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A remark on the Hochschild dimension of liberated quantum groups

Tomasz Brzeziński, Ulrich Krähmer, Réamonn Ó Buachalla, and Karen R. Strung

Abstract. Let A be a Hopf algebra equipped with a projection onto the coordinate Hopf algebra $\mathcal{O}(G)$ of a semisimple algebraic group G. It is shown that if A admits a suitably non-degenerate comodule V and the induced G-module structure of V is non-trivial, then the third Hochschild homology group of A is non-trivial.

1. Introduction

For a field $\mathbb{F}$, let $\mathcal{O}(G)$ denote the Hopf algebra of coordinate (polynomial) functions on an algebraic group G. Let furthermore $HH_*(A)$ denote the Hochschild homology of an associative (unital) algebra A over $\mathbb{F}$ with coefficients in A. In this note we prove the following theorem.

Theorem. *Let G be a semisimple algebraic group over a field $\mathbb{F}$ of characteristic* 0*, let $\pi: A \to \mathcal{O}(G)$ be a Hopf algebra map, and V be a right A-comodule with a non-degenerate symmetric or antisymmetric invariant bilinear form. If the representation of G on V induced by π is non-trivial, then $HH_3(A) \neq 0$.*

This theorem is best seen in the context of the *liberation* procedure [1] for compact quantum matrix groups in the sense of Woronowicz [14]. Although this procedure is not formally defined, its origins can be traced back to the work of Wang [13] on free quantum groups or even earlier to [7]. At the algebraic level, the idea is to construct for a given representation V of an algebraic group G and a non-degenerate bilinear form on V a universal Hopf algebra map $\pi: \mathcal{A}(G) \to \mathcal{O}(G)$ as in the above theorem, see, e.g., [3, Theorem 1]. Following this philosophy, Wang constructed free quantum orthogonal and unitary groups $A_o(N)$, $A_u(N)$ and interpreted the C^*-algebra completions in terms of a free product of C^*-algebras in [13]. The former is a universal C^*-algebra generated by N^2 elements a_{ij} subject to relations

$$\sum_k a_{ik}a_{jk} = \sum_k a_{ki}a_{kj} = \delta_{ij}, \quad a_{ij}^* = a_{ij}.$$

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Collins, Härtel and Thom [6] studied the Hochschild homology of $A_o(N)$ showing that for all $N \geq 2$ the third Hochschild homology group with coefficients in $\mathbb{C}$ is one-dimensional and that $A_o(N)$ is a Calabi–Yau algebra of dimension 3 (the homology groups with arbitrary coefficients vanish in degrees above 3 and satisfy Poincaré duality in the sense of Van den Bergh [12]). Our theorem shows that this non-triviality of third Hochschild homology groups has a general representation-theoretic explanation.

The liberation procedure can be extended to intermediate phases leading, for example, to half-liberated matrix quantum groups [1] or half-commutative Hopf algebras [2]; the theorem can be applied to these examples, too.

The proof of the theorem uses elementary noncommutative geometry: by choosing a basis $e_1, \dots, e_N$ in an N-dimensional comodule over a Hopf algebra A, one obtains an invertible matrix $v \in \mathrm{GL}_N(A)$ with $\rho(e_j) = \sum_i e_i \otimes v_{ij}$ and hence a class $[v] \in K_1(A)$. The Chern–Connes character assigns to $[v]$ classes in the odd cyclic homology groups $HC_{2d+1}(A)$. The main point is that assuming the existence of a symmetric or antisymmetric non-degenerate invariant pairing on V, the class in the cyclic homology group $HC_3(A)$ is in the image of the natural map $HH_3(A) \to HC_3(A)$ (Lemma 2). Under π_*, these classes in the K-theory respectively cyclic and Hochschild homology of $\mathcal{O}(G)$ are well known to be non-trivial (see the final Section 3.5), hence the theorem follows.

2. Preliminaries

In this section, we fix notation and terminology on Hopf algebras and homological algebra. All the material is standard, see, e.g., [11] respectively [4] for more background and details. The theory of self-dual comodules is a slightly more specialised topic, hence we include more details here.

