J. Noncommut. Geom. 19 (2025), 429–450
DOI 10.4171/JNCG/567

© 2024 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Deformations of modified r-matrices and cohomologies of related algebraic structures

Jun Jiang and Yunhe Sheng

Abstract. Modified r-matrices are solutions of the modified classical Yang–Baxter equation, introduced by Semenov-Tian-Shansky, and play important roles in mathematical physics. In this paper, first we introduce a cohomology theory for modified r-matrices. Then we study three kinds of deformations of modified r-matrices using the established cohomology theory, including algebraic deformations, geometric deformations and linear deformations. We give the differential graded Lie algebra that governs algebraic deformations of modified r-matrices. For geometric deformations, we prove the rigidity theorem and study when is a neighborhood of a modified r-matrix smooth in the space of all modified r-matrix structures. In the study of trivial linear deformations, we introduce the notion of a Nijenhuis element for a modified r-matrix. Finally, applications are given to study deformations of the complement of the diagonal Lie algebra and compatible Poisson structures.

1. Introduction

In the seminal work [26], Semenov-Tian-Shansky showed that solutions of the modified classical Yang–Baxter equation, which we call modified r-matrices in this paper, play an important role in studying solutions of Lax equations [24–26]. Furthermore, modified r-matrices are intimately related to particular factorization problems in the corresponding Lie algebras and Lie groups. This factorization problem was considered by Reshetikhin and Semenov-Tian-Shansky in the framework of the enveloping algebra of a Lie algebra with a modified r-matrix to study quantum integrable systems [23]. Any modified r-matrix induces a post-Lie algebra [1], and a factorization theorem for group-like elements of the completion of the Lie enveloping algebra of a post-Lie algebra was established by Ebrahimi-Fard, Mencattini and Munthe-Kaas in [9, 10]. Recently, the global factorization theorem for a Rota–Baxter Lie group was given in [15]. Moreover, modified r-matrices are also useful for the construction of flat metrics and Frobenius manifolds [27], and compatible Poisson structures [18]. Note that in the associative algebra context, such objects are called modified Rota–Baxter algebras by Zhang, Gao and Guo [30, 31].

A classical approach to study a mathematical structure is to associate to it invariants. Among these, cohomology theories occupy a central position as they enable for example to control deformations or extension problems. Note that the cohomology theory for a

Mathematics Subject Classification 2020: 17B38 (primary); 17B56 (secondary).
Keywords: modified classical Yang–Baxter equation, modified r-matrix, cohomology, deformation.

skew-symmetric classical r-matrix was studied in [29] under the general framework of relative Rota–Baxter operators (also called $\mathcal{O}$-operators [17]). The first purpose of this paper is to study the cohomology theory for a modified r-matrix. In [26], Semenov-Tian-Shansky showed that a modified r-matrix $R : \mathfrak{g} \to \mathfrak{g}$ on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ induces a new Lie algebra $\mathfrak{g}_R$ in which the Lie bracket $[\cdot,\cdot]_R$ is given by

$$[x, y]_R = [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}.$$

In [2], Bordemann showed that the induced Lie algebra $\mathfrak{g}_R$ represents on $\mathfrak{g}$. We use the corresponding Chevalley–Eilenberg cohomology [4] of the Lie algebra $\mathfrak{g}_R$ with coefficients in $\mathfrak{g}$ to define the cohomology of the modified r-matrix R. It is well known that there is a one-to-one correspondence between the modified r-matrix R and the Rota–Baxter operator B of weight 1 via the relation $R = \mathrm{Id} + 2B$. The cohomology theory of the latter was given in [16] and the Van Est type theorem was established. We also show that the cohomology of the modified r-matrix $R = \mathrm{Id} + 2B$ and the cohomology of the Rota–Baxter operator B are isomorphic.

The concept of a formal deformation of an algebraic structure began with the seminal work of Gerstenhaber [13, 14] for associative algebras. Nijenhuis and Richardson extended this study to Lie algebras [20,21]. There is a well-known slogan, often attributed to Deligne, Drinfeld and Kontsevich: every reasonable deformation theory is controlled by a differential graded Lie algebra, determined up to quasi-isomorphism. This slogan has been made into a rigorous theorem by Lurie and Pridham [19,22]. It is also meaningful to deform *maps* compatible with given algebraic structures. Recently, the deformation theory of morphisms was developed in [3, 11, 12], the deformation theories of $\mathcal{O}$-operators on Lie algebras and associative algebras were developed in [6,29]. The second purpose of the paper is to study deformation theories of modified r-matrices. We study three kinds of deformations of a modified r-matrix R:

- (algebraic deformations) first we consider an algebraic deformation $R + R'$ for a certain linear map R', and show that this kind of deformations are governed by a differential graded Lie algebra. This fulfill the general slogan for the deformation theory proposed by Deligne, Drinfeld and Kontsevich;
- (geometric deformations) then we consider a smooth geometric deformation R_t such that $R_0 = R$ using the approach developed by Crainic, Schätz and Struchiner in [5]. We show that the tangent space $T_R\mathrm{Orb}_R$ of the orbit Orb_R is the space of 2-coboundaries $B^2(R)$. Consequently, the condition $H^2(R) = 0$ will imply a certain rigidity theorem, and the condition $H^3(R) = 0$ will imply the space of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R. We also give the necessary and sufficient condition on a 2-cocycle giving a geometric deformation using the Kuranishi map;
- (linear deformation) next we study a linear deformation $R + t\widehat{R}$. In particular, trivial linear deformations lead to the concept of Nijenhuis elements for a modified r-matrix. If $x \in \mathfrak{g}$ is a Nijenhuis element, then ad_x is a Nijenhuis operator on the Lie algebra $\mathfrak{g}_R$.

Note that certain particular deformations of classical r-matrices are considered in [28] in the study of integrable infinite-dimensional systems.

The paper is organized as follows. In Section 2, we define the cohomology of a modified r-matrix R using the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}_R$ with coefficients in $\mathfrak{g}$. In Section 3, we construct a differential graded Lie algebra that governs algebraic deformations of a modified r-matrix. In Section 4, we study geometric deformations of a modified r-matrix. In Section 5, we study linear deformations of a modified r-matrix. In Section 6, we study deformations of the complement of the diagonal Lie algebra and compatible Poisson structures as applications.

2. Cohomologies of modified r-matrices

In this section, we establish the cohomology theory of a modified r-matrix R using the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}_R$ with coefficients in $\mathfrak{g}$.

Definition 2.1 ([26]). Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra. A linear map $R : \mathfrak{g} \to \mathfrak{g}$ is called a *modified r-matrix* if it is a solution of the following *modified classical Yang–Baxter equation*:

$$[R(x), R(y)]_\mathfrak{g} = R([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}) - [x, y]_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g}. \tag{1}$$

Definition 2.2. Let R and R' be modified r-matrices on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. A *homomorphism* from R to R' is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\varphi \circ R = R' \circ \varphi.$$

Remark 2.3. The notion of a modified Rota–Baxter operator of weight -1 on an associative algebra was introduced in [8]. More precisely, it is a linear map $P : A \to A$ on an associative algebra $(A, \cdot_A)$ satisfying

$$P(u) \cdot_A P(v) = P(P(u) \cdot_A v + u \cdot_A P(v)) - u \cdot_A v, \quad \forall u, v \in A.$$

It is straightforward to see that if a linear map $P : A \to A$ is a modified Rota–Baxter operator of weight -1 on an associative algebra $(A, \cdot_A)$, then P is a modified r-matrix on the Lie algebra $(A, [\cdot,\cdot]_A)$, where $[\cdot,\cdot]_A$ is the commutator Lie bracket.

Remark 2.4. Let $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. Under the condition $R^2 = \mathrm{Id}$, the following structures are equivalent:

- R is a modified r-matrix;
- R is a Nijenhuis operator;
- R is a product structure;
- there is a vector space direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of $\mathfrak{g}$ into subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that R is given by

$$R(x, u) = (x, -u), \quad \forall x \in \mathfrak{g}_1,\ u \in \mathfrak{g}_2.$$

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra and R be a modified r-matrix. Semenov-Tian-Shansky showed that $(\mathfrak{g}, [\cdot,\cdot]_R)$ is a Lie algebra which plays important roles in the study of integrable systems [26], where

$$[x, y]_R = [R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g}. \tag{2}$$

Recall that a matched pair of Lie algebras consists of Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$, $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h})$, a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ of $\mathfrak{g}$ on $\mathfrak{h}$ and a representation $\varrho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{h}$ on $\mathfrak{g}$, such that some compatibility conditions are satisfied. Bordemann further showed that the induced Lie algebra $\mathfrak{g}_R$ represents on $\mathfrak{g}$ which leads to a matched pair of Lie algebras $((\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}), (\mathfrak{g}, [\cdot,\cdot]_R))$ [2]. Here we give a direct proof to be self-contained.

Proposition 2.5. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. Define a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by*

$$\rho(x)y = [R(x), y]_\mathfrak{g} - R([x, y]_\mathfrak{g}), \quad \forall x, y \in \mathfrak{g}. \tag{3}$$

Then ρ is a representation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ on the vector space $\mathfrak{g}$.

