

## Second order cumulants: Second order even elements and $R$ -diagonal elements

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**Abstract.** We introduce  $R$ -diagonal and even operators of second order. We give a formula for the second order free cumulants of the square  $x^2$  of a second order even element in terms of the second order free cumulants of  $x$ . Similar formulas are proved for the second order free cumulants of  $aa^*$ , when  $a$  is a second order  $R$ -diagonal operator. We also show that if  $r$  is second order  $R$ -diagonal and  $b$  is second order free from  $r$ , then  $rb$  is also second order  $R$ -diagonal. We present a large number of examples, in particular, the limit distribution of products of Ginibre matrices. We prove the conjectured formula of Dartois and Forrester for the fluctuations moments of the product of two independent complex Wishart matrices and generalize it to any number of factors.

### 1. Introduction

Roughly forty years ago, Voiculescu devised a new kind of independence, called *free independence* [27], for non-commuting random variables, which also gives a precise meaning to being in ‘general position’, but without the assumption that the random variables commute. This has been particularly fruitful in dealing with matrix-valued random variables.

The context for free probability theory is a non-commutative probability space  $(\mathcal{A}, \varphi)$ . Here  $\mathcal{A}$  is a unital algebra over  $\mathbb{C}$ , and  $\varphi: \mathcal{A} \rightarrow \mathbb{C}$  is a linear functional with  $\varphi(1) = 1$ . Elements of  $\mathcal{A}$  are our random variables and  $\varphi$  is our expectation. If  $a_1, \dots, a_s \in \mathcal{A}$ , then  $\{\varphi(a_{i_1} \cdots a_{i_n}) \mid i_1, \dots, i_n \in \{1, \dots, s\}\}$  are the mixed moments of  $\{a_1, \dots, a_s\}$ . If  $a_1, \dots, a_s$  are freely independent, then the mixed moments of  $\{a_1, \dots, a_s\}$  are determined, according to a *universal rule*, by the individual moments of each  $a_k$ ,  $1 \leq k \leq s$ .

In this article, we shall further assume that  $\mathcal{A}$  is an involutive algebra, i.e., there is a conjugate linear map  $a \mapsto a^*$  such that  $(ab)^* = b^*a^*$ . Furthermore, we shall assume that  $\varphi(a^*a) \geq 0$  for  $a \in \mathcal{A}$  and that  $\varphi$  is a trace, although many results remain valid

in greater generality. Such a pair  $(\mathcal{A}, \varphi)$  is frequently called a *non-commutative \*-probability space*. If  $\mathcal{A}$  is a  $C^*$ -algebra and  $a \in \mathcal{A}$  is self-adjoint, then  $a$  has a spectral measure,  $\mu_a$ , (relative to  $\varphi$ ) given by  $\varphi(p(a)) = \int_{\mathbb{R}} p(t) d\mu_a(t)$  for polynomials  $p$ . If  $\mathcal{A}$  is a  $C^*$ -algebra and  $a_1$  and  $a_2$  are freely independent and self-adjoint, then the universal rule determines the spectral measure of  $a_1 + a_2$  from that of  $a_1$  and  $a_2$  (see [25, Lecture 12]).

The importance of free probability theory in random matrix theory is that it allows to understand the eigenvalue distribution of many random matrix ensembles that are constructed from others.

By a random matrix ensemble we mean a sequence of random matrices  $\{X_N\}_N$  where  $X_N$  is a  $N \times N$  random matrix. The eigenvalues of  $X_N$  are random. If  $X_N = X_N^*$ , the eigenvalues are real and random. If  $X_N = X_N^*$  and for all  $z$  in the complex upper half plane  $\mathbb{C}^+$ , the limit  $\lim_N E(\frac{1}{N} \text{Tr}((z - X_N)^{-1}))$  exists, we say that the ensemble has a *limit eigenvalue distribution*.

To be concrete, if  $X_N$  and  $Y_N$  both have limit eigenvalue distributions, we may ask about the limit eigenvalue distribution  $X_N + Y_N$ . It turns out that Voiculescu’s free independence rule allows us to calculate the limit eigenvalue distribution of  $X_N + Y_N$ . There are many results, starting with [26], giving sufficient conditions for two ensembles to be asymptotically free (see [22, Chapter 4]).

One way to understand such universal rule is via the free cumulants. In classical probability, the characteristic function of a measure and its logarithm play a prominent role. In free probability, the logarithm of the characteristic function gets replaced by Voiculescu’s  $R$ -transform given by

$$R(z) = G^{(-1)}(z) - z^{-1},$$

where  $G(z) = E((z - X)^{-1})$  is the *Cauchy transform* of  $X$ ,  $z$  is in the complex upper half plane  $\mathbb{C}^+$  and the inverse  $G^{(-1)}$  is defined on a suitable domain in  $\mathbb{C}$ . If  $R$  has a power series expansion  $\sum_{n \geq 1} \kappa_n z^{n-1}$ , which is always the case with a bounded random variable, then the coefficients,  $\{\kappa_n\}_{n \geq 1}$ , of  $R$  are called the *free cumulants* of  $X$ .

After one understands the limit eigenvalue distribution of an ensemble  $\{X_N\}_N$ , the next object of study is the covariance of traces of resolvents

$$\lim_N \text{cov}(\text{Tr}((z - X_N)^{-1}), \text{Tr}((w - X_N)^{-1})),$$

which we call the *second order Cauchy transform* of the limit distribution. In this paper, we shall be concerned with the combinatorial properties of this function; see [10] for the properties of this analytic function.

The limiting covariance of the random variables  $\text{Tr}((z - X_N)^{-1})$  and  $\text{Tr}((w - X_N)^{-1})$  is referred to in the statistical literature as the linear spectral statistics of the ensemble, see [4, 12].

In [21], an extension of free independence was initiated, called second order freeness. The purpose was to find the analogue of universal rule above for linear spectral statistics. In [8], the second order analogue of the  $R$ -transform and second order cumulants were presented. It is these second order cumulants that are the subject of this article.

In particular, we wish to apply them to the study of  $R$ -diagonal operators and even operators.  $R$ -diagonal operators are certain non-normal operators in a  $*$ -probability space. It was the study of these operators that led to the solution of the invariant subspace problem for operators in  $\text{II}_1$  factors [15]. In random matrix theory, they appear in the famous single ring theorem [14] which is a generalization of the circular law for Ginibre matrices. This distribution is the limiting distribution of random matrix ensembles of the form  $U_n T_n V_n$ , where  $T_n$  is positive,  $U_n$  and  $V_n$  are Haar unitaries with independent entries.

Even operators are self-adjoint operators for which all odd moments are zero. There is a close relation between even and  $R$ -diagonal operators. If  $a$  is  $R$ -diagonal, then its distribution can be recovered from its Hermitization  $\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ , which is an even operator.

The contribution of this article is to extend the idea of  $R$ -diagonality and evenness to the case of second order freeness. In particular, for a second order  $R$ -diagonal operator we have, letting,  $\beta_n^{(a)} := \kappa_{2n}(a, a^*, \dots, a, a^*)$  and

$$\beta_{p,q}^{(a)} := \kappa_{2p,2q}(a, a^*, \dots, a, a^*)$$

be the determining series of  $a$ , we have that

$$\kappa_{p,q}(aa^*, \dots, aa^*) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \beta_{(\mathcal{V}, \pi)}, \tag{1.1}$$

where  $\mathcal{PS}_{\text{NC}}(p, q)$  is the set of non-crossing partitioned permutations on a  $(p, q)$ -annulus introduced in Section 2.3.

There is a similar construction of the determining series of a second order even operator so that the same result holds (see equation (3.1)).

In recent work of Borot, Charbonnier, Garcia-Failde, Leid, and Shadrin, [6], the theory of higher order freeness has been connected to topological recursion. In particular, a third order version of equation (2.3) is now known. As it only involves products of the terms in equation (2.3), we expect that much of our formalism extends to higher orders.

Formula (1.1) and its counterpart for even elements imply a functional equation between the series of  $*$ -cumulants of  $a$  and the cumulant of  $aa^*$ , and thus is useful to relate their fluctuation moments. As an application of our main results, at the end of the paper, in Section 8, we work out the  $*$ -cumulants of a number of examples.

These include second order Haar unitaries and products of semicircular and circular operators. In particular, we highlight three important applications to random matrix ensembles.

- (i) We prove the conjectured formula of Dartois and Forrester [9] for the fluctuations moments of the product of two independent complex Wishart matrices and generalize it to any number of factors. See Remark 8.14.
- (ii) We obtain the second order  $*$ -cumulants and second order Cauchy transform of Wishart ensembles with a given covariance: i.e.,  $WAW^*$ , where  $W$  is a Ginibre matrix and  $A$  is a deterministic matrix.
- (iii) We extend the calculation of Dubach and Peled [11] of some fluctuation moments of products of Ginibre matrices, to the case of general  $*$ -moments.

### Outline of the paper

In Section 2, we review the basic notations of second order freeness and the definition and properties of even and  $R$ -diagonal operators of first order. In Section 3, we present the definitions and statements of our main results. In Section 4, we recall the results about non-crossing annular permutations that we need and prove the main technical results needed in Sections 5 and 6. In Section 5, we prove Theorem 3.5, our main result on the determining series of second order  $R$ -diagonal operators. In Section 6, we prove Theorem 3.6, our main result on the determining series of second order even operators. In Section 7, we extend a result of Arizmendi and Vargas on cumulants of products to the second order case. In Section 8, we present many examples of computations of second order  $*$ -cumulants using Theorems 3.5 and 3.6.

As we shall see in Section 8, some properties of first order freeness have a simple extension to the second order case while others do not. For example, contrary to the first order case, the powers of a second order  $R$ -diagonal operator are not necessarily second order  $R$ -diagonal operators (see Example 8.18). Also, creating freeness of second order by conjugating by second order free unitary is more delicate. Recall from [24] that for  $a$  and  $b$  (first order) free elements such that  $b$  is  $R$ -diagonal (or even) we also have that  $bab^*$  is free from  $a$ . Example 8.19 shows that this is not true at the second order level. Another interesting example that shows that second order is more delicate is the relation between the square  $s^2$  of a semicircular  $s$  and the square  $cc^*$  of a circular element. As shown in [25], they both correspond to a free Poisson or Marchenko–Pastur distribution. Interestingly,  $s^2$  and  $cc^*$  have different fluctuation moments. This is shown in Examples 8.6 and 8.8.

## 2. Second order cumulants and second order freeness

### 2.1. Non-crossing partitions and free cumulants

Before reviewing second order cumulants, let us recall some of the basics of free probability which can be found in [25]. We are going through the details here because the construction for the second order case (not in [25]) follows a similar pattern.  $R$ -diagonal operators are defined in terms of free cumulants, so let us begin with these. We let  $[n] = \{1, 2, 3, \dots, n\}$  and  $\mathcal{P}(n)$  be the partitions of  $[n]$ . Recall that  $\pi = \{V_1, \dots, V_k\}$  is a partition of  $[n]$  if  $V_1 \cup \dots \cup V_k = [n]$  and  $V_i \cap V_j = \emptyset$ . The subsets  $V_1, \dots, V_k$  are the *blocks* of  $\pi$ . For  $\pi, \sigma \in \mathcal{P}(n)$ , we write  $\pi \leq \sigma$  if each block of  $\pi$  is contained in some block of  $\sigma$ . This defines a partial order on  $\mathcal{P}(n)$  and with this partial order  $\mathcal{P}(n)$  becomes a lattice.

A partition  $\pi$  of  $[n]$  has a *crossing* if we can find two blocks  $V_r$  and  $V_s$  (assuming  $r \neq s$ ) and  $i < j < k < l$  with  $i, k \in V_r$  and  $j, l \in V_s$ . A partition is *non-crossing* if it has no crossings. Let denote the non-crossing partitions of  $[n]$  by  $\text{NC}(n)$ . We shall regard each partition  $\pi$  on  $[n]$  as a permutation of  $[n]$  as follows. The cycles of the permutation  $\pi$  are the blocks of the partition  $\pi$  with the elements of each block written in increasing order. Using the metric condition given below, Biane [5] identified those permutations which come from a non-crossing partition.

For  $\pi$  a partition or permutation, let  $\#(\pi)$  denote the number of blocks or cycles accordingly. If  $\pi$  is a partition or permutation of  $[n]$ , we let  $|\pi| = n - \#(\pi)$ . For any permutations  $\pi$  and  $\sigma$ , we always have the triangle inequality  $|\pi\sigma| \leq |\pi| + |\sigma|$ . We let  $\gamma_n$  be the permutation of  $[n]$  with the single cycle  $(1, 2, \dots, n)$ . We can now state Biane’s characterization as follows. A permutation  $\pi$  produces a non-crossing partition if and only if  $|\gamma_n| = |\pi| + |\pi^{-1}\gamma_n|$ ; i.e., if the triangle inequality becomes an equality. Note that we simultaneously think of  $\pi$  as a permutation and as a partition; we shall do this frequently in this paper.

Now suppose that  $(\mathcal{A}, \varphi)$  is a non-commutative probability space and for each  $n$  we have  $\kappa_n: \mathcal{A}^{\otimes n} \rightarrow \mathbb{C}$  an  $n$ -linear functional. Given a partition  $\pi$  of  $[n]$ , we can define an  $n$ -linear map  $\kappa_\pi: \mathcal{A}^{\otimes n} \rightarrow \mathbb{C}$  by the following rule. For  $a_1, \dots, a_n \in \mathcal{A}$ ,

$$\kappa_\pi(a_1, \dots, a_n) = \prod_{\substack{V \in \pi \\ V = (i_1, \dots, i_l)}} \kappa_l(a_{i_1}, \dots, a_{i_l}).$$

We are assuming here that  $i_1 < i_2 < \dots < i_l$ .

This construction can be then used to define the sequence  $\{\kappa_n\}_n$  through the moment-cumulant formula:

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \text{NC}(n)} \kappa_\pi(a_1, \dots, a_n). \tag{2.1}$$

This defines the free cumulants inductively and recursively (see [24, Lecture 11]). Let us recall the definition of  $R$ -diagonal elements from [24, Lecture 15]. Recall that we are assuming that  $\mathcal{A}$  is a  $*$ -algebra. For  $\varepsilon \in \{-1, 1\}$ , we let  $a^{(\varepsilon)} = a$  if  $\varepsilon = 1$  and  $a^{(\varepsilon)} = a^*$  if  $\varepsilon = -1$ .

**Definition 2.1.** Let  $a \in \mathcal{A}$ . We say that  $a$  is  $R$ -diagonal if for every  $n$  and every  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ , we have that

$$\kappa_n(a^{(\varepsilon_1)}, a^{(\varepsilon_2)}, \dots, a^{(\varepsilon_n)}) = 0$$

whenever either  $n$  is odd or there is  $1 \leq i < n$  with  $\varepsilon_i = \varepsilon_{i+1}$ . We can express this by saying that all  $*$ -cumulants are zero except possibly  $\kappa_{2l}(a, a^*, \dots, a, a^*)$  and  $\kappa_{2l}(a^*, a, \dots, a^*, a)$ , i.e., we get 0 unless the ‘ $a$ ’s alternate with the ‘ $a^*$ ’s.

**Definition 2.2.** Let  $a \in \mathcal{A}$ . We say that a self-adjoint element  $a$  is *even* if all its odd free cumulants vanish, i.e.,  $\kappa_{2l-1}(a, \dots, a) = 0$  for  $l \geq 1$ . By the moment-cumulant formula, this is equivalent to having all odd moments vanish.

### 2.2. Non-crossing annular permutations

The context for second order freeness is a second order probability space  $(\mathcal{A}, \varphi, \varphi_2)$ . Here  $\mathcal{A}$  is a unital algebra over  $\mathbb{C}$  and  $\varphi: \mathcal{A} \rightarrow \mathbb{C}$  is a linear functional with  $\varphi(1) = 1$  as before. We assume in addition that  $\varphi$  is *tracial*, i.e.,  $\varphi(ab) = \varphi(ba)$ . We assume that  $\varphi_2: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}$  is bilinear, symmetric, tracial in each variable, and is such that  $\varphi_2(1, a) = \varphi_2(a, 1) = 0$  for all  $a \in \mathcal{A}$ . In keeping with our previous assumption, we continue to assume that  $\mathcal{A}$  is a  $*$ -algebra. For an element  $a \in \mathcal{A}$ , we call the double indexed sequence  $\{\varphi_2(a^m, a^n)\}_{m,n}$  the *fluctuation moments* of  $a$ . As mentioned above, the point of second order freeness is that it gives a universal rule for computing fluctuation moments of sums and products of random variables given their individual moments and fluctuation moments. When variables satisfy these universal rules, we say that the elements are second order free.

Recall from [21] that unital subalgebras  $\mathcal{A}_1, \dots, \mathcal{A}_s$  of  $\mathcal{A}$ , where  $(\mathcal{A}, \varphi, \varphi_2)$  is a second order probability space, are *second order free* if they are free with respect to  $\varphi$  and for all  $m, n \geq 1$  and all  $a_1, \dots, a_m, b_1, \dots, b_n \in \mathcal{A}$  such that

- (i)  $\varphi(a_1) = \dots = \varphi(a_m) = \varphi(b_1) = \dots = \varphi(b_n) = 0$ ,
- (ii)  $a_i \in \mathcal{A}_{k_i}$  with  $k_1 \neq k_2 \neq \dots = k_m \neq k_1$ ,
- (iii)  $b_j \in \mathcal{A}_{l_j}$  with  $l_1 \neq l_2 \neq \dots = l_n \neq l_1$ ,

then for  $m, n \geq 2$ , we have

$$\varphi_2(a_1 \cdots a_m, b_1 \cdots b_n) = \delta_{m,n} \sum_{p=1}^m \prod_{i=1}^m \varphi(a_i b_{n+p-i}),$$

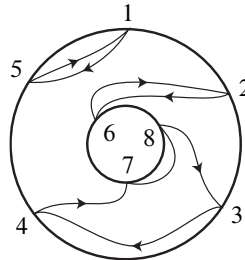
where all indices are interpreted modulo  $m$ .

Here we shall present the cumulant approach from [8, Chapter 7] to second order freeness and for this we must start with an explanation of *non-crossing annular permutations* [19] and [22, Chapter 5].

Let  $m, n \geq 1$  be integers. As before for  $\pi \in S_{m+n}$  (the symmetric group on  $[m+n]$ ), we let  $\#(\pi)$  be the number of cycles in the cycle decomposition of  $\pi$  and  $|\pi| = m+n - \#(\pi)$ . The following permutation will play a special role. Let  $\gamma_{m,n}$  be the permutation with the two cycles  $(1, \dots, m)(m+1, \dots, m+n)$ . A subset of  $[p+q]$  of the form  $\{r, \gamma_{p,q}(r), \gamma_{p,q}^2(r), \dots, \gamma_{p,q}^{k-1}(r)\}$  is a *cyclic interval*.

As before, we always have by the triangle inequality that  $|\gamma_{m,n}| \leq |\pi| + |\pi^{-1}\gamma_{m,n}|$ . If we have equality, then  $\pi$  can be written as  $\pi_1 \times \pi_2$  with  $\pi_1$  a non-crossing partition of  $[m]$  and  $\pi_2$  a non-crossing partition of  $[m+1, m+n] = \{m+1, \dots, m+n\}$ . This is Biane’s characterization. In particular, the equality means that no cycle of  $\pi$  connects the two cycles of  $\gamma_{m,n}$ .

If  $\pi$  does have a cycle that connects the two cycles of  $\gamma_{m,n}$ , then we call  $\pi$  a *non-crossing annular permutation* if  $|\pi| + |\pi^{-1}\gamma_{m,n}| = |\gamma_{m,n}| + 2$ . It was shown in [19] that this is equivalent to being able to draw the cycles of  $\pi$  in an annulus without the blocks crossing, see Figure 1. We denote the non-crossing annular permutations by  $S_{NC}(m, n)$ .



**Figure 1.** A non-crossing annular permutation of the  $(5, 3)$ -annulus. From a diagrammatic point of view, we go around the cycles in clockwise order with the interior of the cycle lying between the two circles in such a way that the cycles do not intersect.

### 2.3. Partitioned permutations

To introduce second order cumulants, we will use a moment-cumulant formula like the one in (2.1), except we shall use the partitioned permutations introduced in [20].

**Definition 2.3.** Let  $\pi \in S_n$  be a permutation and  $\mathcal{U} \in \mathcal{P}(n)$  a partition of  $[n]$ . We write  $\pi \leq \mathcal{U}$  to mean that each cycle of  $\pi$  is contained in some block of  $\mathcal{U}$ . We say the pair  $(\mathcal{U}, \pi)$  is a *partitioned permutation* if  $\pi \leq \mathcal{U}$ . We set  $|\mathcal{U}| = n - \#(\mathcal{U})$ , where as usual  $\#(\mathcal{U})$  is the number of blocks of  $\mathcal{U}$ . We let  $\mathcal{PS}(n)$  be the set of partitioned permutations of  $[n]$ .

When considering partitioned permutations it is sometimes convenient to have a notation to distinguish between a permutation  $\pi$  and the partition coming from its cycle decomposition. In these situations, we shall write  $0_\pi$  to denote this partition.

Recall that  $\mathcal{P}(n)$  is a lattice under the order  $\mathcal{U} \leq \mathcal{V}$  if every block of  $\mathcal{U}$  is contained in some block of  $\mathcal{V}$ . Note that if  $\mathcal{U}$  and  $\mathcal{V}$  are partitions of  $[n]$ , then we have the triangle inequality  $|\mathcal{U} \vee \mathcal{V}| \leq |\mathcal{U}| + |\mathcal{V}|$ . We can define a binary operation on  $\mathcal{PS}(n)$  by setting  $(\mathcal{U}, \pi)(\mathcal{V}, \sigma) = (\mathcal{U} \vee \mathcal{V}, \pi\sigma)$ . For this to make sense, we need to make the easy observation that  $\pi\sigma \leq \mathcal{U} \vee \mathcal{V}$ . More importantly, we can put a length function on  $\mathcal{PS}(n)$  by setting  $|(\mathcal{U}, \pi)| = 2|\mathcal{U}| - |\pi|$ . This length function satisfies the triangle inequality

$$|(\mathcal{U} \vee \mathcal{V}, \pi\sigma)| \leq |(\mathcal{U}, \pi)| + |(\mathcal{V}, \sigma)|, \tag{2.2}$$

see [20, §2.3].

Given a partitioned permutation  $(\mathcal{U}, \pi)$ , we shall call the partitioned permutations  $(\mathcal{V}, \sigma)$  and  $(\mathcal{W}, \tau)$  an *exact factorization* of  $(\mathcal{U}, \pi)$  if we have  $(\mathcal{V}, \sigma)(\mathcal{W}, \tau) = (\mathcal{U}, \pi)$  (i.e.,  $\tau = \sigma^{-1}\pi$  and  $\mathcal{U} = \mathcal{V} \vee \mathcal{W}$ ) and  $|(\mathcal{U}, \pi)| = |(\mathcal{V}, \sigma)| + |(\mathcal{W}, \tau)|$ , i.e., the inequality in (2.2) becomes an equality.

In [8, Proposition 5.11], all the exact factorizations of  $(1_n, \gamma_n)$  and  $(1_{m+n}, \gamma_{m,n})$  were found. Here  $1_n$  denotes the partition on  $[n]$  with 1 block. All factorizations of  $(1_n, \gamma_n)$  are of the form  $(0_\pi, \pi)(0_{\pi^{-1}\gamma_n}, \pi^{-1}\gamma_n)$  with  $\pi \in \text{NC}(n)$  and  $\pi^{-1}\gamma_n$  the Kreweras complement of  $\pi$ .

