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Scale-invariant tangent-point energies for knots

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Abstract. We investigate minimizers and critical points for scale-invariant tangent-point energies $\mathrm{TP}^{p,q}$ of closed curves. We show that (a) minimizing sequences in ambient isotopy classes converge to locally critical embeddings at all but finitely many points and (b) locally critical embeddings are regular.

Technically, the convergence theory (a) is based on a gap estimate for fractional Sobolev spaces with respect to the tangent-point energy. The regularity theory (b) is based on constructing a new energy $\mathcal{E}^{p,q}$ and proving that the derivative γ' of a parametrization of a $\mathrm{TP}^{p,q}$-critical curve γ induces a critical map with respect to $\mathcal{E}^{p,q}$ acting on torus-to-sphere maps.

Keywords: knots, harmonic maps, regularity theory, existence.

1. Introduction and main results

When modeling and simulating topological effects in the sciences, like e.g. protein knotting, one has to make a choice of how to incorporate the avoidance of interpenetration of matter, i.e. self-intersections. Either one explicitly models partial differential equations that incorporate effects such as self-repulsion through penalization, or one constructs a comprehensive variational energy that includes self-repulsive behavior, and hopes that minimizing the energy (or following the steepest descent) delivers a realistic description. Several such self-repulsive energy functionals have been proposed and studied extensively over the last forty years (for an overview we refer the interested reader to [11, 76]; applications are discussed, e.g., in [1–3, 44, 61]) – and all have one thing in common: modeling topological resistance, i.e. self-repulsion, they are necessarily nonlocal functionals. Consequently, questions of most central interest such as existence and regularity

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of minimizing configurations are very challenging. This holds especially true for the geometrically most interesting case of *scale-invariant* knot energies, i.e. energies for which a curve γ and any scaled version $\lambda\gamma$ have the same value for any $\lambda > 0$.

O'Hara and Möbius knot energies

The first *knot energies* have been introduced by Fukuhara [28] and O'Hara [55–57], and are known as O'Hara energies. Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a parametrization of a closed regular Lipschitz curve, i.e., γ is both an immersion and an embedding. For any $\alpha p \geq 4$ and $p \geq 2$, the *O'Hara energy* $\mathcal{O}^{\alpha,p}(\cdot)$ is given by

$$\mathcal{O}^{\alpha,p}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \left(\frac{1}{|\gamma(x) - \gamma(y)|^\alpha} - \frac{1}{d_\gamma(\gamma(x), \gamma(y))^\alpha} \right)^{p/2} |\gamma'(x)|\, |\gamma'(y)|\, dx\, dy$$

where d_γ is the intrinsic distance on the submanifold $\gamma(\mathbb{R}/\mathbb{Z}) \subset \mathbb{R}^3$.

These energies are scale-invariant functionals if $\alpha p = 4$. In this scaling invariant regime, until recently, only the case of the *Möbius energy* $\mathcal{O}^{2,2}$ was understood at all. This was due to the celebrated work by Freedman, He, and Wang [27]. They discussed existence of minimizers within prime knot classes and established $C^{1,1}$-regularity of local minimizers. One can then bootstrap to smoothness [34] and even analyticity [15].[1]

The techniques employed in [27] by Freedman et al. crucially rely on the Möbius invariance of $\mathcal{O}^{2,2}$, and largely fall apart for $\mathcal{O}^{4/p,p}$ when $p \neq 2$ since Möbius invariance does not hold anymore [14]. Indeed, there was no progress on either existence or regularity of scale-invariant knot energies besides the Möbius energy for a long time, until in the two recent works [13, 14], three of the present authors established the regularity theory for all scale-invariant O'Hara energies $\mathcal{O}^{\alpha,p}$ (for critical points and minimizers) via a new approach. Namely, they showed that critical knots γ induce via their derivative γ' a sort of fractional harmonic map between $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ and $\mathbb{S}^2$. Then, extending the tools developed for fractional harmonic maps [24, 68], they obtained a regularity theory via arguments based on compensation effects and harmonic analysis.

Tangent-point energies for curves

In this work we are interested in scale-invariant *tangent-point* energies. As in the case of O'Hara energies, the scale-invariant situation is the most interesting and challenging one, and up to now it was completely out of reach. Due to the lack of Möbius invariance,[2] the geometric techniques of Freedman, He, and Wang [27] cannot be applied. Let us stress that Möbius invariance of an energy does not have any impact for applications and it

[1] Higher regularity via bootstrapping is possible because $p = 2$ implies the functional is a Hilbert-space functional; in particular, such arguments are independent of the presence of scale-invariance [10, 59, 60, 73, 79].

[2] As in the case of O'Hara energies [14], one can check this assertion by numerically computing the energy of a stadium curve before and after applying a Möbius transformation.

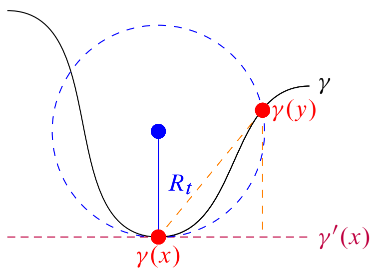

Fig. 1. The tangent-point radius R_t is the radius of the smallest sphere tangent to γ at $\gamma(x)$ and traversing $\gamma(y)$. It tends to zero when $\gamma(x) \to \gamma(y)$ while $\gamma(x)$ and $\gamma(y)$ belong to two different strands of γ. Its reciprocal converges to the local curvature as $y \to x$.

might be considered a curiosity mostly of geometric-topological interest. Actually, from the point of view of applications, one can argue that the tangent-point energies might be preferable to O'Hara energies because they are numerically simpler to compute [1–3, 61], and they have a natural generalization to embedded surfaces [75] which seems to be more convenient than higher-dimensional analogues of O'Hara-type energies [42, 58].

The "classical" tangent-point energy has been studied first by Buck and Orloff [21]. It amounts to the double integral over the reciprocal of

$$R_t(x, y) := \frac{|\gamma(x) - \gamma(y)|^2}{2\left|\frac{\gamma'(x)}{|\gamma'(x)|} \wedge (\gamma(x) - \gamma(y))\right|},$$

which is the smallest radius of a sphere passing through $\gamma(x)$ and $\gamma(y)$ while being tangential at $\gamma(x)$ see Figure 1. Later on, Gonzalez and Maddocks [31] obtained a family of energies by taking the integrand to suitable powers. Decoupling these powers as proposed in [12], we arrive at the two-parameter family

$$\begin{aligned} \mathrm{TP}^{p,q}(\gamma) &:= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\left|\frac{\gamma'(x)}{|\gamma'(x)|} \wedge (\gamma(x) - \gamma(y))\right|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)|\,|\gamma'(y)|\,dx\,dy \\ &= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)|^{1-q} |\gamma'(y)|\,dx\,dy \end{aligned}$$

for any embedded $\gamma \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$. As a standing assumption, we will always restrict the parameters to satisfy

$$q + 2 \le p < 2q + 1. \tag{1.1}$$

If $p \ge 2q + 1$, the energy is infinite even for some smooth diffeomorphisms. If $p < q + 2$, then $\mathrm{TP}^{p,q}$ is not self-repulsive. The subfamily studied by Gonzalez and Maddocks can be recovered by letting $p = 2q$; the Buck–Orloff functional corresponds to $\mathrm{TP}^{4,2}$. While the O'Hara energies $\mathcal{O}^{\alpha,p}$ for $\alpha \to 0$ and $p = 4/\alpha$ converge to Gromov's *distortion* functional [14, 32, 56], the tangent-point energies $\mathrm{TP}^{2q,q}$ converge to Federer's *reach* as $q \to \infty$ [26, 31].

Strzelecki and von der Mosel [74] obtained the first and so far only fundamental result concerning the scale-invariant case $p = q + 2$. They showed in particular that the images of curves with finite $\mathrm{TP}^{q+2,q}$-energy form a topological 1-manifold. However, this could be a nonsmooth object, e.g., a nondifferentiable curve (see Example 4.7) – or even worse: a doubly-traversed line which has zero energy (see Example 4.1)! So there is an issue with even defining the notion of *minimizing* embedded curves of the tangent-point energies. While the energy of the doubly-traversed line is zero and thus the global minimizer, it is certainly not a smooth manifold and therefore should not count as an acceptable minimizer. Let us remark that none of this was an issue for O'Hara energies which would be infinite on any periodic parametrization of a straight segment – *the tangent-point energies are more extrinsic than the O'Hara energies*. Lastly, let us mention that the Lagrangian in the tangent-point energies is at least formally related to the nonlocal mean curvature introduced by Caffarelli–Roquejoffre–Savin [22] and thus to the nonlocal Willmore energy recently discussed in [8, 29].

Main results

With the example of the doubly-traversed line in mind, in order to discuss minimizers in the class of knots (i.e. closed embedded curves), we restrict our interest to those curves which appear as limits of diffeomorphisms.

Let us introduce the localized energy for $A \subset \mathbb{R}/\mathbb{Z}$ by

$$\mathrm{TP}^{p,q}(\gamma; A) := |\gamma'|^{2-q} \int_A \int_A \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p}\, dx\, dy$$

where we assume γ to be parametrized by arclength, $|\gamma'| \equiv \text{const}$.

Following the spirit of an analogous strategy for Willmore surfaces [63, Definition I.1], we introduce the following terminology.

Definition 1.1 (Homeomorphisms with locally small tangent-point energy). A Lipschitz map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is called a *homeomorphism* (onto its image) *with locally ε-small tangent-point energy at $x \in \mathbb{R}/\mathbb{Z}$* if there exists an open interval $B_r(x) \subset \mathbb{R}/\mathbb{Z}$ and a sequence of C^1-homeomorphisms $\gamma_k : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with $|\gamma_k'| \equiv c > 0$ such that

(1) γ_k converges uniformly to γ on $\mathbb{R}/\mathbb{Z}$,

(2) $\sup_k \mathrm{TP}^{p,q}(\gamma_k; \mathbb{R}/\mathbb{Z}) < \infty$,

(3) $\sup_k \mathrm{TP}^{p,q}(\gamma_k; B_r(x)) < \varepsilon$ for some $r > 0$.

Our first main result states that *sequences of curves with uniformly bounded tangent-point energy converge to homeomorphisms with locally ε-small tangent-point energy outside of at most finitely many points*. More precisely, we will prove the following assertion.

Theorem 1.2. *Let $p = q + 2$, $q > 1$, $\Lambda > 0$, and $\varepsilon > 0$. Then there exists an integer $K = K(q, \varepsilon, \Lambda)$ such that any sequence $(\gamma_k)_{k\in\mathbb{N}} \subset C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ of closed embedded curves with*

$$\sup_{k\in\mathbb{N}} \mathrm{TP}^{q+2,q}(\gamma_k) < \Lambda$$

converges uniformly – after possibly translating, rescaling, and reparametrizing γ_k *and passing to a subsequence – to a Lipschitz map* $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ *with the following properties:*

- embeddedness: *γ is a bi-Lipschitz homeomorphism,*
- arclength parametrization: $|\gamma'(x)| = 1$ *for a.e.* $x \in \mathbb{R}/\mathbb{Z}$*,*
- lower semicontinuity: $\mathrm{TP}^{q+2,q}(\gamma) \leq \liminf_{k\to\infty} \mathrm{TP}^{q+2,q}(\gamma_k)$*,*
- subcritical Sobolev space: $\gamma \in W^{1+s,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ *for any* $0 < s < 1/q$*.*

Moreover – and this is crucial – we locally control the critical Sobolev norm outside of a singular set Σ containing at most K points: *for any* $x_0 \in (\mathbb{R}/\mathbb{Z}) \setminus \Sigma$ *there exists some* $r_{x_0} > 0$ *such that*

- $\sup_k \mathrm{TP}^{q+2,q}(\gamma_k; B_{r_{x_0}}(x_0)) < \varepsilon$*,*
- γ_k *weakly converges to* γ *in the Sobolev space* $W^{1+1/q,q}(B_{r_{x_0}}(x_0), \mathbb{R}^3)$*,*
- $\sup_k [\gamma_k']^q_{W^{1+1/q,q}(B_{r_{x_0}}(x_0))} \lesssim \sup_k \mathrm{TP}^{q+2,q}(\gamma_k; B_{r_{x_0}}(x_0)) < \varepsilon$*.*

Remark 1.3 (Pull-tight). In analogy to harmonic maps and O'Hara energies, we expect examples for $p = q + 2$ where the singular set Σ in Theorem 1.2 is nonempty. The idea is as follows. Take a (closed) smooth curve containing a piece of a straight line and replace the latter by a small *nontrivially* knotted arc. Shrinking this arc to zero ("pull-tight") produces a sequence of curves with uniformly bounded energy [56, Theorem 3.1]. In the limit curve we observe a change of topology along with a loss of energy, ruling out strong convergence in the Sobolev norm.

For the Möbius energy $\mathcal{O}^{2,2}$, one can use Möbius invariance to rewrite a minimizing sequence into one that avoids "concentration of topology in a small set"; see [27] for this notion, and [54] for a survey. Even more is true in this special case: the Möbius energy can be decomposed into several Möbius invariant energies that control different features [38–41, 43, 54].

Remark 1.4. In light of two-dimensional analogues [37, 53], one might conjecture that the limit curve γ from Theorem 1.2 does not necessarily belong to a Sobolev space on $\mathbb{R}/\mathbb{Z}$, but this is certainly not clear to us.

Theorem 1.2 is much simpler to prove if $p > q + 2$ [6, 12]. In our limiting range $p = q + 2$, Theorem 1.2 can be understood as a one-dimensional counterpart to the fundamental theorem of Müller and Šverák [53], who showed that surfaces with small second fundamental form with respect to the L^2-norm can be conformally parametrized. We also refer to earlier works by Toro [77, 78] as well as [45, 46, 48, 64, 65, 70]. Indeed, Theorem 1.2 is strongly inspired by the "weak closure theorem" for the Willmore energy [64, Theorem 3.55].

As a particular consequence of Theorem 1.2, homeomorphisms with locally small tangent-point energy as described in Definition 1.1 appear as limits of smooth minimizing sequences (minimizing, e.g., with respect to isotopy classes, see Section 3). Since the convergence of minimizing sequences is only weak, in general the limits of minimizing sequences may not be minimizers – indeed they may not belong to the same isotopy class due to bubbling effects (also called pull-tights). This is why we introduce, once more in analogy to Willmore surfaces [63, Definition I.2], the notion of locally critical embeddings.

Definition 1.5 (Locally critical embedding). A homeomorphism γ with locally ε-small tangent-point energy at $x \in \mathbb{R}/\mathbb{Z}$ as in Definition 1.1 is a *locally* $\mathrm{TP}^{p,q}$*-critical embedding in* $B_r(x)$ if

$$\delta\mathrm{TP}^{p,q}(\gamma,\varphi) = 0 \quad \text{for all } \varphi \in C_c^\infty(B_r(x),\mathbb{R}^3).$$

The notion of locally critical embedding as in Definition 1.5 can be justified by the following theorem which states that *any minimizing sequence of curves* (with respect to isotopy classes, cf. Section 3) *converges away from finitely many points to a locally critical embedding*. This holds for any $\varepsilon > 0$, but the number of the points increases as $\varepsilon \to 0$. Recall that $q > 1$ due to (1.1).

Theorem 1.6. *Let $p = q + 2$. Let $[\gamma_0]$ be an ambient isotopy class and let $(\gamma_k)_{k\in\mathbb{N}} \subset [\gamma_0]$ be a minimizing sequence for*

$$\Lambda := \inf_{\eta\in[\gamma_0]} \mathrm{TP}^{p,q}(\eta)$$

in the sense that $\gamma_k \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ are homeomorphisms and $\gamma_k(\mathbb{R}/\mathbb{Z})$ belongs to the ambient isotopy class $[\gamma_0]$ for all $k \in \mathbb{N}$.

Then, up to reparametrization, translation, rescaling and passing to a subsequence, γ_k uniformly converges to a limit map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ which for any $\varepsilon > 0$ is a locally $\mathrm{TP}^{p,q}$*-critical embedding in the sense of Definition 1.5 except for a finite exception set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ whose cardinality is bounded in terms of Λ and ε.*

Our last main result concerns regularity: *the limit of minimizing sequences is regular outside a finite singular set* Σ. Indeed, we have regularity theory for critical points as in Definition 1.5.

Theorem 1.7. *Let $q \geq 2$. There exists $\varepsilon > 0$ such that the following holds. Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with finite global and ε-small local tangent-point energy around $B_r(x_0)$ as in Definition 1.1 that is a locally* $\mathrm{TP}^{p,q}$*-critical embedding in $B_r(x_0)$ as in Definition 1.5. Then $\gamma \in C^{1,\alpha}(B_{r/2}(x_0),\mathbb{R}^3)$ for some uniform constant $\alpha = \alpha(q) > 0$.*

From the previous two results we draw the following conclusion.

Corollary 1.8. *Assume $p \geq q + 2$ and $q \geq 2$. Let $[\gamma_0]$ be an ambient isotopy class and let $\gamma_k \subset [\gamma_0]$ be a minimizing sequence for*

$$\Lambda := \inf_{\gamma\in[\gamma_0]} \mathrm{TP}^{p,q}(\gamma)$$

in the sense that $(\gamma_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ is a sequence of homeomorphisms and $\gamma_k(\mathbb{R}/\mathbb{Z})$ belongs to the knot class $[\gamma_0]$ for all $k \in \mathbb{N}$.

Then, up to reparametrization, translation, and rescaling and taking a subsequence, γ_k converges to a limit map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ which is a locally $\mathrm{TP}^{p,q}$-critical point in the sense of Definition 1.5 outside of finitely many points (whose number is bounded in terms of Λ). In particular, in view of Theorem 1.7, the limit is $C^{1,\alpha}$ outside of finitely many points.

Remark 1.9. In Theorem 1.7 we restrict to the scale-invariant case $p = q + 2$. For the non-scaling-invariant case $q \geq 2$ and $p > q + 2$, $C^{1,\alpha}$-regularity is a consequence of previous work by [74], with α depending on $p - q - 2 > 0$. Let us remark that a slight adaptation of our arguments, similar in spirit to adaptations carried out in [50], implies $C^{1,\alpha}$-regularity with α independent of $p - q - 2 \geq 0$.

Also, in Theorem 1.7 we restrict our attention to $q \geq 2$ (which includes the "classical" tangent-point energy $T^{4,2}$), but we expect that it is only a minor technical difficulty to obtain the same result for $q > 1$.

Lastly, we consider the target space $\mathbb{R}^3$ throughout this paper to keep the notation simple, but again we expect our results to carry over to curves of arbitrary codimension without more than minor technical difficulties.

Outline and comments on the proofs

In Section 2 we introduce the Sobolev spaces that are essential for this article. In Section 3 we review the notion of ambient isotopy and adapt this concept to $W^{1+s,1/s}$-curves. While this is, to the best of our knowledge, a new result, the main ideas are related to the well-established theory of homotopy groups of Sobolev maps, e.g. in [4, 71].

In Section 4 we prove our first main theorem, Theorem 1.2, which states that sequences of diffeomorphisms with uniformly bounded tangent-point energy converge outside of a finite singular set. The argument is based on a gap estimate, vaguely reminiscent of and substantially inspired by arguments due to Müller–Šverák [53] and Hélein [35] who showed that limits of conformally parametrized two-dimensional maps with a sufficiently small L^2-bound on the second fundamental form are either point maps or bi-Lipschitz. A further crucial ingredient is an adaptation of the "straightness" analysis developed by Strzelecki and von der Mosel [74] (which in their case leads to the fact that finite energy curves are topological 1-manifolds).

In Section 5 we prove our second main theorem, Theorem 1.6, which asserts that minimizing sequences converge to critical points. This is based on Theorem 1.2 combined with a fractional Luckhaus-type lemma, Lemma 5.3, and the theory of isotopy classes for Sobolev maps from Section 3.

In Section 6 we prove the regularity theory, Theorem 1.7. We follow the spirit of [14], building a bridge to harmonic map theory. Namely, we introduce an energy $\mathcal{E}^q$ such that the arclength parametrization γ of a $\mathrm{TP}^{p,q}$-critical knot induces via its derivative γ' an $\mathcal{E}^q$-critical map in the class of maps from $\mathbb{R}/\mathbb{Z}$ to the sphere $\mathbb{S}^2$. The energy $\mathcal{E}^q$ is structurally similar to the $W^{1/q,q}$-seminorm whose critical points are called *$W^{1/q,q}$-harmonic maps*. For $q = 2$ techniques for regularity theory of $W^{1/2,2}$ harmonic maps between manifolds were introduced in the pioneering work by Da Lio and Rivière [23,24]; this was extended to $W^{1/q,q}$ harmonic maps into spheres in [68]. Here, we extend the techniques of [68] to obtain the regularity for derivatives γ' of the arclength parametrization of critical knots γ.

Notation

When $A \leq CB$ for some constant C, we write $A \lesssim B$ or $B \gtrsim A$. We use the notation $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$. Throughout this work, constants will depend on "unimportant" factors like p and q and may change from line to line.

Balls (i.e. intervals) in $\mathbb{R}$ will be denoted by $B_\rho(x)$. We will allow ourselves an abuse of notation to denote *geodesic balls* in $\mathbb{R}/\mathbb{Z}$ by the same notation. All our arguments are local in nature, so that we only need to work with balls which correspond to Euclidean balls.

2. Preliminaries on Sobolev maps

In this section we recall some basic notation and properties of Sobolev maps.

For $s \in (0,1)$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}$ open, the Sobolev space $W^{s,p}(\Omega)$ is defined as all maps $f \in L^p(\Omega)$ such that

$$[f]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}}\, dx\, dy \right)^{1/p} < \infty.$$

For $s \in (1,2)$ the Sobolev space $W^{s,p}(\Omega)$ is defined to be the space of $f \in L^p(\Omega)$ with $f' \in L^p(\Omega)$ and

$$[f']_{W^{s-1,p}(\Omega)} < \infty.$$

One important observation [5, Lemma 2.1] is that small $W^{1+s,1/s}$-norm implies a bi-Lipschitz estimate if $|\gamma'| > 0$.

Lemma 2.1. *Let $s \in (0,1)$. For any $\lambda_1 > \lambda_2 > 0$ there exists $\varepsilon = \varepsilon(\lambda_1, \lambda_2, s) > 0$ such that the following holds. For any $-\infty < a < b < \infty$ and for any $\gamma \in \mathrm{Lip}([a,b], \mathbb{R}^3)$ such that*

$$\inf_{[a,b]} |\gamma'| \geq \lambda_1 \quad \textit{and} \quad [\gamma']_{W^{s,1/s}((a,b))} < \varepsilon,$$

we have

$$|\gamma(x) - \gamma(y)| \geq \lambda_2 |x-y|.$$

Proof. First we prove that

$$\frac{|\gamma(y) - \gamma(x)|^2}{|x-y|^2} \geq (\lambda_1)^2 - \frac{1}{2|x-y|^2} \int_{[x,y]} \int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^2\, dz_1\, dz_2. \quad (2.1)$$

The estimate (2.1) is a consequence of the fundamental theorem of calculus, which says that for any $x \neq y$ we have

$$\gamma(y) - \gamma(x) = \int_x^y \gamma'(z)\, dz.$$

Then

$$\begin{aligned}
|\gamma(y) - \gamma(x)|^2 &= \int_x^y \int_x^y \langle \gamma'(z_1), \gamma'(z_2)\rangle \, dz_1\, dz_2 \\
&= \frac{1}{2}\int_x^y \int_x^y [|\gamma'(z_1)|^2 + |\gamma'(z_2)|^2]\, dz_1\, dz_2 - \frac{1}{2}\int_x^y \int_x^y |\gamma'(z_1) - \gamma'(z_2)|^2 \, dz_1\, dz_2 \\
&\geq |x-y|^2 \,(\lambda_1)^2 - \frac{1}{2}\int_{[x,y]}\int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^2\, dz_1\, dz_2.
\end{aligned}$$

This establishes (2.1).

The claim of Lemma 2.1 follows from (2.1) once we show that for any $s \in (0,1)$ there exists a constant $C = C(s)$ such that

$$\frac{1}{2|x-y|^2}\int_{[x,y]}\int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^2\, dz_1\, dz_2 \leq C(s)\, [\gamma']^2_{W^{s,1/s}((a,b))}. \tag{2.2}$$

Indeed, once (2.2) is established, we choose $\varepsilon > 0$ such that

$$(\lambda_1)^2 - C(s)\varepsilon^2 > (\lambda_2)^2.$$

Then – under the assumptions of Lemma 2.1 – we conclude that

$$\frac{|\gamma(y) - \gamma(x)|^2}{|x-y|^2} \geq (\lambda_2)^2,$$

which is what we wanted to show.

It remains to establish (2.2), and for this we consider three cases.

For $s = 1/2$, (2.2) is a consequence of the following observation:

$$\begin{aligned}
&\frac{1}{2|x-y|^2}\int_{[x,y]}\int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^2\, dz_1\, dz_2 \\
&\leq \frac{1}{2}\int_{[x,y]}\int_{[x,y]} \frac{|\gamma'(z_1) - \gamma'(z_2)|^2}{|z_1 - z_2|^2}\, dz_1\, dz_2 = \frac{1}{2}[\gamma']^2_{W^{1/2,2}([x,y])} \leq \frac{1}{2}[\gamma']^2_{W^{1/2,2}([a,b])}.
\end{aligned}$$

For $s \in (1/2, 1)$, we additionally observe that by Lemma A.3 there exists a constant $C = C(s)$ such that

$$[\gamma']^2_{W^{1/2,2}((a,b))} \leq C(s)\, [\gamma']^2_{W^{s,1/s}((a,b))}.$$

This establishes (2.2) for all $s \in [1/2, 1)$.

If $s \in (0, 1/2)$, then $\frac{1}{2s} > 1$. Hence, by Jensen's inequality,

$$
\begin{aligned}
|x-y|^{-2} & \int_{[x,y]} \int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^2 \, dz_1 \, dz_2 \\
&= \left(|x-y|^{-2} \int_{[x,y]} \int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^2 \, dz_1 \, dz_2 \right)^{\frac{1}{2s} \cdot 2s} \\
&\leq \left(|x-y|^{-2} \int_{[x,y]} \int_{[x,y]} |\gamma'(z_1) - \gamma'(z_2)|^{1/s} \, dz_1 \, dz_2 \right)^{2s} \\
&\leq \left(\int_{[x,y]} \int_{[x,y]} \frac{|\gamma'(z_1) - \gamma'(z_2)|^{1/s}}{|z_1 - z_2|^2} \, dz_1 \, dz_2 \right)^{2s}.
\end{aligned}
$$

This establishes (2.2) for $s \in (0, 1/2)$, concluding the proof. ■

Let us also remark the following consequence of Lemma 2.1, which states that any closed curve has at least a certain (computable) amount of $W^{1+s,1/s}$-energy – i.e. if the $W^{1+s,1/s}$-energy is below a certain threshold, then the curve cannot be closed.

Corollary 2.2. *Let $s \in (0, 1)$ and $-\infty < a < b < \infty$. For any $\lambda > 0$ there exists $\varepsilon = \varepsilon(\lambda, a, b, s) > 0$ such that the following holds. Whenever $\gamma \in \mathrm{Lip}((a, b), \mathbb{R}^3) \cap C^0([a, b])$ with $\gamma(a) = \gamma(b)$ and $\inf |\gamma'| \geq \lambda$, then $[\gamma']_{W^{s,1/s}([a,b])} \geq \varepsilon$.*

Proof. If $[\gamma']_{W^{s,1/s}([a,b])} < \varepsilon$ for small enough ε we know from Lemma 2.1 that γ is bi-Lipschitz, and thus

$$|\gamma(a) - \gamma(b)| = \lim_{x \to a^+} \lim_{y \to b^-} |\gamma(x) - \gamma(y)| \geq c|b - a|. \quad ■$$

3. Ambient isotopy for Sobolev curves

A homotopy theory for Sobolev maps was introduced and established a long time ago. The spirit is that for maps in VMO (see e.g. the phenomenal work [19, 20]) homotopy classes exist, and by Sobolev embedding, homotopy groups for $W^{s,n/s}(\Sigma^n, \mathcal{N})$ coincide with the classical homotopy groups for continuous maps. In particular, this leads to a beautiful theory of density [4, 71]. There are many extensions, e.g. to more general Sobolev spaces [16–18, 33, 52, 62].

We begin here to introduce the fundamental results on isotopy classes (for curves) in fractional Sobolev spaces following the spirit of homotopy classes. To the best of our knowledge, the results in this section are new, in particular our main result, Theorem 3.7, which says that small Sobolev variations of smooth curves do not change their isotopy class. However, there is some overlap with [9] where an isotopy theory for closed sets with controlled bi-Lipschitz constant is developed.

We begin by defining ambient isotopy (by which we mean C^1-ambient isotopy).

Definition 3.1. Two sets $X, Y \subset \mathbb{R}^3$ are called *ambient isotopic* if there exists an *ambient isotopy*, that is, $I \in C^1([0, 1] \times \mathbb{R}^3)$, such that

- $I(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism for all $t \in [0,1]$,
- $I(0,p) = p$ for all $p \in \mathbb{R}^3$,
- $I(1,\cdot) : X \to Y$ is a homeomorphism.

When working with parametrized curves, the following result is very useful: smooth enough isotopy coincides with ambient isotopy [36, Chapter 8, Theorem 1.6, p. 181].

Theorem 3.2. *Let $\gamma_0, \gamma_1 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be two diffeomorphisms, and assume there exists a C^1-isotopy between them, that is, $\Gamma \in C^1([0,1] \times \mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that $\Gamma(0,\cdot) = \gamma_0(\cdot)$ and $\Gamma(1,\cdot) = \gamma_1(\cdot)$ and $\Gamma(t,\cdot) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is a diffeomorphism for all $t \in [0,1]$. Then the images $\gamma_0(\mathbb{R}/\mathbb{Z})$ and $\gamma_1(\mathbb{R}/\mathbb{Z})$ are ambient isotopic.*

We begin by defining ambient isotopy classes for regular $W^{1+s,1/s}$-homeomorphisms. Observe that in view of the formal analogy to homotopy classes, having the techniques by Brezis and Nirenberg [19, 20], one might hope for an "$s = 0$" theory (i.e. $\gamma' \in \mathrm{VMO}$), but we will not pursue that question here. Also, we will make no attempt to consider the higher-dimensional version, but rather focus on curves.

Definition 3.3 (Regular Sobolev homeomorphism). A homeomorphism γ in the class $W^{1+s,1/s}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ is called *regular* if

$$0 < \inf |\gamma'| \le \sup |\gamma'| < \infty$$

where inf and sup are the essential infimum and supremum, respectively.

The isotopy class is derived from smooth approximating maps, whose existence is the content of the following lemma.

Lemma 3.4. *Let $\gamma \in W^{1+s,1/s}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be a regular homeomorphism. Then there exists a sequence of smooth diffeomorphisms $\gamma_k : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with*

$$\tfrac{1}{2} \inf |\gamma'| \le \inf |\gamma_k'| \le \sup |\gamma_k'| \le \sup |\gamma'| \quad \textit{for all } k \in \mathbb{N}$$

such that

$$\|\gamma_k - \gamma\|_{L^\infty(\mathbb{R}/\mathbb{Z})} + [\gamma_k' - \gamma']_{W^{1+s,1/s}(\mathbb{R}/\mathbb{Z})} \xrightarrow{k\to\infty} 0.$$

Proof. Set

$$\lambda := \inf |\gamma'|.$$

Fix some $\varepsilon_0 > 0$ to be specified later.

By absolute continuity of the integral there exists $\delta_0 = \delta_0(\gamma) \in (0,1)$ such that

$$\sup_{B_{10\delta_0} \subset \mathbb{R}/\mathbb{Z}} [\gamma']_{W^{s,1/s}(B_{10\delta_0})} < \varepsilon_0.$$

Since γ is a continuous and injective map, the following infimum is attained and strictly positive:

$$\varepsilon_1 := \inf_{|x-y| \ge \frac{1}{2}\delta_0} |\gamma(x) - \gamma(y)| > 0.$$

Let $\eta \in C_c^\infty(B_1(0), [0,1])$, $\int \eta = 1$, be the usual mollifier kernel, and $\eta_\delta := \delta^{-1}\eta(\cdot/\delta)$. Set

$$\gamma_\delta(x) := \eta_\delta * \gamma(x) = \int_{-1}^{1} \eta(z)\gamma(x+\delta z)\, dz.$$

By periodicity of γ, γ_δ is 1-periodic, and thus is well-defined $\gamma_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$. Moreover, $[\gamma_\delta{}' - \gamma']_{W^{1+s,1/s}(\mathbb{R}/\mathbb{Z})} \xrightarrow{\delta \to 0} 0$, and by Sobolev embedding $\|\gamma_\delta - \gamma\|_{L^\infty(\mathbb{R}/\mathbb{Z})} \xrightarrow{\delta\to 0} 0$.

Let $\delta_1 \in (0, \delta_0)$ be such that

$$\|\gamma_\delta - \gamma\|_{L^\infty(\mathbb{R}/\mathbb{Z})} < \tfrac{1}{10}\varepsilon_1 \quad \text{for all } \delta \in [0, \delta_1]. \tag{3.1}$$

We need to show (for the right choice of ε_0) that for $\delta \in (0, \delta_1)$, γ_δ is a diffeomorphism as requested in the claim.

It is easy to see from the definition of γ_δ that

$$|\gamma_\delta'(x)| \le \sup|\gamma'| \quad \forall \delta > 0.$$

First we observe that for almost every $x \in \mathbb{R}/\mathbb{Z}$ and almost every $z \in \mathbb{R}/\mathbb{Z}$ we have

$$|\gamma_\delta'(x)| \ge |\gamma'(z)| - |\gamma_\delta'(x) - \gamma'(z)| \ge \lambda - |\gamma_\delta'(x) - \gamma'(z)|.$$

Since this holds for almost every $z \in \mathbb{R}/\mathbb{Z}$, we can take the integral mean over $B_\delta(x)$ and find

$$|\gamma_\delta'(x)| \ge \lambda - \fint_{B_\delta(x)} |\gamma_\delta'(x) - \gamma'(z)|\, dz.$$

Now we have $\gamma_\delta' = \eta_\delta * (\gamma')$, and thus

$$\begin{aligned}\fint_{B_\delta(x)} |\gamma_\delta'(x) - \gamma'(z)|\, dz &\lesssim \fint_{B_\delta(x)} \fint_{B_\delta(x)} |\gamma'(z_1) - \gamma'(z)|\, dz_1\, dz \\ &\lesssim [\gamma']_{W^{s,1/s}(B_\delta(x))} < \varepsilon_0.\end{aligned}$$

That is, we have shown

$$|\gamma_\delta'(x)| \ge \lambda - C\varepsilon_0 \quad \text{for almost every } x \in \mathbb{R}/\mathbb{Z}.$$

So if we choose $\varepsilon_0 < \frac{\lambda}{2C}$, we have

$$\inf|\gamma_\delta'| \ge \lambda/2 \quad \forall \delta \in (0, \delta_0).$$

Now choosing ε_0 possibly even smaller (depending on λ), we deduce from Lemma 2.1 that

$$\frac{\lambda}{4}|x-y| \le |\gamma_\delta(x) - \gamma_\delta(y)| \quad \forall \delta \in [0, \delta_0],\ |x-y| < \delta_0. \tag{3.2}$$

On the other hand,

$$|\gamma_\delta(x) - \gamma_\delta(y)| \ge |\gamma(x) - \gamma(y)| - 2\|\gamma - \gamma_\delta\|_{L^\infty}.$$

In view of (3.1), we thus have

$$|\gamma_\delta(x)-\gamma_\delta(y)| \geq \big(1-\tfrac{2}{10}\big)\varepsilon_1 \geq 0.8\varepsilon_1|x-y|_{\mathbb{R}/\mathbb{Z}} \quad \forall |x-y|>\tfrac{1}{2}\delta_0,\ \delta\in[0,\delta_1]. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$|\gamma_\delta(x)-\gamma_\delta(y)| \geq \tfrac{1}{100}\min\{\lambda,\varepsilon_1\}\,|x-y|_{\mathbb{R}/\mathbb{Z}} \quad \forall x,y\in\mathbb{R}/\mathbb{Z},\ \delta\in[0,\delta_1].$$

Consequently, γ_δ is a one-to-one map with $\inf|\gamma_\delta'|>0$.

Hence, γ_δ is a smooth homeomorphism with nonvanishing derivative (i.e. an immersion), thus γ_δ is a diffeomorphism. The proof is concluded by choosing $\gamma_k := \gamma_{\delta_1/k}$. ■

Now that we have approximating smooth diffeomorphisms, we argue that they are all eventually of the same ambient isotopy type.

Proposition 3.5. *Let $\gamma\in W^{1+s,1/s}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ be a regular homeomorphism. Then there exists $\varepsilon=\varepsilon(\gamma,s)>0$ such that for any $\tilde\gamma_1,\tilde\gamma_2\in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ with*

$$\|\tilde\gamma_i-\gamma\|_{L^\infty}+[\tilde\gamma_i'-\gamma']_{W^{s,1/s}}<\varepsilon \tag{3.4}$$

and

$$\tfrac{1}{2}\inf|\gamma'|\leq\inf|\tilde\gamma_i'|\leq\sup|\tilde\gamma_i'|\leq 2\sup|\gamma'|,\quad i=1,2,$$

$\tilde\gamma_1$ is ambient isotopic to $\tilde\gamma_2$.

Proof. The strategy of the proof is very similar to the proof of Lemma 3.4.

Set

$$\lambda := \tfrac{1}{2}\inf|\gamma'|.$$

Throughout the proof we will fix interdependent constants $\varepsilon_0,\varepsilon_1,\rho_0,\delta_0,\delta_1,\tilde\lambda>0$. Then ε needs to be small in dependence on all these constants.

Firstly, depending on ε_0,γ,s we can, by absolute continuity of the integral, choose $\rho_0>0$ such that

$$\sup_{B_{10\rho_0}\subset\mathbb{R}/\mathbb{Z}}[\gamma']_{W^{s,1/s}(B_{10\rho_0})}<\tfrac{1}{2}\varepsilon_0.$$

Since γ is continuous and injective, the following infimum is attained and larger than 0:

$$\varepsilon_1 := \tfrac{1}{10}\inf_{|x-y|\geq\frac{1}{2}\rho_0}|\gamma(x)-\gamma(y)|>0.$$

Observe that if $\varepsilon<\frac{1}{100}\min\{\varepsilon_0,\varepsilon_1\}$ in (3.4) then

$$\sup_{B_{10\rho_0}\subset\mathbb{R}/\mathbb{Z}}[\tilde\gamma_i']_{W^{s,1/s}(B_{10\rho_0})}<\varepsilon_0,\quad i=1,2,$$

and

$$\inf_{|x-y|\geq\frac{1}{2}\rho_0}|\tilde\gamma_i(x)-\tilde\gamma_i(y)|>\varepsilon_1>0.$$

Let $\eta \in C_c^\infty(B_1(0),[0,1])$, $\int \eta = 1$, be the usual mollifier kernel, and $\eta_\delta := \delta^{-1}\eta(\cdot/\delta)$. Set

$$\gamma_\delta := \eta_\delta * \gamma \quad \text{and} \quad \tilde{\gamma}_{i,\delta} := \eta_\delta * \tilde{\gamma}_i.$$

Let δ_1 be such that

$$\|\gamma_\delta - \gamma\|_{L^\infty} < \tfrac{1}{100}\varepsilon_1 \quad \forall \delta \in [0,\delta_1].$$

Again we observe that if $\varepsilon < \frac{1}{100}\min\{\varepsilon_0,\varepsilon_1\}$ in (3.4) then

$$\|\tilde{\gamma}_{i,\delta} - \tilde{\gamma}_i\|_{L^\infty} < \tfrac{1}{10}\varepsilon_1 \quad \forall \delta \in [0,\delta_1],\ i = 1,2.$$

Let $\delta_0 := \min\{\delta_1,\rho_0\}$. As in the proof of Lemma 3.4, for the right choice of ε_0, we find that for each $\delta \in [0,\delta_0]$, γ_δ and $\tilde{\gamma}_{i,\delta}$ are diffeomorphisms satisfying

$$|\tilde{\gamma}_{i,\delta}(x) - \tilde{\gamma}_{i,\delta}(y)|, |\gamma_\delta(x) - \gamma_\delta(y)| \geq \tilde{\lambda}|x-y| \quad \forall x,y \in \mathbb{R}/\mathbb{Z},\ \delta \in [0,\delta_0]$$

for $\tilde{\lambda} := \frac{1}{100}\min\{\lambda,\varepsilon_1\}$.

Since $\tilde{\gamma}_{i,\delta} \xrightarrow{\delta\to 0} \tilde{\gamma}_i$, and $\tilde{\gamma}_i$ are C^1-diffeomorphisms, we see from Theorem 3.2 that $\tilde{\gamma}_{i,\delta_0}$ is ambient isotopic to $\tilde{\gamma}_i$ for $i = 1,2$.

Now we show that $\tilde{\gamma}_{i,\delta_0}$ is ambient isotopic to γ_{δ_0} for $i = 1,2$ if ε in (3.4) is small enough. Indeed, set

$$\Gamma_i(\cdot,t) := t\tilde{\gamma}_{i,\delta_0} + (1-t)\gamma_{\delta_0}.$$

This is clearly a smooth homotopy; we only need to show that for each fixed $t \in [0,1]$ it is a diffeomorphism. But observe that

$$\begin{aligned}|\Gamma_i(x,t) - \Gamma_i(y,t)| &\geq |\gamma_{\delta_0}(x) - \gamma_{\delta_0}(y)| - |x-y|\,\|\gamma'_{\delta_0} - \tilde{\gamma}'_{i,\delta_0}\|_{L^\infty}\\ &\geq (\tilde{\lambda} - \|\gamma'_{\delta_0} - \tilde{\gamma}'_{i,\delta_0}\|_{L^\infty})|x-y|.\end{aligned}$$

Now

$$\|\gamma'_{\delta_0} - \tilde{\gamma}'_{i,\delta_0}\|_{L^\infty} \leq \frac{1}{\delta_0}\|\eta'\|_{L^1}\,\|\gamma - \tilde{\gamma}_i\|_{L^\infty} \leq \frac{\varepsilon}{\delta_0}\|\eta'\|_{L^1}.$$

That is,

$$|\Gamma_i(x,t) - \Gamma_i(y,t)| \geq (\tilde{\lambda} - \|\eta'\|_{L^1}\varepsilon/\delta_0)|x-y| \quad \forall x,y \in \mathbb{R}/\mathbb{Z}.$$

So if we assume that in (3.4) we have $\varepsilon < \frac{1}{100}\min\{\varepsilon_1, \frac{\tilde{\lambda}}{100\|\eta'\|_{L^1}}\delta_0\}$, then we have found that $\Gamma(t,\cdot)$ is globally bi-Lipschitz, and thus a diffeomorphism for each $t \in [0,1]$. This and Theorem 3.2 imply that γ_{δ_0} and $\tilde{\gamma}_{i,\delta_0}$ are ambient isotopic, for each $i = 1,2$.

