta}|^2 t^{1/2}} d\boldsymbol{\eta} \\ &= \frac{c_N}{t^{N/4}} \int_{\mathbb{R}^N} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} |\boldsymbol{\eta}|^{2m} e^{-|\boldsymbol{\eta}|^4 + \mathbf{i}\langle \frac{\boldsymbol{x}}{t^{1/4}}, \boldsymbol{\eta}\rangle_N} d\boldsymbol{\eta}. \end{aligned}$$

Since

$$\int_{\mathbb{R}^N} \sum_{m=0}^{\infty} \left| \frac{(-\lambda)^m t^{m/2}}{m!} |\boldsymbol{\eta}|^{2m} e^{-|\boldsymbol{\eta}|^4 + \mathrm{i}\langle \frac{\boldsymbol{x}}{t^{1/4}}, \boldsymbol{\eta}\rangle_N} \right| d\boldsymbol{\eta} = \int_{\mathbb{R}^N} \sum_{m=0}^{\infty} \frac{|\lambda|^m t^{m/2}}{m!} |\boldsymbol{\eta}|^{2m} e^{-|\boldsymbol{\eta}|^4} d\boldsymbol{\eta}$$
$$= \int_{\mathbb{R}^N} e^{-|\boldsymbol{\eta}|^4 + |\lambda||\boldsymbol{\eta}|^2 t^{1/2}} d\boldsymbol{\eta} < +\infty,$$

the Lebesgue convergence theorem implies that termwise integration is possible, and hence,

$$G_{N,\lambda}(\boldsymbol{x}, t) = \frac{c_N}{t^{N/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^N} |\boldsymbol{\eta}|^{2m} e^{-|\boldsymbol{\eta}|^4 + \mathrm{i}\langle \frac{\boldsymbol{x}}{t^{1/4}}, \boldsymbol{\eta}\rangle_N} d\boldsymbol{\eta}. \tag{2.3}$$

Taking account of $\partial_{z_i}^2 e^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{\eta}\rangle_N} = -\eta_i^2 e^{\mathrm{i}\langle \boldsymbol{z}, \boldsymbol{\eta}\rangle_N}$ $(i = 1, \ldots, N)$, we obtain the desired result by some properties of the Fourier transform and (2.3). ■

Set $\mathbb{Z}_+^N := \{\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N \mid \alpha_i \geq 0 (i = 1, \ldots, N)\}$ and $\mathbb{Z}_+ := \mathbb{Z}_+^1$. Hereafter, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $D_{\boldsymbol{x}}^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$.

For $G_{N,\lambda}$, we have the pointwise estimates as follows.

Theorem 2.2. *There exist $C, \nu, \mu, K > 0$ such that for all $\alpha \in \mathbb{Z}_+^N$, $m \in \mathbb{Z}_+$, $\lambda \in \mathbb{R}$, and $(\boldsymbol{x}, t) \in \mathbb{R}^N \times (0, \infty)$*

$$|D_{\boldsymbol{x}}^\alpha (-\Delta_{\boldsymbol{x}})^m G_{N,\lambda}(\boldsymbol{x}, t)| \leq C \nu^{|\alpha|+2m} \Gamma\left(\frac{|\alpha| + 2m + N}{4}\right) t^{-(N+|\alpha|+2m)/4}$$
$$\cdot \left(1 + \frac{|\boldsymbol{x}|}{t^{1/4}}\right)^{(|\alpha|+2m)/3} e^{-\mu(|\boldsymbol{x}|^4/t)^{1/3} + K|\lambda|^2 t}.$$

In the case $\lambda = 0$, this estimate is originally obtained in Eidel'man [12, Section 3 in Chapter I] and Cui [9, Theorem 3.2]. However, in these references, the precise dependence of the constant on $\alpha \in \mathbb{Z}_+^N$ and $m \in \mathbb{Z}_+$ is not stated. Thus, we give the proof of this theorem in Appendix B below. The following corollary is a direct consequence of Theorem 2.2.

Corollary 2.3. *There exist $C, \nu, \mu > 0$ such that for all $\alpha \in \mathbb{Z}_+^N$, $m \in \mathbb{Z}_+$, and $\boldsymbol{x} \in \mathbb{R}^N$*

$$|D_{\boldsymbol{x}}^\alpha (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x})| \leq C \nu^{|\alpha|+2m} \Gamma\left(\frac{|\alpha| + 2m + N}{4}\right) (1 + |\boldsymbol{x}|)^{(|\alpha|+2m)/3} e^{-\mu|\boldsymbol{x}|^{4/3}}.$$

If $N \geq 2$, for an orthogonal matrix P_N of size N satisfying $P_N \boldsymbol{x} = (|\boldsymbol{x}|, 0, \ldots, 0)^T \in \mathbb{R}^N$, set $\boldsymbol{\xi} = P_N \boldsymbol{\zeta}$ in (2.2). Since $|\det P_N| = 1$ and $|\boldsymbol{\xi}| = |\boldsymbol{\zeta}|$, we obtain

$$g_N(\boldsymbol{x}) = c_N \int_{\mathbb{R}^N} e^{-|\boldsymbol{\zeta}|^4 + \mathrm{i}\langle \boldsymbol{x}, P_N \boldsymbol{\zeta}\rangle_N} d\boldsymbol{\zeta} = c_N \int_{\mathbb{R}^N} e^{-|\boldsymbol{\zeta}|^4 + \mathrm{i}|\boldsymbol{x}|\zeta_1} d\boldsymbol{\zeta}.$$

Applying the change of variable on the polar coordinate, $g_N(\boldsymbol{x})$ is represented as

$$\begin{aligned} g_N(\boldsymbol{x}) &= c_N\omega_{N-2}\int_0^\infty \rho^{N-1}e^{-\rho^4}\left\{\int_0^\pi e^{\mathrm{i}|\boldsymbol{x}|\rho\cos\theta}\sin^{N-2}\theta d\theta\right\}d\rho \\ &= (2\pi)^{-N/2}|\boldsymbol{x}|^{1-N}\int_0^\infty (|\boldsymbol{x}|\rho)^{N/2}e^{-\rho^4}J_{(N-2)/2}(|\boldsymbol{x}|\rho)d\rho, \end{aligned} \tag{2.4}$$

where $J_{(N-2)/2}$ is the $(N-2)/2$-th Bessel function. If $N=1$, we have

$$\begin{aligned} g_1(x) &= c_1\int_{\mathbb{R}} e^{-\xi^4+\mathrm{i}x\xi}d\xi = 2c_1\int_0^\infty e^{-\xi^4}\cos(x\xi)d\xi \\ &= (2\pi)^{-1/2}\int_0^\infty (|x|\xi)^{1/2}e^{-\xi^4}J_{-1/2}(|x|\xi)d\xi. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), set

$$\varphi_N(|\boldsymbol{x}|) := (2\pi)^{-N/2}|\boldsymbol{x}|^{1-N}\int_0^\infty (|\boldsymbol{x}|\rho)^{N/2}e^{-\rho^4}J_{(N-2)/2}(|\boldsymbol{x}|\rho)d\rho$$

for $N \geq 1$, which means $g_N(\boldsymbol{x}) = \varphi_N(|\boldsymbol{x}|)$. According to Ferrero–Gazzola–Grunau [17, Section 2], $\varphi_N(r)$ with $r = |\boldsymbol{x}| \geq 0$ satisfies

$$\begin{aligned} \varphi_N(r) &= \frac{1}{2^{N+1}\pi^{N/2}}\sum_{\ell=0}^\infty \frac{(-1)^\ell\Gamma(\ell/2+N/4)}{2^{2\ell}\Gamma(\ell+1)\Gamma(\ell+N/2)}r^{2\ell}, \\ \varphi_N'(r) &= -r\varphi_{N+2}(r). \end{aligned} \tag{2.6}$$

In addition, we have the following lemma.

Lemma 2.4. *For $g_N(\boldsymbol{x}) = \varphi_N(r)$ with $r = |\boldsymbol{x}| \geq 0$,*

$$\begin{aligned} \Delta^m g_N(\boldsymbol{x}) &= \left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}\right)^m\varphi_N(r) \\ &= \frac{(-1)^m}{2^{N+1}\pi^{N/2}}\sum_{\ell=0}^\infty \frac{(-1)^\ell\Gamma((\ell+m)/2+N/4)}{2^{2\ell}\Gamma(\ell+1)\Gamma(\ell+N/2)}r^{2\ell}. \end{aligned}$$

Proof. For $m=1$, the direct calculation yields that

$$\left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}\right)\varphi_N(r) = \frac{-1}{2^{N+1}\pi^{N/2}}\sum_{\ell=0}^\infty \frac{(-1)^\ell\Gamma((\ell+1)/2+N/4)}{2^{2\ell}\Gamma(\ell+1)\Gamma(\ell+N/2)}r^{2\ell}.$$

The result follows by induction. ∎

We show the following lemma. It is necessary to estimate some integrations of $G_{N,\lambda}$ in the next section.

Lemma 2.5. *Let $\alpha \in \mathbb{Z}_+^N$ with $|\alpha| \le 2$. Then, there exists $\gamma > 0$ such that for $|\lambda|t^{1/2} \le \gamma$*

$$\sum_{m=0}^{\infty} \frac{|\lambda|^m t^{m/2}}{m!} \int_{\mathbb{R}^N} \left| D_{\boldsymbol{x}}^{\alpha} (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) \right| d\boldsymbol{x} < \infty.$$

Proof. By Corollary 2.3, there exist $C, \nu, \mu > 0$ independent of m such that

$$\begin{aligned} &\frac{|\lambda|^m t^{m/2}}{m!} \int_{\mathbb{R}^N} \left| D_{\boldsymbol{x}}^{\alpha} (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) \right| d\boldsymbol{x} \\ &\quad \le \frac{C(|\lambda|\nu^2 t^{1/2})^m}{m!} \Gamma\left(\frac{m}{2} + \frac{|\alpha| + N}{4}\right) \int_{\mathbb{R}^N} (1 + |\boldsymbol{x}|)^{(|\alpha|+2m)/3} e^{-\mu|\boldsymbol{x}|^{4/3}} d\boldsymbol{x}. \end{aligned}$$

Applying the change of variable on the polar coordinate, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} (1 + |\boldsymbol{x}|)^{(|\alpha|+2m)/3} e^{-\mu|\boldsymbol{x}|^{4/3}} d\boldsymbol{x} \\ &\quad \le 2^{(|\alpha|+2m)/3} C_1 \left\{ 1 + \left(\frac{1}{\mu}\right)^{(|\alpha|+2m)/4} \Gamma\left(\frac{m}{2} + \frac{|\alpha| + 3N}{4}\right) \right\} \end{aligned}$$

for a constant $C_1 > 0$ independent of m. Here, it follows from the Schwarz inequality that

$$\Gamma\left(\frac{m}{2} + \frac{|\alpha| + kN}{4}\right) \le \{\Gamma(m)\}^{1/2} \left\{ \Gamma\left(\frac{|\alpha| + kN}{2}\right) \right\}^{1/2} \le (m!)^{1/2} \left\{ \Gamma\left(\frac{|\alpha| + kN}{2}\right) \right\}^{1/2}$$

for $m \in \mathbb{N}$ and $k = 1, 3$. Therefore, we see that

$$\frac{|\lambda|^m t^{m/2}}{m!} \int_{\mathbb{R}^N} \left| D_{\boldsymbol{x}}^{\alpha} (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) \right| d\boldsymbol{x} \le C_2 \frac{(4^{1/3}|\lambda|\nu^2 t^{1/2})^m}{(m!)^{1/2}} + C_3 \left(\frac{4^{1/3}|\lambda|\nu^2 t^{1/2}}{\mu^{1/2}} \right)^m$$

for constants $C_2, C_3 > 0$ independent of m. Choose $\gamma > 0$ satisfying

$$\gamma < \frac{\mu^{1/2}}{4^{1/3}\nu^2}.$$

Then, with the help of d'Alembert test, we can judge that the series

$$\sum_{m=1}^{\infty} \frac{(4^{1/3}|\lambda|\nu^2 t^{1/2})^m}{(m!)^{1/2}}, \quad \sum_{m=1}^{\infty} \left(\frac{4^{1/3}|\lambda|\nu^2 t^{1/2}}{\mu^{1/2}} \right)^m$$

converge uniformly for $|\lambda|t^{1/2} \le \gamma$ so that we obtain the desired result. ■

Lemma 2.5 and the Lebesgue convergence theorem lead to the following lemma.

Lemma 2.6. *Let $\alpha \in \mathbb{Z}_+^N$ with $|\alpha| \le 2$ and $h \in L^\infty(\mathbb{R}^N)$. Then, there exists $\gamma > 0$ such that for $|\lambda|t^{1/2} \le \gamma$*

$$\begin{aligned} &\int_{\mathbb{R}^N} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} D_{\boldsymbol{x}}^{\alpha} (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) h(\boldsymbol{x}) d\boldsymbol{x} \\ &\quad = \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^N} D_{\boldsymbol{x}}^{\alpha} (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) h(\boldsymbol{x}) d\boldsymbol{x}. \end{aligned}$$

With regard to $(-\partial_{x_N}^2)^\ell (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}', 0)$, we have the following representation.

Lemma 2.7. *Let g_N be given by* (2.2). *Then,*

$$(-\partial_{x_N}^2)^{\ell}(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}',0) = c_1 \sum_{j=0}^{m} \binom{m}{j} \mathscr{F}_{N-1}^{-1}\left[\sum_{k=0}^{\infty} \frac{(-2)^k}{k!} L_{2(k+j+\ell)} |\cdot|^{2(k+m-j)} e^{-|\cdot|^4}\right](\boldsymbol{x}')$$

for $\ell, m \in \mathbb{Z}_+$, where

$$L_\sigma := 2\int_0^\infty \xi^\sigma e^{-\xi^4}\,d\xi = \frac{1}{2}\Gamma\left(\frac{\sigma+1}{4}\right) \tag{2.7}$$

for $\sigma \geq 0$.

Proof. Fix any $\boldsymbol{x} \in \mathbb{R}^N$. Taking account of

$$|\boldsymbol{\xi}|^4 = |\boldsymbol{\xi}'|^4 + \xi_N^4 + 2|\boldsymbol{\xi}'|^2\xi_N^2$$

for $\boldsymbol{\xi}' = (\xi_1, \ldots, \xi_{N-1})$, we see that for $\boldsymbol{x}' = (x_1, \ldots, x_{N-1})$

$$\begin{aligned}
&(-\partial_{x_N}^2)^{\ell}(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) \\
&= c_N \int_{\mathbb{R}^N} \xi_N^{2\ell} |\boldsymbol{\xi}|^{2m} e^{-|\boldsymbol{\xi}|^4 + \mathrm{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_N}\,d\boldsymbol{\xi} \\
&= c_N \sum_{j=0}^{m}\binom{m}{j}\int_{\mathbb{R}^N} |\boldsymbol{\xi}'|^{2(m-j)} \xi_N^{2(j+\ell)} e^{-(|\boldsymbol{\xi}'|^4 + \xi_N^4 + 2|\boldsymbol{\xi}'|^2\xi_N^2) + \mathrm{i}\{\langle \boldsymbol{x}',\boldsymbol{\xi}'\rangle_{N-1} + x_N\xi_N\}}\,d\boldsymbol{\xi} \\
&= c_N \sum_{j=0}^{m}\binom{m}{j}\int_{\mathbb{R}^N} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2(k+m-j)} \xi_N^{2(k+j+\ell)} e^{-|\boldsymbol{\xi}'|^4 + \mathrm{i}\langle \boldsymbol{x}',\boldsymbol{\xi}'\rangle_{N-1}} e^{-\xi_N^4 + \mathrm{i}x_N\xi_N}\,d\boldsymbol{\xi}.
\end{aligned}$$

Note that by Fubini's theorem

$$\begin{aligned}
(-\partial_{x_N}^2)^{\ell}(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) = c_N \sum_{j=0}^{m}\binom{m}{j}\int_{\mathbb{R}^{N-1}} |\boldsymbol{\xi}'|^{2(m-j)} e^{-|\boldsymbol{\xi}'|^4 + \mathrm{i}\langle \boldsymbol{x}',\boldsymbol{\xi}'\rangle_{N-1}} \\
\cdot \left\{\int_{\mathbb{R}} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4 + \mathrm{i}x_N\xi_N}\,d\xi_N\right\} d\boldsymbol{\xi}'.
\end{aligned}$$

Here, it follows that, for each $\boldsymbol{\xi}' \in \mathbb{R}^{N-1}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \sum_{k=0}^{n} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4 + \mathrm{i}x_N\xi_N} \right| d\xi_N \\
&\quad \leq \int_{\mathbb{R}} \xi_N^{2(j+\ell)} e^{-\xi_N^4 + 2|\boldsymbol{\xi}'|^2\xi_N^2}\,d\xi_N < +\infty.
\end{aligned}$$

The Lebesgue convergence theorem implies that

$$\begin{aligned}\int_{\mathbb{R}} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4 + \mathrm{i} x_N \xi_N} d\xi_N \\ = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \int_{\mathbb{R}} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4 + \mathrm{i} x_N \xi_N} d\xi_N.\end{aligned}$$

Substituting $x_N = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4} d\xi_N &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \int_{\mathbb{R}} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4} d\xi_N \\ &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} L_{2(k+j+\ell)}.\end{aligned}$$

Since

$$| \cdot |^{2(m-j)} e^{-|\cdot|^4 + \mathrm{i}\langle \boldsymbol{x}', \cdot \rangle_{N-1}} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} L_{2(k+j+\ell)} | \cdot |^{2k} \in L^1(\mathbb{R}^{N-1})$$

by Fubini's theorem, we obtain the desired result. ■

3. Asymptotic expansion of a solution to linear parabolic equations

The purpose of this section is to derive the asymptotic expansion of a solution $u(\boldsymbol{x}, t)$ to (1.3) as $t \to +0$. Throughout this section, we assume that Ω_0 is a compact set in $\mathbb{R}^N$ and $\partial\Omega_0$ is of class C^5. Recalling Proposition 2.1 and Lemma 2.6, the solution $u(\boldsymbol{x}, t)$ to (1.3) is given by

$$\begin{aligned}u(\boldsymbol{x}, t) &= (G_{N,\lambda}(\cdot, t) * \chi_{\Omega_0})(\boldsymbol{x}) \\ &= \frac{1}{t^{N/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^N} (-\Delta_{\boldsymbol{z}})^m g_N\Big(\frac{\boldsymbol{x} - \boldsymbol{y}}{t^{1/4}}\Big) \chi_{\Omega_0}(\boldsymbol{y}) d\boldsymbol{y} \\ &= \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\{\boldsymbol{z} \in \mathbb{R}^N \mid \boldsymbol{x} - t^{1/4}\boldsymbol{z} \in \Omega_0\}} (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z}) d\boldsymbol{z}\end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^N$.

3.1. Notations related to $\partial\Omega_0$

In this subsection, we give some notations related to $\partial\Omega_0$. For $\boldsymbol{x}_* \in \mathbb{R}^N$ and $\delta > 0$, we denote a neighborhood of $\boldsymbol{x}_*$ by

$$\begin{aligned}&Q_{\boldsymbol{x}_*,\delta} := \{\boldsymbol{x} \in \mathbb{R}^N \mid |x_i - x_{*,i}| < \delta (i = 1, \dots, N)\}, \qquad &Q_{\boldsymbol{x}_*} := Q_{\boldsymbol{x}_*,1}, \\ &Q'_{\boldsymbol{x}_*,\delta} := \{\boldsymbol{x} \in Q_{\boldsymbol{x}_*,\delta} \mid x_N = x_{*,N}\}, \qquad &Q'_{\boldsymbol{x}_*} := Q'_{\boldsymbol{x}_*,1}.\end{aligned}$$

Since $\partial\Omega_0$ is compact, we can choose $\delta_0 > 0$ such that the family $\{Q_{\boldsymbol{x}*,\delta_0}\}_{\boldsymbol{x}*\in\partial\Omega_0}$ is an open covering of $\partial\Omega_0$. We may set $\delta_0 = 1$. Let $\boldsymbol{n}(\boldsymbol{x})$ be the unit outward normal to $\partial\Omega_0$ at $\boldsymbol{x} \in \partial\Omega_0$ and P_N an orthogonal matrix of size N such that $P_N\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{e}_N$, where $\boldsymbol{e}_N = (\boldsymbol{0}', 1)^T$ with $\boldsymbol{0}' = (0,\dots,0) \in \mathbb{R}^{N-1}$. Then, set $\tilde{\Omega}_0 := \{\tilde{\boldsymbol{y}} \in \mathbb{R}^N \mid P_N^{-1}\tilde{\boldsymbol{y}} \in \Omega_0\}$. Without loss of generality, we may assume that for each $\boldsymbol{x} \in \partial\Omega_0$ there exists a function $f|_{\boldsymbol{x}} : Q'_{\boldsymbol{0}} \to \mathbb{R}$ satisfying the following properties.