2.1. The comodule V

Let A be a Hopf algebra with coproduct

$$\Delta\colon A \to A \otimes A, \quad a \mapsto a_{(1)} \otimes a_{(2)},$$

counit $\varepsilon\colon A \to \mathbb{F}$, and antipode $S\colon A \to A$ over a field $\mathbb{F}$, and let V be an N-dimensional right A-comodule with coaction

$$\rho\colon V \to V \otimes A, \quad e \mapsto e_{(0)} \otimes e_{(1)}.$$

We fix a vector space basis $\{e_1, \dots, e_N\}$ of V and denote by $\{v_{ij}\}$ the matrix coefficients of V with respect to this basis,

$$\rho(e_j) = \sum_i e_i \otimes v_{ij}.$$

Then we have

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}, \quad \varepsilon(v_{ij}) = \delta_{ij}, \tag{2.1}$$

and the matrix $v \in M_N(A)$ with entries v_{ij} is invertible with inverse matrix v^{-1} having the ij-entry $S(v_{ij})$,

$$\sum_k S(v_{ik})v_{kj} = \sum_k v_{ik}S(v_{kj}) = \varepsilon(v_{ij}) = \delta_{ij}.$$

2.2. The pairing $\langle -, - \rangle$

The comodule V is *self-dual* if there is a non-degenerate bilinear form

$$\langle -, - \rangle : V \otimes V \to \mathbb{F},$$

which is a morphism of A-comodules, where $\mathbb{F}$ carries the trivial coaction

$$\mathbb{F} \to \mathbb{F} \otimes A \cong A, \quad 1 = 1_{\mathbb{F}} \mapsto 1 = 1_A,$$

that is, if

$$\langle d_{(0)}, e_{(0)} \rangle d_{(1)} e_{(1)} = \langle d, e \rangle 1_A$$

holds for all $d, e \in V$.

In terms of the basis $\{e_i\}$, the bilinear form $\langle -, - \rangle$ is determined by the matrix $E \in M_N(\mathbb{F})$ with entries $\langle e_i, e_j \rangle$ and is non-degenerate if and only if $E \in \mathrm{GL}_N(\mathbb{F})$. Analysing when it is A-colinear yields the following.

Lemma 1. *The comodule V is self-dual if and only if there exists $E \in \mathrm{GL}_N(\mathbb{F})$ with*

$$v^{-1} = E^{-1} v^T E,$$

where $v \in M_N(A)$ is as in (2.1).

Proof. Assume that $\langle -, - \rangle$ is any bilinear form on V. In terms of the basis $\{e_j \otimes e_s\}$ of $V \otimes V$, applying the A-coaction on $V \otimes V$ and then the map $\langle -, - \rangle \otimes \mathrm{id}_A$ gives

$$e_j \otimes e_s \mapsto \sum_{ir} e_i \otimes e_r \otimes v_{ij} v_{rs} \mapsto \sum_{ir} E_{ir} v_{ij} v_{rs}.$$

Applying instead $\langle -, - \rangle$ and then the (trivial) coaction on $\mathbb{F}$ gives E_{js}, so $\langle -, - \rangle$ is A-colinear if and only if

$$E_{js} = \sum_{ir} E_{ir} v_{ij} v_{rs}$$

holds for all $1 \leq j, s \leq N$.

If this holds, then multiplying by $S(v_{sk})$ from the right and summing over s yields

$$\sum_s E_{js} S(v_{sk}) = \sum_{irs} E_{ir} v_{ij} v_{rs} S(v_{sk}) = \sum_i E_{ik} v_{ij}.$$

If E is invertible, multiplying from the left by $(E^{-1})_{lj}$ and summing over j finally yields

$$(v^{-1})_{lk} = S(v_{lk}) = \sum_{sj}(E^{-1})_{lj} E_{js} S(v_{sk})$$
$$= \sum_{ij}(E^{-1})_{lj} E_{ik} v_{ij} = (E^{-1} v^T E)_{lk}.$$

Conversely, if there is an $E \in \mathrm{GL}_N(\mathbb{F})$ with this property, simply define $\langle -, - \rangle$ by setting $\langle e_i, e_j \rangle := E_{ij}$ and then the above shows that this renders V self-dual. ■

2.3. The Lie algebra $\mathfrak{g}_A$

The dual vector space $A' = \mathrm{Hom}_{\mathbb{F}}(A, \mathbb{F})$ is an algebra with respect to the convolution product

$$(fg)(a) := f(a_{(1)})g(a_{(2)}), \quad f, g \in A', a \in A,$$

and the subspace

$$\mathfrak{g}_A := \{f \in A' \mid f(ab) = \varepsilon(a) f(b) + f(a)\varepsilon(b), \ \forall a, b \in A\}$$

of *primitive elements* in A' is a Lie algebra with Lie bracket given by the commutator $[f, g] := fg - gf$, for all $f, g \in \mathfrak{g}_A$.