In particular, $\tilde{\gamma}_{1,\delta_0}$ is ambient isotopic to $\tilde{\gamma}_{2,\delta_0}$. Since we have already shown that $\tilde{\gamma}_{i,\delta_0}$ is ambient isotopic to $\tilde{\gamma}_i$ for each $i = 1,2$, we conclude that $\tilde{\gamma}_1$ is ambient isotopic to $\tilde{\gamma}_2$. ■

Since by Lemma 3.4 any $W^{1+s,1/s}$-regular Sobolev homeomorphism has an approximation by regular diffeomorphisms, and by Proposition 3.5 these approximating diffeomorphisms are eventually all of the same ambient isotopy type, the following definition is justified.

Definition 3.6. • Let $\gamma_1, \gamma_2 \in W^{1+s,1/s}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be two regular homeomorphisms. We say that γ_1 and γ_2 are (Sobolev-)*ambient isotopic*, written $\gamma_1 \sim \gamma_2$, if the following properties are met: There exist approximating diffeomorphisms $\tilde{\gamma}_{1,k}$ and $\tilde{\gamma}_{2,k}$ converging in $L^\infty \cap W^{1+s,1/s}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ to γ_1 and γ_2, respectively, and satisfying

$$\tfrac{1}{2} \inf |\gamma_i'| \le \inf |\tilde{\gamma}_{i,k}'| \le \sup |\tilde{\gamma}_{i,k}'| \le 2 \sup |\gamma_i'|, \quad i = 1, 2.$$

Moreover, $\gamma_{1,k}$ and $\gamma_{2,\ell}$ are ambient isotopic for all but finitely many k and ℓ.

• Equivalently, let $[\gamma_0]$ be an ambient isotopy class. We say that a regular $W^{1+s,1/s}$-Sobolev homeomorphism γ_1 belongs to $[\gamma_0]$ if there exist approximating diffeomorphisms $\tilde{\gamma}_{1,k}$ as above such that $\tilde{\gamma}_{1,k} \in [\gamma_0]$ for eventually all $k \in \mathbb{N}$.

Our main result in this section is that two curves which differ only locally and in a set where they have small critical Sobolev norm, have the same ambient isotopy type.

Theorem 3.7. *There exists a uniform $\varepsilon > 0$ such that the following holds. Let $\gamma_1, \gamma_2 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be diffeomorphisms with*

- $\frac{3}{4} \le |\gamma_i'| \le \frac{5}{4}$ *in $\mathbb{R}/\mathbb{Z}$ for $i = 1, 2$*

and assume that there is a ball $B_\rho \subset \mathbb{R}/\mathbb{Z}$ such that

- $\frac{3}{4}|x - y| \le |\gamma_i(x) - \gamma_i(y)| \le \frac{5}{4}|x - y|$ *for all $x, y \in B_{10\rho}$,*
- $[\gamma_i']_{W^{s,1/s}(B_{10\rho})} < \varepsilon$, $i = 1, 2$,
- $\operatorname{dist}(\gamma_i(\mathbb{R}/\mathbb{Z} \setminus B_{9\rho}), \gamma_i(B_{8\rho})) \ge \frac{1}{1000}\rho$ *for $i = 1, 2$,*
- $\gamma_1(x) = \gamma_2(x)$ *for all $x \in \mathbb{R}/\mathbb{Z} \setminus B_{3\rho}$,*
- $\|\gamma_1 - \gamma_2\|_{L^\infty(\mathbb{R}/\mathbb{Z})} < \frac{1}{100}\frac{1}{1000}\rho$.

Then $\gamma_1(\mathbb{R}/\mathbb{Z})$ and $\gamma_2(\mathbb{R}/\mathbb{Z})$ are ambient isotopic as sets in $\mathbb{R}^3$.

Proof. Let $\eta \in C_c^\infty(B_1(0), [0, 1])$ with $\int_{\mathbb{R}} \eta = 1$. Denote $\eta_\delta := \delta^{-1}\eta(\cdot/\delta)$.

Let $\theta \in C_c^\infty(B_{5\rho}, [0, 1])$ be such that $\theta \equiv 1$ in $B_{4\rho}$. We can construct θ such that $\|\theta^{(k)}\|_{L^\infty} \lesssim 1/\rho^k$ for all $k \in \mathbb{N}$.

Set

$$\gamma_{i,\delta}(x) := \int_{B_1(0)} \eta(z)\gamma_i(x + \delta\theta(x)z)\, dz.$$

In the following we need to choose first some $\sigma > 0$, and then obtain some $\varepsilon > 0$ depending on σ.

γ_i is ambient isotopic to $\gamma_{i,\sigma\rho}$ for some uniform constant $\sigma \in (0, 1/2)$. $\gamma_{i,\delta}$ would be the usual convolution if $\theta \equiv 1$.

First observe that $\gamma_{i,\delta}(x) = \gamma_i(x)$ for $x \in \mathbb{R}/\mathbb{Z} \setminus B_{5\rho}$. Moreover,

$$\gamma_{i,\delta}(x) - \gamma_i(x) = \int_{B_1(0)} \eta(z)\big(\gamma_i(x + \delta\theta(x)z) - \gamma_i(x)\big)\, dz \lesssim \|\gamma_i'\|_{L^\infty}\delta\|\theta\|_{L^\infty} \xrightarrow{\delta \to 0} 0.$$

Also we have

$$\gamma_{i,\delta}'(x) = \int_{B_1(0)} \eta(z)\, \gamma_i'(x + \delta\theta(x)z)(1 + \delta\theta'(x)z)\, dz.$$

Thus (recall $|\theta'| \lesssim 1/\rho$)

$$|\gamma'_{i,\delta}(x)| \le \|\gamma'_i\|_{L^\infty}(1 + C\delta/\rho),$$

that is, for a certain (essentially uniform) $\sigma \in (0, 1)$ we can ensure that

$$\|\gamma'_{i,\delta}\|_{L^\infty} \le 11/8 \quad \forall \delta < \sigma\rho.$$

It is the other direction that is more tricky. For any $x, z \in B_{5\rho}$ we have

$$\big||\gamma'_{i,\delta}(x)| - 1\big| \le \big||\gamma'_{i,\delta}(x)| - |\gamma'_i(z)|\big| + 1/4 \le |\gamma'_{i,\delta}(x) - \gamma'_i(z)| + 1/4.$$

In particular, for any $z \in B_1(0)$,

$$\big||\gamma'_{i,\delta}(x)| - 1\big| \le |\gamma'_{i,\delta}(x) - \gamma'_i(x + \delta\theta(x)z)| + 1/4.$$

Integrating in z, as the left-hand side is a constant, we obtain

$$\begin{aligned}
&\big||\gamma'_{i,\delta}(x)| - 1\big| \\
&\le \fint_{B_1(0)} \left| \int_{B_1(0)} \eta(z_2)\gamma'_i(x+\delta\theta(x)z_2)(1+\delta\theta'(x)z_2)dz_2 - \gamma'_i(x+\delta\theta(x)z) \right| dz + 1/4 \\
&= \fint_{B_1(0)} \left| \int_{B_1(0)} \eta(z_2)\big(\gamma'_i(x+\delta\theta(x)z_2)(1+\delta\theta'(x)z_2) - \gamma'_i(x+\delta\theta(x)z)\big)\, dz_2 \right| dz + 1/4 \\
&\le \fint_{B_1(0)} \int_{B_1(0)} |\gamma'_i(x+\delta\theta(x)z_2)(1+\delta\theta'(x)z_2) - \gamma'_i(x+\delta\theta(x)z)|\, dz_2\, dz + 1/4 \\
&\le \fint_{B_1(0)} \int_{B_1(0)} |\gamma'_i(x+\delta\theta(x)z_2) - \gamma'_i(x+\delta\theta(x)z)|\, dz\, dz_2 + 1/4 + C\delta/\rho.
\end{aligned}$$

Now observe that $\theta(x)$ is a fixed, nonnegative number. If $\theta(x) = 0$, then the double integral is zero. If $\theta(x) > 0$, by substitution we have

$$\begin{aligned}
&\fint_{B_1(0)} \int_{B_1(0)} |\gamma'_i(x + \delta\theta(x)z_2) - \gamma'_i(x + \delta\theta(x)z)|\, dz\, dz_2 \\
&\qquad \lesssim \frac{1}{(\delta\theta(x))^2} \int_{B_{\delta\theta(x)}(x)} \int_{B_{\delta\theta(x)}(x)} |\gamma'_i(\tilde{x}) - \gamma'_i(\tilde{y})|\, d\tilde{x}\, d\tilde{y} \\
&\qquad \lesssim [\gamma'_i]_{W^{1/q,q}(B_{\delta\theta(x)}(x))} \le [\gamma'_i]_{W^{1/q,q}(B_{10\rho})} < \varepsilon.
\end{aligned}$$

That is, we have shown that for each $x \in B_{5\rho}$ we have

$$\big||\gamma'_{i,\delta}(x)| - 1\big| \le 1/4 + C(\delta/\rho + \varepsilon).$$

The constant C is uniform, so if $\delta < \sigma\rho$ and $\varepsilon \ll 1$ (uniform constant), we get

$$5/8 \le |\gamma'_{i,\delta}(x)| \le 11/8 \quad \forall x \in \mathbb{R}/\mathbb{Z}. \tag{3.5}$$

(Observe that this estimate is trivial for all x where $\gamma_{i,\delta} = \gamma_i$.)

Next we estimate the Sobolev norm $[\gamma'_{i,\delta}]_{W^{1/q,q}(B_{9\rho})}$. Observe that

$$
\begin{aligned}
&|\gamma'_{i,\delta}(x) - \gamma'_{i,\delta}(y)| \\
&\quad \le \int_{B_1(0)} |\gamma'_i(x + \delta\theta(x)z)(1 + \delta\theta'(x)z) - \gamma'_i(y + \delta\theta(y)z)(1 + \delta\theta'(y)z)|\, dz \\
&\quad \le (1 + C\delta/\rho) \int_{B_1(0)} |\gamma'_i(x + \delta\theta(x)z) - \gamma'_i(y + \delta\theta(y)z)|\, dz \\
&\qquad + \|\gamma'_i\|_{L^\infty} \int_{B_1(0)} |\delta\theta'(x) - \delta\theta'(y)|\, dz.
\end{aligned}
$$

First we observe

$$\int_{B_1(0)} |\delta\theta'(x) - \delta\theta'(y)|\, dz \lesssim \frac{\delta}{\rho^2}|x - y|,$$

and consequently, for any $q > 1$,

$$\int_{B_{9\rho}} \int_{B_{9\rho}} \frac{(\int_{B_1(0)} |\delta\theta'(x) - \delta\theta'(y)|\, dz)^q}{|x - y|^2}\, dx\, dy \lesssim \left(\frac{\delta}{\rho^2}\right)^q \rho^q = \left(\frac{\delta}{\rho}\right)^q.$$

We thus arrive at

$$
\begin{aligned}
&[\gamma'_{i,\delta}]^q_{W^{1/q,q}(B_{9\rho})} \le C\left(\frac{\delta}{\rho}\right)^q \\
&\qquad + (1 + C\delta/\rho)^q \int_{B_1(0)} \int_{B_{9\rho}} \int_{B_{9\rho}} \frac{|\gamma'_i(x + \delta\theta(x)z) - \gamma'_i(y + \delta\theta(y)z)|^q}{|x - y|^2}\, dx\, dy\, dz.
\end{aligned}
$$

Observe that $|1 + \delta\theta'(x)z| \ge 1 - C\delta/\rho \ge 1 - C\sigma$ (for $\delta \le \sigma\rho$). Moreover,

$$|x + \delta\theta(x)z - (y + \delta\theta(y)z)| \lesssim |x - y| + \frac{\delta}{\rho}|x - y| \le 2|x - y|.$$

So we can use the change of variables formula to obtain

$$
\begin{aligned}
&\int_{B_{9\rho}} \int_{B_{9\rho}} \frac{|\gamma'_i(x + \delta\theta(x)z) - \gamma'_i(y + \delta\theta(y)z)|^q}{|x - y|^2}\, dx\, dy \\
&\lesssim \frac{1}{(1 - C\sigma)^2} \int_{B_{9\rho}} \int_{B_{9\rho}} \frac{|\gamma'_i(x + \delta\theta(x)z) - \gamma'_i(y + \delta\theta(y)z)|^q}{|x + \delta\theta(x)z - (y + \delta\theta(y)z)|^2} |1 + \delta\theta'(x)z|\, dx \\
&\qquad\qquad \times |1 + \delta\theta'(y)z|\, dy \\
&\lesssim \frac{1}{(1 - C\sigma)^2} \int_{B_{10\rho}} \int_{B_{10\rho}} \frac{|\gamma'_i(x) - \gamma'_i(y)|^q}{|x - y|^2}\, dx\, dy.
\end{aligned}
$$

In conclusion, for σ (and ε) small enough we have shown

$$[\gamma'_{i,\delta}]_{W^{1/q,q}(B_{9\rho})} \lesssim \varepsilon_0 \quad \forall \delta \le \sigma\rho. \tag{3.6}$$

Here ε_0 can be the small constant of Lemma 2.1, and we infer that $\gamma_{i,\delta}$ is uniformly bi-Lipschitz in $B_{9\rho}$. Global injectivity follows from the assumptions in the theorem.

Outside $B_{9\rho}$ the curve $\gamma_{i,\delta}$ equals γ_i for all $\delta \leq \sigma\rho$. Since we have control over the distance of $\gamma_i(\mathbb{R}/\mathbb{Z} \setminus B_{9\rho})$ to $\gamma(B_{8\rho})$ this gives a global bi-Lipschitz control.

Consequently,

$$\gamma_{i,\delta} : [0, \sigma\rho] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$$

is an isotopy with uniformly bounded bi-Lipschitz estimate, and by Theorem 3.2, γ_i and $\gamma_{i,\sigma\rho}$ are ambient isotopic.

$\gamma_{1,\sigma\rho}$ and $\gamma_{2,\sigma\rho}$ are ambient isotopic. Let $\Gamma : [0, 1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be the convex combination

$$\Gamma(t, x) := t\gamma_{1,\sigma\rho}(x) + (1 - t)\gamma_{2,\sigma\rho}(x).$$

From the estimates above, we have

$$\|\partial_x \Gamma(t, \cdot)\|_{L^\infty} \leq 11/8 \quad \forall t \in [0, 1].$$

We need to get a uniform bi-Lipschitz estimate for Γ.

First, since $\gamma_1 = \gamma_2$ in $\mathbb{R}/\mathbb{Z} \setminus B_{3\rho}$ and $\sigma \in (0, 1/2)$,

$$\gamma_{1,\sigma\rho}(x) = \gamma_{2,\sigma\rho}(x) \quad \forall x \in \mathbb{R}/\mathbb{Z} \setminus B_{7\rho/2}.$$

Thus,

$$\Gamma(t, x) = \gamma_{1,\sigma\rho}(x) \quad \forall x \in \mathbb{R}/\mathbb{Z} \setminus B_{7\rho/2}.$$

In view of (3.6) and (3.5) combined with Lemma 2.1, we know that $\gamma_{1,\sigma\rho}$ is uniformly bi-Lipschitz in $B_{9\rho}$, namely

$$\inf_{t\in[0,1]} \inf_{x,y\in B_{9\rho}\setminus B_{7\rho/2}} \frac{|\Gamma(t, x) - \Gamma(t, y)|}{|x - y|} > 0. \tag{3.7}$$

Secondly, since $\theta \equiv 0$ in $\mathbb{R}/\mathbb{Z} \setminus B_{5\rho}$, we have

$$\gamma_{1,\sigma\rho}(x) = \gamma_{2,\sigma\rho}(x) = \gamma_1(x) = \gamma_2(x) \quad \forall x \in \mathbb{R}/\mathbb{Z} \setminus B_{5\rho}.$$

Thus,

$$\Gamma(t, x) = \gamma_1(x) \quad \forall x \in \mathbb{R}/\mathbb{Z} \setminus B_{5\rho}.$$

Since γ_1 is a regular diffeomorphism, this implies that

$$\inf_{t\in[0,1]} \inf_{x,y\in\mathbb{R}/\mathbb{Z}\setminus B_{5\rho}} \frac{|\Gamma(t, x) - \Gamma(t, y)|}{|x - y|} > 0. \tag{3.8}$$

Combining (3.7) and (3.8) we have

$$\inf_{t\in[0,1]} \inf_{x,y\in\mathbb{R}/\mathbb{Z}\setminus B_{7\rho/2}} \frac{|\Gamma(t, x) - \Gamma(t, y)|}{|x - y|} > 0. \tag{3.9}$$

Observe that we have injectivity of the curve.

Since we have control over the distance from $\gamma_i(\mathbb{R}/\mathbb{Z} \setminus B_{9\rho})$ to $\gamma(B_{8\rho})$ and over $\|\gamma_1 - \gamma_2\|_{L^\infty}$, it remains to show the bi-Lipschitz estimate for $|\Gamma(t, x) - \Gamma(t, y)|$ only for

$x, y \in B_{4\rho}$. For such x, y,

$$\begin{aligned}
|\Gamma(t,x)-\Gamma(t,y)| &\geq |\gamma_{1,\sigma\rho}(x)-\gamma_{1,\sigma\rho}(y)| - |\Gamma(t,x)-\gamma_{1,\sigma\rho}(x)-(\Gamma(t,y)-\gamma_{1,\sigma\rho}(y))|\\
&\geq \tfrac{1}{2}|x-y| - \big|(t-1)\gamma_{1,\sigma\rho}(x)+(1-t)\gamma_{2,\sigma\rho}(x)-\big((t-1)\gamma_{1,\sigma\rho}(y)+(1-t)\gamma_{2,\sigma\rho}(y)\big)\big|\\
&= \tfrac{1}{2}|x-y| - (1-t)|\gamma_{1,\sigma\rho}(y)-\gamma_{1,\sigma\rho}(x)-(\gamma_{2,\sigma\rho}(y)-\gamma_{2,\sigma\rho}(x))|\\
&\geq \tfrac{1}{2}|x-y| - \int_{[x,y]} |\gamma_{1,\sigma\rho}'(z)-\gamma_{2,\sigma\rho}'(z)|\,dz\\
&\geq (1/2 - \|\gamma_{1,\sigma\rho}'-\gamma_{2,\sigma\rho}'\|_{L^\infty(B_{4\rho})})\,|x-y|.
\end{aligned}$$

Since $x, y \in B_{4\rho}$, we have $\theta(x) = \theta(y) = 1$. Thus,

$$\begin{aligned}
\|\gamma_{1,\sigma\rho}'-\gamma_{2,\sigma\rho}'\|_{L^\infty(B_{4\rho})} &= \|\eta_{\sigma\rho} * \gamma_1' - \eta_{\sigma\rho} * \gamma_2'\|_{L^\infty(B_{4\rho})}\\
&\lesssim \frac{1}{\sigma\rho}\|\gamma_1'-\gamma_2'\|_{L^1(B_{6\rho})}\\
&\lesssim \frac{\rho}{\sigma\rho}[\gamma_1'-\gamma_2']_{W^{1/q,q}(B_{6\rho})} \leq 2\varepsilon/\sigma.
\end{aligned}$$

In the last step we have used Poincaré's inequality (and the fact that $\gamma_1' = \gamma_2'$ close to $\partial B_{6\rho}$).

So if we choose ε such that $\varepsilon \ll \sigma$, we obtain

$$|\Gamma(t,x)-\Gamma(t,y)| \geq \tfrac{1}{4}|x-y| \quad \forall x, y \in B_{4\rho}.$$

Combining this with (3.9), we find that Γ is an isotopy. In view of Theorem 3.2, $\gamma_{1,\sigma}(\mathbb{R}/\mathbb{Z})$ and $\gamma_{2,\sigma}(\mathbb{R}/\mathbb{Z})$ are ambient isotopic. Since in turn $\gamma_{i,\sigma}$ and γ_i are ambient isotopic for $i = 1, 2$, we have proven that γ_1 is ambient isotopic to γ_2. ■

4. Homeomorphisms appear as limits: Proof of Theorem 1.2

It is easy to construct a Lipschitz parametrization of curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with vanishing tangent-point energy $\mathrm{TP}^{p,q}(\gamma) = 0$, $p \geq q+2$, but with no reasonable regularity, namely $\gamma \notin C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ and $\gamma \notin W^{1+\frac{p-q-1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.

Example 4.1. For any Lipschitz map $\tilde{\gamma} : \mathbb{R}/\mathbb{Z} \to [0, 1/2]$ with $|\tilde{\gamma}'| \equiv 1$, if we set $\gamma(x) := (\tilde{\gamma}(x), 0, 0) \in \mathbb{R}^3$ then

$$|\gamma'(x) \wedge (\gamma(x)-\gamma(y))| = 0.$$

In particular, if for any $x \in \mathbb{R}/\mathbb{Z}$ there are only finitely many $y \in \mathbb{R}/\mathbb{Z}$ such that $\tilde{\gamma}(x) = \tilde{\gamma}(y)$,

$$\frac{|\gamma'(x) \wedge (\gamma(x)-\gamma(y))|^q}{|\gamma(x)-\gamma(y)|^p} = 0 \quad \text{for } \mathscr{L}^2\text{-a.e. } (x,y) \in (\mathbb{R}/\mathbb{Z})^2,$$

and thus $\mathrm{TP}^{p,q}(\gamma) = 0$.

For example, take $\tilde{\gamma}$ to be

$$\tilde{\gamma}(t) := \begin{cases} t, & t < 1/2, \\ 1/2 - t, & t \in [1/2, 1]. \end{cases}$$

Then γ' has a jump discontinuity at $t = 1/2$ and at $t = 0$. Thus $\gamma' \notin C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ and $\gamma' \notin W^{\frac{p-q-1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ whenever $\gamma \notin W^{1+\frac{p-q-1}{q},q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ for any $p \geq q + 2$ and $q \in (1, \infty)$.

It is easy to extend this example to a map γ with countably many points of nondifferentiability but still with $\mathrm{TP}^{p,q}(\gamma) = 0$.

See also the example of a k-covered circle in [74, after Theorem 1.1].

Example 4.1 shows that there is no hope to classify a reasonable energy space of Lipschitz maps $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with finite tangent-point energy. Rather we investigate the space of diffeomorphisms with finite tangent-point energy, which turns out to be more manageable – this is the content of the following Theorem 4.2, which is the main theorem of this section. In particular, Theorem 4.2 implies Theorem 1.2.

Theorem 4.2. *For any $\Lambda > 0$ and $\varepsilon > 0$ there exists an $L = L(\varepsilon, \Lambda) \in \mathbb{N}$ such that the following holds.*

Let $\gamma_k \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma_k'| \equiv 1$ be homeomorphisms such that

$$\sup_k \|\gamma_k\|_{L^\infty} + \sup_k \mathrm{TP}^{p,q}(\gamma_k) \leq \Lambda.$$

Then there exists a subsequence $(\gamma_{k_i})_{i \in \mathbb{N}}$ and $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that the following hold for some finite set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ with $\#\Sigma \leq L$:

(1) *γ_{k_i} converges uniformly to some $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.*

(2) *For any $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \Sigma$ there exists a radius $\rho(x_0) > 0$ such that γ_{k_i} weakly converges to γ in $W^{1+\frac{p-q-1}{q},q}(B_\rho(x_0))$.*

(3) *$|\gamma'| = 1$ a.e.*

(4) *γ is uniformly bi-Lipschitz in $B_\rho(x_0)$ with the estimate*

$$(1 - \varepsilon)|x - y| \leq |\gamma(x) - \gamma(y)| \leq |x - y| \quad \forall x, y \in B_\rho(x_0).$$

(5) *We have lower semicontinuity:*

$$\mathrm{TP}^{p,q}(\gamma) \leq \liminf_{k \to \infty} \mathrm{TP}^{p,q}(\gamma_k).$$

(6) *γ is a bi-Lipschitz homeomorphism.*

(7) *$\gamma \in W^{1+s,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ for any $0 < s < 1/q$.*

We will prove a more detailed version of Theorem 4.2 in Proposition 4.10.

In order to prove Theorem 4.2, we proceed in several steps.

- First we prove in Section 4.1 that for the approximating sequence γ_k the local tangent-point energy is uniformly small away from a finite set Σ (we will refer to it as the "singular set") of points of energy concentration.
- In Section 4.2 we obtain the Sobolev estimate for smooth curves whenever the tangent-point energy is locally small (see Theorem 4.5), and as a consequence a bi-Lipschitz estimate. This estimate is obtained by a gap estimate. In particular, this method characterizes the energy space for the tangent-point energies in the scale-invariant case.
- In Section 4.3 we adapt an argument due to Strzelecki and von der Mosel [74] to obtain a uniform estimate on global injectivity of the approximating sequence γ_k away from the singular points.
- In Section 4.4 we then obtain, in Proposition 4.10, the convergence outside the singular set, which implies Theorem 4.2.

4.1. Locally uniform smallness

In the first step we ensure that away from a discrete set we have locally uniformly small energy in the approximating sequence.

Proposition 4.3. *Let $p = q + 2$, $q > 1$. For any $\varepsilon > 0$ and $\Lambda > 0$ there exists $L = L(\varepsilon, \Lambda)$ such that the following holds. For any sequence $\gamma_k \in \operatorname{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma_k'| \equiv 1$ such that*

$$\sup_k \operatorname{TP}^{p,q}(\gamma_k) \le \Lambda,$$

there exists a subsequence γ_{k_i} and a set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ consisting of at most L points such that for any $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \Sigma$ there exists a radius $\rho = \rho_{x_0} > 0$ and an index $K \in \mathbb{N}$ such that

$$\sup_{i \ge K} \int_{B_{\rho_{x_0}}(x_0)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_{k_i}'(x) \wedge (\gamma_{k_i}(x) - \gamma_{k_i}(y))|^q}{|\gamma_{k_i}(x) - \gamma_{k_i}(y)|^p} \, dy \, dx < \varepsilon.$$

Proposition 4.3 follows for any integral energy from a relatively standard covering argument; see e.g. [67, Proposition 4.3 and Theorem 4.4]. We give the details for the convenience of the reader.

Proof of Proposition 4.3. Pick $\delta \ll \frac{\varepsilon}{2\Lambda}$ and let $m \in \mathbb{N}$. Then cover $\mathbb{R}/\mathbb{Z}$ by at most $2\lceil(\delta 2^{-m})^{-1}\rceil$ intervals $B(x_i, \delta 2^{-m})$ such that every point $x \in \mathbb{R}/\mathbb{Z}$ is covered at most two times. Then we have

$$\begin{aligned} &\sum_i \int_{B(x_i, \delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx \\ &\qquad \le 2 \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx < 2\Lambda = \frac{2\Lambda}{\varepsilon} \varepsilon. \end{aligned}$$

Hence for every γ_k there exist at most $L := L(\varepsilon, \Lambda) := \lfloor 2\Lambda/\varepsilon \rfloor$ intervals $B(x_i, \delta 2^{-m})$ such that

$$\int_{B(x_i,\delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx \geq \varepsilon.$$

Now assume that we have already shown for $i \in \{1, \dots, n\}$ that

$$\sup_k \int_{B(x_i,\delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx < \varepsilon.$$

If there exist more than L intervals $B(x_i, \delta 2^{-m})$, $i > n$, with

$$\sup_k \int_{B(x_i,\delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx \geq \varepsilon,$$

there must exist at least one $B(x_{n+1}, \delta 2^{-m})$ among them and a subsequence of γ_k such that

$$\sup_k \int_{B(x_{n+1},\delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx < \varepsilon.$$

By repeating this step, we find a subsequence of γ_k for which

$$\sup_k \int_{B(x_i,\delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^{q+2}} \, dy \, dx < \varepsilon$$

holds for all given intervals apart from L many $B(x_{i,m}, \delta 2^{-m})$.

Applying this method iteratively for $m \to \infty$, we can construct a series of subsequences such that for each subsequence $\gamma_{k,m}$ we have

$$\sup_k \int_{B(x_i,\delta 2^{-m})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_{k,m}'(x) \wedge (\gamma_{k,m}(x) - \gamma_{k,m}(y))|^q}{|\gamma_{k,m}(x) - \gamma_{k,m}(y)|^{q+2}} \, dy \, dx < \varepsilon,$$

where $B(x_i, \delta 2^{-m}) \subset \mathbb{R}/\mathbb{Z} \setminus \bigcup_{j \leq m-1} B(x_{i,j}, \delta 2^{-j})$.

Now we choose a diagonal subsequence γ_{k_i} (one element per $\gamma_{k,m}$). Since

$$\bigcup_m \Big(\mathbb{R}/\mathbb{Z} \setminus \bigcup_{i \leq L} B(x_{i,m}, \delta 2^{-m}) \Big) = \mathbb{R}/\mathbb{Z} \setminus \bigcap_m \bigcup_{i \leq L} B(x_{i,m}, \delta 2^{-m}) = \mathbb{R}/\mathbb{Z} \setminus \{y_1, \dots, y_L\}$$

for at most $y_1, \dots, y_L \in \mathbb{R}/\mathbb{Z}$, there indeed exists for any $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \{y_1, \dots, y_L\}$ a $\rho_{x_0} > 0$ such that $B(x_0, \rho_{x_0}) \subset B(x_\ell, \delta 2^{-K})$ for a $K \in \mathbb{N}$ and x_ℓ being the center of one of the at most $2\lceil (\delta 2^{-K})^{-1} \rceil$ intervals $B(x_i, \delta 2^{-K})$. Therefore

$$\sup_{i \geq K} \int_{B(x_0,\rho_{x_0})} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_{k_i}'(x) \wedge (\gamma_{k_i}(x) - \gamma_{k_i}(y))|^q}{|\gamma_{k_i}(x) - \gamma_{k_i}(y)|^{q+2}} \, dy \, dx < \varepsilon. \qquad \blacksquare$$

4.2. *Small local energy implies local Sobolev space estimates*

The main novel ingredient underlying our argument for Theorem 4.2 is a gap estimate for Sobolev spaces with respect to the tangent-point energy.

As discussed in Example 4.1, it is impossible to control the Sobolev norm of γ in terms of the tangent-point energy of γ, $\mathrm{TP}^{p,q}(\gamma)$, without assuming a priori bi-Lipschitz estimates (as done in [12]; see also [7]). This is however not a viable method for the scale-invariant case $p = q + 2$ because the bi-Lipschitz constant is not uniformly controlled as a sequence γ_k converges to γ. We turn this argument around and first a priori assume that the Sobolev norm is finite, and then conclude that this is an estimate which is uniform for sequences γ_k converging to γ.

The first step is the following gap estimate.[3]

Lemma 4.4. *Let* $p \in [q + 2, 2q + 1)$ *and* $q > 1$. *Let* $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ *with* $|\gamma'| \equiv 1$. *Then for any ball* $B \subset \mathbb{R}/\mathbb{Z}$ *of diameter less than* $1/2$,

$$[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} \le C(p,q)\,\mathrm{TP}^{p,q}(\gamma; B) + C(p,q)[\gamma']^{2q}_{W^{\frac{p-q-1}{q},q}(B)} \tag{4.1}$$

whenever the right-hand side is finite.

Proof. The assumption that B has diameter less than $1/2$ implies that B is a geodesic ball and thus convex with respect to the $\mathbb{R}/\mathbb{Z}$-metric. To simplify matters even more, we assume without loss of generality that the ball B is centered at 0 so that $|x - y|$ is actually the Euclidean distance.

Recall the Lagrange identity for $v, w \in \mathbb{R}^3$ with $|v| = 1$:

$$|v \wedge w|^2 = |w|^2 - |v \cdot w|^2.$$

Moreover, observe that $|\gamma'| \equiv 1$ implies

$$|\gamma(x) - \gamma(y)| \le |x - y|.$$

Then

$$\begin{aligned}
&\frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p}\\
&\quad\ge \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|x - y|^p}\\
&\quad= \frac{\big(|\gamma'(x) \wedge (\gamma(y) - \gamma(x) - \gamma'(x)(y - x))|^2\big)^{q/2}}{|x - y|^p}\\
&\quad= \frac{\big(|\gamma(y) - \gamma(x) - \gamma'(x)(y - x)|^2 - |\gamma'(x) \cdot (\gamma(y) - \gamma(x) - \gamma'(x)(y - x))|^2\big)^{q/2}}{|x - y|^p}.
\end{aligned}$$

We have

$$\gamma'(x) \cdot (\gamma'(z) - \gamma'(x)) = \tfrac{1}{2}\big(2\gamma'(x) \cdot \gamma'(z) - 1 - 1\big) = -\tfrac{1}{2}|\gamma'(x) - \gamma'(z)|^2, \tag{4.2}$$

[3] Lemma 4.4 is called a *gap estimate*, because it implies the following: For $\varepsilon := (\frac{1}{2C(p,q)})^{1/q}$ we have either $[\gamma']^q_{W^{1/q,q}(B)} \le 2C(p,q)\mathrm{TP}^{q+2,q}(\gamma, B)$ or $[\gamma']_{W^{1/q,q}(B)} \ge \varepsilon$.

so

$$\left|\gamma'(x)\cdot(\gamma(y)-\gamma(x)-\gamma'(x)(y-x))\right| = |y-x|\left|\fint_{(x,y)} \gamma'(x)\cdot(\gamma'(z)-\gamma'(x))\,dz\right|$$
$$= \frac{1}{2}|y-x| \fint_{(x,y)} |\gamma'(x)-\gamma'(z)|^2\,dz.$$

Consequently,

$$\frac{|\gamma'(x)\wedge(\gamma(x)-\gamma(y))|^q}{|\gamma(x)-\gamma(y)|^p} \geq \frac{\left(|\gamma(y)-\gamma(x)-\gamma'(x)(y-x)|^2-\frac{1}{4}|y-x|^2\left|\fint_{(x,y)}|\gamma'(x)-\gamma'(z)|^2\,dz\right|^2\right)^{q/2}}{|x-y|^p}.$$

Observe that from our computations we know that in particular

$$|\gamma(y)-\gamma(x)-\gamma'(x)(y-x)|^2 \geq \frac{1}{4}|y-x|^2\left|\fint_{(x,y)}|\gamma'(x)-\gamma'(z)|^2\,dz\right|^2.$$

Also observe that for any $r>0$ there exist $c_r\in(0,1)$ and $C_r>0$ such that for any $a\geq b\geq 0$,

$$(a-b)^r \geq c_r\, a^r - C_r\, b^r.$$

From this, and by Jensen's inequality,

$$\frac{|\gamma'(x)\wedge(\gamma(x)-\gamma(y))|^q}{|\gamma(x)-\gamma(y)|^p} \geq c\,\frac{|\gamma(y)-\gamma(x)-\gamma'(x)(y-x)|^q}{|x-y|^p} - C\,\frac{|y-x|^q\left|\fint_{(x,y)}|\gamma'(x)-\gamma'(z)|^2\,dz\right|^q}{|x-y|^p}$$
$$\geq c\,\frac{\left|\frac{\gamma(y)-\gamma(x)-\gamma'(x)(y-x)}{|x-y|}\right|^q}{|x-y|^{p-q}} - C\,\frac{\fint_{(x,y)}|\gamma'(x)-\gamma'(z)|^{2q}\,dz}{|x-y|^{p-q}}.$$

Integrating over x and y in B we obtain

$$\int_B\int_B \frac{\left|\frac{\gamma(y)-\gamma(x)-\gamma'(x)(y-x)}{|x-y|}\right|^q}{|x-y|^{p-q}}\,dx\,dy \leq C_q\left(\mathrm{TP}^{p,q}(\gamma;B)+\int_B\int_B \frac{\fint_{(x,y)}|\gamma'(x)-\gamma'(z)|^{2q}\,dz}{|x-y|^{p-q}}\,dx\,dy\right). \tag{4.3}$$

Recall that $p\in[q+2,2q+1)$, so $\frac{p-q-1}{q}\in(0,1)$. From Lemma A.1,

$$[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} \approx \int_B\int_B \frac{\left|\frac{\gamma(y)-\gamma(x)-\gamma'(x)(y-x)}{|x-y|}\right|^q}{|x-y|^{p-q}}\,dx\,dy.$$

From Lemma A.2,

$$[\gamma']^{2q}_{W^{\frac{p-q-1}{2q},2q}(B)} \approx \int_B \int_B \frac{\fint_{(x,y)} |\gamma'(x)-\gamma'(z)|^{2q}\,dz}{|x-y|^{p-q}}\,dx\,dy.$$

Also, from the Sobolev inequality and Lemma A.3, we have

$$[\gamma']_{W^{\frac{p-q-1}{2q},2q}(B)} \lesssim (\operatorname{diam} B)^{\frac{p-q-2}{2q}} [\gamma']_{W^{\frac{p-q-1}{q},q}(B)} \le [\gamma']_{W^{\frac{p-q-1}{q},q}(B)}.$$

Thus, (4.3) implies

$$[\gamma']^{q}_{W^{\frac{p-q-1}{q},q}(B)} \lesssim \mathrm{TP}^{p,q}(\gamma;B) + [\gamma']^{2q}_{W^{\frac{p-q-1}{q},q}(B)},$$

which is (4.1). The proof is complete. ∎

The gap estimate leads to the following control of the Sobolev norm. We stress that we need to assume *a priori* that γ already belongs to the Sobolev space in question, which rules out the irregular curves of Example 4.1.

Theorem 4.5. *Let $q_0, p_0 > 1$, $p_0 \ge q_1 + 2$, and $p_1 < \infty$ be such that $p_1 - 2q_0 < 1$. Furthermore, assume that $p_0 < p_1$ as well as $q_0 < q_1$. Let $\varepsilon > 0$. Then there exists $\delta = \delta(q_0, p_0, q_1, p_1, \varepsilon) > 0$ and a constant $C = C(q_0, p_0, q_1, p_1) > 0$ such that the following holds for any $p \in [p_0, p_1]$ and $q \in [q_0, q_1]$.*

Let $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma'| \equiv 1$, and assume that for some ball $B \subset \mathbb{R}/\mathbb{Z}$ with diam $B < 1/2$, *we have*

$$\mathrm{TP}^{p,q}(\gamma;B) < \delta$$

and

$$\text{either} \quad \gamma \in C^1(B) \quad \text{or} \quad [\gamma']^{q}_{W^{\frac{p-q-1}{q},q}(B)} < \infty. \tag{4.4}$$

Then

$$[\gamma']^{q}_{W^{\frac{p-q-1}{q},q}(B)} \le C(q_0, p_0, q_1, p_1)\,\mathrm{TP}^{p,q}(\gamma;B) \tag{4.5}$$

and we have the bi-Lipschitz estimate

$$(1-\varepsilon)|x-y| \le |\gamma(x)-\gamma(y)| \le |x-y| \quad \forall x,y \in B. \tag{4.6}$$

Proof. In view of [12, Theorem 1.1, Remark 1.6], $\gamma \in C^1(B)$ and $\mathrm{TP}^{p,q}(\gamma;B) < \infty$ implies

$$[\gamma']^{q}_{W^{\frac{p-q-1}{q},q}(B)} < \infty,$$

so that in (4.4) we can assume the Sobolev space estimate holds (observe that we do not need to assume a priori injectivity here, because if $|\gamma'(x_0)| > 0$ and γ is C^1 around x_0 then it is locally injective around x_0, meaning we can apply [12, Theorem 1.1, Remark 1.6] locally) .

Assume that $B = B_R(x_0)$ for some $x_0 \in \mathbb{R}/\mathbb{Z}$ and $R \in (0, 1/4)$. Set

$$\lambda := \mathrm{TP}^{p,q}(\gamma; B) < \delta.$$

For $r \in (0, R]$ set $B_r := B_r(x_0)$.

By Lemma 4.4 we have for any $r \in (0, R]$,

$$[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B_r)} \le C_1 \mathrm{TP}^{p,q}(\gamma; B_r) + C_2 [\gamma']^{2q}_{W^{\frac{p-q-1}{q},q}(B_r)}.$$

Set

$$f(r) := [\gamma']^q_{W^{\frac{p-q-1}{q},q}(B_r)}.$$

Then

$$f(r) \le C_1\lambda + C_2(f(r))^2 \quad \forall r \in (0, R).$$

Setting $p(t) := C_2 t^2 - t + C_1\lambda$, we have

$$p(f(r)) \ge 0 \quad \forall r \in (0, R). \tag{4.7}$$

The roots of the polynomial p are

$$t_{\lambda;1} := \frac{1}{2C_2} - \sqrt{\frac{1}{(2C_2)^2} - \frac{C_1}{C_2}\lambda}, \quad t_{\lambda;2} := \frac{1}{2C_2} + \sqrt{\frac{1}{(2C_2)^2} - \frac{C_1}{C_2}\lambda}.$$

Let δ be small enough that

$$\frac{1}{(2C_2)^2} - \frac{C_1}{C_2}\delta > \frac{1}{(4C_2)^2}.$$

Then whenever $\lambda \in (0, \delta)$ we have $t_{\lambda;1} < t_{\lambda;2}$ and moreover

$$t_{\lambda;1} \le 2C_1\,\lambda \quad \forall \lambda \in (0, \delta). \tag{4.8}$$

The polynomial p is negative only on the interval $(t_{\lambda,1}, t_{\lambda,2})$. From (4.7) we deduce that for each $r \in (0, R)$ either $f(r) \le t_{\lambda,1}$ or $f(r) \ge t_{\lambda,2}$. Since $f(0) = 0$ and f is continuous, we conclude that necessarily

$$f(r) \le t_{\lambda,1} \quad \text{for each } r < R.$$

That is, in view of (4.8),

$$[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} \le 2C_1\lambda.$$

Recalling the definition of λ, we infer (4.5).

Choosing $\delta > 0$ possibly even smaller, we also obtain (4.6) as a consequence of (4.5) and Lemma 2.1. ∎

Let us also remark, for the sake of completeness, that the argument in the proof of Lemma 4.4 also gives a real classification of the energy space, if one assumes a priori bi-Lipschitz estimates (cf. [12, Proposition 2.4]).

Lemma 4.6. *Let $p \in [q+2, 2q+1)$ and $q > 1$. Let $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma'| \equiv 1$ be bi-Lipschitz, i.e.*

$$(1-\lambda)|x-y| \le |\gamma(x)-\gamma(y)| \le |x-y|.$$

Then for any ball $B \subset \mathbb{R}/\mathbb{Z}$ of diameter less than $1/2$,

$$\mathrm{TP}^{p,q}(\gamma; B) \le C(p,q,\lambda)[\gamma']^q_{W^{\frac{p-q-1}{q},q}(B)} + C(p,q,\lambda)[\gamma']^{2q}_{W^{\frac{p-q-1}{q},q}(B)}.$$

With the help of Lemma 4.6 we obtain the following.