(A1) $f|_{\boldsymbol{x}} \in C^5(\overline{Q'_{\boldsymbol{0}}})$ for any $\boldsymbol{x} \in \partial\Omega_0$ and $\|f|_{\boldsymbol{x}}\|_{C^5(\overline{Q'_{\boldsymbol{0}}})}$ is uniformly bounded for $\boldsymbol{x} \in \partial\Omega_0$.

(A2) $f|_{\boldsymbol{x}}(\boldsymbol{0}') = 0$ and $\nabla_{\boldsymbol{x}'} f|_{\boldsymbol{x}}(\boldsymbol{0}') = \boldsymbol{0}'$.

(A3) $\partial\tilde{\Omega}_0 \cap Q_{P_N\boldsymbol{x}} = \{\tilde{\boldsymbol{y}} \in \mathbb{R}^N \mid \tilde{y}_N - (P_N\boldsymbol{x})_N = f|_{\boldsymbol{x}}(\tilde{\boldsymbol{y}}' - (P_N\boldsymbol{x})')(\tilde{\boldsymbol{y}}' \in Q'_{P_N\boldsymbol{x}})\}$, where $(P_N\boldsymbol{x})_i$ is the i-th component of $P_N\boldsymbol{x}$ and

$$(P_N\boldsymbol{x})' = ((P_N\boldsymbol{x})_1,\dots,(P_N\boldsymbol{x})_{N-1}).$$

Note that $(\tilde{\boldsymbol{y}}', (P_N\boldsymbol{x})_N + f|_{\boldsymbol{x}}(\tilde{\boldsymbol{y}}' - (P_N\boldsymbol{x})'))(\tilde{\boldsymbol{y}}' \in Q'_{P_N\boldsymbol{x}})$ is a graph representation of $\partial\tilde{\Omega}_0$ in a neighborhood of $P_N\boldsymbol{x}$ for $\boldsymbol{x} \in \partial\Omega_0$. Hereafter, for simplicity, we denote $f|_{\boldsymbol{x}}$ by f. We also define a function $\psi = \psi(\boldsymbol{z}', v, t)\colon \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as

$$\psi(\boldsymbol{z}', v, t) := t^{-1/4}\{-v + f(t^{1/4}\boldsymbol{z}')\}.$$

Let $g = (g_{ij})$, $A = (h_{ij})$, H, κ_i, and Δ_g be the induced metric, the second fundamental form, the mean curvature, the principal curvatures, and the Laplace–Beltrami operator of $\partial\Omega_0$, respectively. $|A|^2$ is the norm of the second fundamental form, that is,

$$|A|^2 = \sum_{i,j,k,\ell\in\Lambda} g^{ij}g^{k\ell}h_{ik}h_{j\ell},$$

where $g^{-1} = (g^{ij})$ is the inverse matrix of $g = (g_{ij})$. For the representation of these quantities by f, see Appendix A.

Define the signed distance function to $\partial\Omega_0$ as

$$d(\boldsymbol{y}, \partial\Omega_0) := \begin{cases} \inf_{\boldsymbol{x}\in\partial\Omega_0} |\boldsymbol{y} - \boldsymbol{x}| & (\boldsymbol{y} \in \Omega_0), \\ -\inf_{\boldsymbol{x}\in\partial\Omega_0} |\boldsymbol{y} - \boldsymbol{x}| & (\boldsymbol{y} \in \mathbb{R}^N \setminus \Omega_0). \end{cases} \tag{3.1}$$

Set $(\partial\Omega_0)^\delta := \{\boldsymbol{y} \in \mathbb{R}^N \mid |d(\boldsymbol{y}, \partial\Omega_0)| < \delta\}$. Then, we take $\delta_0 \in (0, 1/2)$ such that for any $\boldsymbol{y} \in (\partial\Omega_0)^{\delta_0}$ there is a unique $\boldsymbol{x}_0 \in \partial\Omega_0$ satisfying $|d(\boldsymbol{y}, \partial\Omega_0)| = |\boldsymbol{y} - \boldsymbol{x}_0|$.

Furthermore, in the following, we often use the notation $F(\boldsymbol{x}, t) = O_D(b(t))$. This means that there exists $C > 0$, which is obtained uniformly for $\boldsymbol{x} \in D$, such that $|F(\boldsymbol{x}, t)| \le C|b(t)|$.

3.2. Asymptotic expansion

We first show the asymptotic expansion of $u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t)$ by using the graph representation.

Theorem 3.1. *Let γ be a constant obtained in Lemma 2.5. Then, there exists $\mu_* > 0$ such that for all $\boldsymbol{x} \in \partial\Omega_0$, $v \in (-\delta_0, \delta_0)$, $\lambda \in \mathbb{R}$, and $t > 0$ satisfying $|\lambda| t^{1/2} \leq \gamma$*

$$u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_0^{\psi(z',v,t)} (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z}) dz_N d\boldsymbol{z}' + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}). \tag{3.2}$$

Furthermore, for $j \in \{1, \ldots, N\}$, $\lambda \in \mathbb{R}$, and $t > 0$ satisfying $|\lambda| t^{1/2} \leq \gamma$

$$\begin{aligned} &\langle \nabla_{\boldsymbol{x}} u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), P_N^{-1} \boldsymbol{e}_j \rangle_N \\ &= -\frac{1}{t^{1/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\psi(z',v,t)} \partial_{z_j} (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z}) dz_N d\boldsymbol{z}' \\ &\quad + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}). \end{aligned} \tag{3.3}$$

Remark 3.2. Note that the term $O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}})$ is not only uniform for $\boldsymbol{x} \in \partial\Omega_0$, but also for $v \in (-\delta_0, \delta_0)$. See the estimate of I_2 in the proof below.

Proof of Theorem 3.1. Step 1. We first prove (3.2). Using the fact that $G_{N,\lambda}(\cdot, t)$ is radially symmetric, we have

$$\begin{aligned} u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) &= \int_{\Omega_0} G_{N,\lambda}(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}) - \boldsymbol{y}, t)\, d\boldsymbol{y} \\ &= \int_{\Omega_0} G_{N,\lambda}(\boldsymbol{y} - (\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x})), t)\, d\boldsymbol{y}. \end{aligned}$$

Choose an orthogonal matrix P_N of size N such that $P_N \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{e}_N$. Setting $\tilde{\boldsymbol{y}} = P_N \boldsymbol{y}$ and taking account of $|P_N^{-1} \boldsymbol{z}| = |\boldsymbol{z}|$ for $\boldsymbol{z} \in \mathbb{R}^N$, we obtain

$$\begin{aligned} u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) &= \int_{\tilde{\Omega}_0} G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N \boldsymbol{x} + v\boldsymbol{e}_N), t)\, d\tilde{\boldsymbol{y}} \\ &= \left(\int_{\tilde{\Omega}_0 \cap Q_{P_N \boldsymbol{x}}} + \int_{\tilde{\Omega}_0 \setminus Q_{P_N \boldsymbol{x}}} \right) G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N \boldsymbol{x} + v\boldsymbol{e}_N), t)\, d\tilde{\boldsymbol{y}} \\ &=: I_1 + I_2. \end{aligned}$$

It follows from Theorem 2.2 with $|\alpha| = m = 0$ that for $t > 0$ satisfying $|\lambda| t^{1/2} \leq \gamma$

$$\begin{aligned} I_2 &\leq C_1 t^{-N/4} e^{K|\lambda|^2 t} \int_{\mathbb{R}^N \setminus B_N(P_N \boldsymbol{x} + v\boldsymbol{e}_N, 1/2)} e^{-\mu(|\tilde{\boldsymbol{y}} - (P_N \boldsymbol{x} + v\boldsymbol{e}_N)|^4/t)^{1/3}} d\tilde{\boldsymbol{y}} \\ &\leq C_2 \int_{\mathbb{R}^N \setminus B_N(\boldsymbol{0}, t^{-1/4}/2)} e^{-\mu|\tilde{\boldsymbol{\eta}}|^{4/3}} d\tilde{\boldsymbol{\eta}} = O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}), \end{aligned}$$

where $B_N(\boldsymbol{x}, r)$ is the N-dimensional ball with the center $\boldsymbol{x}$ and the radius r. We also see that for $t > 0$ satisfying $|\lambda|t^{1/2} \le \gamma$

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{f(\tilde{\boldsymbol{y}}' - (P_N\boldsymbol{x})') + (P_N\boldsymbol{x})_N} G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N), t) d\tilde{y}_N d\tilde{\boldsymbol{y}}' - I_1 \right| \\
&\quad \le Ct^{-N/4} e^{K|\lambda|^2 t} \int_{\mathbb{R}^N \setminus B_N(P_N\boldsymbol{x} + v\boldsymbol{e}_N, 1/2)} e^{-\mu(|\tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N)|^4/t)^{1/3}} d\tilde{\boldsymbol{y}} \\
&\quad = O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}),
\end{aligned}$$

where f is a function satisfying (A1)–(A3). These facts imply that

$$\begin{aligned}
u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = &\int_{\mathbb{R}^{N-1}} \int_{-\infty}^{f(\tilde{\boldsymbol{y}}' - (P_N\boldsymbol{x})') + (P_N\boldsymbol{x})_N} G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N), t)\, d\tilde{y}_N d\tilde{\boldsymbol{y}}' \\
&+ O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}).
\end{aligned}$$

Applying the change of variable

$$\boldsymbol{\eta} = \tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N)$$

and recalling (2.1), Proposition 2.1, and Lemma 2.6, we have

$$\begin{aligned}
&u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) \\
&\quad = \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\psi(\boldsymbol{\eta}', v, 1)} G_{N,\lambda}(\boldsymbol{\eta}, t) d\eta_N d\boldsymbol{\eta}' + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}) \\
&\quad = \frac{1}{2} + \int_{\mathbb{R}^{N-1}} \int_0^{\psi(\boldsymbol{\eta}', v, 1)} G_{N,\lambda}(\boldsymbol{\eta}, t) d\eta_N d\boldsymbol{\eta}' + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}) \\
&\quad = \frac{1}{2} + \frac{1}{t^{N/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_0^{\psi(\boldsymbol{\eta}', v, 1)} (-\Delta_{\boldsymbol{z}})^m g_N\left(\frac{\boldsymbol{\eta}}{t^{1/4}}\right) d\eta_N d\boldsymbol{\eta}' \\
&\qquad + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}).
\end{aligned}$$

By the change of variable $\boldsymbol{z} = t^{-1/4}\boldsymbol{\eta}$, we obtain (3.2).

Step 2. We derive (3.3). Using again the fact that $G_{N,\lambda}(\cdot, t)$ is radially symmetric and setting

$$\tilde{\boldsymbol{y}} = P_N \boldsymbol{y}$$

with an orthogonal matrix P_N introduced in Step 1, we see that

$$\nabla_{\boldsymbol{x}} u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = -\int_{\tilde{\Omega}_0} P_N^T \nabla_{\boldsymbol{z}} G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N), t)\, d\tilde{\boldsymbol{y}},$$

where P_N^T is the transposed matrix of P_N and

$$\nabla_{\boldsymbol{z}} G_{N,\lambda} = (\partial_{z_1} G_{N,\lambda}(\boldsymbol{z}, t), \ldots, \partial_{z_N} G_{N,\lambda}(\boldsymbol{z}, t)).$$

This implies that

$$\begin{aligned}
&\langle \nabla_{\boldsymbol{x}} u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), P_N^{-1}\boldsymbol{e}_j\rangle_N \\
&= -\int_{\widetilde{\Omega}_0} \langle P_N^T \nabla_{\boldsymbol{z}} G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N), t), P_N^{-1}\boldsymbol{e}_j\rangle_N \, d\tilde{\boldsymbol{y}} \\
&= -\int_{\widetilde{\Omega}_0} \partial_{z_j} G_{N,\lambda}(\tilde{\boldsymbol{y}} - (P_N\boldsymbol{x} + v\boldsymbol{e}_N), t) \, d\tilde{\boldsymbol{y}}.
\end{aligned}$$

Applying an argument similar to the above, we are led to (3.3). ■

By the Taylor expansion of $f(\boldsymbol{z}')$ at $\boldsymbol{z}' = \boldsymbol{0}'$, we see that

$$\begin{aligned}
&\psi(\boldsymbol{z}', v, t) \\
&= t^{-1/4}\Big\{ -v + f(\boldsymbol{0}') + \langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1} f(\boldsymbol{0}') t^{1/4} + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\boldsymbol{0}')}{2} t^{1/2} \\
&\qquad + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^3 f(\boldsymbol{0}')}{6} t^{3/4} + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^4 f(\boldsymbol{0}')}{24} t \\
&\qquad + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^5 f(\theta t^{1/4}\boldsymbol{z}')}{120} t^{5/4} \Big\} \\
&= -vt^{-1/4} + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\boldsymbol{0}')}{2} t^{1/4} + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^3 f(\boldsymbol{0}')}{6} t^{1/2} \\
&\quad + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^4 f(\boldsymbol{0}')}{24} t^{3/4} + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^5 f(\theta t^{1/4}\boldsymbol{z}')}{120} t
\end{aligned} \tag{3.4}$$

for some $\theta \in (0, 1)$. Note that $\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^n f$ is defined by

$$\begin{aligned}
\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^n f &= \Big(\sum_{i\in\Lambda} z_i \partial_{z_i}\Big)^n f \\
&= \sum_{d_1+\cdots+d_{N-1}=n} \binom{n}{d_1, \ldots, d_{N-1}} (z_1\partial_{z_1})^{d_1} \cdots (z_{N-1}\partial_{z_{N-1}})^{d_{N-1}} f,
\end{aligned}$$

where

$$\binom{n}{d_1, \ldots, d_{N-1}} = \frac{n!}{d_1! \cdots d_{N-1}!}.$$

According to (A2) and Appendix A below, $\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\boldsymbol{0}')$ is represented as

$$\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\boldsymbol{0}') = \sum_{i\in\Lambda} \kappa_i \zeta_i^2, \tag{3.5}$$

where $\boldsymbol{\zeta}' = P_{N-1}^{-1}\boldsymbol{z}'$ for an orthogonal matrix P_{N-1} of size $N-1$. This implies that

$$\big(\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f\big)^2(\boldsymbol{0}') = \sum_{i\in\Lambda} \kappa_i^2 \zeta_i^4 + 2\sum_{\substack{i_1,i_2\in\Lambda \\ i_1<i_2}} \kappa_{i_1}\kappa_{i_2}\zeta_{i_1}^2\zeta_{i_2}^2, \tag{3.6}$$

$$\big(\langle z', \nabla_{z'}\rangle_{N-1}^2 f\big)^3(\mathbf{0}') = \sum_{i\in\Lambda} \kappa_i^3 \zeta_i^6 + 3 \sum_{\substack{i_1,i_2\in\Lambda\\ i_1\neq i_2}} \kappa_{i_1}^2 \kappa_{i_2} \zeta_{i_1}^4 \zeta_{i_2}^2 + 6 \sum_{\substack{i_1,i_2,i_3\in\Lambda\\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} \zeta_{i_1}^2 \zeta_{i_2}^2 \zeta_{i_3}^2. \tag{3.7}$$

Since P_{N-1} is an orthogonal matrix, for any $F\colon \mathbb{R}^{N-1} \to \mathbb{R}$, it holds that

$$\int_{\mathbb{R}^{N-1}} F(z')dz' = \int_{\mathbb{R}^{N-1}} \widetilde{F}(\boldsymbol{\zeta}')\, d\boldsymbol{\zeta}',$$

where $\widetilde{F}(\boldsymbol{\zeta}') := F(P_{N-1}\boldsymbol{\zeta}') = F(z')$. Hereafter, we use the variable z' in the sense of the variable $\boldsymbol{\zeta}'$ as the above.

We introduce some notations. Set

$$B := \{i \in \Lambda \mid \beta_i \neq 0\}.$$

Then, for $\beta = (\beta_1, \ldots, \beta_{N-1}) \in \mathbb{Z}_+^{N-1}$, $(\beta_i)_{i\in B}$ denotes a multi-index in which only positive components are chosen. For example, if $\beta_{i_1} = p\,(> 0)$, $\beta_{i_2} = q\,(> 0)$, and $\beta_i = 0$ for $i \neq i_1, i_2$, $(\beta_i)_{i\in B}$ is represented as

$$(\beta_i)_{i\in B} = (p, q)_{i_1,i_2}.$$

Using the above notations, we define moments related to g_N as follows:

$$M_0 := \int_{\mathbb{R}^{N-1}} g_N(z', 0)\, dz', \quad M^{\ell,m}_{(\beta_i)_{i\in B}} := \int_{\mathbb{R}^{N-1}} \big\{(-\partial_{z_N}^2)^\ell (-\Delta_z)^m g_N(z', 0)\big\} (z')^\beta dz'$$

for $\ell, m \in \mathbb{Z}_+$ and $\beta = (\beta_1, \ldots, \beta_{N-1}) \in \mathbb{Z}_+^{N-1}$. We calculate the explicit values of some moments.

Lemma 3.3. *Let $\ell, m \in \mathbb{Z}_+$ and $\beta = (\beta_1, \ldots, \beta_{N-1}) \in \mathbb{Z}_+^{N-1}$ with $|\beta| := \beta_1 + \cdots + \beta_{N-1} \leq 6$. Then,* (i)*,* (ii)*, and* (iii) *hold as follows.*

(i) $M_0 = c_1 L_0$.