Three possible factorizations of  $(1_{m+n}, \gamma_{m,n})$  were found. First if  $\pi \in S_{\text{NC}}(m, n)$ , and then

$$\begin{aligned} (0_\pi, \pi)(0_{\pi^{-1}\gamma_{m,n}}, \pi^{-1}\gamma_{m,n}) &= (1_{m+n}, \gamma_{m,n}), \\ |(0_\pi, \pi)| + |(0_{\pi^{-1}\gamma_{m,n}}, \pi^{-1}\gamma_{m,n})| &= |(1_{m+n}, \gamma_{m,n})|. \end{aligned}$$

We let  $\mathcal{PS}_{\text{NC}}(m, n)'$  to be the set of  $(\mathcal{U}, \pi)$ 's, where  $\pi = \pi_1 \times \pi_2 \in \text{NC}(m) \times \text{NC}([m + 1, m + n])$  and  $\mathcal{U} \in \mathcal{P}(m + n)$  is obtained from  $\pi$  as follows: one block of  $\mathcal{U}$  is the union of a cycle of  $\pi_1$  and a cycle of  $\pi_2$ ; all other blocks of  $\mathcal{U}$  are cycles of  $\pi$ . For such a  $(\mathcal{U}, \pi)$ , we have

$$\begin{aligned} (\mathcal{U}, \pi)(0_{\pi^{-1}\gamma_{m,n}}, \pi^{-1}\gamma_{m,n}) &= (1_{m+n}, \gamma_{m,n}), \\ |(\mathcal{U}, \pi)| + |(0_{\pi^{-1}\gamma_{m,n}}, \pi^{-1}\gamma_{m,n})| &= |(1_{m+n}, \gamma_{m,n})|. \end{aligned}$$

Finally, if  $\mathcal{V}$  is obtained by joining a cycle of  $\pi_1^{-1}\gamma_m$  with a cycle of  $\pi_2^{-1}\gamma_n$  and then having all other blocks of  $\mathcal{V}$  cycles of  $\pi^{-1}\gamma_{m,n}$ , then

$$\begin{aligned} (0_\pi, \pi)(\mathcal{V}, \pi^{-1}\gamma_{m,n}) &= (1_{m+n}, \gamma_{m,n}), \\ |(0_\pi, \pi)| + |(\mathcal{V}, \pi^{-1}\gamma_{m,n})| &= |(1_{m+n}, \gamma_{m,n})|. \end{aligned}$$

We write  $\mathcal{PS}_{\text{NC}}(m, n)$  to denote the union  $S_{\text{NC}}(m, n) \cup \mathcal{PS}_{\text{NC}}(m, n)'$ .



### 2.4. Second order cumulants

Let  $(\mathcal{A}, \varphi, \varphi_2)$  be a second order probability space. We can now describe the second order cumulants in terms of the moment-cumulant formula. Recall that the first order cumulants are given by the moment-cumulant formula, see [25, Proposition 11.4],

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in \text{NC}(n)} \kappa_\pi(a_1, \dots, a_n).$$

We use the same method to construct the second order cumulants:

$$\begin{aligned} &\varphi_2(a_1 \cdots a_m, a_{m+1} \cdots a_{m+n}) \\ &= \sum_{\pi \in S_{\text{NC}}(m, n)} \kappa_\pi(a_1, \dots, a_{m+n}) + \sum_{(\mathcal{U}, \pi) \in \mathcal{PS}_{\text{NC}}(m, n)'} \kappa_{(\mathcal{U}, \pi)}(a_1, \dots, a_{m+n}). \end{aligned}$$

The  $\kappa_\pi$ 's in the first term are first order cumulants, summed over  $S_{\text{NC}}(m, n)$ . The  $\kappa_{(\mathcal{U}, \pi)}$ 's in the second term are second order cumulants. Recall that first order cumulants are defined as multiplicative functions (see [25, p. 164]); we shall do the same for the second order cumulants. Indeed, if  $\pi = (1, 2, 4)(3)(5, 6)(7) \in \text{NC}(4) \times \text{NC}(3)$  and  $\mathcal{U} = \{(1, 2, 4), (3, 5, 6), (7)\} \in \mathcal{P}(7)$ , then

$$\kappa_{(\mathcal{U}, \pi)}(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \kappa_3(a_1, a_2, a_4) \kappa_{1,2}(a_3, a_5, a_6) \kappa_1(a_7).$$

In [22, Definition 5.31], there is a full explanation. Note that a simplification of notation is used there in that  $(\mathcal{U}, \pi)$  is denoted by  $\sigma$  and is called a *marked permutation* in that the cycles of  $\pi_1$  and  $\pi_2$  joined by  $\mathcal{U}$  are called *marked*. In our example, the marked cycles are (3) and (5, 6).

If  $a$  is an element of a second order non-commutative probability space  $(\mathcal{A}, \varphi, \varphi_2)$ , we call the set  $\{\varphi(a^n)\}_{n \geq 1}$  the *moment sequence* of  $a$  and the set  $\{\varphi_2(a^m, a^n)\}_{m, n \geq 1}$  the *fluctuation moment sequence* of  $a$ .

The moment-cumulant formulas may be stated in terms of the formal power series, (see [22, Theorem 39]). Let us define the cumulant series

$$R(z) = \frac{1}{z} \sum_{n \geq 1} \kappa_n^a z^n, \quad R(z, w) = \frac{1}{zw} \sum_{p, q \geq 1} \kappa_{p, q}^a z^p w^q,$$

where  $\kappa_n^a = \kappa_n(a, \dots, a)$  and  $\kappa_{p, q}^a = \kappa_{p, q}(a, \dots, a)$ . Let us also denote the moment series

$$G(z) = \frac{1}{z} \sum_{n \geq 0} \varphi(a^n) z^{-n}, \quad G(z, w) = \frac{1}{zw} \sum_{n, m \geq 1} \varphi(a^p, a^q) z^{-p} w^{-q}.$$

Then we have the relations

$$\frac{1}{G(z)} + R(G(z)) = z$$

and

$$G(z, w) = G'(z)G'(w)R(G(z), G(w)) + \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G(z) - G(w)}{z - w} \right). \tag{2.3}$$

Note that

$$\frac{\partial^2}{\partial z \partial w} \log \left( \frac{G(z) - G(w)}{z - w} \right) = \frac{G'(z)G'(w)}{(G(z) - G(w))^2} - \frac{1}{(z - w)^2},$$

and hence we can write (2.3) without the logarithm. In addition, by letting  $\zeta = G(z)$  and  $\omega = G(w)$ , we can write (2.3) as a transformation of differential forms<sup>1</sup>

$$[(z - w)^{-2} + G(z, w)] dz dw = [(\zeta - \omega)^{-2} + R(\zeta, \omega)] d\zeta d\omega.$$

In addition, if  $\tilde{G} = \varphi \circ G$  with  $\varphi$  a Möbius function (fractional linear transformation), then

$$\frac{G'(z)G'(w)}{(G(z) - G(w))^2} = \frac{\tilde{G}'(z)\tilde{G}'(w)}{(\tilde{G}(z) - \tilde{G}(w))^2}.$$

So if we let  $F = 1/G$ , then we can expand  $F(z)$  as a formal power series in  $\frac{1}{z}$  starting with  $\varphi(a)$  (see [22, Lemma 3.20]) and then we can expand  $\frac{F(z)-F(w)}{z-w}$  as a formal power series starting with 1, and hence  $\log \frac{F(z)-F(w)}{z-w}$  also makes sense as a formal power series. As a function of two variables, its significance is that the analyticity of  $\log \frac{F(z)-F(w)}{z-w}$  is equivalent to the univalence of  $F$ . See [16, §1] for further discussion.

### 2.5. Second order freeness

Recall from the introduction that second order freeness gives a universal rule for computing the moments and fluctuation moments of  $a_1 + a_2$  provided one knows the moments and fluctuation moments of  $a_1$  and  $a_2$  separately, and  $a_1$  and  $a_2$  are second order free. The rule is given in [22, Definition 5.24]. For deciding when two or more elements are second order free, it is very convenient to use the rule of vanishing of mixed cumulants, [22, Theorem 5.34]. Given subalgebras  $\mathcal{A}_1, \dots, \mathcal{A}_s$  of a second order non-commutative probability space  $(\mathcal{A}, \varphi, \varphi_2)$  and elements  $a_1, \dots, a_n$  with  $a_i \in \mathcal{A}_{j_i}$ , we say that the cumulant  $\kappa_n(a_1, \dots, a_n)$  is *mixed* if there are  $i_1$  and  $i_2$  in  $[n]$  such that  $j_{i_1} \neq j_{i_2}$ ; the same applies for second order cumulants  $\kappa_{m,n}(a_1, \dots, a_{m+n})$ . We say that *mixed cumulants vanish* if

$$\kappa_n(a_1, \dots, a_n) = 0 \quad \text{and} \quad \kappa_{m,n}(a_1, \dots, a_{m+n}) = 0$$

whenever the cumulant is mixed. The formulation of second order freeness we shall use is that subalgebras  $\mathcal{A}_1, \dots, \mathcal{A}_s$  are second order free if all mixed cumulants vanish.

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<sup>1</sup>We are grateful for Gaëtan Borot for bringing this to our attention.

**2.6. Cumulants with products as entries**

A crucial tool in this paper will be the formula for writing a cumulant with products as entries as a sum of cumulants with all entries single operators [18]. For example, for first order cumulants we have

$$\begin{aligned} \kappa_2(a_1a_2, a_3a_4) &= \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\ &\quad + \kappa_3(a_1, a_3, a_4)\kappa_1(a_2) + \kappa_3(a_1, a_2, a_4)\kappa_1(a_3) \\ &\quad + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) \\ &\quad + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) + \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) \\ &\quad + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) + \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4). \end{aligned}$$

Note that on the right-hand side of the equation above each entry of a cumulant is a single ‘ $a$ ’. The general result is given in Theorem 2.4 below.

**Theorem 2.4** ([25, Theorem 11.12]). *Let  $n_1, \dots, n_r$  be positive integers and  $n = n_1 + \dots + n_r$ . Let  $a_1, \dots, a_n \in (\mathcal{A}, \varphi)$ . Then*

$$\kappa_r(a_1 \cdots a_{n_1}, \dots, a_{n_1+\dots+n_{r-1}+1} \cdots a_{n_1+\dots+n_r}) = \sum_{\pi \in \text{NC}(n)} \kappa_\pi(a_1, \dots, a_n),$$

where the sum is over all  $\pi$ ’s such that  $\pi \vee \tau_{\vec{n}} = 1_n$  and  $\tau_{\vec{n}}$  is the partition with blocks  $(1, \dots, n_1), \dots, (n_1 + \dots + n_{r-1} + 1, \dots, n_1 + \dots + n_r)$ .

In our example above,  $n_1 = n_2 = 2$ ,  $\tau_{\vec{n}} = \{(1, 2), (3, 4)\}$ , and of the 14 non-crossing partitions of [4], there are ten that satisfy the condition  $\pi \vee \tau_{\vec{n}} = 1_4$ ; the four that do not being  $\{(1), (2), (3), (4)\}, \{(1), (2), (3, 4)\}, \{(1, 2), (3), (4)\}$ , and  $\{(1, 2), (3, 4)\}$ .

We shall need the second order version of this expansion. For example, if  $a_1, a_2, a_3$  are in a second order probability space  $(\mathcal{A}, \varphi, \varphi_2)$ , we have

$$\begin{aligned} \kappa_{1,1}(a_1a_2, a_3) &= \kappa_3(a_1, a_3, a_2) + \kappa_{2,1}(a_1, a_2, a_3) \\ &\quad + \kappa_{1,1}(a_1, a_3)\kappa_1(a_2) + \kappa_1(a_1)\kappa_{1,1}(a_2, a_3). \end{aligned}$$

To state the theorem in the second order case, we need a new concept, namely that of a permutation that separates points.

**Definition 2.5.** Suppose we have a permutation  $\pi \in S_n$  and a subset  $A \subseteq [n]$ ; we say that  $\pi$  separates the points of  $A$ , if no two points of  $A$  are in the same cycle of  $\pi$ .

If  $\sigma \in S_n$  and  $A \subseteq [n]$  is such that  $\sigma(A) = A$ , i.e.,  $\sigma$  leaves  $A$  invariant, we denote by  $\sigma|_A$  the restriction of  $\sigma$  to  $A$ . We can extend this to the case when  $\sigma$  does not leave  $A$  invariant as follows. Let  $\sigma|_A$  be the permutation of  $A$  given by  $\sigma|_A(k) = \sigma^r(k)$  where  $r \geq 1$  is the smallest integer such that  $\sigma^r(k) \in A$  but  $\sigma^s(k) \notin A$  for all  $1 \leq s < r$ .

We call the permutation  $\sigma|_A$  the *induced permutation* of  $\sigma$  on  $A$ . When  $\sigma$  leaves  $A$  invariant, this reduces to the restriction of  $\sigma$  to  $A$ .

Using the idea of an induced permutation, the condition of separating points can be written as follows:  $\sigma$  separates the points of  $A$  if and only if  $\sigma|_A = \text{id}_A$ .

**Remark 2.6.** It was shown in [23, Lemma 14] that the condition  $\pi \vee \tau_{\vec{n}} = 1_n$  of Theorem 2.4 is equivalent to the condition that  $\pi^{-1}\gamma_n$  separates the points of  $N = \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_r\}$ . For example, one of the permutations not appearing in the expansion of  $\kappa_2(a_1a_2, a_3a_4)$  is  $\pi = (1, 2)(3, 4)$ . Now  $\pi^{-1}\gamma_4 = (1)(2, 4)(3)$  and  $N = \{2, 4\}$ . So  $\pi^{-1}\gamma_4$  does not separate the points of  $N$ . In the second order case, the condition  $\pi \vee \tau_{\vec{n}} = 1_n$  becomes that  $\pi^{-1}\gamma_{m,n}$  separates the points of  $N$ .

**Proposition 2.7** ([23, Theorem 3]). *Suppose  $n_1, \dots, n_r, n_{r+1}, \dots, n_{r+s}$  are positive integers,  $p = n_1 + \dots + n_r$ ,  $q = n_{r+1} + \dots + n_{r+s}$ , and*

$$N = \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_{r+s}\}.$$

*Given a second probability space  $(\mathcal{A}, \varphi, \varphi_2)$  and  $a_1, \dots, a_{n_1}, a_{n_1+1}, \dots, a_{n_1+n_2}, \dots, a_{n_1+\dots+n_{r+s}} \in \mathcal{A}$ , let  $A_1 = a_1 \cdots a_{n_1}$ ,  $A_2 = a_{n_1+1} \cdots a_{n_1+n_2}$ ,  $\dots$ ,  $A_{r+s} = a_{n_1+\dots+n_{r+s-1}+1} \cdots a_{n_1+\dots+n_{r+s}}$ . Then*

$$\kappa_{r,s}(A_1, \dots, A_r, A_{r+1}, \dots, A_{r+s}) = \sum_{(V, \pi)} \kappa_{(V, \pi)}(a_1, \dots, a_{p+q}), \tag{2.4}$$

*where the summation is over those  $(V, \pi) \in \mathcal{PS}_{\text{NC}}(p, q)$  such that  $\pi^{-1}\gamma_{p,q}$  separates the points of  $N$ .*

**Remark 2.8.** If we let  $O = \{1, n_1 + 1, n_1 + n_2 + 1, \dots, n_1 + \dots + n_{r-1} + 1\} = \gamma_n(N)$  and  $n = n_1 + \dots + n_r$ , then the condition that  $\pi^{-1}\gamma_n$  separates the points of  $N$  is equivalent to  $\gamma_n\pi^{-1}$  separates the points of  $O$ . Indeed, if  $n_k$  and  $n_l$  are in the same orbit of  $\pi^{-1}\gamma_n$ , i.e., there is  $s$  such that  $(\pi^{-1}\gamma_n)^s(n_k) = n_l$ , then  $(\gamma_n\pi^{-1})^s(n_k + 1) = n_l + 1$ , and conversely.

### 2.7. Preliminaries on even and $R$ -diagonal operators

In this subsection, we review the definitions and basic properties of the main subject of this paper: even and  $R$ -diagonal operators. We shall restate the first order results of [24] so that the connection with our results becomes apparent.

Recall that a probability measure determined by moments is symmetric if and only if all of its odd moments vanish. If  $(\mathcal{A}, \varphi)$  is a non-commutative  $*$ -probability space and  $x = x^* \in \mathcal{A}$ , we say that  $x$  is *even* if  $\varphi(x^{2k-1}) = 0$  for  $k = 1, 2, 3, \dots$ . If  $x$  is even, then it is determined by its even moments, which are just the moments of  $x^2$ .

The corresponding relation between the free cumulants of  $x$  and the free cumulants of  $x^2$  was found by Nica and Speicher [24].

**Definition 2.9.** Let  $\pi \in \text{NC}(n)$  be a non-crossing partition. We will regard  $\pi$  as the permutation whose cycles are the blocks of  $\pi$  with the elements arranged in increasing order. Then let  $\hat{\pi}$  be the permutation of  $[2n]$  given by  $\hat{\pi}^2(2k) = 2\pi(k)$  and  $\hat{\pi}(2k) = \gamma_{2n}(2k)$ , where  $\gamma_{2n}$  is the permutation of  $[2n]$  with the one cycle  $(1, 2, 3, \dots, 2n)$ . We call  $\hat{\pi}$  the *double* of  $\pi$ .

**Definition 2.10.** Let us recall from [3] the notion of an even partition. A partition is *even* if all of its blocks have an even number of elements. Likewise, we say a permutation is *even* if all of its cycles have an even number of elements. Note that this is not the usual convention used in group theory.

The theorem of Nica and Speicher is as follows.

**Theorem 2.11** ([24]). *Let  $x = x^* \in (\mathcal{A}, \varphi)$  be even. Then the free cumulants of  $x^2$  can be calculated from the free cumulants of  $x$  as follows:*

$$\kappa_n(x^2, \dots, x^2) = \sum_{\pi \in \text{NC}(n)} \kappa_{\hat{\pi}}(x, \dots, x). \tag{2.5}$$

For  $a \in \mathcal{A}$ , let  $a^{(1)} = a$  and  $a^{(-1)} = a^*$ . Consider the  $*$ -cumulants  $\kappa_m(a^{(\varepsilon_1)}, a^{(\varepsilon_2)}, \dots, a^{(\varepsilon_m)})$  of  $a$ . Recall (see [24]) that  $a$  is  $R$ -diagonal if all  $*$ -cumulants of  $a$  are 0 except those of the form

$$\kappa_{2n}(a, a^*, a, a^*, \dots, a, a^*) = \kappa_{2n}(a^*, a, a^*, a, \dots, a^*, a).$$

The relation between the cumulants of  $a^*a$  and the  $*$ -cumulants of  $a$  is the same as between the square  $x^2$  of an even operator and  $x$ ; see equation (2.5).

**Theorem 2.12** ([24]). *Let  $(\mathcal{A}, \varphi)$  be a non-commutative  $*$ -probability space and  $a \in \mathcal{A}$  an  $R$ -diagonal operator. Then*

$$\kappa_n(a^*a, \dots, a^*a) = \sum_{\pi \in \text{NC}(n)} \kappa_{\hat{\pi}}(a^*, a, a^*, \dots, a^*, a).$$

### 3. Statement of results

Let us present the main results of the paper.

**Definition 3.1.** Let  $(A, \varphi, \varphi_2)$  be a second order  $*$ -non-commutative probability space.

(1) An element  $x \in (\mathcal{A}, \varphi, \varphi_2)$  is called *second order even* (or *even* for short) if  $x = x^*$  and  $x$  is such that odd moments vanish, i.e.,  $\varphi_2(x^p, x^q) = 0$  unless  $p$  and  $q$  are both even and  $\varphi(x^{2n+1}) = 0$  for all  $n \geq 0$ .

(2) An element  $a \in (\mathcal{A}, \varphi, \varphi_2)$  is called *second order  $R$ -diagonal* if it is  $R$ -diagonal (i.e., as in Definition 2.1) and the only non-vanishing second order cumulants are of the form

$$\kappa_{2p,2q}(a, a^*, \dots, a, a^*) = \kappa_{2p,2q}(a^*, a, \dots, a^*, a).$$

**Remark 3.2.** From the moment-cumulant formulas it is clear that an element  $x \in \mathcal{A}$  is even if and only if it self-adjoint and  $\kappa_{p,q}(x, \dots, x) = 0$  unless  $p$  and  $q$  are even and  $\kappa_{2n+1}(x, \dots, x) = 0$  for  $n \geq 0$ .

The first of our results shows that, as in the first order case,  $R$ -diagonality is preserved when multiplying by a free element.

**Theorem 3.3.** *Let  $\{a, a^*\}$  and  $\{b, b^*\}$  be second order free and suppose that  $a$  is second order  $R$ -diagonal. Then  $ab$  is second order  $R$ -diagonal.*

As we will see, the combinatorics of even and  $R$ -diagonal operators are controlled by even annular non-crossing partitions. As already noted, if  $\pi \in \text{NC}(n)$  is even, then for every  $k$ ,  $k$  and  $\pi(k)$  have the opposite parity, because in the gaps, i.e., between  $k$  and  $\pi(k)$ , there must always be an even number of elements.

In the case of non-crossing annular permutations  $\pi \in S_{\text{NC}}(2p, 2q)$ , something a little weaker than this happens. If  $k$  and  $\pi(k)$  are in the same circle (i.e., the same cycle of  $\gamma_{2p,2q}$ ), they have the opposite parity. However, if  $k$  and  $\pi(k)$  are in different circles, they may have the same parity. See Lemmas 4.6 and 4.7.

Since for  $x \in \mathcal{A}$  even or  $R$ -diagonal the distributions of  $xx^*$  may be calculated from the even (resp. alternating) cumulants of  $x$ , let us introduce some notations to encode this information.

**Definition 3.4.** Let  $(A, \varphi, \varphi_2)$  be a second order non-commutative probability space.

(1) Let  $a$  be second order  $R$ -diagonal. Define

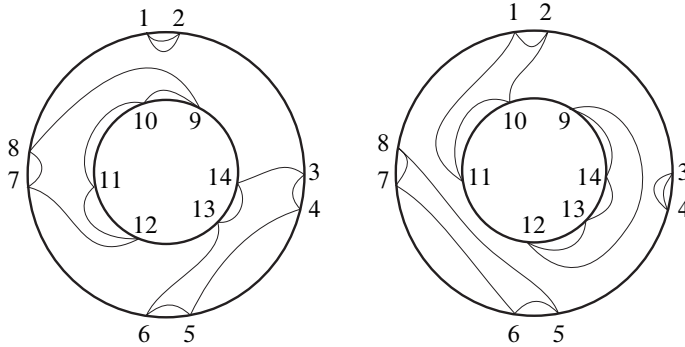
$$\beta_n^{(a)} := \kappa_{2n}(a, a^*, \dots, a, a^*) \quad \text{and} \quad \beta_{p,q}^{(a)} := \kappa_{2p,2q}(a, a^*, \dots, a, a^*).$$

The sequences  $(\beta_n^{(a)})_{n \geq 1}$  and  $(\beta_{p,q}^{(a)})_{p,q \geq 1}$  are called the (first and second order) *determining sequences* of  $a$ .