Example 4.7. There exists a homeomorphism $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ which is bi-Lipschitz, whose derivative is not everywhere continuous, but has finite tangent-point energy $\mathrm{TP}^{q+2,q}(\gamma)$ for any $q > 1$. Moreover, there exists a sequence of C^∞-diffeomorphisms γ_k converging uniformly to γ with uniformly bounded tangent-point energy, i.e.

$$\sup_{k\in\mathbb{N}} \mathrm{TP}^{q+2,q}(\gamma_k) < \infty.$$

Indeed, denote by $N = (0,0,1)$ the north pole of $\mathbb{S}^2$.

Let $u \in W^{1/q,q}([-1/4, 1/4], \mathbb{S}^2) \setminus C^0([-1/4, 1/4], \mathbb{R}^3)$ be such that

$$\langle u, N\rangle \ge 1/4$$

and u is constant for $|x| \le -1/8$ and for $|x| \ge 1/8$.

For example, for any $\eta \in C_c^\infty((-1/8, 1/8), [0,1])$ with $\eta \equiv 1$ in $[-1/16, 1/16]$ we could set

$$u(x) = \left(\frac{1}{\sqrt{2}}\sin(\eta(x)\log\log 1/|x|), \frac{1}{\sqrt{2}}\cos(\eta(x)\log\log 1/|x|), \frac{1}{\sqrt{2}}\right).$$

Now for $x \in [-1/4, 1/4]$, let

$$\gamma(x) = \int_{-1/4}^{x} u(z)\,dz.$$

Then γ is bi-Lipschitz in $[-1/4, 1/4]$ because

$$|\gamma(x)-\gamma(y)| \ge \langle \gamma(x)-\gamma(y), N\rangle = \int_{[x,y]} \langle u, N\rangle \ge \frac{1}{4}|x-y|.$$

Observe that γ' is constant around $x \approx -1/4$ and $x \approx 1/4$, so γ can be smoothly extended to a closed curve on $[-1/2, 1/2]$ which is a smooth 1-D manifold outside of $[-1/4, 1/4]$. By Lemma 4.6 the curve γ has finite tangent-point energy $\mathrm{TP}^{q+2,q}$ but γ is not C^1 since γ' is discontinuous.

On the other hand, in view of Section 3 any regular homeomorphism $\gamma \in W^{1+1/q,q}$ can be approximated by smooth homeomorphisms with uniformly controlled bi-Lipschitz constant, so that in view of Lemma 4.6 the tangent-point energy $\mathrm{TP}^{q+2,q}$ is uniformly bounded.

4.3. The Strzelecki–von der Mosel argument: Locally small energy implies global injectivity

In this section we provide a reformulation of a powerful argument due to Strzelecki and von der Mosel [74] (see also [12, Appendix]). They used it to show that the image of a curve with finite tangent-point energy ($p \geq q + 2$) is a topological 1-manifold embedded in $\mathbb{R}^3$. Recall that this manifold could be the twice covered straight line, as in Example 4.1.

We rework their argument to provide us with uniform injectivity for intervals with small energy in Theorem 4.9.

The following is essentially a reformulation (with a slight refinement) of [74, Lemma 2.1].

Lemma 4.8 (Strzelecki–von der Mosel). *Let $p \geq q + 2$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds.*

Let $\gamma \in \operatorname{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma'| \equiv 1$, and assume that for some $x_0 \in \mathbb{R}/\mathbb{Z}$ and $\rho > 0$ we have

$$\int_{B_\rho(x_0)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(y) - \gamma(x))|^q}{|\gamma(x) - \gamma(y)|^p}\, dy\, dx < \delta. \tag{4.9}$$

Moreover, assume that there is $y_0 \in \mathbb{R}/\mathbb{Z}$ with $d := |\gamma(y_0) - \gamma(x_0)| \leq \rho$.

Then

$$\gamma(\mathbb{R}/\mathbb{Z}) \cap B_{2d}(\gamma(x_0)) \subset B_{\varepsilon d}(L(\gamma(x_0), \gamma(y_0))),$$

where $L(\gamma(x_0), \gamma(y_0))$ is the straight line containing $\gamma(x_0)$ and $\gamma(y_0)$ defined by

$$L(\gamma(x_0), \gamma(y_0)) = \{(1 - t)\gamma(x_0) + t\gamma(y_0) : t \in \mathbb{R}\}.$$

Proof. For $r > 0$ and $p, v \in \mathbb{R}^3$ we define

$$A(r, p) := \{x \in \mathbb{R}/\mathbb{Z} : |\gamma(x) - p| < r\},$$
$$X(r, v) := \{x \in \mathbb{R}/\mathbb{Z} : |\gamma'(x) \wedge v| \geq r\}.$$

Fix $x_0, y_0 \in \mathbb{R}/\mathbb{Z}$ and $d := |\gamma(x_0) - \gamma(y_0)|$ as in the assumption. Set

$$v := \frac{\gamma(x_0) - \gamma(y_0)}{|\gamma(x_0) - \gamma(y_0)|}.$$

Step 1: We show the following: There exists $\sigma_0 \in (0, 1)$ depending only on p and q such that the following holds. For any $\sigma < \sigma_0$ there exists $\delta_0 = \delta_0(\sigma) > 0$ such that whenever $\delta < \delta_0$ in (4.9) then

$$|B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c| > \tfrac{3}{2}\sigma^2 d. \tag{4.10}$$

In order to establish (4.10), we first observe that for any $x \in A(\sigma^2 d, \gamma(x_0))$ and $y \in A(\sigma^2 d, \gamma(y_0))$ we have

$$|\gamma(x) - \gamma(y)| \in [(1 - 2\sigma^2)d, (1 + 2\sigma^2)d].$$

Moreover, for $x \in A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)$ and $y \in A(\sigma^2 d, \gamma(y_0))$,

$$\begin{aligned}\left|\gamma'(x) \wedge \frac{\gamma(x)-\gamma(y)}{|\gamma(x)-\gamma(y)|}\right| &\geq \frac{1}{(1+2\sigma^2)d}|\gamma'(x) \wedge (\gamma(x)-\gamma(y))| \\ &\geq \frac{1}{(1+2\sigma^2)d}\big(|\gamma'(x) \wedge (\gamma(x_0)-\gamma(y_0))| - 2\sigma^2 d\big) \\ &= \frac{1}{1+2\sigma^2}\left(\left|\gamma'(x) \wedge \frac{\gamma(x_0)-\gamma(y_0)}{|\gamma(x_0)-\gamma(y_0)|}\right| - 2\sigma^2\right) \\ &\geq \frac{1}{1+2\sigma^2}(\sigma - 2\sigma^2) = \sigma\frac{1-2\sigma}{1+2\sigma^2}.\end{aligned}$$

Hence, whenever $x \in A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)$ and $y \in A(\sigma^2 d, \gamma(y_0))$,

$$\frac{|\gamma'(x) \wedge (\gamma(x)-\gamma(y))|^q}{|\gamma(x)-\gamma(y)|^p} \geq \frac{\sigma^q}{d^{p-q}}\left(\frac{1-2\sigma}{1+2\sigma^2}\right)^q \frac{1}{(1+2\sigma^2)^{p-q}}.$$

From (4.9) we find

$$\begin{aligned}\delta > \frac{\sigma^q}{d^{p-q}}\left(\frac{1-2\sigma}{1+2\sigma^2}\right)^q &\frac{1}{(1+2\sigma^2)^{p-q}} \\ &\times |B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)|\,|A(\sigma^2 d, \gamma(y_0))|.\end{aligned}$$

Observe that since $|\gamma'| \equiv 1$, we have $|A(\sigma^2 d, \gamma(y_0))| \geq 2\sigma^2 d$. Then

$$|B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)| \leq \frac{d^{p-q-1}}{\sigma^{q+2}}\left(\frac{1+2\sigma^2}{1-2\sigma}\right)^q (1+2\sigma^2)^{p-q}\frac{1}{2}\delta. \tag{4.11}$$

Since $d \leq \rho$, $\sigma \in (0,1)$, and $|\gamma'| \equiv 1$, we have

$$|B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0))| \geq 2\sigma^2 d. \tag{4.12}$$

Taking σ_0 small enough, combining (4.11) and (4.12) we obtain, for any $\sigma \in (0, \sigma_0)$ and any $\delta \in (0, \delta_0)$ where $\delta_0 = \delta_0(\sigma)$ is small enough,

$$\begin{aligned}&|B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c| \\ &\qquad \geq 2\sigma^2 d - |B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)| \geq \tfrac{3}{2}\sigma^2 d.\end{aligned}$$

This establishes (4.10).

Step 2: We are going to show that if σ is small enough and

$$\gamma(\mathbb{R}/\mathbb{Z}) \cap B_{2d}(\gamma(x_0)) \not\subset B_{20\sqrt{\sigma}d}(L(\gamma(x_0), \gamma(y_0))), \tag{4.13}$$

then necessarily for a uniform constant $C = C(p,q)$,

$$1 < 2C\delta\sigma^{-4-q}. \tag{4.14}$$

Once we have proved that, we can argue by contradiction: Choose σ small enough so that in particular $20\sqrt{\sigma} < \varepsilon$. Pick $\delta_0 = \delta_0(\sigma)$ from Step 1, and $\delta < \delta_0$ (depending on σ and thus ε) such that (4.14) is false. Then (4.13) would lead to a contradiction and thus is false, meaning

$$\gamma(\mathbb{R}/\mathbb{Z}) \cap B_{2d}(\gamma(x_0)) \subset B_{20\sqrt{\sigma}d}(L(\gamma(x_0),\gamma(y_0))) \subset B_{\varepsilon d}(L(\gamma(x_0),\gamma(y_0))),$$

as claimed in the lemma.

So we have to show that (4.13) implies (4.14), which we will do now.

If (4.13) is true, there must be a point $z_0 \in \mathbb{R}/\mathbb{Z}$ such that

$$\gamma(z_0) \in B_{2d}(\gamma(x_0)) \setminus B_{20\sqrt{\sigma}d}(L(\gamma(x_0),\gamma(y_0))). \tag{4.15}$$

Denote the angle between $\gamma(z_0) - \gamma(x_0)$ and $\gamma(y_0) - \gamma(x_0)$ by α; see Figure 2.

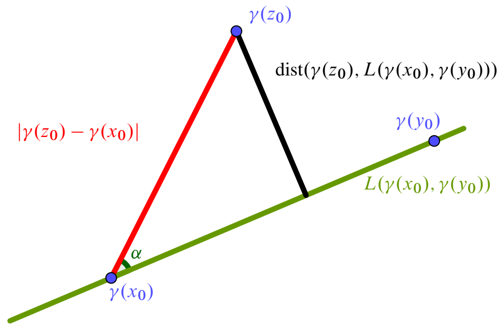

Fig. 2. Definition of α.

Observe that

$$\left|\frac{\gamma(z_0)-\gamma(x_0)}{|\gamma(z_0)-\gamma(x_0)|} \wedge \frac{\gamma(y_0)-\gamma(x_0)}{|\gamma(y_0)-\gamma(x_0)|}\right| = |\sin(\alpha)| = \frac{\operatorname{dist}(\gamma(z_0), L(\gamma(x_0),\gamma(y_0)))}{|\gamma(z_0)-\gamma(x_0)|}$$
$$\overset{(4.15)}{\geq} \frac{20\sqrt{\sigma}d}{2d} = 10\sqrt{\sigma}.$$

Denote by β the angle between $\gamma'(x)$ and $\gamma(y_0) - \gamma(x_0)$ and by θ the angle between $\gamma'(x)$ and $\gamma(z_0) - \gamma(x_0)$. Then we have

$$|\alpha| \leq |\beta| + |\theta|.$$

Thus, a short computation leads to

$$\begin{aligned}\left|\gamma'(x) \wedge \frac{\gamma(z_0)-\gamma(x_0)}{|\gamma(z_0)-\gamma(x_0)|}\right| &= |\sin(\theta)| \geq |\sin(\alpha)| - |\sin(\beta)| \\ &\geq \left|\frac{\gamma(z_0)-\gamma(x_0)}{|\gamma(z_0)-\gamma(x_0)|} \wedge \frac{\gamma(y_0)-\gamma(x_0)}{|\gamma(y_0)-\gamma(x_0)|}\right| - \left|\gamma'(x) \wedge \frac{\gamma(y_0)-\gamma(x_0)}{|\gamma(y_0)-\gamma(x_0)|}\right| \\ &\geq 10\sqrt{\sigma} - \left|\gamma'(x) \wedge \frac{\gamma(y_0)-\gamma(x_0)}{|\gamma(y_0)-\gamma(x_0)|}\right|.\end{aligned}$$

Consequently, for any $x \in X(\sigma, v)^c$,

$$\left| \gamma'(x) \wedge \frac{\gamma(z_0) - \gamma(x_0)}{|\gamma(z_0) - \gamma(x_0)|} \right| \geq 10\sqrt{\sigma} - \sigma.$$

For all σ small enough this implies that for any $x \in X(\sigma, v)^c$,

$$\left| \gamma'(x) \wedge \frac{\gamma(z_0) - \gamma(x_0)}{|\gamma(z_0) - \gamma(x_0)|} \right| \geq 9\sqrt{\sigma}. \tag{4.16}$$

Observe next that for any $x \in B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0))$ and $z \in A(\sigma^2 d, \gamma(z_0))$ (it is important that $z \in \mathbb{R}/\mathbb{Z}$ and need not to lie in $B_\rho(x_0)$) we have

$$|\gamma(z) - \gamma(x)| \geq |\gamma(z_0) - \gamma(x_0)| - 2\sigma^2 d \geq \sqrt{\sigma}(20 - 2\sigma^{3/2})d$$

and

$$|\gamma(z) - \gamma(x)| \leq |\gamma(z_0) - \gamma(x_0)| + 2\sigma^2 d \leq (2 + 2\sigma^2)d.$$

That is, for all σ small enough,

$$|\gamma(z) - \gamma(x)| \in (19\sqrt{\sigma}d, (2 + 2\sigma^2)d). \tag{4.17}$$

We combine (4.17) and (4.16), and find that for any $x \in B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c$ and $z \in A(\sigma^2 d, \gamma(z_0))$,

$$\begin{aligned}
\left| \gamma'(x) \wedge \frac{\gamma(x) - \gamma(z)}{|\gamma(x) - \gamma(z)|} \right| &= \frac{1}{|\gamma(x) - \gamma(z)|} |\gamma'(x) \wedge (\gamma(x) - \gamma(z))| \\
&\geq \frac{1}{|\gamma(x) - \gamma(z)|} (|\gamma'(x) \wedge (\gamma(x_0) - \gamma(z_0))| - 2\sigma^2 d) \\
&\geq \frac{1}{|\gamma(x) - \gamma(z)|} \left(|\gamma(x_0) - \gamma(z_0)| \left| \gamma'(x) \wedge \frac{\gamma(x_0) - \gamma(z_0)}{|\gamma(x_0) - \gamma(z_0)|} \right| - 2\sigma^2 d \right) \\
&\geq \frac{d}{|\gamma(x) - \gamma(z)|} (180\sigma - 2\sigma^2) \gtrsim \frac{\sigma}{2 + 2\sigma^2} (180 - 2\sigma).
\end{aligned}$$

Again, for all small enough σ this implies for any $x \in B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c$ and $z \in A(\sigma^2 d, \gamma(z_0))$,

$$\left| \gamma'(x) \wedge \frac{\gamma(x) - \gamma(z)}{|\gamma(x) - \gamma(z)|} \right| \geq 3\sigma. \tag{4.18}$$

Integrating this inequality in $x \in B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c$ and $z \in A(\sigma^2 d, \gamma(z_0))$, we deduce from (4.9), (4.17), and (4.18) that

$$\delta > \frac{(3\sigma)^q}{((2 + 2\sigma^2)d)^{p-q}} |B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c| \, |A(\sigma^2 d, \gamma(z_0))|.$$

For small enough σ we can simplify this to

$$\delta > \frac{(3\sigma)^q}{(3d)^{p-q}} |B_\rho(x_0) \cap A(\sigma^2 d, \gamma(x_0)) \cap X(\sigma, v)^c| \, |A(\sigma^2 d, \gamma(z_0))|.$$

With (4.10) in mind, we can find $\delta_0 = \delta_0(\sigma)$ such that if $\delta < \delta_0$ then

$$\delta > \frac{(3\sigma)^q}{(3d)^{p-q}} \frac{3}{2} \sigma^2 d |A(\sigma^2 d, \gamma(z_0))|.$$

On the other hand, since $|\gamma'| \equiv 1$, we have $|A(\sigma^2 d, \gamma(z_0))| \geq 2\sigma^2 d$. That is,

$$2\sigma^2 d \leq |A(\sigma^2 d, \gamma(z_0))| < C\delta d^{p-q-1} \sigma^{-2-q}.$$

That is,

$$d^{-(p-2-q)} < 2C\delta\sigma^{-4-q}.$$

If $p \geq q+2$ we have (observe that $d \leq 1$ since $\operatorname{diam}(\gamma(\mathbb{R}/\mathbb{Z})) \leq 1$),

$$1 < 2C\delta\sigma^{-4-q}.$$

That is, under the assumption (4.13) we have shown (4.14), which, as explained above, implies the claim of Lemma 4.8. ∎

Theorem 4.9. *Let $2q+1 > p \geq q+2$. There exists $\delta > 0$ such that the following holds. Let $\gamma \in \operatorname{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be a homeomorphism with $|\gamma'| \equiv 1$, and assume that for some $x_0 \in \mathbb{R}/\mathbb{Z}$ and $\rho > 0$,*

$$\text{either} \quad \gamma \in C^1(B_\rho(x_0)) \quad \text{or} \quad [\gamma']_{W^{\frac{p-q-1}{q},q}(B_\rho(x_0))} < \infty. \tag{4.19}$$

Also assume

$$\int_{B_\rho(x_0)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(y)-\gamma(x))|^q}{|\gamma(x)-\gamma(y)|^p}\, dy\, dx < \delta. \tag{4.20}$$

If for any $z_0 \in \mathbb{R}/\mathbb{Z}$ we have

$$|\gamma(x_0) - \gamma(z_0)| < \tfrac{1}{10}\rho,$$

then there exists $\bar{x} \in B_\rho(x_0)$ such that $\gamma(\bar{x}) = \gamma(z_0)$. In particular, $z_0 \in B_\rho(x_0)$.

Proof. Fix $\sigma, \varepsilon > 0$ to be specified later. Take δ small enough so that Lemma 2.1 and Theorem 4.5 are applicable (in view of (4.19)), so that

$$(1-\sigma)|x-y| \leq |\gamma(x)-\gamma(y)| \leq |x-y| \quad \forall x, y \in B_\rho(x_0). \tag{4.21}$$

Moreover, we can assume that δ is small enough so that Lemma 4.8 is applicable.

Starting from x_0, we are going to construct a sequence $(x_k)_{k=0}^\infty \subset B_\rho(x_0)$ and a sequence $(d_k)_{k=0}^\infty \subset (0,\infty)$ such that for all $k \geq 0$,

- $|x_k - x_{k+1}| \leq \frac{1}{1-\sigma} d_k$,
- $|\gamma(x_k) - \gamma(z_0)| = d_k$,
- $d_{k+1} \leq \frac{1}{100} d_k$,
- $B_{10d_k}(x_k) \subset B_\rho(x_0)$.

Once we have constructed this sequence, we see that x_k is convergent to some $\bar{x} := \lim_{k\to\infty} x_k \in B_\rho(x_0)$ and $\gamma(\bar{x}) = \lim_{k\to\infty} \gamma(x_k) = \gamma(z_0)$. Since γ is injective, this implies $\bar{x} = z_0$ and thus $z_0 \in B_\rho(x_0)$.

Let x_k be already given; we need to construct x_{k+1}. Set

$$d_k := |\gamma(x_k) - \gamma(z_0)| \quad \text{and} \quad \eta_k(t) := \gamma(x_k + t d_k).$$

Observe that for $|t| < 1$, by (4.21),

$$|\eta_k(t) - \gamma(x_k)| < d_k.$$

On the other hand, whenever $|t| \geq (1-\sigma)^{-1}$, by (4.21) we have

$$|\eta_k(t) - \gamma(x_k)| \geq d_k.$$

The intermediate value theorem yields $t_- \in [-(1-\sigma)^{-1}, -1]$ and $t_+ \in [1, (1-\sigma)^{-1}]$ such that

$$|\eta_k(t_\pm) - \gamma(x_k)| = d_k.$$

Without loss of generality (otherwise we interchange the role of t_- and t_+ below) we may assume that

$$|\eta_k(t_-) - \gamma(z_0)| \geq \tfrac{1}{3} d_k. \tag{4.22}$$

Indeed, if the inequality was false for both t_+ and t_-, we would have

$$|\eta_k(t_-) - \eta_k(t_+)| \leq d_k,$$

which violates the bi-Lipschitz assumption (4.21).

Set

$$y_k := x_k + t_+ d_k.$$

Denote the line through $\gamma(x_k)$ and $\gamma(y_k)$ by L_{x_k,y_k}, more precisely let

$$L_{x_k,y_k}(t) := (1-t)\gamma(x_k) + t\gamma(y_k), \quad t \in \mathbb{R}.$$

Since $B_{d_k}(x_k) \subset B_\rho(x_0)$, we can apply Lemma 4.8 to find that

$$|L_{x_k,y_k}(t_1) - \gamma(z_0)| = \inf_{t\in\mathbb{R}} |L_{x_k,y_k}(t) - \gamma(z_0)| \leq \varepsilon d_k \tag{4.23}$$

for some $t_1 \in [-1, 1]$. By Pythagoras' theorem,

$$d_k^2(1-\varepsilon^2) \leq |L_{x_k,y_k}(t_1) - \gamma(x_k)|^2 = t_1^2 d_k^2.$$

That is, if ε is chosen small enough, then $|t_1| \geq 1 - 2\varepsilon$.

We now argue that $t_1 \geq 1 - 2\varepsilon$. Indeed, if we had $t_1 \leq -1 + 2\varepsilon$, then from the bi-Lipschitz estimate,

$$|\eta_k(t_1) - \eta_k(t_+)| \geq (1-\sigma)(2-2\varepsilon)d_k.$$

On the other hand,

$$|L_{x_k,y_k}(t_1) - \eta_k(t_+)| = |1-t_1|\,|\gamma(x_k) - \gamma_k(y_k)| \geq (2-2\varepsilon)d_k.$$

Moreover,

$$|\eta_k(t_1) - \gamma(x_k)| \le d_k \quad \text{and} \quad |L_{x_k,y_k}(t_1) - \gamma(x_k)| = |t_1| d_k \le d_k.$$

So we have

$$L_{x_k,y_k}(t_1), \eta_k(t_1) \in B_{d_k}(\gamma(x_k)) \setminus B_{(1-\sigma)(2-2\varepsilon)d_k}(\eta_k(t_+)).$$

But also

$$|\eta_k(t_+) - \gamma(x_k)| = d_k.$$

From elementary geometry this implies that for small enough ε and σ we have

$$|L_{x_k,y_k}(t_1) - \eta_k(t_1)| < \tfrac{1}{4} d_k.$$

But then from the projection assumption (4.23),

$$\begin{aligned}|\eta_k(t_-) - \gamma(z_0)| &\le |\eta_k(t_-) - \eta_k(t_1)| + |\eta_k(t_1) - L_{x_k,y_k}(t_1)| + |L_{x_k,y_k}(t_1) - \gamma(z_0)| \\ &\le |t_- - t_1| d_k + \tfrac{1}{4} d_k + \varepsilon d_k < \tfrac{1}{2} d_k\end{aligned}$$

for small enough ε and σ. This contradicts (4.22). That is, we have shown $t_1 \in [1-2\varepsilon, 1]$.

Now we find that

$$|\gamma(y_k) - L_{x_k,y_k}(t_1)| = (1-t_1)|\gamma(x_k) - \gamma(y_k)| \le 2\varepsilon d_k.$$

Consequently, by (4.23), for ε small enough,

$$|\gamma(y_k) - \gamma(z_0)| \le 3\varepsilon d_k < \tfrac{1}{100} d_k.$$

So if we set $x_{k+1} := y_k = x_k + t_+ d_k \in B_\rho(x_0)$, we see from (4.21) that

$$|x_{k+1} - x_k| \le (1-\sigma)^{-1} d_k.$$

Also, by the definition, $d_{k+1} := |\gamma(x_{k+1}) - \gamma(z_0)| < \frac{1}{100} d_k$. Lastly,

$$\begin{aligned}B_{10 d_{k+1}}(x_{k+1}) &\subset B_{\frac{1}{10} d_k}(x_{k+1}) \\ &\subset B_{\frac{1}{10} + (1-\sigma)^{-1}) d_k}(x_k) \subset B_{10 d_k}(x_k) \subset B_\rho(x_0).\end{aligned} \tag{4.24}$$

We have thus constructed x_{k+1} with the required properties. ∎

4.4. Convergence

Proposition 4.10. *Let $q + 2 \le p < 2q + 1$. For any $\Lambda > 0$ and $\varepsilon \in (0,1)$ there exists an $L = L(\varepsilon, \Lambda) \in \mathbb{N}$ such that the following holds.*

Let $\gamma_k : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with $|\gamma_k'| \equiv 1$ be C^1-homeomorphisms with

$$\sup_k \|\gamma_k\|_{L^\infty} + \sup_k \mathrm{TP}^{q+2,q}(\gamma_k) \le \Lambda.$$

Then there exists a subsequence $(\gamma_{k_i})_{i \in \mathbb{N}}$ and $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that the following hold for some finite set $\Sigma \subset \mathbb{R}/\mathbb{Z}$ with $\#\Sigma \le L$:

(1) γ_{k_i} *converges uniformly to* γ.

(2) *For any* $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \Sigma$ *there exists a radius* $\rho = \rho(x_0) > 0$ *such that* γ_{k_i} *weakly converges to* γ *in* $W^{1+\frac{p-q-1}{q},q}(B_\rho(x_0))$.

(3) $|\gamma'| = 1$ *a.e.*

(4) γ_{k_i} *and* γ *are uniformly bi-Lipschitz in* $B_\rho(x_0)$ *with*

$$(1-\varepsilon)|x-y| \le |\gamma_{k_i}(x) - \gamma_{k_i}(y)| \le |x-y| \quad \forall x, y \in B_\rho(x_0)\ \forall i \tag{4.25}$$

and

$$(1-\varepsilon)|x-y| \le |\gamma(x) - \gamma(y)| \le |x-y| \quad \forall x, y \in B_\rho(x_0). \tag{4.26}$$

(5) *For any* $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \Sigma$ *and any* $y_0 \in \mathbb{R}/\mathbb{Z}$ *with* $|\gamma_{k_i}(x_0) - \gamma_{k_i}(y_0)| \le \frac{1}{100}\rho(x_0)$ *or* $|\gamma(x_0) - \gamma(y_0)| \le \frac{1}{100}\rho(x_0)$ *we have* $|x_0 - y_0| \le \rho(x_0)$.

(6) *In particular, if* $\gamma(x) = \gamma(y)$, *then either* $x = y$ *or* $\{x, y\} \subset \Sigma$.

(7) *We have lower semicontinuity:*

$$\mathrm{TP}^{p,q}(\gamma) \le \liminf_{k\to\infty} \mathrm{TP}^{p,q}(\gamma_k). \tag{4.27}$$

(8) $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ *is a homeomorphism.*

(9) γ *is globally bi-Lipschitz.*

(10) $\gamma \in W^{1+s,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ *for any* $0 < s < 1/q$.

Proof. (1) By the Arzelà–Ascoli theorem, up to taking a subsequence, we may assume that (1) holds.

(2) Fix $\delta > 0$ to be specified later. By Proposition 4.3, up to taking a further subsequence there is a discrete singular set Σ with $\#\Sigma \le L$ such that for any $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \Sigma$ there exists $\rho_{x_0} > 0$ such that

$$\limsup_{k\to\infty} \int_{B_{\rho_{x_0}}(x_0,10)} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^p}\, dx\, dy < \delta. \tag{4.28}$$

From Theorem 4.5 we find that for each $x_0 \in \mathbb{R}/\mathbb{Z} \setminus \Sigma$,

$$\sup_k\, [\gamma_k']^q_{W^{\frac{p-q-1}{q},q}(B_{\rho_{x_0}}(x_0))} \le C\delta. \tag{4.29}$$

By reflexivity of $W^{\frac{p-q-1}{q},p}(B_{\rho_{x_0}}(x_0))$ and the Banach–Alaoglu theorem, combined with Rellich's theorem, we find that γ_k' weakly converges to γ' in $W^{\frac{p-q-1}{q},p}(B_{\rho_{x_0}}(x_0))$ and the convergence is pointwise a.e. in $B_{\rho_{x_0}}(x_0)$ and strong in $L^1(B_{\rho_{x_0}}(x_0))$. Observe that by uniqueness of the weak limit we need not pass to a further subsequence here. This establishes (2).

(3) In particular, γ_k' a.e. converges to γ' in $\mathbb{R}/\mathbb{Z} \setminus \Sigma$, and Σ is an $\mathscr{L}^1$-zero set. This implies

$$|\gamma'(x)| = \lim_{k\to\infty} |\gamma_k'(x)| = 1 \quad \text{a.e. } x \in \mathbb{R}/\mathbb{Z}.$$

We have established (3).

(4) Having chosen δ in (4.28) small enough, we get a small $W^{\frac{p-q-1}{q},p}(B_{\rho_{x_0}}(x_0))$-norm from (4.29), and from Lemma 2.1 we obtain (4.25). From the uniform convergence $\gamma_k \to \gamma$ we obtain (4.26). This establishes (4).

(5) The statement in (5) for γ_k is a consequence of Theorem 4.9. By uniform convergence this carries over to γ.

(6) Assume $x, y \in \mathbb{R}/\mathbb{Z}$ with $\gamma(x) = \gamma(y)$. If $x \notin \Sigma$, we infer from (5) that $y \in B_{\rho(x)}(x)$. But by (4.26) this implies $x = y$. Similarly, if $y \notin \Sigma$ we obtain $y = x$. So if $\gamma(x) = \gamma(y)$ then either $x = y$ or x and y both belong to Σ.

(7) In order to prove (4.27) observe that by (6) we have

$$|\gamma(x) - \gamma(y)| = 0 \implies (x, y) \in \{(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} : x = y\} \cup \Sigma \times \Sigma.$$

In particular,

$$\{(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} : |\gamma(x) - \gamma(y)| = 0\} \quad \text{is an } \mathscr{L}^2\text{-zero set.}$$

Consequently, the pointwise convergence of γ_k to γ and the $\mathscr{L}^1$-almost everywhere convergence of γ_k' to γ' imply that

$$\frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p} = \lim_{k\to\infty} \frac{|\gamma_k'(x) \wedge (\gamma_k(x) - \gamma_k(y))|^q}{|\gamma_k(x) - \gamma_k(y)|^p} \quad \mathscr{L}^2\text{-a.e. in } \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}.$$

From Fatou's lemma we thus have

$$\mathrm{TP}^{p,q}(\gamma) \le \liminf_{k\to\infty} \mathrm{TP}^{p,q}(\gamma_k),$$

and (7) is established.

(8) By now we know that γ has finite tangent-point energy. By [74, Theorem 1.1] this implies that $\gamma(\mathbb{R}/\mathbb{Z})$ is a topological 1-manifold. On the other hand, $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ has only finitely many self-intersection points, namely $\gamma(\Sigma)$. This means that there are no intersection points at all.

Indeed, assume there are distinct $x_1, x_2 \in \Sigma$ with $\gamma(x_1) = \gamma(x_2)$. Then there exists $\varepsilon > 0$ such that $[x_i - \varepsilon, x_i + \varepsilon] \cap \Sigma = \{x_i\}$, $i = 1, 2$, since Σ is a discrete set. Now $\gamma : [x_i - \varepsilon, x_i + \varepsilon] \to \mathbb{R}^3$ is a one-to-one map and it is (even Lipschitz) continuous.

Now denote by C a cross,

$$C := \{z = (z_1, z_2) \in [-1, 1]^2 : z_1 = 0 \text{ or } z_2 = 0\}$$

and define $f : C \to \mathbb{R}^3$ by

$$f(z) := \begin{cases} \gamma(x_1 + \varepsilon z_1), & z = (z_1, 0), \\ \gamma(x_2 + \varepsilon z_2), & z = (0, z_2). \end{cases}$$

Then f is injective and continuous. Since C is a compact set, we conclude that $f : C \to f(C) \subset \gamma(\mathbb{R}/\mathbb{Z})$ is a homeomorphism. Since $\gamma(\mathbb{R}/\mathbb{Z})$ is a one-dimensional topological manifold, around any $p_0 \in \gamma(\mathbb{R}/\mathbb{Z})$ there exists a homeomorphism h :

$\gamma(\mathbb{R}/\mathbb{Z}) \cap B_{\tilde{\delta}}(p_0) \to \mathbb{R}$ for some small $\tilde{\delta} > 0$. Taking $p_0 := \gamma(x_1)$ we see that $h \circ f : C \cap B_\delta(0) \to \mathbb{R}$ is a homeomorphism for a smaller $\delta > 0$. But the cross C is not homeomorphic to any subset in $\mathbb{R}$, and we have a contradiction from our assumption that $x_1 \neq x_2$. In conclusion, γ is injective. Since $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is continuous and $\mathbb{R}/\mathbb{Z}$ compact we conclude that $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is a homeomorphism. Thus (8) is established.

(9)

$$\inf_{x \neq y \in \mathbb{R}/\mathbb{Z}} \frac{|\gamma(x) - \gamma(y)|}{|x - y|} = 0.$$

Then there exist (convergent) sequences $\mathbb{R}/\mathbb{Z} \ni x_k \xrightarrow{k\to\infty} \bar{x}$ and $\mathbb{R}/\mathbb{Z} \ni y_k \xrightarrow{k\to\infty} \bar{y}$ with

$$\lim_{k\to\infty} \frac{|\gamma(x_k) - \gamma(y_k)|}{|x_k - y_k|} = 0. \tag{4.30}$$

We make several observations:

- $\bar{x} = \bar{y}$. Indeed, if $\bar{x} \neq \bar{y}$, then the continuity of γ combined with (4.30) implies that $0 = \gamma(\bar{x}) - \gamma(\bar{y})$. Since γ is injective, this implies $\bar{x} = \bar{y}$, a contradiction.

- $\bar{x} \in \Sigma$. Indeed, if $\bar{x} \notin \Sigma$, then for all k sufficiently large, $x_k, y_k \in B_{\rho_{\bar{x}}}(\bar{x})$, thus by (4.26),

$$\frac{|\gamma(x_k) - \gamma(y_k)|}{|x_k - y_k|} \geq 1 - \varepsilon \quad \forall k \gg 1.$$

This contradicts (4.30).

- For all but finitely many $k \in \mathbb{N}$ (up to interchanging x_k and y_k) we have $x_k < \bar{x} < y_k$ for all $k \in \mathbb{N}$. Indeed, let $K > 0$ be such that

$$\frac{|\gamma(x_k) - \gamma(y_k)|}{|x_k - y_k|} < \frac{1}{4} \quad \forall k \geq K. \tag{4.31}$$

Also, combining Lemma 2.1 and Theorem 4.5, let $\delta = \delta(\varepsilon) > 0$ be such that for any ball $B \subset \mathbb{R}/\mathbb{Z}$ of diameter $\leq 1/2$,

$$\mathrm{TP}^{p,q}(\gamma; B) < \delta \text{ and } [\gamma']_{W^{\frac{p-q-1}{q},q}(B)} < \infty$$
$$\text{implies} \quad |\gamma(x) - \gamma(y)| \geq \tfrac{1}{2}|x - y| \; \forall x, y \in B. \tag{4.32}$$

By absolute continuity of the integral, and since $\mathrm{TP}^{p,q}(\gamma) < \infty$, there exists a $\bar{\rho} > 0$ such that

$$\mathrm{TP}^{p,q}(\gamma; B) < \delta \quad \text{for all balls } B \subset \mathbb{R}/\mathbb{Z} \text{ with } \operatorname{diam} B < \bar{\rho}. \tag{4.33}$$

Now assume by contradiction that $x_k, y_k \in B_{\bar{\rho}/2}(\bar{x})$ and $x_k < y_k \leq \bar{x}$ for some $k > K$. Take a sequence $(\tilde{y}_{k;i})_{i\in\mathbb{N}}$ such that $x_k < \tilde{y}_{k;i} < \bar{x}$ with $\tilde{y}_{k;i} \xrightarrow{i\to\infty} y_k$. There exists an open ball $B_{k,i} \subset \overline{B_{k,i}} \subset \mathbb{R}/\mathbb{Z} \setminus \Sigma$ of radius $< \bar{\rho}/2$ such that $x_k, \tilde{y}_{k;i} \subset B_{k,i}$. By a covering argument, from (2), we obtain $\gamma' \in W^{\frac{p-q-1}{q},q}(B_{k,i})$ (without any estimate for the norm). However, since the ball $B_{k,i}$ is small enough, we have $\mathrm{TP}^{p,q}(\gamma; B_{k,i}) < \delta$ by (4.33), so from (4.32) we obtain

$$\frac{|\gamma(x_k)-\gamma(\tilde{y}_{k;i})|}{|x_k-\tilde{y}_{k;i}|}\geq\frac{1}{2}.$$

This holds for all $i\in\mathbb{N}$, so letting $i\to\infty$ we get

$$\frac{|\gamma(x_k)-\gamma(y_k)|}{|x_k-y_k|}\geq\frac{1}{2},$$

a contradiction to (4.31).

- There exists $K\in\mathbb{N}$ such that for any $k\geq K$,

$$|\gamma(x_k)-\gamma(y_k)|\geq\tfrac{1}{20}\max\{|x_k-\bar{x}|,|y_k-\bar{x}|\}. \tag{4.34}$$

Indeed, let $\delta>0$ be from Theorem 4.9, and let $R>0$ be such that (4.20) is satisfied for any $x_0\in\mathbb{R}/\mathbb{Z}$ and any $\rho\in(0,R)$ – such an $R>0$ exists by absolute continuity of the integral and since $\mathrm{TP}^{p,q}(\gamma)<\infty$. We can assume by taking R possibly smaller that $B_R(\bar{x})\cap\Sigma=\{\bar{x}\}$.

Let $K\in\mathbb{N}$ be such that $|x_k-y_k|<\frac{1}{100}R$ for all $k\geq K$. Set $\rho:=\frac{1}{2}|x_k-\bar{x}|<R$. Observe that $B_\rho(x_k)\cap\Sigma=\emptyset$ and thus $\gamma'\in W^{\frac{p-q-1}{q},q}(B_\rho(x_k))$ by a covering argument and (2). Since $\bar{x}\notin B_\rho(x_k)$ and $x_k<\bar{x}<y_k$, we find that $y_k\notin B_\rho(x_k)$. Applying Theorem 4.9 in $B_\rho(x_k)$, we conclude that

$$|\gamma(x_k)-\gamma(y_k)|\geq\tfrac{1}{20}|x_k-\bar{x}|.$$

By a similar argument,

$$|\gamma(x_k)-\gamma(y_k)|\geq\tfrac{1}{20}|y_k-\bar{x}|.$$

To deduce (9), observe that by the triangular inequality

$$\max\{|x_k-\bar{x}|,|y_k-\bar{x}|\}\geq\tfrac{1}{2}|x_k-y_k|.$$

Combining this with (4.34) implies

$$|\gamma(x_k)-\gamma(y_k)|\geq\tfrac{1}{40}|x_k-y_k|,$$

which is a contradiction to (4.30). This establishes (9).

(10) The set Σ of points where we do not know already that γ is Sobolev, is at most finite. For simplicity of notation assume that $0\in\Sigma$. Take $r>0$ small enough such that $B_r\cap\Sigma=\{0\}$, and that for δ from Theorem 4.5,

$$\mathrm{TP}^{p,q}(\gamma;B_r)<\delta.$$

For small $\sigma>0$ let

$$X_\sigma:=([-r,-\sigma]\cup[\sigma,r])\times([-r,-\sigma]\cup[\sigma,r]).$$

Since $\mathcal{L}^2([-r,r]^2\setminus X_\sigma)\xrightarrow{\sigma\to0}0$, it suffices to show that

$$\limsup_{\sigma\to0+}\iint_{X_\sigma}\frac{|\gamma'(x)-\gamma'(y)|^q}{|x-y|^{1+sq}}\,dx\,dy<\infty.$$

Since we already know that $\gamma' \in W^{\frac{p-q-1}{q},q}([-r,-\sigma]) \cup W^{\frac{p-q-1}{q},q}([\sigma,r])$, we can use Theorem 4.5 to obtain

$$\iint_{X_\sigma} \frac{|\gamma'(x)-\gamma'(y)|^q}{|x-y|^{1+sq}}\,dx\,dy$$
$$\lesssim \iint_{[\sigma,r]^2\cup[-r,-\sigma]^2} \frac{|\gamma'(x)-\gamma'(y)|^q}{|x-y|^{p-q}}\,dx\,dy + \|\gamma'\|_{L^\infty} \int_{(-r,0)}\int_{(0,r)} \frac{1}{|x-y|^{1+sq}}\,dx\,dy.$$

The second integral is finite for $s < 1/q$. ■

Remark 4.11. It is unclear to us whether Proposition 4.10 (10) holds for $s = 1/q$. If one were able to prove this, then there is a chance to remove the singular set for the regularity theory in Corollary 1.8 with the removability argument as in [50].

5. Weak limits of minimizing sequences are critical: Proof of Theorem 1.6

We would like to compare our minimizing sequence γ_k with the variation $\gamma + t\varphi$, where φ is a locally supported test function. Computing the Euler–Lagrange equations then proves Corollary 1.8. For notational convenience, we restrict ourselves to the case $p = q+2$ instead of $p \geq q+2$. The case $p > q+2$ follows in the same way, but it can also be obtained by simpler, more direct methods.

Theorem 5.1 (Minimizing sequence converging to a critical point). *There exists $\varepsilon_0 > 0$ such that the following holds.*

Let $q > 1$, $\gamma_k \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$, $\gamma \in \mathrm{Lip}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$, $|\gamma_k'| = |\gamma'| = 1$ a.e., with

$$\sup_k \mathrm{TP}^{q+2,q}(\gamma_k) < \infty.$$

Assume that γ_k is approximately minimizing, in the sense that

$$\mathrm{TP}^{q+2,q}(\gamma_k) \leq \mathrm{TP}^{q+2,q}(\tilde\gamma) + 1/k$$

for any $\tilde\gamma$ ambient isotopic to γ_k.