Proof. For all $x, y, z \in \mathfrak{g}$, by (1) and (3), we have

$$\begin{aligned}
&[\rho(x), \rho(y)]z \\
&\quad = \rho(x)\rho(y)z - \rho(y)\rho(x)z \\
&\quad = \rho(x)([R(y), z]_\mathfrak{g} - R([y, z]_\mathfrak{g})) - \rho(y)([R(x), z]_\mathfrak{g} - R([x, z]_\mathfrak{g})) \\
&\quad = [R(x), [R(y), z]_\mathfrak{g}]_\mathfrak{g} - [R(x), R([y, z]_\mathfrak{g})]_\mathfrak{g} - R([x, [R(y), z]_\mathfrak{g}]_\mathfrak{g}) \\
&\qquad + R([x, R([y, z]_\mathfrak{g})]_\mathfrak{g}) - [R(y), [R(x), z]_\mathfrak{g}]_\mathfrak{g} + [R(y), R([x, z]_\mathfrak{g})]_\mathfrak{g} \\
&\qquad + R([y, [R(x), z]_\mathfrak{g}]_\mathfrak{g}) - R([y, R([x, z]_\mathfrak{g})]_\mathfrak{g}) \\
&\quad = [[R(x), R(y)]_\mathfrak{g}, z]_\mathfrak{g} - R([R(x), [y, z]_\mathfrak{g}]_\mathfrak{g}) - R([x, [R(y), z]_\mathfrak{g}]_\mathfrak{g}) + [x, [y, z]_\mathfrak{g}]_\mathfrak{g} \\
&\qquad + R([R(y), [x, z]_\mathfrak{g}]_\mathfrak{g}) - [y, [x, z]_\mathfrak{g}]_\mathfrak{g} + R([y, [R(x), z]_\mathfrak{g}]_\mathfrak{g}) \\
&\quad = [[R(x), R(y)]_\mathfrak{g}, z]_\mathfrak{g} + [[x, y]_\mathfrak{g}, z]_\mathfrak{g} - R([[R(x), y]_\mathfrak{g}, z]_\mathfrak{g}) - R([[x, R(y)]_\mathfrak{g}, z]_\mathfrak{g}),
\end{aligned}$$

and

$$\begin{aligned}
&\rho([x, y]_R)z \\
&\quad = \rho([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g})z \\
&\quad = [R([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}), z]_\mathfrak{g} - R([[R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}, z]_\mathfrak{g}) \\
&\quad = [[R(x), R(y)]_\mathfrak{g}, z]_\mathfrak{g} + [[x, y]_\mathfrak{g}, z]_\mathfrak{g} - R([[R(x), y]_\mathfrak{g}, z]_\mathfrak{g}) - R([[x, R(y)]_\mathfrak{g}, z]_\mathfrak{g}).
\end{aligned}$$

Therefore we have $\rho([x, y]_R) = [\rho(x), \rho(y)]$, which means that ρ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_R)$ on the vector space $\mathfrak{g}$. ∎

Let $\mathrm{d}_{\mathsf{CE}}^R : \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{k+1} \mathfrak{g}, \mathfrak{g})$ be the corresponding Chevalley–Eilenberg coboundary operator of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ with coefficients in the representation $(\mathfrak{g}, \rho)$. More precisely, for all $f \in \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$ and $x_1, \dots, x_{k+1} \in \mathfrak{g}$, we have

$$\begin{aligned}
&\mathrm{d}_{\mathsf{CE}}^R f(x_1, \dots, x_{k+1}) \\
&\quad = \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \\
&\qquad + \sum_{i<j} (-1)^{i+j} f([x_i, x_j]_R, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \\
&\quad = \sum_{i=1}^{k+1} (-1)^{i+1} [R(x_i), f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}} \\
&\qquad - \sum_{i=1}^{k+1} (-1)^{i+1} R([x_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{g}}) \\
&\qquad + \sum_{i<j} (-1)^{i+j} f([R(x_i), x_j]_{\mathfrak{g}} + [x_i, R(x_j)]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}).
\end{aligned} \tag{4}$$

Now, we define the cohomology of a modified r-matrix $R : \mathfrak{g} \to \mathfrak{g}$. Define the space of 0-cochains $C^0(R)$ to be 0 and define the space of 1-cochains $C^1(R)$ to be $\mathfrak{g}$. For $n \geq 2$, define the space of n-cochains $C^n(R)$ by $C^n(R) = \mathrm{Hom}(\wedge^{n-1} \mathfrak{g}, \mathfrak{g})$.

Definition 2.6. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra and R be a modified r-matrix. The cohomology of the cochain complex $(\bigoplus_{i=0}^{+\infty} C^i(R), \mathrm{d}_{\mathsf{CE}}^R)$ is defined to be the *cohomology for the modified r-matrix R*.

Denote the set of n-cocycles by $Z^n(R)$, the set of n-coboundaries by $B^n(R)$ and the n-th cohomology group by

$$H^n(R) = Z^n(R)/B^n(R), \quad n \geq 0.$$

It is obvious that $x \in \mathfrak{g}$ is closed if and only if

$$\mathrm{ad}_x \circ R = R \circ \mathrm{ad}_x,$$

and $f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ is closed if and only if

$$\begin{aligned}
&[R(x), f(y)]_{\mathfrak{g}} - R([x, f(y)]_{\mathfrak{g}}) - [R(y), f(x)]_{\mathfrak{g}} + R([y, f(x)]_{\mathfrak{g}}) \\
&\qquad = f([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}),
\end{aligned} \tag{5}$$

for all $x, y \in \mathfrak{g}$.

At the end of this section, we recall the cohomology theory of Rota–Baxter operators given in [16], and establish its relation with the cohomology theory of modified r-matrices.

Definition 2.7. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $B : \mathfrak{g} \to \mathfrak{g}$ is called a *Rota–Baxter operator of weight* λ if

$$[B(x), B(y)]_{\mathfrak{g}} = B([B(x), y]_{\mathfrak{g}} + [x, B(y)]_{\mathfrak{g}} + \lambda[x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

The following result is well known.

Proposition 2.8. *Let* $\mathfrak{g}$ *be a Lie algebra and* $B \in \mathfrak{gl}(\mathfrak{g})$*. The linear map* $\mathrm{Id} + 2B$ *is a modified* r*-matrix on* $\mathfrak{g}$ *if and only if* B *is a Rota–Baxter operator of weight* 1 *on* $\mathfrak{g}$*.*

Let B be a Rota–Baxter operator of weight 1 on a Lie algebra $\mathfrak{g}$. Consider the cochain complex $(\bigoplus_{k=1}^{+\infty} C^k(B), \mathrm{d}_{\mathsf{CE}}^B)$, where $C^1(B) = \mathfrak{g}$ and $C^k(B) = \mathrm{Hom}(\wedge^{k-1}\mathfrak{g}, \mathfrak{g})$ for $k \geq 2$, and $\mathrm{d}_{\mathsf{CE}}^B$ is defined by

$$\begin{aligned}
\mathrm{d}_{\mathsf{CE}}^B f(u_1, \ldots, u_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} B([f(u_1, \ldots, \widehat{u}_i, \ldots, u_{k+1}), u_i]_{\mathfrak{g}}) \\
&\quad + \sum_{i=1}^{k+1} (-1)^{i+1} [B(u_i), f(u_1, \ldots, \widehat{u}_i, \ldots, u_{k+1})]_{\mathfrak{g}} \\
&\quad + \sum_{i<j} (-1)^{i+j} f([B(u_i), u_j]_{\mathfrak{g}} - [B(u_j), u_i]_{\mathfrak{g}} \\
&\quad + [u_i, u_j]_{\mathfrak{g}}, u_1, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_{k+1}),
\end{aligned}$$

where $f \in C^{k+1}(B)$ and $u_i \in \mathfrak{g}$, $1 \leq i \leq k+1$.

It was proved in [16] that $(\mathrm{d}_{\mathsf{CE}}^B)^2 = 0$. The cohomology of the cochain complex $(\bigoplus_{k=1}^{+\infty} C^k(B), \mathrm{d}_{\mathsf{CE}}^B)$ is defined to be the cohomology of the Rota–Baxter operator B.

Theorem 2.9. *With the above notations, we have*

$$\mathrm{d}_{\mathsf{CE}}^R = 2\mathrm{d}_{\mathsf{CE}}^B.$$

Consequently, for $k \geq 1$*, the* k*-th cohomology group* $H^k(B)$ *of a Rota–Baxter operator* B *is isomorphic with the* k*-th cohomology group* $H^k(R)$ *of the modified* r*-matrix* $R = \mathrm{Id} + 2B$*.*

Proof. For $k \geq 1$, define linear maps $\Phi_k : C^k(B) \to C^k(R)$ by $\Phi_k = 2^{k-2}\mathrm{Id}$. Then the following diagram is commutative:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^B} & \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^B} & \cdots & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^B} & \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}) & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^B} & \cdots \\
 & & \downarrow{\scriptstyle \frac{1}{2}\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & & & \downarrow{\scriptstyle 2^{k-1}\mathrm{Id}} & & \\
0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^R} & \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^R} & \cdots & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^R} & \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g}) & \xrightarrow{\mathrm{d}_{\mathsf{CE}}^R} & \cdots .
\end{array}$$