The right A-comodule V is naturally a left A'-module via

$$f \triangleright e := e_{(0)} f(e_{(1)}), \quad f \in A', e \in V.$$

As A itself is also a right A-comodule via Δ, A becomes analogously a left A'-module via

$$f \triangleright a := a_{(1)} f(a_{(2)}), \quad f \in A', a \in A.$$

In particular, this defines an action of the Lie algebra $\mathfrak{g}_A$ of primitive elements $f \in A'$ by $\mathbb{F}$-linear derivations on A,

$$\begin{aligned} f \triangleright (ab) &= a_{(1)} b_{(1)} f(a_{(2)} b_{(2)}) \\ &= a_{(1)} b_{(1)} (\varepsilon(a_{(2)}) f(b_{(2)}) + f(a_{(2)}) \varepsilon(b_{(2)})) \\ &= a(f \triangleright b) + (f \triangleright a) b. \end{aligned} \tag{2.2}$$

2.4. Hochschild (co)homology

We denote by

$$b_n \colon A^{\otimes n+1} \to A^{\otimes n},$$
$$\beta_n \colon \mathrm{Hom}_{\mathbb{F}}(A^{\otimes n}, A) \to \mathrm{Hom}_{\mathbb{F}}(A^{\otimes n+1}, A)$$

the Hochschild (co)boundary maps of the algebra A and by

$$HH_n(A) := \ker b_n / \operatorname{im} b_{n+1}, \quad H^n(A, A) := \ker \beta_n / \operatorname{im} \beta_{n-1}$$

the Hochschild (co)homology of A with coefficients in A. In particular, an $\mathbb{F}$-linear derivation of A is the same as a Hochschild 1-cocycle, so by (2.2), the action of primitive elements $f \in \mathfrak{g}_A$ on A defines a linear map $\mathfrak{g}_A \to H^1(A, A)$.

Recall finally that there are well-defined cup and cap products (see, e.g., [4, Section XI.6])

$$\smile : H^i(A, A) \times H^j(A, A) \to H^{i+j}(A, A),$$
$$\frown : HH_i(A) \times H^j(A, A) \to HH_{i-j}(A)$$

which at the level of (co)cycles are given by

$$(\varphi \smile \psi)(a_1, \dots, a_i, b_1, \dots, b_j) = \varphi(a_1, \dots, a_i)\psi(b_1, \dots, b_j)$$

and

$$(a_0 \otimes \cdots \otimes a_i) \frown \varphi = a_0\varphi(a_1, \dots, a_j) \otimes a_{j+1} \otimes \cdots \otimes a_i,$$

and finally that the cup product is graded commutative, that is, that for all $[\varphi] \in H^i(A, A)$ and $[\psi] \in H^j(A, A)$, we have

$$[\varphi] \smile [\psi] = (-1)^{ij}[\psi] \smile [\varphi]. \tag{2.3}$$

3. Proof of the theorem

In this section, we prove the main theorem. We construct explicitly a suitable Hochschild 3-cycle on a Hopf algebra A and then show that it is non-trivial by pairing it with the Lie algebra of primitive elements in the dual Hopf algebra A'.

3.1. The Hochschild 3-cycle c_V

The starting point of the proof of the main result of this paper is the following remark which we expect to be well known to experts.

Lemma 2. *Assume $(V, \langle -, - \rangle)$ is a self-dual comodule over A. If $\langle -, - \rangle$ is symmetric or antisymmetric, then*

$$c_V := \sum_{ijkl} (v^{-1})_{ji} \otimes v_{ik} \otimes (v^{-1})_{kl} \otimes v_{lj} + \sum_{ij} 1 \otimes v_{ij} \otimes 1 \otimes (v^{-1})_{ji} \in A^{\otimes 4}$$

is a Hochschild 3-cycle, i.e., $b_3 c_V = 0$. If V is simple, then the converse implication holds as well.