Assume that γ_k uniformly converges to γ in $\mathbb{R}/\mathbb{Z}$ and for a geodesic ball $B_{100\rho} \subset \mathbb{R}/\mathbb{Z}$, e.g. $\rho < \frac{1}{1000}$,

$$\sup_k \int_{B_{100\rho}} \int_{\mathbb{R}/\mathbb{Z}} \mu(\gamma_k,x,y)\,dy\,dx < \varepsilon_0 \tag{5.1}$$

where

$$\mu(\sigma,x,y) := \frac{|\sigma'(x)\wedge(\sigma(x)-\sigma(y))|^q}{|\sigma(x)-\sigma(y)|^{q+2}}\,|\sigma'(x)|^{1-q}\,|\sigma'(y)|.$$

Then for any $\varphi \in C_c^\infty(B_\rho,\mathbb{R}^3)$ there exists $t_0 > 0$ such that for all $t \in (-t_0,t_0)$,

$$\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus(B_\rho{}^c)^2} \mu(\gamma,x,y)\,dx\,dy \leq \iint_{(\mathbb{R}/\mathbb{Z})^2\setminus(B_\rho{}^c)^2} \mu(\gamma+t\varphi,x,y)\,dx\,dy.$$

For the proof of Theorem 5.1 we need to obtain first a fractional version of the Luckhaus lemma.

5.1. A fractional Luckhaus lemma in one dimension

The Luckhaus lemma [49, Lemma 1] is an important tool for harmonic maps, usually given in the form below; see [72, Section 2.6, Lemma 1]. It essentially provides a way to glue together two maps u and v along the boundary $\partial B_1(0)$ with explicit dependence on the size δ of the glued region.

Lemma 5.2. *Let $\mathcal{N}$ be any compact subset of $\mathbb{R}^n$, $n \geq 2$. Assume $u, v \in W^{1,2}(\mathbb{S}^{n-1}, \mathcal{N})$ and $\delta \in (0,1)$. Then there exists $w \in W^{1,2}(\mathbb{S}^{n-1} \times [0,\delta], \mathbb{R}^n)$ with $w = u$ in a neighborhood of $\mathbb{S}^{n-1} \times \{0\}$, $w = v$ in a neighborhood of $\mathbb{S}^{n-1} \times \{\delta\}$,*

$$\int_{\mathbb{S}^{n-1}\times[0,\delta]} |\nabla w|^n \leq C\delta \int_{\mathbb{S}^{n-1}} (|\nabla u|^2 + |\nabla v|^2) + C\delta^{-1} \int_{\mathbb{S}^{n-1}} |u-v|^2,$$

and

$$\begin{aligned}&\operatorname{dist}^2(w(\mathbb{S}^{n-1} \times [0,\delta]), \mathcal{N}) \\ &\quad \leq C\delta^{1-n} \left(\int_{\mathbb{S}^{n-1}} (|\nabla u|^2 + |\nabla v|^2)\right)^{1/2} \left(\int_{\mathbb{S}^{n-1}} |u-v|^2\right)^{1/2} + C\delta^{-n} \int_{\mathbb{S}^{n-1}} |u-v|^2.\end{aligned}$$

We will need a version of this lemma for fractional Sobolev spaces in one dimension. Working in one dimension has advantages and disadvantages: The advantage is that the boundary of a ball consists of two points, and the possibility of explicit computations. The main disadvantage is that there may be no reasonable trace spaces for $W^{s,p}([0,1])$ when $sp \leq 1$. In any case, the following might be interesting on its own.

Lemma 5.3. *Assume $u, v : \mathbb{R} \to \mathbb{R}^3$ are locally integrable, have a Lebesgue point at $x = \pm 1$, and*

$$\int_{\mathbb{R}} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}\, dy + \int_{\mathbb{R}} \frac{|v(x)-v(y)|^p}{|x-y|^{1+sp}}\, dy < \infty, \quad x = -1, 1.$$

Then for any $\delta \leq 1/2$ there exists $w : (-2,2) \to \mathbb{R}^3$ with the following properties:

- *$w(x) = u(x)$ for $|x| > 1$ and $w(x) = v(x/(1-\delta))$ for $|x| < 1-\delta$, namely we can choose*

$$w(x) = \begin{cases} u(x), & |x| > 1, \\ (1-\eta(x))u(-1) + \eta(x)v(-1), & x \in [-1, -1+\delta], \\ v(x/(1-\delta)), & |x| < 1-\delta, \\ (1-\eta(x))u(1) + \eta(x)v(1), & x \in [1-\delta, 1], \end{cases}$$

  where $\eta : \mathbb{R} \to [0,1]$ is any smooth map such that $\eta \equiv 0$ for $|x| > 1-\delta/4$, $\eta \equiv 1$ for $|x| \leq 1 - \frac{1}{2}\delta$, and $|\eta'| \lesssim 1/\delta$.

- *For any $r \in (1,2)$ we have the estimate*

$$
\begin{aligned}
&[w]^p_{W^{s,p}(-r,r)} \\
&\quad \le \int_{1<|y|<r} \int_{r>|x|>1} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}\,dx\,dy + (1-\delta)^{1-sp}[v]^p_{W^{s,p}(-1,1)} \\
&\qquad + 2(1-\delta) \int_{|y|<1} \int_{r>|x|>1} \frac{|u(x)-v(y)|^p}{|x-(1-\delta)y|^{1+sp}}\,dx\,dy \\
&\qquad + C\delta \Big(\int_{r>|y|>1} \frac{|u(1)-u(y)|^p}{|1-y|^{1+sp}}\,dy + \int_{|y|>1} \frac{|u(-1)-u(y)|^p}{|-1-y|^{1+sp}}\,dy \Big) \\
&\qquad + C\delta(1-\delta)^{-sp} \Big(\int_{|y|<1} \frac{|v(1)-v(y)|^p}{|1-y|^{1+sp}}\,dy + \int_{|y|<1} \frac{|v(-1)-v(y)|^p}{|-1-y|^{1+sp}}\,dy \Big) \\
&\qquad + C\delta^{1-sp}\,(|u(1)-v(1)|^p + |u(-1)-v(-1)|^p) \\
&\qquad + C\delta^2(|u(1)-u(-1)|^p + |v(1)-v(-1)|^p). \qquad (5.2)
\end{aligned}
$$

- *If we set $K := u(-2,2) \cup v(-2,2)$, then*

$$\operatorname{dist}(w((-2,2)), K) \le |u(-1)-v(-1)| + |u(1)-v(1)|.$$

Proof. We can find η with the properties specified above such that $|\eta'| \lesssim 1/\delta$. Then (for simplicity we assume $r=2$ here, the case of general r is the same)

$$
\begin{aligned}
&\int_{(-2,2)} \int_{(-2,2)} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy \\
&\qquad = \int_{|y|>1} \int_{|x|>1} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}\,dx\,dy \\
&\qquad\quad + (1-\delta)^{1-sp} \int_{|y|<1} \int_{|x|<1} \frac{|v(x)-v(y)|^p}{|x-y|^{1+sp}}\,dx\,dy \\
&\qquad\quad + \mathrm{III} + 2\mathrm{IV} + 2\mathrm{V} + 2\mathrm{VI}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{III} &:= \int_{|y|\in(1-\delta,1)} \int_{|x|\in(1-\delta,1)} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy, \\
\mathrm{IV} &:= \int_{|y|<1-\delta} \int_{|x|>1} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy, \\
\mathrm{V} &:= \int_{|y|\in(1-\delta,1)} \int_{|x|<1-\delta} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy, \\
\mathrm{VI} &:= \int_{|y|\in(1-\delta,1)} \int_{|x|>1} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy.
\end{aligned}
$$

To estimate III we observe that

$$\begin{aligned}
\mathrm{III} \lesssim& \iint_{x,y\in(1-\delta,1)} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+\iint_{x,y\in(-1,-1+\delta)} \frac{|w(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy,\\
&+\big(|u(-1)-u(1)|^p+|v(1)-v(-1)|^p\big)\delta^2\\
\lesssim& |v(1)-u(1)|^p \iint_{x,y\in(1-\delta,1)} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+|v(-1)-u(-1)|^p \iint_{x,y\in(-1,-1+\delta)} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+(|u(-1)-u(1)|^p+|v(1)-v(-1)|^p)\delta^2\\
\lesssim& \delta^{1-sp}|v(1)-u(1)|^p+\delta^{1-sp}|v(-1)-u(-1)|^p\\
&+\delta^2(|u(-1)-u(1)|^p+|v(1)-v(-1)|^p)
\end{aligned}$$

Also

$$\begin{aligned}
\mathrm{IV} =& \int_{|y|<1-\delta}\int_{|x|>1} \frac{|u(x)-v(y/(1-\delta))|^p}{|x-y|^{1+sp}}\,dx\,dy\\
=& (1-\delta)\int_{|y|<1}\int_{|x|>1} \frac{|u(x)-v(y)|^p}{|x-(1-\delta)y|^{1+sp}}\,dx\,dy.
\end{aligned}$$

The tricky terms (that need to vanish as $\delta\to 0$) are the remaining ones:

$$\begin{aligned}
\mathrm{V} =& \int_{Y_1}\int_X \frac{|v(x/(1-\delta))-(1-\eta(y))u(1)-\eta(y)v(1)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+\int_{Y_2}\int_X \frac{|v(x/(1-\delta))-(1-\eta(y))u(-1)-\eta(y)v(-1)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
=& \int_{Y_1}\int_X \frac{|v(x/(1-\delta))-v(1)+(1-\eta(y))(v(1)-u(1))|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+\int_{Y_2}\int_X \frac{|v(x/(1-\delta))-v(-1)+(1-\eta(y))(v(-1)-u(-1))|^p}{|x-y|^{1+sp}}\,dx\,dy\\
=& \int_{Y_1}\int_X \frac{|v(x/(1-\delta))-v(1)+(\eta(x)-\eta(y))(v(1)-u(1))|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+\int_{Y_2}\int_X \frac{|v(x/(1-\delta))-v(-1)+(\eta(x)-\eta(y))(v(-1)-u(-1))|^p}{|x-y|^{1+sp}}\,dx\,dy,
\end{aligned}$$

where $X=(-1+\delta,1-\delta)$, $Y_1=(1-\delta,1)$ and $Y_2=(-1,-1+\delta)$. That is,

$$\begin{aligned}
V \lesssim& \int_{Y_1}\int_X \frac{|v(x/(1-\delta))-v(1)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+\int_{Y_2}\int_X \frac{|v(x/(1-\delta))-v(-1)|^p}{|x-y|^{1+sp}}\,dx\,dy\\
&+\delta^{1-sp}(|u(1)-v(1)|^p+|u(-1)-v(-1)|^p).
\end{aligned}$$

Now for $y \in Y_1$ and $x \in X$ we have $|x-y| \geq |x-(1-\delta)|$, thus

$$\int_{Y_1}\int_X \frac{|v(x/(1-\delta))-v(1)|^p}{|x-y|^{1+sp}}\,dx\,dy \leq \int_{Y_1}\int_X \frac{|v(x/(1-\delta))-v(1)|^p}{|x-(1-\delta)|^{1+sp}}\,dx\,dy$$
$$= \delta\int_X \frac{|v(x/(1-\delta))-v(1)|^p}{|x-(1-\delta)|^{1+sp}}\,dx$$
$$= \delta(1-\delta)^{-sp}\int_{|x|<1}\frac{|v(x)-v(1)|^p}{|x-1|^{1+sp}}\,dx.$$

Arguing similarly for the other part, we obtain

$$\begin{aligned}\mathrm{V} \leq\ & C\delta(1-\delta)^{-sp}\int_{|x|<1}\frac{|v(x)-v(-1)|^p}{|-1-x|^{1+sp}}\,dx \\ &+ C\delta(1-\delta)^{-sp}\int_{|x|<1}\frac{|v(x)-v(1)|^p}{|x-1|^{1+sp}}\,dx \\ &+ \delta^{1-sp}\big(|u(1)-v(1)|^p + |u(-1)-v(-1)|^p\big).\end{aligned}$$

Now for VI,

$$\mathrm{VI} = \int_{|y|\in(1-\delta,1)}\int_{|x|>1}\frac{|u(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy.$$

Again we have two parts, of which we only estimate

$$\begin{aligned}&\int_{y\in(1-\delta,1)}\int_{|x|>1}\frac{|u(x)-w(y)|^p}{|x-y|^{1+sp}}\,dx\,dy \\ &\quad= \int_{y\in(1-\delta,1)}\int_{|x|>1}\frac{|u(x)-(1-\eta(y))u(1)-\eta(y)v(1)|^p}{|x-y|^{1+sp}}\,dx\,dy \\ &\quad= \int_{y\in(1-\delta,1)}\int_{|x|>1}\frac{|u(x)-u(1)+\eta(y)(u(1)-v(1))|^p}{|x-y|^{1+sp}}\,dx\,dy \\ &\quad\precsim \int_{y\in(1-\delta,1)}\int_{|x|>1}\frac{|u(x)-u(1)|^p}{|x-y|^{1+sp}}\,dx\,dy + |u(1)-v(1)|^p\delta^{1-sp} \\ &\quad\precsim \int_{y\in(1-\delta,1)}\int_{|x|>1}\frac{|u(x)-u(1)|^p}{|x-1|^{1+sp}}\,dx\,dy + |u(1)-v(1)|^p\delta^{1-sp} \\ &\quad= \delta\int_{|x|>1}\frac{|u(x)-u(1)|^p}{|x-1|^{1+sp}}\,dx + |u(1)-v(1)|^p\delta^{1-sp}.\end{aligned}$$

For the last inequality we have used the fact that for any $\delta \in (0,1)$, if $y \in (1-\delta,1)$ and $x > 1$ or $x < -1$, then $|x-y| \geq c|x-1|$ for some uniform constant c. ∎

5.2. *Proof of Theorem* 5.1

Armed with the Luckhaus lemma, we can now prove Theorem 5.1. The idea is to work with γ_k' and $\gamma' + t\varphi'$, using the Luckhaus lemma to glue $\gamma' + t\varphi'$ to γ_k' where the gluing happens in an annulus $B_R \setminus B_{R(1-\delta)}$. The important observation is that even under the assumption of only weak convergence the norms on the annulus $B_R \setminus B_{R(1-\delta)}$ vanish in the limit.

We face an additional technicality that we want to glue derivatives. Up to adding a corrector term after integration, this leads to a curve $\sigma_{k,\delta}$ which coincides with γ_k outside of a ball B_R. The map $\sigma_{k,\delta}$ may not coincide with $\gamma + t\varphi$ inside the ball B_R, but its derivative $\sigma'_{k,\delta}$ is essentially γ' inside the ball, which is good enough for our purposes.

Proof of Theorem 5.1. We may assume that the ball B_ρ is centered at zero. Since ρ is small, we may and do assume for simplicity that the distance $|x-y|$ corresponds to the Euclidean distance.

For any $\varepsilon > 0$ there is $\varepsilon_0 > 0$ such that (5.1) implies

$$[\gamma']_{W^{1/q,q}(B_{100\rho})} + \sup_k [\gamma'_k]_{W^{1/q,q}(B_{100\rho})} < \varepsilon. \tag{5.3}$$

Indeed, this follows, similar to the arguments in Section 4, from the fact that γ_k is smooth and (5.1) implies by Theorem 4.5 a uniform bound of γ'_k in $W^{1/q,q}(B_{100\rho})$, so that a subsequence of γ'_k converges to γ' weakly in $W^{1/q,q}(B_{100\rho})$.

Let $\varphi \in C_c^\infty(B_\rho)$. By Fubini's theorem, there exists $R \in (\rho, 2\rho)$ such that $\pm R$ are Lebesgue points of γ'_k and γ',

$$\int_{\mathbb{R}} \frac{|\gamma'(x)-\gamma'(y)|^q}{|x-y|^2}dy + \sup_k \int_{\mathbb{R}} \frac{|\gamma'_k(x)-\gamma'_k(y)|^q}{|x-y|^2}dy < \infty, \quad x = -R, R, \tag{5.4}$$

and we can also assume that $\gamma'_k(\pm R)$ converges to $\gamma'(\pm R)$ for this R, and $\gamma'_k \xrightarrow{k\to\infty} \gamma'$ weakly in $W^{1/q,q}(B_{100\rho})$ and strongly in $L^q(B_{100\rho})$.

This also implies

$$\int_{\mathbb{R}} \frac{|(\gamma+t\varphi)'(x)-(\gamma+t\varphi)'(y)|^q}{|x-y|^2}\, dy < \infty, \quad x = -R, R, \ \forall t \in \mathbb{R}. \tag{5.5}$$

Construction of a comparison map $\sigma_{k,\delta}$. Fix $\delta > 0$. Apply the Luckhaus lemma (Lemma 5.3) to $\gamma' + t\varphi'$ (within B_R) and γ'_k (in $B_R{}^c$). Then we obtain

$$g_{k,\delta}(x) = \begin{cases} \gamma'_k(x), & x < -R, \\ (\gamma+t\varphi)'(x/(1-\delta)), & |x| < (1-\delta)R, \\ \gamma'_k(x), & x > R. \end{cases}$$

For $(1-\delta)R \le |x| \le R$ we have an interpolation between $\gamma'(\pm R)$ and $\gamma'_k(\pm R)$ as in Lemma 5.3.

Observe that for $t_0 \ll 1$, $|(\gamma+t\varphi)'|$ is as close to 1 a.e. as we want, so we also get the estimate

$$\operatorname{dist}(g_{k,\delta}(x), \mathbb{S}^2) \le \varepsilon \quad \text{for } k \gg 1,\ t_0 \ll 1.$$

Pick $\theta \in C^\infty(\mathbb{R}, [0,1])$ with $\theta \equiv 0$ for $x < -R/2$, $\theta \equiv 1$ for $x \geq R/2$, and $|\theta'| \lesssim 1/R$. We set

$$\sigma_{k,\delta}(x) := \int_{-1}^{x} g_{k,\delta}(z)\, dz + \gamma_k(-1) + \theta(x)\left(\int_{-R}^{R} [\gamma_k'(z) - g_{k,\delta}(z)]\, dz\right). \tag{5.6}$$

Properties of $\sigma_{k,\delta}$. We need to show that $\sigma_{k,\delta}$ (for all small t, small δ and large k) is a comparison function for γ_k.

First we show

$$\sigma_{k,\delta} \equiv \gamma_k \quad \text{on } \mathbb{R}/\mathbb{Z} \setminus [-R, R]. \tag{5.7}$$

Indeed, observe that for $x < -R/2$,

$$\sigma_{k,\delta}(x) = \int_{-1}^{x} g_{k,\delta}(z)\, dz + \gamma_k(-1),$$

and thus

$$\sigma_{k,\delta}'(x) = g_{k,\delta}(x) = \gamma_k'(x) \quad \text{whenever } x < -R.$$

Since moreover $\sigma_{k,\delta}(-1) = \gamma_k(-1)$, we have

$$\sigma_{k,\delta} \equiv \gamma_k \quad \text{on } [-1, -R).$$

Moreover, for $x > R$,

$$\begin{aligned} \sigma_{k,\delta}(x) &= \int_{-1}^{x} g_{k,\delta}(z)\, dz + \gamma_k(-1) + \left(\int_{-R}^{R} [\gamma_k'(z) - g_{k,\delta}(z)]\, dz\right) \\ &= \int_{-1}^{x} \gamma_k'(z)\, dz + \gamma_k(-1). \end{aligned}$$

Again, this implies that $\sigma_{k,\delta}'(x) = \gamma_k'(x)$ for $x > R$ and since $\int_{-1}^{1} \gamma_k' = 0$, we find that $\sigma_{k,\delta}(1) = \gamma_k(-1) = \gamma_k(1)$.

Therefore, (5.7) is established.

Next, we show that there exists $t_0 > 0$, $\delta_0 > 0$ and $K_0 \in \mathbb{N}$ such that

$$1/2 \leq |\sigma_{k,\delta}'| \leq 3/2 \quad \forall |t| < t_0,\ \delta \in (0, \delta_0),\ k \geq K_0. \tag{5.8}$$

Indeed, there exists $t_0 > 0$ such that $|\gamma' + t\varphi'| \in (1 - \frac{1}{1000}, 1 + \frac{1}{1000})$ a.e. for all $|t| < t_0$. From Lemma 5.3 we then have

$$\big||g_{k,\delta}(x)| - 1\big| \leq \frac{1}{1000} + |\gamma_k'(-R) - \gamma'(-R)| + |\gamma_k'(R) - \gamma'(R)|.$$

Since by assumption $\gamma_k'(\pm R) \xrightarrow{k\to\infty} \gamma'(\pm R)$, we can find $K_0 \in \mathbb{N}$ such that

$$\big||g_{k,\delta}(x)| - 1\big| \leq \frac{2}{1000} \quad \forall |t| < t_0,\ k \geq K_0. \tag{5.9}$$

On the other hand, the term involving θ converges in $C^\infty(\mathbb{R})$ to zero. Namely, we have (recall that $\varphi(\pm R) = 0$)

$$\begin{aligned}
&\left|\int_{-R}^{R} [\gamma_k'(z) - g_{k,\delta}(z)]\, dz\right| \\
&\quad \le \left|\int_{-R}^{R} \gamma_k'(z)\, dz - \int_{-(1-\delta)R}^{(1-\delta)R} (\gamma + t\varphi)'(z/(1-\delta))\, dz\right| + 2\delta R \|g_{k,\delta}\|_{L^\infty} \\
&\quad \overset{(5.9)}{\le} \left|\int_{-R}^{R} \gamma_k'(z)\, dz - \int_{-(1-\delta)R}^{(1-\delta)R} (\gamma + t\varphi)'(z/(1-\delta))\, dz\right| + 4\delta R \\
&\quad = \left|\gamma_k(R) - \gamma_k(-R) - \big((1-\delta)(\gamma + t\varphi)(R) - (1-\delta)(\gamma + t\varphi)(-R)\big)\right| + 4\delta R \\
&\quad = \left|\gamma_k(R) - \gamma_k(-R) - \big((1-\delta)\gamma(R) - (1-\delta)\gamma(-R)\big)\right| + 4\delta R \\
&\quad \le \max_{x \in \{-R,R\}} \big(|\gamma_k(x) - \gamma(x)| + \delta|\gamma(x)|\big) + 4\delta R. \qquad (5.10)
\end{aligned}$$

Since γ_k uniformly converges to γ, we deduce that for any $L \in \mathbb{N}$,

$$\begin{aligned}
&\forall \tilde{\varepsilon} > 0\, \exists \delta_0 > 0, K_0 \in \mathbb{N}: \\
&\quad \left\|\theta(\cdot) \int_{-R}^{R} [\gamma_k'(z) - g_{k,\delta}(z)]\, dz\right\|_{C^L(\mathbb{R})} < \tilde{\varepsilon} \quad \forall \delta \in (0, \delta_0),\ k \ge K_0. \qquad (5.11)
\end{aligned}$$

(5.9) and (5.11) readily imply (5.8).

Next we estimate the Sobolev norm of $\sigma_{k,\delta}$. From the Luckhaus lemma, (5.2), we obtain the estimate (observe that we work on the ball so the constants depend on R by scaling)

$$\begin{aligned}
&[g_{k,\delta}]^q_{W^{1/q,q}(B_{100\rho})} \\
&\lesssim [\gamma_k']^q_{W^{1/q,q}(B_{100\rho})} + [\gamma']^q_{W^{1/q,q}(B_{100\rho})} + |t|[\varphi']^q_{W^{1/q,q}(B_{100\rho})} \\
&\quad + C(R)\delta\bigg(\int_{\mathbb{R}} \frac{|(\gamma + t\varphi)'(R) - (\gamma + t\varphi)'(y)|^q}{|R-y|^2}\, dy \\
&\qquad\qquad + \int_{\mathbb{R}} \frac{|(\gamma + t\varphi)'(-R) - (\gamma + t\varphi)'(y)|^q}{|-R-y|^2)}\, dy\bigg) \\
&\quad + C(R)\delta\bigg(\int_{\mathbb{R}} \frac{|\gamma_k'(R) - \gamma_k'(y)|^q}{|R-y|^2} dy + \int_{\mathbb{R}} \frac{|\gamma_k'(-R) - \gamma_k'(y)|^q}{|-R-y|^2}\, dy\bigg) \\
&\quad + C(R)\big(|\gamma_k'(R) - \gamma'(R)|^q + |\gamma_k'(-R) - \gamma'(-R)|^q\big) \\
&\quad + C(R)\delta^{-2}\|\gamma_k' - \gamma\|^q_{L^q(B_{100\rho})} + C(R)\delta^2.
\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}
&2(1-\delta) \int_{|y|<R} \int_{100\rho>|x|>R} \frac{|\gamma_k'(x) - (\gamma + t\varphi)'(y)|^q}{|x - (1-\delta)y|^2}\, dx\, dy \\
&\quad = 2(1-\delta) \int_{|y|<R} \int_{100\rho>|x|>R} \frac{|(\gamma_k + t\varphi)'(x) - (\gamma + t\varphi)'(y)|^q}{|x - (1-\delta)y|^2}\, dx\, dy \\
&\quad \lesssim (1-\delta)[(\gamma_k + t\varphi)']^q_{W^{1/q,q}(B_{100\rho})} + (1-\delta)\delta^{-2}\|\gamma_k' - \gamma'\|^q_{L^q(B_R)}.
\end{aligned}$$

In view of the convergence $\gamma_k'(\pm R) \xrightarrow{k\to\infty} \gamma'(\pm R)$, the L^q-convergence of γ_k' to γ', (5.4), and (5.5) together with (5.3), we find that there are δ_0, t_0 such that

$$\forall \delta \in (0,\delta_0)\,\exists K(\delta): \quad [g_{k,\delta}]^q_{W^{1/q,q}(B_{100\rho})} < 4\varepsilon \quad \forall k \geq K(\delta),\ |t| < t_0.$$

From (5.11) we thus conclude that taking possibly δ_0 and t_0 smaller and $K(\delta)$ larger,

$$\forall \delta \in (0,\delta_0)\,\exists K(\delta): \quad [\sigma'_{k,\delta}]^q_{W^{1/q,q}(B_{100\rho})} < 4\varepsilon \quad \forall k \geq K(\delta),\ |t| < t_0. \tag{5.12}$$

The comparison. In view of uniform convergence, (5.1), and (5.3), we can apply Theorem 4.9 and Lemma 2.1 to conclude that the assumptions of Theorem 3.7 are satisfied for $\sigma_{k,\delta}$ and γ_k for all $k \geq K(\delta)$ for some large number K.

Theorem 3.7 implies $\sigma_{k,\delta}$ and γ_k are ambient isotopic (technically, we can mollify $\sigma_{k,\delta}$ around $B_{5\rho}$; then this mollification is ambient isotopic to γ_k, so we get the estimate and remove the mollification again – observe that γ_k is smooth), and thus

$$\mathrm{TP}^{q+2,q}(\gamma_k) \leq \mathrm{TP}^{q+2,q}(\sigma_{k,\delta}) + 1/k \quad \forall k \geq K(\delta).$$

Since $\gamma_k \equiv \sigma_{k,\delta}$ on ${B_R}^c$, this implies

$$\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2} \mu(\gamma_k,x,y)\,dx\,dy \leq \iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2} \mu(\sigma_{k,\delta},x,y)\,dx\,dy + 1/k. \tag{5.13}$$

We claim pointwise convergence

$$\mu(\gamma,x,y) = \lim_{k\to\infty} \mu(\gamma_k,x,y) \quad \text{a.e. } (x,y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus ({B_{2\rho}}^c)^2. \tag{5.14}$$

Observe that by the definition of μ this pointwise convergence is immediate for a.e. $(x,y) \in (\mathbb{R}/\mathbb{Z})^2$ with $\gamma(x) \neq \gamma(y)$. On the other hand, for any $(x,y) \in (\mathbb{R}/\mathbb{Z})^2 \setminus ({B_{2\rho}}^c)^2$ $= (B_{2\rho} \times \mathbb{R}/\mathbb{Z}) \cup (\mathbb{R}/\mathbb{Z} \times B_{2\rho})^2$, we know that $\gamma(x) = \gamma(y)$ if and only if $x = y$. Since $x = y$ is a set of measure zero in $\mathbb{R}/\mathbb{Z}$, we conclude that (5.14) must hold.

Applying Fatou's lemma to (5.13), in view of (5.14) we conclude that

$$\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2} \mu(\gamma,x,y)\,dx\,dy \leq \liminf_{k\to\infty} \iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2} \mu(\sigma_{k,\delta},x,y)\,dx\,dy.$$

The convergence of $\sigma_{k,\delta}$ as $k \to \infty$. We now fix $\delta \in (0,\delta_0)$ and consider the limit as $k \to \infty$. Set

$$g_\delta(x) := \begin{cases} \gamma'(x), & x < -R, \\ \gamma'(-R), & x \in (-R, -(1-\delta)R), \\ (\gamma + t\varphi)'(x/(1-\delta)), & |x| < (1-\delta)R, \\ \gamma'(R), & x \in ((1-\delta)R, R), \\ \gamma'(x), & x > R, \end{cases}$$

and

$$\sigma_\delta(x) := \int_{-1}^{x} g_\delta(z)\,dz + \gamma(-1) + \theta(x)\Big(\int_{-R}^{R} [\gamma'(z) - g_\delta(z)]\,dz\Big).$$

From the Sobolev convergence of γ_k we observe that (up to a subsequence)

$$g_{k,\delta} \xrightarrow{k\to\infty} g_\delta \quad \text{a.e. in } (-R,R).$$

We conclude that

$$\sigma_{k,\delta} \xrightarrow{k\to\infty} \sigma_\delta \quad \text{a.e. in } \mathbb{R}/\mathbb{Z}.$$

Indeed, for $x \in (-R,R)$ we use dominated convergence and the a.e. convergence of γ_k' to γ' in $B_{100\rho}$. For $x \not\in (-R,R)$ we use $\sigma_{k,\delta}(x) = \gamma_k(x)$, $\sigma_\delta(x) = \gamma(x)$, and the uniform convergence of γ_k to γ. Since on the other hand $|\sigma_{k,\delta}'|$ is uniformly bounded, we conclude that

$$\sigma_{k,\delta} \xrightarrow{k\to\infty} \sigma_\delta \quad \text{uniformly in } \mathbb{R}/\mathbb{Z}. \tag{5.15}$$

We also observe that

$$\sigma_\delta \xrightarrow{\delta\to 0} \gamma + t\varphi \quad \text{uniformly in } \mathbb{R}/\mathbb{Z}. \tag{5.16}$$

This is obvious outside of $(-R,R)$ since both sides are equal. Also, similar to (5.10) we have

$$\left\| \theta(\cdot)\left(\int_{-R}^{R} [\gamma'(z) - g_\delta(z)]\, dz\right)\right\|_{L^\infty} \xrightarrow{\delta\to 0} 0.$$

Also

$$g_\delta(z) - (\gamma + t\varphi)'(z) \xrightarrow{\delta\to 0} 0 \quad \text{a.e. } z \in \mathbb{R}/\mathbb{Z}.$$

By dominated convergence, we conclude that

$$\left|\int_{-1}^{x} g_\delta(z)\, dz - \int_{-1}^{x} (\gamma + t\varphi)'(z)\, dz\right| \le \int_{-1}^{1} |g_\delta - (\gamma + t\varphi)'| \xrightarrow{\delta\to 0} 0$$

uniformly in x, so (5.16) holds.

In particular, choosing δ_0 possibly smaller, we may assume that for any $|t| \ll 1$,

$$\sigma_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3 \quad \text{is injective.} \tag{5.17}$$

Next we claim that

$$\limsup_{k\to\infty} \left| \iint_{(\mathbb{R}/\mathbb{Z})^2 \setminus (B_R{}^c)^2} \mu(\sigma_{k,\delta}, x, y)\, dx\, dy - \iint_{(\mathbb{R}/\mathbb{Z})^2 \setminus (B_R{}^c)^2} \mu(\sigma_\delta, x, y)\, dx\, dy \right| \le C\delta. \tag{5.18}$$

First observe that

$$|\sigma_{k,\delta}'(x)| \xrightarrow{k\to\infty} |\sigma_\delta'(x)| \quad \forall \delta < \delta_0, \text{uniformly for } x \in \mathbb{R}/\mathbb{Z}. \tag{5.19}$$

Indeed, by the support of φ, for $|x| > R$,

$$|\sigma_{k,\delta}'(x)| = |\gamma_k'(x)| = 1 = |\gamma'(x)| = |\sigma_\delta'(x)|. \tag{5.20}$$

Also for $|x| \in ((1-\delta)R, R)$,

$$\begin{aligned}|g_{k,\delta}(x)| &= |\eta(x)\gamma'(\pm R) + (1-\eta(x))\gamma_k'(\pm R)| \\ &\xrightarrow{k\to\infty} |\eta(x)\gamma'(\pm R) + (1-\eta(x))\gamma'(\pm R)| \\ &= |\gamma'(\pm R)| = |g_\delta(x)|.\end{aligned}$$

and for $|x| < (1-\delta)R$ we have

$$|g_{k,\delta}(x)| = |g_\delta(x)|.$$

By the definition of σ_δ and in view of (5.11) we have established (5.19).

Next we claim that we have injectivity uniformly in k in the following sense: for any $\delta > 0$ there exists a small $c(\delta) > 0$ (taking $K(\delta)$ possibly larger than before)

$$\inf_{k\geq K(\delta)} \inf_{\substack{(x,y)\in(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2 \\ |x-y|\geq\delta/100}} |\sigma_{k,\delta}(x) - \sigma_{k,\delta}(y)| \gtrsim c(\delta). \tag{5.21}$$

We allow the constant $c(\delta)$ to also depend on R and ρ, but we stress only the dependence on δ. In view of the uniform convergence (5.15), (5.21) follows once we establish

$$\inf_{\substack{(x,y)\in(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2 \\ |x-y|\geq\delta/100}} |\sigma_\delta(x) - \sigma_\delta(y)| \gtrsim c(\delta),$$

which is a consequence of injectivity of σ_δ (see (5.17)).

Thus, with the help of (5.21), setting $D_\delta := (\mathbb{R}/\mathbb{Z})^2 \setminus ({B_R}^c)^2 \cap |x-y| \gtrsim \delta$ we can make a brute force estimate

$$\begin{aligned}&\left|\iint_{D_\delta} \mu(\sigma_{k,\delta}, x, y)\,dx\,dy - \iint_{D_\delta} \mu(\sigma_\delta, x, y)\,dx\,dy\right| \\ &\qquad\lesssim C(\delta, \|\gamma_k'\|_{L^\infty}, \|\gamma_k\|_{L^\infty}) \int_{\mathbb{R}/\mathbb{Z}} [|\gamma_k'(x) - \gamma'(x)|^q + |\gamma_k(x) - \gamma(x)|^q]\,dx.\end{aligned}$$

Since $\gamma_k \xrightarrow{k\to\infty} \gamma$ a.e., by dominated convergence (recall $|\gamma_k'| \leq 1$), we have

$$\lim_{k\to\infty} \left|\iint_{D_\delta} \mu(\sigma_{k,\delta}, x, y)\,dx\,dy - \iint_{D_\delta} \mu(\sigma_\delta, x, y)\,dx\,dy\right| = 0.$$

That is,

$$\begin{aligned}&\limsup_{k\to\infty} \left|\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2} \mu(\sigma_{k,\delta}, x, y)\,dx\,dy - \iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2} \mu(\sigma_\delta, x, y)\,dx\,dy\right| \\ &\quad\lesssim \limsup_{k\to\infty} \left|\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2, |x-y|\leq\delta/100} \mu(\sigma_{k,\delta}, x, y) - \mu(\sigma_\delta, x, y)\,dx\,dy\right|.\end{aligned} \tag{5.22}$$

Also observe that $\mu(\sigma_{k,\delta}, x, y) = \mu(\sigma_\delta, x, y)$ for all $x, y \in B_{(1-\delta)R}$ and $k \in \mathbb{N}$.

In the light of (5.20) we have to consider the following terms (observe that μ is not symmetric in x, y):

$$\limsup_{k\to\infty}\left|\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus(B_R{}^c)^2}\mu(\sigma_{k,\delta},x,y)\,dx\,dy-\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus(B_R{}^c)^2}\mu(\sigma_\delta,x,y)\,dx\,dy\right|$$

$$\overset{(5.8)}{\lesssim}\limsup_{k\to\infty}\left|\sum_{\ell=1}^{5}\iint_{A_\ell}\left(\frac{|\sigma_{k,\delta}'(x)\wedge(\sigma_{k,\delta}(x)-\sigma_{k,\delta}(y))|^q}{|\sigma_{k,\delta}(x)-\sigma_{k,\delta}(y)|^{q+2}}-\frac{|\sigma_\delta'(x)\wedge(\sigma_\delta(x)-\sigma_\delta(y))|^q}{|\sigma_\delta(x)-\sigma_\delta(y)|^{q+2}}\right)\cdot|\sigma_\delta'(x)|^{1-q}|\sigma_\delta'(y)|\,dx\,dy\right|$$

where

$$\begin{aligned}
A_1 &:= (\pm R, \pm R(1+\delta))\times(\pm R(1-\delta),\pm R),\\
A_2 &:= (\pm R(1-\delta),\pm R)\times(\pm R,\pm R(1+\delta)),\\
A_3 &:= (\pm R(1-2\delta),\pm R(1-\delta))\times(\pm R(1-\delta),\pm R),\\
A_4 &:= (\pm R(1-\delta),\pm R)\times(\pm R(1-2\delta),\pm R(1-\delta)),\\
A_5 &:= (\pm R(1-\delta),\pm R)\times(\pm R(1-\delta),\pm R).
\end{aligned}$$

Observe that in each of these regimes $\theta\equiv 1$ or $\theta\equiv 0$, that is,

$$\limsup_{k\to\infty}\left|\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus(B_R{}^c)^2}\mu(\sigma_{k,\delta},x,y)\,dx\,dy-\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus(B_R{}^c)^2}\mu(\sigma_\delta,x,y)\,dx\,dy\right|$$

$$\lesssim\limsup_{k\to\infty}\left|\sum_{\ell=1}^{5}\iint_{A_\ell}\left(\frac{|g_{k,\delta}(x)\wedge\int_x^y g_{k,\delta}(z)\,dz|^q}{|\sigma_{k,\delta}(x)-\sigma_{k,\delta}(y)|^{q+2}}-\frac{|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q}{|\sigma_\delta(x)-\sigma_\delta(y)|^{q+2}}\right)\cdot|\sigma_\delta'(x)|^{1-q}|\sigma_\delta'(y)|\,dx\,dy\right|$$

$$=\limsup_{k\to\infty}\left|\sum_{\ell=1}^{5}\iint_{A_\ell}\left(\frac{|g_{k,\delta}(x)\wedge\int_x^y g_{k,\delta}(z)\,dz|^q}{|\int_x^y g_{k,\delta}(z)\,dz|^{q+2}}-\frac{|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q}{|\int_x^y g_\delta(z)\,dz|^{q+2}}\right)\cdot|\sigma_\delta'(x)|^{1-q}|\sigma_\delta'(y)|\,dx\,dy\right|$$

$$\lesssim\limsup_{k\to\infty}\sum_{\ell=1}^{5}\iint_{A_\ell}\frac{\left||g_{k,\delta}(x)\wedge\int_x^y g_{k,\delta}(z)\,dz|^q-|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q\right|}{|\int_x^y g_{k,\delta}(z)\,dz|^{q+2}}\cdot|\sigma_\delta'(x)|^{1-q}|\sigma_\delta'(y)|\,dx\,dy$$

$$+\limsup_{k\to\infty}\sum_{\ell=1}^{5}\iint_{A_\ell}\frac{\left||\int_x^y g_\delta(z)\,dz|^{q+2}-|\int_x^y g_{k,\delta}(z)\,dz|^{q+2}\right|}{|\int_x^y g_\delta(z)\,dz|^{q+2}}\frac{|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q}{|\int_x^y g_{k,\delta}(z)\,dz|^{q+2}}\cdot|\sigma_\delta'(x)|^{1-q}|\sigma_\delta'(y)|dx\,dy$$

By the uniform bi-Lipschitz estimate we continue as follows:

$$\lesssim \limsup_{k\to\infty}\sum_{\ell=1}^{5}\iint_{A_\ell}\frac{\left||g_{k,\delta}(x)\wedge\int_x^y g_{k,\delta}(z)\,dz|^q-|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q\right|}{|x-y|^{q+2}}\,dx\,dy$$

$$+\limsup_{k\to\infty}\sum_{\ell=1}^{5}\iint_{A_\ell}\frac{\left||\int_x^y g_\delta(z)\,dz|^{q+2}-|\int_x^y g_{k,\delta}(z)\,dz|^{q+2}\right|}{|x-y|^{q+2}}\,\frac{|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q}{|\int_x^y g_\delta(z)\,dz|^{q+2}}$$

$$\cdot\,|\sigma_\delta'(x)|^{1-q}|\sigma_\delta'(y)|\,dx\,dy.$$

The second set of integrals converges to zero by dominated convergence since $g_{k,\delta}$ converges a.e. to g_δ. To establish (5.18) we argue as in the proof Lemma 5.3 to find that the first terms satisfy

$$\limsup_{k\to\infty}\sum_{\ell=1}^{5}\iint_{A_\ell}\frac{\left||g_{k,\delta}(x)\wedge\int_x^y g_{k,\delta}(z)\,dz|^q-|g_\delta(x)\wedge\int_x^y g_\delta(z)\,dz|^q\right|}{|x-y|^{q+2}}\,dx\,dy\lesssim\delta.$$

The convergence of σ_δ as $\delta\to 0$. By now we have shown that for any $\delta\in(0,\delta_0)$,

$$\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2}\mu(\gamma,x,y)\,dx\,dy\le\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2}\mu(\sigma_\delta,x,y)\,dx\,dy+C\delta.$$

Observe that $\sigma_\delta\xrightarrow{\delta\to 0}\gamma+t\varphi$, in view of (5.11) – indeed, essentially repeating the Luckhaus lemma argument above, we see that

$$\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2}\mu(\sigma_\delta,x,y)\,dx\,dy\xrightarrow{\delta\to 0}\iint_{(\mathbb{R}/\mathbb{Z})^2\setminus({B_R}^c)^2}\mu(\gamma+t\varphi,x,y)\,dx\,dy.\quad\blacksquare$$

6. The regularity theory for critical points: Proof of Theorem 1.7

This section is dedicated to proving $C^{1,\alpha}$-regularity of locally critical points for scale-invariant tangent-point energies $\mathrm{TP}^{q+2,q}$ with $q\ge 2$. Our main goal is the following.

Proposition 6.1 (Local decay estimate). *Let $q\ge 2$ and γ be a locally critical embedding in the sense of Definition 1.5 with small tangent-point energy $\mathrm{TP}^{q+2,q}$ around a geodesic ball $B_r(x_0)\subset\mathbb{R}/\mathbb{Z}$, and assume $|\gamma'|\equiv\mathrm{const}>0$ almost everywhere. Let $u:=\gamma'/|\gamma'|$, that is, $u:\mathbb{R}/\mathbb{Z}\to\mathbb{S}^2$ is such that $\int_{\mathbb{R}/\mathbb{Z}}u=0$, and let $\tilde u:\mathbb{R}\to\mathbb{R}^3$ be an $L^\infty\cap W^{1/q,q}$-extension of $u|_{B_r(x_0)}$ from $B_r(x_0)$ to $\mathbb{R}$. Then there exist $\varepsilon,\tau,\theta\in(0,1)$ and $N_0\in\mathbb{N}$ such that the following holds.*

If $N\ge N_0$, $\rho>0$, and $y\in B_{r/2}(x_0)$ are such that $B_{2^N\rho}:=B_{2^N\rho}(y)\subset B_r(x_0)$, and $[\tilde u]_{W^{1/q,q}(B_{2^N\rho})}\le\varepsilon$, then

$$[u]^q_{W^{1/q,q}(B_\rho)}\le\tau[u]^q_{W^{1/q,q}(B_{2^N\rho})}+\sum_{l=1}^{\infty}2^{-\theta(N+l)}[\tilde u]^q_{W^{1/q,q}(B_{2^{N+l}\rho})}+\rho.$$

Proposition 6.1 implies Theorem 1.7 by the usual Dirichlet growth-type iteration techniques.