(2) Let  $x$  be a second order even element. Define  $\beta_n^{(x)} := \kappa_{2n}^{(x)} := \kappa_{2n}(x, \dots, x)$  and letting  $\kappa_{p,q}^{(x)} = \kappa_{p,q}(x, \dots, x)$ , we set

$$\beta_{p,q}^{(x)} := \kappa_{2p,2q}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{\text{all}+}(2p,2q)} \kappa_{\pi}^{(x)}, \tag{3.1}$$

where  $S_{\text{NC}}^{\text{all}+}(2p, 2q)$  denotes the set of even non-crossing annular permutations  $\pi$  with only through cycles such that whenever  $k$  and  $\pi(k)$  are in different cycles of  $\gamma_{2p,2q}$



**Figure 2.** Two examples of even non-crossing annular permutations. We call the one on the left *parity reversing* because for every  $k$ ,  $k$  and  $\pi(k)$  have the opposite parity. We call the one on the right *parity preserving* because when  $k$  and  $\pi(k)$  are on opposite circles, they have the same parity.

we have that  $k$  and  $\pi(k)$  have the same parity. See Figure 2, Definition 4.8, and Notation 4.20 for more details. We will call the families  $(\beta_n^{(x)})_{n \geq 1}$  and  $(\beta_{p,q}^{(x)})_{p,q \geq 1}$  the (first and second order) *determining sequences* of  $x$ .

Recall from [25] that in the first order case the cumulants and determining sequence are related by the following equations. For  $x$  an even operator, we have (see [25, (11.15)])

$$\kappa^{(x^2)} = \beta^{(x)} * \zeta \quad \text{and} \quad \beta^{(x)} = \kappa^{(x^2)} * \mu, \tag{3.2}$$

where  $\zeta$  and  $\mu$  are respectively the zeta and Möbius functions of the lattice of non-crossing partitions [25, Lecture 10]. The convolution here is over the lattice of non-crossing partitions; equation (3.2) means that

$$\kappa_n^{(x^2)} = \sum_{\pi \in \text{NC}(n)} \beta_\pi^{(x)}.$$

For an  $R$ -diagonal operator,  $a$ , we have

$$\kappa^{(aa^*)} = \beta^{(a)} * \zeta \quad \text{and} \quad \beta^{(a)} = \kappa^{(aa^*)} * \mu.$$

See [25, Proposition 15.6]. As above, this means that

$$\kappa_n^{(aa^*)} = \sum_{\pi \in \text{NC}(n)} \beta_\pi^{(a)}.$$

The second of our results gives a formula for the second order cumulants of  $aa^*$  in terms of the determining sequence of  $a$ .

**Theorem 3.5.** *Let  $a$  be a second order  $R$ -diagonal operator with determining sequences  $(\beta_n^{(a)})_{n \geq 1}$  and  $(\beta_{p,q}^{(a)})_{p,q \geq 1}$ . Then we have*

$$\kappa_{p,q}(aa^*, \dots, aa^*) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \beta_{(\mathcal{V}, \pi)}^{(a)}. \tag{3.3}$$

The proof of the last theorem will rely on the formula for cumulants with products as arguments (Proposition 2.7), and the observation that  $\{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p, 2q)^- \mid \gamma_{2p,2q}\pi^{-1} \text{ separates the points of } O\}$  is in bijection with  $\mathcal{PS}_{\text{NC}}(p, q)$ , where  $O = \{1, 3, 5, 7, \dots, 2p + 2q - 1\}$ . By  $\mathcal{PS}_{\text{NC}}(2p, 2q)$  we mean all  $(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p, 2q)$  such that the cycles of  $\pi$  alternate between even and odd numbers.

Similarly, we give a formula for the second order cumulants of  $x^2$  in terms of the determining sequence of  $x$ . Analogous formulas for the first order case have been used in [2] to prove that properties of  $x$  are transferred to  $x^2$  (e.g., free infinite divisibility or representations as the multiplication with a free Poisson).

**Theorem 3.6.** *Let  $x$  be an even element with determining sequences  $(\beta_n^{(x)})_{n \geq 1}$  and  $(\beta_{p,q}^{(x)})_{p,q \geq 1}$ . Then the second order cumulants of  $x^2$  are given by*

$$\kappa_{p,q}(x^2, \dots, x^2) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \beta_{(\mathcal{V}, \pi)}^{(x)}. \tag{3.4}$$

Note that even though formulas (3.3) and (3.4) are the same, the definition of second order determining sequences for even operators and for  $R$ -diagonal ones are quite different. Furthermore, note that using the second order zeta function,  $\zeta$  (see [8, §5.4]), we may respectively rewrite equations (3.3) and (3.4) as

$$\kappa^{(aa^*)} = \beta^{(a)} * \zeta \quad \text{and} \quad \kappa^{(x^2)} = \beta^{(x)} * \zeta. \tag{3.5}$$

Then by [8, Theorem 6.3], we may also rewrite Theorems 3.5 and 3.6 (together with their first order counterparts) in terms of generating functions as follows.

**Theorem 3.7.** *Let  $x$  be either a second order even or second order  $R$ -diagonal element with determining sequences  $(\beta_n)_{n \geq 1}$  and  $(\beta_{p,q})_{p,q \geq 1}$ . Denote by*

$$\kappa_n = \kappa_n^{(xx^*)} \quad \text{and} \quad \kappa_{p,q} := \kappa_{p,q}^{(xx^*)}$$

(or  $\kappa_n = \kappa_n^{x^2}$  and  $\kappa_{p,q}^{(x^2)}$  in case  $x$  is even) and define the formal power series

$$B(z) = \frac{1}{z} \sum_{n \geq 1} \beta_n z^n, \quad B(z, w) = \frac{1}{zw} \sum_{m,n \geq 1} \beta_{m,n} z^m w^n$$

and

$$C(z) = \frac{1}{z} + \frac{1}{z} \sum_{n \geq 1} \kappa_n z^{-n}, \quad C(z, w) = \frac{1}{zw} \sum_{n,m \geq 1} \kappa_{m,n} z^{-m} w^{-n}.$$



Then we have as a formal power series the relations

$$\frac{1}{C(z)} + B(C(z)) = z,$$

and

$$C(z, w) = C'(z)C'(w)B(C(z), C(w)) + \frac{\partial^2}{\partial z \partial w} \log \left( \frac{C(z) - C(w)}{z - w} \right).$$

Finally, we prove the following theorem which gives moments and cumulants of products of free random variables. For  $\pi \in S_{\text{NC}}(p, q)$ , we let  $\text{Kr}(\pi) = \pi^{-1}\gamma_{p,q}$  denote the Kreweras complement of  $\pi$ . Also we shall say that a permutation  $\pi$  is  $k$ -alternating if  $\pi(i) = i + 1 \pmod k$ . The set of  $k$ -alternating elements of  $S_{\text{NC}}(kp, kq)$  is denoted by  $S_{\text{NC}}^{k\text{-alt}}(kp, kq)$ . A permutation is  $k$ -equal if every cycle is of size  $k$ . The set of  $k$ -alternating and  $k$ -equal permutations in  $S_{\text{NC}}(kp, kq)$  is denoted by  $S_{\text{NC}}^{k\text{-e-a}}(kp, kq)$ , see Definition 7.1.

**Theorem 3.8** (Second order moments and cumulants of products of free variables). *Let  $a_1, \dots, a_k$  be operators which are second order free and such that  $\kappa_{p,q}^{(a_i)} = 0$  for all  $p$  and  $q$ . Denote by  $a := a_1 a_2 \cdots a_k$ , then*

$$\varphi(a^p, a^q) = \sum_{\pi \in S_{\text{NC}}^{k\text{-alt}}(kp, kq)} \kappa_{\text{Kr}(\pi)}(a_1, a_2, \dots, a_k, \dots, a_1, a_2, \dots, a_k). \quad (3.6)$$

Furthermore,

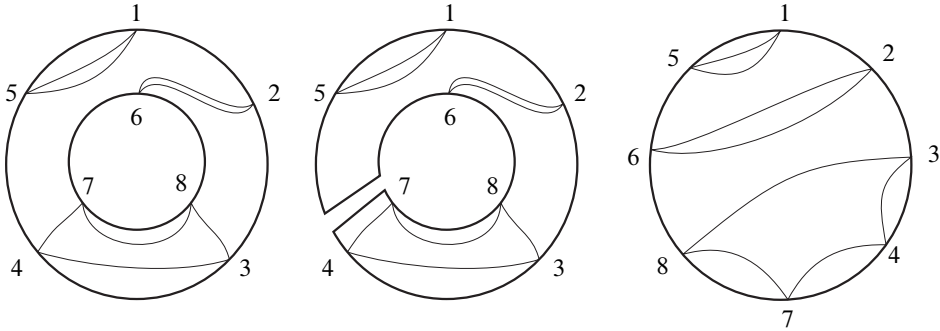
$$\kappa_{p,q}^{(a)} = \sum_{\pi \in S_{\text{NC}}^{k\text{-e-a}}(kp, kq)} \kappa_{\text{Kr}(\pi)}(a_1, a_2, \dots, a_k, \dots, a_1, a_2, \dots, a_k). \quad (3.7)$$

This last theorem is of importance in wireless communication because it includes the important case of products of complex Wishart random matrices [28].

### 4. Combinatorial lemmas on even annular permutations

In this section, we prove Propositions 4.19 and 4.23, which are the combinatorial results needed to prove Theorems 3.5 and 3.6. The case of an even operator, Theorem 3.6, requires an analysis of whether parity is preserved when crossing to the other circle, cf. Figure 2.

**Definition 4.1.** Recall that  $\gamma_{p,q}$  is the permutation in  $S_{p+q}$  with the two cycles  $(1, 2, 3, \dots, p)(p + 1, \dots, p + q)$  and  $S_{\text{NC}}(p, q)$  (the non-crossing annular permutations) is the set of permutations  $\pi$  in  $S_{p+q}$  such that at least one cycle of  $\pi$  meets both cycles of  $\gamma_{p,q}$  and  $|\pi| + |\pi^{-1}\gamma_{p,q}| = |\gamma_{p,q}| + 2$ , or equivalently,  $\#(\pi) + \#(\gamma_{p,q}\pi^{-1}) = p + q$ . A cycle of  $\pi$  that meets both cycles of  $\gamma_{p,q}$  is a *through cycle*.



**Figure 3.** An unfolding of  $\pi \in S_{NC}(5, 3)$ . We cut a channel between the two circles following a cycle of  $\pi$ . In this example, we go along the path from 4 to 7. This turns the annulus into a disc. See Remark 4.2 and Figure 6 for another example.

For  $\pi$ , an even non-crossing permutation on  $[n]$ ,  $k$  and  $\pi(k)$  always have the opposite parity; in the annular case something a little weaker holds. See Lemmas 4.6 and 4.7.

**Remark 4.2.** Let  $\pi \in S_{NC}(p, q)$ . Then we can *unfold*  $\pi$  into a non-crossing partition on  $[p + q]$ . This unfolding is not unique, but it is useful in reducing the annular case to the disc case. First illustrate this with an example. Suppose  $\pi = (1, 5)(2, 6)(3, 4, 7, 8)$ . Then  $\pi \in S_{NC}(5, 3)$ . Let  $\tilde{\gamma} = (1, 2, 3, 4, 7, 8, 6, 5)$ . Then  $\pi$  is non-crossing with respect to  $\tilde{\gamma}$  in that  $|\pi| + |\pi^{-1}\tilde{\gamma}| = |\tilde{\gamma}|$ , as in Section 2.1. See Figure 3. The next lemma shows that we can do this for every element of  $S_{NC}(p, q)$ .

**Lemma 4.3.** Let  $\pi \in S_{NC}(p, q)$  and  $k$  be such that  $k$  and  $\pi(k)$  are in different cycles of  $\gamma_{p,q}$ . Let  $\tilde{\gamma} = \gamma_{p,q} \cdot (k, \gamma_{p,q}^{-1}\pi(k))$ , i.e., the product of  $\gamma_{p,q}$  and the transposition  $(k, \gamma_{p,q}^{-1}\pi(k))$ . Then  $\tilde{\gamma}$  has one cycle and  $|\pi| + |\pi^{-1}\tilde{\gamma}| = |\tilde{\gamma}|$ , i.e.,  $\pi$  is non-crossing with respect to  $\tilde{\gamma}$ .

*Proof.* Since  $k$  and  $\gamma_{p,q}^{-1}\pi(k)$  are in different cycles of  $\gamma_{p,q}$ ,  $\tilde{\gamma}$  has only one cycle. Also  $\#(\pi) + \#(\pi^{-1}\tilde{\gamma}) = \#(\pi) + \#(\pi^{-1}\gamma_{p,q}) + 1 = p + q + 1$ , since  $k$  and  $\gamma_{p,q}^{-1}\pi(k)$  are in the same cycle of  $\gamma_{p,q}^{-1}\pi$ . Thus  $|\pi| + |\pi^{-1}\tilde{\gamma}| = |\tilde{\gamma}|$ . ■

**Remark 4.4.** Let  $\pi$  and  $k$  be as in Lemma 4.3. Then we can write  $\tilde{\gamma}$  as the cycle

$$(1, 2, 3, \dots, k, \pi(k), \gamma_{p,q}\pi(k), \dots, p + q, p + 1, \dots, \gamma_{p,q}^{-1}\pi(k), \gamma_{p,q}(k), \dots, p).$$

Suppose  $c$  is a non-through cycle of  $\pi$ , say  $c \subset [p]$ . Write  $c = (i_1, i_2, \dots, i_k)$ . Since  $c$  does not meet  $[p + 1, p + q]$ , at most one of the gaps  $[i_1 + 1, i_2 - 1], \dots, [i_k + 1, i_1 - 1]$  can contain a point on a through block.

**Lemma 4.5.** *Let  $\pi \in S_{\text{NC}}(p, q)$  and  $\bar{\pi}$  be the partition of  $[p]$  obtained as follows. Each cycle of  $\pi$  contained in  $[p]$  becomes a block of  $\bar{\pi}$ . All the remaining points form one more block. Then  $\bar{\pi}$  is non-crossing.*

*Proof.* Let  $k$  be such that  $k \in [p]$  and  $\pi(k) \in [p + 1, p + q]$ . Let  $\tilde{\gamma} = \gamma_{p,q}(k, \gamma_{p,q}^{-1}\pi(k))$ . Then by Lemma 4.3,  $\pi$  is non-crossing with respect to  $\tilde{\gamma}$ . So also is  $\sigma$ , the partition which is all singletons except for the block  $(\pi(k), \gamma_{p,q}\pi(k), \dots, \gamma_{p,q}^{-1}\pi(k))$ , i.e.,

$$\sigma = \{\{1\}, \{2\}, \dots, \{k\}, \{\pi(k), \dots, \gamma_{p,q}^{-1}\pi(k)\}, \{\gamma_{p,q}(k), \dots, \{p\}\}.$$

Hence,  $\pi \vee \sigma$  is also non-crossing, and thus so is  $\bar{\pi} = (\pi \vee \sigma)|_{[p]}$ . ■

**Lemma 4.6.** *Suppose  $\pi \in S_{\text{NC}}(p, q)$  with  $p$  and  $q$  even, has all cycles of even length. Then any cycle of  $\pi$  which is contained in one of the two cycles of  $\gamma_{p,q}$  alternates between even and odd elements. Moreover, if there is  $k$  such that  $k$  and  $\pi(k)$  lie in different cycles of  $\gamma_{p,q}$  and have opposite parities, then all cycles of  $\pi$  alternate between even and odd elements.*

*Proof.* Let  $c = (i_1, \dots, i_{2k})$  be a cycle of  $\pi$  which lies in one cycle of  $\gamma_{p,q}$ , and  $i_l$  and  $i_{l+1}$  adjacent points in this cycle. Consider the cyclic interval  $[i_l + 1, i_{l+1} - 1]$ , see Section 2.2 for the definition. If no point of this cyclic interval lies in a through cycle of  $\pi$ , then  $\pi$  restricts to a partition of this interval with all cycles even, thus there must be an even number of elements in this interval, and hence  $i_l$  and  $i_{l+1}$  must have opposite parities. Now suppose that there is an element of  $[i_l + 1, i_{l+1} - 1]$  that is in a through cycle of  $\gamma_{p,q}\pi^{-1}$ . Consider each of the other gaps in  $c$ :  $[i_{l+1} + 1, i_{l+2} - 1], \dots, [i_{l-1} + 1, i_l - 1]$ . By Remark 4.4, none contains an element which is on a through cycle of  $\pi$ . Thus  $\pi$  restricts to a partition of each of these intervals with each restriction having blocks of even size. Thus each gap has an even number of elements. This means  $\pi$  alternated between even and odd numbers as we move around a circle. By hypothesis,  $\pi$  alternates as we cross over. Thus  $\pi$  alternates between even and odd numbers. ■

**Lemma 4.7.** *Suppose  $\pi \in S_{\text{NC}}(p, q)$  with  $p$  and  $q$  even, has all cycles of even length. If there is  $k$  such that  $k$  and  $\pi(k)$  lie in different cycles of  $\gamma_{p,q}$  and have the same parity, then all through cycles of  $\pi$  alternate between even and odd elements except when they cross between the cycles of  $\gamma_{p,q}$ , i.e., if  $j$  and  $\pi(j)$  are in the same cycle of  $\gamma_{p,q}$ , then they have opposite parities, and if they are in different cycles of  $\gamma_{p,q}$ , they have the same parity.*

*Proof.* Suppose  $k \in [p]$  and  $\pi(k)$  lie in different cycles of  $\gamma_{p,q}$  and have the same parity. Let  $\gamma_p$  be the permutation in  $S_{p+q}$  which in cycle notation is  $(1, 2, 3, \dots, p)$ , let  $\tilde{\pi} = \gamma_p\pi\gamma_p^{-1}$ , and let  $\tilde{k} = \gamma_p(k)$ . Then all the cycles of  $\tilde{\pi}$  have even lengths, and  $\tilde{k}$

and  $\tilde{\pi}(\tilde{k})$  have opposite parities and lie in different cycles of  $\gamma_{p,q}$ . So by Lemma 4.6, all cycles of  $\tilde{\pi}$  alternate between odd and even elements.

If  $j$  and  $\pi(j)$  are in the same cycle of  $\gamma_{p,q}$ , they have opposite parities, as either they are in  $[p + 1, \dots, p + q]$  and  $\tilde{\pi}(j) = \pi(j)$ , or they are both in  $[p]$ , and if we let  $l = \gamma_p(j)$ , then  $l$  and  $\tilde{\pi}(l) = \gamma_p(\pi(j))$  have opposite parities and thus  $j = \gamma_p^{-1}(l)$  and  $\pi(j) = \gamma_p^{-1}(\tilde{\pi}(l))$  have opposite parities.

Alternatively, suppose that  $j$  and  $\pi(j)$  are in different cycles of  $\gamma_{p,q}$ . First suppose that  $k \in [j]$  and  $\pi(j) \in [p + 1, p + q]$ . Let  $l = \gamma_p(j)$ . Then  $l$  and  $\tilde{\pi}(l) = \pi(j)$  have opposite parities by Lemma 4.6. Thus  $j$  and  $\pi(j)$  have the same parity. On the other hand, suppose that  $\pi(j) \in [p]$  and  $j \in [p + 1, p + q]$ . Then  $j$  and  $\tilde{\pi}(j) = \gamma_p(\pi(j))$  have opposite parities, and thus  $j$  and  $\pi(j)$  have the same parity. ■

**Definition 4.8.** Suppose  $\pi \in S_{\text{NC}}(2p, 2q)$  is even. We say that  $\pi$  is *parity reversing* if for all  $k$ ,  $\pi(k)$  and  $k$  have the opposite parity. We denote the elements of  $S_{\text{NC}}(2p, 2q)$  that have cycles of even length and that are parity reversing by  $S_{\text{NC}}^-(2p, 2q)$ . The remaining even elements of  $S_{\text{NC}}(2p, 2q)$ , we call *parity preserving* because for any  $k$  such that  $k$  and  $\pi(k)$  are in different cycles of  $\gamma_{2p,2q}$  we have that  $k$  and  $\pi(k)$  have the same parity, cf. Lemma 4.7. We denote the parity preserving elements by  $S_{\text{NC}}^+(2p, 2q)$ . See Figure 2.

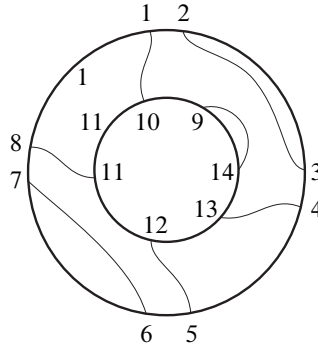
**Remark 4.9.** If  $\pi \in S_{\text{NC}}(2p, 2q)$  and  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$  (cf. Remark 2.8), then we cannot have  $\pi(2k) = 2l - 1$  with  $2k$  and  $2l - 1$  on different circles. If  $\pi(2k) = 2l - 1$ , then we would have both  $\gamma_{2p,2q}(2k) \neq 2l - 1$ , because they are on different circles, and  $\gamma_{2p,2q}\pi^{-1}(2l - 1) = \gamma_{2p,2q}(2k)$ ; since  $\gamma_{2p,2q}(2k) \in O$ , this is contrary to our assumption about  $\gamma_{2p,2q}\pi^{-1}$  separating the points of  $O$ . Thus either  $\pi(2k)$  is on the same circle as  $2k$  or  $\pi(2k) = 2l$ . We have shown that if  $l$  and  $\pi(l)$  are on different circles, then either both are even or  $l$  is odd. This means that there are no pairings in  $S_{\text{NC}}^-(2p, 2q)$  such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . See Figure 4.

**Lemma 4.10.** Suppose  $\pi \in S_{\text{NC}}^-(2p, 2q)$ . If  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ , then for all  $k$ ,  $\gamma_{2p,2q}\pi^{-1}(2k - 1) = 2k - 1$ , or equivalently, for all  $k$ ,  $\pi(2k) = \gamma_{2p,2q}(2k)$ .

*Proof.* Recall that for  $\pi \in S_{\text{NC}}^-(2p, 2q)$ ,  $\pi(2k - 1)$  is even for all  $k$ . Therefore, we have  $\gamma_{2p,2q}\pi^{-1}(2k - 1)$  is odd for all  $k$ . Since  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ , we must have that  $\gamma_{2p,2q}\pi^{-1}(2k - 1) = 2k - 1$  for all  $k$ . Let  $2l = \gamma_{2p,2q}^{-1}(2k - 1)$ , then  $\pi(2l) = 2k - 1 = \gamma_{2p,2q}(2l)$ . ■

**Definition 4.11.** Suppose  $\pi \in S_{\text{NC}}^-(2p, 2q)$  is such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Let  $\check{\pi}$  be the permutation defined by  $2\check{\pi}(k) = \pi^2(2k)$  for  $1 \leq k \leq p + q$ .

**Lemma 4.12.** Suppose  $\pi \in S_{\text{NC}}^-(2p, 2q)$  is such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Then  $\gamma_{p,q}\check{\pi}^{-1}(k) = l$  if and only if  $\gamma_{2p,2q}\pi^{-1}(2k) = 2l$ .



**Figure 4.** A pairing in  $S_{NC}^-(8, 6)$ . Note that whenever there is a through string,  $(1, 10)$  in this example, we must have two elements of  $O$ ,  $\{1, 11\}$  in this example, in the same cycle of  $\gamma_{2p,2q}\pi^{-1}$ . See Remark 4.9.

*Proof.* By Lemma 4.10, for all  $1 \leq k \leq p + q$ ,  $\pi(2k) = \gamma_{2p,2q}(2k)$ . Thus for any  $1 \leq k \leq p + q$ ,  $\pi(2\gamma_{p,q}^{-1}(k)) = \gamma_{2p,2q}(2\gamma_{p,q}^{-1}(k)) = 2k - 1 = \gamma_{2p,2q}^{-1}(2k)$ . Hence

$$\begin{aligned} \gamma_{p,q}\check{\pi}^{-1}(k) = l &\Leftrightarrow \check{\pi}(\gamma_{p,q}^{-1}(l)) = k \Leftrightarrow 2\check{\pi}(\gamma_{p,q}^{-1}(l)) = 2k \Leftrightarrow \pi^2(2\gamma_{p,q}^{-1}(l)) = 2k \\ &\Leftrightarrow \pi(\gamma_{2p,2q}^{-1}(2l)) = 2k \Leftrightarrow \gamma_{2p,2q}\pi^{-1}(2k) = 2l. \quad \blacksquare \end{aligned}$$

**Lemma 4.13.** Suppose  $\pi \in S_{NC}^-(2p, 2q)$  is such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Then  $\check{\pi} \in S_{NC}(p, q)$ .