In fact, for any $f\in \mathrm{Hom}(\wedge^k\mathfrak{g},\mathfrak{g})$, $x_i\in\mathfrak{g}$, $1\le i\le k+1$, we have

$$
\begin{aligned}
&\mathrm{d}^R_{\mathsf{CE}}(\Phi_k f)(x_1,\dots,x_{k+1})\\
&=2^{k-1}\Big(\sum_{i=1}^{k+1}(-1)^{i+1}\big([x_i,f(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]_{\mathfrak{g}}\\
&\quad+2[B(x_i),f(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]_{\mathfrak{g}}\big)\\
&\quad-\sum_{i=1}^{k+1}(-1)^{i+1}[x_i,f(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]_{\mathfrak{g}}\\
&\quad-\sum_{i=1}^{k+1}(-1)^{i+1}2B([x_i,f(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]_{\mathfrak{g}})\\
&\quad+\sum_{i<j}(-1)^{i+j}2f([x_i,x_j]_{\mathfrak{g}},x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1})\\
&\quad+\sum_{i<j}(-1)^{i+j}2f([B(x_i),x_j]_{\mathfrak{g}}+[x_i,B(x_j)]_{\mathfrak{g}},x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1})\Big)\\
&=2^k\Big(\sum_{i=1}^{k+1}(-1)^{i+1}([B(x_i),f(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]_{\mathfrak{g}}\\
&\quad-B([x_i,f(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]_{\mathfrak{g}}))\\
&\quad+\sum_{i<j}(-1)^{i+j}f([B(x_i),x_j]_{\mathfrak{g}}+[x_i,B(x_j)]_{\mathfrak{g}}\\
&\quad+[x_i,x_j]_{\mathfrak{g}},x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1})\Big)\\
&=\Phi_{k+1}(\mathrm{d}^B_{\mathsf{CE}}f)(x_1,\dots,x_{k+1}),
\end{aligned}
$$

which implies that $\mathrm{d}^R_{\mathsf{CE}}=2\mathrm{d}^B_{\mathsf{CE}}$ and $H^k(B)\cong H^k(R)$, $k\ge 1$. ■

Example 2.10. Consider the Lie algebra $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{R})$. It is well known that the Cartan subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ is $H=\mathrm{span}\{E_{ii}-E_{i+1i+1}\mid 1\le i\le n-1\}$. Denote the Borel subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ by $B(\mathfrak{sl}(n,\mathbb{R}))$. It is well known that $B(\mathfrak{sl}(n,\mathbb{R}))=H\oplus\mathrm{span}\{E_{ij}\mid i<j\}$. Thus $\mathfrak{sl}(n,\mathbb{R})=B(\mathfrak{sl}(n,\mathbb{R}))\oplus A$ as vector spaces, where $A=\mathrm{span}\{E_{ij}\mid i>j\}$. Define a linear map $R:\mathfrak{sl}(n,\mathbb{R})\to\mathfrak{sl}(n,\mathbb{R})$ by

$$R(x+u)=x-u,\quad \forall x\in B(\mathfrak{sl}(n,\mathbb{R})),\ u\in A.$$

By Remark 2.4, we obtain that R is a modified r-matrix on the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$. Assume that $a=x+u\in\mathfrak{sl}(n,\mathbb{R})$ where $x\in B(\mathfrak{sl}(n,\mathbb{R}))$ and $u\in A$, such that $\mathrm{d}^R_{\mathsf{CE}}a=0$, that is

$$\mathrm{d}^R_{\mathsf{CE}}a(y)=[R(y),a]-R([y,a])=0,\quad \forall y\in\mathfrak{sl}(n,\mathbb{R}).$$

- For any $y\in A$, $[R(y),a]-R([y,a])=0$ implies that $x\in H$.
- For any $y\in B(\mathfrak{sl}(n,\mathbb{R}))$, $[R(y),a]-R([y,a])=0$ implies that $u=0$.

Thus $\mathrm{d}^R_{\mathsf{CE}}a=0$ if and only if $a\in H$. Therefore, $H^1(R)\cong\mathbb{R}^{n-1}$.

Example 2.11. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, where the Lie bracket is given by $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$ with respect to the basis $\{e, f, h\}$. Then $R : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})$ defined by

$$R(e, f, h) = (e, f, h) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a modified r-matrix. Let

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})$$

satisfy $\mathrm{d}^R_{\mathsf{CE}} T = 0$. Then we obtain

$$0 = [e, T(f)] + [f, T(e)] - R([e, T(f)]) + R([f, T(e)]),$$
$$0 = [e, T(h)] - [h, T(e)] - R([e, T(h)]) + R([h, T(e)]) + 4T(e)$$

and

$$0 = -[f, T(h)] - [h, T(f)] - R([f, T(h)]) + R([h, T(f)]).$$

Thus we have $t_{11} = t_{21} = t_{31} = 0$ and $t_{22} = t_{13} = 0$. By Example 2.10, we have $B^2(R) = \mathrm{Im}\, \mathrm{d}^R_{\mathsf{CE}} \cong \frac{\mathfrak{g}}{\ker \mathrm{d}^R_{\mathsf{CE}}} = \frac{\mathfrak{g}}{H^1(R)} \cong \mathbb{R}^2$. Thus $H^2(R) \simeq \mathbb{R}^2$.

3. Algebraic deformations of modified r-matrices

In this section, we construct a differential graded Lie algebra that governs algebraic deformations of a modified r-matrix.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. Further, we consider the graded vector space $C^*(\mathfrak{g}) = \bigoplus_{k=1}^{+\infty} \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$. Define a skew-symmetric bracket operation

$$[\![\cdot, \cdot]\!] : \mathrm{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^q \mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{p+q} \mathfrak{g}, \mathfrak{g})$$

by

$$\begin{aligned} &[\![f, g]\!](x_1, x_2, \dots, x_{p+q}) \\ &= \sum_{\sigma \in S(q,1,p-1)} (-1)^{\sigma} f([g(x_{\sigma(1)}, \dots, x_{\sigma(q)}), x_{\sigma(q+1)}]_{\mathfrak{g}}, x_{\sigma(q+2)}, \dots, x_{\sigma(p+q)}) \\ &\quad - (-1)^{pq} \sum_{\sigma \in S(p,1,q-1)} (-1)^{\sigma} \\ &\quad \cdot g([f(x_{\sigma(1)}, \dots, x_{\sigma(p)}), x_{\sigma(p+1)}]_{\mathfrak{g}}, x_{\sigma(p+2)}, \dots, x_{\sigma(p+q)}) \\ &\quad + (-1)^{pq} \sum_{\sigma \in S(p,q)} (-1)^{\sigma} [f(x_{\sigma(1)}, \dots, x_{\sigma(p)}), g(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)})]_{\mathfrak{g}}, \end{aligned} \quad (6)$$

for all $f \in \mathrm{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g})$, $g \in \mathrm{Hom}(\wedge^q \mathfrak{g}, \mathfrak{g})$.

Then we have the following theorem characterizing modified r-matrices.

Theorem 3.1. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. Then $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$ is a graded Lie algebra and its Maurer–Cartan elements are precisely Rota–Baxter operators of weight* 0.

Moreover, a linear map $R \in \mathfrak{gl}(\mathfrak{g})$ is a modified r-matrix on the Lie algebra $\mathfrak{g}$ if and only if R satisfies the equation

$$[\![R, R]\!] = 2\pi, \tag{7}$$

where we denote $[\cdot,\cdot]_{\mathfrak{g}}$ by π.

Proof. By [29, Corollary 6.1], $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$ is a graded Lie algebra.

For $R \in \mathfrak{gl}(\mathfrak{g})$, we have

$$[\![R, R]\!](x, y) = 2(R([R(x), y]_{\mathfrak{g}}) - R([R(y), x]_{\mathfrak{g}}) - [R(x), R(y)]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

By this equality, we can deduce that on the one hand R is a Rota–Baxter operator of weight 0 if and only if $[\![R, R]\!] = 0$, i.e., R is a Maurer–Cartan element. On the other hand, R is a modified r-matrix on the Lie algebra $\mathfrak{g}$ if and only if R satisfies (7). ■

Proposition 3.2. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then $[\![R, R]\!]$ is in the center of the graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$.*

Proof. Denote the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ by π. Since R is a modified r-matrix on the Lie algebra $\mathfrak{g}$, we have $[\![R, R]\!] = 2\pi$ via Theorem 3.1. For all $f \in \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$, by (6), we have

$$\begin{aligned}
&[\![2\pi, f]\!](x_1, \ldots, x_k, x_{k+1}, x_{k+2})\\
&\quad = 2\Big(\sum_{\sigma \in S(k,1,1)} (-1)^{|\sigma|} \pi(\pi(f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}), x_{\sigma(k+2)})\\
&\qquad - \sum_{\sigma \in S(2,1,k-1)} (-1)^{|\sigma|} f(\pi(\pi(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}), x_{\sigma(4)}, \ldots, x_{\sigma(k+2)})\\
&\qquad + \sum_{\sigma \in S(2,k)} (-1)^{|\sigma|} \pi(\pi(x_{\sigma(1)}, x_{\sigma(2)}), f(x_{\sigma(3)}, \ldots, x_{\sigma(k+2)}))\Big)\\
&\quad = 0,
\end{aligned}$$

which implies that $[\![R, R]\!]$ is in the center of $C^*(\mathfrak{g})$. ■

We denote $[\![R, \cdot]\!]$ by d_R. Now we obtain the differential graded Lie algebra that governs algebraic deformations of a modified r-matrix.

Theorem 3.3. *With the above notations, $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$ is a differential graded Lie algebra.*

Moreover, $R + R'$ is still a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ if and only if R' is a Maurer–Cartan element of the differential graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$.

Proof. It follows from the graded Jacobi identity that d_R is a graded derivation on the graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$. By Proposition 3.2, we have

$$\mathrm{d}_R^2 f = [\![R, [\![R, f]\!]]\!] = [\![[\![R, R]\!], f]\!] - [\![R, [\![R, f]\!]]\!],$$

which implies that

$$\mathrm{d}_R^2 f = [\![R, [\![R, f]\!]]\!] = \frac{1}{2}[\![[\![R, R]\!], f]\!] = 0.$$

Therefore, $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$ is a differential graded Lie algebra.

Let R' be a linear map from $\mathfrak{g}$ to $\mathfrak{g}$. Then $R + R'$ is a modified r-matrix if and only if

$$[\![R + R', R + R']\!] = 2\pi,$$

that is

$$0 = [\![R, R']\!] + \frac{1}{2}[\![R', R']\!] = \mathrm{d}_R R' + \frac{1}{2}[\![R', R']\!].$$

Thus $R + R'$ is still a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ if and only if R' is a Maurer–Cartan element of the differential graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$. ■

At the end of this section, we establish the relationship between the coboundary operator $\mathrm{d}_{\mathsf{CE}}^R$ and the differential d_R.