Proof of Theorem 1.7. First note that by definition, u and $\tilde{u}$ coincide locally around $B_r(x_0)$, so for any ball $B \subset B_r(x_0)$ we have

$$[u]_{W^{1/q,q}(B)} = [\tilde{u}]_{W^{1/q,q}(B)}.$$

By iterating the decay estimate on small balls [13, Lemma A.8], we obtain a $\sigma > 0$ such that

$$\sup_{0<\rho<r/2,\, y\in B_{r/2}(x)} \rho^{-\sigma}[u]^q_{W^{1/q,q}(B_\rho(y))} \lesssim C(u).$$

By Jensen's inequality, we conclude that

$$\sup_{0<\rho<r/2,\, y\in B_{r/2}(x)} \rho^{-\sigma} \fint_{B_\rho(y)} |u(z) - (u)_{B_\rho(y)}|^q \, dz \lesssim C(u),$$

where $(u)_{B_\rho(y)}$ denotes the mean value of u in $B_\rho(y)$, and hence u belongs to the Campanato space $\mathcal{L}^{q,1+\sigma}(B_r(x_0))$. The characterization of Campanato spaces with the use of Hölder spaces [30, Theorem 1.2] implies that $u \in C^{\sigma/q}_{\text{loc}}(B_{r/2}(x_0))$, which concludes the proof of Theorem 1.7. ∎

In order to obtain Proposition 6.1 – inspired by the investigations of critical O'Hara energies in [14] by comparison to the theory of fractional harmonic maps, cf. [24, 68] – we proceed as follows:

- In Section 6.1 we relate critical knots of the tangent-point energies $\mathrm{TP}^{p,q}$ with $p \in [q+2, 2q+1)$ and $q > 1$ to fractional harmonic maps: We first define a suitable energy $\mathcal{E}^{p,q}$ such that the unit tangents $u := \gamma'/|\gamma'|$ of locally $\mathrm{TP}^{p,q}$-critical embeddings γ with constant-speed parametrization are locally critical maps of the energy $\mathcal{E}^{p,q}$ in the class of maps $v : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$. We then establish that the new energy $\mathcal{E}^{p,q}$ is locally comparable to a $W^{\frac{p-q-1}{q},q}$-seminorm; see Section 4. Consequently, the equations that the critical maps u satisfy are indeed structurally similar to the Euler–Lagrange equations of fractional harmonic maps into the sphere $\mathbb{S}^2$ as treated in [68].
- In Section 6.2 we derive the Euler–Lagrange equations of the new energies $\mathcal{E}^{p,q}$ for $p \in [q+2, 2q+1)$, $q > 1$, and study the highest order and remainder terms of the Lagrangian.
- In Section 6.3 we finally treat the actual decay estimate of Proposition 6.1.

Before continuing, we need to introduce some notation for integration on $\mathbb{R}/\mathbb{Z}$ [14, Remark 2.2]:

(1) We denote by $\rho(x, y)$ the distance of $x, y \in \mathbb{R}/\mathbb{Z}$ on $\mathbb{R}/\mathbb{Z}$, in particular

$$\rho(x, y) = |x - y| \bmod \tfrac{1}{2}.$$

(2) If x and y are not antipodal, which means $|x-y| \neq \frac{1}{2}$, we denote by $x \triangleright y$ the shortest geodesic from x to y. Hence, for any $\mathbb{Z}$-periodic f we define

$$\oint_{x \triangleright y} f := \int_x^{\tilde{y}} f(z)\,dz,$$

where $\tilde{y} \in y + \mathbb{Z}$ such that $|x - \tilde{y}| < 1/2$.

(3) Furthermore, we write

$$\sigma(x \triangleright y) = \operatorname{sgn} \oint_{x \triangleright y} 1.$$

Hence if $x \triangleright y$ is positively oriented, we have $\sigma(x \triangleright y) = 1$, and if $x \triangleright y$ is negatively oriented, we get $\sigma(x \triangleright y) = -1$.

(4) Now given a $\mathbb{Z}$-periodic function f, we define

$$\fint_{x \triangleright y} f := \frac{\sigma(x \triangleright y)}{\rho(x,y)} \oint_{x \triangleright y} f.$$

We also have to deal with the fact that the critical embeddings of interest are only locally known to be of class $W^{1+1/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, which motivates the use of the extension $\tilde{u}$ as described below. This is a mere technical inconvenience, and we recommend the first-time reader to mentally identify u and $\tilde{u}$ in the arguments to come.

Remark 6.2. Let $p \in [q+2, 2q+1)$, $q > 1$, and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with locally small tangent-point energy according to Definition 1.1 that is a locally $\mathrm{TP}^{p,q}$-critical embedding in $B_r(x_0)$ as in Definition 1.5. Then Theorem 1.2 implies that γ is globally bi-Lipschitz and of class $W^{1+s,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ for any $0 < s < 1/q$. However, γ is not known to globally belong to the class $W^{1+1/q,q}$ or even $W^{1+\frac{p-q-1}{q},q}$; we only have the local statement $\gamma \in W^{1+\frac{p-q-1}{q},q}(B_r(x_0), \mathbb{R}^3)$ due to Theorem 4.5.

Although we aim to mostly work with the local $W^{\frac{p-q-1}{q},q}$-Gagliardo seminorm of γ', we also have to take into account the global behavior of γ' outside of $B_r(x_0)$ on account of the nonlocality of the proposed problem. For this reason, when necessary, we may interpret $B_r(x_0)$ as an interval in $[-1,2]$ and extend $\gamma'|_{B_r(x_0)}$ from $B_r(x_0)$ to a function $\tilde{u} \in W^{\frac{p-q-1}{q},q}(\mathbb{R}, \mathbb{R}^3)$ that is uniformly bounded. Such an extension exists since $\|\gamma'\|_{L^\infty} \leq 1$ and by standard construction of extensions (see e.g. [25, Theorem 5.4]). Note that in this setting for any ball $B \subset B_r(x_0)$ we have

$$[\tilde{u}]_{W^{\frac{p-q-1}{q},q}(B)} = [\gamma']_{W^{\frac{p-q-1}{q},q}(B)}.$$

6.1. *A new energy $\mathcal{E}^{p,q}$*

Our first objective is to construct a new energy $\mathcal{E}^{p,q}$, which coincides with $\mathrm{TP}^{p,q}$ for sufficiently regular curves γ, but only depends on the first derivative γ'. We then show that any locally critical embedding of the tangent-point energies $\mathrm{TP}^{p,q}$ parametrized by arclength produces a locally critical $\mathbb{S}^2$-valued map of the new energy $\mathcal{E}^{p,q}$.

We recall that the tangent-point energies are for any $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ given by

$$\mathrm{TP}^{p,q}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)|^{1-q} |\gamma'(y)| \, dy \, dx.$$

Now we transform the wedge product in the numerator by Lagrange's identity and the fundamental theorem of calculus to

$$\begin{aligned}
&|\gamma'(x) \wedge (\gamma(y) - \gamma(x))|^2 \\
&= \big|\gamma'(x) \wedge \big(\gamma(x) - \gamma(y) - \gamma'(x)(y-x)\big)\big|^2 \\
&= |\gamma'(x)|^2 |\gamma(y) - \gamma(x) - \gamma'(x)(y-x)|^2 - \Big(\gamma'(x) \cdot \big(\gamma(y) - \gamma(x) - \gamma'(x)(y-x)\big)\Big)^2 \\
&= |y-x|^2 \bigg(|\gamma'(x)|^2 \bigg| \fint_{x \triangleright y} \gamma'(z) \, dz - \gamma'(x) \bigg|^2 - \bigg| |\gamma'(x)|^2 - \fint_{x \triangleright y} \gamma'(x) \cdot \gamma'(z) \, dz \bigg|^2 \bigg) \\
&= |y-x|^2 \bigg(|\gamma'(x)|^2 \bigg| \fint_{x \triangleright y} [\gamma'(z) - \gamma'(x)] \, dz \bigg|^2 - \frac{1}{4} \bigg| \fint_{x \triangleright y} |\gamma'(x) - \gamma'(z)|^2 \, dz \\
&\qquad\qquad + |\gamma'(x)|^2 - \fint_{x \triangleright y} |\gamma'(z)|^2 \, dz \bigg|^2 \bigg).
\end{aligned}$$

Additionally, observe that

$$\begin{aligned}
\frac{|\gamma(y) - \gamma(x)|^2}{|y-x|^2} &= \fint_{x \triangleright y} \fint_{x \triangleright y} \gamma'(s) \cdot \gamma'(t) \, ds \, dt \\
&= \fint_{x \triangleright y} |\gamma'(z)|^2 \, dz - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |\gamma'(s) - \gamma'(t)|^2 \, ds \, dt. \qquad (6.1)
\end{aligned}$$

Therefore, we can rewrite $\mathrm{TP}^{p,q}(\gamma)$ in terms of the first derivative γ' as

$$\begin{aligned}
&\int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\big(|\gamma'(x)|^2 | \fint_{x \triangleright y} [\gamma'(z) - \gamma'(x)] \, dz|^2 - \frac{1}{4} | \fint_{x \triangleright y} |\gamma'(x) - \gamma'(z)|^2 \, dz + |\gamma'(x)|^2 - \fint_{x \triangleright y} |\gamma'(z)|^2 \, dz |^2 \big)^{q/2}}{|y-x|^{p-q}} \\
&\cdot \bigg(\fint_{x \triangleright y} |\gamma'(z)|^2 \, dz - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |\gamma'(s) - \gamma'(t)|^2 \, ds \, dt \bigg)^{-p/2} |\gamma'(x)|^{1-q} |\gamma'(y)| \, dy \, dx.
\end{aligned}$$

This motivates introducing the following real-valued energy $\mathcal{E}^{p,q}$ for any maps $u : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$:

$$\begin{aligned}
\mathcal{E}^{p,q}(u) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \bigg(&|u(x)|^2 \Big| \fint_{x \triangleright y} [u(z) - u(x)] \, dz \Big|^2 \\
&- \frac{1}{4} \bigg(\fint_{x \triangleright y} |u(x) - u(z)|^2 \, dz + |u(x)|^2 - \fint_{x \triangleright y} |u(z)|^2 \, dz \bigg)^2 \bigg)^{q/2} \\
\cdot \bigg(\fint_{x \triangleright y} |u(z)|^2 \, dz - &\frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u(s) - u(t)|^2 \, ds \, dt \bigg)^{-p/2} |u(x)|^{1-q} |u(y)| \, \frac{dy \, dx}{\rho(x,y)^{p-q}}.
\end{aligned}$$

For $\eta \in C^\infty(\mathbb{R}, [0,\infty))$, we set moreover

$$\begin{aligned}
\mathcal{E}_\eta^{p,q}(u) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \Bigg(& |u(x) - \eta(x)(u)_{\mathbb{R}/\mathbb{Z}}|^2 \left| \fint_{x \triangleright y} [u(z) - u(x)]\, dz \right|^2 \\
& - \frac{1}{4} \bigg(\fint_{x \triangleright y} |u(x) - u(z)|^2\, dz + |u(x) - \eta(x)(u)_{\mathbb{R}/\mathbb{Z}}|^2 \\
& \qquad - \fint_{x \triangleright y} |u(z) - \eta(z)(u)_{\mathbb{R}/\mathbb{Z}}|^2\, dz \bigg)^2 \Bigg)^{q/2} \\
& \cdot \left(\fint_{x \triangleright y} |u(z) - \eta(z)(u)_{\mathbb{R}/\mathbb{Z}}|^2\, dz - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u(s) - u(t)|^2\, ds\, dt \right)^{-p/2} \\
& \cdot |u(x) - \eta(x)(u)_{\mathbb{R}/\mathbb{Z}}|^{1-q} |u(y) - \eta(y)(u)_{\mathbb{R}/\mathbb{Z}}| \, \frac{dy\, dx}{\rho(x,y)^{p-q}}.
\end{aligned}$$

We define the localized version $\mathcal{E}_{\eta,D}^{p,q}(u)$ for any $D \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ by the same formula with the integration domain $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ replaced by D.

We observe that the energies $\mathrm{TP}^{p,q}$ and $\mathcal{E}^{p,q}$ coincide for our embeddings of interest, recalling that $(\gamma')_{\mathbb{R}/\mathbb{Z}} = 0$.

Lemma 6.3. *Let $p \in [q+2, 2q+1)$ and $q > 1$. For any embedding $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with finite tangent-point energies $\mathrm{TP}^{p,q}$ and constant speed parametrization as well as $\eta \in C^\infty(\mathbb{R}, [0,\infty))$, we have*

$$\mathrm{TP}^{p,q}(\gamma) = \mathcal{E}_\eta^{p,q}(\gamma') = \mathcal{E}^{p,q}(\gamma'),$$

and, in particular, for any subset $D \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$,

$$\iint_D \frac{|\gamma'(x) \wedge (\gamma(x) - \gamma(y))|^q}{|\gamma(x) - \gamma(y)|^p} |\gamma'(x)|^{1-q} |\gamma'(y)|\, dy\, dx = \mathcal{E}_{\eta,D}^{p,q}(\gamma').$$

It remains to show that locally $\mathrm{TP}^{p,q}$-critical embeddings for the tangent-point energies (see Definition 1.5) indeed induce locally $\mathcal{E}^{p,q}$-critical maps into the sphere $\mathbb{S}^2$.

The main result of this section is the analogue of [14, Theorem 2.1]: (locally) critical knots $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ induce (locally) $\mathcal{E}_\eta^{p,q}$-critical maps $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ by setting $u := \gamma'/|\gamma'|$.

Theorem 6.4. *Let $p \in [q+2, 2q+1)$, $q > 1$, and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with locally small tangent-point energy $\mathrm{TP}^{p,q}$ around the open interval $B_r(x_0)$, in the sense of Definition 1.1, and assume $|\gamma'| \equiv \mathrm{const}$. Denote the unit tangent field of γ by $u := \gamma'/|\gamma'| : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$.*

If γ is a locally $\mathrm{TP}^{p,q}$-critical embedding in $B_r(x_0)$, in the sense of Definition 1.5, then there exists some $\eta \in C_c^\infty(B_r(x_0), [0,\infty))$ such that the map $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ is an $\mathcal{E}_\eta^{p,q}$-critical map in $B_r(x_0)$ in the class of maps $v : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$.

Namely, for any $\varphi \in C_c^\infty(B_r(x_0), \mathbb{R}^3)$, if we set

$$u_\varepsilon := \frac{u + \varepsilon\varphi}{|u + \varepsilon\varphi|},$$

we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}^{p,q}_{\eta}(u_\varepsilon) = 0.$$

Proof. We argue similarly to the proof of [14, Theorem 2.1], but additional problems appear since we only have the criticality in a ball, not globally (which is what forces us to introduce $\eta \in C_c^\infty(B_r(x_0))$, while in [14, Theorem 2.1] we can choose $\eta \equiv 1$).

For simplicity we assume that $x_0 \in (0,1)$ and $r \ll \min\{x_0, 1-x_0\}$ (we can always assume that $x_0 = 1/2$ by the periodicity of the problem).

Pick $\theta \in C^\infty(\mathbb{R})$ with $\theta \equiv 0$ for $x < x_0 - r/2$ and $\theta \equiv 1$ for $x \geq x_0 + r/2$, and with $|\theta'| \lesssim 1/r$. We can also assume that $\theta' \geq 0$. Below we will choose $\eta := \theta'$.

Similar to (5.6), for φ and u_ε as in the statement, set

$$\gamma_\varepsilon(x) := \gamma(0) + \int_0^x u_\varepsilon(z)\,dz + \theta(x)\bigg(\int_{x_0-r}^{x_0+r} [\gamma'(z) - u_\varepsilon(z)]\,dz\bigg).$$

We observe that $\gamma_\varepsilon(x) = \gamma(x)$ for $x < x_0 - r$ and $x > x_0 + r$. Indeed, for $x < x_0 - r$ we have

$$\gamma_\varepsilon(x) = \gamma(0) + \int_0^x \gamma'(z)\,dz = \gamma(x),$$

and for $x > x_0 + r$ we have

$$\begin{aligned}
\gamma_\varepsilon(x) &= \gamma(0) + \int_0^{x_0-r} \gamma'(z)\,dz + \int_{x_0-r}^{x_0+r} u_\varepsilon(z)\,dz + \int_{x_0+r}^{x} \gamma'(z)\,dz \\
&\quad + \theta(x)\bigg(\int_{x_0-r}^{x_0+r} [\gamma'(z) - u_\varepsilon(z)]\,dz\bigg) \\
&= \gamma(0) + \int_0^{x_0-r} \gamma'(z)\,dz + \int_{x_0-r}^{x_0+r} u_\varepsilon(z)\,dz + \int_{x_0+r}^{x} \gamma'(z)\,dz \\
&\quad + 1 \cdot \bigg(\int_{x_0-r}^{x_0+r} [\gamma'(z) - u_\varepsilon(z)]\,dz\bigg) \\
&= \gamma(0) + \int_0^{x_0-r} \gamma'(z)\,dz + \int_{x_0+r}^{x} \gamma'(z)\,dz + \int_{x_0-r}^{x_0+r} \gamma'(z)\,dz \\
&= \gamma(0) + \int_0^{x} \gamma'(z)\,dz = \gamma(x).
\end{aligned}$$

Moreover, we find that for almost every x,

$$u_\varepsilon(x) = \gamma'(x) + \varepsilon(\varphi(x) - \langle \varphi(x), \gamma'(x)\rangle \gamma'(x)) + O(\varepsilon^2),$$

and, using again the support of φ,

$$\begin{aligned}
&\gamma_\varepsilon(x) = \gamma(x) \\
&\ + \varepsilon\bigg(\int_0^x \big(\varphi(z) - \langle \varphi(z), \gamma'(z)\rangle \gamma'(z)\big)\,dz - \theta(x)\bigg(\int_0^1 \big(\varphi(z) - \langle \varphi(z), \gamma'(z)\rangle \gamma'(z)\big)\,dz\bigg)\bigg) \\
&\ + O(\varepsilon^2)
\end{aligned}$$

as well as

$$\begin{aligned}\gamma_\varepsilon'(x) &= \gamma'(x) \\ &\quad + \varepsilon\Big(\varphi(x) - \langle\gamma'(x), \varphi(x)\rangle\gamma'(x) - \theta'(x)\Big(\int_0^1 \big(\varphi(z) - \langle\varphi(z), \gamma'(z)\rangle\gamma'(z)\big)\, dz\Big)\Big) \\ &\quad + O(\varepsilon^2).\end{aligned}$$

By Lemma 6.3, for small $\varepsilon > 0$ we have

$$\mathrm{TP}^{p,q}(\gamma_\varepsilon) = \mathcal{E}^{p,q}(\gamma_\varepsilon').$$

Since by assumption γ is critical in $B_r(x_0)$ and γ_ε is a permissible variation, we obtain

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}^{p,q}(\gamma_\varepsilon') = \delta\mathcal{E}^{p,q}(u)[\psi],$$

where the function

$$\psi(x) := \varphi(x) - \langle\gamma'(x), \varphi(x)\rangle\gamma'(x) - \theta'(x)\Big(\int_0^1 (\varphi(z) - \langle\varphi(z), \gamma'(z)\rangle\gamma'(z))\, dz\Big)$$

has support in $B_r(x_0)$. Setting $\eta := \theta'$, we conclude the proof. ■

Remark 6.5. The function η appearing in the previous theorem might resemble a Lagrange multiplier. However, in our setting η can be chosen more freely. The construction of η in the proof above is only one out of many possibilities to define permissible functions η.

6.2. *Euler–Lagrange equations of $\mathcal{E}^{p,q}$*

In this section, we derive the Euler–Lagrange equations of $\mathcal{E}_\eta^{p,q}$ for $p \in [q+2, 2q+1)$, $q > 1$, and suitable $\eta \in C^\infty(\mathbb{R}, [0,\infty))$. We realize that the new energies $\mathcal{E}_\eta^{q,q}$ have a nonlinear and nonlocal Lagrangian. Furthermore, we obtain a decomposition of the Lagrangian into a term of highest order, denoted by Q, and terms of lower order, denoted by R.

The leading order operator Q on a subset $D \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ for $u : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ and $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is given by

$$\begin{aligned}&Q_D^{(p,q)}(u, \varphi) \\ &:= q \iint_D \Big| \fint_{x \triangleright y} [u(z) - u(x)]\, dz \Big|^{q-2} \Big(1 - \frac{1}{2} \fint_{x\triangleright y} \fint_{x\triangleright y} |u(s) - u(t)|^2\, ds\, dt\Big)^{-p/2} \\ &\qquad \cdot \fint_{x\triangleright y} \fint_{x\triangleright y} (u(z_1) - u(x)) \cdot (\varphi(z_2) - \varphi(x))\, dz_1\, dz_2\, \frac{dy\, dx}{\rho(x,y)^{p-q}} \\ &= \frac{q}{2} \iint_D a^{q-2}\Big(1 - \frac{1}{2}c\Big)^{-p/2} a'(\varphi)\, \frac{dy\, dx}{\rho(x,y)^{p-q}}. \qquad (6.2)\end{aligned}$$

(For the definition of a, a', c, etc., see below.)

The remainders, which are, as we shall see, "of lower order", are given as follows:

$$R_D^{1,(p,q)}(u,\varphi) := \frac{q}{2}\iint_D \Big(\big(a-\tfrac14 b^2\big)^{\frac{q-2}{2}} - a^{\frac{q-2}{2}}\Big)\big(1-\tfrac12 c\big)^{-p/2} a'(\varphi)\,\frac{dy\,dx}{\rho(x,y)^{p-q}},$$
$$R_D^{2,(p,q)}(u,\varphi) := -\frac{q}{4}\iint_D \big(a-\tfrac14 b^2\big)^{\frac{q-2}{2}}\big(1-\tfrac12 c\big)^{-p/2} bb'(\varphi)\,\frac{dy\,dx}{\rho(x,y)^{p-q}},$$
$$R_D^{3,(p,q)}(u,\varphi) := \frac{p}{4}\iint_D \big(a-\tfrac14 b^2\big)^{q/2}\big(1-\tfrac12 c\big)^{-\frac{p+2}{2}} c'(\varphi)\,\frac{dy\,dx}{\rho(x,y)^{p-q}},$$
$$R_{\eta,D}^{4,(p,q)}(u,\varphi) := -\frac{p}{2}\iint_D \big(a-\tfrac14 b^2\big)^{q/2}\big(1-\tfrac12 c\big)^{-\frac{p+2}{2}} d'(\varphi)\,\frac{dy\,dx}{\rho(x,y)^{p-q}},$$
$$R_{\eta,D}^{5,(p,q)}(u,\varphi) := q\iint_D \big(a-\tfrac14 b^2\big)^{\frac{q-2}{2}}\big(1-\tfrac12 c\big)^{-p/2}\big(a-\tfrac12 b\big) e'(\varphi)\,\frac{dy\,dx}{\rho(x,y)^{p-q}},$$
$$R_{\eta,D}^{6,(p,q)}(u,\varphi) := \frac{q}{4}\iint_D \big(a-\tfrac14 b^2\big)^{\frac{q-2}{2}}\big(1-\tfrac12 c\big)^{-p/2} bd'(\varphi)\,\frac{dy\,dx}{\rho(x,y)^{p-q}},$$
$$R_{\eta,D}^{7,(p,q)}(u,\varphi) := \iint_D \big(a-\tfrac14 b^2\big)^{q/2}\big(1-\tfrac12 c\big)^{-p/2}\big((1-q)e'(\varphi)+f'(\varphi)\big)\,\frac{dy\,dx}{\rho(x,y)^{p-q}}.$$

Here

$$a := \left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^2,$$
$$b := \fint_{x\rhd y}|u(z)-u(x)|^2\,dz,$$
$$c := \fint_{x\rhd y}\fint_{x\rhd y}|u(s)-u(t)|^2\,ds\,dt,$$

and

$$a'(\varphi) := 2\fint_{x\rhd y}\fint_{x\rhd y}(u(z_1)-u(x))\cdot(\varphi(z_2)-\varphi(x))\,dz_1\,dz_2,$$
$$b'(\varphi) := 2\fint_{x\rhd y}(u(z)-u(x))\cdot(\varphi(z)-\varphi(x))\,dz,$$
$$c'(\varphi) := 2\fint_{x\rhd y}\fint_{x\rhd y}(u(s)-u(t))\cdot(\varphi(s)-\varphi(t))\,ds\,dt,$$
$$d'(\varphi) := -2\fint_{x\rhd y}\eta(z)u(z)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}}\,dz,$$
$$e'(\varphi) := -\eta(x)u(x)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}},$$
$$f'(\varphi) := -\eta(y)u(y)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}}.$$

Note that when considering the entire domain $D=\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$, we drop the label D in the definition of $Q_D^{(p,q)}(u,\varphi)$ and in the remainders $R_D^{k,(p,q)}(u,\varphi)$ and $R_{\eta,D}^{k,(p,q)}(u,\varphi)$.

Lemma 6.6 (Euler–Lagrange equations). *Let $p\in[q+2,2q+1)$, $q>1$, and $u:\mathbb{R}/\mathbb{Z}\to\mathbb{S}^2$ with $\int_{\mathbb{R}/\mathbb{Z}}u=0$ be a locally $\mathcal{E}_\eta^{p,q}$-critical map around the interval $B_r(x_0)$*

in the class of maps $v : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ *and let* $\eta \in C_c^\infty(B_r(x_0), [0,\infty))$*. Then for any test function* $\varphi \in W_0^{\frac{p-q-1}{q},q}(B_r(x_0), \mathbb{R}^3)$*, which is also tangential, i.e.* $\varphi \in T_u\mathbb{S}^2$*, if we set* $u_\varepsilon = u + \varepsilon\varphi$*, then*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}_\eta^{p,q}(u_\varepsilon) = Q^{(p,q)}(u,\varphi) + \sum_{k=1}^{3} R^{k,(p,q)}(u,\varphi) + \sum_{k=4}^{7} R_\eta^{k,(p,q)}(u,\varphi) = 0.$$

Proof. Recalling the definition of $\mathcal{E}_\eta^{p,q}$, we set

$$F(a,b,c,d,e) := \big(e^2 a - \tfrac{1}{4}(b + e^2 - d)^2\big)^{q/2}\big(d - \tfrac{1}{2}c\big)^{-p/2},$$

which implies

$$\mathcal{E}^{p,q}(u) = \int_{\mathbb{R}/\mathbb{Z}}\int_{\mathbb{R}/\mathbb{Z}} F(a(0),b(0),c(0),d(0),e(0))e(0)^{1-q} f(0)\frac{dy\,dx}{\rho(x,y)^{p-q}},$$

where

$$\begin{aligned}
a(\varepsilon) &:= \left|\fint_{x\triangleright y}[u_\varepsilon(z) - u_\varepsilon(x)]\,dz\right|^2,\\
b(\varepsilon) &:= \fint_{x\triangleright y}|u_\varepsilon(z) - u_\varepsilon(x)|^2\,dz,\\
c(\varepsilon) &:= \fint_{x\triangleright y}\fint_{x\triangleright y}|u_\varepsilon(s) - u_\varepsilon(t)|^2\,ds\,dt,\\
d(\varepsilon) &:= \fint_{x\triangleright y}|u_\varepsilon(z) - \eta(z)(u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|^2\,dz,\\
e(\varepsilon) &:= |u_\varepsilon(x) - \eta(x)(u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|,\\
f(\varepsilon) &:= |u_\varepsilon(y) - \eta(y)(u_\varepsilon)_{\mathbb{R}/\mathbb{Z}}|.
\end{aligned}$$

First we note that $d(0) = e(0) = f(0) = 1$ since $|u| \equiv 1$ and $(u)_{\mathbb{R}/\mathbb{Z}} = 0$. Furthermore, observe that

$$\begin{aligned}
a'(0) \equiv a'(0)(u,\varphi) &= 2\fint_{x\triangleright y}\fint_{x\triangleright y}(u(z_1) - u(x))\cdot(\varphi(z_2) - \varphi(x))\,dz_1\,dz_2,\\
b'(0) \equiv b'(0)(u,\varphi) &= 2\fint_{x\triangleright y}(u(z) - u(x))\cdot(\varphi(z) - \varphi(x))\,dz,\\
c'(0) \equiv c'(0)(u,\varphi) &= 2\fint_{x\triangleright y}\fint_{x\triangleright y}(u(s) - u(t))\cdot(\varphi(s) - \varphi(t))\,ds\,dt,
\end{aligned}$$

and due to $u\cdot\varphi \equiv 0$ and $(u)_{\mathbb{R}/\mathbb{Z}} = 0$,

$$\begin{aligned}
d'(0) \equiv d'(0)(u,\varphi,\eta) &= -2\fint_{x\triangleright y}\eta(z)u(z)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}}\,dz,\\
e'(0) \equiv e'(0)(u,\varphi,\eta) &= -\eta(x)u(x)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}},\\
f'(0) \equiv f'(0)(u,\varphi,\eta) &= -\eta(y)u(y)\cdot(\varphi)_{\mathbb{R}/\mathbb{Z}}.
\end{aligned}$$

Hence, by the product rule we obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} F(a(\varepsilon), b(\varepsilon), c(\varepsilon), d(\varepsilon), e(\varepsilon)) d(\varepsilon)^{1-q} e(\varepsilon) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&= \frac{q}{2} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} a(0)^{\frac{q-2}{2}} \big(1 - \tfrac{1}{2}c(0)\big)^{-p/2} a'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad + \frac{q}{2} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \Big(\big(a(0) - \tfrac{1}{4}b(0)^2\big)^{\frac{q-2}{2}} - a(0)^{\frac{q-2}{2}}\Big) \big(1 - \tfrac{1}{2}c(0)\big)^{-p/2} a'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad - \frac{q}{4} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \big(a(0) - \tfrac{1}{4}b(0)^2\big)^{\frac{q-2}{2}} \big(1 - \tfrac{1}{2}c(0)\big)^{-p/2} b(0) b'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad + \frac{p}{4} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \big(a(0) - \tfrac{1}{4}b(0)^2\big)^{q/2} \big(1 - \tfrac{1}{2}c(0)\big)^{-\frac{p+2}{2}} c'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad - \frac{p}{2} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \big(a(0) - \tfrac{1}{4}b(0)^2\big)^{q/2} \big(1 - \tfrac{1}{2}c(0)\big)^{-\frac{p+2}{2}} d'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad + q \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \big(a(0) - \tfrac{1}{4}b(0)^2\big)^{\frac{q-2}{2}} \big(1 - \tfrac{1}{2}c(0)\big)^{-p/2} \big(a(0) - \tfrac{1}{2}b(0)\big) e'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad + \frac{q}{4} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \big(a(0) - \tfrac{1}{4}b(0)^2\big)^{\frac{q-2}{2}} \big(1 - \tfrac{1}{2}c(0)\big)^{-p/2} b(0) d'(0) \frac{dy\,dx}{\rho(x,y)^{p-q}} \\
&\quad + \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \big(a(0) - \tfrac{1}{4}b(0)^2\big)^{q/2} \big(1 - \tfrac{1}{2}c(0)\big)^{-p/2} ((1-q)e'(0) + f'(0)) \frac{dy\,dx}{\rho(x,y)^{p-q}}.
\end{aligned}$$

■

Remark 6.7. For a given homeomorphism $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with locally small tangent-point energy $\mathrm{TP}^{p,q}$ and its unit tangent field $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$, we set

$$k(x,y) := 1 - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u(s) - u(t)|^2 \, ds\, dt = 1 - \tfrac{1}{2} c(0).$$

Note that in all terms of the Euler–Lagrange equation, either $k(x,y)^{-p/2}$ or $k(x,y)^{-\frac{p+2}{2}}$ appears as a factor. The motivation behind this definition is the observation that $k(x,y)^{-r}$ for any $r > 0$ is bounded: On the one hand, it is easy to see that

$$1 - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u(s) - u(t)|^2 \, ds\, dt \le 1;$$

on the other hand, there exists a constant $c > 0$ such that

$$0 < c \le 1 - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |u(s) - u(t)|^2 \, ds\, dt.$$

This can be shown by recalling that u denotes the unit tangent field of γ and by applying the fundamental theorem of calculus as in (6.1) as well as the global bi-Lipschitz continuity of γ due to Theorem 1.2, from which we conclude that

$$1 - \frac{1}{2} \fint_{x \triangleright y} \fint_{x \triangleright y} |\gamma'(s) - \gamma'(t)|^2 \, ds\, dt = \frac{|\gamma(y) - \gamma(x)|^2}{|y - x|^2} \ge (1 - \varepsilon)^2 > 0$$

for any $\varepsilon > 0$ small and $x \ne y$.

As next steps we are going to show that Q is indeed the leading order operator and the remainder terms R are of "lower order". Namely, in Proposition 6.8 we essentially show that $Q(u,\varphi)$ controls the Sobolev norm $[u]^{q-1}_{W^{\frac{p-q-1}{q},q}(B_\rho)}$ for a good choice of $\varphi \in C_c^\infty(B(\rho))$, in particular, whenever $B(\rho)$ is a ball compactly contained in $B_r(x_0)$ and $[\varphi]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \leq 1$. Then Proposition 6.9 shows that each of the remainder terms R essentially satisfies the following estimate:

$$|R^{k,(p,q)}(u,(\varphi u\wedge)_{ij})| \lesssim [u]^{\tilde{q}}_{W^{\tilde{q},q}(B_{2\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma l}[u]^{q-1}_{W^{\frac{p-q-1}{q},q}(B_{2^{l+1}\rho})} + C(r,u)\rho^\sigma \tag{6.3}$$

for some $\tilde{q} > q-1$ and some $\sigma > 0$. Such terms on the right-hand side can be absorbed by an iteration argument, as discussed in the proof of Theorem 1.7.

Next we show that the leading order term $Q^{p,q}$ controls the Sobolev norm.

Proposition 6.8. *Let $p \in [q+2, 2q+1)$, $q > 1$, and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with locally small tangent-point energy $\mathrm{TP}^{p,q}$ around the interval $B_r(x_0)$, in the sense of Definition 1.1. Furthermore, denote the unit tangent field of γ by $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\int_{\mathbb{R}/\mathbb{Z}} u = 0$, let $y_0 \in B_{r/2}(x_0)$, and choose $\rho > 0$ such that $B_\rho := B_\rho(y_0) \subset B_r(x_0)$. Then we have*

$$\begin{aligned}[u]^q_{W^{\frac{p-q-1}{q},q}(B_\rho)} &\lesssim \iint_{B_\rho{}^2} k(x,y)^{-p/2} \frac{|\fint_{x\rhd y}[u(z)-u(x)]\,dz|^q}{\rho(x,y)^{p-q}}\,dy\,dx \\ &\approx Q^{(p,q)}_{B_\rho\times B_\rho}(u,u)\end{aligned}$$

with constants only depending on p and q.

Proof. We begin by recalling the definition of the main term $Q^{(p,q)}_{B_\rho\times B_\rho}$ in (6.2) and test it with u, so that the expression simplifies to

$$\begin{aligned}&Q^{(p,q)}_{B_\rho\times B_\rho}(u,u)\\ &= q\iint_{B_\rho^2}\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^{q-2}\fint_{x\rhd y}\fint_{x\rhd y}(u(z_1)-u(x))\cdot(u(z_2)-u(x))\,dz_1\,dz_2\\ &\qquad\qquad\cdot k(x,y)^{-p/2}\frac{dy\,dx}{\rho(x,y)^{p-q}}\\ &= q\iint_{B_\rho^2} k(x,y)^{-p/2}\frac{|\fint_{x\rhd y}[u(z)-u(x)]\,dz|^q}{\rho(x,y)^{p-q}}\,dy\,dx.\end{aligned}$$

Note that the factor $k(x,y)^{-p/2}$ is strictly positive and bounded by Remark 6.7. Furthermore, we have

$$|u(y)-u(x)| \leq \left|u(y)-\fint_{y\rhd x}u(z)\,dz\right| + \left|\fint_{x\rhd y}u(z)\,dz - u(x)\right|,$$

which implies, by the previous arguments,

$$\begin{aligned}
[u]^q_{W^{\frac{p-q-1}{q},q}(B_\rho)} &= \iint_{B_\rho{}^2} \frac{|u(y)-u(x)|^q}{\rho(x,y)^{p-q}}\,dy\,dx \\
&\lesssim \iint_{B_\rho{}^2} \frac{|\fint_{x\rhd y}[u(z)-u(x)]\,dz|^q}{\rho(x,y)^{p-q}}\,dy\,dx + \iint_{B_\rho{}^2} \frac{|\fint_{y\rhd x}[u(z)-u(y)]\,dz|^q}{\rho(x,y)^{p-q}}\,dy\,dx \\
&\lesssim \iint_{B_\rho{}^2} k(x,y)^{-p/2} \frac{|\fint_{x\rhd y}[u(z)-u(x)]\,dz|^q}{\rho(x,y)^{p-q}}\,dy\,dx \approx Q^{(p,q)}_{B_\rho\times B_\rho}(u,u). \qquad \blacksquare
\end{aligned}$$

It remains to obtain the "lower order" property for the remainder terms R. We recall that for any $v \in \mathbb{R}^3$ the linear map $v\wedge : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the $\mathbb{R}^{3\times 3}$-matrix

$$v\wedge = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

Proposition 6.9. *Let $p \in [q+2, 2q+1)$, $q \geq 2$, and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with locally small tangent-point energy $\mathrm{TP}^{p,q}$ around the interval $B_r(x_0)$, in the sense of Definition 1.1. Denote the unit tangent field of γ by $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\fint_{\mathbb{R}/\mathbb{Z}} u = 0$ and take $\eta \in C_c^\infty(B_{r/2}(x_0), [0,\infty))$. Furthermore, let $y_0 \in B_{r/2}(x_0)$, choose $\rho > 0$ such that $B_{4\rho}(y_0) \subset B_r(x_0)$, and define $B_\rho := B_\rho(y_0)$. Let $\varphi \in C_c^\infty(B_\rho, \mathbb{R})$ be such that $[\varphi]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \leq 1$. Then the following holds for any $j = 1, 2, 3$:*

For the first remainder $k = 1$ and $2 < q < 4$, we have

$$\begin{aligned}
|R^{1,(p,q)}(u,(\varphi u\wedge)_{ij})| \lesssim{}& [u]^{2q-3}_{W^{\frac{p-q-1}{q},q}(B_{2\rho})} + \sum_{l=1}^{\infty} 2^{-(l+1)\frac{p-q}{q}} [\tilde{u}]^{q-1}_{W^{\frac{p-q-1}{q},q}(B_{2^{l+1}\rho})} \\
&+ \rho\, r^{-(p-q+1)}.
\end{aligned}$$

For $q = 2$ we have $R^{1,(p,q)} \equiv 0$. For $k = 1$ and $q \geq 4$, and for $k = 2, 3$ and any $q \geq 2$, we have

$$\begin{aligned}
|R^{k,(p,q)}(u,(\varphi u\wedge)_{ij})| \lesssim{}& [u]^{q+1}_{W^{\frac{p-q-1}{q},q}(B_{2\rho})} + \sum_{l=1}^{\infty} 2^{-(l+1)\frac{p-q}{q}} [\tilde{u}]^{q-1}_{W^{\frac{p-q-1}{q},q}(B_{2^{l+1}\rho})} \\
&+ \rho\, r^{-(p-q+1)},
\end{aligned}$$

and for $k = 4, 5, 6, 7$ and $q \geq 2$, we have

$$\sum_{k=4}^{7} |R^{k,(p,q)}_\eta(u,(\varphi u\wedge)_{ij})| \lesssim \rho\Big(\mathcal{E}^{p,q}(u) + r^{-(p-q)} + r^{\frac{p-q-2}{2}} [u]^q_{W^{\frac{p-q-1}{q},q}(B_r(x_0))}\Big),$$

where $\tilde{u}$ denotes a $W^{\frac{p-q-1}{q},q}$-extension of $u|_{B_r(x_0)}$ from $B_r(x_0)$ to $\mathbb{R}$ as discussed in Remark 6.2. The constants in these inequalities depend on p and q and may also depend on global properties of u such as $\|u\|_{L^\infty}$, $[u]_{W^{\frac{p-q-1}{q},q}(B_r(x_0))}$ and $[\tilde{u}]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})}$.

A very similar statement holds for $q \in (1,2)$, only the tail's exponents change, but one still obtains an estimate as in (6.3). We leave the details to the reader.

Proof of Proposition 6.9. We begin with some general observations on factors appearing in the integrands of the remainder terms. First, the remainders contain a factor of the form $k(x,y)^{-r}$ for some $r>0$, which is strictly positive and bounded by Remark 6.7. Next we consider the factors $(a-\frac{1}{4}b^2)^{\frac{q-2}{2}}$ and $(a-\frac{1}{4}b^2)^{q/2}$ appearing for $k=2,\dots,7$. Recall that

$$a=\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^2 \quad \text{and} \quad b=\fint_{x\rhd y}|u(z)-u(x)|^2\,dz.$$

As by (4.2) we have

$$\frac{1}{2}\fint_{x\rhd y}|u(z)-u(x)|^2\,dz=\left|\fint_{x\rhd y}u(x)\cdot(u(z)-u(x))\,dz\right|\le\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|, \tag{6.4}$$

it follows that $0\le a-\frac{1}{4}b^2$ and thus the factors can be simplified, when necessary, to

$$\left(a-\tfrac{1}{4}b^2\right)^{\frac{q-2}{2}}\le a^{\frac{q-2}{2}}=\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^{q-2},$$
$$\left(a-\tfrac{1}{4}b^2\right)^{q/2}\le a^{q/2}=\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^{q}.$$

For $k=1$ we have to study the factor $((a-\frac{1}{4}b^2)^{\frac{q-2}{2}}-a^{\frac{q-2}{2}})$ instead. For $q=2$ this factor equals 0, whereas for $2<q<4$ it can be estimated by

$$\left(a-\tfrac{1}{4}b^2\right)^{\frac{q-2}{2}}-a^{\frac{q-2}{2}}\le 2^{2-q}b^{q-2}=2^{2-q}\left(\fint_{x\rhd y}|u(z)-u(x)|^2\,dz\right)^{q-2},$$

since $x^r-y^r\le(x-y)^r$ for any $x\ge y\ge 0$ and $0\le r<1$, and for $q\ge 4$ by

$$\begin{aligned}&\left(a-\tfrac{1}{4}b^2\right)^{\frac{q-2}{2}}-a^{\frac{q-2}{2}}\\ &\quad\lesssim b^2a^{\frac{q-4}{2}}=\left(\fint_{x\rhd y}|u(z)-u(x)|^2\,dz\right)^2\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^{q-4}\end{aligned}$$

since $|x^r-y^r|\le c(r)|x-y|\,|x^{r-1}+y^{r-1}|$ for any $x\ge y\ge 0$ and $r\ge 0$. Last but not least, we note that the test functions $(\varphi u\wedge)_{ij}$ are tangential, i.e. $(\varphi u\wedge)_{ij}\in T_u\mathbb{S}^2$ for any $j=1,2,3$ due to the fact that $u\wedge u=0$.