(ii) *If $|\beta|$ is odd, $M^{\ell,m}_{(\beta_i)_{i\in B}} = 0$.*

(iii) *For $i_0, i_1, i_2, i_3 \in \Lambda$ with $i_1 < i_2 < i_3$,*

$$M^{0,0}_{(\beta_i)_{i\in B}} = \begin{cases} 4c_1 L_2 & ((\beta_i)_{i\in B} = (2)_{i_0}),\\ -12 c_1 L_0 & ((\beta_i)_{i\in B} = (4)_{i_0}),\\ -4 c_1 L_0 & ((\beta_i)_{i\in B} = (2,2)_{i_1,i_2}),\end{cases}$$

$$M^{1,0}_{(\beta_i)_{i\in B}} = \begin{cases} -60 c_1 L_0 & ((\beta_i)_{i\in B} = (6)_{i_0}),\\ -12 c_1 L_0 & ((\beta_i)_{i\in B} = (4,2)_{i_1,i_2}),\\ -4 c_1 L_0 & ((\beta_i)_{i\in B} = (2,2,2)_{i_1,i_2,i_3}),\end{cases}$$

$$M^{0,1}_{(2)_{i_0}} = -c_1 L_0.$$

Proof. Step 1. We first prove (i). It follows from Lemma 2.7 that for $\beta \in \mathbb{Z}_+^{N-1}$

$$\int_{\mathbb{R}^{N-1}} \{(-\partial_{z_N}^2)^\ell(-\Delta_z)^m g_N(z',0)\}(z')^\beta\,dz'$$
$$= c_1\sum_{j=0}^m \binom{m}{j}\int_{\mathbb{R}^{N-1}} \mathcal{F}_{N-1}^{-1}\left[\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+j+\ell)}|\cdot|^{2(k+m-j)}e^{-|\cdot|^4}\right](z')(z')^\beta\,dz'.$$

Setting $(\ell,m)=(0,0)$ and $\beta=(0,\dots,0)\in\mathbb{Z}_+^{N-1}$, we have

$$M_0 = c_1\mathcal{F}_{N-1}\left[\mathcal{F}_{N-1}^{-1}\left[\sum_{k=0}^\infty \frac{(-2)^k}{m!}L_{2k}|\cdot|^{2k}e^{-|\cdot|^4}\right]\right](\mathbf{0}')$$
$$= c_1\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2k}|\xi'|^{2k}e^{-|\xi'|^4}\Big|_{\xi'=\mathbf{0}'} = c_1L_0.$$

Step 2. We derive the precise form of $M^{\ell,m}_{(\beta_i)_{i\in B}}$ and prove (ii). It is observed that

$$\int_{\mathbb{R}^{N-1}} \mathcal{F}_{N-1}^{-1}\left[\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+j+\ell)}|\cdot|^{2(k+m-j)}e^{-|\cdot|^4}\right](z')(z')^\beta\,dz'$$
$$=(-\mathbf{i})^{-|\beta|}\int_{\mathbb{R}^{N-1}} \mathcal{F}_{N-1}^{-1}\left[\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+j+\ell)}|\cdot|^{2(k+m-j)}e^{-|\cdot|^4}\right](z')(-\mathbf{i}z')^\beta\,dz'$$
$$=\mathbf{i}^{|\beta|}\int_{\mathbb{R}^{N-1}} \mathcal{F}_{N-1}^{-1}\left[\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+j+\ell)}D_{\xi'}^\beta\big(|\xi'|^{2(k+m-j)}e^{-|\xi'|^4}\big)\right](z')\,dz'$$
$$=\mathbf{i}^{|\beta|}\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+j+\ell)}D_{\xi'}^\beta\big(|\xi'|^{2(k+m-j)}e^{-|\xi'|^4}\big)(\mathbf{0}').$$

This implies that

$$M^{\ell,m}_{(\beta_i)_{i\in B}} = c_1\mathbf{i}^{|\beta|}\sum_{j=0}^m\binom{m}{j}\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+j+\ell)}D_{\xi'}^\beta\big(|\xi'|^{2(k+m-j)}e^{-|\xi'|^4}\big)(\mathbf{0}').$$

In particular, we see that

$$M^{0,0}_{(\beta_i)_{i\in B}} = c_1\mathbf{i}^{|\beta|}\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2k}D_{\xi'}^\beta\big(|\xi'|^{2k}e^{-|\xi'|^4}\big)(\mathbf{0}'), \tag{3.8}$$

$$M^{1,0}_{(\beta_i)_{i\in B}} = c_1\mathbf{i}^{|\beta|}\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+1)}D_{\xi'}^\beta\big(|\xi'|^{2k}e^{-|\xi'|^4}\big)(\mathbf{0}'), \tag{3.9}$$

$$M^{0,1}_{(\beta_i)_{i\in B}} = c_1\mathbf{i}^{|\beta|}\Big\{\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2k}D_{\xi'}^\beta\big(|\xi'|^{2(k+1)}e^{-|\xi'|^4}\big)(\mathbf{0}')$$
$$+\sum_{k=0}^\infty \frac{(-2)^k}{k!}L_{2(k+1)}D_{\xi'}^\beta\big(|\xi'|^{2k}e^{-|\xi'|^4}\big)(\mathbf{0}')\Big\}. \tag{3.10}$$

Since $M^{\ell,m}_{(\beta_i)_{i\in B}}$ is real-valued by its definition and $\mathbf{i}^{|\beta|} = \pm\mathbf{i}$ if $|\beta|$ is odd, we easily obtain (ii).

Step 3. We show that (iii) holds. By virtue of the Leibniz rule, we have

$$D^{\beta}_{\boldsymbol{\xi}'}\big(|\boldsymbol{\xi}'|^{2(k+j)}e^{-|\boldsymbol{\xi}'|^4}\big) = \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\big(D^{\gamma}|\boldsymbol{\xi}'|^{2(k+j)}\big)\big(D^{\beta-\gamma}e^{-|\boldsymbol{\xi}'|^4}\big)$$

for $j = 0, 1$, where $\gamma = (\gamma_1,\dots,\gamma_{N-1}) \in \mathbb{Z}^{N-1}_{+}$ and

$$\binom{\beta}{\gamma} = \binom{\beta_1}{\gamma_1}\cdots\binom{\beta_{N-1}}{\gamma_{N-1}}, \quad \binom{\beta_i}{\gamma_i} = \frac{\beta_i!}{\gamma_i!(\beta_i-\gamma_i)!}.$$

Note that $\gamma \le \beta$ is defined as $\gamma_i \le \beta_i\,(i = 1,\dots,N-1)$. Since

$$|\boldsymbol{\xi}'|^{2(k+j)} = (\xi_1^2+\cdots+\xi_{N-1}^2)^{k+j} = \sum_{d_1+\cdots+d_{N-1}=k+j}\binom{k+j}{d_1,\dots,d_{N-1}}\xi_1^{2d_1}\cdots\xi_{N-1}^{2d_{N-1}},$$

we see that

$$\begin{aligned}
&D^{\gamma}_{\boldsymbol{\xi}'}(|\boldsymbol{\xi}'|^{2(k+j)})(\mathbf{0}')\\
&= \sum_{d_1+\cdots+d_{N-1}=k+j}\binom{k+j}{d_1,\dots,d_{N-1}}\big(\partial^{\gamma_1}_{\xi_1}\xi_1^{2d_1}\big)\cdots\big(\partial^{\gamma_{N-1}}_{\xi_{N-1}}\xi_{N-1}^{2d_{N-1}}\big)\Big|_{\boldsymbol{\xi}'=\mathbf{0}'}\\
&= \begin{cases}\binom{k+j}{\gamma_1/2,\dots,\gamma_{N-1}/2}\gamma_1!\cdots\gamma_{N-1}! & (\gamma_i \text{ is even}, \gamma_1+\cdots+\gamma_{N-1} = 2(k+j)),\\ 0 & (\text{otherwise}).\end{cases}
\end{aligned}$$

This implies that

$$\begin{aligned}
&D^{\beta}_{\boldsymbol{\xi}'}\big(|\boldsymbol{\xi}'|^{2(k+j)}e^{-|\boldsymbol{\xi}'|^4}\big)(\mathbf{0}')\\
&\quad= \sum_{\substack{2\sigma_1\le\beta_1,\dots,2\sigma_{N-1}\le\beta_{N-1},\\ \sigma_1+\cdots+\sigma_{N-1}=k+j}}\binom{\beta_1}{2\sigma_1}\cdots\binom{\beta_{N-1}}{2\sigma_{N-1}}\binom{k+j}{\sigma_1,\dots,\sigma_{N-1}}\\
&\qquad\cdot(2\sigma_1)!\cdots(2\sigma_{N-1})!\big(\partial^{\beta_1-2\sigma_1}_{\xi_1}\cdots\partial^{\beta_{N-1}-2\sigma_{N-1}}_{\xi_{N-1}}e^{-|\boldsymbol{\xi}'|^4}\big)(\mathbf{0}')\\
&\quad= \sum_{\substack{2\sigma_1\le\beta_1,\dots,2\sigma_{N-1}\le\beta_{N-1},\\ \sigma_1+\cdots+\sigma_{N-1}=k+j}}\frac{\beta_1!\cdots\beta_{N-1}!}{(\beta_1-2\sigma_1)!\cdots(\beta_{N-1}-2\sigma_{N-1})!}\binom{k+j}{\sigma_1,\dots,\sigma_{N-1}}\\
&\qquad\cdot\big(\partial^{\beta_1-2\sigma_1}_{\xi_1}\cdots\partial^{\beta_{N-1}-2\sigma_{N-1}}_{\xi_{N-1}}e^{-|\boldsymbol{\xi}'|^4}\big)(\mathbf{0}').
\end{aligned}\tag{3.11}$$

Here, it follows that

$$\begin{aligned}
&\partial_{\xi_i}e^{-|\boldsymbol{\xi}'|^4}\big|_{\boldsymbol{\xi}'=\mathbf{0}} = \partial_{\xi_{i_1}}\partial_{\xi_{i_2}}e^{-|\boldsymbol{\xi}'|^4}\big|_{\boldsymbol{\xi}'=\mathbf{0}} = \partial_{\xi_{i_1}}\partial_{\xi_{i_2}}\partial_{\xi_{i_3}}e^{-|\boldsymbol{\xi}'|^4}\big|_{\boldsymbol{\xi}'=\mathbf{0}} = 0,\\
&\partial_{\xi_{i_1}}\partial_{\xi_{i_2}}\partial_{\xi_{i_3}}\partial_{\xi_{i_4}}e^{-|\boldsymbol{\xi}'|^4}\big|_{\boldsymbol{\xi}'=\mathbf{0}} = -8(\delta_{i_1i_2}\delta_{i_3i_4}+\delta_{i_1i_3}\delta_{i_2i_4}+\delta_{i_1i_4}\delta_{i_2i_3}),\\
&\partial_{\xi_{i_1}}\partial_{\xi_{i_2}}\partial_{\xi_{i_3}}\partial_{\xi_{i_4}}\partial_{\xi_{i_5}}e^{-|\boldsymbol{\xi}'|^4}\big|_{\boldsymbol{\xi}'=\mathbf{0}} = \partial_{\xi_{i_1}}\partial_{\xi_{i_2}}\partial_{\xi_{i_3}}\partial_{\xi_{i_4}}\partial_{\xi_{i_5}}\partial_{\xi_{i_6}}e^{-|\boldsymbol{\xi}'|^2}\big|_{\boldsymbol{\xi}'=\mathbf{0}} = 0.
\end{aligned}$$

Therefore, we obtain

$$\left(\partial_{\xi_1}^{\beta_1-2\sigma_1}\cdots\partial_{\xi_{N-1}}^{\beta_{N-1}-2\sigma_{N-1}}e^{-|\boldsymbol{\xi}'|^4}\right)(\mathbf{0}') = \begin{cases} 1 & (\beta_i-2\sigma_i=0\ (i=1,\dots,N-1)),\\ -4! & (\beta_{i_0}-2\sigma_{i_0}=4,\ \beta_i-2\sigma_i=0(i\neq i_0)),\\ -8 & (\beta_i-2\sigma_i=2(i=i_1,i_2),\ \beta_i-2\sigma_i=0(i\neq i_1,i_2)),\\ 0 & (\text{otherwise}) \end{cases} \tag{3.12}$$

for $|\beta|\le 6$ and $i_0,i_1,i_2,i_3\in\Lambda$ with $i_1<i_2<i_3$. Thus, (3.11) and (3.12) imply that

$$D_{\xi'}^{\beta}\left(|\boldsymbol{\xi}'|^{2(k+j)}e^{-|\boldsymbol{\xi}'|^4}\right)(\mathbf{0}') = \begin{cases} 2 & ((\beta_i)_{i\in B}=(2)_{i_0},\ (k,j)=(1,0),(0,1)),\\ -4! & ((\beta_i)_{i\in B}=(4)_{i_0},\ (k,j)=(0,0)),\\ 4! & ((\beta_i)_{i\in B}=(4)_{i_0},\ (k,j)=(2,0),(1,1)),\\ -8 & ((\beta_i)_{i\in B}=(2,2)_{i_1,i_2},\ (k,j)=(0,0)),\\ 8 & ((\beta_i)_{i\in B}=(2,2)_{i_1,i_2},\ (k,j)=(2,0),(1,1)),\\ -6! & ((\beta_i)_{i\in B}=(6)_{i_0},\ (k,j)=(1,0),(0,1))\\ 6! & ((\beta_i)_{i\in B}=(6)_{i_0},\ (k,j)=(3,0),(2,1)),\\ -6\cdot 4! & ((\beta_i)_{i\in B}=(4,2)_{i_1,i_2},\ (k,j)=(1,0),(0,1)),\\ 6\cdot 4! & ((\beta_i)_{i\in B}=(4,2)_{i_1,i_2},\ (k,j)=(3,0),(2,1)),\\ -2\cdot 4! & ((\beta_i)_{i\in B}=(2,2,2)_{i_1,i_2,i_3},\ (k,j)=(1,0),(0,1)),\\ 2\cdot 4! & ((\beta_i)_{i\in B}=(2,2,2)_{i_1,i_2,i_3},\ (k,j)=(3,0),(2,1)),\\ 0 & (\text{otherwise}) \end{cases} \tag{3.13}$$

for $|\beta|\le 6$. Furthermore, by (3.8), (3.9), (3.10), and (3.13), we have

$$M_{(\beta_i)_{i\in B}}^{0,0} = \begin{cases} 4c_1L_2 & ((\beta_i)_{i\in B}=(2)_{i_0}),\\ -4!c_1(L_0-2L_4) & ((\beta_i)_{i\in B}=(4)_{i_0}),\\ -8c_1(L_0-2L_4) & ((\beta_i)_{i\in B}=(2,2)_{i_1,i_2}), \end{cases}$$

$$M_{(\beta_i)_{i\in B}}^{1,0} = \begin{cases} -6!c_1\left(2L_4-\frac{8}{3!}L_8\right) & ((\beta_i)_{i\in B}=(6)_{i_0}),\\ -6\cdot 4!c_1\left(2L_4-\frac{8}{3!}L_8\right) & ((\beta_i)_{i\in B}=(4,2)_{i_1,i_2}),\\ -2\cdot 4!c_1\left(2L_4-\frac{8}{3!}L_8\right) & ((\beta_i)_{i\in B}=(2,2,)_{i_1,i_2,i_3}), \end{cases}$$

$$M_{(2)_{i_0}}^{0,1} = -2c_1(L_0-2L_4).$$

Since it follows from (2.7) that

$$L_4=\frac{1}{2}\Gamma\left(\frac{5}{4}\right)=\frac{1}{8}\Gamma\left(\frac{1}{4}\right)=\frac{1}{4}L_0,\quad L_8=\frac{1}{2}\Gamma\left(\frac{9}{4}\right)=\frac{5}{32}\Gamma\left(\frac{1}{4}\right)=\frac{5}{16}L_0,$$

we are led to (iii). ∎

Combining Lemma 3.3 with Theorem 3.1, we obtain the following precise asymptotic expansion of the solution to (1.3).

Theorem 3.4. *Let γ be a constant obtained in Lemma 2.5 and take $v = Vt$ in Theorem 3.1 for $V \in \mathbb{R}$. Then, an asymptotic expansion of a solution to (1.3) is given by*

$$\begin{aligned} &u(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) \\ &\quad = \frac{1}{2} + c_1\Gamma\left(\frac{3}{4}\right)Ht^{1/4} \\ &\quad\quad - \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\left\{V + \frac{1}{2}\left(\Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3\in\Lambda \\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H\right)\right\}t^{3/4} \\ &\quad\quad + O_{\partial\Omega_0}(t) \end{aligned} \tag{3.14}$$

for any $\boldsymbol{x} \in \partial\Omega_0$, $\lambda \in \mathbb{R}$, and $t > 0$ satisfying $|\lambda|t^{1/2} \le \gamma$ and $|V|t \le \delta_0$.

Proof. Step 1. Set

$$\begin{aligned} \Xi^{(0)} &:= \int_{\mathbb{R}^{N-1}} \int_0^{\psi(z',Vt,t)} g_N(z)dz_N dz', \\ \Xi^{(1)} &:= \int_{\mathbb{R}^{N-1}} \int_0^{\psi(z',Vt,t)} (-\Delta_z)g_N(z)dz_N dz'. \end{aligned}$$

Then, we prove that

$$u(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} + \Xi^{(0)} - \lambda t^{1/2}\Xi^{(1)} + O_{\partial\Omega_0}(t). \tag{3.15}$$

According to (3.2), we see that

$$\begin{aligned} &u(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) \\ &\quad = \frac{1}{2} + \Xi^{(0)} - \lambda t^{1/2}\Xi^{(1)} \\ &\quad\quad + \sum_{m=2}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_0^{\psi(z',Vt,t)} (-\Delta_z)^m g_N(z)dz_N dz' + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}). \end{aligned}$$

By calculation similar to the proof of Lemma 2.5, we are able to find $C > 0$ and $\gamma > 0$ such that

$$\begin{aligned} &\left|\sum_{m=2}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_0^{\psi(z',v,t)} (-\Delta_z)^m g_N(z)dz_N dz'\right| \\ &\quad \le \sum_{m=2}^{\infty} \frac{|\lambda|^m t^{m/2}}{m!} \int_{\mathbb{R}^N} |(-\Delta_z)^m g_N(z)|dz \le C|\lambda|^2 t \end{aligned}$$

for $|\lambda|t^{1/2} \le \gamma$. This implies (3.15).

Step 2. We derive the precise asymptotics for $\Xi^{(0)}$ and $\Xi^{(1)}$. Applying the Taylor's expansion of $g_N(z)$ and $(-\Delta_z)g_N(z)$ with respect to the variable z_N and taking account of

$$\partial_{z_N} g_N(z',0) = \partial_{z_N}^3 g_N(z',0) = 0, \quad \partial_{z_N}(-\Delta_z)g_N(z',0) = 0, \tag{3.16}$$

which is obtained with the help of (2.6), we have

$$g_N(z) = g_N(z',0) + \frac{1}{2}\partial_{z_N}^2 g_N(z',0)z_N^2 + \frac{1}{4!}\partial_{z_N}^4 g_N(z',\theta_0 z_N)z_N^4,$$
$$(-\Delta_z)g_N(z) = (-\Delta_z)g_N(z',0) + \frac{1}{2}\partial_{z_N}^2(-\Delta_z)g_N(z',\theta_1 z_N)z_N^2$$

for some $\theta_0, \theta_1 \in (0,1)$. These imply that

$$\begin{aligned}\Xi^{(0)} &= \int_{\mathbb{R}^{N-1}} g_N(z',0)\psi(z',Vt,t)dz' - \frac{1}{6}\int_{\mathbb{R}^{N-1}}(-\partial_{z_N}^2)g_N(z',0)\{\psi(z',Vt,t)\}^3 dz' \\ &\quad + \frac{1}{4!}\int_{\mathbb{R}^{N-1}}\int_0^{\psi(z',Vt,t)}(-\partial_{z_N}^2)^2 g_N(z',\theta_0 z_N)z_N^4 dz_N dz', \\ \Xi^{(1)} &= \int_{\mathbb{R}^{N-1}}(-\Delta_z)g_N(z',0)\psi(z',Vt,t)\,dz' \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^{N-1}}\int_0^{\psi(z',Vt,t)}(-\partial_{z_N}^2)(-\Delta_z)g_N(z',\theta_1 z_N)z_N^2 dz_N dz'.\end{aligned}$$

Using Corollary 2.3, we observe that

$$\left|\int_{\mathbb{R}^{N-1}}\int_0^{\psi(z',Vt,t)}(-\partial_{z_N}^2)^2 g_N(z',\theta_0 z_N)z_N^4 dz_N dz'\right| = O_{\partial\Omega_0}(t^{5/4}),$$
$$\left|\int_{\mathbb{R}^{N-1}}\int_0^{\psi(z',Vt,t)}(-\partial_{z_N}^2)(-\Delta_z)g_N(z',\theta_1 z_N)z_N^2 dz_N dz'\right| = O_{\partial\Omega_0}(t^{3/4}).$$