Proposition 3.4. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then we have*

$$\mathrm{d}_{\mathsf{CE}}^R(f) = (-1)^{n-1}[\![R, f]\!], \quad \forall f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}).$$

Proof. For any $f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g})$ and x_i, $1 \le i \le n$, by (6), we have

$$\begin{aligned}
&(-1)^{n-1}[\![R, f]\!](x_1, \dots, x_n)\\
&\quad = (-1)^{n-1}\Big(\sum_{\sigma \in S(n-1,1)} (-1)^{\sigma} R([f(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}), x_{\sigma(n)}]_{\mathfrak{g}})\\
&\qquad - (-1)^{n-1} \sum_{\sigma \in S(1,1,n-2)} (-1)^{\sigma} f([R(x_{\sigma(1)}), x_{\sigma(2)}]_{\mathfrak{g}}, x_{\sigma(3)}, \dots, x_{\sigma(n)})\\
&\qquad + (-1)^{n-1} \sum_{\sigma \in S(1,n-1)} (-1)^{\sigma} [R(x_{\sigma(1)}), f(x_{\sigma(2)}, \dots, x_{\sigma(n)})]_{\mathfrak{g}}\Big)\\
&\quad = \sum_{i=1}^{n} (-1)^{i+1} R([f(x_1, \dots, \hat{x}_i, \dots, x_n), x_i]_{\mathfrak{g}})\\
&\qquad + \sum_{i<j} (-1)^{i+j} \big(f([R(x_i), x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n)\\
&\qquad - f([R(x_j), x_i]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n)\big)\\
&\qquad + \sum_{i=1}^{n} (-1)^{i+1} [R(x_i), f(x_1, \dots, \hat{x}_i, \dots, x_n)]_{\mathfrak{g}}\\
&\quad = \mathrm{d}_{\mathsf{CE}}^R(f)(x_1, \dots, x_n).
\end{aligned}$$

We finish the proof. ■

4. Geometric deformations of modified r-matrices

In this section, we study geometric deformations of modified r-matrices following the approach developed by Crainic, Schatz and Struchiner. We show that the condition $H^2(R) = 0$ will imply a certain rigidity theorem, and the condition $H^3(R) = 0$ will imply the space of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R. We also give the necessary and sufficient condition on a 2-cocycle giving a geometric deformation using the Kuranishi map.

Definition 4.1. Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. A *geometric deformation* of R is a smooth one parameter family of modified r-matrices R_t on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ such that $R_0 = R$.

Definition 4.2. Two geometric deformations R_t and R'_t of R are called *equivalent* if there exists a smooth family of modified r-matrices isomorphisms $\varphi_t : R_t \to R'_t$ such that $\varphi_0 = \mathrm{Id}$, where φ_t are inner automorphisms of the Lie algebra $\mathfrak{g}$.

Let R_t be a geometric deformation of R. Denote $\frac{d}{dt}|_{t=0} R_t$ by $\dot{R}_0$. Then there is the following proposition.

Proposition 4.3. *With the above notations, $\dot{R}_0$ is a 2-cocycle in $C^2(R)$. Moreover, if R_t and R'_t are equivalent geometric deformations of R, then $[\dot{R}_0] = [\dot{R}'_0]$ in $H^2(R)$.*

Proof. Since R_t is a geometric deformation of R, for any $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}
&[R(x), \dot{R}_0(y)]_\mathfrak{g} + [\dot{R}_0(x), R(y)]_\mathfrak{g}\\
&\quad = \frac{d}{dt}|_{t=0}[R_t(x), R_t(y)]_\mathfrak{g}\\
&\quad = \frac{d}{dt}|_{t=0}(R_t([R_t(x), y]_\mathfrak{g} + [x, R_t(y)]_\mathfrak{g}) - [x, y]_\mathfrak{g})\\
&\quad = \dot{R}_0([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}) + R([\dot{R}_0(x), y]_\mathfrak{g} + [x, \dot{R}_0(y)]_\mathfrak{g}).
\end{aligned} \tag{8}$$

Thus by (5) and (8), we have $\mathrm{d}^R_{\mathsf{CE}}(\dot{R}_0) = 0$.

Assume that φ_t is an isomorphism from R_t to R'_t, that is

$$\varphi_t(R_t(x)) = R'_t(\varphi_t(x)), \quad \forall x \in \mathfrak{g}.$$

Denote $\frac{d}{dt}|_{t=0}\varphi_t$ by $\dot{\varphi}_0$. Then we have $\dot{\varphi}_0(R(x)) + \dot{R}_0(x) = R(\dot{\varphi}_0(x)) + \dot{R}'_0(x)$. Since φ_t are inner automorphisms of the Lie algebra $\mathfrak{g}$, it follows that $\dot{\varphi}_0$ is an inner derivation of the Lie algebra $\mathfrak{g}$. Thus there exists $y \in \mathfrak{g}$ such that $\dot{\varphi}_0 = \mathrm{ad}_y$. Therefore, we have

$$[y, R(x)]_\mathfrak{g} + \dot{R}_0(x) = R([y, x]_\mathfrak{g}) + \dot{R}'_0(x),$$

which implies $\dot{R}_0 - \dot{R}'_0 = \mathrm{d}^R_{\mathsf{CE}}(y)$. Thus $[\dot{R}_0] = [\dot{R}'_0]$ in $H^2(R)$. ∎

Next, we consider under which conditions does a cocycle $f \in Z^2(R)$ determine a geometric deformation R_t. Define the Kuranishi map $K : Z^2(R) \to H^3(R)$ by

$$K(f) = \big[[\![f, f]\!]\big], \quad \forall f \in Z^2(R).$$

Now we give a necessary condition of the above question. The sufficient condition need some preparations and will be given at the end of this section.

Proposition 4.4. *Assume that there exists a geometric deformation R_t of R on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ such that $\dot{R}_0 = f \in Z^2(R)$, then $K(f) = 0$.*

Proof. Consider the Taylor expansion of R_t around $t = 0$, then we have

$$R_t(x) = R(x) + tf(x) + \frac{t^2}{2}g(x) + o(t^3).$$

Since $[R_t(x), R_t(y)]_{\mathfrak{g}} = R_t([R_t(x), y]_{\mathfrak{g}} + [x, R_t(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}$ and $f \in Z^2(R)$, we have

$$\frac{t^2}{2}\big(\mathrm{d}^R_{\mathsf{CE}}(g)(x, y) + 2[f(x), f(y)]_{\mathfrak{g}} - 2f([f(x), y]_{\mathfrak{g}} + [x, f(y)]_{\mathfrak{g}})\big) + o(t^3) = 0. \quad (9)$$

Thus by (6) and (9), we obtain $[\![f, f]\!] = \mathrm{d}^R_{\mathsf{CE}}g$, which implies that $K(f) = 0$. ■

Let $E \xrightarrow{\pi} M$ be a vector bundle. Assume that there is a smooth action $\cdot : G \times E \to E$ of a Lie group G on E preserving the zero-section $Z : M \to E$. It follows that M inherits a G-action. We also denote the action of G on M by $\cdot : G \times M \to M$. For all $x \in M$, define a smooth map $\mu_x : G \to M$ by $\mu_x(g) = g \cdot x$. Denote the tangent map from $\mathfrak{g}$ to T_xM by $D(\mu_x)_{e_G}$, where e_G is the unit of G.

Definition 4.5 ([5]). A section $s : M \to E$ is called *equivariant* if s satisfies

$$s(g \cdot x) = g \cdot s(x), \quad \forall g \in G, \; x \in M.$$

Denote the zero set of a section $s : M \to E$ by $z(s) = \{x \in M \mid s(x) = 0\}$. A zero $x \in M$ of s is called *non-degenerate* if the sequence

$$\mathfrak{g} \xrightarrow{D(\mu_x)_{e_G}} T_xM \xrightarrow{D^v(s)_x} E_x$$

is exact, where $D^v(s)_x$ is the vertical derivative of s at x.

Proposition 4.6 ([5]). *Let s be an equivariant section of the vector bundle $E \xrightarrow{\pi} M$ and x be a non-degenerate zero of s. Then there is an open neighborhood U of x and a smooth map $p : U \to G$ such that for all $m \in U$ with $s(m) = 0$, one has $p(m) \cdot x = m$. In particular, the orbit of x under the action of G and the zero set of s coincide in an open neighborhood of x.*

Proposition 4.7 ([5]). *Let E and F be vector bundles over a smooth manifold M. Let $s \in \Gamma(E)$ be a section and $\phi \in \Gamma(\mathrm{Hom}(E, F))$ be a vector bundle map such that $\phi \circ s = 0$. Suppose that $x \in M$ is $s(x) = 0$ such that*

$$T_x M \xrightarrow{D^v(s)_x} E_x \xrightarrow{\phi_x} F_x$$

is exact. Then $s^{-1}(0)$ is locally a manifold around x of dimension $\dim \ker(D^v(s)_x)$.

Denote the group whose elements are inner automorphisms of a Lie algebra $\mathfrak{g}$ by $\mathrm{InnAut}(\mathfrak{g})$. Then its Lie algebra is the Lie algebra of inner derivations of $\mathfrak{g}$ and denote it by $\mathrm{InnDer}(\mathfrak{g})$. Define an action of $\mathrm{InnAut}(\mathfrak{g})$ on $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ by

$$\cdot : \mathrm{InnAut}(\mathfrak{g}) \times \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}), \quad A \cdot f = AfA^{-1},$$

for all $A \in \mathrm{InnAut}(\mathfrak{g})$, $f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$. Assume that R is a modified r-matrix on a Lie algebra $\mathfrak{g}$, then the orbit $\mathrm{Orb}_R = \{A \cdot R \mid A \in \mathrm{InnAut}(\mathfrak{g})\}$ of R is a manifold. Define a map $\mu_R : \mathrm{InnAut}(\mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ by $\mu_R(A) = A \cdot R$. Then $T_R \mathrm{Orb}_R$ is $D(\mu_R)_{e_G}(\mathrm{InnDer}(\mathfrak{g}))$, where $D(\mu_R)_{e_G}$ is the tangent map of μ_R at e_G.