Proof. It is straightforward to verify that

$$b_3 c_V = \sum_{ij} 1 \otimes \big((v^{-1})_{ij} \otimes v_{ji} - v_{ij} \otimes (v^{-1})_{ji}\big),$$

and Lemma 1 yields

$$b_3 c_V = \sum_{ijsr} 1 \otimes v_{ij} \otimes \big(E_{ir} v_{rs} E^{-1}_{sj} - E^T_{ir} v_{rs} (E^{-1})^T_{sj}\big),$$

which vanishes if $E^T = \pm E$. If V is simple, then the v_{ij} are linearly independent (by the Jacobson density theorem) and the above computation shows first that

$$EvE^{-1} = E^T v (E^{-1})^T \Leftrightarrow E^{-1} E^T v = v E^{-1} E^T.$$

Again by the Jacobson density theorem and the fact that the only matrices commuting with all others are scalar multiples of the identity matrix, this implies that $E^{-1}E^T$ is a constant, so $E^T = \lambda E$ for some $\lambda \in \mathbb{F}$ which is necessarily ± 1. ∎

3.2. The cap product $c_V \frown \varphi$

Let us take any $f_1, f_2, f_3 \in \mathfrak{g}_A$, i.e., primitive elements of A', and let φ be the cup product of the associated derivations of A,

$$\varphi\colon A^{\otimes 3} \to A, \quad a_1 \otimes a_2 \otimes a_3 \mapsto (f_1 \triangleright a_1)(f_2 \triangleright a_2)(f_3 \triangleright a_3).$$

We now show that the cap product between c_V and φ is a scalar multiple of the identity 1_A.

Lemma 3. *Let $F_i\colon V \to V$, $e \mapsto f_i \triangleright e = e_{(0)} f_i(e_{(1)})$ be the linear map defined by the action of f_i, $i = 1, 2, 3$. Then,*

$$c_V \frown \varphi = -\mathrm{tr}(F_1 F_2 F_3).$$

Proof. If $\partial\colon A \to A$ is any derivation and $v \in \mathrm{GL}_N(A)$, then the Leibniz rule implies

$$\partial(v^{-1}_{rk}) = -\sum_{ij} v^{-1}_{ri} \partial(v_{ij}) v^{-1}_{jk}$$

and of course $\partial(1) = 0$. Thus

$$\begin{aligned} c_V \frown \varphi &= \sum_{ijkl} v^{-1}_{ji} (f_1 \triangleright v_{ik})(f_2 \triangleright v^{-1}_{kl})(f_3 \triangleright v_{lj}) \\ &= -\sum_{ijklmn} v^{-1}_{ji} (f_1 \triangleright v_{ik}) v^{-1}_{km} (f_2 \triangleright v_{mn}) v^{-1}_{nl} (f_3 \triangleright v_{lj}) \\ &= -\sum_{ijklmnpqr} v^{-1}_{ji} v_{ip} F_{1,pk} v^{-1}_{km} v_{mq} F_{2,qn} v^{-1}_{nl} v_{lr} F_{3,rj} \\ &= -\sum_{jkn} F_{1,jk} F_{2,kn} F_{3,nj}. \end{aligned}$$ ∎

3.3. Evaluation in ε

The following is true for any algebra that admits a 1-dimensional representation.

Lemma 4. *The 0-cycle $1 \in A$ has a non-trivial class in $HH_0(A) = A/[A, A]$.*

Proof. The counit ε inevitably vanishes on all commutators but maps 1_A to $1_{\mathbb{F}}$. ∎

3.4. The Casimir operator

In view of Lemma 4, Lemma 3 implies $[c_V] \neq 0$ as long as there are $f_1, f_2, f_3 \in \mathfrak{g}_A$ with $\mathrm{tr}(F_1 F_2 F_3) \neq 0$.

This is in particular the case when A admits a Hopf algebra map to the coordinate Hopf algebra of a semisimple algebraic group G which acts non-trivially on V: using the graded commutativity (2.3) of $\smile$, we observe that

$$\mathrm{tr}(F_1[F_2, F_3]) = \mathrm{tr}(F_1 F_2 F_3) - \mathrm{tr}(F_1 F_3 F_2) = 2\mathrm{tr}(F_1 F_2 F_3).$$

Now recall that if $\mathfrak{g}$ is the Lie algebra of G, then, as G and hence $\mathfrak{g}$ are semisimple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and, therefore, the (quadratic) Casimir operator $\mathcal{C}$ of $\mathfrak{g}$ can be expressed as a finite sum

$$\mathcal{C} = \sum_{m=1}^{M} f_{m1}[f_{m2}, f_{m3}], \quad f_{mi} \in \mathfrak{g}.$$

Under the map $\pi^*: \mathfrak{g} \to \mathfrak{g}_A$ dual to π these f_{mi} yield primitive elements in $\mathfrak{g}_A$ and hence classes $[\varphi] \in H^3(A, A)$ which add up to a class whose pairing with $[c_V]$ is $-\frac{1}{2}\mathrm{tr}(\mathcal{C})$. If G acts non-trivially on V, this is non-zero, so $[c_V] \neq 0$.