After these first considerations, we proceed with studying the full remainder terms.

We begin with the first remaining term $R_D^{1,(p,q)}$ for some general $D\subset\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ and $2\le q<4$. By the introductory comments on occurring factors, for any $j=1,2,3$ we get

$$\begin{aligned}&|R_D^{1,(p,q)}(u,(\varphi u\wedge)_{ij})|\\ &\lesssim\iint_D\left|\fint_{x\rhd y}|u(z)-u(x)|^2\,dz\right|^{q-2}\\ &\quad\cdot\left|\left(\fint_{x\rhd y}[u(z_1)-u(x)]\,dz_1\right)\cdot\left(\fint_{x\rhd y}[(\varphi u\wedge)_{ij}(z_2)-(\varphi u\wedge)_{ij}(x)]\,dz_2\right)\right|\frac{dx\,dy}{\rho(x,y)^{p-q}}.\end{aligned}$$

If we consider the exemplary case $j = 1$, the dot product reads

$$\begin{aligned}
&\left(\fint_{x\triangleright y}[u(z_1)-u(x)]\,dz_1\right)\cdot\left(\fint_{x\triangleright y}[(\varphi u\wedge)_{i1}(z_2)-(\varphi u\wedge)_{i1}(x)]\,dz_2\right)\\
&\quad=\left(\fint_{x\triangleright y}[u_2(z_1)-u_2(x)]\,dz_1\right)\left(\fint_{x\triangleright y}[\varphi(z_2)u_3(z_2)-\varphi(x)u_3(x)]\,dz_2\right)\\
&\qquad+\left(\fint_{x\triangleright y}[u_3(z_1)-u_3(x)]\,dz_1\right)\left(\fint_{x\triangleright y}[\varphi(z_2)(-u_2)(z_2)-\varphi(x)(-u_2)(x)]\,dz_2\right).
\end{aligned}$$

By adding $0=\varphi(x)u_3(z_2)-\varphi(x)u_3(z_2)$ to the second factor in the first summand, respectively, $0=\varphi(x)(-u_2)(z_2)-\varphi(x)(-u_2)(z_2)$ to the second factor in the second summand, the dot product turns to

$$\begin{aligned}
&\left(\fint_{x\triangleright y}[u(z_1)-u(x)]\,dz_1\right)\cdot\left(\fint_{x\triangleright y}[(\varphi u\wedge)_{i1}(z_2)-(\varphi u\wedge)_{i1}(x)]\,dz_2\right)\\
&\quad=\left(\fint_{x\triangleright y}[u(z_1)-u(x)]\,dz_1\right)\cdot\left(\fint_{x\triangleright y}[(\varphi(z_2)-\varphi(x))\,(u\wedge)_{i1}(z_2)]\,dz_2\right)\\
&\qquad+\varphi(x)\left(\fint_{x\triangleright y}[u(z_1)-u(x)]\,dz_1\right)\cdot\left(\fint_{x\triangleright y}[(u\wedge)_{i1}(z_2)-(u\wedge)_{i1}(x)]\,dz_2\right).
\end{aligned}$$

But since $(u\wedge)_{i1}\in T_u\mathbb{S}^2$, the last summand in the previous equation vanishes. Now by the same arguments for $j=2,3$, we deduce

$$\begin{aligned}
&|R_D^{1,(p,q)}(u,(\varphi u\wedge)_{ij})|\\
&\quad\lesssim\|u\|_{L^\infty}\iint_D\left(\fint_{x\triangleright y}|u(z)-u(x)|^2\,dz\right)^{q-2}\left(\fint_{x\triangleright y}|u(z_1)-u(x)|\,dz_1\right)\\
&\qquad\cdot\left(\fint_{x\triangleright y}|\varphi(z_2)-\varphi(x)|\,dz_2\right)\frac{dy\,dx}{\rho(x,y)^{p-q}}.
\end{aligned}\tag{6.5}$$

Using $\|u\|_{L^\infty}\le 1$ simplifies this inequality. Now to take advantage of the local behavior of u and φ, we split the integration domain into

$$\begin{aligned}
&|R^{1,(p,q)}_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}}|\\
&\quad\le|R^{1,(p,q)}_{B_{2\rho}\times B_{2\rho}}|+|R^{1,(p,q)}_{B_{2\rho}\times(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})}|+|R^{1,(p,q)}_{(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})\times B_{2\rho}}|+|R^{1,(p,q)}_{(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})\times(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})}|.
\end{aligned}$$

The first term can be estimated by (6.5), Hölder's inequality for $1=\frac{q-2}{q}+\frac{1}{q}+\frac{1}{q}$, Jensen's inequality, and the identification of Lemma A.2, so that for any $j=1,2,3$,

$$\begin{aligned}
&|R^{1,(p,q)}_{B_{2\rho}\times B_{2\rho}}(u,(\varphi u\wedge)_{ij})|\\
&\quad\lesssim\iint_{B_{2\rho}^2}\left(\fint_{x\triangleright y}|u(z)-u(x)|^2\,dz\right)^{q-2}\left(\fint_{x\triangleright y}|u(z_1)-u(x)|\,dz_1\right)\\
&\qquad\cdot\left(\fint_{x\triangleright y}|\varphi(z_2)-\varphi(x)|\,dz_2\right)\frac{dy\,dx}{\rho(x,y)^{p-q}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\iint_{B_{2\rho}^2} \frac{\fint_{x \rhd y} |u(z_0) - u(x)|^{2q} \, dz_0}{\rho(x,y)^{p-q}} \, dy \, dx \right)^{\frac{q-2}{q}} \\
&\quad \cdot \left(\iint_{B_{2\rho}^2} \frac{\fint_{x \rhd y} |u(z_1) - u(x)|^{q} \, dz_1}{\rho(x,y)^{p-q}} \, dy \, dx \right)^{1/q} \\
&\quad \cdot \left(\iint_{B_{2\rho}^2} \frac{\fint_{x \rhd y} |\varphi(z_2) - \varphi(x)|^{q} \, dz_2}{\rho(x,y)^{p-q}} \, dx \, dy \right)^{1/q} \\
&\approx [u]^{2q-4}_{W^{\frac{p-q-1}{2q},2q}(B_{2\rho})} [u]_{W^{\frac{p-q-1}{q},q}(B_{2\rho})} [\varphi]_{W^{\frac{p-q-1}{q},q}(B_{2\rho})} \lesssim [u]^{2q-3}_{W^{\frac{p-q-1}{q},q}(B_{2\rho})},
\end{aligned}
$$

where we have applied the Sobolev embedding (Lemma A.3) and the assumption on φ, i.e. $[\varphi]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \leq 1$, in the last inequality. For the second term of the splitting, we subdivide the integration domain $B_{2\rho} \times (\mathbb{R}/\mathbb{Z} \setminus B_{2\rho})$ into $B_{2\rho} \times (B_r(x_0) \setminus B_{2\rho})$, to use the local fractional Sobolev regularity of u in $B_r(x_0)$, and the rest $B_{2\rho} \times (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0))$. At this point recall that $\tilde{u}$ denotes a $W^{\frac{p-q-1}{q},q}$-extension of $u|_{B_r(x_0)}$ from $B_r(x_0)$ to $\mathbb{R}$ (Remark 6.2). By inequality (6.5) and the disjoint support estimate of Lemma C.1 we then get

$$
\begin{aligned}
&|R^{1,(p,q)}_{B_{2\rho} \times (B_r(x_0) \setminus B_{2\rho})}(u, (\varphi u \wedge)_{ij})| \\
&\qquad \lesssim \int_{B_{2\rho}} \int_{B_r(x_0) \setminus B_{2\rho}} \left(\fint_{x \rhd y} |\tilde{u}(z) - \tilde{u}(x)|^2 \, dz \right)^{q-2} \left(\fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(x)| \, dz_1 \right) \\
&\qquad\quad \cdot \left(\fint_{x \rhd y} |\varphi(z_2) - \varphi(x)| \, dz_2 \right) \frac{dy \, dx}{\rho(x,y)^{p-q}} \\
&\qquad \lesssim \sum_{l=1}^{\infty} 2^{-(l+1)\frac{p-q}{q}} [\tilde{u}]^{2q-3}_{W^{\frac{p-q-1}{q},q}(B_{2^l \rho})} [\varphi]_{W^{\frac{p-q-1}{q},q}(B_\rho)} \\
&\qquad \lesssim [\tilde{u}]^{q-2}_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \sum_{l=1}^{\infty} 2^{-(l+1)\frac{p-q}{q}} [\tilde{u}]^{q-1}_{W^{\frac{p-q-1}{q},q}(B_{2^l \rho})} [\varphi]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \\
&\qquad \lesssim \sum_{l=1}^{\infty} 2^{-(l+1)\frac{p-q}{q}} [\tilde{u}]^{q-1}_{W^{\frac{p-q-1}{q},q}(B_{2^l \rho})},
\end{aligned} \tag{6.6}
$$

where the constant depends on $[\tilde{u}]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} < \infty$ as well as $[\varphi]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \leq 1$. Since $\rho(x,y) \geq r/4$ for $x \in B_{2\rho}$ and $y \in \mathbb{R}/\mathbb{Z} \setminus B_r(x_0)$, we estimate the remainder term by (6.5) so that

$$
\begin{aligned}
&|R^{1,(p,q)}_{B_{2\rho} \times (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0))}(u, (\varphi u \wedge)_{ij})| \\
&\qquad \lesssim \int_{B_{2\rho}} \int_{\mathbb{R}/\mathbb{Z} \setminus B_r(x_0)} \left(\fint_{x \rhd y} |u(z) - u(x)|^2 \, dz \right)^{q-2} \left(\fint_{x \rhd y} |u(z_1) - u(x)| \, dz_1 \right) \\
&\qquad\quad \cdot \left(\fint_{x \rhd y} |\varphi(z_2) - \varphi(x)| \, dz_2 \right) \frac{dy \, dx}{\rho(x,y)^{p-q}} \\
&\qquad \lesssim (r/4)^{-(p-q+1)} \|u\|^{2q-3}_{L^\infty} \|\varphi\|_{L^1}.
\end{aligned} \tag{6.7}
$$

Note that $\varphi \in W^{\frac{p-q-1}{q},q}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset B_\rho$ together with the Sobolev inequalities of Theorems A.4 and A.5 leads to

$$\|\varphi\|_{L^1} \lesssim \rho\,[\varphi]_{W^{\frac{p-q-1}{q},q}(\mathbb{R})} \lesssim \rho. \tag{6.8}$$

The estimates on $R^{1,(p,q)}_{B_{2\rho}\times(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})}$ also work for $R^{1,(p,q)}_{(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})\times B_{2\rho}}$ by symmetry. In the case of $R^{1,(p,q)}_{(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})\times(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})}$, we have to consider

$$\begin{aligned}(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho})\times(\mathbb{R}/\mathbb{Z}\setminus B_{2\rho}) = {}&(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))\times(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))\\ &\cup(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))\times(B_r(x_0)\setminus B_{2\rho})\\ &\cup(B_r(x_0)\setminus B_{2\rho})\times(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))\\ &\cup(B_r(x_0)\setminus B_{2\rho})\times(B_r(x_0)\setminus B_{2\rho}).\end{aligned}$$

$R^{1,(p,q)}_D$ can be estimated for the first three domains as in (6.7), due to $\operatorname{supp}\varphi\subset B_\rho$, which implies that either $\fint_{x\rhd y}[\varphi(z_2)-\varphi(x)]\,dz_2 = 0$ or $\rho(x,y)\geq r$. For the fourth domain we proceed with the help of the tail estimate of Lemma C.1 similar to (6.6). Therefore, the statement follows for the first remaining term in the case of $2\leq q<4$.

For $q\geq 4$ in $R^{1,(p,q)}$ the same methods work as well, in particular, for $D\subset\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$,

$$\begin{aligned}&|R^{1,(p,q)}_D(u,(\varphi u\wedge)_{ij})|\\ &\quad\lesssim\|u\|_{L^\infty}\iint_D\left(\fint_{x\rhd y}|u(z)-u(x)|^2\,dz\right)^2\left(\fint_{x\rhd y}|u(z_1)-u(x)|\,dz_1\right)^{q-3}\\ &\qquad\qquad\cdot\left(\fint_{x\rhd y}|\varphi(z_2)-\varphi(x)|\,dz_2\right)\frac{dy\,dx}{\rho(x,y)^{p-q}}.\end{aligned}$$

We only need to change the exponents in Hölder's inequality to $1=\frac{2}{q}+\frac{q-3}{q}+\frac{1}{q}$, since we deal with the factor

$$\left(\fint_{x\rhd y}|u(z)-u(x)|^2\,dz\right)^2\left|\fint_{x\rhd y}[u(z)-u(x)]\,dz\right|^{q-4}$$

instead of $(\fint_{x\rhd y}|u(z)-u(x)|^2\,dz)^{q-2}$, and slightly adapt the disjoint support estimate of Lemma C.1 as well as (6.7), to achieve the desired result.

For the second and third remainders we have to deal for $j=1,2,3$ with the scalar products

$$\begin{aligned}&\fint_{x\rhd y}(u(z)-u(x))\cdot((\varphi u\wedge)_{ij}(z)-(\varphi u\wedge)_{ij}(x))\,dz,\\ &\fint_{x\rhd y}\fint_{x\rhd y}(u(s)-u(t))\cdot((\varphi u\wedge)_{ij}(s)-(\varphi u\wedge)_{ij}(t))\,ds\,dt,\end{aligned}$$

in place of

$$\fint_{x\rhd y}\fint_{x\rhd y}(u(z_1)-u(x))\cdot((\varphi u\wedge)_{ij}(z_2)-(\varphi u\wedge)_{ij}(x))\,dz_1\,dz_2.$$

However, the techniques presented for the first remainder, in particular adding zeros and using $(u\wedge)_{ij} \in T_u\mathbb{S}^2$ for $j = 1, 2, 3$, work out similarly and lead for any $D \subset \mathbb{R}/\mathbb{Z} \setminus \mathbb{R}/\mathbb{Z}$ to

$$\begin{aligned}&|R_D^{2,(p,q)}(u, (\varphi u\wedge)_{ij})|\\ &\quad\lesssim \|u\|_{L^\infty} \iint_D \left(\fint_{x\triangleright y} |u(z) - u(x)|^2\, dz\right)\left(\fint_{x\triangleright y} |u(z_1) - u(x)|\, dz_1\right)^{q-1}\\ &\qquad\qquad \cdot \left(\fint_{x\triangleright y} |\varphi(z_2) - \varphi(x)|\, dz_2\right) \frac{dy\, dx}{\rho(x,y)^{p-q}}\end{aligned}$$

as well as

$$\begin{aligned}&|R_D^{3,(p,q)}(u, (\varphi u\wedge)_{ij})|\\ &\lesssim \|u\|_{L^\infty} \iint_D \left(\fint_{x\triangleright y} |u(z_1) - u(x)|\, dz_1\right)^{q+1}\left(\fint_{x\triangleright y} |\varphi(z_2) - \varphi(x)|\, dz_2\right) \frac{dy\, dx}{\rho(x,y)^{p-q}}.\end{aligned}$$

Hence, by proceeding as for the first remainder, the statement follows for $k = 2, 3$ from adjusting the exponents in Hölder's inequality and the tail estimate in Lemma C.1.

Let us now turn to the cases $k = 4, \ldots, 7$. The remainder terms for $k = 4$ and $k = 7$ are easier to handle since

$$\begin{aligned}\left|\fint_{x\triangleright y} \eta(z)u(z) \cdot ((\varphi u\wedge)_{ij})_{\mathbb{R}/\mathbb{Z}}\, dz\right| &\lesssim \|\eta\|_{L^\infty} \|u\|_{L^\infty}^2 \|\varphi\|_{L^1},\\ |\eta(x)u(x) \cdot ((\varphi u\wedge)_{ij})_{\mathbb{R}/\mathbb{Z}}| &\lesssim \|\eta\|_{L^\infty} \|u\|_{L^\infty}^2 \|\varphi\|_{L^1},\\ |\eta(y)u(y) \cdot ((\varphi u\wedge)_{ij})_{\mathbb{R}/\mathbb{Z}}| &\lesssim \|\eta\|_{L^\infty} \|u\|_{L^\infty}^2 \|\varphi\|_{L^1}.\end{aligned} \tag{6.9}$$

Note that $\|u\|_{L^\infty}, \|\eta\|_{L^\infty} \lesssim 1$, and by (6.8), $\|\varphi\|_{L^1} \lesssim \rho$. Thus the remaining factors emerge in the energy $\mathcal{E}^{p,q}(u)$, in particular

$$|R_\eta^{k,(p,q)}(u, (\varphi u\wedge)_{ij})| \lesssim \rho\, \mathcal{E}^{p,q}(u),$$

since in case $k = 4$ we can extend the integrand with an extra factor $1 - \frac{1}{2}c$ due to its boundedness (see Remark 6.7), and in case $k = 7$ the necessary factors of the energy are already given. It remains to study the cases $k = 5$ and $k = 6$, whose factors in the integrand are not necessarily comparable with the energy. We begin with $k = 5$ and observe by $\operatorname{supp} \eta \subset B_{r/2}(x_0)$ that

$$\begin{aligned}R_\eta^{5,(p,q)}(u, (\varphi u\wedge)_{ij}) = -q \int_{B_{r/2}(x_0)} \int_{\mathbb{R}/\mathbb{Z}} &\left(a - \tfrac{1}{4}b^2\right)^{\frac{q-2}{2}} k(x,y)^{-p/2}\left(a - \tfrac{1}{2}b\right)\\ &\cdot \left(\eta(x)u(x) \cdot ((\varphi u\wedge)_{ij})_{\mathbb{R}/\mathbb{Z}}\right) \frac{dy\, dx}{\rho(x,y)^{p-q}}.\end{aligned}$$

We split the integration domain into

$$B_{r/2}(x_0) \times \mathbb{R}/\mathbb{Z} = (B_{r/2}(x_0) \times B_r(x_0)) \cup (B_{r/2}(x_0) \times (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0))).$$

For the first domain, by the previous comments on the factors appearing in the integrand, inequalities (6.9) and (6.8), Jensen's inequality, Lemma A.2, and Sobolev embedding (Lemma A.3), we have

$$\begin{aligned} &|R^{5,(p,q)}_{\eta,B_{r/2}(x_0)\times B_r(x_0)}(u,(\varphi u\wedge)_{ij})| \\ &\quad\lesssim \|\eta\|_{L^\infty}\|u\|^2_{L^\infty}\|\varphi\|_{L^1}\left(\int_{B_r(x_0)}\int_{B_r(x_0)}\frac{\fint_{x\rhd y}|u(z)-u(x)|^{2q}\,dz}{\rho(x,y)^{p-q}}dy\,dx\right)^{1/2} \\ &\quad\lesssim \rho[u]^q_{W^{\frac{p-q-1}{2q},2q}(B_r(x_0))} \\ &\quad\lesssim \rho\, r^{\frac{p-q-2}{2}}[u]^q_{W^{\frac{p-q-1}{q},q}(B_r(x_0))}. \end{aligned} \tag{6.10}$$

For the second domain we notice that $\rho(x,y)\geq r/2$ for $x\in B_{r/2}(x_0)$ and $y\in\mathbb{R}/\mathbb{Z}\setminus B_r(x_0)$, which leads by (6.9), estimates like (6.7), and (6.8) to

$$\begin{aligned} |R^{5,(p,q)}_{\eta,B_{r/2}(x_0)\times(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))}(u,(\varphi u\wedge)_{ij})| &\lesssim (r/2)^{-(p-q)}\|u\|^{q+2}_{L^\infty}\|\eta\|_{L^\infty}\|\varphi\|_{L^1} \\ &\lesssim \rho r^{-(p-q)}. \end{aligned} \tag{6.11}$$

For $k=6$ we distinguish the following parts of the integration domain:

$$\begin{aligned} \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} = B_r(x_0)&\times B_r(x_0) \\ &\cup B_r(x_0)\times(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0)) \\ &\cup (\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))\times B_r(x_0) \\ &\cup (\mathbb{R}/\mathbb{Z}\setminus B_r(x_0))\times(\mathbb{R}/\mathbb{Z}\setminus B_r(x_0)). \end{aligned}$$

For the first integration domain we get the same estimate as in (6.10), and for the others we find by $\operatorname{supp}\eta\subset B_{r/2}(x_0)$ that either $\rho(x,y)\geq r/2$ or

$$\fint_{x\rhd y}\eta(z)u(z)\cdot((\varphi u\wedge)_{ij})_{\mathbb{R}/\mathbb{Z}}\,dz=0.$$

Therefore, we deduce the same estimate as in (6.11). All the remainder cases have thus been estimated. ∎

6.3. *Regularity theory for $\mathcal{E}^q$-critical points: Proof of Proposition 6.1*

In this section we finally apply the grand machinery of showing Hölder regularity for (essentially) fractional harmonic maps, which correspond to the first derivatives of our critical knots of interest. We establish a proof for the scale-invariant tangent-point energies, i.e. for $\mathrm{TP}^{p,q}$ with $p=q+2$ and $q\geq 2$, along the lines of [14, 68], but face some major obstacles due to the local definition of critical points for scale-invariant tangent-point energies (Definition 1.5). Our main goal is to show the decay estimate of Proposition 6.1.

We begin by estimating the Gagliardo seminorm of u by an operator $\Gamma_{\beta,B}u$ resembling the Riesz potential, which is introduced below.

First we recall that the term containing the highest order in the Euler–Lagrange equation of $\mathcal{E}^q$, $q \geq 2$, for maps $u, \varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ and some small interval $B \subset \mathbb{R}/\mathbb{Z}$ is given by

$$\begin{aligned} Q_{B\times B}(u,\varphi) &:= Q^{(q+2,q)}_{B\times B}(u,\varphi) \\ &= q \int_B \int_B \left| \fint_{x\triangleright y} [u(z) - u(x)]\, dz \right|^{q-2} k(x,y)^{-\frac{q+2}{2}} \\ &\quad \cdot \fint_{x\triangleright y} \fint_{x\triangleright y} (u(z_1) - u(x)) \cdot (\varphi(z_2) - \varphi(x))\, dz_1\, dz_2\, \frac{dy\, dx}{\rho(x,y)^2}. \end{aligned}$$

Note that for our functions of interest the factor $k(x,y)^{-\frac{q+2}{2}}$ is strictly positive and bounded (see Remark 6.7).

As in [14, 68], we now define a vector-valued potential for $0 < \beta < 1$ by

$$\begin{aligned} \Gamma_{\beta,B\times B}\, u(z) &:= \int_B \int_B \left| \fint_{x\triangleright y} [u(z_0) - u(x)]\, dz_0 \right|^{q-2} k(x,y)^{-\frac{q+2}{2}} \\ &\cdot \fint_{x\triangleright y} \fint_{x\triangleright y} (u(z_1) - u(x))(|z - z_2|^{\beta-1} - |z - x|^{\beta-1})\, dz_1\, dz_2\, \frac{dy\, dx}{\rho(x,y)^2}, \end{aligned} \tag{6.12}$$

following the definition of the Riesz potential $\mathcal{I}_\beta$ of order β, which is defined by

$$\mathcal{I}_\beta f(x) = \int_{\mathbb{R}} |z - x|^{\beta-1} f(z)\, dz.$$

The inverse of the Riesz potential $\mathcal{I}_\beta$ is called the fractional Laplacian of order β, which for $\beta \in (0,2)$ has the form

$$(-\Delta)^{\beta/2} f(x) = c \int_{\mathbb{R}} \frac{f(y) - f(x)}{|x - y|^{1+\beta}}\, dy$$

for some $c > 0$ (see [25]). In our situation, we observe that

$$Q_{B\times B}(u,\varphi) = q \int_{\mathbb{R}} \Gamma_{\beta,B\times B} u(z) \cdot (-\Delta)^{\beta/2} \varphi(z)\, dz. \tag{6.13}$$

Note that

$$\mathcal{I}_\alpha \Gamma_{\beta,B\times B}\, u = \Gamma_{\alpha+\beta,B\times B} u \quad \text{for any } \alpha, \beta > 0. \tag{6.14}$$

Our first intermediate result is the following.

Proposition 6.10 (Left-hand side estimates). *Let $q \geq 2$ with $1/q - 1/\mu > 0$ small, and let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with locally small tangent-point energy $\mathrm{TP}^{q+2,q}$ around an interval $B_r(x_0)$ in $\mathbb{R}/\mathbb{Z}$, in the sense of Definition 1.1. Moreover, denote the unit tangent field of γ by $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\int_{\mathbb{R}/\mathbb{Z}} u = 0$ and for any $y_0 \in B_{r/2}(x_0)$,* *choose $\rho > 0$ such that $B_{2^L\rho} := B_{2^L\rho}(y_0) \subset B_r(x_0)$ for large $L \in \mathbb{N}$. Then for any $\delta > 0$,*

$$[u]^q_{W^{1/q,q}(B_\rho)} \lesssim [u]_{W^{1/q,q}(B_{2^L\rho})} \|\chi_{B_{2^K\rho}} \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} + \delta [u]^q_{W^{1/q,q}(B_{2^L\rho})}$$
$$+ C_\delta ([u]^q_{W^{1/q,q}(B_{2^L\rho})} - [u]^q_{W^{1/q,q}(B_\rho)})$$

for any $L, K \in \mathbb{N}$ large enough with $L \gg K$. The constant in this inequality only depends on q.

Proof. Let $\eta \in C_c^\infty(B_{2\rho})$ with $\eta \equiv 1$ on B_ρ and $|\nabla^k \eta| \le C(k)\rho^{-k}$. Recall that $(u)_A = \frac{1}{|A|}\int_A u$ and set

$$\psi(x) := \eta(x)(u(x) - (u)_{B_{2\rho}\setminus B_\rho}).$$

Then for any $x, y \in B_\rho$ we have

$$\left| \fint_{x \triangleright y} [\psi(z) - \psi(x)]\, dz \right|^2 = \left| \fint_{x \triangleright y} [u(z) - u(x)]\, dz \right|^2$$

and therefore by Proposition 6.8, for any $L \ge 2$,

$$[u]^q_{W^{1/q,q}(B_\rho)} \lesssim \int_{B_{2^L\rho}} \int_{B_{2^L\rho}} \left| \fint_{x\triangleright y} [u(z) - u(x)]\, dz \right|^{q-2} k(x,y)^{-\frac{q+2}{2}}$$
$$\cdot \left| \fint_{x\triangleright y} \psi(z) - \psi(x)\, dz \right|^2 \frac{dy\, dx}{\rho(x,y)^2}.$$

Now we decompose

$$\psi(z) - \psi(x) = (u(z) - u(x)) - (1 - \eta(z))(u(z) - u(x))$$
$$+ (\eta(z) - \eta(x))(u(x) - (u)_{B_{2\rho}\setminus B_\rho}),$$

which leads to

$$[u]^q_{W^{1/q,q}(B_\rho)} \lesssim \mathrm{I} - \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I} := \iint_{{B_{2^L\rho}}^2} \left| \fint_{x\triangleright y} [u(z) - u(x)]\, dz \right|^{q-2} k(x,y)^{-\frac{q+2}{2}}$$
$$\cdot \fint_{x\triangleright y} \fint_{x\triangleright y} (u(z_1) - u(x)) \cdot (\psi(z_2) - \psi(x))\, dz_1\, dz_2 \frac{dx\, dy}{\rho(x,y)^2},$$

$$\mathrm{II} := \iint_{{B_{2^L\rho}}^2} \left| \fint_{x\triangleright y} [u(z) - u(x)]\, dz \right|^{q-2} k(x,y)^{-\frac{q+2}{2}}$$
$$\cdot \fint_{x\triangleright y} \fint_{x\triangleright y} ((1 - \eta(z_1))(u(z_1) - u(x))) \cdot (\psi(z_2) - \psi(x))\, dz_1\, dz_2 \frac{dx\, dy}{\rho(x,y)^2},$$

$$\mathrm{III} := \iint_{B_{2L_\rho}{}^2} \left| \fint_{x \rhd y} [u(z) - u(x)]\, dz \right|^{q-2} k(x,y)^{-\frac{q+2}{2}} \cdot \fint_{x \rhd y} \fint_{x \rhd y} ((\eta(z_1) - \eta(x))(u(x) - (u)_{B_{2\rho} \setminus B_\rho})) \cdot (\psi(z_2) - \psi(x))\, dz_1\, dz_2 \cdot \frac{dx\, dy}{\rho(x,y)^2}.$$

We start with the terms involving the cutoff function. By the boundedness of $k(x,y)^{-\frac{q+2}{2}}$ (Remark 6.7), Hölder's inequality for $1 = \frac{q-2}{q} + \frac{1}{q} + \frac{1}{q}$, Jensen's inequality and Lemma A.2 we have

$$\begin{aligned}
&|\mathrm{II}| \\
&\lesssim \iint_{(B_{2L_\rho})^2} \left(\fint_{x \rhd y} |u(z) - u(x)|\, dz \right)^{q-2} \cdot \fint_{x \rhd y} (1 - \eta(z_1))\, |u(z_1) - u(x)|\, dz_1 \fint_{x \rhd y} |\psi(z_2) - \psi(x)|\, dz_2 \frac{dx\, dy}{\rho(x,y)^2} \\
&\lesssim \left(\iint_{(B_{2L_\rho})^2} \frac{\fint_{x \rhd y} |u(z) - u(x)|^q\, dz}{\rho(x,y)^2}\, dx\, dy \right)^{\frac{q-2}{q}} \\
&\quad \cdot \left(\iint_{(B_{2L_\rho})^2} \frac{\fint_{x \rhd y} (1 - \eta(z_1))^q |u(z_1) - u(x)|^q\, dz_1}{\rho(x,y)^2}\, dx\, dy \right)^{1/q} \\
&\quad \cdot \left(\iint_{(B_{2L_\rho})^2} \frac{\fint_{x \rhd y} |\psi(z_2) - \psi(x)|^q\, dz_2}{\rho(x,y)^2}\, dx\, dy \right)^{1/q} \\
&\lesssim [u]^{q-2}_{W^{1/q,q}(B_{2L_\rho})} [\psi]_{W^{1/q,q}(B_{2L_\rho})} \left(\iint_{(B_{2L_\rho})^2} (1 - \eta(z))^q |u(z) - u(x)|^q \frac{dz\, dx}{\rho(z,x)^2} \right)^{1/q}.
\end{aligned}$$

Then $[\psi]_{W^{1/q,q}(B_{2L_\rho})} \lesssim [u]_{W^{1/q,q}(B_{2L_\rho})}$ by Proposition A.7, and the assumption $\eta \equiv 1$ on B_ρ and Young's inequality lead to

$$\begin{aligned}
|\mathrm{II}| &\lesssim [u]^{q-1}_{W^{1/q,q}(B_{2L_\rho})} \left(\int_{B_{2L_\rho}} \int_{B_{2L_\rho} \setminus B_\rho} |u(z) - u(x)|^q \frac{dz\, dx}{\rho(z,x)^2} \right)^{1/q} \\
&\lesssim \delta [u]^q_{W^{1/q,q}(B_{2L_\rho})} + C_\delta \left([u]^q_{W^{1/q,q}(B_{2L_\rho})} - [u]^q_{W^{1/q,q}(B_\rho)} \right).
\end{aligned}$$

Regarding III, we estimate along the lines of II, first of all

$$\begin{aligned}
|\mathrm{III}| &\lesssim [u]^{q-2}_{W^{1/q,q}(B_{2L_\rho})} [\psi]_{W^{1/q,q}(B_{2L_\rho})} \\
&\quad \cdot \left(\int_{B_{2L_\rho}} \int_{B_{2L_\rho}} |\eta(x) - \eta(z)|^q |u(x) - (u)_{B_{2\rho} \setminus B_\rho}|^q \frac{dz\, dx}{\rho(z,x)^2} \right)^{1/q},
\end{aligned}$$

and next with the help of Proposition A.6 for $L \geq 2$ as well as Young's inequality,

$$\begin{aligned}|\mathrm{III}| &\lesssim [u]^{q-1}_{W^{1/q,q}(B_{2L_\rho})}\Big(\int_{B_{2L_\rho}}\int_{B_{2L_\rho}} |\eta(x)-\eta(z)|^q |u(x)-(u)_{B_{2\rho}\setminus B_\rho}|^q \frac{dz\,dx}{\rho(z,x)^2}\Big)^{1/q}\\ &\lesssim [u]^{q-1}_{W^{1/q,q}(B_{2L_\rho})}\big([u]^q_{W^{1/q,q}(B_{2L_\rho})} - [u]^q_{W^{1/q,q}(B_\rho)}\big)^{1/q}\\ &\lesssim \delta[u]^q_{W^{1/q,q}(B_{2L_\rho})} + C_\delta\big([u]^q_{W^{1/q,q}(B_{2L_\rho})} - [u]^q_{W^{1/q,q}(B_\rho)}\big).\end{aligned}$$

Hence

$$|\mathrm{II}| + |\mathrm{III}| \lesssim \delta[u]^q_{W^{1/q,q}(B_{2L_\rho})} + C_\delta\big([u]^q_{W^{1/q,q}(B_{2L_\rho})} - [u]^q_{W^{1/q,q}(B_\rho)}\big).$$

For the remaining term I, by definition of $Q_{B\times B}$ in (6.2) and Proposition A.7 for $L \geq 1$ we see that

$$\begin{aligned}|\mathrm{I}| &\lesssim [\psi]_{W^{1/q,q}(\mathbb{R})} \sup_{\varphi\in C_c^\infty(B_{2\rho},\mathbb{R}^3),\,[\varphi]_{W^{1/q,q}(\mathbb{R})}\leq 1} |Q_{B_{2L_\rho}\times B_{2L_\rho}}(u,\varphi)|\\ &\lesssim [u]_{W^{1/q,q}(B_{2L_\rho})} \sup_{\varphi\in C_c^\infty(B_{2\rho},\mathbb{R}^3),\,[\varphi]_{W^{1/q,q}(\mathbb{R})}\leq 1} |Q_{B_{2L_\rho}\times B_{2L_\rho}}(u,\varphi)|.\end{aligned}$$

By using the identity (6.13) for $\mu > q$ with $1/q - 1/\mu > 0$ small and introducing cutoff functions $\eta_{B_R} \in C_c^\infty(B_{2R})$ with $\eta_{B_R} \equiv 1$ on B_R and $\|\nabla^k \eta_{B_R}\|_{L^\infty} \lesssim R^{-k}$ for $R > 0$, we get, for any $K \geq 1$ and some $\varphi \in C_c^\infty(B_{2\rho}, \mathbb{R}^3)$ with $[\varphi]_{W^{1/q,q}(\mathbb{R})} \leq 1$,

$$\begin{aligned}|Q_{B_{2L_\rho}\times B_{2L_\rho}}(u,\varphi)| &= q\left|\int_{\mathbb{R}} \Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u(z)\cdot(-\Delta)^{\frac{1}{2\mu}}\varphi(z)\,dz\right|\\ &\lesssim \left|\int_{\mathbb{R}} \eta_{B_{2^{K-1}\rho}}(z)\Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u(z)\cdot(-\Delta)^{\frac{1}{2\mu}}\varphi(z)\,dz\right|\\ &\quad + \sum_{k=K}^\infty \left|\int_{\mathbb{R}} (\eta_{B_{2^k\rho}} - \eta_{B_{2^{k-1}\rho}})(z)\Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u(z)\cdot(-\Delta)^{\frac{1}{2\mu}}\varphi(z)\,dz\right|.\end{aligned}$$

We estimate the first term by Hölder's inequality, Sobolev's inequality (Theorem A.5) and $[\varphi]_{W^{1/q,q}(\mathbb{R})} \leq 1$ as

$$\begin{aligned}&\left|\int_{\mathbb{R}} \eta_{B_{2^{K-1}\rho}}(z)\Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u(z)\cdot(-\Delta)^{\frac{1}{2\mu}}\varphi(z)\,dz\right|\\ &\qquad\lesssim \|\eta_{B_{2^{K-1}\rho}}\Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \|(-\Delta)^{\frac{1}{2\mu}}\varphi\|_{L^\mu}\\ &\qquad\lesssim \|\chi_{B_{2^K\rho}}\Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} [\varphi]_{W^{1/q,q}(\mathbb{R})}\\ &\qquad\lesssim \|\chi_{B_{2^K\rho}}\Gamma_{1/\mu,B_{2L_\rho}\times B_{2L_\rho}} u\|_{L^{\frac{\mu}{\mu-1}}}.\end{aligned}$$

Using integration by parts and the property $\mathcal{I}_\alpha \Gamma_{\beta,B_{2L_\rho}\times B_{2L_\rho}} u = \Gamma_{\alpha+\beta,B_{2L_\rho}\times B_{2L_\rho}} u$ (see (6.14)), the second part can be rewritten as

$$\sum_{k=K}^{\infty}\left|\int_{\mathbb{R}}(\eta_{B_{2^k\rho}}-\eta_{B_{2^{k-1}\rho}})(z)\Gamma_{1/\mu,B_{2L\rho}\times B_{2L\rho}}u(z)\cdot(-\Delta)^{\frac{1}{2\mu}}\varphi(z)\,dz\right|$$
$$=\sum_{k=K}^{\infty}\left|\int_{\mathbb{R}}(-\Delta)^{1/q-1/\mu}(\mathcal{I}_{2/q-2/\mu}\Gamma_{1/\mu,B_{2L\rho}\times B_{2L\rho}}u)(z)\right.$$
$$\left.\cdot\big((\eta_{B_{2^k\rho}}-\eta_{B_{2^{k-1}\rho}})(-\Delta)^{\frac{1}{2\mu}}\varphi\big)(z)\,dz\right|$$
$$=\sum_{k=K}^{\infty}\left|\int_{\mathbb{R}}\Gamma_{2/q-1/\mu,B_{2L\rho}\times B_{2L\rho}}u(z)\right.$$
$$\left.\cdot(-\Delta)^{1/q-1/\mu}\big((\eta_{B_{2^k\rho}}-\eta_{B_{2^{k-1}\rho}})(-\Delta)^{\frac{1}{2\mu}}\varphi\big)(z)\,dz\right|.$$

Then we estimate by Hölder's inequality, Proposition A.8 for $1/q-1/\mu>0$ small enough, the localization argument of Proposition B.1, and Sobolev's inequality (Theorem A.5):

$$\sum_{k=K}^{\infty}\left|\int_{\mathbb{R}}\Gamma_{2/q-1/\mu,B_{2L\rho}\times B_{2L\rho}}u(z)\cdot(-\Delta)^{1/q-1/\mu}\big((\eta_{B_{2^k\rho}}-\eta_{B_{2^{k-1}\rho}})(-\Delta)^{\frac{1}{2\mu}}\varphi\big)(z)\,dz\right|$$
$$\lesssim\sum_{k=K}^{\infty}\|\Gamma_{2/q-1/\mu,B_{2L\rho}\times B_{2L\rho}}u\|_{L^{(1-2/q+1/\mu)^{-1}}}$$
$$\cdot\left\|(-\Delta)^{1/q-1/\mu}\big((\eta_{B_{2^k\rho}}-\eta_{B_{2^{k-1}\rho}})(-\Delta)^{\frac{1}{2\mu}}\varphi\big)\right\|_{L^{(2/q-1/\mu)^{-1}}}$$
$$\lesssim\sum_{k=K}^{\infty}[u]^{q-1}_{W^{1/q,q}(B_{2L\rho})}\,2^{-\sigma k}\,\|(-\Delta)^{\frac{1}{2\mu}}\varphi\|_{L^\mu}\lesssim\sum_{k=K}^{\infty}2^{-\sigma k}\,[u]^{q-1}_{W^{1/q,q}(B_{2L\rho})}$$

for some $\sigma>0$. The statement of the proposition follows by choosing K large enough. ∎

In the next step we need estimates involving the operator $\Gamma_{1/\mu,B\times B}u$, which appears in the left-hand side of estimates of Proposition 6.10, to obtain the decay estimate of Proposition 6.1. We start by splitting the operator $\Gamma_{1/\mu,B\times B}u$ by projecting it into the linear space spanned by u and the linear space orthogonal to u. More precisely, since $|u|=1$ a.e., we have

$$\|\chi_{B_{2K\rho}}\Gamma_{1/\mu,B_{2L\rho}\times B_{2L\rho}}u\|_{L^{\frac{\mu}{\mu-1}}}$$
$$\lesssim\|\chi_{B_{2K\rho}}u\cdot\Gamma_{1/\mu,B_{2L\rho}\times B_{2L\rho}}u\|_{L^{\frac{\mu}{\mu-1}}}+\|\chi_{B_{2K\rho}}u\wedge\Gamma_{1/\mu,B_{2L\rho}\times B_{2L\rho}}u\|_{L^{\frac{\mu}{\mu-1}}}.\tag{6.15}$$

Here we recall that $v\wedge$ for any $v\in\mathbb{R}^3$ is given by the $\mathbb{R}^{3\times3}$-matrix

$$v\wedge=\begin{pmatrix}0&-v_3&v_2\\v_3&0&-v_1\\-v_2&v_1&0\end{pmatrix}.$$

We then treat each part of the splitting separately. However, both estimates are based on effects of integration by compensation using nonlinear commutators as well as information from the Euler–Lagrange equations.

Lemma 6.11 (Right-hand side estimates I). *Let $q \geq 2$ with $1/q - 1/\mu > 0$ small, and let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a homeomorphism with small tangent-point energy $\mathrm{TP}^{q+2,q}$ around the interval $B_r(x_0)$ in $\mathbb{R}/\mathbb{Z}$, in the sense of Definition 1.1. Denote the unit tangent field of γ by $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\int_{\mathbb{R}/\mathbb{Z}} u = 0$ and let $\tilde{u}$ be a $W^{1/q,q}$-extension of $u|_{B_r(x_0)}$ from $B_r(x_0)$ to $\mathbb{R}$ as discussed in Remark 6.2. Moreover, for $y_0 \in B_{r/2}(x_0)$, choose $\rho > 0$ such that $B_{2^{2L}\rho} := B_{2^{2L}\rho}(y_0) \subset B_r(x_0)$ for large $L \in \mathbb{N}$. Then*

$$\|\chi_{B_{2^K\rho}} u \cdot \Gamma_{1/\mu, B_{2^L\rho} \times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \lesssim [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{k=1}^{\infty} 2^{-\sigma(L+k)} [\tilde{u}]^q_{W^{1/q,q}(B_{2^{2L+k}\rho})}$$

for any $K, L \in \mathbb{N}$ large enough with $L \gg K$. The constant in this inequality only depends on q.