Proposition 4.8. *With the above notations, $T_R \mathrm{Orb}_R$ is $B^2(R)$.*

Proof. Since $T_R \mathrm{Orb}_R = D(\mu_R)_{e_G}(\mathrm{InnDer}(\mathfrak{g}))$, for any $v \in T_R \mathrm{Orb}_R$, there exists $x \in \mathfrak{g}$ such that

$$\begin{aligned} v &= \frac{d}{dt}|_{t=0} \exp(t \mathrm{ad}_x) \cdot R \\ &= \frac{d}{dt}|_{t=0} (\exp(t \mathrm{ad}_x) R \exp(-t \mathrm{ad}_x)) \\ &= \mathrm{ad}_x R - R \mathrm{ad}_x \\ &= -\mathrm{d}^R_{\mathsf{CE}} x. \end{aligned}$$

Thus we have $T_R \mathrm{Orb}_R = B^2(R)$. ■

Theorem 4.9. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. If $H^2(R) = 0$, then there exists an open neighborhood $U \subset \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ of R and a smooth map $p : U \to \mathrm{InnAut}(\mathfrak{g})$ such that $p(R') \cdot R = R'$ for every modified r-matrix $R' \in U$.*

Proof. Denote by $M = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ and $E = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$. Then E is a trivial vector bundle over M with fiber $\mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$. Define an action of $\mathrm{InnAut}(\mathfrak{g})$ on the manifold E by

$$\cdot : \mathrm{InnAut}(\mathfrak{g}) \times E \to E, \quad A \cdot (f, \alpha) = (AfA^{-1}, A\alpha \circ A^{-1}),$$

for $(f, \alpha) \in E$, $A \in \mathrm{InnAut}(\mathfrak{g})$, where $A\alpha \circ A^{-1}(x, y) = A\alpha(A^{-1}x, A^{-1}y)$ for any $x, y \in \mathfrak{g}$. Define a section $s : M \to E$ by

$$s(f) = (f, S(f)), \quad \forall f \in M,$$

where $S : \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \to \mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g})$ is given by

$$S(f)(x,y) = [f(x), f(y)]_{\mathfrak{g}} - f([f(x),y]_{\mathfrak{g}} + [x,f(y)]_{\mathfrak{g}}) + [x,y]_{\mathfrak{g}},$$

for all $f \in \mathrm{Hom}(\mathfrak{g},\mathfrak{g})$, $x, y \in \mathfrak{g}$. Then for any $A \in \mathrm{InnAut}(\mathfrak{g})$, $f \in M$ and $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} AS(f)\circ A^{-1}(x,y) &= A([f(A^{-1}x), f(A^{-1}y)]_{\mathfrak{g}} - f([f(A^{-1}x), A^{-1}y]_{\mathfrak{g}} \\ &\quad + [A^{-1}x, f(A^{-1}y)]_{\mathfrak{g}}) + [A^{-1}x, A^{-1}y]_{\mathfrak{g}}) \\ &= [Af(A^{-1}x), Af(A^{-1}y)]_{\mathfrak{g}} - AfA^{-1}([Af(A^{-1}x), y]_{\mathfrak{g}} \\ &\quad + [x, Af(A^{-1}y)]_{\mathfrak{g}}) + [x,y]_{\mathfrak{g}} \\ &= S(AfA^{-1})(x,y). \end{aligned}$$

Thus we have

$$A \cdot s(f) = (AfA^{-1}, AS(f)\circ A^{-1}) = s(A\cdot f),$$

which implies that s is an equivariant section.

Since R is a modified r-matrix on the Lie algebra $\mathfrak{g}$, it follows that $R \in z(s)$. Moreover, since E is a trivial vector bundle, we have $D^v(s)_R = D(S)_R : T_R M \to E_R$. For any $g \in \mathrm{Hom}(\mathfrak{g},\mathfrak{g})$, $x, y \in \mathfrak{g}$,

$$\begin{aligned} D(S)_R(g)(x,y) &= \frac{d}{dt}|_{t=0} S(R+tg)(x,y) \\ &= \frac{d}{dt}|_{t=0}\Big([R(x)+tg(x), R(y)+tg(y)]_{\mathfrak{g}} \\ &\quad - (R+tg)([R(x)+tg(x), y]_{\mathfrak{g}} + [x, R(y)+tg(y)]_{\mathfrak{g}}) + [x,y]_{\mathfrak{g}}\Big) \\ &= [g(x), R(y)]_{\mathfrak{g}} + [R(x), g(y)]_{\mathfrak{g}} - R([g(x),y]_{\mathfrak{g}} + [x, g(y)]_{\mathfrak{g}}) \\ &\quad - g([R(x),y]_{\mathfrak{g}} + [x,R(y)]_{\mathfrak{g}}) \\ &= \mathrm{d}^R_{\mathsf{CE}}(g)(x,y). \end{aligned} \tag{10}$$

By Proposition 4.8 and $H^2(R) = 0$, we have that R is a non-degenerate zero of s. By Proposition 4.6, there exists an open neighborhood $U \subset \mathrm{Hom}(\mathfrak{g},\mathfrak{g})$ of R and a smooth map $p : U \to \mathrm{InnAut}(\mathfrak{g})$ such that $p(R)\cdot R = R'$ for every modified r-matrix $R' \in U$. ∎

Theorem 4.10. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. If $H^3(R) = 0$, then the space of modified matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R whose dimension is* $\dim Z^2(R)$.

Proof. Denote by $M = \mathrm{Hom}(\mathfrak{g},\mathfrak{g})$, $E = \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \times \mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g})$ and $F = \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \times \mathrm{Hom}(\wedge^3\mathfrak{g},\mathfrak{g})$. Then E and F are trivial vector bundles over M with fiber $\mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g})$ and $\mathrm{Hom}(\wedge^3\mathfrak{g},\mathfrak{g})$ respectively. Define a smooth map $\phi : E \to F$ by

$$\phi(f,\alpha) = (f, [\![f,\alpha]\!]), \quad \forall f \in M,\ \alpha \in \mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g}).$$

Thus ϕ is a vector bundle map.

Moreover, denote the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ by π, define $s(f) = \pi - \frac{1}{2}[\![f, f]\!]$. By Proposition 3.2, we know that π lies in the center, we have

$$\phi \circ s(f) = \Big(f, [\![f, \pi]\!] - \frac{1}{2}[\![f, [\![f, f]\!]]\!]\Big) = (f, 0),$$

which implies $\phi \circ s = 0$. Moreover, denote $\phi_R : E_R \to F_R$ by $\phi(R, \cdot)$, then $\phi_R = \mathrm{d}^R_{\mathsf{CE}}$. By (10) and $H^3(R) = 0$, we have that

$$T_R M \xrightarrow{D^v(s)_R} E_R \xrightarrow{\phi_R} F_R$$

is exact. By Proposition 4.7, we obtain that the space of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R, whose dimension is $\dim Z^2(R)$. ■

At the end of this section, we give the sufficient condition on a 2-cocycle to give a geometric deformation. Recall that the necessary condition is given in Proposition 4.4 using the Kuranishi map.

Corollary 4.11. *With the above notations, if $H^3(R) = 0$, then any $f \in Z^2(R)$ gives rise to a geometric deformation of R.*

Proof. Since R is a modified r-matrix and $H^3(R) = 0$, we have that the space W of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R, whose dimension is $\dim Z^2(R)$. Assume $\gamma(t) \in W$, by Proposition 4.3, we have $\dot{\gamma}(0) \in Z^2(R)$. Moreover, $\dim W = \dim Z^2(R)$, then $T_R W = Z^2(R)$. Thus any $f \in Z^2(R)$ gives rise to a geometric deformation of R. ■

5. Linear deformations of modified r-matrices

In this section, we study linear deformations of a modified r-matrix using the established cohomology theory. In particular, a trivial linear deformation leads to a Nijenhuis element for a modified r-matrix R.

Definition 5.1. Let R be a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $\widehat{R} : \mathfrak{g} \to \mathfrak{g}$ be a linear map. If there exists a positive number $\varepsilon \in \mathbb{R}$ such that $R_t = R + t\widehat{R}$ is still a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ for all $t \in (-\varepsilon, \varepsilon)$, we say that $\widehat{R}$ generates a *linear deformation* of the modified r-matrix R.

Definition 5.2. Let $R : \mathfrak{g} \to \mathfrak{g}$ be a modified r-matrix on $\mathfrak{g}$. Two linear deformations $R^1_t = R + t\widehat{R}_1$ and $R^2_t = R + t\widehat{R}_2$ are said to be *equivalent* if there exists an $x \in \mathfrak{g}$ such that

$$\varphi_t = \mathrm{Id}_{\mathfrak{g}} + t\,\mathrm{ad}_x,$$

satisfies the following conditions:

(i) $\varphi_t([y, z]_{\mathfrak{g}}) = [\varphi_t(y), \varphi_t(z)]_{\mathfrak{g}}, \forall y, z \in \mathfrak{g}$,

(ii) $R^2_t \circ \varphi_t = \varphi_t \circ R^1_t$.