Then, it follows from (3.4), (3.5), (3.7) and Lemma 3.3 that

$$\begin{aligned}\Xi^{(0)} &= \frac{1}{2}\int_{\mathbb{R}^{N-1}} g_N(z',0)\sum_{i_0\in\Lambda} z_{i_0}^2\kappa_{i_0}dz't^{1/4} - \int_{\mathbb{R}^{N-1}} g_N(z',0)Vdz't^{3/4} \\ &\quad + \frac{1}{24}\int_{\mathbb{R}^{N-1}} g_N(z',0)\Bigg(\sum_{i_0\in\Lambda} z_{i_0}^4\partial_{z_{i_0}}^4 f + 6\sum_{\substack{i_1,i_2\in\Lambda\\ i_1<i_2}} z_{i_1}^2 z_{i_2}^2\partial_{z_{i_1}}^2\partial_{z_{i_2}}^2 f\Bigg)dz't^{3/4} \\ &\quad - \frac{1}{48}\int_{\mathbb{R}^{N-1}}(-\partial_{z_N}^2)g_N(z',0)\Bigg(\sum_{i_0\in\Lambda} z_{i_0}^6\kappa_{i_0}^3 + 3\sum_{\substack{i_1,i_2\in\Lambda\\ i_1\neq i_2}} z_{i_1}^4 z_{i_2}^2\kappa_{i_1}^2\kappa_{i_2} \\ &\qquad\qquad + 6\sum_{\substack{i_1,i_2,i_3\in\Lambda\\ i_1<i_2<i_3}} z_{i_1}^2 z_{i_2}^2 z_{i_3}^2\kappa_{i_1}\kappa_{i_2}\kappa_{i_3}\Bigg)dz't^{3/4} + O_{\partial\Omega_0}(t)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i_0\in\Lambda} M^{0,0}_{(2)_{i_0}}\kappa_{i_0} t^{1/4} - M_0 V t^{3/4} \\
&\quad + \frac{1}{24}\Bigg(\sum_{i_0\in\Lambda} M^{0,0}_{(4)_{i_0}}\partial^4_{z_{i_0}} f + 6\sum_{\substack{i_1,i_2\in\Lambda\\ i_1<i_2}} M^{0,0}_{(2,2)_{i_1,i_2}}\partial^2_{z_{i_1}}\partial^2_{z_{i_2}} f\Bigg) t^{3/4} \\
&\quad - \frac{1}{48}\Bigg(\sum_{i_0\in\Lambda} M^{1,0}_{(6)_{i_0}}\kappa^3_{i_0} + 3\sum_{\substack{i_1,i_2\in\Lambda\\ i_1\neq i_2}} M^{1,0}_{(4,2)_{i_1,i_2}}\kappa^2_{i_1}\kappa_{i_2} \\
&\qquad\quad + 6\sum_{\substack{i_1,i_2,i_3\in\Lambda\\ i_1<i_2<i_3}} M^{1,0}_{(2,2,2)_{i_1,i_2,i_3}}\kappa_{i_1}\kappa_{i_2}\kappa_{i_3}\Bigg) t^{3/4} + O_{\partial\Omega_0}(t) \\
&= 2c_1 L_2 \sum_{i_0\in\Lambda}\kappa_{i_0} t^{1/4} \\
&\quad - c_1 L_0\Bigg\{V + \frac{1}{2}\Bigg(\sum_{i_0\in\Lambda}\partial^4_{z_{i_0}} f + \sum_{\substack{i_1,i_2\in\Lambda\\ i_1<i_2}}\partial^2_{z_{i_1}}\partial^2_{z_{i_2}} f - \frac{5}{2}\sum_{i_0\in\Lambda}\kappa^3_{i_0} - \frac{3}{2}\sum_{\substack{i_1,i_2\in\Lambda\\ i_1\neq i_2}}\kappa^2_{i_1}\kappa_{i_2} \\
&\qquad\qquad - \sum_{\substack{i_1,i_2,i_3\in\Lambda\\ i_1<i_2<i_3}}\kappa_{i_1}\kappa_{i_2}\kappa_{i_3}\Bigg)\Bigg\} t^{3/4} + O_{\partial\Omega_0}(t),
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{(1)} &= \frac{1}{2}\int_{\mathbb{R}^{N-1}} (-\Delta_z) g_N(z',0)\sum_{i_0\in\Lambda} z^2_{i_0}\kappa_{i_0}\, dz'\, t^{1/4} + O_{\partial\Omega_0}(t^{3/4}) \\
&= -\frac{1}{2}c_1 L_0\sum_{i_0\in\Lambda}\kappa_{i_0} t^{1/4} + O_{\partial\Omega_0}(t^{3/4}).
\end{aligned}
$$

Step 3. From the results in Step 1 and Step 2, we prove (3.14). Taking account of $L_0 = \Gamma(1/4)/2$ and $L_2 = \Gamma(3/4)/2$, we obtain

$$
\begin{aligned}
&u(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) \\
&= \frac{1}{2} + \Xi^{(0)} - \lambda t^{1/2}\Xi^{(1)} + O_{\partial\Omega_0}(t) \\
&= \frac{1}{2} + c_1\Gamma\left(\frac{3}{4}\right)\sum_{i_0\in\Lambda}\kappa_{i_0} t^{1/4} \\
&\quad - \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\Bigg\{V + \frac{1}{2}\Bigg(\sum_{i_0\in\Lambda}\partial^4_{z_{i_0}} f + 2\sum_{\substack{i_1,i_2\in\Lambda\\ i_1<i_2}}\partial^2_{z_{i_1}}\partial^2_{z_{i_2}} f - \frac{5}{2}\sum_{i_0\in\Lambda}\kappa^3_{i_0} - \frac{3}{2}\sum_{\substack{i_1,i_2\in\Lambda\\ i_1\neq i_2}}\kappa^2_{i_1}\kappa_{i_2} \\
&\qquad\qquad - \sum_{\substack{i_1,i_2,i_3\in\Lambda\\ i_1<i_2<i_3}}\kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda\sum_{i_0\in\Lambda}\kappa_{i_0}\Bigg)\Bigg\} t^{3/4} + O_{\partial\Omega_0}(t).
\end{aligned}
$$

Referring to Appendix A below, we see that

$$H = \sum_{i \in \Lambda} \kappa_i, \quad |A|^2 = \sum_{i \in \Lambda} \kappa_i^2,$$

$$\begin{aligned}\Delta_g H + H|A|^2 - \frac{1}{2}H^3 &= \sum_{i_0 \in \Lambda} \partial_{z_{i_0}}^4 f + 2 \sum_{\substack{i_1, i_2 \in \Lambda \\ i_1 < i_2}} \partial_{z_{i_1}}^2 \partial_{z_{i_2}}^2 f - \frac{5}{2} \sum_{i \in \Lambda} \kappa_i^3 \\ &\quad - \frac{3}{2} \sum_{\substack{i_1, i_2 \in \Lambda \\ i_1 \neq i_2}} \kappa_{i_1}^2 \kappa_{i_2} - 3 \sum_{\substack{i_1, i_2, i_3 \in \Lambda \\ i_1 < i_2 < i_3}} \kappa_{i_1} \kappa_{i_2} \kappa_{i_3}.\end{aligned}$$

Consequently, we have (3.14). ∎

4. A thresholding algorithm for the flow by (1.1)

In this section, we first introduce a threshold function based on the asymptotic expansion obtained in Theorem 3.4 and propose the thresholding algorithm. After that, we justify its algorithm. Throughout this section, we assume that $\Omega_0 \subset \mathbb{R}^N$ is a compact set and $\partial\Omega_0$ is of class C^5.

4.1. A thresholding algorithm

Let u be a solution to (1.3). Set

$$u_a(\boldsymbol{x}, t) := u(\boldsymbol{x}, a^4 t)$$

for $\boldsymbol{x} \in \mathbb{R}^N$ and $a > 0$. Then, it follows from Theorem 3.4 that for $V \in \mathbb{R}$

$$\begin{aligned}&u_a(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) \\ &= \frac{1}{2} + c_1 \Gamma\left(\frac{3}{4}\right) aHt^{1/4} \\ &\quad - \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\frac{V}{a} + \frac{a^3}{2}\left(\Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2 \sum_{\substack{i_1, i_2, i_3 \in \Lambda \\ i_1 < i_2 < i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H\right)\right\} t^{3/4} \\ &\quad + O_{\partial\Omega_0}(t).\end{aligned}$$

Hence, we see that

$$\begin{aligned}&u_{3a}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - 3u_{2a}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) + 3u_a(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - \frac{1}{2} \\ &= -\frac{11c_1}{12a}\Gamma\left(\frac{1}{4}\right) \\ &\quad \cdot \left\{V + \frac{18a^4}{11}\left(\Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2 \sum_{\substack{i_1, i_2, i_3 \in \Lambda \\ i_1 < i_2 < i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H\right)\right\} t^{3/4} \\ &\quad + O_{\partial\Omega_0}(t).\end{aligned}$$

Choosing $a > 0$ such that $18a^4/11 = 1$, we are led to

$$\begin{aligned} &u_{3a}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - 3u_{2a}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) + 3u_a(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - \frac{1}{2} \\ &\quad = -\frac{11c_1}{12a}\Gamma\Big(\frac{1}{4}\Big)\Big(V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3\in\Lambda \\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H\Big)t^{3/4} \\ &\qquad + O_{\partial\Omega_0}(t). \end{aligned} \tag{4.1}$$

From the above observation, let us introduce a threshold function and a new set generated by its function. Define a threshold function $U(\boldsymbol{x}, t)$ as

$$U(\boldsymbol{x}, t) := u_{3a}(\boldsymbol{x}, t) - 3u_{2a}(\boldsymbol{x}, t) + 3u_a(\boldsymbol{x}, t) \tag{4.2}$$

for $a > 0$ satisfying $18a^4/11 = 1$ and set

$$\Omega(t) := \Big\{\boldsymbol{x} \in \mathbb{R}^N \mid U(\boldsymbol{x}, t) \geq \frac{1}{2}\Big\}. \tag{4.3}$$

For any $\boldsymbol{x} \in \partial\Omega_0$ and small $t > 0$, we define $V = V(\boldsymbol{x}, t)$ by

$$\boldsymbol{x} + V(\boldsymbol{x}, t)t\boldsymbol{n}(\boldsymbol{x}) \in \partial\Omega(t). \tag{4.4}$$

Then, setting $\boldsymbol{y}(\boldsymbol{x}, t) := \boldsymbol{x} + V(\boldsymbol{x}, t)t\boldsymbol{n}(\boldsymbol{x})$, we notice that

$$V(\boldsymbol{x}, t) = -\frac{d(\boldsymbol{y}(\boldsymbol{x}, t), \partial\Omega_0)}{t}, \quad |d(\boldsymbol{y}(\boldsymbol{x}, t), \partial\Omega_0)| = |\boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{x}|$$

for all $\boldsymbol{x} \in \partial\Omega_0$ and small $t > 0$. Hence, we can regard V as the outward normal velocity from $\partial\Omega_0$ to $\partial\Omega(t)$. Here, $d(\boldsymbol{y}, \partial\Omega_0)$ is defined by (3.1).

We assume that

$$\|V\|_{L^\infty(\partial\Omega_0\times(0,t_0))} < \infty \quad \text{for some } t_0 > 0, \tag{4.5}$$

and $U(\boldsymbol{x} + V(\boldsymbol{x}, t)t\boldsymbol{n}(\boldsymbol{x}), t) = 1/2$. Then, by (4.1),

$$V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3\in\Lambda \\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H = O_{\partial\Omega_0}(t^{1/4}),$$

where $g = g(\boldsymbol{x})$ and $\kappa_i = \kappa_i(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial\Omega_0$. This implies that V must be closed to the L^2-gradient $-\nabla_{L^2}\mathcal{E}_\lambda^N(\partial\Omega_0)$. We emphasize that the assumption (4.5) is actually valid as we show in the next subsection.

Based on the above argument, let us derive a thresholding algorithm for (1.1). First, we solve the initial value problem (1.3) for the initial function $\chi_{\Omega_0}(\boldsymbol{x})$ and let u^0 be the corresponding solution. Define a threshold function $U^0(\boldsymbol{x}, t)$ as (4.2) and a set $\Omega^0(t)$ as (4.3). Fix a time step $h > 0$ and define $\Omega_1 := \Omega^0(h)$. As the second step, we solve the problem (1.3) with Ω_1 replacing Ω_0 and define $\Omega^1(t)$ as the set of (4.3) with u replaced by the solution to the problem (1.3) with the new initial function $\chi_{\Omega_1}(\boldsymbol{x})$. Repeat this procedure to obtain a sequence $\{\Omega_k\}_{k\in\mathbb{Z}_+}$ of compact sets in $\mathbb{R}^N$. Then, setting

$$\widehat{\Omega}^h(t) = \Omega_k \quad \text{for } t \in [kh, (k+1)h)(k \in \mathbb{Z}_+)$$

and letting $h \to 0$, we can expect that at least formally there is a limit flow $\{\partial\widehat{\Omega}(t)\}_{t\geq 0}$ of $\{\partial\widehat{\Omega}^h(t)\}_{t\geq 0}$ as $h \to 0$, whose boundary moves by (1.1) with $\widehat{\Gamma}(t) = \partial\widehat{\Omega}(t)$. Indeed, since $\widehat{V}^h(\boldsymbol{x}, t)$, given by

$$\boldsymbol{x} + \widehat{V}^h(\boldsymbol{x}, t)h\boldsymbol{n}(\boldsymbol{x}) \in \partial\Omega_{k+1} \quad \text{for } t \in [kh, (k+1)h)\ (k \in \mathbb{Z}_+) \text{ and } \boldsymbol{x} \in \partial\Omega_k,$$

can be regarded as a normal velocity of $\partial\widehat{\Omega}^h(t)$, the above observation implies that the limit flow of $\{\partial\widehat{\Omega}^h(t)\}_{t\geq 0}$ formally moves by (1.1).

Remark 4.1. Set

$$E_\alpha(\boldsymbol{x}, t) := \frac{1}{(t^\alpha)^N}\rho\left(\frac{|\boldsymbol{x}|}{t^\alpha}\right), \quad \rho(s) := \frac{1}{(4\pi)^{N/2}}\exp\left(-\frac{s^2}{4}\right),$$
$$\psi_\alpha(\boldsymbol{z}', v, t) := t^{-\alpha}\{-v + f(t^\alpha \boldsymbol{z}')\}.$$

Then, the expansion of $\psi_\alpha(\boldsymbol{z}', Vt, t)$ is given by

$$\begin{aligned}\psi_\alpha(\boldsymbol{z}', Vt, t) = -Vt^{1-\alpha} &+ \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\boldsymbol{0}')}{2}t^\alpha + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^3 f(\boldsymbol{0}')}{6}t^{2\alpha} \\ &+ \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^4 f(\boldsymbol{0}')}{24}t^{3\alpha} + \frac{\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^5 f(\theta t^\alpha \boldsymbol{z}')}{120}t^{4\alpha}\end{aligned}$$

for some $\theta \in (0, 1)$. Let $\nabla_{L^2}\mathcal{E}_0^N(\Gamma)$ be the L^2-gradient of $\mathcal{E}_0^N(\Gamma)$ given by (1.2) with $\lambda = 0$, that is,

$$\nabla_{L^2}\mathcal{E}_0^N(\Gamma) = \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3\in\Lambda \\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3},$$

and set

$$w_\alpha(\boldsymbol{x}, t) := (E_\alpha(\cdot, t) * \chi_{\Omega_0})(\boldsymbol{x}).$$

Since $E_{1/2}(\boldsymbol{x}, t)$ is the Gauss kernel, $w_{1/2}$ is a solution to

$$\begin{cases} w_t = \Delta w & \text{in } \mathbb{R}^N \times (0, \infty), \\ w(\boldsymbol{x}, 0) = \chi_{\Omega_0}(\boldsymbol{x}) & \text{in } \mathbb{R}^N. \end{cases}$$

Here, the precise asymptotic expansion of $w_{1/2}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t)$ until the term of $t^{3/2}$ is represented as

$$\begin{aligned} &w_{1/2}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) \\ &\quad = \frac{1}{2} - c_1\sqrt{\pi}(V - H)t^{1/2} \\ &\qquad + c_1\sqrt{\pi}\left\{\frac{1}{2}\nabla_{L^2}\mathcal{E}_0^N(\Gamma) + \frac{1}{4}V(2|A|^2 + H^2) - \frac{1}{4}V^2 H + \frac{1}{12}V^3\right\}t^{3/2} + O_{\partial\Omega_0}(t^2). \end{aligned}$$

We remark that the precise asymptotic expansion until the term of $t^{1/2}$ is obtained in Evans [15] and its expansion is the basis of the BMO algorithm for the mean curvature flow (cf. Bence–Merriman–Osher [5]). On the other hand, considering the asymptotic expansion of $w_{1/4}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t)$, we have

$$\begin{aligned} w_{1/4}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} &+ c_1\sqrt{\pi}Ht^{1/4} \\ &- c_1\sqrt{\pi}\Big(V - \frac{1}{2}\nabla_{L^2}\mathcal{E}_0^N(\Gamma)\Big)t^{3/4} + O_{\partial\Omega_0}(t). \end{aligned} \tag{4.6}$$

The difference between (4.6) and our expansion

$$\begin{aligned} u(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} &+ c_1\Gamma\Big(\frac{3}{4}\Big)Ht^{1/4} \\ &- \frac{c_1}{2}\Gamma\Big(\frac{1}{4}\Big)\Big(V + \frac{1}{2}\nabla_{L^2}\mathcal{E}_0^N(\Gamma)\Big)t^{3/4} + O_{\partial\Omega_0}(t), \end{aligned}$$

which is given by (3.14) with $\lambda = 0$, is the sign before $\nabla_{L^2}\mathcal{E}_0^N(\Gamma)$. This difference is due to the fact that $w_{1/4}$ satisfies the second-order parabolic equation

$$w_t = (2\sqrt{t})^{-1}\Delta w,$$

whereas u fulfills the fourth-order parabolic equation $u_t = -\Delta^2 u$. The thresholding algorithms of the Willmore flow based on (4.6) are derived for $N = 2$ in Esedoḡlu–Ruuth–Tsai [14] and for $N = 3$ in Grzhibovskis–Heintz [20]. Since the difference explained above is essentially related to the parabolicity, the same threshold function as ours, that is, U given by (4.2), cannot be chosen if $w_{1/4}$ is used.

4.2. Properties of evolving sets

We give a justification of the argument in the previous subsection; more precisely, we prove that the assumption (4.5) is actually valid. In order to do so, we prepare several propositions and lemmas. In the following, we use the notation defined as in (4.2), (4.3), and (4.4).

Proposition 4.2. *There exist $K_* > 0$ and $t_* > 0$ such that*

$$\partial\Omega(t) \subset \{\boldsymbol{x} \in \mathbb{R}^N \mid |d(\boldsymbol{x}, \partial\Omega_0)| \le K_* t^{1/4}\}$$

for $t \in (0, t_*)$, where $d(\cdot, \partial\Omega_0)$ is the signed distance function to $\partial\Omega_0$ given by (3.1).

Proof. Step 1. Set $D_+(t, r) := \{\boldsymbol{x} \in \mathbb{R}^N \mid d(\boldsymbol{x}, \partial\Omega_0) > rt^{1/4}\}$ for $r > 0$. We show that there exist $t_+ > 0$ and $K_* > 0$ such that $D_+(t, K_*) \subset \Omega(t)$ for all $t \in (0, t_+)$. Recalling the definition (3.1) of the signed distance function, we can find

$$t_+ = t_+(r) > 0$$

satisfying $\emptyset \neq D_+(t,r) \subset \Omega_0$ for $t \in (0,t_+)$. Fix any $\boldsymbol{x} \in D_+(t,r)$. Taking account of $B_N(\boldsymbol{x}, rt^{1/4}/2) \subset \Omega_0$ for $t \in (0,t_+)$ and recalling (2.1), we see that for any $a > 0$

$$\begin{aligned}
|u_a(\boldsymbol{x},t) - 1| &= |(G_{N,\lambda}(\,\cdot\,, a^4 t) * \chi_{\Omega_0})(\boldsymbol{x}) - 1| \\
&\le \left| \int_{B_N(\boldsymbol{x}, rt^{1/4}/2)} G_{N,\lambda}(\boldsymbol{x}-\boldsymbol{y}, a^4 t)\, d\boldsymbol{y} - 1 \right| \\
&\quad + \int_{\Omega_0 \setminus B_N(\boldsymbol{x}, rt^{1/4}/2)} |G_{N,\lambda}(\boldsymbol{x}-\boldsymbol{y}, a^4 t)|\, d\boldsymbol{y} \\
&\le 2 \int_{\mathbb{R}^N \setminus B_N(\boldsymbol{x}, rt^{1/4}/2)} |G_{N,\lambda}(\boldsymbol{x}-\boldsymbol{y}, a^4 t)|\, d\boldsymbol{y}.
\end{aligned}$$

Here, it follows from Theorem 2.2 with $|\alpha| = 0$ and $m = 0$ that for $t \in (0,t_+)$

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, rt^{1/4}/2)} |G_{N,\lambda}(\boldsymbol{x}-\boldsymbol{y}, a^4 t)|\, d\boldsymbol{y} \\
&\quad \le \frac{C_1 e^{K|\lambda|^2 t}}{a^N t^{N/4}} \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, rt^{1/4}/2)} e^{-\mu(|\boldsymbol{x}-\boldsymbol{y}|^4/(a^4 t))^{1/3}}\, d\boldsymbol{y} \\
&\quad \le C_2 \int_{\mathbb{R}^N \setminus B(\boldsymbol{0}, r/2a)} e^{-\mu|\boldsymbol{z}|^{4/3}}\, d\boldsymbol{z}.
\end{aligned}$$

Using the polar coordinate, we have

$$\int_{\mathbb{R}^N \setminus B(\boldsymbol{0}, r/2a)} e^{-\mu|\boldsymbol{z}|^{4/3}}\, d\boldsymbol{z} = \omega_{N-1} \int_{r/2a}^{\infty} e^{-\mu\rho^{4/3}} \rho^{N-1}\, d\rho \le C_3 e^{-\mu_1 (r/a)^{4/3}},$$

where $C_3 := (3/4)(2/\mu)^{3N/4}\Gamma(3N/4)$ and $\mu_1 := \mu/2^{7/3}$. Thus, it is seen that

$$|u_a(\boldsymbol{x},t) - 1| \le C_4 e^{-\mu_1 (r/a)^{4/3}}.$$

By means of this inequality, we obtain

$$U(\boldsymbol{x},t) = u_{3a}(\boldsymbol{x},t) - 3u_{2a}(\boldsymbol{x},t) + 3u_a(\boldsymbol{x},t) \ge 1 - 7C_4 e^{-\mu_1 (r/3a)^{4/3}}$$

for $t \in (0,t_+)$. Taking $r = K_* > 0$ such that $7C_4 e^{-\mu_1 (K_*/3a)^{4/3}} < 1/2$, we conclude that

$$U(\boldsymbol{x},t) > \frac{1}{2}$$

for $t \in (0, t_+(K_*))$. This implies that $\boldsymbol{x} \in \operatorname{int}\Omega(t)$ so that $D_+(t, K_*) \subset \operatorname{int}\Omega(t)$ for $t \in (0, t_+(K_*))$.