Proof. First observe that, since $|u| = 1$ a.e. in $B_r(x_0)$,

$$u(z) \cdot \int_x^y \int_x^y (u(z_1) - u(x))(|z - z_2|^{1/\mu - 1} - |z - x|^{1/\mu-1})\, dz_1\, dz_2 =$$
$$-\frac{1}{2} \int_x^y \int_x^y (u(z_1) - u(x)) \cdot (u(z_1) + u(x) - 2u(z)) \big(|z - z_2|^{1/\mu-1} - |z - x|^{1/\mu-1}\big)\, dz_1\, dz_2$$

for almost all $x, y, z \in B_r(x_0)$. Therefore, by the definition of Γ in (6.12) and the boundedness of $k(x, y)^{-\frac{q+2}{2}}$ (Remark 6.7),

$$\begin{aligned} &|\chi_{B_{2^K\rho}} u(z) \cdot \Gamma_{1/\mu, B_{2^L\rho} \times B_{2^L\rho}} u(z)| \\ &\lesssim \chi_{B_{2^L\rho}}(z) \int_{B_{2^L\rho}} \int_{B_{2^L\rho}} \left(\fint_{x \triangleright y} |u(z_0) - u(x)|\, dz_0 \right)^{q-2} \\ &\qquad \cdot \fint_{x \triangleright y} |u(z_1) - u(x)|\, |u(z_1) + u(x) - 2u(z)|\, dz_1 \\ &\qquad \cdot \fint_{x \triangleright y} \big| |z - z_2|^{1/\mu-1} - |z - x|^{1/\mu-1} \big|\, dz_2\, \frac{dy\, dx}{\rho(x, y)^2}. \end{aligned}$$

Observe that u is evaluated only in $B_r(x_0)$ here as $B_{2^K\rho} \subset B_{2^L\rho} \subset B_r(x_0)$ ($L \gg K$). Since u and $\tilde{u}$ coincide on $B_r(x_0)$ by construction (see Remark 6.2), we can continue with $\tilde{u}$ from now on as it is globally $W^{1/q,q}$-regular, unlike u, which is only in $W^{1/q,q}(B_r(x_0), \mathbb{R}^3)$.

For the next step, we need the notion of the uncentered Hardy–Littlewood maximal function, which is given by

$$\mathcal{M} f(x) = \sup_{B_r(x) \ni y} \frac{1}{|B_r(y)|} \int_{B_r(y)} |f(z)|\, dz.$$

By Proposition D.3, for small $\delta > 0$,

$$\Big(\fint_{x \rhd y} |\tilde{u}(z_0) - \tilde{u}(x)|\, dz_0 \Big)^{q-2} \lesssim |x-y|^{(1/\mu - \delta)(q-2)} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(x) \big)^{q-2}.$$

Now we decompose the integral $\fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(x)|\, |\tilde{u}(z_1) + \tilde{u}(x) - 2\tilde{u}(z)|\, dz_1$ into four terms:

$$\fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(x)|\, |\tilde{u}(z_1) + \tilde{u}(x) - 2\tilde{u}(z)|\, dz_1$$

$$\leq \fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(y)|^2\, dz_1 \tag{6.16}$$

$$+ \fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(y)|\, dz_1\, |\tilde{u}(y) + \tilde{u}(x) - 2\tilde{u}(z)| \tag{6.17}$$

$$+ |\tilde{u}(y) - \tilde{u}(x)| \fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(y)|\, dz_1 \tag{6.18}$$

$$+ |\tilde{u}(y) - \tilde{u}(x)|\, |\tilde{u}(y) + \tilde{u}(x) - 2\tilde{u}(z)|. \tag{6.19}$$

For the first term (6.16), by duality we obtain, for some $\varphi \in C_c^\infty(\mathbb{R})$ with $\|\varphi\|_{L^\mu} \leq 1$,

$$\Big\| \int_{B_{2L_\rho}} \int_{B_{2L_\rho}} |x-y|^{(1/\mu-\delta)(q-2)} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(x) \big)^{q-2}$$
$$\cdot \fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(y)|^2\, dz_1 \fint_{x \rhd y} \big| |\cdot - z_2|^{1/\mu - 1} - |\cdot - x|^{1/\mu - 1} \big|\, dz_2 \, \frac{dy\, dx}{\rho(x,y)^2} \Big\|_{L^{\frac{\mu}{\mu-1}}(B_{2K_\rho})}$$
$$\lesssim \int_{\mathbb{R}} \int_{B_{2L_\rho}} \int_{B_{2L_\rho}} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(x) \big)^{q-2} |\varphi(z)|$$
$$\cdot \fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(y)|^2\, dz_1$$
$$\cdot \fint_{x \rhd y} \big| |z - z_2|^{1/\mu - 1} - |z - x|^{1/\mu - 1} \big|\, dz_2\, |x-y|^{(1/\mu-\delta)(q-2)-2}\, dy\, dx\, dz.$$

Hence, for an admissible $\varepsilon \in (0,1)$ and

$$\Big(\frac{1}{\mu} - \delta \Big)(q-1) + \frac{1}{2q} - 1 < \varepsilon < \frac{1}{\mu},$$

we can apply Lemma D.6 to the previous inequality and achieve

$$\begin{aligned}
&\Bigg\| \int_{B_{2L\rho}} \int_{B_{2L\rho}} |x-y|^{(1/\mu-\delta)(q-2)} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(x)\big)^{q-2} \\
&\cdot \fint_{x \triangleright y} |\tilde{u}(z_1)-\tilde{u}(y)|^2 \, dz_1 \fint_{x \triangleright y} \big| |\cdot - z_2|^{1/\mu-1} - |z-x|^{1/\mu-1} \big| \, dz_2 \, \frac{dy\,dx}{\rho(x,y)^2} \Bigg\|_{L^{\frac{\mu}{\mu-1}}(B_{2K\rho})} \\
&\precsim \int_{\mathbb{R}} \mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} \big(\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z)\big)^{q-2} \\
&\qquad \cdot \mathcal{M}\big(\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u} \mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u}\big)(z) \mathcal{I}_{1/\mu-\varepsilon}|\varphi|(z)\, dz \\
&\quad + \int_{\mathbb{R}} \big(\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z)\big)^{q-2} \\
&\qquad \cdot \mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} \mathcal{M}\big(\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u} \mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u}\big)(z) \mathcal{I}_{1/\mu-\varepsilon}|\varphi|(z)\, dz \\
&\quad + \int_{\mathbb{R}} \mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} \big(\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z)\big)^{q-2} \\
&\qquad \cdot \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z) \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u}(z) \mathcal{I}_{1/\mu-\varepsilon}|\varphi|(z)\, dz \\
&\quad + \int_{\mathbb{R}} \big(\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z)\big)^{q-2} \\
&\cdot \mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u}\big)(z) \mathcal{I}_{1/\mu-\varepsilon}|\varphi|(z)\, dz. \qquad (6.20)
\end{aligned}$$

The integrals appearing on the right-hand side do make sense as they can be traced back to $\tilde{u} \in W^{1/q,q}(\mathbb{R})$ by applying Hölder's inequality and the following estimates: First, by the Hardy–Littlewood maximal inequality and Sobolev's inequality (Theorem A.5) we have

$$\begin{aligned}
&\|(\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u})^{q-2}\|_{L^{((1/\mu-\delta)(q-2))^{-1}}} \\
&\quad = \left(\int_{\mathbb{R}} |\chi_{B_{2L\rho}}(z) \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z)|^{(1/\mu-\delta)^{-1}} \, dz \right)^{(1/\mu-\delta)(q-2)} \\
&\quad \precsim \left(\int_{\mathbb{R}} |(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(z)|^{(1/\mu-\delta)^{-1}} \, dz \right)^{(1/\mu-\delta)(q-2)} \precsim [\tilde{u}]^{q-2}_{W^{1/q,q}(\mathbb{R})} < \infty,
\end{aligned}$$

and with the additional help of Theorem A.4,

$$\begin{aligned}
&\|\mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} (\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u})^{q-2}\|_{L^{(1-\frac{1}{2q}-(1/\mu-\delta)-\varepsilon)^{-1}}} \\
&\quad \precsim \|(\chi_{B_{2L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u})^{q-2}\|_{L^{((1/\mu-\delta)(q-2))^{-1}}} \precsim [\tilde{u}]^{q-2}_{W^{1/q,q}(\mathbb{R})} < \infty.
\end{aligned}$$

In a similar fashion with a slightly adapted Hölder's inequality we observe

$$\begin{aligned}
&\big\|\mathcal{M}\big(\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}\, \mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} u\big)\big\|_{L^{\frac{1}{\frac{1}{2q}+(1/\mu-\delta)}}} \\
&\qquad \precsim \|(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}\|_{L^{\frac{1}{1/\mu-\delta}}} \|(-\Delta)^{\frac{1}{4q}} \tilde{u}\|_{L^{2q}} \precsim [\tilde{u}]^2_{W^{1/q,q}(\mathbb{R})} < \infty,
\end{aligned}$$

and hence also

$$\Big\| \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u} \Big\|_{L^{\frac{1}{\frac{1}{2q}+(1/\mu-\delta)}}} \lesssim [\tilde{u}]^2_{W^{1/q,q}(\mathbb{R})},$$

$$\Big\| \mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} \big(\mathcal{M}(\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u} \mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u}) \big) \Big\|_{L^{\frac{1}{1-(1/\mu-\delta)(q-2)-\varepsilon}}} \lesssim [\tilde{u}]^2_{W^{1/q,q}(\mathbb{R})},$$

$$\Big\| \mathcal{I}_{(1/\mu-\delta)(q-1)+\frac{1}{2q}+\varepsilon-1} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/(2q)}{2}} \tilde{u} \big) \Big\|_{L^{\frac{1}{1-(1/\mu-\delta)(q-2)-\varepsilon}}} \lesssim [\tilde{u}]^2_{W^{1/q,q}(\mathbb{R})},$$

as well as

$$\big\| \mathcal{I}_{1/\mu-\varepsilon} |\varphi| \big\|_{L^{1/\varepsilon}} \lesssim \|\varphi\|_{L^\mu}.$$

By combining these observations and recalling $\|\varphi\|_{L^\mu} \le 1$, we conclude that

$$\Bigg\| \int_{B_{2^L\rho}} \int_{B_{2^L\rho}} |x-y|^{(1/\mu-\delta)(q-2)} \big(\chi_{B_{2^L\rho}} \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(x)\big)^{q-2} \cdot \fint_{x \rhd y} |\tilde{u}(z_1)-\tilde{u}(y)|^2 \, dz_1 \fint_{x\rhd y} \big| |z-z_2|^{1/\mu-1} - |z-x|^{1/\mu-1} \big| \, dz_2 \, \frac{dy\,dx}{\rho(x,y)^2} \Bigg\|_{L^{\frac{\mu}{\mu-1}}(B_{2^K\rho})} \lesssim [\tilde{u}]^q_{W^{1/q,q}(\mathbb{R})}.$$

With a view to the desired decay estimate of Proposition 6.1, we need to take advantage of the local Sobolev regularity of u, i.e. $u \in W^{1/q,q}(B_r(x_0))$. Recall that u and $\tilde{u}$ coincide on $B_r(x_0)$ by construction (see Remark 6.2), and hence so do their Gagliardo seminorms on subsets of $B_r(x_0)$. For this reason, we localize the previous estimates with the help of the factor $\chi_{B_{2^L\rho}}$ to get for some $\sigma = \sigma(q) > 0$ the upper bound

$$[u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{k=1}^{\infty} 2^{-\sigma(L+k)} [\tilde{u}]^q_{W^{1/q,q}(B_{2^{2L+k}\rho})}.$$

In particular, here we have applied Proposition B.3 to (6.20), the localized maximal inequality of Proposition B.4 and the localized Sobolev inequality of Lemma B.5.

We proceed with the remaining terms (6.17)–(6.19),

$$\begin{aligned} U(x,y,z) &:= (6.17) + (6.18) + (6.19) \\ &= \fint_{x\rhd y} |\tilde{u}(z_1)-\tilde{u}(y)| \, dz_1 \, |\tilde{u}(y)+\tilde{u}(x)-2\tilde{u}(z)| \\ &\quad + |\tilde{u}(y)-\tilde{u}(x)| \fint_{x\rhd y} |\tilde{u}(z_1)-\tilde{u}(y)| \, dz_1 \\ &\quad + |\tilde{u}(y)-\tilde{u}(x)| \, |\tilde{u}(y)+\tilde{u}(x)-2\tilde{u}(z)|, \end{aligned}$$

in the following similar manner. We begin by estimating $\fint_{x\rhd y} |\tilde{u}(z_1)-\tilde{u}(y)| \, dz_1$, respectively, $|\tilde{u}(y)-\tilde{u}(x)|$, by Proposition D.3 and Lebesgue's differentiation theorem by

$$|x-y|^{1/\mu-\delta} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(x) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}} \tilde{u}(y) \big).$$

Together with Lemma D.4 we get the upper bound

$$\int_{B_{2^L\rho}}\int_{B_{2^L\rho}} |x-y|^{(1/\mu-\delta)(q-1)}\big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u}(x)+\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u}(y)\big)^{q-1}$$
$$\cdot |x-y|^{1/q+\varepsilon}\big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}}\tilde{u}(x)+\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}}\tilde{u}(y)+\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}}\tilde{u}(z)\big)$$
$$\cdot k_{1/\mu-\frac{1}{2q}-\varepsilon,1/\mu}(x,y,z)\,\frac{dy\,dx}{\rho(x,y)^2}.$$

For some $\varphi \in C_c^\infty(\mathbb{R})$ with $\|\varphi\|_{L^\mu} \le 1$ we arrive at

$$\Big\| \int_{B_{2^L\rho}}\int_{B_{2^L\rho}} |x-y|^{(1/\mu-\delta)(q-1)}\big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u}(x)+\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u}(y)\big)^{q-1}$$
$$\cdot U(x,y,\cdot)\fint_{x\triangleright y}\big||\cdot-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\,\frac{dy\,dx}{\rho(x,y)^2}\Big\|_{L^{\frac{\mu}{\mu-1}}(B_{2^K\rho})}$$
$$\lesssim \iiint_{\mathbb{R}^3}\big((\chi_{B_{2^L\rho}}\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u})^{q-1}(x)+(\chi_{B_{2^L\rho}}\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u})^{q-1}(y)\big)$$
$$\cdot\big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}}\tilde{u}(x)+\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}}\tilde{u}(y)+\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}}\tilde{u}(z)\big)\,k_{1/\mu-\frac{1}{2q}-\varepsilon,1/\mu}(x,y,z)$$
$$\cdot|\varphi(z)|\,\frac{dx\,dy\,dz}{|x-y|^{2-(1/\mu-\delta)(q-1)-1/q-\varepsilon}}$$
$$\lesssim \|(-\Delta)^{\frac{1/\mu-\delta}{2}}\tilde{u}\|^{q-1}_{L^{1/\mu-\delta}}\,\|(-\Delta)^{\frac{1}{4q}}\tilde{u}\|_{L^{2q}}\|\varphi\|_{L^\mu}\lesssim[\tilde{u}]^q_{W^{1/q,q}(\mathbb{R})},$$

where we have applied Proposition D.5 and then estimated analogously to the case (6.16). When considering $u \in W^{1/q,q}(B_r(x_0))$, we make use of the factor $\chi_{B_{2^L\rho}}$ again and localize the estimate as described in the case above. ∎

It remains to treat the second term appearing on the right-hand side of the projection estimate (6.15).

Lemma 6.12 (Right-hand side estimates II). *Let $q \ge 2$ with $1/q - 1/\mu > 0$ small, and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a locally $\mathrm{TP}^{q+2,q}$-critical embedding in the interval $B_r(x_0) \subset \mathbb{R}/\mathbb{Z}$, in the sense of Definition 1.5. Denote the unit tangent field of γ by $u : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ such that $\int_{\mathbb{R}/\mathbb{Z}} u = 0$ and let $\tilde{u}$ be a $W^{1/q,q}$-extension of $u|_{B_r(x_0)}$ from $B_r(x_0)$ to $\mathbb{R}$ as discussed in Remark 6.2. Moreover, for $y_0 \in B_{r/2}(x_0)$, choose $\rho > 0$ such that $B_{2^{2L}\rho} := B_{2^{2L}\rho}(y_0) \subset B_r(x_0)$ for large $L \in \mathbb{N}$. Then*

$$\|\chi_{B_{2^K\rho}} u \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}}$$
$$\lesssim [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + [u]^{2q-3}_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(K+l)}[\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{2L+l}\rho})}$$
$$+ 2^{-\sigma K}[u]^{q-1}_{W^{1/q,q}(B_{2^{2L}\rho})} + \rho\big(\mathcal{E}^q(u) + r^{-3} + [u]^q_{W^{1/q,q}(B_r(x_0))}\big)$$

for any large enough $K \in \mathbb{N}$ and $L \in \mathbb{N}$ with $L \gg K$. The constant, besides depending on q, may also depend on global properties of u such as $\|u\|_{L^\infty}$, $\|\tilde{u}\|_{L^\infty}$, $[u]_{W^{1/q,q}(B_r(x_0))}$, and $[\tilde{u}]_{W^{1/q,q}(\mathbb{R})}$.

Proof. We prove the statement along the lines of [14, Lemma 3.8]. First, by duality we have

$$\|\chi_{B_{2^K\rho}} u \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \lesssim \int_{\mathbb{R}} u(z) \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u(z) \cdot \psi(z)\, dz$$

for some map $\psi \in C_c^\infty(B_{2^K\rho}, \mathbb{R}^3)$ with $\|\psi\|_{L^\mu} \le 1$. Hence, it is sufficient to show that for $\psi \in C_c^\infty(B_{2^K\rho})$ scalar,

$$\begin{aligned} &\left|\int_{\mathbb{R}} u(z) \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u(z) \cdot \psi(z)\, dz\right| \\ &\quad \lesssim [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + [u]^{2q-3}_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{l=1}^\infty 2^{-\sigma(K+l)} [\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{2L+l}\rho})} \\ &\qquad + 2^{-\sigma K}[u]^{q-1}_{W^{1/q,q}(B_{2^{2L}\rho})} + \rho\big(\mathcal{E}^q(u) + r^{-3} + [u]^q_{W^{1/q,q}(B_r(x_0))}\big). \end{aligned}$$

Note that u on the left-hand side is only considered on $B_r(x_0)$ due to supp $\psi \subset B_{2^K\rho}$, the local definition of Γ for $B_{2^L\rho}$ in (6.12), and $B_{2^K\rho} \subset B_{2^L\rho} \subset B_r(x_0)$ ($L \gg K$). Therefore, as u coincides with $\tilde{u}$ on $B_r(x_0)$ by construction (Remark 6.2), we can continue with $\tilde{u}$ from now on as it is globally $W^{1/q,q}$-regular in contrast to u, which is only in $W^{1/q,q}(B_r(x_0), \mathbb{R}^3)$. However, if we later obtain $W^{1/q,q}$-seminorms of $\tilde{u}$ restricted to subsets of $B_r(x_0)$, we may switch back to the original function u.

With the help of usual cutoff functions, i.e. $\eta_{B_R} \in C_c^\infty(B_{2R})$ with $\eta_{B_R} \equiv 1$ on B_R and $\|\nabla^k \eta_{B_R}\|_{L^\infty} \lesssim R^{-k}$ for $R > 0$, we decompose the integral on the left-hand side as

$$\int_{\mathbb{R}} \tilde{u}(z) \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} \tilde{u}(z) \cdot \psi(z)\, dz = \mathrm{I} + \sum_{l=1}^\infty \mathrm{II}_l,$$

where

$$\begin{aligned} \mathrm{I} &:= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2\mu}} (\eta_{B_{2^{2K}\rho}} \mathcal{I}_{1/\mu}\psi)(z) \tilde{u}(z) \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} \tilde{u}(z)\, dz, \\ \mathrm{II}_l &:= \int_{\mathbb{R}} (-\Delta)^{1/q-1/\mu} \big((-\Delta)^{\frac{1}{2\mu}} (\eta_{B_{2^{2K+l+1}\rho} \setminus B_{2^{2K+l}\rho}} \mathcal{I}_{1/\mu}\psi)\tilde{u}\big)(z) \\ &\qquad\qquad \wedge \mathcal{I}_{2/q-2/\mu} \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} \tilde{u}(z)\, dz. \end{aligned}$$

We first focus on the terms II_l. For that, we define

$$\varphi(z) := (-\Delta)^{1/q-1/\mu} \big((-\Delta)^{\frac{1}{2\mu}} (\eta_{B_{2^{2K+l+1}\rho} \setminus B_{2^{2K+l}\rho}} \mathcal{I}_{1/\mu}\psi)\tilde{u}\big)(z).$$

By (6.13) and (6.14), and the boundedness of $k(x,y)^{-\frac{q+2}{2}}$ (Remark 6.7), we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \varphi(z) \wedge \mathcal{I}_{2/q-2/\mu} \Gamma_{1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)\, dz\right| \\
&= \left|\int_{\mathbb{R}} \varphi(z) \wedge \Gamma_{2/q-1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)\, dz\right| \\
&= \Bigg| \iint_{B^2_{2L_\rho}} \left| \fint_{x \rhd y} [\tilde{u}(z_0) - \tilde{u}(x)]\, dz_0 \right|^{q-2} k(x,y)^{-\frac{q+2}{2}} \\
&\qquad \cdot \Bigg(\fint_{x \rhd y} \fint_{x \rhd y} ((\tilde{u} \wedge)_{ij}(z_1) - (\tilde{u} \wedge)_{ij}(x)) \\
&\qquad \cdot (\mathcal{I}_{2/q-1/\mu}\varphi(z_2) - \mathcal{I}_{2/q-1/\mu}\varphi(x))\, dz_1\, dz_2 \Bigg)_{i=1}^3 \frac{dx\, dy}{\rho(x,y)^2} \Bigg| \\
&\lesssim \iint_{B^2_{2L_\rho}} \left(\fint_{x \rhd y} |\tilde{u}(z_0) - \tilde{u}(x)|\, dz_0 \right)^{q-2} \fint_{x \rhd y} |\tilde{u}(z_1) - \tilde{u}(x)|\, dz_1 \\
&\qquad \cdot \fint_{x \rhd y} |\mathcal{I}_{2/q-1/\mu}\varphi(z_2) - \mathcal{I}_{2/q-1/\mu}\varphi(x)|\, dz_2 \frac{dx\, dy}{\rho(x,y)^2} \\
&\lesssim [\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2L_\rho})} [\mathcal{I}_{2/q-1/\mu}\varphi]_{W^{1/q,q}(B_{2L_\rho})} \lesssim [u]^{q-1}_{W^{1/q,q}(B_{2L_\rho})} \|\varphi\|_{L^{\frac{1}{2/q-1/\mu}}}, \qquad (6.21)
\end{aligned}$$

where we have applied Lemma A.2 and Sobolev's inequality of Theorem A.5 at the end. It remains to estimate $\|\varphi\|_{L^{\frac{1}{2/q-1/\mu}}}$. For this purpose we introduce for any $\alpha > 0$ the three-term commutator

$$H_\alpha(f,g) = (-\Delta)^{\alpha/2}(fg) - f(-\Delta)^{\alpha/2} g - g(-\Delta)^{\alpha/2} f. \qquad (6.22)$$

Taking advantage of the estimate in Appendix B.1, the uniform boundedness of $\tilde{u}$ (Remark 6.2), Hölder's inequality, Sobolev inequalities of Theorems A.4 and A.5 for $1/q - 1/\mu > 0$ very small, and Proposition B.1, we estimate

$$\begin{aligned}
&\|\varphi\|_{L^{\frac{1}{2/q-1/\mu}}} \\
&\quad \lesssim \|\tilde{u}\|_{L^\infty} \|(-\Delta)^{\frac{1}{2}(\frac{1}{\mu}+2(\frac{1}{q}-\frac{1}{\mu}))}(\eta_{B_{2^{2K+l+1}\rho} \setminus B_{2^{2K+l}\rho}} \mathcal{I}_{1/\mu}\psi)\|_{L^{\frac{1}{2/q-1/\mu}}} \\
&\qquad + \|(-\Delta)^{1/q-1/\mu}\tilde{u}\|_{L^{\frac{1}{2/q-2/\mu}}} \|(-\Delta)^{\frac{1}{2\mu}}(\eta_{B_{2^{2K+l+1}\rho} \setminus B_{2^{2K+l}\rho}} \mathcal{I}_{1/\mu}\psi)\|_{L^\mu} \\
&\qquad + \left\| H_{2/q-2/\mu}\big(\tilde{u}, (-\Delta)^{\frac{1}{2\mu}}(\eta_{B_{2^{2K+l+1}\rho} \setminus B_{2^{2K+l}\rho}} \mathcal{I}_{1/\mu}\psi)\big)\right\|_{L^{\frac{1}{2/q-1/\mu}}} \\
&\quad \lesssim (\|\tilde{u}\|_{L^\infty} + \|(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\|_{L^\mu})\, 2^{-(K+l+1)\frac{\mu-1}{\mu}} \|\psi\|_{L^\mu} \\
&\qquad + \|(-\Delta)^{1/q-1/\mu}\tilde{u}\|_{L^{\frac{1}{2/q-2/\mu}}} \\
&\qquad \cdot \|(-\Delta)^{\frac{1}{2}(\frac{1}{\mu}+2(\frac{1}{q}-\frac{1}{\mu}))}(\eta_{B_{2^{2K+l+1}\rho} \setminus B_{2^{2K+l}\rho}} \mathcal{I}_{1/\mu}\psi)\|_{L^{\frac{1}{2/q-1/\mu}}} \\
&\quad \lesssim 2^{-(K+l)\sigma}(\|\tilde{u}\|_{L^\infty} + [\tilde{u}]_{W^{1/q,q}(\mathbb{R})})
\end{aligned}$$

for some small $\sigma = \sigma(q) > 0$.

Now we switch to the term I. We again start by introducing

$$\varphi(z) := \eta_{B_{22K_\rho}} \mathcal{I}_{1/\mu} \psi(z),$$

and observe that, by Proposition B.1 and our assumption $\|\psi\|_{L^\mu} \le 1$,

$$\|(-\Delta)^{\frac{1}{2\mu}} \varphi\|_{L^\mu} \lesssim 1. \tag{6.23}$$

Next we split I into three terms with respect to the three-term commutator (6.22) as

$$\begin{aligned}
\mathrm{I}_1 &:= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2\mu}} (\varphi \tilde{u})(z) \wedge \Gamma_{1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)\, dz, \\
\mathrm{I}_2 &:= -\int_{\mathbb{R}} \varphi(z) (-\Delta)^{\frac{1}{2\mu}} \tilde{u}(z) \wedge \Gamma_{1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)\, dz, \\
\mathrm{I}_3 &:= -\int_{\mathbb{R}} H_{1/\mu}(\varphi, \tilde{u})(z) \wedge \Gamma_{1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)\, dz.
\end{aligned}$$

For the term I_1, by (6.13) we have

$$\begin{aligned}
|\mathrm{I}_1| &= q \big| \big(Q^{(q)}_{B_{2L_\rho} \times B_{2L_\rho}} (u, (\varphi \tilde{u} \wedge)^T_{ij}) \big)_{i=1}^3 \big| \\
&= q \Bigg| \iint_{B^2_{2L_\rho}} \bigg| \fint_{x \rhd y} [\tilde{u}(z_0) - \tilde{u}(x)]\, dz_0 \bigg|^{q-2} k(x,y)^{-\frac{q+2}{2}} \\
&\quad \cdot \bigg(\fint_{x \rhd y} \fint_{x \rhd y} (\tilde{u}(z_1) - \tilde{u}(x)) \cdot \big((\varphi \tilde{u} \wedge)^T_{ij}(z_2) - (\varphi \tilde{u} \wedge)^T_{ij}(x) \big)\, dz_1\, dz_2 \bigg)_{i=1}^3 \frac{dy\, dx}{\rho(x,y)^2} \Bigg|.
\end{aligned}$$

Note that $\tilde{u}$ is considered only on $B_{2L_\rho} \subset B_r(x_0)$ here. Hence, we can change $\tilde{u}$ back to u, as they coincide on $B_r(x_0)$ by construction (cf. Remark 6.2). Then we simplify the expression by the skew-symmetry of $\varphi u \wedge$ and split I_1 by the triangle inequality into

$$\begin{aligned}
&\sum_{m=1}^4 \Bigg| \iint_{D_m} \bigg| \fint_{x \rhd y} [u(z_0) - u(x)]\, dz_0 \bigg|^{q-2} k(x,y)^{-\frac{q+2}{2}} \\
&\quad \cdot \bigg(\fint_{x \rhd y} \fint_{x \rhd y} (u(z_1) - u(x)) \cdot \big((\varphi u \wedge)_{ij}(z_2) - (\varphi u \wedge)_{ij}(x) \big)\, dz_1\, dz_2 \bigg)_{j=1}^3 \frac{dy\, dx}{\rho(x,y)^2} \Bigg| \\
&\lesssim \sum_{m=1}^4 \sum_{j=1}^3 |Q^{(q)}_{D_m}(u, (\varphi u \wedge)_{ij})|,
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}, \\
D_2 &= (\mathbb{R}/\mathbb{Z} \setminus B_{2L_\rho}) \times B_{2L_\rho}, \\
D_3 &= B_{2L_\rho} \times (\mathbb{R}/\mathbb{Z} \setminus B_{2L_\rho}), \\
D_4 &= (\mathbb{R}/\mathbb{Z} \setminus B_{2L_\rho}) \times (\mathbb{R}/\mathbb{Z} \setminus B_{2L_\rho}).
\end{aligned}$$

For the term with integration domain D_1, we employ the assumption that γ is a locally $\mathrm{TP}^{q+2,q}$-critical embedding in $B_r(x_0)$. Hence Section 6.1 implies that u is an $\mathcal{E}^q_\eta$-critical map in $B_r(x_0)$ for some $\eta \in C_c^\infty(B_{r/2}(x_0), [0,\infty))$. As u fulfills the Euler–Lagrange equations (Lemma 6.6) we observe that for any $(\varphi u\wedge)_{ij} \in T_u\mathbb{S}^2$, $j = 1, 2, 3$,

$$|Q^{(q)}(u, (\varphi u\wedge)_{ij})| \lesssim \sum_{k=1}^{3} |R^{k,(q)}(u, (\varphi u\wedge)_{ij})| + \sum_{k=4}^{7} |R^{k,(q)}_\eta(u, (\varphi u\wedge)_{ij})|,$$

where we set $R^{k,(q)} = R^{k,(q+2,q)}$ for $q \geq 2$. Therefore, Proposition 6.9 leads to

$$\begin{aligned}|Q^{(q)}(u, (\varphi u\wedge)_{ij})| &\lesssim [u]^{2q-3}_{W^{1/q,q}(B_{2^3K_\rho})} + [u]^{q+1}_{W^{1/q,q}(B_{2^3K_\rho})}\\ &\quad + \sum_{l=1}^{\infty} 2^{-\sigma(3K+l)}[\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{3K+l}\rho})} + \rho\big(\mathcal{E}^q(u) + r^{-3} + [u]^q_{W^{1/q,q}(B_r(x_0))}\big)\end{aligned}$$

for $0 < \sigma := 2/q \leq 1$. We estimate the terms with domains D_m for $m = 2, 3, 4$ up to positive constants by methods similar to (6.5) by

$$\|u\|_{L^\infty} \iint_{D_m} \left(⨍_{x\triangleright y} |u(z_0) - u(x)|\, dz_0\right)^{q-1} \left(⨍_{x\triangleright y} |\varphi(z_1) - \varphi(x)|\, dz_1\right) \frac{dx\, dy}{\rho(x,y)^2}.$$

Since u is not known to be in $W^{1/q,q}$ outside of $B_r(x_0)$, we need to distinguish, for the domain D_2, the cases $(\mathbb{R}/\mathbb{Z} \setminus B_r(x_0)) \times B_{2^L\rho}$ and $(B_r(x_0) \setminus B_{2^L\rho}) \times B_{2^L\rho}$. The first case is handled along the lines of (6.7) since $\rho(x,y) \geq r/4$, and to the second case we can apply due to $\operatorname{supp}\varphi \subset B_{2^{2K}\rho}$ an adapted version of Lemma C.1, which gives in total with $\tilde{u} \in W^{1/q,q}(\mathbb{R})$ the upper bound (up to a positive constant)

$$\rho r^{-3} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)}[\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{L+2K+l}\rho})}.$$

We treat the term with domain D_3 similarly by symmetry. For D_4 we can deduce the same bound in the following manner. First we also distinguish several subdomains according to the only locally known fractional Sobolev regularity of u in $B_r(x_0)$. In particular, we have to study the cases

$$\begin{aligned}(\mathbb{R}/\mathbb{Z} \setminus B_{2^L\rho}) \times (\mathbb{R}/\mathbb{Z} \setminus B_{2^L\rho}) &= (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0)) \times (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0))\\ &\quad \cup (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0)) \times (B_r(x_0) \setminus B_{2^L\rho})\\ &\quad \cup (B_r(x_0) \setminus B_{2^L\rho}) \times (\mathbb{R}/\mathbb{Z} \setminus B_r(x_0))\\ &\quad \cup (B_r(x_0) \setminus B_{2^L\rho}) \times (B_r(x_0) \setminus B_{2^L\rho}).\end{aligned}$$

In the first three cases the double integral either equals 0 due to $\operatorname{supp}\varphi \subset B_{2^{2K}\rho}$ or can be estimated with the help of $\rho(x,y) \geq r$ similarly to (6.7). For the last case we use an adapted version of Lemma C.1.

For the term I_2, we first observe that by the definition of $\Gamma_{\beta,B\times B}$ in (6.12),

$$\mathrm{I}_2 \approx -\int_{B_{2L\rho}}\int_{B_{2L\rho}} \left| \fint_{x\rhd y} [\tilde{u}(z_0)-\tilde{u}(x)]\,dz_0 \right|^{q-2} k(x,y)^{-\frac{q+2}{2}} \\ \cdot \fint_{x\rhd y} (\tilde{u}(z_1)-\tilde{u}(x))^T\,dz_1 \\ \cdot \fint_{x\rhd y} \big(\mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\wedge)(z_2)-\mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\wedge)(x)\big)\,dz_2\,\frac{dy\,dx}{\rho(x,y)^2}.$$

Now since $\tilde{u}\wedge$ is orthogonal to $\tilde{u}$, we rewrite

$$\fint_{x\rhd y} (\tilde{u}(z_1)-\tilde{u}(x))^T\,dz_1 \fint_{x\rhd y} \big(\mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\wedge)(z_2)-\mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\wedge)(x)\big)\,dz_2 \\ = \fint_{x\rhd y} (\tilde{u}(z_1)-\tilde{u}(x))^T\,dz_1 \fint_{x\rhd y} \tilde{\Phi}(z_2,x)\,dz_2,$$

where

$$\tilde{\Phi}(z_2,x) := \mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\wedge)(z_2)-\mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u}\wedge)(x) \\ -\tfrac{1}{2}(\tilde{u}\wedge(z_2)-\tilde{u}\wedge(x))(\varphi(x)+\varphi(y)).$$

Therefore, by Remark 6.7 we have

$$|\mathrm{I}_2| \lesssim \int_{B_{2L\rho}}\int_{B_{2L\rho}} \left(\fint_{x\rhd y} |\tilde{u}(z_0)-\tilde{u}(x)|\,dz_0\right)^{q-1} \fint_{x\rhd y} |\Phi(z_2,x)|\,dz_2\,\frac{dy\,dx}{\rho(x,y)^2}, \tag{6.24}$$

where

$$\Phi(z_2,x) := \mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u})(z_2)-\mathcal{I}_{1/\mu}(\varphi(-\Delta)^{\frac{1}{2\mu}}\tilde{u})(x) \\ -\tfrac{1}{2}(\tilde{u}(z_2)-\tilde{u}(x))(\varphi(x)+\varphi(y)).$$

Now we are in a position to apply an adapted version of [68, Lemma 6.6]. For that, we define

$$U := (-\Delta)^{\frac{1}{2\mu}}\tilde{u},$$

and find

$$\Phi(z_2,x) = \mathcal{I}_{1/\mu}(\varphi U)(z_2)-\mathcal{I}_{1/\mu}(\varphi U)(x)-\tfrac{1}{2}(\tilde{u}(z_2)-\tilde{u}(x))(\varphi(x)+\varphi(y)) \\ \lesssim \int_{\mathbb{R}} (|z_2-z|^{1/\mu-1}-|x-z|^{1/\mu-1})U(z)\varphi(z)\,dz \\ -\frac{1}{2}\int_{\mathbb{R}} (|z_2-z|^{1/\mu-1}-|x-z|^{1/\mu-1})U(z)(\varphi(x)+\varphi(y))\,dz \\ \lesssim -\frac{1}{2}\int_{\mathbb{R}} (|z_2-z|^{1/\mu-1}-|x-z|^{1/\mu-1})U(z)(\varphi(x)+\varphi(y)-2\varphi(z))\,dz.$$

Together with Lemma D.4 we obtain, for $\varepsilon < 1/\mu - \frac{1}{2q} < 1$ small enough,

$$\begin{aligned}
&\left| \fint_{x \rhd y} \Phi(z_2, x)\, dz_2 \right| \\
&\quad \lesssim \int_{\mathbb{R}} \left(\fint_{x \rhd y} \left| |z_2 - z|^{1/\mu - 1} - |x - z|^{1/\mu - 1} \right| dz_2 \right) |U(z)|\, |\varphi(x) + \varphi(y) - 2\varphi(z)|\, dz \\
&\quad \lesssim \int_{\mathbb{R}} |x - y|^{1/q + \varepsilon} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}} \varphi(x) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}} \varphi(y) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}} \varphi(z) \big) \\
&\qquad\qquad \cdot k_{1/\mu - \frac{1}{2q} - \varepsilon, 1/\mu}(x, y, z) |U(z)|\, dz.
\end{aligned}$$

Furthermore, by Proposition D.3 for small $\delta > 0$ we have

$$\left(\fint_{x \rhd y} |\tilde{u}(z_0) - \tilde{u}(x)|\, dz_0 \right)^{q-1} \lesssim |x - y|^{(1/\mu - \delta)(q-1)} \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu - \delta}{2}} \tilde{u}(x) \big)^{q-1}.$$

We conclude that

$$\begin{aligned}
|\mathrm{I}_2| \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} & |x - y|^{1/q + \varepsilon + (1/\mu - \delta)(q-1) - 2} \chi_{B_{2L_\rho}}(x) \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1/\mu - \delta}{2}} \tilde{u}(x) \big)^{q-1} \\
& \cdot \big(\mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}} \varphi(x) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}} \varphi(y) + \mathcal{M}\mathcal{M}(-\Delta)^{\frac{1}{4q}} \varphi(z) \big) \\
& \cdot k_{1/\mu - \frac{1}{2q} - \varepsilon, 1/\mu}(x, y, z)\, |U(z)|\, dy\, dz\, dx.
\end{aligned}$$

Arguing along the lines of the proof of Lemma 6.11, we therefore get, under assumption (6.23),

$$\begin{aligned}
|\mathrm{I}_2| &\lesssim \|(-\Delta)^{\frac{1/\mu - \delta}{2}} \tilde{u}\|^{q-1}_{L^{1/\mu - \delta}} \, \|(-\Delta)^{\frac{1}{4q}} \varphi\|_{L^{2q}} \, \|U\|_{L^\mu} \\
&\lesssim [\tilde{u}]^{q-1}_{W^{1/q,q}(\mathbb{R})} \|(-\Delta)^{\frac{1}{2\mu}} \varphi\|_{L^\mu} \, [\tilde{u}]_{W^{1/q,q}(\mathbb{R})} \lesssim [\tilde{u}]^q_{W^{1/q,q}(\mathbb{R})},
\end{aligned}$$

where we have applied Sobolev's inequality in the last step. To take $u \in W^{1/q,q}(B_r(x_0))$ into consideration, we localize this estimate by introducing the factor $\chi_{B_{2L_\rho}}$ in (6.24) and using Proposition B.3, the localized maximal inequality of Proposition B.4, and the localized Sobolev inequality of Lemma B.5.

Using integration by parts, equality (6.14), and Hölder's inequality, the last term I_3 can be bounded by

$$\begin{aligned}
\mathrm{I}_3 &= -\int_{\mathbb{R}} H_{1/\mu}(\varphi, \tilde{u})(z) \wedge \Gamma_{1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)\, dz \\
&\lesssim \int_{\mathbb{R}} |(-\Delta)^{1/q - 1/\mu} H_{1/\mu}(\varphi, \tilde{u})(z)|\, |\mathcal{I}_{2/q - 2/\mu} \Gamma_{1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}(z)|\, dz \\
&\lesssim \|(-\Delta)^{1/q - 1/\mu} H_{1/\mu}(\varphi, \tilde{u})\|_{L^{(2/q - 1/\mu)^{-1}}} \, \|\Gamma_{2/q - 1/\mu, B_{2L_\rho} \times B_{2L_\rho}} \tilde{u}\|_{L^{(1 - 2/q + 1/\mu)^{-1}}}.
\end{aligned}$$

Then applying the three-term commutator estimate of Appendix B.1, Proposition A.8, Sobolev's inequality of Theorem A.5 and assumption (6.23), we obtain

$$\begin{aligned}
\mathrm{I}_3 &\lesssim \|(-\Delta)^{\frac{1}{2\mu}} \varphi\|_{L^\mu} \|(-\Delta)^{\frac{1}{2\mu}} \tilde{u}\|_{L^\mu} [\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2L_\rho})} \\
&\lesssim [\tilde{u}]_{W^{1/q,q}(\mathbb{R})} [\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2L_\rho})}.
\end{aligned}$$

Also this estimate can get localized, in particular by the localized Sobolev inequality of Lemma B.5 and the localized version of the three-term-commutator estimate of Appendix B.1.