Theorem 5.3. *Let $\widehat{R} : \mathfrak{g} \to \mathfrak{g}$ generate a linear deformation of the modified r-matrix R. Then $\widehat{R}$ is a 2-cocycle.*

Let R^1_t and R^2_t be equivalent linear deformations of R generated by $\widehat{R}_1$ and $\widehat{R}_2$ respectively. Then $[\widehat{R}_1] = [\widehat{R}_2]$ in $H^2(R)$.

Proof. Since $R_t = R + t\widehat{R}$ is a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$, we have

$$[R_t(x), R_t(y)]_\mathfrak{g} = R_t([R_t(x), y]_\mathfrak{g} + [x, R_t(y)]_\mathfrak{g}) - [x, y]_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g}.$$

Consider the coefficients of t and t^2 respectively, we have

$$\begin{aligned}&[\widehat{R}(x), R(y)]_\mathfrak{g} + [R(x), \widehat{R}(y)]_\mathfrak{g}\\ &\quad = R([\widehat{R}(x), y]_\mathfrak{g} + [x, \widehat{R}(y)]_\mathfrak{g}) + \widehat{R}([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}), \quad \forall x, y \in \mathfrak{g}, \quad (11)\end{aligned}$$

and

$$[\widehat{R}(x), \widehat{R}(y)]_\mathfrak{g} = \widehat{R}([\widehat{R}(x), y]_\mathfrak{g} + [x, \widehat{R}(y)]_\mathfrak{g}). \quad (12)$$

By (11), we deduce that $\widehat{R}$ is a 2-cocycle of the modified r-matrix R.

If R^1_t and R^2_t are equivalent linear deformations of R, then there exists $x \in \mathfrak{g}$ such that

$$(\mathrm{Id}_\mathfrak{g} + t\mathrm{ad}_x)(R + t\widehat{R}_1)(u) = (R + t\widehat{R}_2)(\mathrm{Id}_\mathfrak{g} + t\mathrm{ad}_x)(u), \quad \forall u \in \mathfrak{g},$$

which implies

$$\widehat{R}_1(u) - \widehat{R}_2(u) = [R(u), x]_\mathfrak{g} - R([u, x]_\mathfrak{g}), \quad \forall u \in \mathfrak{g}. \quad (13)$$

By (13), we have

$$\widehat{R}_1 - \widehat{R}_2 = \mathrm{d}^R_{\mathsf{CE}} x,$$

where $\mathrm{d}^R_{\mathsf{CE}}$ is given by (4). Thus $[\widehat{R}_1] = [\widehat{R}_2]$ in $H^2(R)$. ∎

Definition 5.4. A linear deformation of a modified r-matrix R generated by $\widehat{R}$ is *trivial* if there exists an $x \in \mathfrak{g}$ such that $\mathrm{Id} + t\mathrm{ad}_x$ is an isomorphism from $R_t = R + t\widehat{R}$ to R.

Definition 5.5. Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. An element $x \in \mathfrak{g}$ is called a *Nijenhuis element* associated to R if x satisfies

$$[[x, y]_\mathfrak{g}, [x, z]_\mathfrak{g}]_\mathfrak{g} = 0, \quad (14)$$

$$[x, [x, R(y)]_\mathfrak{g}]_\mathfrak{g} = [x, R([x, y]_\mathfrak{g})]_\mathfrak{g}, \quad (15)$$

for all $y, z \in \mathfrak{g}$.

Let $\widehat{R}$ generate a trivial linear deformation of a modified r-matrix R on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. Then there exists $x \in \mathfrak{g}$ such that

$$\begin{aligned}(\mathrm{Id} + t\mathrm{ad}_x)[y, z]_\mathfrak{g} &= [y + t[x, y]_\mathfrak{g}, z + t[x, z]_\mathfrak{g}]_\mathfrak{g},\\ R(y + t[x, y]_\mathfrak{g}) &= (\mathrm{Id} + t\mathrm{ad}_x)(R(y) + t\widehat{R}(y)),\end{aligned}$$

for all $y,z\in\mathfrak{g}$. Therefore, we have

$$[[x,y]_\mathfrak{g},[x,z]_\mathfrak{g}]_\mathfrak{g}=0,\quad [x,\widehat{R}(y)]_\mathfrak{g}=0,\quad R([x,y]_\mathfrak{g})=[x,R(y)]_\mathfrak{g}+\widehat{R}(y).$$

Thus a trivial linear deformation gives rise to a Nijenhuis element.

Theorem 5.6. *Let R be a modified r-matrix on a Lie algebra $\mathfrak{g}$. Then for any Nijenhuis element $x\in\mathfrak{g}$, $R_t=R+t\mathrm{d}_{\mathsf{CE}}^R x$ is a trivial linear deformation of the modified r-matrix R.*

Proof. Denote by $\widehat{R}=\mathrm{d}_{\mathsf{CE}}^R x$. To show that R_t is a linear deformation of R, it suffices to show that (11) and (12) hold. Note that (11) means that $\widehat{R}$ is closed, which holds naturally since now $\widehat{R}=\mathrm{d}_{\mathsf{CE}}^R x$ is exact. Thus, we need to verify that equation (12) holds. For any $y,z\in\mathfrak{g}$, by (4), we obtain $\widehat{R}(y)=[R(y),x]_\mathfrak{g}-R([y,x]_\mathfrak{g})$. Moreover, by (1), (14) and (15), it follows that

$$\begin{aligned}
&[R([y,x]_\mathfrak{g}),R([z,x]_\mathfrak{g})]_\mathfrak{g}\\
&\overset{(1),(14)}{=}R\big([R([y,x]_\mathfrak{g}),[z,x]_\mathfrak{g}]_\mathfrak{g}+[[y,x]_\mathfrak{g},R([z,x]_\mathfrak{g})]_\mathfrak{g}\big)\\
&\quad=\ R\big([R([y,x]_\mathfrak{g}),[z,x]_\mathfrak{g}]\big)+R\big([[y,x]_\mathfrak{g},R([z,x]_\mathfrak{g})]_\mathfrak{g}\big)\\
&\quad=\ R\big([[R([y,x]_\mathfrak{g}),z]_\mathfrak{g},x]_\mathfrak{g}\big)+R\big([z,[R([y,x]_\mathfrak{g}),x]_\mathfrak{g}]\big)\\
&\qquad+R\big([[y,R([z,x]_\mathfrak{g})]_\mathfrak{g},x]_\mathfrak{g}\big)+R\big([y,[x,R([z,x]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}\big)\\
&\overset{(14),(15)}{=}R\big([[R([y,x]_\mathfrak{g}),z]_\mathfrak{g},x]_\mathfrak{g}\big)+R\big([x,[z,[x,R(y)]_\mathfrak{g}]_\mathfrak{g}]\big)\\
&\qquad+R\big([[y,R([z,x]_\mathfrak{g})]_\mathfrak{g},x]_\mathfrak{g}\big)-R\big([x,[y,[x,R(z)]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g}\big),\\
&-[[R(y),x]_\mathfrak{g},R([z,x]_\mathfrak{g})]_\mathfrak{g}\\
&\quad=\ -[[R(y),R([z,x]_\mathfrak{g})]_\mathfrak{g},x]_\mathfrak{g}-[R(y),[x,R([z,x]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}\\
&\quad\overset{(1)}{=}\ -[R([R(y),[z,x]_\mathfrak{g}]_\mathfrak{g}),x]_\mathfrak{g}-[R([y,R([z,x]_\mathfrak{g})]_\mathfrak{g}),x]_\mathfrak{g}\\
&\qquad+[[y,[z,x]_\mathfrak{g}]_\mathfrak{g},x]_\mathfrak{g}-[R(y),[x,R([z,x]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}\\
&\overset{(14),(15)}{=}-[x,[x,R([R(y),z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}-[R([z,[R(y),x]_\mathfrak{g}]_\mathfrak{g}),x]_\mathfrak{g}\\
&\qquad-[R([y,R([z,x]_\mathfrak{g})]_\mathfrak{g}),x]_\mathfrak{g}+[[y,[z,x]_\mathfrak{g}]_\mathfrak{g},x]_\mathfrak{g}+[x,[x,[R(y),R(z)]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g}\\
&\qquad+[[x,[R(y),x]_\mathfrak{g}]_\mathfrak{g},R(z)]_\mathfrak{g}\\
&\overset{(1),(15)}{=}[x,[x,R([y,R(z)]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}-[x,[x,[y,z]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g}+[[y,[z,x]_\mathfrak{g}]_\mathfrak{g},x]_\mathfrak{g}\\
&\qquad-[R([z,[R(y),x]_\mathfrak{g}]_\mathfrak{g}),x]_\mathfrak{g}-[R([y,R([z,x]_\mathfrak{g})]_\mathfrak{g}),x]_\mathfrak{g}\\
&\qquad-[[x,R([x,y]_\mathfrak{g}]_\mathfrak{g}),R(z)]_\mathfrak{g},
\end{aligned}$$

and

$$\begin{aligned}
&-[R([y,x]_\mathfrak{g}),[R(z),x]_\mathfrak{g}]_\mathfrak{g}\\
&\overset{(1)}{=}[R([[R(z),y]_\mathfrak{g},x]_\mathfrak{g}),x]_\mathfrak{g}+[R([y,[R(z),x]_\mathfrak{g}]_\mathfrak{g}),x]_\mathfrak{g}+[R([z,R([y,x]_\mathfrak{g})]_\mathfrak{g}),x]_\mathfrak{g}\\
&\quad-[[z,[y,x]_\mathfrak{g}]_\mathfrak{g},x]_\mathfrak{g}+[R(z),[x,R([y,x]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}.
\end{aligned}$$