Step 2. Set $D_-(t,r) := \{\boldsymbol{x} \in \mathbb{R}^N \mid d(\boldsymbol{x}, \partial\Omega_0) < -rt^{1/4}\}$ for $r > 0$. We prove that there exists $t_- > 0$ such that $D_-(t, K_*) \subset \mathbb{R}^N \setminus \Omega(t)$ for all $t \in (0, t_-)$, where K_* is a constant as in Step 1. Recalling (3.1) again, we are able to find a $t_- = t_-(r)$ such that

$$\emptyset \neq D_-(t,r) \subset \mathbb{R}^N \setminus \Omega_0$$

for $t \in (0, t_-)$. Fix any $\boldsymbol{x} \in D_-(t, r)$. Since $\Omega_0 \subset \mathbb{R}^N \setminus B_N(\boldsymbol{x}, rt^{1/4}/2)$ for $t \in (0, t_-)$, it follows that for any $a > 0$ and $t \in (0, t_-)$

$$\begin{aligned}|u_a(\boldsymbol{x}, t)| &= |(G_{N,\lambda}(\cdot, a^4 t) * \chi_{\Omega_0})(\boldsymbol{x})| \\ &\leq \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, rt^{1/4}/2)} |G_{N,\lambda}(\boldsymbol{x} - \boldsymbol{y}, a^4 t)| d\boldsymbol{y} \\ &\leq C_4 e^{-\mu_1 (r/a)^{4/3}}.\end{aligned}$$

Using this inequality, we have

$$U(\boldsymbol{x}, t) = u_{3a}(\boldsymbol{x}, t) - 3u_{2a}(\boldsymbol{x}, t) + 3u_a(\boldsymbol{x}, t) \leq 7C_4 e^{-\mu_1 (r/3a)^{4/3}}$$

for $t \in (0, t_-)$. Taking $r = K_* > 0$ as in Step 1, we obtain

$$U(\boldsymbol{x}, t) < \frac{1}{2}$$

for $t \in (0, t_-(K_*))$. This implies that $\boldsymbol{x} \in \mathbb{R}^N \setminus \Omega(t)$ so that $D_-(t, K_*) \subset \mathbb{R}^N \setminus \Omega(t)$ for $t \in (0, t_-(K_*))$.

Step 3. Define $t_* := \min\{t_+(K_*), t_-(K_*)\}$. Then, we see that

$$D_+(t, K_*) \subset \mathrm{int}\Omega(t), \quad D_-(t, K_*) \subset \mathbb{R}^N \setminus \Omega(t)$$

for $t \in (0, t_*)$. This leads to the desired result. ∎

Furthermore, we are able to derive the following refinement of Proposition 4.2.

Proposition 4.3. *For any $\varepsilon \in (0, 1)$, there exists $t_{*,\varepsilon} > 0$ such that*

$$\partial\Omega(t) \subset \{\boldsymbol{x} \in \mathbb{R}^N \mid |d(\boldsymbol{x}, \partial\Omega_0)| \leq \varepsilon t^{1/4}\}$$

for $t \in (0, t_{*,\varepsilon})$.

In order to prove this proposition, we need several preparations. Let us consider the case where $\lambda = 0$ and $\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N \leq 0\}$. Taking account of $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{e}_N$ and $f \equiv 0$ in this case, we see that for $\boldsymbol{x} \in \partial\Omega_0$, $v \in \mathbb{R}$, and $t > 0$

$$u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} + \int_{\mathbb{R}^{N-1}} \int_0^{-vt^{-1/4}} g_N(\boldsymbol{z})\, d\boldsymbol{z} = u(v\boldsymbol{e}_N, t).$$

Since $u(v\boldsymbol{e}_N, t)$ is the solution to (1.3) with $\lambda = 0$ and $\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N \leq 0\}$, $u(v\boldsymbol{e}_N, t)$ can be the solution to

$$\begin{cases} \tilde{u}_t = -\partial_{y_N}^4 \tilde{u} & \text{in } \mathbb{R} \times (0, \infty), \\ \tilde{u}(y_N, 0) = \chi_{(-\infty,0]}(y_N) = \begin{cases} 1 & \text{in } (-\infty, 0], \\ 0 & \text{in } \mathbb{R}^N \setminus (-\infty, 0]. \end{cases} \end{cases} \tag{4.7}$$

On the other hand, the solution $\tilde{u}(y_N, t)$ to (4.7) is represented as

$$\tilde{u}(y_N, t) = \frac{1}{2} + \int_0^{-y_N t^{-1/4}} g_1(z)\, dz.$$

By the uniqueness of the solution to (4.7), we obtain

$$u(v\boldsymbol{e}_N, t) = \tilde{u}(v, t) = \frac{1}{2} + \int_0^{-vt^{-1/4}} g_1(z)\, dz.$$

As a result, it follows that for $\boldsymbol{x} \in \partial\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = 0\}$

$$u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} + \int_{\mathbb{R}^{N-1}} \int_0^{-vt^{-1/4}} g_N(\boldsymbol{z})\, d\boldsymbol{z} = \frac{1}{2} + \int_0^{-vt^{-1/4}} g_1(r)\, dr. \quad (4.8)$$

We state several properties of the integration of

$$g_1(z) = \varphi_1(|z|)$$

based on Ferrero–Gazzola–Grunau [17] and Gazzola–Grunau [19]. For $n \in \mathbb{Z}_+$, set

$$\Phi_{N,n}(r) := \sum_{\ell=0}^{n} (-1)^\ell b_{N,\ell} r^{2\ell}, \quad b_{N,\ell} := \frac{1}{2^{N+1}\pi^{N/2}} \cdot \frac{\Gamma(\ell/2 + N/4)}{2^{2\ell}\Gamma(\ell+1)\Gamma(\ell+N/2)}.$$

The following lemma holds.

Lemma 4.4 ([17, Lemma A.2]). *Set $\delta_{N,n} := b_{N,n+1}/b_{N,n}$ and assume that $n \in \mathbb{Z}_+$ is even. Then, for $0 \le r \le 1/\sqrt{\delta_{N,n}}$,*

$$\begin{aligned}
&\Phi_{N,n-1}(r) \le \varphi_N(r) \le \Phi_{N,n},\\
&\max\{|\varphi_N(r) - \Phi_{N,n}(r)|, |\varphi_N(r) - \Phi_{N,n-1}(r)|, |\Phi_{N,n-1}(r) - \Phi_{N,n}(r)|\} \le b_{N,n} r^{2n}.
\end{aligned}$$

Set

$$\Psi(r) := \int_0^r \varphi_1(\eta)\, d\eta.$$

Since φ_1 is represented as a power series and converges locally uniformly in $\mathbb{R}$, we readily see that

$$\Psi(r) = \sum_{\ell=0}^{\infty} (-1)^\ell b_{1,\ell} \int_0^r \eta^{2\ell}\, d\eta = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} b_{1,\ell} r^{2\ell+1}.$$

For $n \in \mathbb{Z}_+$, define $\Psi_n(r)$ as

$$\Psi_n(r) := \int_0^r \Phi_{1,n}(\eta)\, d\eta = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{2\ell+1} b_{1,\ell} r^{2\ell+1}.$$

As a direct consequence of Lemma 4.4, we obtain the following corollary.

Corollary 4.5. *Assume that $n \in \mathbb{Z}_+$ is even. Then, for $0 \le r \le 1/\sqrt{\delta_{1,n}}$,*

$$\Psi_{n-1}(r) \le \int_0^r \varphi_1(\eta)\,d\eta \le \Psi_n(r),$$
$$\max\left\{\left|\int_0^r \varphi_1(\eta)\,d\eta - \Psi_n(r)\right|, \left|\int_0^r \varphi_1(\eta)\,d\eta - \Psi_{n-1}(r)\right|, \left|\Psi_n(r) - \Psi_{n-1}(r)\right|\right\} \le \frac{b_{1,n}}{2n+1} r^{2n+1}.$$

According to [17, Theorem 2.3], φ_N changes its sign infinitely many times. Let $\{r_k^\pm\}_{k\in\mathbb{N}}$ be a sequence satisfying

$$\varphi_1(r_k^\pm) = 0, \quad 0 < r_1^+ < r_1^- < r_2^+ < r_2^- < \cdots < r_k^+ < r_k^- < \cdots,$$
$$\varphi_1(r) \begin{cases} > 0 & \text{for } r \in \bigcup_{k\in\mathbb{Z}_+}(r_k^-, r_{k+1}^+), \\ < 0 & \text{for } r \in \bigcup_{k\in\mathbb{N}}(r_k^+, r_k^-), \end{cases}$$

where $r_0^- = 0$. Applying Lemma 4.4 with $n = 16$, we obtain

$$3.453 < r_1^+ < 3.454, \quad 6.784 < r_1^- < 6.785. \tag{4.9}$$

The following lemma can be proved by using [19, Theorem 1 and Remark 1].

Lemma 4.6. (i) $\Psi(r) > 0$ *for* $r > 0$.

(ii) $\Psi(r)$ *takes local maximum (resp., local minimum) at r_k^+ (resp., r_k^-).*

(iii) $\Psi(r)$ *is strictly increasing (resp., decreasing) in each interval (r_k^-, r_{k+1}^+) for $k \in \mathbb{Z}_+$ (resp., (r_k^+, r_k^-) for $k \in \mathbb{N}$), where $r_0^- = 0$.*

Furthermore, we are able to prove the following lemma.

Lemma 4.7. $\{\Psi(r_k^+)\}_{k\in\mathbb{N}}$ *(resp., $\{\Psi(r_k^-)\}_{k\in\mathbb{N}}$) is strictly decreasing (resp., increasing).*

Proof. It follows from [19, Theorem 1] that for each $k \in \mathbb{N}$

$$\int_{r_k^-}^{r_{k+1}^-} \varphi_1(r)\,dr > 0, \quad \int_{r_k^+}^{r_{k+1}^+} \varphi_1(r)\,dr < 0. \tag{4.10}$$

This implies that

$$\Psi(r_{k+1}^-) = \Psi(r_k^-) + \int_{r_k^-}^{r_{k+1}^-} \varphi_1(r)\,dr > \Psi(r_k^-),$$
$$\Psi(r_{k+1}^+) = \Psi(r_k^+) + \int_{r_k^+}^{r_{k+1}^+} \varphi_1(r)\,dr < \Psi(r_k^+)$$

for each $k \in \mathbb{N}$. ∎

With the help of (4.9), Corollary 4.5, and some numerical computations, we observe that

$$\Psi(r_1^+/3) > 0.32584, \quad 0.5522 < \Psi(r_1^+) < 0.5523, \quad 0.4938 < \Psi(r_1^-) < 0.4939. \tag{4.11}$$

For $r \geq 0$, set

$$I(r) := \Psi(r/(3a)) - 3\Psi(r/(2a)) + 3\Psi(r/a).$$

Using (4.11), we can prove the following lemma.

Lemma 4.8. $I(r) > 0$ *for* $r > 0$.

Proof. We divide the proof into several cases.

Case 1: $0 < r/a \leq r_1^+$. We readily see that

$$I(r) = \left(\int_0^{r/(3a)} + 3\int_{r/(2a)}^{r/a} \right) \varphi_1(\eta)\, d\eta > 0.$$

Case 2: $r_1^+ < r/a \leq r_1^-$. Note that $r_1^+/3 < r/(3a) < r/(2a) \leq r_1^-/2 < r_1^+$ by (4.9). Then, it follows from Lemma 4.6 (iii) and (4.11) that

$$I(r) > \Psi(r_1^+/3) - 3\Psi(r_1^+) + 3\Psi(r_1^-) > 0.32584 + 3(-0.5523 + 0.4938) = 0.1329 > 0.$$

Case 3: $r_1^- < r/a \leq 3r_1^-$. First, we derive the lower bound of $\Psi(r/(3a))$. Note that $r_1^+/3 < r_1^-/3 < r/(3a) \leq r_1^-$. If $r/(3a) < r_1^+$, we have

$$\Psi(r/(3a)) = \Psi(r_1^+/3) + \int_{r_1^+/3}^{r/(3a)} \varphi_1(\eta)\, d\eta > \Psi(r_1^+/3).$$

If $r/(3a) \geq r_1^+$, Lemma 4.6 (iii) implies that $\Psi(r/(3a)) \geq \Psi(r_1^-)$. Second, let us consider the upper bound of $\Psi(r/(2a))$. If $r/(2a) \leq r_1^+$, it follows from Lemma 4.6 (iii) that $\Psi(r/(2a)) \leq \Psi(r_1^+)$. If $r/(2a) > r_1^+$, we can choose $k_* \in \mathbb{N}$ such that $r_{k_*}^+ < r/(2a) \leq r_{k_*+1}^+$. Then, by (4.10), we see that

$$\Psi(r/(2a)) = \Psi(r_1^+) + \sum_{k=1}^{k_*-1} \int_{r_k^+}^{r_{k+1}^+} \varphi_1(\eta)\, d\eta + \int_{r_{k_*}^+}^{r/(2a)} \varphi_1(\eta)\, d\eta < \Psi(r_1^+).$$

Finally, we derive the lower bound of $\Psi(r/a)$. Choose $\ell_* \in \mathbb{N}$ such that $r_{\ell_*}^- < r/a \leq r_{\ell_*+1}^+$. By virtue of (4.10), we obtain

$$\Psi(r/a) = \Psi(r_1^-) + \sum_{\ell=1}^{\ell_*-1} \int_{r_\ell^-}^{r_{\ell+1}^-} \varphi_1(\eta)\, d\eta + \int_{r_{\ell_*}^-}^{r/a} \varphi_1(\eta)\, d\eta > \Psi(r_1^-).$$

Consequently, based on these bounds and (4.11), we obtain

$$\begin{aligned} I(r) &> \min\{\Psi(r_1^+/3), \Psi(r_1^-)\} - 3\Psi(r_1^+) + 3\Psi(r_1^-) \\ &> 0.32584 + 3(-0.5523 + 0.4938) = 0.1329 > 0. \end{aligned}$$

Case 4: $3r_1^- < r/a$. Choose $k_* \in \mathbb{N}$ such that $r_{k_*}^- \leq r/(3a) < r_{k_*+1}^-$. Then, we have

$$\Psi(r/(3a)) \begin{cases} > \Psi(r_{k_*}^-) & \text{if } r/(3a) \in (r_{k_*}^-, r_{k_*+1}^+), \\ \geq \Psi(r_{k_*+1}^-) & \text{if } r/(3a) \in [r_{k_*+1}^+, r_{k_*+1}^-). \end{cases}$$

Since $\Psi(r_k^-) \geq \Psi(r_1^-)$ for $k \in \mathbb{N}$ by Lemma 4.7, we see that $\Psi(r/(3a)) > \Psi(r_1^-)$. To derive the lower bound of $-\Psi(r/(2a)) + \Psi(r/a)$, take $\ell_*, m_* \in \mathbb{N}$ satisfying

$$r_{\ell_*}^+ \leq r/(2a) < r_{\ell_*+1}^+, \quad r_{m_*}^- \leq r/a < r_{m_*+1}^-.$$

Then, it follows from Lemma 4.7 that

$$-\Psi(r/(2a)) + \Psi(r/a) > -\Psi(r_{\ell_*}^+) + \Psi(r_{m_*}^-) \geq -\Psi(r_1^+) + \Psi(r_1^-).$$

Therefore, by virtue of these bounds and (4.11), we see that

$$I(r) > \Psi(r_1^-) + 3\{-\Psi(r_1^+) + \Psi(r_1^-)\} > 0.4938 + 3(-0.5523 + 0.4938) = 0.3183 > 0.$$

This completes the proof. ∎

Now, we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. For $\boldsymbol{y} \in \partial\Omega(t)$, take $\boldsymbol{x} \in \partial\Omega_0$ such that $|\boldsymbol{y} - \boldsymbol{x}| = |d(\boldsymbol{y}, \partial\Omega_0)|$. Set $v = d(\boldsymbol{y}, \partial\Omega_0)$. Then, we have $\boldsymbol{y} = \boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x})$. Applying Theorem 3.1, we obtain

$$u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), a^4 t) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{a^{2m}(-\lambda)^m t^{m/2}}{m!} \cdot \int_{\mathbb{R}^{N-1}} \int_0^{\psi_a(\boldsymbol{z}', v, t)} (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z}) dz_N d\boldsymbol{z}' + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}),$$

where

$$\psi_a(\boldsymbol{z}', v, t) := \frac{1}{a t^{1/4}}\{-v + f(a t^{1/4} \boldsymbol{z}')\} \tag{4.12}$$

for $a > 0$ and a function f satisfying (A1)–(A3). Then, we have

$$\begin{aligned} &U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) \\ &= \frac{1}{2} + \sum_{m=0}^{\infty} \frac{a^{2m}(-\lambda)^m t^{m/2}}{m!} \\ &\quad \cdot \left(3^{2m} \int_{\mathbb{R}^{N-1}} \int_0^{\psi_{3a}(\boldsymbol{z}', v, t)} - 3 \cdot 2^{2m} \int_{\mathbb{R}^{N-1}} \int_0^{\psi_{2a}(\boldsymbol{z}', v, t)} + 3 \int_{\mathbb{R}^{N-1}} \int_0^{\psi_a(\boldsymbol{z}', v, t)} \right) \\ &\quad \cdot (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z}) dz_N d\boldsymbol{z}' + O_{\partial\Omega_0}(e^{-\mu_* t^{-1/3}}). \end{aligned}$$

Since $\boldsymbol{y} \in \partial\Omega(t)$ and $v = d(\boldsymbol{y}, \partial\Omega_0)$, we see that

$$U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2}.$$

Furthermore, Proposition 4.2 implies that $|v| \leq K_* t^{1/4}$ for $t \in (0, t_*)$.