*Proof.* We have  $\#(\check{\pi}) = \#(\pi)$ . The cycles of  $\gamma_{2p,2q}\pi^{-1}$  are the singletons  $(2k - 1)$ , of which there are  $p + q$ , and the cycles consisting of even numbers. But by Lemma 4.12, the orbits of even numbers correspond to the orbits of  $\gamma_{p,q}\check{\pi}^{-1}$ . Thus  $\#(\gamma_{2p,2q}\pi^{-1}) = p + q + \#(\gamma_{p,q}\check{\pi}^{-1})$ . Hence

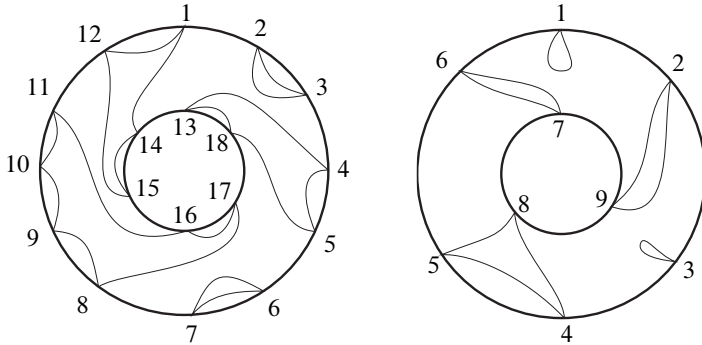
$$\#(\check{\pi}) + \#(\gamma_{p,q}\check{\pi}^{-1}) = \#(\pi) + \#(\gamma_{p,q}\pi^{-1}) - (p + q) = p + q.$$

By Remark 4.9,  $\pi$  has a through block with more than two elements, so  $\check{\pi}$  will have a through block, and thus  $\check{\pi} \in S_{NC}(p, q)$ . ■

The next lemma will be crucial in proving Theorem 3.5 on the determining sequences of a second order  $R$ -diagonal element.

**Lemma 4.14.** The map  $\pi \mapsto \check{\pi}$  is a bijection from  $S_{NC}^-(2p, 2q) \cap \{\pi \mid \gamma_{2p,2q}\pi^{-1} \text{ separates the points of } O\}$  to  $S_{NC}(p, q)$ .

*Proof.* Suppose that  $\pi_1$  and  $\pi_2$  are in  $S_{NC}^-(2p, 2q) \cap \{\pi \mid \gamma_{2p,2q}\pi^{-1} \text{ sep. } O\}$  and  $\check{\pi}_1 = \check{\pi}_2$ . (Here and below ‘sep.  $O$ ’ means ‘separates the points of  $O$ ’.) Therefore,  $\pi_1^2(2k) = \pi_2^2(2k)$  for all  $k$ . Since  $\pi_i(2k) = \gamma_{2p,2q}(2k)$  for  $i = 1, 2$ , we only have to show that  $\pi_1$  and  $\pi_2$  agree on the odd numbers. But  $\pi_1(\gamma_{2p,2q}(2k)) = \pi_1^2(2k) =$



**Figure 5.** On the left we see the permutation  $\pi = (1, 14, 15, 12)(2, 3)(4, 5, 18, 13)(6, 7)(8, 9, 10, 11, 16, 17)$ ;  $\pi$  is in  $S_{\text{NC}}^-(12, 6)$  and  $\gamma_{12,6}\pi^{-1}$  separates the points of  $O = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$ . On the right is  $\tilde{\pi} = (1)(2, 9)(3)(4, 5, 8)(6, 7)$ . Note that  $\pi^2|_E = (2)(4, 18)(6)(8, 10, 16)(12, 14)$ , where  $\pi^2|_E$  is the restriction of  $\pi^2$  to the even numbers.

$\pi_2^2(2k) = \pi_2(\gamma_{2p,2q}(2k))$ . Hence  $\pi_1$  and  $\pi_2$  agree on the odd numbers as well. This proves injectivity.

Let  $\sigma \in S_{\text{NC}}(p, q)$ . Define  $\pi$  by  $\pi(2k) = \gamma_{2p,2q}(2k)$  and  $\pi(\gamma_{2p,2q}(2k)) = 2\sigma(k)$ . Then  $\pi^2(2k) = 2\sigma(k)$ . So if  $(i_1, \dots, i_k)$  is a cycle of  $\sigma$ , then the corresponding cycle of  $\pi$  is  $(2i_1, \gamma_{2p,2q}(2i_1), 2i_2, \dots, 2i_k, \gamma_{2p,2q}(2i_k))$ . Thus  $\pi$  is even and, by construction, is a parity reversing permutation. Also  $\#(\pi) = \#(\sigma)$ . Furthermore, we have that

$$\gamma_{2p,2q}(\pi^{-1}(2k - 1)) = 2k - 1.$$

Thus every odd number is a singleton of  $\gamma_{2p,2q}\pi^{-1}$ . As in the proof of Lemma 4.12, we have  $\gamma_{p,q}\sigma^{-1}(k) = l \Leftrightarrow \gamma_{2p,2q}\pi^{-1}(2k) = 2l$ , thus

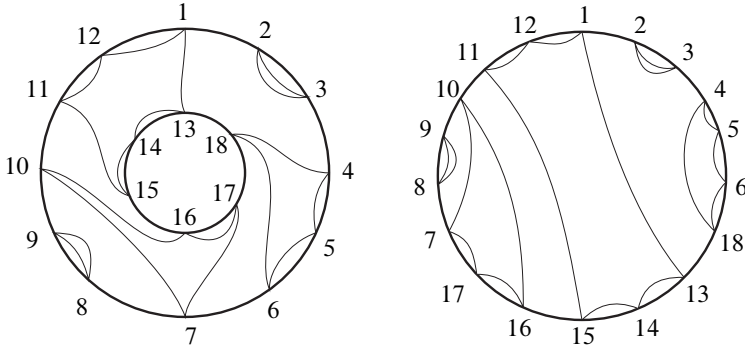
$$\#(\pi) + \#(\gamma_{2p,2q}\pi^{-1}) = \#(\sigma) + \#(\gamma_{p,q}\sigma^{-1}) + p + q = 2p + 2q.$$

Therefore,  $\pi \in S_{\text{NC}}^-(2p, 2q)$ . Since, by construction,  $\pi(2k) = \gamma_{2p,2q}(2k)$ , we have  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Thus  $\pi \in S_{\text{NC}}^-(2p, 2q) \cap \{\pi \mid \gamma_{2p,2q}\pi^{-1} \text{ sep. } O\}$  and  $\tilde{\pi} = \sigma$ . ■

See Figure 5 for an example of the relation between  $\pi$  and  $\tilde{\pi}$ .

**Lemma 4.15.** *Let  $\pi \in S_{\text{NC}}^+(2p, 2q)$  be such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Suppose  $k$  is such that  $2k$  and  $\pi(2k)$  are in the same cycle of  $\gamma_{2p,2q}$ . Then  $\pi(2k) = \gamma_{2p,2q}(2k)$ .*

*Proof.* First let us suppose that  $2k$  is in a through cycle of  $\pi$ . Let  $l$  be such that  $2k, \pi(2k), \dots, \pi^{l-1}(2k)$  are in the same cycle of  $\gamma_{2p,2q}$  but  $\pi^l(2k)$  is in a different cycle. Let  $\tilde{\gamma} = \gamma_{2p,2q}(\pi^{l-1}(2k), \gamma_{2p,2q}^{-1}\pi^l(2k))$ , then  $\tilde{\gamma}$  has one cycle and, by Lemma 4.3,



**Figure 6.** On the left we see the permutation  $\pi = (1, 13, 14, 15, 11, 12)(2, 3)(4, 5, 6, 18), (7, 10, 16, 17)(8, 9)$ ;  $\pi$  is in  $S_{\text{NC}}^+(12, 6)$  and  $\gamma_{12,6}\pi^{-1}$  separates the points of  $O = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$ . On the right is  $\pi$ , drawn relative to  $\gamma_{12,6}(6, 17)$ .

$\pi$  is non-crossing with respect to  $\tilde{\gamma}$ . See Figure 6. Let us suppose  $2k \in [2p]$ ; the case where  $2k \in [2p + 1, 2p + 2q]$  is identical. Then we may write

$$\begin{aligned} \tilde{\gamma} = & (1, 2, 3, \dots, 2k, \dots, \pi^{l-1}(2k), \pi^l(2k), \gamma_{2p,2q}\pi^l(2k), \dots, \\ & 2p + 2q, 2p + 1, \dots, \gamma_{2p,2q}^{-1}\pi^l(2k), \gamma_{2p,2q}\pi^{l-1}(2k), \dots, p). \end{aligned}$$

The cyclic interval  $I = (2k, \gamma_{2p,2q}(2k), \dots, \pi^{l-1}(2k))$  lies in the cycle  $(1, 2, 3, \dots, 2p)$  of  $\gamma_{2p,2q}$ , and the endpoints of  $I$  lie in the same cycle of  $\pi$ , see [19, Remark 3.4 (2)]. Let  $c$  be a cycle of  $\pi$  containing a point of  $I$  but not  $2k$ . Since  $\pi$  is non-crossing with respect to  $\tilde{\gamma}$ ,  $c$  must be contained in  $I$  and thus not be a through cycle. Thus the gap, if it exists, between  $2k$  and  $\pi(2k)$  is a union of non-through cycles of  $\pi$ . Hence  $\pi(2k) = 2j - 1$  for some  $j$ . Then  $\gamma_{2p,2q}(2k)$  and  $\pi(2k)$  are both in  $O$  and in the same cycle of  $\gamma_{2p,2q}\pi^{-1}$ . Hence they must be equal.

Now suppose that  $2k$  is not in a through cycle of  $\pi$ . Then by Lemma 4.6,  $\pi(2k) = \gamma_{2p,2q}(2l)$  for some  $l$ . So  $\gamma_{2p,2q}\pi^{-1}(\gamma_{2p,2q}(2l)) = \gamma_{2p,2q}(2k)$ . Then we have that  $\gamma_{2p,2q}(2k)$  and  $\gamma_{2p,2q}(2l)$  are in the same orbit of  $\gamma_{2p,2q}\pi^{-1}$  and thus  $\gamma_{2p,2q}(2k) = \gamma_{2p,2q}(2l)$ . Hence  $\pi(2k) = \gamma_{2p,2q}(2l) = \gamma_{2p,2q}(2k)$  as required. ■

**Notation 4.16.** Let  $V \subset [p + q]$  and let  $\hat{V} \subset [2p + 2q]$  be the set  $\hat{V} = \{2k \mid k \in V\} \cup \{2k + 1 \mid k \in V\}$ . We call  $\hat{V}$  the *double* of  $V$ .

**Lemma 4.17.** Let  $\pi \in S_{\text{NC}}^+(2p, 2q)$  be such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Let  $V$  be the union of all through cycles of  $\pi$ . Then there is  $U \subset [p + q]$  such that  $V = \hat{U}$ .

*Proof.* We must show that  $V$  is the disjoint union of cyclic intervals of the form

$$\{2i, \gamma_{2p,2q}(2i), \gamma_{2p,2q}^2(2i), \dots, \gamma_{2p,2q}^{2r+1}(2i)\} \tag{*}$$

for if this is so, then we can take  $U$  to be the corresponding union of cyclic intervals  $\{i, \gamma_{p,q}(i), \dots, \gamma_{p,q}^r(i)\}$ . To prove that  $V$  has this structure, it suffices to show that for each through cycle  $c$  of  $\pi$  we have that both  $c \cap [2p]$  and  $c \cap [2p + 1, 2p + 2q]$  have this structure, i.e., they are a disjoint union of cyclic intervals of the form  $(*)$ ; this reduces to showing that if  $2k$  is in a through cycle of  $\pi$  and both  $2k$  and  $\pi(2k)$  are in the same cycle of  $\gamma_{2p,2q}$ , then  $\pi(2k) = \gamma_{2p,2q}(2k)$ . This is what was proved in Lemma 4.15. ■

**Notation 4.18.** Recall that  $\mathcal{PS}_{\text{NC}}(p, q)'$  is the set of all partitioned permutations  $(\mathcal{U}, \pi)$  where  $\mathcal{U}$  has a block which is the union of two cycles of  $\pi$ , see Section 2.3. Suppose we are given  $(\mathcal{U}, \sigma) \in \mathcal{PS}_{\text{NC}}(p, q)'$ . Write  $\sigma = \sigma_1 \times \sigma_2 \in \text{NC}(p) \times \text{NC}(q)$ . Let  $U$  be the block of  $\mathcal{U}$  which is the union of a cycle of  $\sigma_1$  and a cycle of  $\sigma_2$ . Let  $S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p, 2q)$  be the set of  $\pi \in S_{\text{NC}}^+(2p, 2q)$  such that

- each non-through cycle of  $\pi$  is the double of some cycle  $c$  of  $\sigma$  (cf. Definition 2.9);
- the union of all the through cycles of  $\pi$  is  $\hat{U}$ .

**Proposition 4.19.**

$$\{\pi \in S_{\text{NC}}^+(2p, 2q) \mid \gamma_{2p,2q}\pi^{-1} \text{ sep. } O\} = \bigcup_{(\mathcal{U}, \sigma) \in \mathcal{PS}_{\text{NC}}(p, q)'} S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p, 2q)$$

and the union is disjoint.

*Proof.* First let us show that the union is disjoint. Suppose  $\pi \in S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p, 2q)$ . Since each non-through cycle of  $\pi$  is the double of a cycle of  $\sigma$ , all the cycles of  $\sigma$ , except the two cycles joined by  $\mathcal{U}$ , are determined by  $\pi$ . Moreover, the union of the two cycles of  $\sigma$  joined by  $\mathcal{U}$  is the union of the through cycles of  $\pi$ ; so these two cycles of  $\sigma$ , one in  $[p]$  and the other in  $[p + 1, p + q]$ , are also determined by  $\pi$ . Thus the union is disjoint.

Let  $\pi \in S_{\text{NC}}^+(2p, 2q)$  be such that  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Let  $\pi_1 \in \text{NC}(p)$  and  $\pi_2 \in \text{NC}([p + 1, p + q])$  be the non-crossing partitions constructed in Lemma 4.5. By Lemma 4.6, each non-through cycle of  $\pi$  is the double of a subset of either  $[p]$  or  $[p + 1, p + q]$ . By Lemma 4.15, the blocks of  $\pi_1$  and  $\pi_2$  which come from the through cycles of  $\pi$  are also the doubles of subsets of  $[p]$  and  $[p + 1, p + q]$ , respectively. Hence there are  $\sigma_1 \in \text{NC}(p)$  and  $\sigma_2 \in \text{NC}([p + 1, p + q])$  such that  $\pi_1 = \hat{\sigma}_1$  and  $\pi_2 = \hat{\sigma}_2$ . Now let  $\mathcal{U} \in \mathcal{P}(p + q)$  be the partition whose blocks are just the blocks of  $\sigma_1$  and  $\sigma_2$ , except we join the blocks of  $\sigma_1$  and  $\sigma_2$  coming from the through cycles of  $\pi$ . Letting  $\sigma = \sigma_1 \times \sigma_2$ , we have  $\pi \in S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p, 2q)$ . ■

**Notation 4.20.** Suppose  $\pi \in S_{\text{NC}}(p, q)$  is such that all cycles contain points of both cycles of  $\gamma_{p,q}$ . Then we say that  $\pi$  has *all through cycles*. The set of non-crossing annular permutations with all through cycles is denoted  $S_{\text{NC}}^{\text{all}}(p, q)$ .



**Lemma 4.21.** *Let  $\pi \in S_{\text{NC}}^{\text{all}}(p, q)$ , then all cycles of  $\gamma_{p,q}\pi^{-1}$  are either singletons or pairs, and the pairs are all through cycles. Moreover, if  $\pi$  is parity preserving (cf. Definition 4.8), then  $\gamma_{p,q}\pi^{-1}$  separates the points of  $O$ .*

*Proof.* Suppose  $\pi$  has  $k$  cycles. Then there are cyclic intervals

$$I_1, I_2, \dots, I_k \subset [p] \quad \text{and} \quad J_1, J_2, \dots, J_k \subset [p + 1, p + q]$$

with

$$I_l = (i_1^{(l)}, i_2^{(l)}, \dots, i_{r_l}^{(l)}) \quad \text{and} \quad J_l = (j_1^{(l)}, j_2^{(l)}, \dots, j_{s_l}^{(l)})$$

such that the  $l$ -th cycle of  $\pi$  is  $(i_1^{(l)}, i_2^{(l)}, \dots, i_{r_l}^{(l)}, j_1^{(l)}, j_2^{(l)}, \dots, j_{s_l}^{(l)})$ . Hence the only through cycles of  $\gamma_{2p,2q}\pi^{-1}$  are of the form  $(\gamma_{p,q}(i_{r_l}^{(l)}), j_1^{(l)})$  and all other cycles are singletons. If in addition  $\pi$  is parity preserving, then  $i_{r_l}^{(l)}$  and  $j_1^{(l)}$  are of the same parity, so  $\gamma_{p,q}(i_{r_l}^{(l)})$  and  $j_1^{(l)}$  are of opposite parities. Thus  $\gamma_{p,q}\pi^{-1}$  separates the points of  $O$ . ■

**Lemma 4.22.** *Let  $O = \{1, 3, 5, \dots, 2n - 1\}$ . Suppose  $\pi \in \text{NC}(2n)$  is even and  $\gamma_{2n}\pi^{-1}|_O = \text{id}_O$ . Then there is  $\sigma \in \text{NC}(n)$  such that  $\pi = \hat{\sigma}$ , in fact, given by  $2\sigma(k) = \pi^2(2k)$ . Conversely, given  $\sigma \in \text{NC}(n)$ ,  $\hat{\sigma}$  is even and  $\gamma_{2n}\hat{\sigma}^{-1}|_O = \text{id}_O$ .*

*Proof.* As  $\pi$  is an even permutation, then the cycles of  $\pi$  must alternate between even and odd numbers (because any gaps in an orbit must skip over an even number of points), hence  $\gamma_{2n}\pi^{-1}$  leaves both  $N$  (the even numbers) and  $O$  invariant. The condition  $\gamma_{2n}\pi^{-1}|_O = \text{id}_O$  then implies that for every  $k$ ,  $\gamma_{2n}\pi^{-1}(2k - 1) = 2k - 1$ . Hence for every  $k$ ,  $\pi(2k) = \gamma_{2n}(2k)$ , and thus if we let  $\sigma \in \text{NC}(n)$  be given by  $2\sigma(k) = \pi^2(2k)$ , we have that  $\pi = \hat{\sigma}$  is the double (cf. Definition 2.9) of  $\sigma$ , as required by Theorem 2.11. ■

The next proposition will be crucial in proving Theorem 3.6; it will serve as the annular analogue of Lemma 4.22.

**Proposition 4.23.** *Let  $(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p, 2q)'$  be such that  $\pi$  is even and  $\gamma_{2p,2q}\pi^{-1}$  separates the points of  $O$ . Then there is  $(\mathcal{U}, \sigma) \in \mathcal{PS}_{\text{NC}}(p, q)'$  such that  $\mathcal{V} = \hat{\mathcal{U}}$  and  $\pi = \hat{\sigma}$ . Moreover, this correspondence is a bijection.*

*Proof.* Since  $(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p, 2q)'$ ,  $\pi = \pi_1 \times \pi_2$  with  $\pi_1 \in \text{NC}(2p)$  and  $\pi_2 \in \text{NC}(2q)$ . Moreover,  $\pi_1$  and  $\pi_2$  are both even, and  $\gamma_{2p}\pi_1^{-1}$  and  $\gamma_{2q}\pi_2^{-1}$  separate the points of  $\{1, 3, 5, \dots, 2p - 1\}$  and  $\{1, 3, 5, \dots, 2q - 1\}$ , respectively. Thus, by Lemma 4.22, there are  $\sigma_1 \in \text{NC}(p)$  and  $\sigma_2 \in \text{NC}(q)$  such that  $\pi_1 = \hat{\sigma}_1$  and  $\pi_2 = \hat{\sigma}_2$ . Thus  $\pi = \hat{\sigma}$ . Now  $\mathcal{U}$  is formed by joining a cycle of  $\sigma_1$  with a cycle of  $\sigma_2$ . So if we form  $\mathcal{V}$  by joining the corresponding cycles of  $\pi$  and  $\pi_2$ , then  $\mathcal{V} = \hat{\mathcal{U}}$ . Since we can recover  $\sigma$  from  $\hat{\sigma}$ , this correspondence is a bijection. ■

### 5. $R$ -diagonal elements of second order

In this section, we prove Theorems 3.3 and 3.5 regarding  $R$ -diagonal elements.

#### 5.1. The proof of Theorem 3.3

Theorem 3.3 asserts that if  $\{a, a^*\}$  and  $\{b, b^*\}$  are second order free and  $a$  is  $R$ -diagonal of second order, then  $ab$  is  $R$ -diagonal of second order.

Let  $c = ab$  and  $c^{(1)} = c$ , and  $c^{(-1)} = c^*$ . So we must show that

- ( $\alpha$ )  $\kappa_n(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}, \dots, c^{(\varepsilon_n)}) = 0$  unless  $n$  is even and  $\varepsilon_i = -\varepsilon_{i+1}$  for  $1 \leq i < n$ ,
- ( $\beta$ )  $\kappa_{p,q}(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}, \dots, c^{(\varepsilon_{p+q})}) = 0$  unless  $p$  and  $q$  are even, and for  $1 \leq i < 2p$  or  $2p + 1 \leq i < 2p + 2q$ , we have  $\varepsilon_i = -\varepsilon_{i+1}$ .

Note that ( $\alpha$ ) is proved in [24] and [25, Proposition 15.8].

Fix  $p$  and  $q$ , we shall show that  $\kappa_{p,q}(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}, \dots, c^{(\varepsilon_{p+q})})$  is necessarily 0 unless  $p$  and  $q$  are even and  $\varepsilon_i = -\varepsilon_{i+1}$  for  $1 \leq i < p$  and  $p + 1 \leq i < p + q$ . To expand this cumulant, we use the formula for cumulants with products as entries as written in Proposition 2.7,

$$\begin{aligned} &\kappa_{p,q}((ab)^{(\varepsilon_1)}, (ab)^{(\varepsilon_2)}, \dots, (ab)^{(\varepsilon_{p+q})}) \\ &= \sum_{\pi \in S_{\text{NC}(2p,2q)}} \kappa_{\pi}(x_1, x_2, \dots, x_{2(p+q)-1}, x_{2p+2q}) \\ &+ \sum_{\substack{\pi \in \text{NC}(2p) \times \text{NC}(2q) \\ \pi = \pi_1 \times \pi_2}} \sum_{\substack{\mathcal{V} \geq \pi \\ |\mathcal{V}| = |\pi| + 1}} \kappa_{(\mathcal{V}, \pi)}(x_1, x_2, \dots, x_{2(p+q)-1}, x_{2p+2q}), \end{aligned}$$

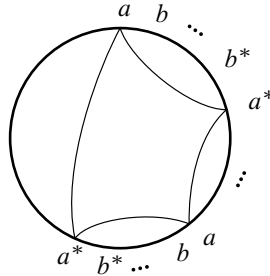
where

$$x_{2i-1} = \begin{cases} a, & \varepsilon_i = 1, \\ b^*, & \varepsilon_i = -1 \end{cases} \quad \text{and} \quad x_{2i} = \begin{cases} a^*, & \varepsilon_i = -1, \\ b, & \varepsilon_i = 1, \end{cases}$$

and the sum is over all  $\pi \in S_{\text{NC}(2p + 2q)}$  in the first sum, and  $\text{NC}(2p) \times \text{NC}(2q)$  in the second, such that  $\gamma_{2p,2q}\pi^{-1}|_O = \text{id}_O$  and  $O = \{1, 3, 5, \dots, 2(p + q) - 1\}$ . Moreover, in the second sum we require the partition  $\mathcal{V}$  to have one block which is the union of a cycle of  $\pi_1$  and a cycle of  $\pi_2$ .