By (14) and the above equations, we have

$$
\begin{aligned}
&[\hat{R}(y), \hat{R}(z)]_{\mathfrak{g}} - \hat{R}([\hat{R}(y), z]_{\mathfrak{g}} + [y, \hat{R}(z)]_{\mathfrak{g}}) \\
&\quad = [[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} \\
&\qquad - [R([[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z]), x]_{\mathfrak{g}} + R([[[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
&\qquad - [R([y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} + R([[y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
&\quad = -[[R(y), x]_{\mathfrak{g}}, R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} + [R([y, x]_{\mathfrak{g}}), R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}} \\
&\qquad - [R([y, x]_{\mathfrak{g}}), [R(z), x]_{\mathfrak{g}}]_{\mathfrak{g}} - [R([[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z]), x]_{\mathfrak{g}} \\
&\qquad + R([[[R(y), x]_{\mathfrak{g}} - R([y, x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
&\qquad - [R([y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}} + R([[y, [R(z), x]_{\mathfrak{g}} - R([z, x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}) \\
&\quad = 0.
\end{aligned}
$$

Thus $R_t = R + t\mathrm{d}^R_{\mathsf{CE}}x$ is a linear deformation of the modified r-matrix R. Since $x \in \mathfrak{g}$ is a Nijenhuis element, we have $(\mathrm{Id} + t\mathrm{ad}_x)[y, z]_{\mathfrak{g}} = [y + t[x, y]_{\mathfrak{g}}, z + t[x, z]_{\mathfrak{g}}]_{\mathfrak{g}}$ and $R \circ (\mathrm{Id} + t\mathrm{ad}_x) = (\mathrm{Id} + t\mathrm{ad}_x) \circ (R + t\mathrm{d}^R_{\mathsf{CE}}x)$. Thus for any Nijenhuis element $x \in \mathfrak{g}$, $R_t = R + t\mathrm{d}^R_{\mathsf{CE}}x$ is a trivial linear deformation of the modified r-matrix R. ■

At the end of this section, we consider the relation between linear deformations of modified r-matrices and linear deformations of the induced Lie algebras. Recall that a skew-symmetric bilinear map $\omega : \wedge^2\mathfrak{g} \to \mathfrak{g}$ generates a linear deformation of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ if $[\cdot,\cdot]_t = [\cdot,\cdot]_{\mathfrak{g}} + t\omega$ defines a Lie algebra structure on $\mathfrak{g}$ for all $t \in (-\varepsilon, \varepsilon)$.

Proposition 5.7. *Let $\hat{R}$ generate a linear deformation of a modified r-matrix R on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then ω defined by*

$$\omega(x, y) = [\hat{R}(x), y]_{\mathfrak{g}} + [x, \hat{R}(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g},$$

generates a linear deformation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ given by the modified r-matrix R, which is exactly the one associated to the linear deformation of the modified r-matrix R.

Proof. It is obvious that

$$[x, y]_{R_t} = [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}} + t([\hat{R}(x), y]_{\mathfrak{g}} + [x, \hat{R}(y)]_{\mathfrak{g}}) = [x, y]_R + t\omega(x, y).$$

Since $[\cdot,\cdot]_{R_t}$ are Lie algebra structures, we have that ω generates a linear deformation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ given by the modified r-matrix R. ■

The notion of a Nijenhuis operator on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ was given in [7], which gives rise to a trivial linear deformation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

Definition 5.8 ([7]). Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $N : \mathfrak{g} \to \mathfrak{g}$ is called *Nijenhuis operator* if

$$[N(x), N(y)]_{\mathfrak{g}} = N([N(x), y]_{\mathfrak{g}} + [x, N(y)]_{\mathfrak{g}}) - N^2([x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}.$$

Theorem 5.9. *Let $x \in \mathfrak{g}$ be a Nijenhuis element associated to a modified matrix R. Then ad_x is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$.*

Proof. For any $x, y, z \in \mathfrak{g}$, by (14) and (15), we have

$$\begin{aligned}
&[\mathrm{ad}_x y, \mathrm{ad}_x z]_R \\
&\quad = [R([x,y]_\mathfrak{g}), [x,z]_\mathfrak{g}]_\mathfrak{g} + [[x,y]_\mathfrak{g}, R([x,z]_\mathfrak{g})]_\mathfrak{g} \\
&\quad = [[R([x,y]_\mathfrak{g}), x]_\mathfrak{g}, z]_\mathfrak{g} + [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [[x, R([x,z]_\mathfrak{g})]_\mathfrak{g}, y]_\mathfrak{g} \\
&\qquad + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g} \\
&\quad = -[[x,[x,R(y)]_\mathfrak{g}]_\mathfrak{g}, z]_\mathfrak{g} + [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [[x,[x,R(z)]_\mathfrak{g}]_\mathfrak{g}, y]_\mathfrak{g} \\
&\qquad + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g} \\
&\quad = -[x, [[x,R(y)]_\mathfrak{g}, z]_\mathfrak{g}]_\mathfrak{g} + [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [x, [[x,R(z)]_\mathfrak{g}, y]_\mathfrak{g}]_\mathfrak{g} \\
&\qquad + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{ad}_x([\mathrm{ad}_x y, z]_R + [y, \mathrm{ad}_x z]_R) - \mathrm{ad}_x^2([y,z]_R) \\
&\quad = [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [[x,y]_\mathfrak{g}, R(z)]_\mathfrak{g} + [x, [R(y), [x,z]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g} \\
&\qquad + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g} - [x, [x, [R(y), z]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g} - [x, [x, [y, R(z)]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g}.
\end{aligned}$$

Thus $[\mathrm{ad}_x y, \mathrm{ad}_x z]_R = \mathrm{ad}_x([\mathrm{ad}_x y, z]_R + [y, \mathrm{ad}_x z]_R) - \mathrm{ad}_x^2([y,z]_R)$, which implies that ad_x is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$. ■

6. Applications

In this section, we give some applications of the above deformation theories, including deformations of complements of the diagonal Lie algebra $\mathfrak{g}_\Delta$ and compatible Poisson structures.

6.1. Deformations of complements

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra, then we have a direct-product Lie algebra structure $[\cdot,\cdot]_\oplus$ on $\mathfrak{g} \oplus \mathfrak{g}$, where

$$[(x_1, y_1), (x_2, y_2)]_\oplus = ([x_1, x_2]_\mathfrak{g}, [y_1, y_2]_\mathfrak{g}), \quad \forall x_i, y_i \in \mathfrak{g},\ i = 1, 2.$$

Define the subspace $\mathfrak{g}_\Delta$ by $\mathfrak{g}_\Delta = \{(x,x) \mid \forall x \in \mathfrak{g}\}$ and the subspace $\mathfrak{g}_{-\Delta} = \{(x,-x) \mid \forall x \in \mathfrak{g}\}$. It is obvious that $\mathfrak{g}_\Delta$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$, while $\mathfrak{g}_{-\Delta}$ is not a Lie subalgebra. To find a complement of $\mathfrak{g}_\Delta$ which is also a Lie subalgebra, it is natural to consider the graph of certain linear map from $\mathfrak{g}_{-\Delta}$ to $\mathfrak{g}_\Delta$. It is known that a complement of $\mathfrak{g}_\Delta$ is isomorphic to a graph of a linear map from $\mathfrak{g}_{-\Delta}$ to $\mathfrak{g}_\Delta$. Let $R \in \mathfrak{gl}(\mathfrak{g})$ be a linear map. Define a linear map $\widehat{R} : \mathfrak{g}_{-\Delta} \to \mathfrak{g}_\Delta$ by

$$\widehat{R}(x, -x) = (-R(x), -R(x)), \quad \forall x \in \mathfrak{g}.$$

Proposition 6.1. *With the above notations, the graph* $\mathcal{G}(\widehat{R}) := \{\widehat{R}u + u \mid u \in \mathfrak{g}_{-\Delta}\}$ *is a Lie subalgebra of* $(\mathfrak{g} \oplus \mathfrak{g}, [\cdot,\cdot]_{\oplus})$ *if and only if* R *is a modified* r*-matrix.*

Proof. For all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}
&[(-R(x), -R(x)) + (x, -x), (-R(y), -R(y)) + (y, -y)]_{\oplus} \\
&\quad = ([x, y]_{\mathfrak{g}}, [x, y]_{\mathfrak{g}}) + ([R(x), R(y)]_{\mathfrak{g}}, [R(x), R(y)]_{\mathfrak{g}}) \\
&\qquad + (-[R(x), y]_{\mathfrak{g}}, [R(x), y]_{\mathfrak{g}}) + (-[x, R(y)]_{\mathfrak{g}}, [x, R(y)]_{\mathfrak{g}}) \\
&\quad = ([x, y]_{\mathfrak{g}} + [R(x), R(y)]_{\mathfrak{g}}, [x, y]_{\mathfrak{g}} + [R(x), R(y)]_{\mathfrak{g}}) \\
&\qquad + (-[R(x), y]_{\mathfrak{g}} - [x, R(y)]_{\mathfrak{g}}, [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}).
\end{aligned}$$

Thus $\mathcal{G}(\widehat{R})$ is a Lie subalgebra if and only if

$$R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}) = [x, y]_{\mathfrak{g}} + [R(x), R(y)]_{\mathfrak{g}},$$

i.e., R is a modified r-matrix. ■

Proposition 6.2. *Let* R *be a modified* r*-matrix. Then* $(\mathfrak{g}_{\Delta}, \mathcal{G}(\widehat{R}))$ *is a matched pair of Lie algebras.*

Proof. It is obvious that $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\Delta} \oplus \mathcal{G}(\widehat{R})$ since $\mathfrak{g}_{\Delta} \cap \mathcal{G}(\widehat{R}) = 0$. Then the conclusion follows from the fact that both $\mathfrak{g}_{\Delta}$ and $\mathcal{G}(\widehat{R})$ are Lie subalgebras. ■

Summarizing the above studies, we have the following conclusion.