Let us prove $v = o_{\partial\Omega_0}(t^{1/4})$ as $t \to +0$. In the proof by contradiction, suppose that there exists $\varepsilon_* > 0$ such that for any $k \in \mathbb{N}$ there are $\{t_k\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{y}_k\}_{k\in\mathbb{N}} \subset \partial\Omega(t_k)$ such that

$$0 < t_k < \frac{1}{k}, \quad |d(\boldsymbol{y}_k, \partial\Omega_0)| > \varepsilon_* t_k^{1/4}.$$

Then, there exist sequences $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}} \subset \partial\Omega_0$ and $\{r_k\}_{k\in\mathbb{N}}$ satisfying

$$\boldsymbol{y}_k = \boldsymbol{x}_k + v_k \boldsymbol{n}_k, \quad \varepsilon_* < |r_k| \le K_*, \quad U(\boldsymbol{x}_k + v_k \boldsymbol{n}_k, t_k) = \frac{1}{2},$$

where $v_k := -r_k t_k^{1/4}$ and $\boldsymbol{n}_k := \boldsymbol{n}(\boldsymbol{x}_k)$. Since $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ are bounded sequences, by taking subsequences if necessary, we may assume that $\boldsymbol{x}_k \to \boldsymbol{x}_* \in \partial\Omega_0$, $r_k \to r_*$ as $k \to \infty$, where $\varepsilon_* \le |r_*| \le K_*$. On the other hand, $\psi_a(\boldsymbol{z}', v_k, t_k)$ is represented as

$$\begin{aligned}\psi_a(\boldsymbol{z}', v_k, t_k) &= (a t_k^{1/4})^{-1}\left\{ - v_k + \frac{(a t_k^{1/4})^2 \langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\theta a t_k^{1/4} \boldsymbol{z}')}{2}\right\} \\ &= \frac{r_k}{a} + \frac{a t_k^{1/4}}{2}\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle_{N-1}^2 f(\theta a t_k^{1/4} \boldsymbol{z}')\end{aligned}$$

for some $\theta \in (0, 1)$. Since $\psi_a(\boldsymbol{z}', v_k, t_k) \to r_*/a$ as $k \to \infty$, it follows from (4.8) that

$$\begin{aligned}\frac{1}{2} &= \lim_{k\to\infty} U(\boldsymbol{x}_k + v_k \boldsymbol{n}_k, t_k) \\ &= \frac{1}{2} + \left(\int_{\mathbb{R}^{N-1}} \int_0^{r_*/3a} -3 \int_{\mathbb{R}^{N-1}} \int_0^{r_*/2a} +3 \int_{\mathbb{R}^{N-1}} \int_0^{r_*/a} \right) g_N(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \frac{1}{2} + \left(\int_0^{r_*/(3a)} -3 \int_0^{r_*/(2a)} +3 \int_0^{r_*/a} \right) g_1(r)\, dr.\end{aligned}$$

By Lemma 4.8 and $g_1(r) = \varphi_1(|r|)$, we see that if $r_* > 0$,

$$\left(\int_0^{r_*/(3a)} -3 \int_0^{r_*/(2a)} +3 \int_0^{r_*/a} \right) g_1(r)\, dr > 0,$$

and if $r_* < 0$,

$$\begin{aligned}&\left(\int_0^{r_*/(3a)} -3 \int_0^{r_*/(2a)} +3 \int_0^{r_*/a} \right) g_1(r)\, dr \\ &= -\left(\int_0^{-r_*/(3a)} -3 \int_0^{-r_*/(2a)} +3 \int_0^{-r_*/a} \right) g_1(r)\, dr < 0.\end{aligned}$$

These facts lead to a contradiction. Therefore, $v = o_{\partial\Omega_0}(t^{1/4})$ as $t \to +0$. ■

We next prove an estimate of the derivative of U, which guarantees that $\partial\Omega(t)$ is a smooth hypersurface. Set $M_{0,a} := 11M_0/(6a)$, where M_0 is as in Lemma 3.3.

Proposition 4.9. *There exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for $t > 0$ small enough, $\boldsymbol{x} \in \partial\Omega_0$, and $v \in (-\delta_0, \delta_0)$ satisfying $|v| \le \varepsilon_0 t^{1/4}$*

$$-\frac{3M_{0,a}}{2}t^{-1/4} - C(\varepsilon_0 + t^{1/4}) \le \langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N \le -\frac{M_{0,a}}{2}t^{-1/4} + C(\varepsilon_0 + t^{1/4}). \tag{4.13}$$

Here, $\delta_0 \in (0, 1/2)$ has been taken at the end of Section 3.1. Furthermore, $\partial\Omega(t)$ is a smooth hypersurface for $t > 0$ small enough.

Proof. *Step 1*. Set

$$\Xi^{(1)}_{N,a} := -\frac{1}{at^{1/4}} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\psi_a(z',v,t)} \partial_{z_N} g_N(z) dz_N dz'.$$

Then, we have

$$|\langle \nabla_{\boldsymbol{x}} u_a(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N - \Xi^{(1)}_{N,a}| \le C_1|\lambda| t^{1/4}. \tag{4.14}$$

Indeed, applying an argument similar to the proof of Lemma 2.5, we obtain

$$\left| \frac{1}{at^{1/4}} \sum_{m=1}^{\infty} \frac{a^{2m}(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\psi_a(z',v,t)} \partial_{z_N} (-\Delta_z)^m g_N(z)\, dz_N dz' \right| \le C_1|\lambda| t^{1/4}$$

for $|\lambda| t^{1/2} \le \gamma/a^2$, where $\psi_a(z', v, t)$ is given by (4.12). This inequality and (3.3) yield (4.14).

Step 2. We prove (4.13). By (4.2) and (4.14), there exists a $C_2 > 0$ such that

$$|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N - (\Xi^{(1)}_{N,3a} - 3\Xi^{(1)}_{N,2a} + 3\Xi^{(1)}_{N,a})| \le C_2|\lambda| t^{1/4} \tag{4.15}$$

for small $t > 0$. Note that $\Xi^{(1)}_{N,a}$ is rewritten as

$$\Xi^{(1)}_{N,a} = -\frac{1}{at^{1/4}} \int_{\mathbb{R}^{N-1}} g_N(z', \psi_a(z', v, t))\, dz'.$$

It follows from (3.16) and Taylor's theorem that

$$g_N(z', \psi_a(z', v, t)) = g_N(z', 0) + \frac{1}{2}\partial^2_{z_N} g_N(z', \theta\psi_a(z', v, t))\{\psi_a(z', v, t)\}^2$$

for some $\theta \in (0, 1)$. This and the definition of M_0 imply that

$$\Xi^{(1)}_{N,a} = -\frac{M_0}{at^{1/4}} - \frac{1}{2at^{1/4}} \int_{\mathbb{R}^{N-1}} \partial^2_{z_N} g_N(z', \theta\psi_a(z', v, t))\{\psi_a(z', v, t)\}^2 dz'. \tag{4.16}$$

In addition, taking account of

$$\psi_a(z', v, t) = -\frac{v}{at^{1/4}} + \frac{at^{1/4}}{2}\langle z', \nabla_{z'}\rangle^2_{N-1} f(\widetilde{\theta} a t^{1/4} z')$$

for some $\widetilde{\theta} \in (0,1)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 g_N(z', \theta\psi_a(z', v, t))\{\psi_a(z', v, t)\}^2 dz' \\
&= \frac{v^2}{(at^{1/4})^2} \int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 g_N(z', \theta\psi_a(z', v, t))\, dz' \\
&\quad - v \int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 g_N(z', \theta\psi_a(z', v, t)) \langle z', \nabla_{z'}\rangle_{N-1}^2 f(\widetilde{\theta} a t_k^{1/4} z')\, dz' \\
&\quad + \frac{(at^{1/4})^2}{4} \int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 g_N(z', \theta\psi_a(z', v, t)) \big\{\langle z', \nabla_{z'}\rangle_{N-1}^2 f(\widetilde{\theta} a t_k^{1/4} z')\big\}^2 dz'. \quad (4.17)
\end{aligned}$$

We assume that $|v| \le \varepsilon_0 t^{1/4}$, where $\varepsilon_0 \in (0,1)$ is suitably chosen later. Then, it follows that

$$|\psi_a(z', v, t)| \le C_3(|z'|^2 t^{1/4} + \varepsilon_0).$$

This fact and Corollary 2.3 with $|\alpha| = 2$ and $m = 0$ yield that

$$\begin{aligned}
|\partial_{z_N}^2 g_N(z', \theta\psi_a(z', v, t))| &\le C_4 \big\{1 + (|z'|^2 + |\psi_a(z', v, t)|^2)^{1/2}\big\}^{2/3} e^{-\mu|z'|^{4/3}} \\
&\le C_4 (1 + |z'| + |\psi_a(z', v, t)|)^{2/3} e^{-\mu|z'|^{4/3}} \\
&\le C_5 (1 + |z'|^{2/3} + |z'|^{4/3} t^{1/6}) e^{-\mu|z'|^{4/3}}.
\end{aligned}$$

Applying this estimate to the right-hand side of (4.17), we are able to find a constant $C_6 > 0$ such that

$$\left| \frac{1}{2} \int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 g_N(z', \theta\psi_a(z', v, t))\{\psi_a(z', v, t)\}^2 dz' \right| \le C_6 \left\{ \frac{\varepsilon_0^2}{a^2} + \varepsilon_0 t^{1/4} + (at^{1/4})^2 \right\}$$

for $t > 0$ small enough. Recalling (4.16), we see that

$$\left| \Xi_{N,3a}^{(1)} - 3\Xi_{N,2a}^{(1)} + 3\Xi_{N,a}^{(1)} + M_{0,a} t^{-1/4} \right| \le C_7 \left(\frac{\varepsilon_0^2}{a^3 t^{1/4}} + \frac{\varepsilon_0}{a} + at^{1/4} \right)$$

for some $C_7 > 0$. Choosing $\varepsilon_0 > 0$ such that

$$\frac{C_7 \varepsilon_0^2}{a^2} \le \frac{M_0}{2},$$

we obtain

$$\begin{aligned}
-\frac{3M_{0,a}}{2} t^{-1/4} - C_7 \left(\frac{\varepsilon_0}{a} + at^{1/4} \right) &\le \Xi_{N,3a}^{(1)} - 3\Xi_{N,2a}^{(1)} + 3\Xi_{N,a}^{(1)} \\
&\le -\frac{M_{0,a}}{2} t^{-1/4} + C_7 \left(\frac{\varepsilon_0}{a} + at^{1/4} \right).
\end{aligned}$$

By this inequality and (4.15), we are led to the desired result.

Step 3. Let us prove that $\partial\Omega(t)$ is a smooth hypersurface for $t > 0$ small enough. By virtue of (4.13), $\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N$ is far from zero for $t > 0$ small enough; in particular, it is negative. This fact and the implicit function theorem imply the desired result. ∎

Theorem 4.10. *There exists $C_0 > 0$ such that*

$$\sup_{\boldsymbol{x}\in\partial\Omega(t)} |d(\boldsymbol{x}, \partial\Omega_0)| \le C_0 t$$

for $t > 0$ small enough.

Proof. Applying (3.14) with $V = 0$, there exists $C_1 > 0$ such that

$$\left| U(\boldsymbol{x}, t) - \frac{1}{2} \right| \le C_1 t^{3/4} \tag{4.18}$$

for $\boldsymbol{x} \in \partial\Omega_0$ and $t > 0$ small enough. For any $\boldsymbol{y} \in \partial\Omega(t)$, let $\boldsymbol{x} \in \partial\Omega_0$ be a point satisfying $|d(\boldsymbol{y}, \partial\Omega_0)| = |\boldsymbol{y} - \boldsymbol{x}|$. Then, $\boldsymbol{y}$ can be represented as $\boldsymbol{y} = \boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x})$, where $v = d(\boldsymbol{y}, \partial\Omega_0)$. This implies that

$$U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) - U(\boldsymbol{x}, t) = \langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + \theta v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N v$$

for some $\theta \in (0, 1)$. Taking account of $U(\boldsymbol{y}, t) = 1/2$ for $\boldsymbol{y} \in \partial\Omega(t)$ and using Proposition 4.9 and (4.18), we see that

$$\begin{aligned}
&|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + \theta v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N v| \le C_1 t^{3/4}, \\
&-C_2 t^{-1/4} \le \langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + \theta v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N \le -C_3 t^{-1/4}
\end{aligned}$$

for $t > 0$ small enough where C_2, C_3 are positive constants independent of $\boldsymbol{x} \in \partial\Omega_0$ and $t > 0$. As a result, it follows that there exists $C_4 > 0$ such that $|v| \le C_4 t$ for $\boldsymbol{x} \in \partial\Omega_0$ and small $t > 0$. This is the desired result. ∎

From this theorem, it follows that V, defined as in (4.4), is bounded on $\partial\Omega_0 \times (0, t_0)$ for some small $t_0 > 0$, and hence, the argument in Section 4.1 is justified; that is, we have the following theorem.

Theorem 4.11. *Let V be as in (4.4). Then, there exist $C > 0$ and $t_0 > 0$ such that*

$$\left| V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2 \sum_{\substack{i_1,i_2,i_3\in\Lambda \\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H \right| \le C t^{1/4}$$

for all $t \in (0, t_0)$ and $\boldsymbol{x} \in \partial\Omega_0$. Especially, this estimate turns to

$$\begin{aligned}
&\left| V + \kappa_{ss} + \frac{1}{2}\kappa^3 - \lambda\kappa \right| \le C t^{1/4} && \text{if } N = 2, \\
&\left| V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 - \lambda H \right| \le C t^{1/4} && \text{if } N = 3
\end{aligned}$$

for all $t \in (0, t_0)$ and $\boldsymbol{x} \in \partial\Omega_0$.

Remark 4.12. (i) If $\partial\Omega_0$ is of class $C^n (n \geq 6)$, we can replace $t^{1/4}$ with $t^{1/2}$ in Theorem 4.11. But we omit the details because the actual calculations are more complicated.

(ii) Based on the boundedness of V, we can improve (4.13) as follows:

$$|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N + M_{0,a} t^{-1/4}| \leq C t^{3/4}.$$

Moreover, we can also estimate

$$|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{\tau}(\boldsymbol{x})\rangle| \leq C t^{3/4},$$

where $\boldsymbol{\tau}(\boldsymbol{x})$ is any unit tangential vector of $\partial\Omega_0$ at $\boldsymbol{x} \in \partial\Omega_0$. Thus, the outer unit normal $-\nabla_{\boldsymbol{x}} U(\cdot, t)/|\nabla_{\boldsymbol{x}} U(\cdot, t)|$ of $\partial\Omega(t)$ is nearly equal to $\boldsymbol{n}(\cdot)$, that of $\partial\Omega_0$, for any small $t > 0$.

5. Numerical experiments

In this section, we present some results of numerical experiments based on our thresholding algorithm which is proposed in Section 4.1. We focus on the case where $N = 2$ and $\lambda = 0$, that is, the Willmore flow for planar curves.

5.1. Numerical scheme

Let $D = (\mathbb{R}/L\mathbb{Z}) \times (\mathbb{R}/L\mathbb{Z})$ be a periodic square region with $L > 0$ and assume that $\Omega_0 \subset D$ holds. Then, we solve the following initial value problem:

$$\begin{cases} u_t = -(-\Delta)^2 u & \text{in } D \times (0, \infty), \\ u(\cdot, 0) = \chi_{\Omega_0} & \text{in } D. \end{cases} \tag{5.1}$$

By the definition of D, the periodic boundary condition is implicitly imposed on the above problem. Note that there are other possibilities regarding the boundary conditions; for example, the Dirichlet boundary condition or the Neumann boundary condition is natural. However, we adopt the periodic boundary condition here since we develop a numerical scheme based on the Fourier transform.

We seek a solution in the form of Fourier series

$$u(\boldsymbol{x}, t) = \sum_{\boldsymbol{\xi} \in \mathbb{Z}^2} u_{\boldsymbol{\xi}}(t) e^{2\pi \mathrm{i} \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle_2 / L},$$

since u is supposed to be a periodic function. Taking account of

$$u_t = \sum_{\boldsymbol{\xi} \in \mathbb{Z}} \dot{u}_{\boldsymbol{\xi}}(t) e^{2\pi \mathrm{i} \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle_2 / L}, \quad (-\Delta)^2 u = \sum_{\boldsymbol{\xi} \in \mathbb{Z}} \frac{16\pi^4}{L^4} |\boldsymbol{\xi}|^4 u_{\boldsymbol{\xi}}(t) e^{2\pi \mathrm{i} \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle_2 / L}$$

and substituting these expressions into the first equation in (5.1), we are led to an infinite system of ordinary differential equations

$$\dot{u}_{\boldsymbol{\xi}}(t) = -\frac{16\pi^4}{L^4} |\boldsymbol{\xi}|^4 u_{\boldsymbol{\xi}}(t), \quad \boldsymbol{\xi} \in \mathbb{Z}^2,$$

where the dot symbol indicates the time derivative. Each ordinary differential equation can be separately and explicitly solved as

$$u_{\boldsymbol{\xi}}(t) = \chi_{\Omega_0,\boldsymbol{\xi}} e^{-16\pi^4|\boldsymbol{\xi}|^4 t/L^4},$$

where $\chi_{\Omega_0,\boldsymbol{\xi}}$ denotes the Fourier coefficient of the characteristic function χ_{Ω_0}. Then, the function U defined by (4.2) can be computed as

$$\begin{aligned} U(\boldsymbol{x},h) &= u_{3a}(\boldsymbol{x},h) - 3u_{2a}(\boldsymbol{x},h) + 3u_a(\boldsymbol{x},h) \\ &= \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2} \chi_{\Omega_0,\boldsymbol{\xi}} \Big(e^{-16\pi^4|\boldsymbol{\xi}|^4(3a)^4h/L^4} - 3e^{-16\pi^4|\boldsymbol{\xi}|^4(2a)^4h/L^4} \\ &\qquad + 3e^{-16\pi^4|\boldsymbol{\xi}|^4a^4h/L^4} \Big) e^{2\pi\mathbf{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_2/L}, \end{aligned} \tag{5.2}$$

where $h > 0$ denotes a time step. Hence, we obtain Ω_1 by $\Omega(h)$, where $\Omega(t)$ is given by (4.3). Repeating the above procedure yields an approximation of the Willmore flow.

In the actual computation, we first generate a uniform mesh $\{\boldsymbol{x}_{ij}\}_{(i,j)\in\Delta_N^2}$ as

$$\boldsymbol{x}_{ij} = \begin{pmatrix} (i+\frac{1}{2})r \\ (j+\frac{1}{2})r \end{pmatrix}, \quad (i,j)\in\Delta_N^2,$$

where $r := L/N$, $\Delta_N := \{0,1,\dots,N-1\}$, and N is a power of 2. Let us denote an approximate value of $u(\cdot,t)$ at (x_i,y_j) by $u_{ij}(t)$. Using the fast Fourier transform, we can compute discrete Fourier coefficients $\chi_{\Omega_0,\boldsymbol{\xi}}$ for $\boldsymbol{\xi}\in\Delta_N^2$ with $O(N^2\log N)$ complexity. Then, we are able to approximate $\{U(\boldsymbol{x}_{ij},h)\}_{(i,j)\in\Delta_N^2}$ given by (5.2) by means of applying the inverse fast Fourier transform to $\{\chi_{\Omega_0,\boldsymbol{\xi}}(u_{\boldsymbol{\xi}}(81a^4h) - 3u_{\boldsymbol{\xi}}(16a^4h) + 3u_{\boldsymbol{\xi}}(a^4h))\}_{\boldsymbol{\xi}\in\Delta_N^2}$.

Remark 5.1. As described above, we use a uniform mesh to perform the fast Fourier transform and its inverse. It is preferable to adopt some mesh refinement techniques to capture the evolution of the interface more precisely. Indeed, Ruuth [39] developed a gradually adaptive mesh refinement technique and applied it to the computation of the curve shortening and mean curvature flows based on the BMO algorithm. This technique has also been applied to the Willmore flow in Esedoḡlu–Ruuth–Tsai [14]. It could be expected that a similar approach would be necessary for our algorithm. However, as a first step of numerical computation of the Willmore flow based on our proposed scheme, we adopt the uniform mesh in this paper.

5.2. Numerical results

In this subsection, we show two numerical results of our algorithm.