Since we have assumed that  $\{a, a^*\}$  and  $\{b, b^*\}$  are second order free, we know that both  $\kappa_{(\mathcal{V}, \pi)}(x_1, \dots, x_{2p+2q})$  and  $\kappa_{\pi}(x_1, \dots, x_{2p+2q})$  are equal to 0 unless

- (i) all cycles of  $\pi$  visit either only elements of  $\{a, a^*\}$ , let us call these  $a$ -cycles, or only elements of  $\{b, b^*\}$ , let us call these  $b$ -cycles; moreover, since  $a$  is  $R$ -diagonal, the  $a$ -cycles must alternate between  $a$  and  $a^*$  and, in particular, must have an even number of elements (see Figure 7),
- (ii)  $\gamma_{2p,2q}\pi^{-1}|_O = \text{id}_O$  (cf. Definition 2.5).



**Figure 7.** If  $\kappa_\pi(x_1, \dots, x_n) \neq 0$ , then  $\pi$  must consist of  $a$  cycles and  $b$  cycles, and the  $a$  cycles must alternate between  $a$  and  $a^*$ .

**Lemma 5.1.** *Suppose  $\pi \in S_{\text{NC}}(2p, 2q) \cup \text{NC}(2p) \times \text{NC}(2q)$  satisfies (i) and (ii). If  $\varepsilon_i = -1$ , then*

$$\pi(2i) = \gamma_{2p,2q}(2i) \quad \text{and} \quad \varepsilon_{\gamma_{p,q}(i)} = 1.$$

*Proof.* If  $\varepsilon_i = -1$ , then  $x_{2i-1} = b^*$  and  $x_{2i} = a^*$ . Thus  $\pi(2i)$  must be odd so that the next element in the cycle containing  $2i$  will be an  $a$ . Let  $2j - 1 = \pi(2i)$ . Then  $\pi^{-1}(2j - 1) = 2i$ , so  $\gamma_{2p,2q}\pi^{-1}(2j - 1)$  is odd. Since  $\gamma_{2p,2q}\pi^{-1}|_O = \text{id}_O$ , we must have that  $\gamma_{2p,2q}\pi^{-1}(2j - 1) = 2j - 1$ , so  $\pi(2i) = 2j - 1 = \gamma_{2p,2q}\pi^{-1}(\pi(2i)) = \gamma_{2p,2q}(2i)$ . Hence  $x_{\gamma_{2p,2q}(2i)} = a$  and so  $\varepsilon_{\gamma_{p,q}(i)} = 1$ . ■

*Proof of Theorem 3.3.* To prove  $(\beta)$ , we have to show that if there is  $i$  such that  $\varepsilon_i = \varepsilon_{i+1}$ , then no  $\pi$  can simultaneously satisfy conditions (i) and (ii).

Lemma 5.1 already says that  $\varepsilon_i = \varepsilon_{i+1} = -1$  is impossible. Suppose that  $\varepsilon_i = \varepsilon_{i+1} = 1$ . Then  $x_{2i+1} = a$  and so  $\pi^{-1}(2i + 1) = 2j$  for some  $j$  and  $\varepsilon_j = -1$ . By Lemma 5.1,  $\gamma_{2p,2q}(2i) = \pi(2j) = \gamma_{2p,2q}(2j)$ ; so  $i = j$ , but  $\varepsilon_j = -1$  and  $\varepsilon_i = 1$ . As this is impossible, we cannot have  $\varepsilon_i = \varepsilon_{\gamma_{p,q}(i)}$ .

The arguments above show that  $\varepsilon_i = -\varepsilon_{i+1}$  for  $1 \leq i < p$  or  $p + 1 \leq i < p + q$ , and that  $\varepsilon_p = -\varepsilon_1$  and  $\varepsilon_{p+q} = -\varepsilon_{p+1}$ . This implies that  $p$  and  $q$  must be even and thus we have proved  $(\beta)$  as desired. ■

A *second order Haar unitary* is the second order limiting distribution of a Haar distributed random unitary matrix. In Proposition 8.2, we show that a second order Haar unitary is second order  $R$ -diagonal; see Section 8.1. Thus we get the following corollary. Furthermore, in Proposition 6.4, part (iii) we will see that any  $R$ -diagonal element can be realized in this way.

**Corollary 5.2.** *Let  $u$  and  $b$  be elements in some second order  $*$ -probability space such that  $u$  is a second order Haar unitary and such that  $u$  and  $b$  are second order  $*$ -free. Then  $ub$  is  $R$ -diagonal.*

**5.2. Cumulants of  $aa^*$  for an  $R$ -diagonal operator**

Now we prove Theorem 3.5, which allows us to calculate the cumulants of a second order  $R$ -diagonal element  $a$  with determining sequences  $(\beta_n)_{n \geq 1}$  and  $(\beta_{p,q})_{p,q \geq 1}$  by the formula

$$\kappa_{p,q}(aa^*, \dots, aa^*) = \sum_{\pi \in \mathcal{PS}_{\text{NC}}(p,q)} \beta_\pi. \tag{5.1}$$

*Proof of Theorem 3.5.* We use the formula for cumulants with products as arguments as in Proposition 2.7. Indeed,

$$\kappa_{p,q}(aa^*, \dots, aa^*) = \sum_{\substack{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p,2q) \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } O}} \kappa_{(\mathcal{V}, \pi)}(a, \dots, a^*). \tag{5.2}$$

Let  $\mathcal{X} = \{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p, 2q) \mid \gamma_{2p,2q}\pi^{-1} \text{ separates the points of } O \text{ and } \kappa_{(\mathcal{V}, \pi)}(a, a^*, \dots, a, a^*) \neq 0\}$ . Below we define a bijection  $\psi: \mathcal{X} \rightarrow \mathcal{PS}_{\text{NC}}(p, q)$  such that  $\kappa_{(\mathcal{V}, \pi)}(a, \dots, a^*) = \beta_{\psi(\mathcal{V}, \pi)}$ . This will show that right-hand sides of equations (5.1) and (5.2) are the same and thus prove the theorem.

Now, since  $a$  is  $R$ -diagonal, the only non-vanishing cumulants  $\kappa_{(\mathcal{V}, \pi)}$  in the sum above have entries which alternate between  $a$  and  $a^*$  and, in particular, each block of  $\pi$  must have an even number of elements.

We have two terms on the right-hand side of equation (5.2). The first term is when  $\pi$  is a permutation in  $S_{\text{NC}}^-(2p, 2q)$ , then by Lemma 4.14 there is a unique element  $\sigma \in S_{\text{NC}}(p, q)$  such that  $\hat{\sigma} = \pi$ . We set  $\psi(\pi) = \sigma$ , and then by construction  $\beta_\sigma = \kappa_{\psi(\sigma)}(a, a^*, \dots, a, a^*)$ . If  $\pi \in S_{\text{NC}}(2p, 2q) \setminus S_{\text{NC}}^-(2p, 2q)$ , we have

$$\kappa_\pi(a, \dots, a^*) = 0.$$

In the second case,  $(\mathcal{V}, \pi)$  is in  $\mathcal{PS}_{\text{NC}}(2p, 2q)'$ ; by Proposition 4.23, there is  $(\check{\mathcal{V}}, \check{\pi}) \in \mathcal{PS}_{\text{NC}}(p, q)'$  such that  $\beta_{(\check{\mathcal{V}}, \check{\pi})} = \kappa_{(\mathcal{V}, \pi)}(a, a^*, \dots, a, a^*)$ . So we set

$$\psi(\mathcal{V}, \pi) = (\check{\mathcal{V}}, \check{\pi}),$$

and then

$$\sum_{\substack{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(2p,2q) \\ \pi^{-1}\gamma_{2p,2q} \text{ sep. } O}} \kappa_{(\mathcal{V}, \pi)}(a, \dots, a^*) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \beta_{(\mathcal{V}, \pi)}. \tag{5.3}$$

as desired. ■

**Remark 5.3.** Note that equation (5.3) can be inverted. So that given a second order  $R$ -diagonal element,  $a$ , the  $*$ -cumulants of  $a$  can be recovered from the cumulants of  $aa^*$ .

## 6. Even elements

### 6.1. Squares of even operators

Let  $(A, \varphi, \varphi_2)$  be a second order non-commutative probability space. Recall that an element  $x \in \mathcal{A}$  is called *even* if  $\varphi(x^{2n+1}) = 0$  for all  $n \geq 0$  and  $\varphi_2(x^p, x^q) = 0$  unless both  $p$  and  $q$  are even. Recall also that for an even operator  $x$ , we set

$$\beta_n = \kappa_{2n}^{(x)} \quad \text{and} \quad \beta_{p,q} = \kappa_{2p,2q}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{\text{all}+}(2p,2q)} \kappa_{\pi}^{(x)}$$

to be the determining sequences of  $x$ .

Now suppose we are given  $(\mathcal{U}, \sigma) \in \mathcal{PS}_{\text{NC}}(p, q)'$ . Write  $\sigma = \sigma_1 \times \sigma_2 \in \text{NC}(p) \times \text{NC}(q)$ . Let  $U$  be the block of  $\mathcal{U}$  which is the union of a cycle of  $\sigma_1$  and a cycle of  $\sigma_2$ . Recall from Notation 4.18 that  $S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p, 2q)$  is the set of  $\pi \in S_{\text{NC}}^+(2p, 2q)$  such that

- each non-through cycle of  $\pi$  is the double of some cycle  $c$  of  $\sigma$ ;
- the union of all the through cycles of  $\pi$  is  $\hat{U}$ .

**Lemma 6.1.** *Let  $x$  be an even element. Then*

$$\sum_{\substack{\pi \in S_{\text{NC}}^+(2p,2q) \\ \gamma_{2p,2q} \pi^{-1} \text{ sep. } O}} \kappa_{\pi}^{(x)} = \sum_{(\mathcal{U}, \sigma) \in \mathcal{PS}(p,q)'} \sum_{\pi \in S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p,2q)} \kappa_{\pi}^{(x)}.$$

*Proof.* By Proposition 4.19, the sets being summed over are the same and thus the sums are equal. ■

**Lemma 6.2.** *Let  $x$  be second order even and  $(\mathcal{U}, \sigma) \in \mathcal{PS}_{\text{NC}}(p, q)'$ . Then*

$$\beta_{(\mathcal{U}, \sigma)}^{(x)} = \kappa_{(\hat{\mathcal{U}}, \hat{\sigma})}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{(\mathcal{U}, \sigma)}(2p,2q)} \kappa_{\pi}^{(x)}.$$

*Proof.* Let the cycles of  $\sigma$  be  $c_1, c_2, \dots, c_k$  with  $c_{k-1} \subset [p]$ ,  $c_k \subset [p+1, p+q]$ . We shall also write  $\mathcal{U} = \{c_1, \dots, c_{k-2}, c_{k-1} \cup c_k\}$ . For this proof, we shall write  $\beta_{c_i}$  for  $\beta_{k_i}^{(x)}$  if the cycle  $c_i$  has  $k_i$  elements. Then  $\beta_{c_i} = \kappa_{\hat{c}_i}^{(x)}$ ; so

$$\beta_{c_1} \cdots \beta_{c_{k-2}} = \kappa_{\hat{c}_1}^{(x)} \cdots \kappa_{\hat{c}_{k-2}}^{(x)}.$$

Also

$$\beta_{c_{k-1}, c_k} = \kappa_{\hat{c}_{k-1}, \hat{c}_k}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{\text{all}+}(\hat{c}_{k-1}, \hat{c}_k)} \kappa_{\pi}^{(x)}.$$

Hence

$$\begin{aligned}
 \beta_{(\mathcal{U},\sigma)}^{(x)} &= \beta_{c_1} \beta_{c_2} \cdots \beta_{c_{k-2}} \beta_{c_{k-1},c_k} \\
 &= \kappa_{\hat{c}_1}^{(x)} \kappa_{\hat{c}_2}^{(x)} \cdots \kappa_{\hat{c}_{k-2}}^{(x)} \left( \kappa_{\hat{c}_{k-1},\hat{c}_k}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{\text{all}+}(\hat{c}_{k-1},\hat{c}_k)} \kappa_{\pi}^{(x)} \right) \\
 &= \kappa_{(\hat{\mathcal{U}},\hat{\sigma})}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{(\mathcal{U},\sigma)}(2p,2q)} \kappa_{\pi}^{(x)}. \quad \blacksquare
 \end{aligned}$$

**Lemma 6.3.** *Let  $x$  be an even element and  $\beta_n = \kappa_{2n}^{(x)}$ . Then*

$$\sum_{\substack{\pi \in S_{\text{NC}}^-(2p,2q) \\ \gamma_{2p,2q} \pi^{-1} \text{ sep. } O}} \kappa_{\pi}^{(x)} = \sum_{\sigma \in S_{\text{NC}}(p,q)} \beta_{\sigma}.$$

*Proof.* The  $\pi$ 's over which we must sum are the ones that are in bijection with  $S_{\text{NC}}(p, q)$  by Lemma 4.14. Moreover, a cycle of  $\pi$  which must have even length, corresponds to a cycle of  $\tilde{\pi}$  of half this length. For such a  $\pi$ , we then have  $\kappa_{\pi}^{(x)} = \beta_{\tilde{\pi}}$ . This proves the lemma.  $\blacksquare$

Finally, we are able to prove Theorem 3.6 which states that the second order cumulants of  $x^2$  are given by

$$\kappa_{p,q}(x^2, \dots, x^2) = \sum_{(\mathcal{V},\pi) \in \mathcal{P}S_{\text{NC}}(p,q)} \beta_{(\mathcal{V},\pi)}^{(x)}.$$

*Proof of Theorem 3.6.* On one hand, Lemma 6.3 says

$$\sum_{\substack{\pi \in S_{\text{NC}}^-(2p,2q) \\ \gamma_{2p,2q} \pi^{-1} \text{ sep. } O}} \kappa_{\pi}^{(x)} = \sum_{\pi \in S_{\text{NC}}(p,q)} \beta_{\pi}^{(x)},$$

and on the other, we have by using Lemma 4.21 and Proposition 4.19, respectively, that

$$\begin{aligned}
 &\sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}S_{\text{NC}}(2p,2q)' \\ \gamma_{2p,2q} \pi^{-1} \text{ sep. } O}} \kappa_{(\mathcal{V},\pi)}^{(x)} + \sum_{\substack{\pi \in S_{\text{NC}}^+(2p,2q) \\ \gamma_{2p,2q} \pi^{-1} \text{ sep. } O}} \kappa_{\pi}^{(x)} \\
 &= \sum_{(\mathcal{U},\sigma) \in \mathcal{P}S_{\text{NC}}(p,q)'} \left\{ \kappa_{(\hat{\mathcal{U}},\hat{\sigma})}^{(x)} + \sum_{\pi \in S_{\text{NC}}^{(\mathcal{U},\sigma)}(2p,2q)} \kappa_{\pi}^{(x)} \right\} = \sum_{(\mathcal{U},\sigma) \in \mathcal{P}S_{\text{NC}}(p,q)'} \beta_{(\mathcal{U},\sigma)}^{(x)},
 \end{aligned}$$

where the last equality follows from Lemma 6.2.

From this we get using the formula for cumulants with products as arguments and separating the sum into parts over  $S_{\text{NC}}^-(2p, 2q)$ ,  $S_{\text{NC}}^+(2p, 2q)$ , and  $\mathcal{P}S_{\text{NC}}(2p, 2q)'$

$$\begin{aligned} \kappa_{p,q}^{(x^2)} &= \sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}S_{\text{NC}}(2p,2q) \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } \mathcal{O}}} \kappa_{(\mathcal{V},\pi)}^{(x)} = \sum_{\substack{\pi \in S_{\text{NC}}(2p,2q) \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } \mathcal{O}}} \kappa_{\pi}^{(x)} + \sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}S_{\text{NC}}(2p,2q)' \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } \mathcal{O}}} \kappa_{(\mathcal{V},\pi)}^{(x)} \\ &= \sum_{\substack{\pi \in S_{\text{NC}}^-(2p,2q) \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } \mathcal{O}}} \kappa_{\pi}^{(x)} + \sum_{\substack{\pi \in S_{\text{NC}}^+(2p,2q) \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } \mathcal{O}}} \kappa_{\pi}^{(x)} + \sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}S_{\text{NC}}(2p,2q)' \\ \gamma_{2p,2q}\pi^{-1} \text{ sep. } \mathcal{O}}} \kappa_{(\mathcal{V},\pi)}^{(x)} \\ &\stackrel{(*)}{=} \sum_{\pi \in S_{\text{NC}}(p,q)} \beta_{\pi}^{(x)} + \sum_{(\mathcal{U},\sigma) \in \mathcal{P}S_{\text{NC}}(p,q)'} \beta_{(\mathcal{U},\sigma)}^{(x)} \\ &= \sum_{(\mathcal{U},\sigma) \in \mathcal{P}S_{\text{NC}}(p,q)} \beta_{(\mathcal{U},\sigma)}^{(x)} \end{aligned}$$

as desired, where the equality (\*) follows from Lemma 6.3. ■

### 6.2. From $R$ -diagonal to even operators

Let  $(\mathcal{A}, \varphi, \varphi_2)$  be a non-commutative probability space, and let  $d$  be a positive integer. Consider the algebra  $M_d(\mathcal{A})$  of  $d \times d$  matrices over a space  $\mathcal{A}$  and the functionals  $\tilde{\varphi} := \varphi \circ \text{tr} : M_d(\mathcal{A}) \rightarrow \mathbb{C}$ , and  $\tilde{\varphi}_2 := \varphi_2 \circ \text{tr} \otimes \text{tr}$  on  $M_d(\mathcal{A}) : M_d(\mathcal{A}) \otimes M_d(\mathcal{A}) \rightarrow \mathbb{C}$  defined by the formulas

$$\begin{aligned} \tilde{\varphi}((a_{i,j})_{i,j=1}^d) &= \frac{1}{d} \sum_{i=1}^d \varphi(a_{ii}) = \varphi(\text{tr}(A)), \\ \tilde{\varphi}_2((a_{i,j})_{i,j=1}^d, (b_{i,j})_{i,j=1}^d) &= \frac{1}{d^2} \sum_{i,j=1}^d \varphi_2(a_{ii}, b_{jj}) = \varphi_2(\text{tr}(A), \text{tr}(B)). \end{aligned}$$

Then  $(M_d(\mathcal{A}), \tilde{\varphi}, \tilde{\varphi}_2)$  is itself a second order non-commutative probability space. Note that if either  $(a_{ij})_{i,j=1}^d$  or  $(b_{ij})_{i,j=1}^d$  is zero on the diagonal, then

$$\tilde{\varphi}_2((a_{ij})_{i,j=1}^d, (b_{ij})_{i,j=1}^d) = 0.$$

We can realize second order even elements as  $2 \times 2$  matrices with  $R$ -diagonal elements in the off-diagonal entries. This generalizes the theorem in [25, Proposition 15.12] where the first order case was considered.

**Proposition 6.4.** *Let  $a$  be second order  $R$ -diagonal in  $(\mathcal{A}, \varphi, \varphi_2)$  and consider the off-diagonal matrix*

$$A := \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

as an element in  $(M_2(\mathcal{A}), \tilde{\varphi}, \tilde{\varphi}_2)$ . Then

- (i)  $A$  is second order even element;
- (ii)  $A$  has the same determining sequence as  $a$ :  $\beta_{p,q}^{(A)} = \beta_{p,q}^{(a)}$  for all  $p, q \geq 1$  and  $\beta_n^{(A)} = \beta_n^{(a)}$  for all  $n$ ;
- (iii) any second order  $R$ -diagonal operator can be realized in the form  $xu$ , where  $u$  is a second order Haar unitary,  $x$  is second order even, and  $u$  and  $x$  are second order  $*$ -free;
- (iv) The second order cumulants of  $a$  and the second order cumulants of  $A$  are related by

$$\kappa_{p,q}^{(a)} = \kappa_{p,q}^{(A)} + \sum_{\pi \in S_{\text{NC}}^{\text{all}+}(m,n)} \kappa_{\pi}^{(A)}.$$

*Proof.*  $A$  is even because  $\text{tr}(A^p) = 0$  for  $p$  odd. On the first order level, claims (i)–(iv) are proved in [25, Proposition 15.12], so we shall only prove the corresponding claims at the second order level. For (ii), the only observation needed is that from Theorems 3.5 and 3.6 the determining sequence of  $A$  and  $a$  can be calculated from the moments and fluctuation moments of  $aa^*$  and  $A^2$  in the same way, (see equations (3.3) and (3.4)). So it is enough to see that the moments and fluctuation moments of  $aa^*$  and  $A^2$  are the same, i.e.,  $\tilde{\varphi}_2(A^{2m}, A^{2n}) = \varphi_2((aa^*)^m, (aa^*)^n)$ , but this is clear from the definition of  $A$ .

Claim (iii) is proved as follows. Let  $U$  be a second order Haar unitary which is second order free from  $A$ . By Theorem 3.3 and Proposition 8.2 (below),  $X = AU^*$  is  $R$ -diagonal and  $XX^* = AU^*UA = A^2$ . This then implies that the determining series of  $X$  coincides with the determining series of  $A$  and also with the determining series of  $a$ , by (ii). So  $X$  and  $a$  are two second order  $R$ -diagonal operators with the same distribution.

(iv) By (iii), the determining sequences coincide, i.e.,  $\beta_{p,q}^{(A)} = \beta_{p,q}^{(a)}$  for all  $p, q \geq 1$  and  $\beta_n^{(A)} = \beta_n^{(a)}$  for all  $n$ . This gives us exactly the desired relation. ■

### 7. Cumulants of products of free random variables

In this section, we will prove the second order analogue of the theorem of Arizmendi and Vargas [3] for the moments and cumulants of  $a_1 a_2 \cdots a_n$  when  $a_i$ 's are second order free. We will use the formula for products as arguments.

**Definition 7.1.** Let a  $\pi$  be a non-crossing permutation in  $S_{\text{NC}}(kp, kq)$ .

- (i)  $\pi$  is called  $k$ -divisible if the size of every cycle of  $\pi$  is a multiple of  $k$ .
- (ii)  $\pi$  is called  $k$ -equal if the size of every cycle of  $\pi$  is  $k$ .



- (iii)  $\pi$  is called  $k$ -alternating if  $\pi(i) \equiv i + 1 \pmod{k}$ .  $S_{\text{NC}}^{k\text{-alt}}(kp, kq)$  is the set of  $k$ -alternating permutations.
- (iv)  $S_{\text{NC}}^{k\text{-e-a}}(kp, kq)$  is the set of  $k$ -equal and  $k$ -alternating permutations.
- (v)  $\pi$  is called  $k$ -preserving if  $\pi(i) \equiv i \pmod{k}$ .
- (vi) If  $\pi$  is  $k$ -preserving and  $\pi^{-1}\gamma_{kp,kq}$  separates the points  $\{k, 2k, \dots, (p + q)k\}$ , then we say that  $\pi$  is  $k$ -completing.

Note that if  $\pi$  is  $k$ -alternating, then it is also  $k$ -divisible. The converse is not true but there is a  $k$ -to-1 correspondence from  $k$ -divisible to  $k$ -alternating partitions, giving by a relabelling in one of the circles.