3.5. The class $\pi_*([c_V])$

Following a suggestion by M. Khalkhali, we end with a brief historical account on the role that $\pi_*([c_V]) \in HH_3(\mathcal{O}(G))$ plays in the cohomology of algebraic groups and Lie algebras. For more details, we refer to Samelson's survey [10].

As we work over a field of characteristic 0, a connected semisimple algebraic group G is a smooth affine variety, and the Hochschild–Kostant–Rosenberg isomorphism [8]

$$HH_n(\mathcal{O}(G)) \cong \Omega^n(G)$$

identifies $\pi_*([c_V])$ with the (Kähler) differential 3-form

$$\omega_V := \sum_{ijkl} g_{ji}^{-1} \mathrm{d}g_{ik} \mathrm{d}g_{kl}^{-1} \mathrm{d}g_{lj} \in \Omega^3(G) \tag{3.1}$$

on G; here $g_{ij} := \pi(v_{ij}) \in \mathcal{O}(G)$ are the matrix coefficients of the linear representation $G \to \mathrm{GL}(V) \cong \mathrm{GL}_N(\mathbb{F})$ defined by π and ρ. Note that $\mathrm{d}g_{ij}^{-1} = \sum_{rs} g_{ir}^{-1}(\mathrm{d}g_{rs})g_{sj}^{-1}$ and that this implies $\mathrm{d}\omega_V = 0$. When $\pi = \mathrm{id}_{\mathcal{O}(G)}$, $V = \mathfrak{g}$, and ρ is the adjoint representation of G, then $\langle -, - \rangle$ can be taken to be the Killing form. Thus every semisimple algebraic group G over a field of characteristic 0 comes equipped with a canonical de Rham cohomology class $[\omega_{\mathfrak{g}}] \in \mathrm{HdR}^3(G)$.

One of the main results of Chevalley and Eilenberg's seminal paper [5] was that this cohomology class is non-trivial. Hopf had shown in [9] that the de Rham cohomology of a compact and connected Lie group is that of a product of odd-dimensional spheres S^{m_i}, and for the classical matrix groups the m_i had been already known earlier. Chevalley and Eilenberg then fully implemented an idea that goes back to Cartan: the cotangent bundle of an algebraic group G admits a natural trivialisation, $T^*G \cong G \times \mathfrak{g}'$. From a Hopf-algebraist's perspective, this stems from the fact that $\Omega^1(G)$ and hence also $\Omega(G) = \Lambda_{\mathcal{O}(G)}\Omega^1(G)$ is a Hopf module over $\mathcal{O}(G)$, hence by the fundamental theorem of Hopf modules [11, Theorem 4.1.1], $\Omega(G)$ is a free $\mathcal{O}(G)$-module with a basis given by the elements that are invariant under the $\mathcal{O}(G)$-coaction. Geometrically, this coaction is the G-action on differential forms given by right translation, hence the basis elements are the right-invariant differential forms. By evaluation in the unit element $e \in G$, these become identified with elements of the exterior algebra $\Lambda_{\mathbb{F}}\mathfrak{g}'$ of the dual of the Lie algebra $\mathfrak{g}$. The de Rham differential is G-equivariant, hence restricts to the right-invariant differential forms, and under the isomorphism becomes the Chevalley–Eilenberg differential on $\Lambda_{\mathbb{F}}\mathfrak{g}'$ that computes the Lie algebra cohomology $H(\mathfrak{g}, \mathbb{F})$ [5, Theorem 9.1]. Furthermore, the differential forms which are not just right- but also left-invariant become identified with the ad-invariant cochains in the Chevalley–Eilenberg complex, and on these the coboundary map is trivial [5, (19.2)]. As G is reductive, this subcomplex of biinvariant differential forms is actually quasiisomorphic to the de Rham complex, so the de Rham cohomology of G can be identified with the algebra of ad-invariant Chevalley–Eilenberg cochains. If we consider compact Lie groups over $\mathbb{F} = \mathbb{R}$, then the statements carry over to smooth functions and differential forms and the de Rham complex is quasiisomorphic to the subcomplex of biinvariant differential forms, which are automatically closed [5, (12.3)].