5.2.1. Self-similar solution. First, let us consider the case where the initial curve is a circle. As is well documented, the solution develops over time, maintaining a circular

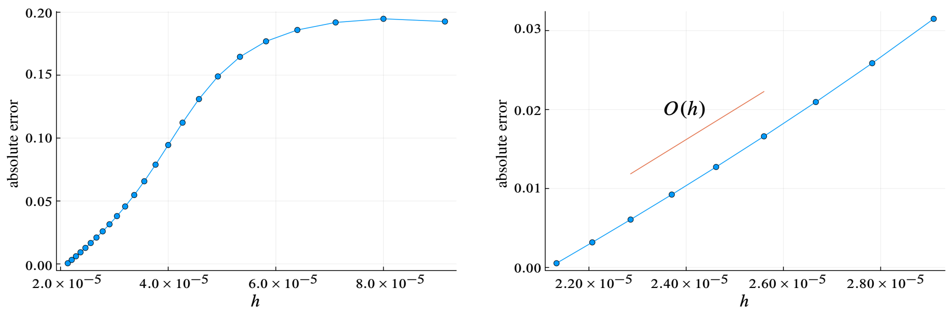

Figure 1. Plot of the absolute error versus the time step size h at $t = 0.00064$. The left is the overview and the right is the close-up around $h = 2.50 \times 10^{-5}$.

shape. Its radius, designated as R, is defined as the solution to the following ordinary differential equation:

$$\dot{R} = \frac{1}{2}\left(\frac{1}{R}\right)^3.$$

That is to say, R is specifically represented as

$$R(t) = \sqrt[4]{R(0)^4 + 2t}.$$

Figure 1 illustrates how the discrepancy between the area of the exact solution and that of the numerical solution diminishes as h is gradually reduced. The initial radius is 0.1, and the final computation time is 0.00064. The remaining parameters were determined as follows:

- $L = 1$ (the domain size),
- $N = 2^{14}$ (the number of mesh).

As illustrated in the graph, the approximation error exhibits a gradual decline as h is reduced. The orange line in the graph represents $O(h)$, indicating that the proposed scheme is first-order accurate. This is a superior value compared to the result by Esedoḡlu–Ruuth–Tsai [14], which suggested one-half-order accuracy.

5.2.2. Other nontrivial examples. In this subsection, we show several nontrivial numerical results. Parameters are chosen as follows:

- $L = 5$ (the domain size),
- $N = 2^{10} = 1024$ (the number of mesh).

Note that the time step h cannot simply be made smaller. As pointed out by Ruuth [39, Section 2.3], the spatial step must be sufficiently smaller than the time step to allow the interface to move.

First, consider the case where the following Cassini's oval gives the initial region:

$$\Omega_0 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1^2 + x_2^2)^2 - 2b^2(x_1^2 - x_2^2) \le a^4 - b^4\}, \quad a = 0.6825, \quad b = 0.678.$$

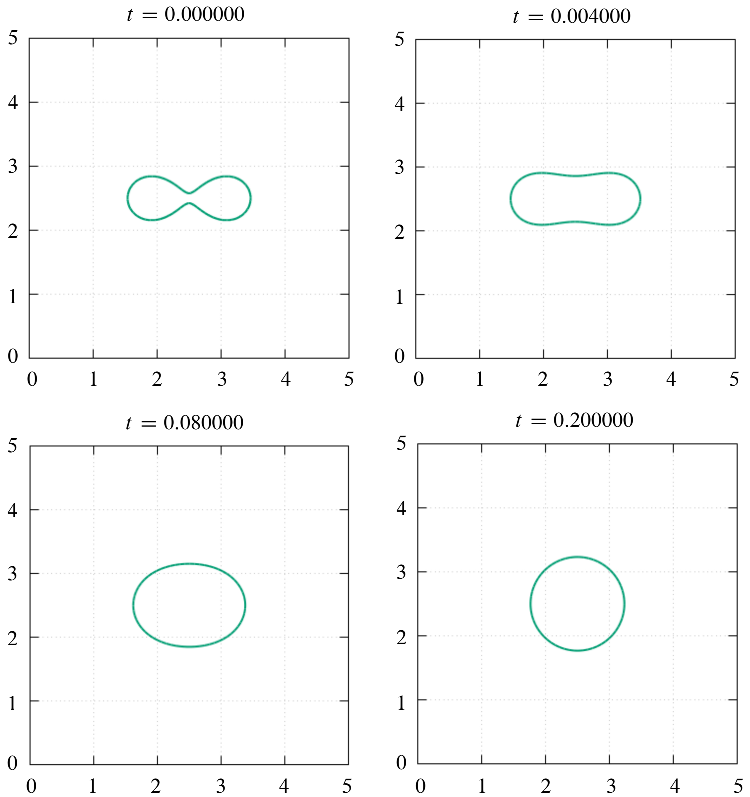

Figure 2. Numerical results of the Willmore flow with initial region given as the Cassini parameters are chosen as $L = 5$, $N = 1024$, and $h = 0.004$.

The results are shown in Figure 2, where the time step h is chosen as $h = 0.004$. It can be seen that the numerical computations are performed stably without any numerical instability. Another more complex initial shape is the following initial region:

$$\Omega_0 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq \max\{0.01, r(\boldsymbol{x})^2\}\}, \tag{5.3}$$

where

$$c = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad r(\boldsymbol{x}) = 0.5 + \frac{16c^5 - 20c^3 + 5c}{3}.$$

The results are depicted in Figure 3, in which the time step h is chosen as

$$h = 0.003.$$

In this case, the numerical computation is also successful without instability.

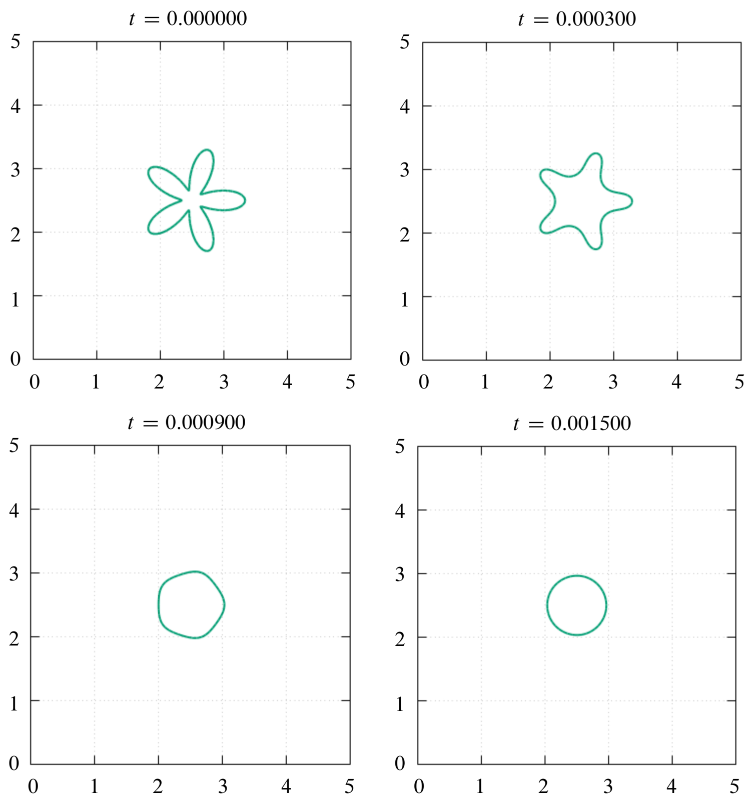

Figure 3. Numerical results of the Willmore flow with initial region given by (5.3). Parameters are chosen as $L = 5$, $N = 1024$, and $h = 0.0003$.

A. Mean curvature and its derivatives

Set $\Phi(\boldsymbol{x}') = (\boldsymbol{x}', f(\boldsymbol{x}'))$, where $\boldsymbol{x}' = (x_1, \ldots, x_{N-1})$. In this subsection, we denote ∂_{x^i} by ∂_i. First, we have

$$\partial_i \Phi = (0, \ldots, 0, \underset{\hat{i}}{1}, 0, \ldots, 0, \partial_i f),$$
$$\partial_1 \Phi \times \cdots \times \partial_{N-1} \Phi = (-\nabla_{\boldsymbol{x}'} f, 1).$$

Then, the outer unit normal $\boldsymbol{n}$ to the hypersurface $\{\Phi(\boldsymbol{x}') \mid \boldsymbol{x}' \in D\}$ for $D \subset \mathbb{R}^{N-1}$ is represented as

$$\boldsymbol{n} = \frac{(-\nabla_{\boldsymbol{x}'} f, 1)}{\sqrt{1 + |\nabla_{\boldsymbol{x}'} f|^2}}.$$

Also, the first fundamental forms are

$$g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle_N = \delta_{ij} + \partial_i f \partial_j f$$

Setting $g = \det(g_{ij}) = 1 + |\nabla_{\boldsymbol{x}'} f|^2$, we see that

$$g^{ij} = \delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla_{\boldsymbol{x}'} f|^2}.$$

Let us derive the second fundamental forms. Since

$$\partial_i \partial_j \Phi = (0, \dots, 0, \partial_i \partial_j f),$$

we are led to

$$h_{ij} = \langle \partial_i \partial_j \Phi, \boldsymbol{n} \rangle_N = \frac{\partial_i \partial_j f}{\sqrt{1 + |\nabla_{\boldsymbol{x}'} f|^2}}.$$

Thus, the mean curvature H is represented as

$$H = \sum_{i,j \in \Lambda} g^{ij} h_{ij} = \sum_{i,j \in \Lambda} \left(\delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla_{\boldsymbol{x}'} f|^2} \right) \frac{\partial_i \partial_j f}{\sqrt{1 + |\nabla_{\boldsymbol{x}'} f|^2}}.$$

Furthermore, we have

$$\begin{aligned} |A|^2 &= \sum_{i,j,k,\ell \in \Lambda} g^{ik} g^{j\ell} h_{ij} h_{k\ell} \\ &= \sum_{i,j,k,\ell \in \Lambda} \left(\delta_{ik} - \frac{\partial_i f \partial_k f}{1 + |\nabla_{\boldsymbol{x}'} f|^2} \right) \left(\delta_{j\ell} - \frac{\partial_j f \partial_\ell f}{1 + |\nabla_{\boldsymbol{x}'} f|^2} \right) \frac{\partial_i \partial_j f \partial_k \partial_\ell f}{1 + |\nabla_{\boldsymbol{x}'} f|^2}. \end{aligned}$$

When $\nabla_{\boldsymbol{x}'} f = \mathbf{0}'$ at some point, H and $|A|^2$ are given by

$$H = \sum_{i,j \in \Lambda} \delta_{ij} \partial_i \partial_j f = \sum_{i \in \Lambda} \partial_i^2 f, \quad |A|^2 = \sum_{i,j,k,\ell \in \Lambda} \delta_{ik} \delta_{j\ell} \partial_i \partial_j f \partial_k \partial_\ell f = \sum_{i,j \in \Lambda} (\partial_i \partial_j f)^2.$$

We derive the Laplace–Beltrami operator Δ_g. The definition of the Laplace–Beltrami operator Δ_g is

$$\Delta_g = \sum_{i,j \in \Lambda} \frac{1}{\sqrt{g}} \partial_i \big(\sqrt{g} g^{ij} \partial_j \big) = \sum_{i,j \in \Lambda} \left\{ g^{ij} \partial_i \partial_j + \frac{1}{\sqrt{g}} \partial_i \big(\sqrt{g} g^{ij} \big) \partial_j \right\}.$$

Since

$$\partial_i g = \partial_i (1 + |\nabla_{\boldsymbol{x}'} f|^2) = 2 \sum_{k \in \Lambda} \partial_k f \partial_i \partial_k f,$$

it follows that

$$\begin{aligned}
\sum_{i\in\Lambda}\partial_i(\sqrt{g}g^{ij}) &= \sum_{i\in\Lambda}(\partial_i\sqrt{g})g^{ij} + \sum_{i\in\Lambda}\sqrt{g}\partial_i g^{ij} \\
&= \frac{1}{2\sqrt{g}}\sum_{i\in\Lambda}(\partial_i g)\left(\delta_{ij} - \frac{\partial_i f\partial_j f}{g}\right) \\
&\quad - \sqrt{g}\sum_{i\in\Lambda}\left(\frac{\partial_i^2 f\partial_j f + \partial_i f\partial_i\partial_j f}{g} - \frac{\partial_i f\partial_j f\partial_i g}{g^2}\right) \\
&= \frac{1}{\sqrt{g}}\sum_{k\in\Lambda}\partial_k f\partial_j\partial_k f + \frac{1}{g\sqrt{g}}\partial_j f\sum_{i,k\in\Lambda}\partial_i f\partial_k f\partial_i\partial_k f \\
&\quad - \frac{1}{\sqrt{g}}\sum_{i\in\Lambda}\left(\partial_i^2 f\partial_j f + \partial_i f\partial_i\partial_j f\right) \\
&= \frac{1}{g\sqrt{g}}\partial_j f\sum_{i,k\in\Lambda}\partial_i f\partial_k f\partial_i\partial_k f - \frac{1}{\sqrt{g}}\partial_j f\sum_{i\in\Lambda}\partial_i^2 f \\
&= -\partial_j f\sum_{i,k\in\Lambda}\left(\delta_{ik} - \frac{\partial_i f\partial_k f}{g}\right)\frac{\partial_i\partial_k f}{\sqrt{g}} \\
&= -H\partial_j f.
\end{aligned}$$

Note that $\nabla_{\boldsymbol{x}'} f = \mathbf{0}'$ at some point gives $\partial_i(\sqrt{g}g^{ij}) = 0$. Here, we have

$$\partial_k\partial_\ell H = \sum_{i,j\in\Lambda}\left\{(\partial_k\partial_\ell g^{ij})h_{ij} + \partial_k g^{ij}\partial_\ell h_{ij} + \partial_\ell g^{ij}\partial_k h_{ij} + g^{ij}(\partial_k\partial_\ell h_{ij})\right\}.$$

The first partial derivatives of g^{ij} and h_{ij} are given by

$$\begin{aligned}
\partial_k g^{ij} &= -\frac{\partial_i\partial_k f\partial_j f + \partial_i f\partial_j\partial_k f}{g} + \frac{\partial_i f\partial_j f\partial_k g}{g^2}, \\
\partial_k h_{ij} &= \frac{\partial_i\partial_j\partial_k f}{\sqrt{g}} - \frac{\partial_i\partial_j f\partial_k g}{2g\sqrt{g}},
\end{aligned}$$

and the second partial derivatives of g^{ij} and h_{ij} are given by

$$\begin{aligned}
\partial_k\partial_\ell g^{ij} = &-\frac{\partial_i\partial_k\partial_\ell f\partial_j f + \partial_i\partial_k f\partial_j\partial_\ell f + \partial_i\partial_\ell f\partial_j\partial_k f + \partial_i f\partial_j\partial_k\partial_\ell f}{g} \\
&+ \frac{(\partial_i\partial_k f\partial_j f + \partial_i f\partial_j\partial_k f)\partial_\ell g}{g^2} \\
&+ \frac{\partial_i\partial_\ell f\partial_j f\partial_k g + \partial_i f\partial_j\partial_\ell f\partial_k g + \partial_i f\partial_j f\partial_k\partial_\ell g}{g^2} \\
&- \frac{2\partial_i f\partial_j f\partial_k g\partial_\ell g}{g^3},
\end{aligned}$$

$$\partial_k\partial_\ell h_{ij} = \frac{\partial_i\partial_j\partial_k\partial_\ell f}{\sqrt{g}} - \frac{\partial_i\partial_j\partial_k f\,\partial_\ell g}{2g\sqrt{g}} - \frac{\partial_i\partial_j\partial_\ell f\,\partial_k g + \partial_i\partial_j f\,\partial_k\partial_\ell g}{2g\sqrt{g}} + \frac{3\partial_i\partial_j f\,\partial_k g\,\partial_\ell g}{4g^2\sqrt{g}}.$$

Note that

$$\partial_k\partial_\ell g = 2\sum_{m\in\Lambda}\big(\partial_k\partial_m f\,\partial_\ell\partial_m f + \partial_m f\,\partial_k\partial_\ell\partial_m f\big).$$

When $\nabla_{\boldsymbol{x}'} f = \mathbf{0}'$ at some point, we have

$$g = 1, \qquad \partial_k g = 0, \qquad \partial_k\partial_\ell g = 2\sum_{m\in\Lambda}\partial_k\partial_m f\,\partial_\ell\partial_m f,$$
$$g_{ij} = g^{ij} = \delta_{ij}, \quad \partial_k g^{ij} = 0, \quad \partial_k\partial_\ell g^{ij} = -(\partial_i\partial_k f\,\partial_j\partial_\ell f + \partial_i\partial_\ell f\,\partial_j\partial_k f)$$

so that we obtain

$$\partial_k\partial_\ell H = -\sum_{i,j\in\Lambda}(\partial_i\partial_k f\,\partial_j\partial_\ell f + \partial_i\partial_\ell f\,\partial_j\partial_k f)\partial_i\partial_j f + \sum_{i,j\in\Lambda}\delta_{ij}\Big(\partial_i\partial_j\partial_k\partial_\ell f - \partial_i\partial_j f\sum_{m\in\Lambda}\partial_k\partial_m f\,\partial_\ell\partial_m f\Big).$$

This implies that

$$\begin{aligned}\Delta_g H &= \sum_{k,\ell\in\Lambda} g^{k\ell}\partial_k\partial_\ell H\\ &= \sum_{i,j\in\Lambda}\sum_{k,\ell\in\Lambda}\delta_{ij}\delta_{k\ell}\Big(\partial_i\partial_j\partial_k\partial_\ell f - \partial_i\partial_j f\sum_{m\in\Lambda}\partial_k\partial_m f\,\partial_\ell\partial_m f\Big)\\ &\quad - \sum_{i,j\in\Lambda}\sum_{k,\ell\in\Lambda}\delta_{k\ell}(\partial_i\partial_k f\,\partial_j\partial_\ell f + \partial_i\partial_\ell f\,\partial_j\partial_k f)\partial_i\partial_j f\\ &= \sum_{i,j\in\Lambda}\partial_i^2\partial_j^2 f - H|A|^2 - 2\sum_{i,j,k\in\Lambda}\partial_i\partial_j f\,\partial_i\partial_k f\,\partial_j\partial_k f.\end{aligned}$$

Consequently, it follows that under the assumption $\nabla_{\boldsymbol{x}'} f = \mathbf{0}'$ at some point

$$\begin{aligned}&\Delta_g H + H|A|^2 - \frac{1}{2}H^3\\ &= \sum_{i,j\in\Lambda}\partial_i^2\partial_j^2 f - H|A|^2 - 2\sum_{i,j,k\in\Lambda}\partial_i\partial_j f\,\partial_i\partial_k f\,\partial_j\partial_k f + H|A|^2 - \frac{1}{2}H^3\\ &= \sum_{i,j\in\Lambda}\partial_i^2\partial_j^2 f - \frac{1}{2}H^3 - 2\sum_{i,j,k\in\Lambda}\partial_i\partial_j f\,\partial_i\partial_k f\,\partial_j\partial_k f.\end{aligned}$$

Since $A = (\partial_i\partial_j f)$ is a symmetric matrix, there exists an orthogonal matrix $P_{N-1} = (p_{ij})$ such that

$$P_{N-1}^{-1} A P_{N-1} = \begin{pmatrix}\kappa_1 & & \\ & \ddots & \\ & & \kappa_{N-1}\end{pmatrix},$$

where $\kappa_i (i = 1, \dots, N-1)$ are the principle curvatures. Taking account of $P_{N-1}^{-1} = {}^t P_{N-1}$, it follows that

$$A = P_{N-1} \begin{pmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_{N-1} \end{pmatrix} P_{N-1}^{-1} = P_{N-1} \begin{pmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_{N-1} \end{pmatrix} {}^t P_{N-1}$$

so that we have

$$\partial_i \partial_j f = \sum_{m \in \Lambda} \kappa_m p_{im} p_{jm}.$$

Since $\sum_k p_{ki} p_{kj} = \delta_{ij}$, this implies that

$$\begin{aligned} &\sum_{i,j,k \in \Lambda} \partial_i \partial_j f \partial_i \partial_k f \partial_j \partial_k f \\ &= \sum_{i,j,k \in \Lambda} \sum_{m_1,m_2,m_3 \in \Lambda} \kappa_{m_1} \kappa_{m_2} \kappa_{m_3} p_{i,m_1} p_{j,m_1} p_{i,m_2} p_{k,m_2} p_{j,m_3} p_{k,m_3} \\ &= \sum_{m_1,m_2,m_3 \in \Lambda} \kappa_{m_1} \kappa_{m_2} \kappa_{m_3} \Big(\sum_{i \in \Lambda} p_{i,m_1} p_{i,m_2} \Big) \Big(\sum_{j \in \Lambda} p_{j,m_1} p_{j,m_3} \Big) \Big(\sum_{k \in \Lambda} p_{k,m_2} p_{k,m_3} \Big) \\ &= \sum_{m_1,m_2,m_3 \in \Lambda} \kappa_{m_1} \kappa_{m_2} \kappa_{m_3} \delta_{m_1 m_2} \delta_{m_1 m_3} \delta_{m_2 m_3} = \sum_{m \in \Lambda} \kappa_m^3. \end{aligned}$$

Thus, we are led to

$$\begin{aligned} \Delta_g H + H|A|^2 - \frac{1}{2} H^3 &= \sum_{i,j \in \Lambda} \partial_i^2 \partial_j^2 f - \frac{1}{2} \Big(\sum_{m \in \Lambda} \kappa_m \Big)^3 - 2 \sum_{m \in \Lambda} \kappa_m^3 \\ &= \sum_{i \in \Lambda} \partial_i^4 f + 2 \sum_{\substack{i,j \in \Lambda \\ i<j}} \partial_i^2 \partial_j^2 f - \frac{5}{2} \sum_{m \in \Lambda} \kappa_m^3 \\ &\quad - \frac{3}{2} \sum_{\substack{m_1,m_2 \in \Lambda \\ m_1 \neq m_2}} \kappa_{m_1}^2 \kappa_{m_2} - 3 \sum_{\substack{m_1,m_2,m_3 \in \Lambda \\ m_1<m_2<m_3}} \kappa_{m_1} \kappa_{m_2} \kappa_{m_3}. \end{aligned}$$

B. Proof of Theorem 2.2

Theorem 2.2 with $\lambda = 0$ is the same as Cui [9, Theorem 3.2]. However, in [9], the author does not state the dependence of the constant C on $\alpha \in \mathbb{Z}^N$ and $m \in \mathbb{N}$. Noting this point, we give a proof of Theorem 2.2 in this subsection.