From now on, we will denote by

$$\gamma = \gamma_{kp,kq} = (1, 2, \dots, kp)(kp + 1, \dots, kp + kq)$$

and denote by  $\text{Kr}(\sigma) = \sigma^{-1}\gamma$ . The following is the analogue of [3, Proposition 3.1].

**Lemma 7.2.** *Let  $\sigma$  be in  $S_{\text{NC}}(kp, kq)$ .*

- (i)  $\sigma$  is  $k$ -alternating if and only if  $\sigma^{-1}\gamma$  is  $k$ -preserving.
- (ii)  $\sigma$  is  $k$ -alternating and  $k$ -equal if and only if  $\sigma^{-1}\gamma$  is  $k$ -completing.

*Proof.* (i) Suppose that  $\sigma$  is  $k$ -alternating. Since  $\gamma(i) \equiv i + 1 \pmod{k}$ , then

$$\sigma^{-1}\gamma(i) = \sigma^{-1}(i + 1) \equiv i \pmod{k}.$$

Conversely, if  $\sigma^{-1}\gamma(i) \equiv i \pmod{k}$ , then  $\sigma(i) = \gamma(i) \equiv i + 1 \pmod{k}$ .

(ii) Note that by (i) if  $\sigma^{-1}\gamma$  is  $k$ -completing, then  $\sigma$  is  $k$ -alternating. We claim that  $\sigma$  is  $k$ -equal. Suppose that this is not the case. That is, suppose that  $\sigma$  is  $k$ -alternating, but not  $k$ -equal. Then  $\sigma$  has at most  $p + q - 1$  cycles. But  $(\sigma^{-1}\gamma)^{-1}\gamma = \gamma^{-1}\sigma\gamma$  has the same number of cycles as  $\sigma$ . Thus  $(\sigma^{-1}\gamma)^{-1}\gamma$  cannot separate more than  $p + q - 1$  points from  $\{k, 2k, \dots, (p + q)k\}$ . Thus  $\sigma^{-1}\gamma$  is not  $k$ -completing, yielding a contradiction.

Conversely, if  $\sigma$  is  $k$ -alternating and  $k$ -equal, then  $\sigma^k$  is the identity permutation. Let  $K = \{k, 2k, 3k, \dots, (p + q)k\}$ . Then  $\sigma(\gamma^i(K)) = \gamma^{i+1}(K)$  for  $1 \leq i \leq k$ . To show that  $\sigma^{-1}\gamma$  is  $k$ -completing, we must show that

$$\gamma^{-1}\sigma^{-r}\gamma(K) \cap K = \emptyset \quad \text{for } 1 \leq r < k.$$

Since  $\sigma(\gamma^i(K)) = \gamma^{i+1}(K)$  for  $1 \leq i \leq k$ , we have

$$\gamma^{-1}\sigma^{-r}\gamma(K) \cap K = \gamma^{-r}(K) \cap K = \emptyset \quad \text{for } 1 \leq i \leq k. \quad \blacksquare$$

Now we are in position to prove Theorem 3.8.

**Theorem 7.3** (Moments and cumulants of products of free variables). *Let  $a_1, \dots, a_k$  be operators which are second order free and such that  $\kappa_{p,q}(a_i) = 0$  for all  $p, q \in \mathbb{N}$ . Let  $a = a_1 a_2 \cdots a_k$ . Then*

$$\varphi_2(a^p, a^q) = \sum_{\pi \in S_{\text{NC}}^{k\text{-alt}}(kp, kq)} \kappa_{\text{Kr}(\pi)}(a_1, a_2, \dots, a_k, \dots, a_1, a_2, \dots, a_k). \tag{7.1}$$

Furthermore,

$$\kappa_{p,q}(a, \dots, a) = \sum_{\pi \in S_{\text{NC}}^{k\text{-e-a}}(kp, kq)} \kappa_{\text{Kr}(\pi)}(a_1, \dots, a_k, \dots, a_1, \dots, a_k). \tag{7.2}$$

*Proof.* We shall use the moment-cumulant formula (Section 2.4) and the second order freeness of the  $a_i$ 's. By hypothesis, the second order cumulants of  $a_i$ 's are zero. In addition, we have further assumed that the  $a_i$ 's are second order free. Thus for all partitioned permutations  $(\mathcal{V}, \pi)$ , we have that  $\kappa_{(\mathcal{V}, \pi)}(a_1, \dots, a_k, \dots, a_1, \dots, a_k) = 0$ . Thus in the moment-cumulant formula, we only need to sum over  $S_{\text{NC}}(kp, kq)$ :

$$\varphi(a^p, a^q) = \sum_{\pi \in S_{\text{NC}}(kp, kq)} \kappa_{\pi}(a_1, a_2, \dots, a_k, \dots, a_1, a_2, \dots, a_k).$$

The contribution of a  $\pi$  will be non-zero only if  $\pi$  joins an  $a_i$  with another  $a_i$ , which implies that  $\pi$  is  $k$ -preserving. Let  $\sigma = \gamma\pi^{-1}$ , then  $\text{Kr}(\sigma) = \pi$  and by Lemma 7.2 (i)  $\sigma$  is  $k$ -alternating. This proves the first formula.

For the second formula, we use the formula for products as arguments, as in Proposition 2.7, for  $a = a_1 a_2 \cdots a_k$ . Again, since the second order cumulants of  $a_i$ 's are zero, the only non-vanishing contribution are from  $\pi$  in  $S_{\text{NC}}(kp, kq)$ . Thus we get

$$\kappa_{p,q}(a, \dots, a) = \sum_{\pi \in S_{\text{NC}}(kp, kq)} \kappa_{\pi}(a_1, \dots, a_k, \dots, a_1, \dots, a_k),$$

where the sum runs over the permutations  $\pi$  in  $S_{\text{NC}}(kp, kq)$  such that  $\pi^{-1}\gamma_{kp, kq}$  separates the points  $\{k, 2k, \dots, (p + q)k\}$ . Since the random variables are second order free, the sum runs only over the  $k$ -preserving partitions in  $S_{\text{NC}}(kp, kq)$ . These two conditions on  $\pi$  mean exactly that  $\pi$  is  $k$ -completing. Finally, by Lemma 7.2 the permutations involved in the sum are exactly the Kreweras complements of  $k$ -equal permutations which are also  $k$ -alternating, and the formula follows. ■

Note that by Lemma 7.2 we can write the set

$$\{\text{Kr}(\pi) \mid \pi \in S_{\text{NC}}(kp, kq), \pi \text{ is } k\text{-equal and } k\text{-alternating}\}$$

as

$$\{\sigma \in S_{\text{NC}}(kp, kq) \mid \sigma \text{ is } k\text{-preserving and } \text{Kr}(\sigma) \text{ is } k\text{-equal}\}.$$

Of particular interest will be the case where  $\kappa_n(a_i) = 1$  for all  $i$  and all  $n$ , i.e., when the  $a_i$ 's are free Poisson operators. This will be explained in Example 8.15.

**Remark 7.4.** There is a natural bijection between  $k$  divisible partitions  $(k + 1)$ -equal ones for  $\text{NC}(n)$  which is given by the ‘fattening procedure’ as described by Arizmendi in [1]. The second order analogue is also true, that is, we have  $S_{\text{NC}}^{k\text{-alt}}(kp, kq)$  is in bijection with  $S_{\text{NC}}^{j\text{-e-a}}(jp, jq)$ , for  $j = k + 1$ ; we leave this observation as an exercise for the reader.

### 8. Examples and applications

We now give some examples on how to use the main theorems and constructions given above.

#### 8.1. A second order Haar unitary is second order $R$ -diagonal

Our first example comes from [23, §5].

**Definition 8.1.** Let  $(\mathcal{A}, \varphi, \varphi_2)$  be a second order probability space and  $u \in \mathcal{A}$  a unitary. We say that  $u$  is a *second order Haar unitary* if  $\varphi(u^k) = \varphi(u^{*k}) = 0$  for  $k \geq 1$  (i.e.,  $u$  is a Haar unitary) and  $\varphi_2(u^k, u^{*l}) = k\delta_{k,l}$ .

Let us first see that a second order Haar unitary is second order  $R$ -diagonal.

**Proposition 8.2.** *Let  $u$  be a second order Haar unitary. Then  $u$  is second order  $R$ -diagonal. Moreover, given  $p$  and  $q$  positive integers and  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_{p+q} \in \{-1, 1\}$ , we have  $\kappa_{p,q}(u^{\varepsilon_1}, u^{\varepsilon_2}, \dots, u^{\varepsilon_{p-1}}, u^{\varepsilon_p}, u^{\varepsilon_{p+1}}, \dots, u^{\varepsilon_{p+q}}) = 0$  unless  $p$  and  $q$  are even and*

$$\begin{aligned} \varepsilon_1 + \varepsilon_2 = \dots = \varepsilon_{p-1} + \varepsilon_p = \varepsilon_{p+1} + \varepsilon_{p+2} = \dots \\ = \varepsilon_{p+q-1} + \varepsilon_{p+q} = 0, \end{aligned} \tag{8.1}$$

i.e., the  $\varepsilon$ 's alternate in sign except possibly between  $p$  and  $p + 1$ .

*Proof.* Let us show first that if  $\varepsilon_1 + \dots + \varepsilon_{p+q} \neq 0$ , then  $\kappa_{p,q}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) = 0$ . Suppose  $\varepsilon_1 + \dots + \varepsilon_{p+q} \neq 0$ . First note that for all  $\pi \in \mathcal{S}_{\text{NC}}(p, q)$ , some cycle of  $\pi$ ,  $(i_1, \dots, i_k)$ , must have  $\varepsilon_1 + \dots + \varepsilon_k \neq 0$  and thus  $\varphi_\pi(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) = 0$  because we assumed that  $u$  is a Haar unitary. Also for  $(\mathcal{U}, \pi) \in \mathcal{PS}_{\text{NC}}(p, q)'$ , we have that  $\mathcal{U}$  must have a block where the  $\varepsilon$ 's do not sum to 0. If this block is a cycle,  $(i_1, \dots, i_k)$ , of  $\pi$ , then  $\varphi(u^{\varepsilon_{i_1}} \dots u^{\varepsilon_{i_k}}) = 0$ . If this block is the union of two cycles  $(i_1, \dots, i_k)$  and  $(j_1, \dots, j_l)$  of  $\pi$ , then  $\varphi_2(u^{\varepsilon_{i_1}} \dots u^{\varepsilon_{i_k}}, u^{\varepsilon_{j_1}} \dots u^{\varepsilon_{j_l}}) = 0$ . In either case, we have

$$\varphi(\mathcal{U}, \pi)(u^{\varepsilon_1}, \dots, u^{\varepsilon_p}, u^{\varepsilon_{p+1}}, \dots, u^{\varepsilon_{p+q}}) = 0.$$

Since we have by [8, Definition 7.4]

$$\begin{aligned} &\kappa_{p,q}(u^{\varepsilon_1}, \dots, u^{\varepsilon_p}, u^{\varepsilon_{p+1}}, \dots, u^{\varepsilon_{p+q}}) \\ &= \sum_{(\mathcal{U}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \mu(\mathcal{V}, \pi) \varphi_{(\mathcal{U}, \pi)}(u^{\varepsilon_1}, \dots, u^{\varepsilon_p}, u^{\varepsilon_{p+1}}, \dots, u^{\varepsilon_{p+q}}), \end{aligned}$$

we have that  $\kappa_{p,q}(u^{\varepsilon_1}, \dots, u^{\varepsilon_p}, u^{\varepsilon_{p+1}}, \dots, u^{\varepsilon_{p+q}}) = 0$ . From now on, we assume that  $\varepsilon_1 + \dots + \varepsilon_{p+q} = 0$  and  $p + q$  is even.

We shall prove the proposition by induction on  $(p, q)$ . If  $p = q = 1$ , we have

$$\kappa_{1,1}(u^{\varepsilon_1}, u^{\varepsilon_2}) = \varphi_2(u^{\varepsilon_1}, u^{\varepsilon_2}) - \kappa_2(u^{\varepsilon_1}, u^{\varepsilon_2}) = 0$$

because  $\varphi_2(u^{\varepsilon_1}, u^{\varepsilon_2}) = \kappa_2(u^{\varepsilon_1}, u^{\varepsilon_2}) = 1$ .

If  $p = q = 2$ , we have to check that  $\kappa_{2,2}(u^{\varepsilon_1}, u^{\varepsilon_2}, u^{\varepsilon_3}, u^{\varepsilon_4}) = 0$  when either  $\varepsilon_1 = \varepsilon_2$  or  $\varepsilon_3 = \varepsilon_4$ ; in fact, one condition implies the other because  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0$ . Thus

$$\begin{aligned} 2 &= \varphi_2(u^{\varepsilon_1}u^{\varepsilon_2}, u^{\varepsilon_3}u^{\varepsilon_4}) \\ &= \sum_{\pi \in S_{\text{NC}}(2,2)} \kappa_{\pi}(u^{\varepsilon_1}, u^{\varepsilon_2}, u^{\varepsilon_3}, u^{\varepsilon_4}) + \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}(p,q)'} \kappa_{(\mathcal{V}, \pi)}(u^{\varepsilon_1}, u^{\varepsilon_2}, u^{\varepsilon_3}, u^{\varepsilon_4}). \end{aligned}$$

In the first term, we will only have contributions from permutations that have cycles of even length that alternate between  $u$  and  $u^*$ . There are no permutations with cycles of length four that alternate because we have two  $u$ 's on one circle and two  $u^*$ 's on the other. There are two permutations with two cycles of length two that alternate between  $u$  and  $u^*$ . Thus the contribution of the first term is 2. The contribution of the second term is just  $\kappa_{2,2}(u^{\varepsilon_1}, u^{\varepsilon_2}, u^{\varepsilon_3}, u^{\varepsilon_4})$  as all other terms contain factors of  $\kappa_{1,1}$  or  $\kappa_{2,1}$  which we have already shown to be 0. Hence  $\kappa_{2,2}(u^{\varepsilon_1}, u^{\varepsilon_2}, u^{\varepsilon_3}, u^{\varepsilon_4}) = 0$  when  $\varepsilon_1 = \varepsilon_2$  or  $\varepsilon_3 = \varepsilon_4$ .

Suppose that the proposition holds whenever  $r < p$  and  $q \leq s$  or  $r \leq p$  and  $s < q$ , and that either  $p$  is odd,  $q$  is odd, or property (8.1) fails. Then we can find in either the first  $p$  positions or in the last  $q$  positions a cyclically adjacent pair of  $\varepsilon$ 's of the same sign. Recall that  $\varphi_2$  is tracial in each variable, so without loss of generality, we may assume that  $\varepsilon_1 = 1, \varepsilon_2 = -1$ , and  $\varepsilon_3 = -1$ .

Now  $u^{\varepsilon_1}u^{\varepsilon_2} = 1$ , so by [8, Proposition 7.8],  $\kappa_{p-1,q}(u^{\varepsilon_1}u^{\varepsilon_2}, u^{\varepsilon_3}, \dots, u^{\varepsilon_{p+q}}) = 0$ . Let  $N = \{2, 3, 4, \dots, p + q\}$ . By Proposition 2.7,

$$\begin{aligned} 0 &= \kappa_{p-1,q}(u^{\varepsilon_1}u^{\varepsilon_2}, u^{\varepsilon_3}, \dots, u^{\varepsilon_{p+q}}) \\ &= \sum_{\substack{\pi \in S_{\text{NC}}(p,q) \\ \pi^{-1}\gamma_{p,q} \\ \text{sep. } N}} \kappa_{\pi}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) + \sum_{\substack{(\mathcal{V}, \pi) \in \mathcal{PS}(p,q)' \\ \pi^{-1}\gamma_{p,q} \\ \text{sep. } N}} \kappa_{(\mathcal{V}, \pi)}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}). \end{aligned} \tag{8.2}$$

Let us examine the condition on  $\pi$  in each of the two terms. The requirement that  $\pi^{-1}\gamma_{p,q}$  separates the points of  $N$  means that either  $\pi = \gamma_{p,q}$  or  $\pi = \gamma_{p,q}(1, r)$  for some  $r \neq 1$ .

For the first term, in order for  $\pi$  to connect the two cycles of  $\gamma$ , we must have  $\pi = \gamma_{p,q}(1, r)$  for some  $r \in [p + 1, p + q]$ . This means that  $\pi = (1, r + 1, r + 2, \dots, p + q, p + 1, \dots, r, 2, 3, 4, \dots, p)$  and thus the first order cumulant  $\kappa_\pi(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}})$  is 0 because  $\varepsilon_2$  and  $\varepsilon_3$  are adjacent.

In the second term, we must have either  $\pi = \gamma_{p,q}(1, r)$  with  $r \in [p]$  or  $\pi = \gamma_{p,q}$ . Let us first suppose the former holds. Then

$$\pi = (1, r + 1, r + 2, \dots, p)(2, 3, 4, \dots, r)(p + 1, p + 2, \dots, p + q)$$

and there are two possibilities for  $\mathcal{V}$ . Either

$$\mathcal{V} = \{(1, r + 1, r + 2, \dots, p, p + 1, \dots, p + q)(2, 3, \dots, r)\}$$

or

$$\mathcal{V} = \{(2, 3, \dots, r, p + 1, \dots, p + q)(1, r + 1, r + 2, \dots, p)\}.$$

In the first case,

$$\begin{aligned} \kappa_{(\mathcal{V}, \pi)}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) &= \kappa_{r-1}(u^{\varepsilon_2}, u^{\varepsilon_3}, \dots, u^{\varepsilon_r}) \\ &\quad \times \kappa_{p-r+1, q}(u^{\varepsilon_1}, u^{\varepsilon_{r+1}}, \dots, u^{\varepsilon_{p+q}}) = 0, \end{aligned}$$

because  $\varepsilon_2$  and  $\varepsilon_3$  are adjacent in the first factor (unless  $r = 2$ , but then we have a singleton, so we get 0 in this case also).

In the second case,

$$\begin{aligned} \kappa_{(\mathcal{V}, \pi)}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) &= \kappa_{r-1, q}(u^{\varepsilon_2}, u^{\varepsilon_3}, \dots, u^{\varepsilon_r}, u^{\varepsilon_{p+1}}, \dots, u^{\varepsilon_{p+q}}) \\ &\quad \times \kappa_{p-r+1}(u^{\varepsilon_1}, u^{\varepsilon_{r+1}}, \dots, u^{\varepsilon_p}) = 0, \end{aligned}$$

by our induction hypothesis because  $\varepsilon_2$  and  $\varepsilon_3$  are adjacent (unless  $r = 2$  in which case  $r - 1$  is odd and so we get 0 in this case also). We have now shown that all the terms in equation (8.2) are zero except possibly the case where  $\pi = \gamma_{p,q}$  and  $\mathcal{V} = 1_{p+q}$ . Since the sum is 0, we must then also have that this term is also 0, i.e.,

$$\kappa_{p,q}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) = \kappa_{(1_{p+q}, \gamma_{p,q})}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) = 0$$

as required. ■

**Notation 8.3.** We put

$$c_{m,n} := \#(S_{\text{NC}}(m, n)) = \frac{2mn}{m+n} \binom{2m-1}{m} \binom{2n-1}{n}.$$

**Proposition 8.4.** *Let  $p = 2m$  and  $q = 2n$  be even integers and  $\varepsilon_1, \dots, \varepsilon_{p+q} \in \{-1, 1\}$  alternating. Then  $\kappa_{p,q}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) = (-1)^{m+n} c_{m,n}$ .*

*Proof.* We have just shown that  $u$  is second order  $R$ -diagonal and since  $uu^* = 1$ , we can use Theorem 3.5 to calculate the cumulants of  $u$ . Indeed, the cumulants of 1 are given by  $\kappa_1 = 1, \kappa_n = 0$  for  $n > 1$  and  $\kappa_{p,q} = 0$  for all  $p, q \geq 1$ . Thus the determining sequence of  $u$  is given by the inversion formula (3.5)  $\delta = \beta^{(u)} * \zeta$ . Hence  $\beta^{(u)}$  is the Möbius function calculated in [8, Theorem 5.24], so

$$\kappa_{p,q}(u^{\varepsilon_1}, \dots, u^{\varepsilon_{p+q}}) = (-1)^{m+n} c_{m,n}. \quad \blacksquare$$

**Proposition 8.5.** *Suppose that  $r$  and  $u$  are second order  $*$ -free and that  $u$  is a second order Haar unitary. Then  $r$  is second order  $R$ -diagonal if and only if  $r$  and  $ur$  have the same second order  $*$ -distribution.*

*Proof.* Suppose that  $r$  is second order  $R$ -diagonal. We have already seen in Proposition 8.2 that  $u$  is second order  $R$ -diagonal. By Theorem 3.3, we have that  $ur$  is also second order  $R$ -diagonal. The second order  $*$ -distributions of  $r$  and  $ur$  are given by their  $*$ -cumulants, and by Theorem 3.5, these are given by the cumulants of  $rr^*$  and  $urur^*$ , respectively. Since both  $\varphi$  and  $\varphi_2$  are tracial, these cumulants are equal. Thus  $r$  and  $ur$  have the same second order  $*$ -distribution.

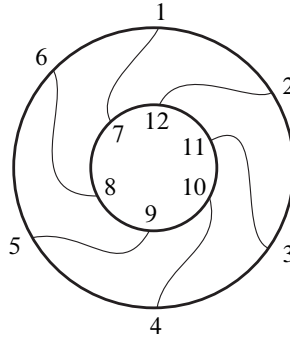
Conversely, suppose that  $r$  and  $ur$  have the same second order  $*$ -distribution. By Theorem 3.3,  $ur$  is second order  $R$ -diagonal. Thus  $r$  is second order  $R$ -diagonal.  $\blacksquare$

### 8.2. Products of semicircular and circular operators

Following [23, §4], a random variable  $s$  in a second order non-commutative probability space is called a *second order semicircular operator* if its first order cumulants satisfy  $\kappa_n(s, s, \dots, s) = 0$  for all  $n \neq 2$  and  $\kappa_2(s, s) = 1$ , and for all  $p$  and  $q$  the second order cumulants  $\kappa_{p,q}$  are 0. This operator appears as the limit of GUE (Gaussian unitary ensemble) random matrices as the size tends to infinity. See [21, Theorem 3.5].

**Example 8.6** (Square of semicircular). It follows from the definition that the first order determining sequence of the semicircular operator is given by  $\beta_1 = 1$  and  $\beta_n = 0$  for  $n > 0$ . The second order determining sequence is given by  $\beta_{k,k} = k$  and  $\beta_{p,q} = 0$  if  $p \neq q$  since the only terms appearing in formula (3.1), are ‘spoke’ diagrams (see Figure 8) which are parity preserving. Theorem 3.6 now gives the second order cumulants of  $s^2$ :

$$\kappa_{p,q}(s^2, \dots, s^2) = \sum_{\pi \in \mathcal{S}_{\text{NC}}(p,q)} \beta_\pi + \sum_{(\mathcal{V}, \pi) \in \mathcal{P}\mathcal{S}_{\text{NC}}(p,q)'} \beta_{(\mathcal{V}, \pi)} = \sum_{\pi \in \mathcal{P}\mathcal{S}_{\text{NC}}(p,q)'} \beta_{(\mathcal{V}, \pi)}$$



**Figure 8.** A *spoke diagram*, this means a non-crossing annular pairing in which all pairs connect the two cycles of  $\gamma_{p,q}$ . This example is *parity preserving* because the connected points have the same parity.

because there are no partitions in  $S_{NC}(p, q)$  with only blocks of size one. Now, the only non-vanishing terms in the last sum are of the form  $\beta_{k,k}\beta_1^{p+q-2k}$ . Indeed, for  $\beta_{(\mathcal{V},\pi)} \neq 0$ , we must have  $\pi = \pi_1 \times \pi_2 \in NC(p) \times NC(q)$ , such that both  $\pi_1$  and  $\pi_2$  have one cycle with  $k$  elements and all others singletons. Then  $\mathcal{V}$  will have one block that is the union of the cycle of  $\pi_1$  with  $k$  elements and the cycle of  $\pi_2$  which also has  $k$  elements; all other blocks  $\mathcal{V}$  are singletons. Moreover, there are  $\binom{p}{k}\binom{q}{k}$  ways of choosing the two cycles with  $k$  elements. For such a  $(\mathcal{V}, \pi)$ , we have  $\beta_{(\mathcal{V},\pi)} = \beta_{k,k}\beta_1^{2p+2q-2k} = k$ . Thus,

$$\begin{aligned} \kappa_{p,q}(s^2, \dots, s^2) &= \sum_{(\mathcal{V},\pi) \in PS_{NC}(p,q)'} \beta_{(\mathcal{V},\pi)} = \sum_{k>0} \binom{p}{k} \binom{q}{k} \beta_{k,k} \\ &= \sum_{k>0} k \binom{p}{k} \binom{q}{k} = p \binom{p+q-1}{p}. \end{aligned}$$

The last equality follows by the Chu–Vandermonde formula.