Theorem 6.3. *Let* R_t *be a geometric deformation of a modified* r*-matrix* R*. Then* $\mathcal{G}(\widehat{R}_t)$ *is a deformation of the complement* $\mathcal{G}(\widehat{R})$*. Moreover,* $(\mathfrak{g}_{\Delta}, \mathcal{G}(\widehat{R}_t))$ *are matched pairs of Lie algebras.*

6.2. Compatible Poisson structures

A *compatible Poisson structure* consists of two Poisson structures π, π' on a manifold M such that $\pi + \pi'$ is also a Poisson structure on the manifold M.

Let R be a modified r-matrix on a Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g}, [\cdot,\cdot]_R)$ is a Lie algebra and we denote by $(\mathfrak{g}^*, \{\cdot,\cdot\}_R)$ the corresponding linear Poisson manifold.

Proposition 6.4. *Let* R *be a modified* r*-matrix on a Lie algebra* $\mathfrak{g}$ *and* $R_t = R + t\widehat{R}$ *be a linear deformation of* R*. For any* $t_1, t_2 \in \mathbb{R}$*,* $\{\cdot,\cdot\}_{R_{t_1}}$ *and* $\{\cdot,\cdot\}_{R_{t_2}}$ *are compatible Poisson structures on* $\mathfrak{g}^*$*.*

Proof. By the fact that $R + \frac{t_1+t_2}{2}\widehat{R}$ is also a modified r-matrix on the Lie algebra $\mathfrak{g}$, we have

$$[x, y]_{R_{t_1}} + [x, y]_{R_{t_2}} = 2\Big(\Big[R(x) + \frac{t_1 + t_2}{2}\widehat{R}(x), y\Big]_{\mathfrak{g}} + \Big[x, R(y) + \frac{t_1 + t_2}{2}\widehat{R}(y)\Big]\Big),$$

which implies that $[\cdot,\cdot]_{R_{t_1}} + [\cdot,\cdot]_{R_{t_2}}$ is also a Lie bracket on the Lie algebra $\mathfrak{g}$ by (2). Thus, for any $t_1, t_2 \in \mathbb{R}$, $\{\cdot,\cdot\}_{R_{t_1}}$ and $\{\cdot,\cdot\}_{R_{t_2}}$ are compatible Poisson structures on $\mathfrak{g}^*$. ■

Acknowledgments. We give warmest thanks to Jianghua Lu and Chenchang Zhu for helpful comments.

Funding. This research is supported by NSFC (11922110) and China Postdoctoral Science Foundation (2023M741349).

References

[1] C. Bai, L. Guo, and X. Ni, Nonabelian generalized Lax pairs, the classical Yang–Baxter equation and PostLie algebras. *Comm. Math. Phys.* **297** (2010), no. 2, 553–596 Zbl 1206.17020 MR 2651910

[2] M. Bordemann, Generalized Lax pairs, the modified classical Yang–Baxter equation, and affine geometry of Lie groups. *Comm. Math. Phys.* **135** (1990), no. 1, 201–216 Zbl 0714.58025 MR 1086757

[3] D. V. Borisov, Formal deformations of morphisms of associative algebras. *Int. Math. Res. Not.* (2005), no. 41, 2499–2523 Zbl 1126.16017 MR 2181057

[4] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.* **63** (1948), 85–124 Zbl 0031.24803 MR 0024908

[5] M. Crainic, F. Schätz, and I. Struchiner, A survey on stability and rigidity results for Lie algebras. *Indag. Math. (N.S.)* **25** (2014), no. 5, 957–976 Zbl 1328.17001 MR 3264783

[6] A. Das, Deformations of associative Rota–Baxter operators. *J. Algebra* **560** (2020), 144–180 Zbl 1458.16008 MR 4104452

[7] I. Dorfman, *Dirac structures and integrability of nonlinear evolution equations*. Nonlinear Science: Theory and Applications, John Wiley & Sons, Ltd., Chichester, 1993 MR 1237398

[8] K. Ebrahimi-Fard, Loday-type algebras and the Rota–Baxter relation. *Lett. Math. Phys.* **61** (2002), no. 2, 139–147 Zbl 1035.17001 MR 1936573

[9] K. Ebrahimi-Fard and I. Mencattini, Post-Lie algebras, factorization theorems and isospectral flows. In *Discrete mechanics, geometric integration and Lie–Butcher series*, pp. 231–285, Springer Proc. Math. Stat. 267, Springer, Cham, 2018 MR 3883647

[10] K. Ebrahimi-Fard, I. Mencattini, and H. Munthe-Kaas, Post-Lie algebras and factorization theorems. *J. Geom. Phys.* **119** (2017), 19–33 Zbl 1402.17021 MR 3661521

[11] Y. Frégier and M. Zambon, Simultaneous deformations and Poisson geometry. *Compos. Math.* **151** (2015), no. 9, 1763–1790 Zbl 1383.17009 MR 3406444

[12] Y. Frégier and M. Zambon, Simultaneous deformations of algebras and morphisms via derived brackets. *J. Pure Appl. Algebra* **219** (2015), no. 12, 5344–5362 Zbl 1356.17021 MR 3390025

[13] M. Gerstenhaber, The cohomology structure of an associative ring. *Ann. of Math. (2)* **78** (1963), 267–288 Zbl 0131.27302 MR 0161898

[14] M. Gerstenhaber, On the deformation of rings and algebras. *Ann. of Math. (2)* **79** (1964), 59–103 Zbl 0123.03101 MR 0171807

[15] L. Guo, H. Lang, and Y. Sheng, Integration and geometrization of Rota–Baxter Lie algebras. *Adv. Math.* **387** (2021), article no. 107834 Zbl 1468.17026 MR 4271483

[16] J. Jiang, Y. Sheng, and C. Zhu, Lie theory and cohomology of relative Rota–Baxter operators. *J. Lond. Math. Soc. (2)* **109** (2024), no. 2, article no. e12836 Zbl 07811234

[17] B. A. Kupershmidt, What a classical r-matrix really is. *J. Nonlinear Math. Phys.* **6** (1999), no. 4, 448–488 Zbl 1015.17015 MR 1722068

[18] L.-C. Li, Classical r-matrices and compatible Poisson structures for Lax equations on Poisson algebras. *Comm. Math. Phys.* **203** (1999), no. 3, 573–592 Zbl 0949.37053 MR 1700162

[19] J. Lurie, DAG X: Formal moduli problems. 2011, http://www.math.harvard.edu/~lurie/papers/DAG-X.pdf, visited on line 7 March 2024

[20] A. Nijenhuis and R. W. Richardson, Jr., Cohomology and deformations in graded Lie algebras. *Bull. Amer. Math. Soc.* **72** (1966), 1–29 Zbl 0136.30502 MR 0195995

[21] A. Nijenhuis and R. W. Richardson, Jr., Commutative algebra cohomology and deformations of Lie and associative algebras. *J. Algebra* **9** (1968), 42–53 Zbl 0175.31302 MR 0225830

[22] J. P. Pridham, Unifying derived deformation theories. *Adv. Math.* **224** (2010), no. 3, 772–826 Zbl 1195.14012 MR 2628795

[23] N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems. *J. Geom. Phys.* **5** (1988), no. 4, 533–550 (1989) Zbl 0711.17008 MR 1075721

[24] A. G. Reyman and M. A. Semenov-Tian-Shansky, Reduction of Hamiltonian systems, affine Lie algebras and Lax equations. *Invent. Math.* **54** (1979), no. 1, 81–100 Zbl 0403.58004 MR 0549548

[25] M. A. Semenov-Tian-Shansky, Integrable systems and factorization problems. In *Factorization and integrable systems (Faro, 2000)*, pp. 155–218, Oper. Theory Adv. Appl. 141, Birkhäuser, Basel, 2003 Zbl 1030.37048 MR 2021098

[26] M. A. Semenov-Tian-Shansky, What is a classical r-matrix? *Funct. Anal. Appl.* **17** (1983), 259–272. Translation from *Funktsional. Anal. i Prilozhen.* **17** (1983), no. 4, 17–33 Zbl 0535.58031 MR 0725413

[27] B. M. Szablikowski, Classical r-matrix like approach to Frobenius manifolds, WDVV equations and flat metrics. *J. Phys. A* **48** (2015), no. 31, article no. 315203 Zbl 1334.37082 MR 3371210

[28] B. M. Szablikowski and M. Błaszak, On deformations of standard R-matrices for integrable infinite-dimensional systems. *J. Math. Phys.* **46** (2005), no. 4, article no. 042702 Zbl 1067.37102 MR 2131246

[29] R. Tang, C. Bai, L. Guo, and Y. Sheng, Deformations and their controlling cohomologies of $\mathcal{O}$-operators. *Comm. Math. Phys.* **368** (2019), no. 2, 665–700 Zbl 1440.17015 MR 3949721

[30] X. Zhang, X. Gao, and L. Guo, Free modified Rota–Baxter algebras and Hopf algebras. *Int. Electron. J. Algebra* **25** (2019), 12–34 Zbl 1406.16034 MR 3904855

[31] X. Zhang, X. Gao, and L. Guo, Modified Rota–Baxter algebras, shuffle products and Hopf algebras. *Bull. Malays. Math. Sci. Soc.* **42** (2019), no. 6, 3047–3072 Zbl 1466.16039 MR 3999078

Received 12 April 2023; revised 19 June 2023.

Jun Jiang
Department of Mathematics, Jilin University, 130012 Changchun, P. R. China; junjiang@jlu.edu.cn

Yunhe Sheng
Department of Mathematics, Jilin University, 130012 Changchun, P. R. China;
shengyh@jlu.edu.cn