First, we prepare some notations, according to [9]. Set

$$P_{2k}(\mathbf{i}\boldsymbol{\zeta}) := \{(\mathbf{i}\zeta_1)^2 + \cdots + (\mathbf{i}\zeta_N)^2\}^k = (-1)^k (\zeta_1^2 + \cdots + \zeta_N^2)^k$$

for $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N$ and $k \in \mathbb{Z}_+$. Since it follows from direct calculations that there exist $\delta \in (0,1)$, $K_1 > 1$, and $K_2 > 1$ such that

$$\mathrm{Re}\{-P_4(\mathbf{i}\boldsymbol{\zeta}) - \lambda P_2(\mathbf{i}\boldsymbol{\zeta})\} \leq -\delta|\mathrm{Re}\boldsymbol{\zeta}|^4 + K_1|\operatorname{Im}\boldsymbol{\zeta}|^4 + K_2|\lambda|^2$$

for $\lambda \in \mathbb{R}$, where $\mathrm{Re}\boldsymbol{\zeta} := (\mathrm{Re}\zeta_1, \dots, \mathrm{Re}\zeta_N)$ and $\operatorname{Im}\boldsymbol{\zeta} := (\operatorname{Im}\zeta_1, \dots, \operatorname{Im}\zeta_N)$, we are led to

$$\left|e^{-P_4(\mathbf{i}\boldsymbol{\zeta})-\lambda P_2(\mathbf{i}\boldsymbol{\zeta})}\right| \leq e^{-\delta|\mathrm{Re}\boldsymbol{\zeta}|^4+K_1|\operatorname{Im}\boldsymbol{\zeta}|^4+K_2|\lambda|^2} \quad \text{for all } \boldsymbol{\zeta} \in \mathbb{C}^N. \tag{B.1}$$

Note that

$$G_N(\boldsymbol{x}, t) = c_N \int_{\mathbb{R}^N} e^{-(P_4(\mathbf{i}\boldsymbol{\xi})+\lambda P_2(\mathbf{i}\boldsymbol{\xi}))t+\mathbf{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_N}\, d\boldsymbol{\xi}.$$

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. The proof follows from the argument in that of [9, Theorem 3.2]. It follows from Cauchy's integral theorem that

$$\begin{aligned}
&D_{\boldsymbol{x}}^{\alpha}(-\Delta_{\boldsymbol{x}})^m G_N(\boldsymbol{x}, t) \\
&= c_N \int_{\mathbb{R}^N} (\mathbf{i}\boldsymbol{\xi} - \boldsymbol{\eta})^{\alpha} \left\{ -\sum_{j=1}^{N} (\mathbf{i}\xi_j - \eta_j)^2 \right\}^m e^{-(P_4(\boldsymbol{\xi}+\mathbf{i}\boldsymbol{\eta})+\lambda P_2(\boldsymbol{\xi}+\mathbf{i}\boldsymbol{\eta}))t+\mathbf{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_N - \langle \boldsymbol{x},\boldsymbol{\eta}\rangle_N}\, d\boldsymbol{\xi},
\end{aligned}$$

where $\boldsymbol{\eta} \in \mathbb{R}^N$ is arbitrary and independent of $\boldsymbol{\xi} \in \mathbb{R}^N$. Then, (B.1) implies that

$$\begin{aligned}
&|D_{\boldsymbol{x}}^{\alpha}(-\Delta_{\boldsymbol{x}})^m G_N(\boldsymbol{x}, t)| \\
&\quad \leq c_N e^{-\langle \boldsymbol{x},\boldsymbol{\eta}\rangle_N} \int_{\mathbb{R}^N} (|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{|\alpha|+2m} |e^{-(P_4(\boldsymbol{\xi}+\mathbf{i}\boldsymbol{\eta})+\lambda P_2(\boldsymbol{\xi}+\mathbf{i}\boldsymbol{\eta}))t}|\, d\boldsymbol{\xi} \\
&\quad \leq c_N e^{-\langle \boldsymbol{x},\boldsymbol{\eta}\rangle_N + K_1|\boldsymbol{\eta}|^4 t + K_2|\lambda|^2 t} \int_{\mathbb{R}^N} (|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{|\alpha|+2m} e^{-\delta|\boldsymbol{\xi}|^4 t}\, d\boldsymbol{\xi} \\
&\quad \leq c_N e^{-\langle \boldsymbol{x},\boldsymbol{\eta}\rangle_N + K_1|\boldsymbol{\eta}|^4 t + K_2|\lambda|^2 t} \sum_{k=0}^{|\alpha|+2m} \binom{|\alpha| + 2m}{k} |\eta|^{|\alpha|+2m-k} \int_{\mathbb{R}^N} |\boldsymbol{\xi}|^k e^{-\delta|\boldsymbol{\xi}|^4 t}\, d\boldsymbol{\xi}.
\end{aligned}$$

Applying the change of variable on the polar coordinate, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\boldsymbol{\xi}|^k e^{-\delta|\boldsymbol{\xi}|^4 t}\, d\boldsymbol{\xi} &= \omega_{N-1} \int_0^{\infty} r^{k+N-1} e^{-\delta r^4 t}\, dr \\
&= \frac{\omega_{N-1}}{4} \Gamma\left(\frac{k+N}{4}\right)(\delta t)^{-(N+k)/4} \\
&\leq \frac{\omega_{N-1}}{4} \max\left\{\Gamma\left(\frac{|\alpha| + 2m + N}{4}\right), \Gamma\left(\frac{N}{4}\right)\right\}(\delta t)^{-(N+k)/4},
\end{aligned}$$

where ω_{N-1} is the area of the $(N-1)$-dimensional unit ball. Taking account of the fact that the minimum of $\Gamma(s)$ exists and is positive for $s > 0$, we see that

$$\begin{aligned}
&|D_{\boldsymbol{x}}^{\alpha}(-\Delta_{\boldsymbol{x}})^m G_N(\boldsymbol{x}, t)| \\
&\quad \leq C\Gamma\left(\frac{|\alpha| + 2m + N}{4}\right) t^{-N/4} e^{-\langle \boldsymbol{x},\boldsymbol{\eta}\rangle_N + K_1|\boldsymbol{\eta}|^4 t + K_2|\lambda|^2 t} \left\{(\delta t)^{-1/4} + |\boldsymbol{\eta}|\right\}^{|\alpha|+2m},
\end{aligned}$$

where $C > 0$ is a constant depending only on N. Since $\boldsymbol{\eta} \in \mathbb{R}^N$ is arbitrary, we can choose $\boldsymbol{\eta} := (4K_1)^{-1/3} t^{-1/3} |\boldsymbol{x}|^{-2/3} \boldsymbol{x}$. Then, it follows that

$$\begin{aligned}
&e^{-\langle \boldsymbol{x},\boldsymbol{\eta}\rangle_N + K_1|\boldsymbol{\eta}|^4 t + K_2|\lambda|^2 t}\left\{(\delta t)^{-1/4} + |\boldsymbol{\eta}|\right\}^{|\alpha|+2m} \\
&\quad = e^{-\mu(|\boldsymbol{x}|^4/t)^{1/3} + K_2|\lambda|^2 t}\left\{\frac{1}{(\delta t)^{1/4}} + \frac{1}{(4K_1)^{1/3}}\left(\frac{|\boldsymbol{x}|}{t}\right)^{1/3}\right\}^{|\alpha|+2m} \\
&\quad = \nu^{(|\alpha|+2m)/4} t^{-(|\alpha|+2m)/4} e^{-\mu(|\boldsymbol{x}|^4/t)^{1/3} + K_2|\lambda|^2 t}\left\{1 + \frac{\delta^{1/4}}{(4K_1)^{1/3}}\left(\frac{|\boldsymbol{x}|}{t^{1/4}}\right)^{1/3}\right\}^{|\alpha|+2m} \\
&\quad \le \nu^{(|\alpha|+2m)/4} t^{-(|\alpha|+2m)/4} e^{-\mu(|\boldsymbol{x}|^4/t)^{1/3} + K_2|\lambda|^2 t}\left\{1 + \left(\frac{|\boldsymbol{x}|}{t^{1/4}}\right)^{1/3}\right\}^{|\alpha|+2m},
\end{aligned}$$

where

$$\mu = \frac{3}{4}(4K_1)^{-1/3} \in (0,1), \quad \nu = \frac{1}{\delta} > 1.$$

In the last inequality, we have used $\delta \in (0,1)$ and $K_1 > 1$. Since $1 + r^{1/3} \le 2^{2/3}(1+r)^{1/3}$ for $r > 0$, we are led to the desired result. ■

Acknowledgments. The authors would like to thank the anonymous referees for useful comments.

Funding. This project is supported by JSPS KAKENHI Grant numbers JP23H00085, JP23K03215, JP20K03748, and JP19K03562 for the first and second authors, JP22KJ0719 for the third author, and JP22K03425 for the fourth author.

References

[1] L. Angiuli, U. Massari, and M. Miranda, Jr., Geometric properties of the heat content. *Manuscripta Math.* **140** (2013), no. 3-4, 497–529 Zbl 1260.49078 MR 3019137

[2] G. Barles and C. Georgelin, A simple proof of convergence for an approximation scheme for computing motions by mean curvature. *SIAM J. Numer. Anal.* **32** (1995), no. 2, 484–500 Zbl 0831.65138 MR 1324298

[3] J. W. Barrett, H. Garcke, and R. Nürnberg, A parametric finite element method for fourth order geometric evolution equations. *J. Comput. Phys.* **222** (2007), no. 1, 441–462 Zbl 1112.65093 MR 2298053

[4] J. W. Barrett, H. Garcke, and R. Nürnberg, Parametric approximation of Willmore flow and related geometric evolution equations. *SIAM J. Sci. Comput.* **31** (2008), no. 1, 225–253 Zbl 1186.65133 MR 2460777

[5] J. Bence, B. Merriman, and S. Osher, Diffusion generated motion by mean curvature. In *Computational crystal growers workshop*, J. Taylored., Selected Lectures in Math., AMS, Province, RI, 1992

[6] E. Bretin, S. Masnou, and É. Oudet, Phase-field approximations of the Willmore functional and flow. *Numer. Math.* **131** (2015), no. 1, 115–171 Zbl 1326.49018 MR 3383330

[7] P. Colli and P. Laurençot, A phase-field approximation of the Willmore flow with volume constraint. *Interfaces Free Bound.* **13** (2011), no. 3, 341–351 Zbl 1233.35110 MR 2846014

[8] P. Colli and P. Laurençot, A phase-field approximation of the Willmore flow with volume and area constraints. *SIAM J. Math. Anal.* **44** (2012), no. 6, 3734–3754 Zbl 1262.35120 MR 3023428

[9] S. Cui, Local and global existence of solutions to semilinear parabolic initial value problems. *Nonlinear Anal.* **43** (2001), no. 3, Ser. A: Theory Methods, 293–323 Zbl 0963.35075 MR 1796979

[10] G. Dziuk, Computational parametric Willmore flow. *Numer. Math.* **111** (2008), no. 1, 55–80 Zbl 1158.65073 MR 2448203

[11] G. Dziuk, E. Kuwert, and R. Schätzle, Evolution of elastic curves in $\mathbb{R}^n$: existence and computation. *SIAM J. Math. Anal.* **33** (2002), no. 5, 1228–1245 Zbl 1031.53092 MR 1897710

[12] S. D. Eidel'man, *Parabolic systems*. North-Holland, Amsterdam, 1969

[13] S. Esedoḡlu and F. Otto, Threshold dynamics for networks with arbitrary surface tensions. *Comm. Pure Appl. Math.* **68** (2015), no. 5, 808–864 Zbl 1334.82072 MR 3333842

[14] S. Esedoḡlu, S. J. Ruuth, and R. Tsai, Threshold dynamics for high order geometric motions. *Interfaces Free Bound.* **10** (2008), no. 3, 263–282 Zbl 1157.65330 MR 2453132

[15] L. C. Evans, Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.* **42** (1993), no. 2, 533–557 Zbl 0802.65098 MR 1237058

[16] M. Fei and Y. Liu, Phase-field approximation of the Willmore flow. *Arch. Ration. Mech. Anal.* **241** (2021), no. 3, 1655–1706 Zbl 1476.35021 MR 4284532

[17] A. Ferrero, F. Gazzola, and H.-C. Grunau, Decay and eventual local positivity for biharmonic parabolic equations. *Discrete Contin. Dyn. Syst.* **21** (2008), no. 4, 1129–1157 Zbl 1172.35029 MR 2399453

[18] J. Fuchs and T. Laux, Strong convergence of the thresholding scheme for the mean curvature flow of mean convex sets. *Adv. Calc. Var.* **17** (2024), no. 2, 421–465 Zbl 1543.53087 MR 4726520

[19] F. Gazzola and H.-C. Grunau, Some new properties of biharmonic heat kernels. *Nonlinear Anal.* **70** (2009), no. 8, 2965–2973 Zbl 1170.35440 MR 2509382

[20] R. Grzhibovskis and A. Heintz, A convolution thresholding scheme for the Willmore flow. *Interfaces Free Bound.* **10** (2008), no. 2, 139–153 Zbl 1147.53054 MR 2453126

[21] H. Ishii, A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature. In *Curvature flows and related topics (Levico, 1994)*, pp. 111–127, GAKUTO Internat. Ser. Math. Sci. Appl. 5, Gakkōtosho, Tokyo, 1995 Zbl 0844.35043 MR 1365304

[22] H. Ishii, G. E. Pires, and P. E. Souganidis, Threshold dynamics type approximation schemes for propagating fronts. *J. Math. Soc. Japan* **51** (1999), no. 2, 267–308 Zbl 0935.53006 MR 1674750

[23] E. Kuwert and R. Schätzle, The Willmore flow with small initial energy. *J. Differential Geom.* **57** (2001), no. 3, 409–441 Zbl 1035.53092 MR 1882663

[24] E. Kuwert and R. Schätzle, Gradient flow for the Willmore functional. *Comm. Anal. Geom.* **10** (2002), no. 2, 307–339 Zbl 1029.53082 MR 1900754

[25] T. Laux and J. Lelmi, De Giorgi's inequality for the thresholding scheme with arbitrary mobilities and surface tensions. *Calc. Var. Partial Differential Equations* **61** (2022), no. 1, article no. 35 Zbl 07453345 MR 4358242

[26] T. Laux and F. Otto, Convergence of the thresholding scheme for multi-phase mean-curvature flow. *Calc. Var. Partial Differential Equations* **55** (2016), no. 5, article no. 129 Zbl 1388.35121 MR 3556529

[27] T. Laux and F. Otto, Brakke's inequality for the thresholding scheme. *Calc. Var. Partial Differential Equations* **59** (2020), no. 1, article no. 39 Zbl 07161205 MR 4056816
[28] T. Laux and F. Otto, The thresholding scheme for mean curvature flow and de Giorgi's ideas for minimizing movements. In *The role of metrics in the theory of partial differential equations*, pp. 63–93, Adv. Stud. Pure Math. 85, The Mathematical Society of Japan, Tokyo, 2020 Zbl 1486.65112 MR 4385030
[29] F. Leoni, Convergence of an approximation scheme for curvature-dependent motions of sets. *SIAM J. Numer. Anal.* **39** (2001), no. 4, 1115–1131 Zbl 1008.65067 MR 1870835
[30] P. Loreti and R. March, Propagation of fronts in a nonlinear fourth order equation. *European J. Appl. Math.* **11** (2000), no. 2, 203–213 Zbl 0960.49030 MR 1757511
[31] P. Mascarenhas, Diffusion generated motion by mean curvature. Campus report, University of California, Los Angeles, 1992
[32] U. F. Mayer and G. Simonett, A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow. *Interfaces Free Bound.* **4** (2002), no. 1, 89–109 Zbl 1005.65095 MR 1877537
[33] T. Metivet, A. Sengers, M. Ismaïl, and E. Maitre, Diffusion-redistanciation schemes for 2D and 3D constrained Willmore flow: application to the equilibrium shapes of vesicles. *J. Comput. Phys.* **436** (2021), article no. 110288 Zbl 07513851 MR 4237459
[34] S. Okabe and G. Wheeler, The p-elastic flow for planar closed curves with constant parametrization. *J. Math. Pures Appl. (9)* **173** (2023), 1–42 Zbl 1518.53077 MR 4572449
[35] A. Rätz and M. Röger, A new diffuse-interface approximation of the Willmore flow. *ESAIM Control Optim. Calc. Var.* **27** (2021), article no. 14 Zbl 1468.35240 MR 4233237
[36] R. C. Reilly, Variational properties of functions of the mean curvatures for hypersurfaces in space forms. *J. Differential Geometry* **8** (1973), 465–477 Zbl 0277.53030 MR 0341351
[37] F. Rupp, The volume-preserving Willmore flow. *Nonlinear Anal.* **230** (2023), article no. 113220 Zbl 1518.53078 MR 4541420
[38] R. E. Rusu, An algorithm for the elastic flow of surfaces. *Interfaces Free Bound.* **7** (2005), no. 3, 229–239 Zbl 1210.35149 MR 2171130
[39] S. J. Ruuth, Efficient algorithms for diffusion-generated motion by mean curvature. *J. Comput. Phys.* **144** (1998), no. 2, 603–625 Zbl 0946.65093 MR 1638032
[40] G. Simonett, The Willmore flow near spheres. *Differential Integral Equations* **14** (2001), no. 8, 1005–1014 Zbl 1161.35429 MR 1827100
[41] L. Vivier, Convergence of an approximation scheme for computing motions with curvature dependent velocities. *Differential Integral Equations* **13** (2000), no. 10-12, 1263–1288 Zbl 0983.65112 MR 1785707

Received 12 February 2024; revised 18 September 2024.

Katsuyuki Ishii
Graduate School of Maritime Sciences, Kobe University, 5-1-1 Fukaeminami-machi, Higashinada-ku, Kobe 658-0022, Japan; ishii@maritime.kobe-u.ac.jp

Yoshihito Kohsaka
Graduate School of Maritime Sciences, Kobe University, 5-1-1 Fukaeminami-machi, Higashinada-ku, Kobe 658-0022, Japan; kohsaka@maritime.kobe-u.ac.jp

Nobuhito Miyake
Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan; miyake@math.kyushu-u.ac.jp

Koya Sakakibara
Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa 920-1192; Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS), RIKEN, 2-1 Hirosawa, Wako 351-0198, Japan; ksakaki@se.kanazawa-u.ac.jp