Another way to derive these cumulants is using the series from Theorem 3.7. Indeed,  $B(z) = 1$  implies that  $C(z) = \frac{1}{z-1}$  and then  $C'(z) = -\frac{1}{(z-1)^2}$ . Also,

$$B(z, w) = \frac{1}{zw} \sum_{p,q \geq 1} \beta_{p,q} z^p w^q = \frac{1}{zw} \sum_{n \geq 1} n z^n w^n = \frac{1}{(1-zw)^2}$$

and

$$\frac{\partial^2}{\partial z \partial w} \log \left( \frac{C(z) - C(w)}{z - w} \right) = \frac{\partial^2}{\partial z \partial w} \log \left( \frac{1}{(z-1)(w-1)} \right) = 0.$$

Hence,

$$\begin{aligned}
 C(z, w) &= C'(z)C'(w)B(C(z), C(w)) \\
 &= \frac{1}{(z-1)^2(w-1)^2} \frac{1}{(1-(z-1)^{-1}(w-1)^{-1})^2} \\
 &= \frac{1}{((z-1)(w-1)-1)^2} = \frac{1}{(zw-z-w)^2} \\
 &= \sum_{p,q \geq 1} p \binom{p+q-1}{p} z^{-(p+1)} w^{-(q+1)}.
 \end{aligned}$$

**Example 8.7** (Circular operator). Consider  $s_1$  and  $s_2$  second order free semicircular operators. We call  $c = \frac{s_1 + is_2}{\sqrt{2}}$  a (second order) circular operator. The operator  $c$  is a second order  $R$ -diagonal. Indeed, since  $s_1$  and  $s_2$  are second order free, their mixed free cumulants vanish, also the second order cumulant of  $s_1$  and  $s_2$  vanish, thus by linearity the same is true for  $c$ . That is,  $\kappa_{p,q}(c^{(\varepsilon_1)}, \dots, c^{(\varepsilon_{p+q})}) = 0$  for all  $\varepsilon_1, \dots, \varepsilon_{p+q} \in \{\pm 1\}$ . On the other hand, it is well known  $\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$  are the only first order vanishing cumulants of  $c$  (see [25, Example 11.23]). Hence  $c$  is second order  $R$ -diagonal.

In [11], Dubach and Peled computed the fluctuation moments of some words in second order  $*$ -free second order circular operators. More precisely, let  $c_1, \dots, c_s$  be second order circular operators and second order  $*$ -free. Dubach and Peled computed the fluctuation moments of the form  $\varphi_2(w^k, (w^*)^l)$ , where  $w = c_{i_1} \cdots c_{i_k}$  is a word in  $c_1, \dots, c_s$ . In the next few examples, we will present the second order free  $*$ -cumulants of  $w$  and the second order free cumulants of  $w^*w$  when  $i_1, \dots, i_k$  are distinct. See Example 8.19 for a short discussion of why our method does not work more generally.

**Example 8.8** ( $cc^*$  for a circular operator  $c$ ). This is a continuation of Examples 8.6 and 8.7. Let  $c$  be a circular element and  $s$  a semicircular operator as in Examples 8.6 and 8.7. Now recall that  $cc^*$  and  $s^2$  both have a Poisson distribution with respect to  $\varphi$ , i.e.,  $\varphi((cc^*)^n) = \varphi((s^2)^n) = \frac{1}{n+1} \binom{2n}{n}$ . So one might expect that  $cc^*$  and  $s^2$  have the same distribution in the second order level. We see that this is not the case.

Indeed, the determining sequences of  $c$  are given by  $\beta_1 = 1$  and  $\beta_n = 0$  for  $n > 0$  and  $\beta_{p,q} = 0$  for all  $p$  and  $q$ . This already shows our claim since  $\beta_{n,n} = n$  in the case for semicircle of Example 8.6. Moreover, from formula (3.3), it is readily seen that the second order cumulants of  $cc^*$  are all zero, i.e.,  $\kappa_{p,q}(cc^*, cc^*, \dots, cc^*) = 0$ .

Since the first order cumulants of  $cc^*$  are all 1, the  $(p, q)$ -fluctuation moments of  $cc^*$  count the number of elements in  $S_{\text{NC}}(p, q)$ . This means that

$$\varphi_2((cc^*)^p, (cc^*)^q) = |S_{\text{NC}}(p, q)| = \frac{pq}{2(p+q)} \binom{2p}{p} \binom{2q}{q}.$$



The last equality comes from the fact that  $|S_{\text{NC}}(p, q)| = \frac{1}{2}|\text{NC}_2(2p, 2q)|$ , where  $\text{NC}_2(2p, 2q)$  is the set of non-crossing annular pairings (see [19, Corollary 6.7]).

One way to interpret the calculation above is that

$$\kappa^{(cc^*)} = \zeta \quad \text{and} \quad \varphi_2((cc^*)^p, (cc^*)^q) = \zeta^{*2}(1_{p+q}, \gamma_{p,q}).$$

**Example 8.9** (Products of free circular operators). Let  $c_1$  and  $c_2$  be a pair of second order circular elements and suppose they are second order free. Let  $h = c_1c_2$ . Let us recall some facts from [25, Lecture 15] about first order case  $R$ -diagonal operators.

Let  $a$  be  $R$ -diagonal. As we are working in a tracial non-commutative probability space, we have from [25, Proposition 15.6] that

$$\kappa_n^{(aa^*)} = \sum_{\pi \in \text{NC}(n)} \beta_\pi^{(a)}$$

which can be written in terms of multiplicative functions (cf. [25, Lecture 10]) as  $\kappa^{(aa^*)} = \beta^{(a)} * \zeta$ . Now suppose that  $a_1$  and  $a_2$  are  $*$ -free and  $R$ -diagonal. We have from [25, Example 15.24] that

$$\begin{aligned} \beta_n^{(a_1a_2)} &= \sum_{\pi \in \text{NC}(n)} \beta_\pi^{(a_1)} \sum_{\sigma \leq K(\pi)} \beta_\sigma^{(a_2)} = \sum_{\pi \in \text{NC}(n)} \beta_\pi^{(a_1)} (\beta^{(a_2)} * \zeta)(0_n, \text{Kr}(\pi)) \\ &= \beta^{(a_1)} * \beta^{(a_2)} * \zeta(1_n) = (\beta^{(a_1)} * \beta^{(a_2)} * \zeta)_n. \end{aligned}$$

Hence  $\beta^{(a_1a_2)} = \beta^{(a_1)} * \beta^{(a_2)} * \zeta$ . Thus  $\kappa^{(a_1a_2a_2^*a_1^*)} = \beta^{(a_1)} * \beta^{(a_2)} * \zeta * \zeta$ .

Now let us return to the case where  $c_1$  and  $c_2$  are  $*$ -free and circular. Since  $\kappa_{2n}(c_1, c_1^*, \dots, c_1, c_1^*) = 0$  for  $n \geq 2$  and  $\kappa_2(c_1, c_1^*) = 1$ , we have  $\beta_1^{(c_1)} = 1$  and  $\beta_n^{(c_1)} = 0$  for  $n \geq 2$ . In the language of multiplicative functions, we have  $\beta^{(c_1)} = \delta$ . Thus  $\kappa^{(c_1c_2c_2^*c_1^*)} = \delta * \delta * \zeta * \zeta = \zeta^{*2}$ .

Now suppose we have  $k$  circular operators,  $c_1, \dots, c_k$ , which are  $*$ -free. Let  $c = c_1 \cdots c_k$ , then  $\kappa^{(cc^*)} = \zeta^{*k}$ , which are in turn given by the Fuss–Catalan numbers (see [25, Example 10.24]).

We have that  $h = c_1c_2$  is  $R$ -diagonal, so the only non-vanishing cumulants of  $h$  are those of the form  $\kappa_{2n}(h, h^*, \dots, h, h^*)$ . These are equal to 1 for all  $n$ . Let us briefly review how to see this:

$$\kappa_{2n}(h, h^*, \dots, h, h^*) = \sum_{\pi} \kappa_\pi(c_1, c_2, c_2^*, c_1^*, \dots, c_1, c_2, c_2^*, c_1^*),$$

where the sum runs over all  $\pi \in \text{NC}(2n)$  such that  $\pi \vee \rho = 1_{4n}$  and  $\rho = \{(1, 2), \dots, (4n - 1, 4n)\}$ . By freeness, all blocks of  $\pi$  can contain only  $\{c_1, c_1^*\}$  or only  $\{c_2, c_2^*\}$ . Moreover, since  $c_1$  and  $c_2$  are circular, the blocks of  $\pi$  must be of size 2 and can only connect  $c_i$  to  $c_i^*$ . Thus  $\pi$  must be a pairing. In the diagram below, we represent  $c_1, c_1^*, c_2$ , and  $c_2^*$  by 1,  $\bar{1}$ , 2, and  $\bar{2}$ , respectively. Let us start with the leftmost 1. If it were to

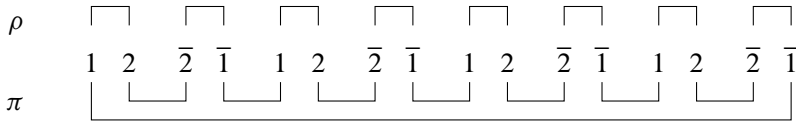
be connected any  $\bar{1}$  other than the rightmost, then the condition  $\pi \vee \rho = 1_{2kn}$  would be violated. Similarly, if the leftmost 2 were connected to any  $\bar{2}$  other than the  $\bar{2}$  immediately to the right of 2, then the condition  $\pi \vee \rho = 1_{2kn}$  would be violated. Continuing in this way, we see that there is only one such  $\pi$  that satisfies the two conditions

$$\kappa_\pi(c_1, c_2, c_2^*, c_1^*, \dots, c_1, c_2, c_2^*, c_1^*) \neq 0$$

and  $\pi \vee \rho = 1_{2kn}$ . Moreover, since  $\kappa_2(c_i, c_i^*) = \kappa_2(c_i^*, c_i) = 1$ , we have that

$$\kappa_{2n}(h, h^*, \dots, h, h^*) = 1$$

as claimed.



Since  $c_1$  and  $c_2$  are second order  $R$ -diagonal, so is  $h$ . In Example 8.7, we saw that the second order cumulants of  $c_1$  and  $c_2$  are all zero and the only non-vanishing cumulants of first order are

$$\kappa_2(c_1, c_1^*) = \kappa_2(c_1^*, c_1) = \kappa_2(c_2, c_2^*) = \kappa_2(c_2^*, c_2) = 1. \tag{8.3}$$

Let us show that all second order cumulants of  $h$  are 0. Since  $h$  is second order  $R$ -diagonal, we have that the only possible non-vanishing cumulants are

$$\kappa_{2p,2q}(h, h^*, \dots, h, h^*) \quad \text{and} \quad \kappa_{2p,2q}(h^*, h, \dots, h^*, h).$$

We shall show the first of these is 0; the proof for the second follows because  $h^*$  is also the product of circular operators. To compute  $\kappa_{2p,2q}(h, h^*, \dots, h, h^*)$ , we use Proposition 2.7 to write

$$\kappa_{2p,2q}(h, h^*, \dots, h, h^*) = \sum_{\substack{\pi \in S_{\text{NC}}(4p,4q) \\ \pi^{-1}\gamma_{4p,4q} \text{ sep. } N}} \kappa_\pi(c_1, c_2, c_2^*, c_1^*, \dots, c_1, c_2, c_2^*, c_1^*)$$

with  $N = \{2, 4, 6, \dots, 4p + 4q\}$ . Note that there are no terms involving second order cumulants as the second order  $*$ -cumulants of  $c_1$  and  $c_2$  vanish. For a  $\pi \in S_{\text{NC}}(4p, 4q)$  to contribute to this sum,  $\pi$  must be a pairing because the first order  $*$ -cumulants of  $c_1$  and  $c_2$  are all 0 except those in equation (8.3). We claim that  $\kappa_\pi(c_1, c_2, c_2^*, c_1^*, \dots, c_1, c_2, c_2^*, c_1^*) = 0$  whenever  $\pi^{-1}\gamma_{4p,4q}$  separates the points of  $N$ . Such a  $\pi$  must have a through string. Suppose it connects  $c_1$  on one circle to  $c_1^*$  on the other circle. This means that there are  $j$  and  $k$  such that  $\pi(4j + 1) = 4k$ . But

then  $\pi^{-1}\gamma_{4p,4q}(4j) = 4k$ , which contradicts the assumption that  $\pi^{-1}\gamma_{4p,4q}$  separates the points of  $N$ . The case where a through string connects  $c_2$  on one circle to  $c_2^*$  on the other is similar. Hence  $h$  is a second order  $R$ -diagonal with *all* second order cumulants equal to 0.

We now calculate the second order cumulants of  $hh^*$ . From the discussion above, we have the determining sequences of  $h$  are given by  $\beta_n = 1$  and  $\beta_{p,q} = 0$  for all  $p$  and  $q$ . From equation (3.3), we have that

$$\begin{aligned} \kappa_{p,q}(hh^*, \dots, hh^*) &= \sum_{(\mathcal{V}, \pi) \in PS_{NC}(p,q)'} \beta_{(\mathcal{V}, \pi)} + \sum_{\pi \in S_{NC}(p,q)} \beta_{\pi} \\ &= \sum_{\pi \in S_{NC}(p,q)} 1 = |S_{NC}(p,q)| = \frac{pq}{2(p+q)} \binom{2p}{p} \binom{2q}{q}. \end{aligned}$$

One way to interpret the calculation above is that

$$\kappa(hh^*) = \zeta^{*2}$$

and consequently,

$$\varphi_2((hh^*)^p, (hh^*)^q) = \zeta^{*3} (1_{p+q} \cdot \gamma_{p,q}) = \frac{2}{3+1} \frac{pq}{p+q} \binom{3p}{p} \binom{3q}{q}.$$

The last equality will be explained in further detail for all powers,  $\zeta^{*k}$ , in Remark 8.13.

Before going to more general examples, let us notice that the determining sequence of  $h$  coincides with the cumulants of  $c_1 c_1^*$ . This is of course not a coincidence, as we will see in Remark 8.12.

**Example 8.10** (Conjugation by a free circular element). Let  $c$  be a second order circular operator as in Example 8.7 which is second order  $*$ -free from  $a$ . We are interested in the second order cumulants of  $cac^*$ . Recall that for the first order cumulants, we have

$$\kappa_n(cac^*, \dots, cac^*) = \varphi(a^n).$$

To see this, use the formula for cumulants with products for entries

$$\kappa_n(cac^*, \dots, cac^*) = \sum_{\pi \in NC(3n)} \kappa_{\pi}(c, a, c^*, \dots, c, a, c^*),$$

here the sum runs over all  $\pi$  such that  $\pi^{-1}\gamma_{3n}$  separates the points of  $\{3, 6, 9, \dots, 3n\}$ . Recall that  $\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$  and all other cumulants vanish. Thus, for

$$\kappa_{\pi}(c, a, c^*, \dots, c, a, c^*) \neq 0,$$

$\pi$  must pair a  $c$  with a  $c^*$ . There is only one way this can be achieved and also satisfy the requirement that  $\pi^{-1}\gamma_{3n}$  separates the points of  $\{3, 6, 9, \dots, 3n\}$ :

$$\pi = \left[ \begin{array}{cccccccccccccccc} c & a & c^* & c & a & c^* & c & a & c^* & c & a & c^* & \cdots & c & a & c^* & c & a & c^* \end{array} \right].$$

As this exposes all the  $a$ 's, we get the sum  $\sum_{\pi \in \text{NC}(n)} \kappa_{\pi}(a, \dots, a) = \varphi(a^n)$ , as claimed. Thus for the cumulant generating function

$$C_{cac^*}(z) = \sum_{n=0}^{\infty} \kappa_n(cac^*, \dots, cac^*)z^n,$$

we have

$$C_{cac^*}(z) = z^{-1}G_a(z^{-1}),$$

where  $G_a(z) = \sum_{n=0}^{\infty} \varphi(a^n)z^{-(n+1)}$  is the Cauchy transform of  $a$ .

Let us show that a similar formula holds in the second order case.

**Theorem 8.11.** *Let  $c_1, \dots, c_k$  be second order circular operators and suppose that  $\{c_1, c_1^*\}, \{c_2, c_2^*\}, \dots, \{c_k, c_k^*\}$ , and  $\{a\}$  are second order free in a second order probability space  $(\mathcal{A}, \varphi, \varphi_2)$ . Then*

$$\kappa(c_1c_2 \cdots c_kac_k^* \cdots c_2^*c_1^*) = \kappa(a) * \zeta^{*k}.$$

*Proof.* It suffices to prove this for  $k = 1$  and use induction; for this we use the formula for products as arguments directly.

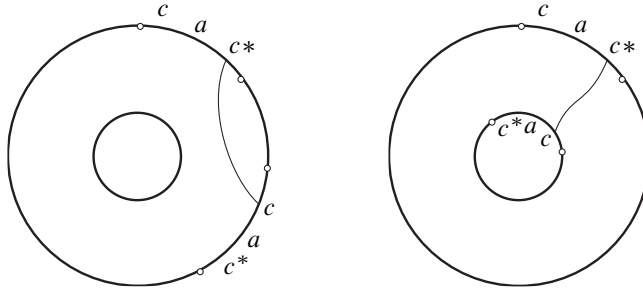
$$\kappa_{p,q}(cac^*, \dots, cac^*) = \sum_{(\nu, \pi) \in \mathcal{P}\mathcal{S}_{\text{NC}}(kp, kq)} \kappa_{(\nu, \pi)}(c, a, c^*, \dots, cac^*),$$

where the sum runs over permutations such that  $\pi^{-1}\gamma_{kp, kq}$  separates the points  $\{3, 6, \dots, 3(p + q)\}$ .

Let us analyze the possible permutations in the sum. Recall that the only non-vanishing  $*$ -cumulants of  $c$  are  $\kappa_2(c, c^*)$  and  $\kappa_2(c^*, c)$ . So in  $\pi$ , any  $c$  must be connected to  $c^*$ . For this, in principle there are two possibilities:

- (i)  $c$  and  $c^*$  are on the same circle, as in Figure 9 (left),
- (ii)  $c$  and  $c^*$  are in opposite circles, as in Figure 9 (right).

We shall show that (ii) is not possible and in case (i) the  $a$  and  $a^*$  must be adjacent as in Example 8.9. In either case, suppose  $c^*$  is in position  $3i$  and  $c$  is in position  $3j + 1$ , then  $3i$  and  $3j + 1$  are not separated by  $\pi^{-1}\gamma$ . Indeed,  $\pi^{-1}\gamma(3j) = \pi^{-1}(3j + 1) = 3i$ . The only possibility is then that  $j = i$ , i.e., each  $c^*$  in a position  $3i$  is connected to  $c$  in the position  $3i + 1$ , in the same circle; this excludes case (ii).



**Figure 9.** The two cases (i) and (ii) above.

Thus, we may write the partitioned permutations  $(\mathcal{V}, \pi)$  in the sum above as  $\pi = \pi_1 \pi_2$ , where  $\pi_1 = (3, 4)(6, 7) \cdots (3p + 3q - 3, 3p + q - 2)$  and  $\pi_2$  is any annular non-crossing permutation on  $[2, 5, 8, \dots, 3p - 1; 3p, \dots, 3p + 3q - 1]$ , and when  $\pi$  has no through blocks,  $\mathcal{V}$  connects a block of  $\pi_2$  on the outer circle to a block of  $\pi_2$  on the inner circle. Thus the restriction of  $(\mathcal{V}, \pi)$  to  $[2, 5, 8, \dots, 3p - 1; 3p, \dots, 3p + 3q - 1]$  is an arbitrary partitioned permutation, and thus the set of restrictions is isomorphic to the set  $\mathcal{PS}_{\text{NC}}(p, q)$ . Since  $\kappa_2(c, c^*) = 1$ , we have  $\kappa_\pi(c, c^*, \dots, c, c^*) = 1$  and thus

$$\kappa_{p,q}(cac^*, \dots, cac^*) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \kappa_{(\mathcal{V}, \pi)}(a, \dots, a) = \varphi_2(a^p, a^q). \tag{8.4}$$

Hence

$$\kappa_{p,q}^{(cac^*)} = \varphi_2(a^p, a^q) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(p,q)} \kappa_{(\mathcal{V}, \pi)}^{(a)} = \kappa^{(a)} * \zeta(1_{p+q}, \gamma_{p,q}).$$

Since this is true for all  $p$  and  $q$ , we have  $\kappa^{(cac^*)} = \kappa^{(a)} * \zeta^{*k}$ , as claimed. ■

We let

$$C_{cac^*}(z, w) = \sum_{m,n \geq 1} \kappa_{m,n}(cac^*, \dots, cac^*) z^m w^n$$

be the generating function for the second order cumulants. Then by equation (8.4), we have

$$C_{cac^*}(z, w) = z^{-1} w^{-1} G_a(z^{-1}, w^{-1}),$$

where

$$G_a(z, w) = \sum_{m,n \geq 1} \varphi_2(a^m, a^n) z^{-(m+1)} w^{-(n+1)}$$

is the second order Cauchy transform of  $a$ .

A particularly important example is the case where the fluctuation moments of  $a$  are 0. In this case,  $G_a(z, w) = 0$  and equation (2.3) is reduced to

$$G_{cac^*}(z, w) = \frac{\partial^2}{\partial z \partial w} \log \left( \frac{G_{cac^*}(z) - G_{cac^*}(w)}{z - w} \right).$$

From the random matrix perspective, this corresponds to the case where  $a$  is a limit of deterministic matrices, and  $cac^*$  then corresponds to  $WAW^*$ , where  $W$  is a Ginibre matrix and  $A$  is deterministic. When  $A$  is positive semidefinite, this is also called a complex Wishart matrix with covariance matrix  $A$ .

**Remark 8.12.** Observe in the last example that in the case where  $a = rr^*$ , the relation,  $\kappa^{(crr^*c^*)} = \beta^{(cr)} * \zeta$  (see equation 3.5), between the determining sequence of  $cr$  and the cumulants of  $crr^*c^*$  is the same as the one between the cumulants of  $rr^*$  and the cumulants of  $crr^*c^*$ ,  $\kappa^{(crr^*c^*)} = \kappa^{(rr^*)} * \zeta$ . Thus  $\beta^{(cr)} = \kappa^{(rr^*)}$ , which in particular explains the last comment in Example 8.9.