Our form ω_V (and in fact the Hochschild cycle c_V) is manifestly biinvariant: replacing the function $g_{ij} \in \mathcal{O}(G)$ by $\sum_r g_{ir}t_{rj}$ or $\sum_s t_{is}g_{sj}$ for a constant matrix $T \in \mathrm{GL}_N(\mathbb{F})$ with entries t_{ij} yields the same form ω_V. In our proof above, we have applied $\pi^*\colon \mathfrak{g} \to \mathfrak{g}_A$ to Lie algebra elements $f_j \in \mathfrak{g}$ and then computed the pairing of $[\varphi]$ with $[c_V]$. By very definition, this amounts to pairing the corresponding Hochschild 3-cocycle on $\mathcal{O}(G)$ with $\pi_*([c_V])$, and under the Hochschild–Kostant–Rosenberg isomorphism, this Hochschild 3-cocycle is the right-invariant multivector field $f_1 \wedge f_2 \wedge f_3$ on G. That is, our computation can indeed be reinterpreted in terms of Lie algebra cohomology as the evaluation of the Chevalley–Eilenberg cocycle

$$\chi_V\colon \mathfrak{g} \otimes_{\mathbb{F}} \mathfrak{g} \otimes_{\mathbb{F}} \mathfrak{g} \to \mathbb{F}, \quad f_1 \otimes f_2 \otimes f_3 \mapsto \mathrm{tr}(F_1F_2F_3), \tag{3.2}$$

where

$$F_j := (\mathrm{d}_e\rho)(f_j) \in \mathfrak{gl}(V) \cong M_N(\mathbb{F})$$

are the values of the f_j under the representation $\mathfrak{g} \to \mathfrak{gl}(V) \cong M_N(\mathbb{F})$ corresponding to the representation $G \to \mathrm{GL}(V)$.

Note that our assumptions on V enter the fact that χ_V is a 3-cocycle.

Lemma 5. *If $E \in \mathrm{GL}_N(\mathbb{F})$ is an invertible matrix and*

$$\mathfrak{g} \subseteq \{F \in M_N(\mathbb{F}) \mid F^T = -EFE^{-1}\}$$

is a Lie subalgebra, then

$$\chi\colon \mathfrak{g} \otimes_{\mathbb{F}} \mathfrak{g} \otimes_{\mathbb{F}} \mathfrak{g} \to \mathbb{F}, \quad \chi(F_1, F_2, F_3) := \mathrm{tr}(F_1F_2F_3)$$

is an ad*-invariant cocycle in $C^3(\mathfrak{g}, \mathbb{F}) = \Lambda_{\mathbb{F}}\mathfrak{g}'$.*

Proof. That χ is an alternating 3-form is seen as follows:

$$\begin{aligned}\operatorname{tr}(F_1 F_2 F_3) &= \operatorname{tr}(F_3^T F_2^T F_1^T) = -\operatorname{tr}(E F_3 F_2 F_1 E^{-1}) \\ &= -\operatorname{tr}(F_3 F_2 F_1) = -\operatorname{tr}(F_2 F_1 F_3).\end{aligned}$$

That χ satisfies the cocycle condition

$$\begin{aligned}0 = {} & -\chi([F_1, F_2], F_3, F_4) + \chi([F_1, F_3], F_2, F_4) - \chi([F_1, F_4], F_2, F_3) \\ & - \chi([F_2, F_3], F_1, F_4) + \chi([F_2, F_4], F_1, F_3) - \chi([F_3, F_4], F_1, F_2)\end{aligned}$$

is shown similarly. The ad-invariance is immediate. ∎

In this way, the condition on V to carry a non-degenerate symmetric or antisymmetric invariant bilinear form can also be motivated from the point of view of Lie algebra cohomology.

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Tomasz Brzeziński
Department of Mathematics, Swansea University, Swansea University Bay Campus, Fabian Way, Swansea SA1 8EN, UK; Faculty of Mathematics, University of Białystok, K. Ciołkowskiego 1M, 15-245 Białystok, Poland; t.brzezinski@swansea.ac.uk

Ulrich Krähmer
Institut für Geometrie, Technische Universität Dresden, 01062 Dresden, Germany; ulrich.kraehmer@tu-dresden.de

Réamonn Ó Buachalla
Mathematical Institute, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic; obuachalla@karlin.mff.cuni.cz

Karen R. Strung
Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague, Czech Republic; strung@math.cas.cz