**Remark 8.13.** If in Theorem 8.11 we let  $a = 1$  and  $h = c_1c_2 \cdots c_kac_k^* \cdots c_2^*c_1^*$ , then we have  $\kappa^{(a)} = \kappa^{(1)} = \delta$  and so

$$\kappa^{(h)} = \delta * \zeta^{*k} = \zeta^{*k}.$$

Thus

$$\varphi_2(h^p, h^q) = \sum_{(\mathcal{V}, \pi) \in \mathcal{P}S_{NC}(p, q)} \kappa_{(\mathcal{V}, \pi)}^{(h)} = \kappa^{(h)} * \zeta(1_{p+q}, \gamma_{p, q}) = \zeta^{*(k+1)}(1_{p+q}, \gamma_{p, q}).$$

By the formula of Bousquet-Mélou and Schaeffer [7] (see the discussion in [8, §5.17, p. 38] for the interpretation in our current notation), we have that for  $l \geq 2$ ,

$$\zeta^{*l}(1_{p+q}, \gamma_{p, q}) = \frac{l-1}{l} \frac{pq}{p+q} \binom{lp}{p} \binom{lq}{q}. \tag{8.5}$$

Thus in our example, when  $h = c_1c_2 \cdots c_kc_k^* \cdots c_2^*c_1^*$ , we have

$$\varphi_2(h^p, h^q) = \frac{k}{k+1} \frac{pq}{p+q} \binom{(k+1)p}{p} \binom{(k+1)q}{q}. \tag{8.6}$$

Note that by Theorem 3.3 we have that  $c_1 \cdots c_k$  is  $R$ -diagonal and the non-vanishing cumulants are given by

$$\begin{aligned} &\kappa_{p, q}(c_1 \cdots c_k, c_k^* \cdots c_1^*, \dots, c_1 \cdots c_k, c_k^* \cdots c_1^*) \\ &= \beta_{p, q}^{(c_1 \cdots c_k)} \kappa^{(c_1 \cdots c_k c_k^* \cdots c_1^*)} * \mu(1_{p+q}, \gamma_{p, q}) = \zeta^{*(k-1)}(1_{p+q}, \gamma_{p, q}) \end{aligned}$$

which we can easily evaluate by formula (8.5) of Bousquet-Mélou and Schaeffer.

**Remark 8.14.** Let us show that equation (8.6) allows us to complete a claim in [9, Remark 5] concerning the second order moments of the product of two Wishart random matrices with shape parameter 1. In [9] the authors consider two independent complex Ginibre matrices,  $X_1$  and  $X_2$ . They let  $S_2 = X_1 X_1^\dagger X_2 X_2^\dagger$ , where  $X^\dagger$  means the conjugate transpose,  $X^*$ , of  $X$ . By writing  $X_1$  in its real and imaginary parts, we see (cf. [21, §3.2]) that the second order joint limit distribution of  $\{X_1, X_2\}$  is that of two second order  $*$ -free circular operators. Let us denote them by  $x_1$  and  $x_2$ , respectively, and set  $s_2 = x_1 x_1^* x_2 x_2^*$ . Now let  $c_1 = x_1^*$  and  $c_2 = x_2$ . Then  $c_1$  and  $c_2$  are two second order  $*$ -free circular operators. Let, as in Remark 8.13,  $h = c_1 c_2 c_2^* c_1^*$ . Then  $s_2^p = c_1^* h^{p-1} c_1 c_2 c_2^*$ , and so  $\varphi_2(h^p, h^q) = \varphi_2(s_2^p, s_2^q) = c_{p,q}^{[0]}$  (in the notation of [9]), because  $\varphi_2$  is tracial in each of its arguments. Thus

$$c_{p,q}^{[0]} = \varphi_2(h^p, h^q) = \frac{2}{3+1} \frac{pq}{p+q} \binom{3p}{p} \binom{3q}{q}.$$

Note that this is now an easy conclusion of a very general result. Moreover, the calculations are much simpler than those required in [9].

**Example 8.15** (Product of  $k$  free circular operators). Let  $h = c_1 c_1^* c_2 c_2^* \cdots c_k c_k^*$ . From equations (7.1) and (7.2) of Theorem 7.3, we get a combinatorial description of the fluctuation moments and cumulants:

$$\varphi_2(h^p, h^q) = |S_{\text{NC}}^{k\text{-alt}}(kp, kq)| \quad \text{and} \quad \kappa_{p,q}(h, \dots, h) = |S_{\text{NC}}^{k\text{-e-a}}(kp, kq)|.$$

To the best of our knowledge a precise formula for these quantities was not known for  $k > 2$ . However, the method of Example 8.14 can be extended to compute the fluctuation moments of  $c_1^* c_1 c_2^* c_2 \cdots c_n^* c_n$ , thereby extending the result of [9] ( $n = 2$ ) to the general case and as a byproduct, giving a formula for the above quantities. First we must relate the fluctuation moments of  $c_1^* c_1 c_2^* c_2 \cdots c_n^* c_n$  to those of  $h^* h$ , where  $h = c_1 c_2 \cdots c_k$ ; in fact, we first show a general result that will imply that these two variables are identically distributed. Here we shall follow an idea of Kargin [17], but avoid the use of the  $S$ -transform, which does not yet exist at the second order level.

Let  $(\mathcal{A}, \varphi, \varphi_2)$  be a (tracial) second order  $*$ -probability space and  $x_1, \dots, x_n \in \mathcal{A}$  are second order  $*$ -free and identically  $*$ -distributed. Let  $y = x_2 \cdots x_n$ . Then  $x_1$  and  $y$  are free. By traciality, we have  $x_1 x_1^*$  and  $x_1^* x_1$  are identically distributed and, likewise,  $y y^*$  and  $y^* y$  are identically distributed.

Suppose  $k \geq 1$ , then

$$\varphi([(x_1 y)^* (x_1 y)]^k) = \varphi([(y y^*) (x_1^* x_1)]^k).$$

Now both  $y y^*$  and  $y^* y$  are free from  $x_1^* x_1$ . So we can replace  $y y^*$  by  $y^* y$  to get

$$\varphi([(y y^*) (x_1^* x_1)]^k) = \varphi([(y^* y) (x_1 x_1^*)]^k) = \varphi([(y^* y) (x_1^* x_1)]^k).$$

Thus we have

$$\varphi([(x_1 y)^*(x_1 y)]^k) = \varphi([(x_1^* x_1)(y^* y)]^k). \tag{8.7}$$

When  $n = 2$ , we have

$$\varphi([(x_1 x_2)^*(x_1 x_2)]^k) = \varphi([(x_1^* x_1)(x_2^* x_2)]^k).$$

**Lemma 8.16.** *For all  $k \geq 1$ , we have*

$$\varphi([(x_1 \cdots x_n)^*(x_1 \cdots x_n)]^k) = \varphi([(x_1^* x_1) \cdots (x_n^* x_n)]^k).$$

*Proof.* We have already checked this when  $n = 2$ . Suppose it holds for  $n = l - 1$  and we let  $y = x_2 \cdots x_l$ . Then  $y^* y$  and  $(x_2^* x_2) \cdots (x_l^* x_l)$  are identically distributed. By equation (8.7),

$$\varphi([(x_1 \cdots x_l)^*(x_1 \cdots x_l)]^k) = \varphi([(x_1^* x_1)(y^* y)]^k).$$

Since  $y^* y$  and  $(x_2^* x_2) \cdots (x_l^* x_l)$  are identically distributed, we have

$$\varphi([(x_1^* x_1)(y^* y)]^k) = \varphi([(x_1^* x_1)(x_2^* x_2) \cdots (x_l^* x_l)]^k).$$

With the previous equation, we now have

$$\varphi([(x_1 \cdots x_l)^*(x_1 \cdots x_l)]^k) = \varphi([(x_1^* x_1)(x_2^* x_2) \cdots (x_l^* x_l)]^k).$$

This completes the inductive step. ■

Now we want to repeat this in the second order case. By traciality, we have that  $x_1 x_1^*$  and  $x_1^* x_1$  are identically distributed at the second order level and, likewise,  $y y^*$  and  $y^* y$  are identically distributed at the second order level. Let  $p, q \geq 1$ . Then

$$\begin{aligned} \varphi_2([(x_1 y)^*(x_1 y)]^p, [(x_1 y)^*(x_1 y)]^q) &= \varphi_2([(y y^*)(x_1^* x_1)]^p, [(y y^*)(x_1^* x_1)]^q) \\ &= \varphi_2([(y^* y)(x_1^* x_1)]^p, [(y^* y)(x_1^* x_1)]^q) \\ &= \varphi_2([(x_1^* x_1)(y^* y)]^p, [(x_1^* x_1)(y^* y)]^q). \end{aligned}$$

This shows that  $(x_1 x_2)^*(x_1 x_2)$  and  $(x_1^* x_1)(x_2^* x_2)$  have identical second order distributions. If, by induction hypothesis, we have that  $y^* y$  and  $(x_2^* x_2) \cdots (x_n^* x_n)$  have identical second order distributions, then, as  $x_1$  is second order  $*$ -free from  $\{x_2, \dots, x_n\}$ , we have that

$$(x_1 \cdots x_n)^*(x_1 \cdots x_n) \quad \text{and} \quad (x_1^* x_1)(x_2^* x_2) \cdots (x_n^* x_n)$$

have identical second order distributions. We state this as a lemma.



**Lemma 8.17.** *Suppose  $x_1, \dots, x_n$  are second order  $*$ -free. Then for all  $p, q \geq 1$ ,*

$$\begin{aligned} & \varphi_2([(x_1 \cdots x_n)^*(x_1 \cdots x_n)]^p, [(x_1 \cdots x_n)^*(x_1 \cdots x_n)]^q) \\ &= \varphi_2([(x_1^*x_1) \cdots (x_n^*x_n)]^p, [(x_1^*x_1) \cdots (x_n^*x_n)]^q). \end{aligned}$$

This shows that  $c_1^*c_1c_2^*c_2 \cdots c_k^*c_k$  and  $h^*h$  have the same fluctuation moments. As in Example 8.13, we have

$$\varphi_2([c_1^*c_1 \cdots c_k^*c_k]^p, [c_1^*c_1 \cdots c_k^*c_k]^q) = \varphi_2(h^p, h^q) = \zeta^{*(k+1)}(1_{p+q}, \gamma_{p,q})$$

and by Bousquet-Mélou and Schaeffer [7], we have

$$\varphi_2([c_1^*c_1 \cdots c_k^*c_k]^p, [c_1^*c_1 \cdots c_k^*c_k]^q) = \frac{k}{k+1} \frac{pq}{p+q} \binom{(k+1)p}{p} \binom{(k+1)q}{q}.$$

We shall point out that Gorin and Sun [13, Theorem 4.13] give an integral formula for the limiting covariance of the height function  $\mathcal{H}_t$  of the eigenvalues of a product of Ginibre matrices. It is not clear how to use such a formula to obtain the combinatorial formulas above for the fluctuation moments.

### 8.3. Counterexamples

We end with two examples that show that not all properties lift to the second order level.

First, contrary to the first order case the powers of a second order  $R$ -diagonal operator is not  $R$ -diagonal as the following example shows.

**Example 8.18** (Powers of Haar unitary). Let  $u$  be a second order Haar unitary, then  $u^n$  is first order  $R$ -diagonal. However, it is not  $R$ -diagonal of second order.

We will show that

$$\kappa_{m,n}(u^p, \dots, u^p, u^{-p}, \dots, u^{-p}) = (p-1)n\delta_{m,n}. \tag{8.8}$$

This shows that  $u^p$  is not second order  $R$ -diagonal, for  $p \neq 1$ , since there are non-alternating cumulants which do not vanish.

To prove (8.8), we use the moment-cumulant formula and the fact that  $u^p$  is a first order Haar unitary. Recall that

$$\begin{aligned} \delta_{m,n}np &= \varphi_2(u^{mp}, u^{-np}) = \sum_{\pi \in \mathcal{S}_{\text{NC}}(m,n)} \kappa_{\pi}(u^p, \dots, u^p, u^{-p}, \dots, u^{-p}) \\ &+ \sum_{(\mathcal{V}, \pi) \in \mathcal{P}_{\text{NC}}(m,n)'} \kappa_{(\mathcal{V}, \pi)}(u^p, \dots, u^p, u^{-p}, \dots, u^{-p}). \end{aligned}$$

Notice that the first sum does not depend on  $p$  since we only consider first order cumulants. Let us call this quantity  $\Sigma_1(m, n)$ , so that

$$\delta_{m,n}np = \Sigma_1(m, n) + \sum_{(\mathcal{V}, \pi) \in P_{\text{NC}}(m, n)'} \kappa_{(\mathcal{V}, \pi)}(u^p, \dots, u^p, u^{-p}, \dots, u^{-p}).$$

For the second sum, notice that if  $(\mathcal{V}, \pi) \neq (1_{m,m}, \gamma_{m,m})$ , then

$$\kappa_{(\mathcal{V}, \pi)}(u^p, \dots, u^p, u^{-p}, \dots, u^{-p}) = 0.$$

Indeed, at least one block of  $\mathcal{V}$ , say  $B$ , is contained in either  $[m]$  or  $[m + 1, m + n]$ , and for this block either  $\kappa_{|B|}(u^p, \dots, u^p) = 0$  or, respectively,  $\kappa_{|B|}(u^{-p}, \dots, u^{-p}) = 0$ , since  $u^p$  is a first order Haar unitary.

This means that

$$\kappa_{m,n}(u^p, \dots, u^p, u^{-p}, \dots, u^{-p}) = np\delta_{m,n} - \Sigma_1(m, n). \tag{8.9}$$

It remains to calculate  $\Sigma_1(m, n)$ , which only depends on  $m$  and  $n$ , and not on  $p$ . In particular, we may take  $p = 1$  in (8.9) to obtain

$$0 = \kappa_{m,n}(u, \dots, u, u^{-1}, \dots, u^{-1}) = n\delta_{m,n} - \Sigma_1(m, n),$$

from which  $\Sigma_1(m, n) = \delta_{m,n}n$ , and then (8.8) follows.

More generally, we may consider a sequence of exponents

$$(\varepsilon, \theta) = (\varepsilon_1, \dots, \varepsilon_m, \theta_1, \dots, \theta_n) \quad \text{with } \varepsilon_i, \theta_j \in \{-1, 1\}.$$

We are interested in the second order cumulant

$$\kappa_{m,n}^{(\varepsilon, \theta, p)} := \kappa_{m,n}(u^{p\varepsilon_1}, \dots, u^{p\varepsilon_m}, u^{p\theta_1}, \dots, u^{p\theta_n}).$$

We will show in equation (8.10) that we may write  $\kappa_{m,n}^{(\varepsilon, \theta, p)}$  in terms of  $p, \kappa_{m,n}^{(\varepsilon, \theta, 1)}$  and  $\kappa_{m,n}^{(\varepsilon, \theta, 2)}$ .

We prove by induction on  $m + n$  that there are functions  $a_{(\mathcal{V}, \pi)}$  and  $b_{(\mathcal{V}, \pi)}$  such that for all  $(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(m, n)$  and all  $p$  we have

$$\begin{aligned} \kappa_{(\mathcal{V}, \pi)}(u^{p\varepsilon_1}, u^{p\varepsilon_2}, \dots, u^{p\varepsilon_m}, u^{p\theta_1}, u^{p\theta_2}, \dots, u^{p\theta_n}) \\ = a_{(\mathcal{V}, \pi)}(\varepsilon, \theta)p + b_{(\mathcal{V}, \pi)}(\varepsilon, \theta). \end{aligned}$$

When  $m = n = 1$ , we set  $\mathcal{V} = \{(1, 2)\}$  and  $\pi = (1)(2)$ . For  $\varepsilon, \theta \in \{-1, 1\}$ , we set

$$a_{(\mathcal{V}, \pi)}(\varepsilon, \theta) = \begin{cases} 1, & \varepsilon + \theta = 0, \\ 0, & \varepsilon + \theta \neq 0 \end{cases} \quad \text{and} \quad b_{(\mathcal{V}, \pi)}(\varepsilon, \theta) = \begin{cases} -1, & \varepsilon + \theta = 0, \\ 0, & \varepsilon + \theta \neq 0, \end{cases}$$

By (8.8), we have  $\kappa_{1,1}(u^{p\varepsilon}, u^{p\theta}) = (p-1)\delta_{\varepsilon,-\theta} = a_{(\mathcal{V},\pi)}(\varepsilon, \theta)p + b_{(\mathcal{V},\pi)}(\varepsilon, \theta)$ . This starts the induction.

To continue, we write  $\varepsilon = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_m$ ; then  $u^{p\varepsilon_1}u^{p\varepsilon_2}\dots u^{p\varepsilon_m} = u^{p\varepsilon}$ . Likewise, we set  $\theta = \theta_1 + \theta_2 + \dots + \theta_n$ , then  $u^{p\theta_1}u^{p\theta_2}\dots u^{p\theta_n} = u^{p\theta}$ . By the moment-cumulant formula,

$$p\varphi_2(u^\varepsilon, u^\theta) = \varphi_2(u^{p\varepsilon}, u^{p\theta}) = \sum_{\pi \in \mathcal{S}_{\text{NC}}(m,n)} \kappa_\pi(u^{p\varepsilon_1}, \dots, u^{p\varepsilon_m}, u^{p\theta_1}, \dots, u^{p\theta_n}) + \sum_{(\mathcal{V},\pi) \in \mathcal{P}_{\text{NC}}(m,n)'} \kappa_{(\mathcal{V},\pi)}(u^{p\varepsilon_1}, \dots, u^{p\varepsilon_m}, u^{p\theta_1}, \dots, u^{p\theta_n}).$$

The sum  $\Sigma^{(\varepsilon,\theta)} := \sum_{\pi \in \mathcal{S}_{\text{NC}}(m,n)} \kappa_\pi(u^{p\varepsilon_1}, \dots, u^{p\theta_n})$  only depends on  $\varepsilon$  and  $\theta$ . On the other hand, suppose  $(\mathcal{V}, \pi) \neq (1_{m+n}, \gamma_{m,n})$ . Write  $\pi = \pi_1 \times \pi_2 \in \text{NC}(m) \times \text{NC}(n)$ . Let  $B$  be the block of  $\mathcal{V}$  that is the union of a cycle of  $\pi_1$  (with  $k$  elements) and a cycle of  $\pi_2$  (with  $l$  elements).

Then there are  $\eta_1, \dots, \eta_{k+l} \in \{-1, 1\}$  and  $\zeta_1^{(1)}, \dots, \zeta_{r_1}^{(1)}, \dots, \zeta_1^{(s)}, \dots, \zeta_{r_s}^{(s)} \in \{-1, 1\}$  such that

$$\begin{aligned} &\kappa_{(\mathcal{V},\pi)}(u^{p\varepsilon_1}, \dots, u^{p\varepsilon_m}, u^{p\theta_1}, \dots, u^{p\theta_n}) \\ &= \kappa_{k,l}(u^{p\eta_1}, \dots, u^{p\eta_{k+l}}) \prod_{i=1}^s \kappa_{r_i}(u^{p\zeta_1^{(i)}} , \dots, u^{p\zeta_{r_i}^{(i)}}). \end{aligned}$$

Thus, by induction this cumulant is a product of a linear function of  $p, \kappa_{k,l}(u^{p\eta_1}, \dots, u^{p\eta_{k+l}})$ , and terms that do not depend on  $p$ , namely, cumulants of first order. Let us denote this linear function by  $\tilde{a}_{(\mathcal{V},\pi)}(\varepsilon, \theta)p + \tilde{b}_{(\mathcal{V},\pi)}(\varepsilon, \theta)$ . Putting this all together, we arrive at the identity

$$\begin{aligned} \kappa_{m,n}(u^{p\varepsilon_1}, \dots, u^{p\theta_n}) &= p\varphi_2(u^\varepsilon, u^\theta) + \Sigma^{(\varepsilon,\theta)} \\ &+ \sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}_{\text{NC}}(m,n)' \\ (\mathcal{V},\pi) \neq (1_{m+n}, \gamma_{m,n})}} (\tilde{a}_{(\mathcal{V},\pi)}(\varepsilon, \theta)p + \tilde{b}_{(\mathcal{V},\pi)}(\varepsilon, \theta)) \\ &= p\left(\varphi_2(u^\varepsilon, u^\theta) + \sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}_{\text{NC}}(m,n)' \\ (\mathcal{V},\pi) \neq (1_{m+n}, \gamma_{m,n})}} \tilde{a}_{(\mathcal{V},\pi)}(\varepsilon, \theta)\right) \\ &+ \Sigma^{(\varepsilon,\theta)} + \sum_{\substack{(\mathcal{V},\pi) \in \mathcal{P}_{\text{NC}}(m,n)' \\ (\mathcal{V},\pi) \neq (1_{m+n}, \gamma_{m,n})}} \tilde{b}_{(\mathcal{V},\pi)}(\varepsilon, \theta), \end{aligned}$$

which proves the inductive step. By substituting  $\kappa_{m,n}^{(\varepsilon,\theta,2)}$  and  $\kappa_{m,n}^{(\varepsilon,\theta,1)}$  into the formula above, we see that

$$\kappa_{m,n}^{(\varepsilon,\theta,p)} = p(\kappa_{m,n}^{(\varepsilon,\theta,2)} - \kappa_{m,n}^{(\varepsilon,\theta,1)}) + (2\kappa_{m,n}^{(\varepsilon,\theta,1)} - \kappa_{m,n}^{(\varepsilon,\theta,2)}) \tag{8.10}$$

as claimed.

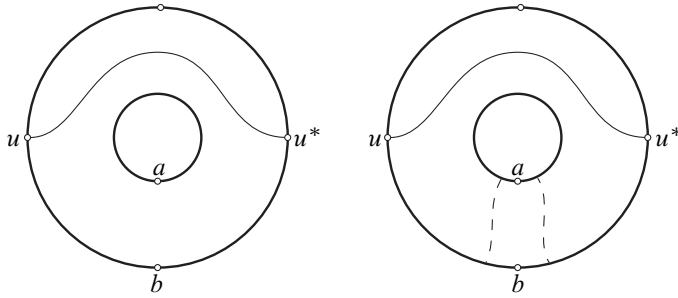


Figure 10. Diagrams in Example 8.19 with non-zero contribution.

Finally, in first order free probability, it was shown in [24] that if  $u$  is a Haar unitary free from  $a$ , then  $ubu^*$  is free from  $a$ , for any  $b$  free from  $u$ . We now prove that, for  $u$  and  $a$  second order free, in general,  $ubu^*$  is not second order free from  $a$ .

**Example 8.19.** Suppose that  $a$  and  $b$  are operators in a second order probability space, and  $u$  is a second order Haar unitary which is second order free from  $a$  and  $b$ . Then, from formula (2.4) of Proposition 2.7 for cumulants with products as arguments, we have

$$\kappa_{1,1}(ubu^*, a) = \sum_{(\mathcal{V}, \pi) \in \mathcal{PS}_{\text{NC}}(3,1)} \kappa_{(\mathcal{V}, \pi)}(u, b, u^*, a),$$

where the sum is over partitions that separate the points  $\{3, 4\}$ . The only partitions satisfying that 3 and 4 are in different cycles as shown in Figure 10. Thus

$$\kappa_{1,1}(ubu^*, a) = \kappa_2(u, u^*)\kappa_2(b, a) + \kappa_2(u, u^*)\kappa_{1,1}(b, a) = \kappa_2(u, u^*)\varphi_2(b, a).$$

Thus, whenever  $\varphi_2(b, a) \neq 0$ , (in particular, whenever  $a$  and  $b$  are not second order free),  $ubu^*$  and  $a$  are not second order free since  $\kappa_{1,1}(ubu^*, a)$  does not vanish. Note that this does not contradict [8, Theorem 2.9] because with the hypotheses there we get the second order freeness of  $a$  and  $b$ .

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