In total, using again the fact that u and $\tilde{u}$ coincide on $B_r(x_0)$ by construction (see Remark 6.2), we obtain an estimate of the form

$$\begin{aligned}
&\|\chi_{B_{2^K\rho}} u \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \\
&\quad \lesssim 2^{-\sigma K}[u]^{q-1}_{W^{1/q,q}(B_{2^{2L}\rho})} + [u]^{2q-3}_{W^{1/q,q}(B_{2^{3K}\rho})} + [u]^{q+1}_{W^{1/q,q}(B_{2^{3K}\rho})} \\
&\qquad + \sum_{l=1}^{\infty} 2^{-\sigma(3K+l)}[\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{3K+l}\rho})} + \rho\big(\mathcal{E}^q(u) + r^{-3} + [u]^q_{W^{1/q,q}(B_r(x_0))}\big) \\
&\qquad + \rho r^{-3} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)}[\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{L+2K+l}\rho})} \\
&\qquad + [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)}[\tilde{u}]^q_{W^{1/q,q}(B_{2^{2L+l}\rho})} \\
&\qquad + [u]^{q-1}_{W^{1/q,q}(B_{2^L\rho})}\Big([u]_{W^{1/q,q}(B_{2^{3K}\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(L+k)}[\tilde{u}]_{W^{1/q,q}(B_{2^{3K+l}\rho})}\Big),
\end{aligned}$$

which can be simplified by choosing $L \geq 3K$ and factoring out the constant $[u]_{W^{1/q,q}(B_r(x_0))}$, wherever it makes sense, to

$$\begin{aligned}
&\|\chi_{B_{2^K\rho}} u \wedge \Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \\
&\quad \lesssim [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + [u]^{2q-3}_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(K+l)}[\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{2L+l}\rho})} \\
&\qquad + 2^{-\sigma K}[u]^{q-1}_{W^{1/q,q}(B_{2^{2L}\rho})} + \rho\big(\mathcal{E}^q(u) + r^{-3} + [u]^q_{W^{1/q,q}(B_r(x_0))}\big). \qquad \blacksquare
\end{aligned}$$

It only remains to prove the decay estimate based on the elaborated left-hand side and right-hand side estimates.

Proof of Proposition 6.1. This proof is in the spirit of [14, proof of Proposition 3.9].

First let K_0 be a large number, which will be specified later, and for $K \geq K_0$ set $L := 10K$ and $N := 20K$. Moreover, let $\varepsilon, \delta > 0$ be small numbers, to be chosen later, and assume

$$[\tilde{u}]_{W^{1/q,q}(B_{2^N\rho})} < \varepsilon. \tag{6.25}$$

We then combine the left-hand side and right-hand side estimates to obtain a recursive estimate. First recall from the left-hand side estimate of Proposition 6.10 that there exists a large constant $C_\delta > 0$ such that

$$\begin{aligned}
[u]^q_{W^{1/q,q}(B_\rho)} \lesssim\ & [u]_{W^{1/q,q}(B_{2^L\rho})}\|\chi_{B_{2^K\rho}}\Gamma_{1/\mu, B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} + \delta[u]^q_{W^{1/q,q}(B_{2^L\rho})} \\
& + C_\delta\big([u]^q_{W^{1/q,q}(B_{2^L\rho})} - [u]^q_{W^{1/q,q}(B_\rho)}\big)
\end{aligned}$$

for K large and $1/q - 1/\mu > 0$ small enough. In the next step we split the operator $\Gamma_{1/\mu, B_{2^K\rho}\times B_{2^L\rho}}$ by (6.15) into

$$\|\chi_{B_{2^K\rho}} \Gamma_{1/\mu,B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \\ \precsim \|\chi_{B_{2^K\rho}} u \cdot \Gamma_{1/\mu,B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} + \|\chi_{B_{2^K\rho}} u \wedge \Gamma_{1/\mu,B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}}.$$

The first term on the right-hand side may be estimated by Lemma 6.11:

$$\|\chi_{B_{2^K\rho}} u \cdot \Gamma_{1/\mu,B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \\ \precsim [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [\tilde{u}]^q_{W^{1/q,q}(B_{2^{2L+l}\rho})},$$

whereas the second term on the right-hand side is by Lemma 6.12 bounded by

$$\|\chi_{B_{2^K\rho}} u \wedge \Gamma_{1/\mu,B_{2^L\rho}\times B_{2^L\rho}} u\|_{L^{\frac{\mu}{\mu-1}}} \\ \precsim [u]^q_{W^{1/q,q}(B_{2^{2L}\rho})} + [u]^{2q-3}_{W^{1/q,q}(B_{2^{2L}\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(K+l)} [\tilde{u}]^{q-1}_{W^{1/q,q}(B_{2^{2L+l}\rho})} \\ + 2^{-\sigma K} [u]^{q-1}_{W^{1/q,q}(B_{2^{2L}\rho})} + \rho\big(\mathcal{E}^q(u) + r^{-3} + [u]^q_{W^{1/q,q}(B_r(x_0))}\big).$$

By setting $\theta := \frac{\sigma}{20} > 0$ we conclude that there exists a large constant $C = C(q, r, \mathcal{E}^q(u),$ $\|u\|_{L^\infty}, \|\tilde{u}\|_{L^\infty}, [u]_{W^{1/q,q}(B_r(x_0))}, [\tilde{u}]_{W^{1/q,q}(\mathbb{R})}) > 0$ such that

$$[u]^q_{W^{1/q,q}(B_\rho)} \leq C [u]^q_{W^{1/q,q}(B_{2^N\rho})} \big([u]_{W^{1/q,q}(B_{2^N\rho})} + \delta + 2^{-\theta N}\big) \\ + C C_\delta \big([u]^q_{W^{1/q,q}(B_{2^N\rho})} - [u]^q_{W^{1/q,q}(B_\rho)}\big) + C\rho \\ + C \sum_{l=1}^{\infty} 2^{-\theta(N+l)} [\tilde{u}]^q_{W^{1/q,q}(B_{2^{N+l}\rho})}.$$

Now we employ the hole-filling technique: We add $C C_\delta [u]^q_{W^{1/q,q}(B_{2^N\rho})}$ to both sides of the inequality and then divide by $2CC_\delta + 1$, so that under the initial bound (6.25) on $[u]^q_{W^{1/q,q}(B_{2^N\rho})}$ we obtain

$$[u]^q_{W^{1/q,q}(B_\rho)} \leq [u]^q_{W^{1/q,q}(B_{2^N\rho})} \frac{\varepsilon + \delta + 2^{-\theta N} + 2CC_\delta}{2CC_\delta + 1} + \rho \\ + \sum_{l=1}^{\infty} 2^{-\theta(N+l)} [\tilde{u}]^q_{W^{1/q,q}(B_{2^{N+l}\rho})}.$$

If we then choose δ and ε small enough, and K_0 large enough, such that

$$\varepsilon + \delta + 2^{-\theta K_0} \leq 1/2,$$

by defining

$$0 < \tau := \frac{\frac{1}{2} + 2CC_\delta}{2CC_\delta + 1} < 1$$

we get the desired estimate for any $N \geq N_0$ with $N_0 = 20K_0$:

$$[u]^q_{W^{1/q,q}(B_\rho)} \leq \tau [u]^q_{W^{1/q,q}(B_{2^N\rho})} + \sum_{l=1}^{\infty} 2^{-\theta(N+l)} [\tilde{u}]^q_{W^{1/q,q}(B_{2^{N+l}\rho})} + \rho. \qquad \blacksquare$$

Appendix A. Gagliardo–Sobolev space

Recall that the seminorm for the fractional Sobolev space $W^{s,p}(B)$ for $s \in (0,1)$, $p \in (1,\infty)$, and $B \subset \mathbb{R}$ is given by

$$[f]_{W^{s,p}(B)} = \left(\int_B \int_B \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}}\, dx\, dy \right)^{1/p}.$$

In this section we gather a few useful facts that we need throughout the paper. Most likely all of them are known to experts and we do not claim any originality here, but we could not find them in the literature.

We begin with two identifications for the fractional Sobolev space.

Lemma A.1 (Identification 1). *Let $s \in (0,1)$ and $p \in (1,\infty)$. Then for any ball $B \subset \mathbb{R}$ or $B = \mathbb{R}$, and any $f \in C_c^\infty(\mathbb{R})$,*

$$\begin{aligned}[f']^p_{W^{s,p}(B)} &:= \int_B \int_B \frac{|f'(x) - f'(y)|^p}{|x-y|^{1+sp}}\, dx\, dy \\ &\approx \int_B \int_B \frac{\left|\frac{f(y)-f(x)-f'(x)(y-x)}{|x-y|}\right|^p}{|x-y|^{1+sp}}\, dx\, dy.\end{aligned}$$

The constant depends on s and p, but not on the set B or the function f.

Proof. The $\lesssim$-estimate. We have

$$|f'(x) - f'(y)|^p \lesssim \left| f'(x) - \frac{1}{y-x} \int_x^y f'(z)\, dz \right|^p + \left| f'(y) - \frac{1}{x-y} \int_y^x f'(z)\, dz \right|^p.$$

By the fundamental theorem of calculus,

$$\begin{aligned}&\left| f'(x) - \frac{1}{y-x} \int_x^y f'(z)\, dz \right|^p + \left| f'(y) - \frac{1}{x-y} \int_y^x f'(z)\, dz \right|^p \\ &\qquad = \frac{|f(y) - f(x) - f'(x)(y-x)|^p}{|y-x|^p} + \frac{|f(x) - f(y) - f'(y)(x-y)|^p}{|y-x|^p}.\end{aligned}$$

Thus,

$$\int_B \int_B \frac{|f'(x) - f'(y)|^p}{|x-y|^{1+sp}}\, dx\, dy \lesssim \int_B \int_B \frac{\left|\frac{f(y)-f(x)-f'(x)(y-x)}{|x-y|}\right|^p}{|x-y|^{1+sp}}\, dx\, dy.$$

The $\gtrsim$-estimate. For the opposite inequality, by the fundamental theorem of calculus and Jensen's inequality,

$$\left|\frac{f(y)-f(x)-f'(x)(y-x)}{|x-y|}\right|^p = \left|\frac{(y-x)\big(\frac{1}{y-x}\int_x^y f'(z)-f'(x)\,dz\big)}{|x-y|}\right|^p \leq \frac{1}{|y-x|}\int_{(x,y)} |f'(z)-f'(x)|^p\,dz.$$

We integrate both sides over B in x and y:

$$\begin{aligned}\int_B\int_B \frac{\left|\frac{f(y)-f(x)-f'(x)(y-x)}{|x-y|}\right|^p}{|x-y|^{1+sp}}\,dy\,dx &\leq \int_B\int_B\int_{(x,y)} \frac{|f'(z)-f'(x)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx\\ &\leq \int_B\int_B\int_{x>z>y} \frac{|f'(z)-f'(x)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx\\ &\quad+\int_B\int_B\int_{y>z>x} \frac{|f'(z)-f'(x)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx.\end{aligned}$$

By the Fubini theorem, for any $x \in B$,

$$\begin{aligned}&\int_B\int_{y>z>x} \frac{|f'(z)-f'(x)|^p}{|x-y|^{2+sp}}\,dz\,dy\\ &\quad\leq \int_{B\cap\{z>x\}} |f'(z)-f'(x)|^p \int_z^\infty \frac{1}{|x-y|^{2+sp}}\,dy\,dz\\ &\quad= \int_{B\cap\{z>x\}} |f'(z)-f'(x)|^p \frac{1}{1+sp}\,\frac{1}{|x-z|^{1+sp}}\,dz\\ &\quad\leq \frac{1}{1+sp}\int_B \frac{|f'(z)-f'(x)|^p}{|x-z|^{1+sp}}\,dz. \qquad (A.1)\end{aligned}$$

Thus,

$$\int_B\int_B\int_{y>z>x} \frac{|f'(z)-f'(x)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx \lesssim [f']^p_{W^{s,q}(B)},$$

and likewise

$$\int_B\int_B\int_{x>z>y} \frac{|f'(z)-f'(x)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx \lesssim [f']^p_{W^{s,q}(B)}. \qquad ■$$

Lemma A.2 (Identification 2). *Let $s \in (0,1)$ and $p \in (1,\infty)$. For any $g \in C_c^\infty(\mathbb{R})$ and any ball $B \subset \mathbb{R}$ or $B = \mathbb{R}$ we have*

$$[g]^p_{W^{s,p}(B)} \approx \int_B\int_B \frac{⨍_{(x,y)} |g(x)-g(z)|^p\,dz}{|x-y|^{1+sp}}\,dx\,dy.$$

The constant depends on s and p but not on the set B or the function g.

Proof. The $\lesssim$-estimate. By Jensen's inequality, we have

$$|g(x)-g(y)|^p \lesssim \left|g(x)-\fint_{(x,y)} g(z)\,dz\right|^p + \left|g(y)-\fint_{(x,y)} g(z)\,dz\right|^p$$
$$\le \fint_{(x,y)} |g(x)-g(z)|^p\,dz + \fint_{(x,y)} |g(y)-g(z)|^p\,dz.$$

Thus,

$$[g]^p_{W^{s,p}(B)} \lesssim \int_B\int_B \frac{\fint_{(x,y)} |g(x)-g(z)|^p\,dz}{|x-y|^{1+sp}}\,dx\,dy.$$

The $\gtrsim$-estimate. The opposite direction is a consequence of Fubini's theorem and the fact that B is convex. Indeed,

$$\int_B\int_B \frac{\fint_{(x,y)} |g(x)-g(z)|^p\,dz}{|x-y|^{1+sp}}\,dy\,dx \le \int_B\int_B\int_{x>z>y} \frac{|g(x)-g(z)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx$$
$$+ \int_B\int_B\int_{y>z>x} \frac{|g(x)-g(z)|^p}{|x-y|^{2+sp}}\,dz\,dy\,dx.$$

Now we argue as in (A.1) to obtain the claim. ∎

Lemma A.3 (Sobolev embedding). *Let $B\subset\mathbb{R}$ be a ball. For $s,t\in(0,1)$ with $t<s$, and $p,q\in(1,\infty)$ with*

$$s-1/p \ge t-1/q,$$

we have

$$[f]_{W^{t,q}(B)} \le C(s,t,p,q)(\operatorname{diam} B)^{s-t-1/p+1/q}\,[f]_{W^{s,p}(B)}. \tag{A.2}$$

If $B=\mathbb{R}$ and $s-1/p=t-1/q$, then

$$[f]_{W^{t,q}(B)} \le C(s,t,p,q)[f]_{W^{s,p}(B)}. \tag{A.3}$$

The constant $C(s,t,p,q)$ does not depend on f or B.

Proof. We first treat (A.3), for $B=\mathbb{R}$ and $s-1/p=t-1/q$.

In $\mathbb{R}$ we can use the abstract Sobolev embedding theorem for Triebel spaces,

$$[g]_{W^{t,q}(\mathbb{R})} \lesssim [g]_{W^{s,p}(\mathbb{R})}.$$

Indeed, by [66, Proposition, p. 14], for $s\in(0,1)$,

$$[g]_{W^{t,q}(\mathbb{R})} \approx [g]_{\dot F^t_{q,q}(\mathbb{R})}.$$

By [66, Section 2.2.3, p. 31] we have

$$[g]_{F^t_{qq}(\mathbb{R})} \lesssim [g]_{F^s_{pp}(\mathbb{R})}.$$

So (A.3) is established.

Next we treat (A.2) for a ball B. Observe that for any $x_0 \in \mathbb{R}$ and $r > 0$ we have

$$[f(x_0 + r\cdot)]_{W^{t,q}(B_1(0))} = r^{t-1/q}[f]_{W^{t,q}(B_r(x_0))}$$

and

$$[f(x_0{+}r\,\cdot)]_{W^{s,p}(B_1(0))} = r^{s-1/p}[f]_{W^{t,q}(B_r(x_0))}.$$

So (A.2) follows by scaling and translation from the case $B = B_1(0)$.

Moreover, we can assume that $(f)_B := \fint_B f = 0$. Indeed, once we have shown (A.2) under the assumption $(f)_B = 0$ we can apply it to $f - (f)_B$ to get the full result.

So from now on we assume $(f)_B = 0$ and $B = B_1(0)$.

Set

$$g(x) := \begin{cases} f(x), & |x| \le 1, \\ f(x/|x|^2), & |x| > 1. \end{cases}$$

We claim that

$$[g]_{W^{t,q}(\mathbb{R})} \approx [f]_{W^{t,q}(B)}, \quad [g]_{W^{s,p}(\mathbb{R})} \approx [f]_{W^{s,p}(B)}. \tag{A.4}$$

Indeed, we have

$$[g]_{W^{t,q}(\mathbb{R})} \ge [g]_{W^{t,q}(B)} = [f]_{W^{t,q}(B)},$$

which establishes the $\gtrsim$-case for (A.4). For the $\lesssim$-case, observe that

$$[g]^q_{W^{t,q}(\mathbb{R})} = [g]^q_{W^{t,q}(B_1(0))} + [g]^q_{W^{t,q}(B_1(0)^c)} + 2\int_{B_1(0)^c}\int_{B_1(0)} \frac{|g(x)-g(y)|^q}{|x-y|^{1+sq}}\,dx\,dy.$$

First we observe that by the substitution $\tilde{x} := x/|x|^2$,

$$[g]^q_{W^{t,q}(B_1(0)^c)} = \int_{B_1(0)}\int_{B_1(0)} \frac{|f(\tilde{x})-f(\tilde{y})|^q}{\left|\tilde{x}/|\tilde{x}|^2 - \tilde{y}/|\tilde{y}|^2\right|^{1+tq}}\,|\tilde{x}|^2\,|\tilde{y}|^2\,d\tilde{x}\,d\tilde{y}.$$

For $\tilde{x}, \tilde{y} \in B_1(0)$ we have $|\tilde{x}-\tilde{y}| \le \left|\tilde{x}/|\tilde{x}|^2 - \tilde{y}/|\tilde{y}|^2\right|$ and thus

$$[g]^q_{W^{t,q}(B_1(0)^c)} \le \int_{B_1(0)}\int_{B_1(0)} \frac{|f(\tilde{x})-f(\tilde{y})|^q}{|\tilde{x}-\tilde{y}|^{1+tq}}\,d\tilde{x}\,d\tilde{y} = [f]^q_{W^{t,q}(B_1(0))}.$$

Similarly,

$$\int_{B_1(0)^c}\int_{B_1(0)} \frac{|g(x)-g(y)|^q}{|x-y|^{1+sq}}\,dx\,dy \le \int_{B_1(0)}\int_{B_1(0)} \frac{|f(x)-f(\tilde{y})|^q}{\left|x-\tilde{y}/|\tilde{y}|^2\right|^{1+tq}}|\tilde{y}|\,dx\,d\tilde{y}.$$

For $x, \tilde{y} \in B_1(0)$ we have $|x-\tilde{y}| \lesssim \left|x-\tilde{y}/|\tilde{y}|^2\right|$ and thus

$$\int_{B_1(0)^c}\int_{B_1(0)} \frac{|g(x)-g(y)|^q}{|x-y|^{1+sq}}\,dx\,dy \lesssim [f]^q_{W^{t,q}(B_1(0))}.$$

This establishes (A.4).

From (A.4) and (A.3) we obtain (A.2) for $s - 1/p = t - 1/q$.

For (A.2) in the case $s - 1/p > t - 1/q$ we define

$$h(x) := \eta(x)g(x),$$

where $\eta \in C_c^\infty(B_2(0), [0,1])$ with $\eta \equiv 1$ in $B_1(0)$ is the typical cutoff function. We apply the inhomogeneous Sobolev inequality ([66, Proposition, p. 14], [66, Section 2.2.3, p. 31]) to h:

$$[h]_{W^{t,q}(\mathbb{R})} \precsim [h]_{W^{s,p}(\mathbb{R})} + \|h\|_{L^p(\mathbb{R})} \le [h]_{W^{s,p}(\mathbb{R})} + \|g\|_{L^p(B_2(0))}.$$

Now it is not too difficult to obtain

$$[h]_{W^{s,p}(\mathbb{R})} \le \|g\|_{L^p(B_2(0))} + [g]_{W^{s,p}(\mathbb{R})}.$$

Moreover,

$$\|g\|_{L^p(B_2(0))} \precsim \|f\|_{L^p(B_1(0))} = \|f - (f)_{B_1(0)}\|_{L^p(B_1(0))} \precsim [f]_{W^{s,p}(B_1(0))},$$

where in the last step we have used the fact that $(f)_{B_1(0)} = 0$ and Jensen's inequality. This establishes (A.2). ■

Theorem A.4 (Classical Sobolev inequality, [68, Theorem 1.5]). *Let $s \ge t \ge 0$ and $p \in (1, \frac{1}{s-t})$ and define $p^*_{s,t} = \frac{p}{1-(s-t)p}$. Then for any $f \in C_c^\infty(\mathbb{R})$,*

$$\|(-\Delta)^{t/2} f\|_{L^{p^*_{s,t}}(\mathbb{R})} \precsim \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R})},$$

or in other words

$$\|\mathcal{I}_{s-t} f\|_{L^{p^*_{s,t}}(\mathbb{R})} \precsim \|f\|_{L^p(\mathbb{R})}.$$

Theorem A.5 (Sobolev inequality, [68, Theorem 1.6]). *Let $s > t \ge 0$ and $p \in (1, \frac{1}{s-t})$ and define $p^*_{s,t} = \frac{p}{1-(s-t)p}$. Then for any $f \in C_c^\infty(\mathbb{R})$,*

$$\|(-\Delta)^{t/2} f\|_{L^{p^*_{s,t}}(\mathbb{R})} \precsim \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}} \, dz \, dy \right)^{1/p}.$$

Also, $s \in (0,1)$, $\delta \in (s,1)$ and $p \le \frac{1}{s-\delta}$, then for $p^*_{s,\delta} = \frac{p}{1-(s-\delta)p}$,

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mathcal{I}_\delta f(x) - \mathcal{I}_\delta f(y)|^{p^*_{s,t}}}{|x-y|^{1+sp^*_{s,\delta}}} \, dz \, dy \right)^{1/p^*_{s,\delta}} \precsim \|f\|_{L^p(\mathbb{R})}.$$

Proposition A.6 ([68, Proposition D.2]). *Let $s \in (0,1)$, $q \in (1,\infty)$, and $\eta \in C_c^\infty(B_{2\rho})$ with $\eta \equiv 1$ on B_ρ. Then for any $L \in \mathbb{N}$, $L > 1$,*

$$\int_{B_{2^L\rho}} \int_{B_{2^L\rho}} \frac{|\eta(x) - \eta(y)|^q |u(y) - (u)_{B_{2\rho} \setminus B_\rho}|}{|x-y|^{1+sp}} \, dx \, dy \precsim [u]^q_{W^{s,p}(B_{2^L\rho})} - [u]^q_{W^{s,p}(B_\rho)}.$$

Proposition A.7 ([68, Proposition D.3]). *Let $s \in (0,1)$, $q \in (1,\infty)$, $\eta \in C_c^\infty(B_{2\rho})$ with $\eta \equiv 1$ on B_ρ, and*

$$\psi(x) := \eta(x)(u(x) - (u)_{B_{2\rho}\setminus B_\rho}).$$

Then for any $L \in \mathbb{N}$, $L > 1$,

$$[\psi]_{W^{s,p}(\mathbb{R})} \lesssim [u]_{W^{s,p}(B_{2^L\rho})}.$$

We also find use of an adapted version of [68, Proposition D.4]:

Proposition A.8. *For $\delta > 0$ small enough, we have*

$$\|\Gamma_{1/q+\delta,B_\rho} u\|_{L^{1/(1-1/q-\delta)}} \lesssim [u]^{q-1}_{W^{1/q,q}(B_\rho)}.$$

Proof. For some $\varphi \in C_c^\infty(\mathbb{R})$ with $\|\varphi\|_{L^{1/(1/q+\delta)}} \leq 1$, by Remark 6.7, Hölder's inequality, the identification for fractional Sobolev spaces (Lemma A.2), and the Sobolev inequality (Theorem A.5) we have

$$\begin{aligned}
&\|\Gamma_{1/q+\delta,B_\rho} u\|_{L^{1/(1-1/q-\delta)}} \lesssim \int_{\mathbb{R}} \Gamma_{1/q+\delta,B_\rho} u(z)\,\varphi(z)\,dz \\
&\lesssim \int_{B_\rho}\int_{B_\rho} \fint_{x\triangleright y} |u(z_1)-u(x)|^{q-1}\,dz_1 \fint_{x\triangleright y} |\mathcal{I}_{1/q-\delta}\varphi(z_2) - \mathcal{I}_{1/q-\delta}\varphi(x)|\,dz_2\,\frac{dy\,dx}{\rho(x,y)^2} \\
&\lesssim \left(\int_{B_\rho}\int_{B_\rho} \frac{\fint_{x\triangleright y} |u(z_1)-u(x)|^q\,dz_1}{\rho(x,y)^2}\,dy\,dx\right)^{\frac{q-1}{q}} \\
&\quad\cdot \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\fint_{x\triangleright y} |\mathcal{I}_{1/q-\delta}\varphi(z_2) - \mathcal{I}_{1/q-\delta}\varphi(x)|^q\,dz_2}{|x-y|^2}\,dx\,dy\right)^{1/q} \\
&\lesssim [u]^{q-1}_{W^{1/q,q}(B_\rho)} \|\varphi\|_{L^{1/(1/q+\delta)}}.
\end{aligned}$$

■

Appendix B. Localization arguments

For the convenience of the reader, we recall some results related to localization. For an overview of these statements we refer to [68], but they can also be found elsewhere in the literature.

Proposition B.1 ([68, Proposition B.2]). *Let $p > 1$, $t \in (0,1)$, and $\delta \geq 0$ small. Then for any $\varphi \in C_c^\infty(B_{2^K})$ and $L > 2$ we have*

$$\|(-\Delta)^{\delta/2}((\eta_{B_{2^{K+L}}} - \eta_{B_{2^{K+L-1}}})(-\Delta)^{t/2}\varphi)\|_{L^{\frac{np}{n+\delta p}}} \lesssim 2^{-L(n(p-1)/p+t)} \|(-\Delta)^{t/2}\varphi\|_{L^p},$$

$$\|(-\Delta)^{\frac{t+\delta}{2}}((\eta_{B_{2^{K+L}}} - \eta_{B_{2^{K+L-1}}})\mathcal{I}_t\varphi)\|_{L^{\frac{np}{n+\delta p}}} \lesssim 2^{-Ln(p-1)/p} \|\varphi\|_{L^p}.$$

Proposition B.2 ([68, Proposition B.3]). *Let $s \in (0,n)$ and $p \in (1, n/s)$. Then for some $\sigma > 0$ and any $L \in \mathbb{N}$,*

$$\|\mathcal{I}_s f\|_{L^{\frac{np}{n-sp}}(B_\rho)} \lesssim \|f\|_{L^p(B_{2^L\rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} \|f\|_{L^p(B_{2^{L+l}\rho})}.$$

Proposition B.3 ([68, Proposition B.4]). *Let* $s_1, s_2, s_3 \in [0, n)$ *and* $p_1, p_2, p_3 \in (1, \infty)$ *be such that*

$$p_i^* := \frac{n p_i}{n - s_i p_i} \in (1, \infty).$$

If moreover

$$\sum_{i=1}^{3} \left(\frac{1}{p_i} - \frac{s_i}{n} \right) = 1,$$

then we have the following pseudo-local behavior for any $L \in \mathbb{N}$ *and some* $\sigma > 0$*:*

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{I}_{s_1}(\chi_{B_\rho} f_1)\, \mathcal{I}_{s_2} f_2\, \mathcal{I}_{s_3} f_3 \\
&\qquad \lesssim \|f_1\|_{L^{p_1}(B_{2^L \rho})} \|f_2\|_{L^{p_2}(B_{2^L \rho})} \|f_3\|_{L^{p_3}(B_{2^L \rho})} \\
&\qquad\quad + \sum_{l=1}^{\infty} 2^{-(L+l)\sigma} \|f_1\|_{L^{p_1}(B_{2^{L+l} \rho})} \|f_2\|_{L^{p_2}(B_{2^{L+l} \rho})} \|f_3\|_{L^{p_3}(B_{2^{L+l} \rho})}. \qquad \text{(B.1)}
\end{aligned}$$

Proposition B.4. *Let* $p > 1$*. Then for some* $\sigma > 0$ *and any* $L \in \mathbb{N}$*,*

$$\|\mathcal{M} f\|_{L^p(B_\rho)} \lesssim \|f\|_{L^p(B_{2^L \rho})} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} \|f\|_{L^p(B_{2^{L+l} \rho})}.$$

Proof. We first split by Fatou's lemma and the Minkowski inequality:

$$\|\mathcal{M}(f)\|_{L^p(B_\rho)} \le \|\mathcal{M}(\chi_{B_{2^L \rho}} f)\|_{L^p(B_\rho)} + \sum_{l=1}^{\infty} \|\mathcal{M}(\chi_{B_{2^{L+l} \rho} \setminus B_{2^{L+l-1} \rho}} f)\|_{L^p(B_\rho)}.$$

For the first term, by the Hardy–Littlewood maximal inequality we get

$$\begin{aligned}
\|\mathcal{M}(\chi_{B_{2^L \rho}} f)\|_{L^p(B_\rho)} &\le \|\mathcal{M}(\chi_{B_{2^L \rho}} f)\|_{L^p(\mathbb{R}^n)} \\
&\le \|\chi_{B_{2^L \rho}} f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(B_{2^L \rho})}.
\end{aligned}$$

For the remaining terms, we observe that, for any $x \in B_\rho$ by the definition of the Hardy–Littlewood maximal function and Hölder's inequality,

$$\begin{aligned}
\mathcal{M}(\chi_{B_{2^{L+l} \rho} \setminus B_{2^{L+l-1} \rho}} f)(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \chi_{B_{2^{L+l} \rho} \setminus B_{2^{L+l-1} \rho}}(y) |f(y)|\, dy \\
&\lesssim (2^{L+l} \rho)^{-n} \int_{B_{2^{L+l} \rho}} \chi_{B_{2^{L+l} \rho} \setminus B_{2^{L+l-1} \rho}}(y) |f(y)|\, dy \\
&\lesssim (2^{L+l} \rho)^{-n} (2^{L+l} \rho)^{n-n/p} \|f\|_{L^p(B_{2^{L+l} \rho})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{M}(\chi_{B_{2^{L+l} \rho} \setminus B_{2^{L+l-1} \rho}} f)\|_{L^p(B_\rho)} &= \left(\int_{B_\rho} |\mathcal{M}(\chi_{B_{2^{L+l} \rho} \setminus B_{2^{L+l-1} \rho}} f)(x)|^p\, dx \right)^{1/p} \\
&\lesssim (2^{L+l} \rho)^{-n} (2^{L+l} \rho)^{n-n/p} \rho^{n/p} \|f\|_{L^p(B_{2^{L+l} \rho})} \\
&\approx 2^{-\frac{n}{p}(L+l)} \|f\|_{L^p(B_{2^{L+l} \rho})}. \qquad \blacksquare
\end{aligned}$$

We also need a localized version of the Sobolev inequality of Theorem A.5:

Lemma B.5 ([68, Lemma C.1]). *Given $0 < t < s < 1$, define $p_s = n/s$ and $p_t = n/t$. Then for any $L \in \mathbb{Z}$ and $K \in \mathbb{N}$, we have*

$$\|\chi_{B_{2^L}}(-\Delta)^{t/2} f\|_{L^{p_t}} \lesssim [f]_{W^{s,p_s}(B_{2^{L+K}})} + \sum_{k=1}^{\infty} 2^{-\sigma(K+k)}[f]_{W^{s,p_s}(B_{2^{L+K+k}})}.$$

B.1. Three-term-commutator estimates

For $\alpha > 0$ the three-term commutator is defined by

$$H_\alpha(f,g) = (-\Delta)^{\alpha/2}(fg) - f(-\Delta)^{\alpha/2} g - g(-\Delta)^{\alpha/2} f.$$

This operator measures the deviation from the Leibniz rule for $|D|^\alpha$. For fractional harmonic maps it was discovered in [23, 24] how $H_\alpha(\cdot,\cdot)$ takes the role of the div-curl term, and in particular it was shown that in $\mathbb{R}^1$, $(-\Delta)^{1/4} H_{1/2}(f,g)$ belongs to the Hardy space if $(-\Delta)^{1/4} f, (-\Delta)^{1/4} g \in L^2(\mathbb{R})$ (see also [47]). There have been multiple extensions since; the following estimate and its localized version on the three-term commutator will be helpful.

Theorem B.6 ([68, Theorem A.1]). *For any $\varepsilon \geq 0$ small and $p \in (1,\infty)$, we have*

$$\|(-\Delta)^{\varepsilon/2} H_\alpha(f,g)\|_{L^p} \lesssim \|(-\Delta)^{\alpha/2} f\|_{L^{p_1}} \|(-\Delta)^{\alpha/2} g\|_{L^{p_2}},$$

where $p_1, p_2 \in (1, n/\alpha]$ are such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha - \varepsilon}{n}.$$

If $\operatorname{supp} f \subset B_{2^K}$*, then for any $L \in \mathbb{N}$,*

$$\|(-\Delta)^{\varepsilon/2} H_\alpha(f,g)\|_{L^p} \lesssim \|(-\Delta)^{\alpha/2} f\|_{L^{p_1}} \Big(\|(-\Delta)^{\alpha/2} g\|_{L^{p_2}(B_{2^{K+L}})} + \sum_{k=1}^{\infty} 2^{-\sigma(L+k)} \|(-\Delta)^{\alpha/2} g\|_{L^{p_2}(B_{2^{K+L+k}})} \Big).$$

Appendix C. Tail estimates

Lemma C.1. *Let $\varphi \in C_c^\infty(B_r)$, $p \geq q+2$, $q \geq 2$, and $L, k \in \mathbb{N}$. Then*

$$\int_{D_1}\int_{D_2} \left(\fint_{x \rhd y} |u(z) - u(x)|^2 \, dz \right)^{q-2} \left(\fint_{x \rhd y} |u(z_1) - u(x)| \, dz_1 \right) \cdot \left(\fint_{x \rhd y} |\varphi(z_2) - \varphi(x)| \, dz_2 \right) \frac{dy\, dx}{\rho(x,y)^{p-q}} \lesssim 2^{-(L+k)\frac{p-q}{q}} [u]^{2q-3}_{W^{\frac{p-q-1}{q},q}(B_{2^{L+k}r})} [\varphi]_{W^{\frac{p-q-1}{q},q}(B_r)}$$

in each of the following cases:

$$
\begin{aligned}
&\text{Case 1:}\quad D_1 \times D_2 = B_{2^L r} \times (B_{2^{L+k} r} \setminus B_{2^{L+k-1} r}),\\
&\text{Case 2:}\quad D_1 \times D_2 = (B_{2^{L+k} r} \setminus B_{2^{L+k-1} r}) \times B_{2^L r},\\
&\text{Case 3:}\quad D_1 \times D_2 = (B_{2^{L+k} r} \setminus B_{2^{L+k-1} r}) \times (B_{2^{L+k} r} \setminus B_{2^{L+k-1} r}).
\end{aligned}
$$

Proof. Since $⨍_{x \triangleright y} |\varphi(z_2) - \varphi(x)|\, dz_2 = 0$ for either $x, y < -r$ or $x, y > r$ due to the support of φ in B_r, we only need to consider the cases

$$
\begin{aligned}
&\text{Case 1:}\quad -2^L r < x < r, \quad 2^{L+k-1} r < y < 2^{L+k} r,\\
&\text{Case 2:}\quad 2^{L+k-1} r < x < 2^{L+k} r, \quad -2^L r < y < r,\\
&\text{Case 3:}\quad -2^{L+k} r < x < -2^{L+k-1} r, \quad 2^{L+k-1} r < y < 2^{L+k} r.
\end{aligned}
$$

The cases

$$
\begin{aligned}
&-r < x < 2^L r, \quad -2^{L+k} r < y < -2^{L+k-1} r,\\
&-2^{L+k} r < x < -2^{L+k-1} r, \quad -r < y < 2^L r,\\
&2^{L+k-1} r < x < 2^{L+k} r, \quad -2^{L+k} r < y < -2^{L+k-1} r,
\end{aligned}
$$

follow analogously.

We first examine Case 1. By using Hölder's inequality, Jensen's inequality and the Sobolev embedding (Lemma A.3), with constants depending on the domain, we get

$$
\begin{aligned}
&\int_{B_{2^L r}} \int_{B_{2^{L+k} r} \setminus B_{2^{L+k-1} r}} \rho(x,y)^{-(p-q)} \Big(\rho(x,y)^{-1} \int_x^y |u(z) - u(x)|^2\, dz \Big)^{q-2}\\
&\qquad \cdot \Big(\rho(x,y)^{-1} \int_x^y |u(z_1) - u(x)|\, dz_1 \Big) \Big(\rho(x,y)^{-1} \int_x^y |\varphi(z_2) - \varphi(x)|\, dz_2 \Big) dy\, dx\\
&\lesssim (2^{L+k} r)^{-(p-q)-q} (2^{L+k} r) \int_{B_{2^{L+k} r}} \Big(\int_{B_{2^{L+k} r}} |u(z) - u(x)|^2\, dz \Big)^{q-2}\\
&\qquad \cdot \Big(\int_{B_{2^{L+k} r}} |u(z_1) - u(x)|\, dz_1 \Big) \Big(\int_{B_{2^{L+k} r}} |\varphi(z_2) - \varphi(x)|\, dz_2 \Big) dx\\
&\lesssim (2^{L+k} r)^{-(p-q)-q+1} (2^{L+k} r)^{(q-1)\frac{q-2}{q}} \Big(\iint_{(B_{2^{L+k} r})^2} |u(z) - u(x)|^{2q}\, dz\, dx \Big)^{\frac{q-2}{q}}\\
&\qquad \cdot (2^{L+k} r)^{(q-1)1/q} \Big(\iint_{(B_{2^{L+k} r})^2} |u(z_1) - u(x)|^q\, dz_1\, dx \Big)^{1/q}\\
&\qquad \cdot (2^{L+k} r)^{(q-1)1/q} \Big(\iint_{(B_{2^{L+k} r})^2} |\varphi(z_2) - \varphi(x)|^q\, dz_2\, dx \Big)^{1/q}\\
&\lesssim (2^{L+k} r)^{-(p-q)} (2^{L+k} r)^{(p-q)\frac{q-2}{q}} [u]^{2q-4}_{W^{\frac{p-q-1}{2q},2q}(B_{2^{L+k} r})} (2^{L+k} r)^{(p-q)/q}\\
&\qquad \cdot [u]_{W^{\frac{p-q-1}{q},q}(B_{2^{L+k} r})} r^{(p-q)/q} [\varphi]_{W^{\frac{p-q-1}{q},q}(B_r)}\\
&\approx 2^{-(L+k)\frac{p-q}{q}} [u]^{2q-3}_{W^{\frac{p-q-1}{q},q}(B_{2^{L+k} r})} [\varphi]_{W^{\frac{p-q-1}{q},q}(B_r)}.
\end{aligned}
$$

Case 2 differs from Case 1 by symmetry only in the integration over y, where we use

$$\int_{B_{2^L r}} dy \lesssim 2^{L+k} r.$$

In Case 3 the estimates also hold since the distance between x and y is even greater than in the previous cases. ■

Appendix D. Mean value arguments

In the following we introduce mean value arguments and compensation effects, which turn out to be crucial for elaborating the right-hand side estimates.

The first statement is a typical mean value argument (see [51, Lemma 3.3]).

Lemma D.1. *Let $\alpha \in \mathbb{R}$ and $a, b \in \mathbb{R}$ with*

$$|a-b| \lesssim \min\{|a|, |b|\}.$$

Then for any $\varepsilon \in [0,1]$,

$$\big||a|^\alpha - |b|^\alpha\big| \lesssim |a-b|^\varepsilon \min\{|a|^{\alpha-\varepsilon}, |b|^{\alpha-\varepsilon}\}.$$

In our situation we will have to deal with the expression

$$\frac{1}{|x-y|}\int_{(x,y)} \big||z-z_2|^{\alpha-1} - |z-x|^{\alpha-1}\big|\, dz_2.$$

The following lemma tells us that it behaves very similarly to

$$\big||z-y|^{\alpha-1} - |z-x|^{\alpha-1}\big|.$$

Lemma D.2. *Let $x, y, z \in \mathbb{R}$ be three distinct points inside a geodesic ball $B \subset \mathbb{R}$ and $\alpha \in (0,1)$. Set*

$$F(x,y,z) := \frac{1}{|x-y|}\int_{(x,y)} \big||z-z_2|^{\alpha-1} - |z-x|^{\alpha-1}\big|\, dz_2.$$

- *If*

$$|x-y| \lesssim \min\{|x-z|, |y-z|\} \tag{D.1}$$

then for any $\varepsilon \in [0,1]$,

$$F(x,y,z) \lesssim |x-y|^\varepsilon \min\{|x-z|^{\alpha-\varepsilon-1}, |y-z|^{\alpha-\varepsilon-1}\}. \tag{D.2}$$

- *If*

$$|x-z| \lesssim \min\{|x-y|, |y-z|\} \tag{D.3}$$

then for any $\varepsilon \in [0,1]$,

$$F(x,y,z) \lesssim |x-z|^{\alpha-1} \lesssim |x-y|^\varepsilon |x-z|^{\alpha-\varepsilon-1}. \tag{D.4}$$

- *If*

$$|y - z| \lesssim \min\{|x - y|, |x - z|\} \tag{D.5}$$

then for any $\varepsilon \in [0, 1]$,

$$F(x, y, z) \lesssim |y - z|^{\alpha-1} \lesssim |x - y|^{\varepsilon} |y - z|^{\alpha-\varepsilon-1}. \tag{D.6}$$

Proof of (D.2). Observe that (D.1) implies

$$|x - z| \approx |y - z|.$$

Case 1: $|z - x| \leq 10|x - y|$. In this case $|x - y| \approx |z - x|$, and for any $z_2 \in (x, y)$ we have $|z - z_2| \lesssim |x - y| \lesssim |z - x|$. We then simply integrate

$$\begin{aligned} F(x, y, z) &\lesssim |x - y|^{-1} \int_{|z-z_2| \lesssim |x-y|} |z - z_2|^{\alpha-1}\, dz_2 \\ &\approx |x - y|^{\alpha-1} \\ &\approx |x - y|^{\varepsilon} \min\{|x - z|^{\alpha-\varepsilon-1}, |y - z|^{\alpha-\varepsilon-1}\}. \end{aligned}$$

Case 2: $|z - x| > 10|x - y|$. In this case, for any $z_2 \in (x, y)$ we have $|z - z_2| \gtrsim |x - y|$ and $|z_2 - x| \leq |x - y| \lesssim |z_2 - z|$. In particular, for any $z_2 \in (x, y)$,

$$|z_2 - x| \lesssim \min\{|z_2 - z|, |z - x|\}.$$

By the typical mean value theorem argument (see Lemma D.1),

$$\left| |z - z_2|^{\alpha-1} - |z - x|^{\alpha-1} \right| \lesssim |z_2 - x|\, |z - x|^{\alpha-2} \lesssim |x - y|\, |z - x|^{\alpha-2}.$$

Integrating this, we obtain

$$\begin{aligned} F(x, y, z) &\lesssim |x - y|\, |z - x|^{\alpha-2} \\ &\lesssim |x - y|^{\varepsilon} |z - x|^{\alpha-\varepsilon-1} \\ &\approx |x - y|^{\varepsilon} \min\{|x - z|^{\alpha-\varepsilon-1}, |y - z|^{\alpha-\varepsilon-1}\}. \end{aligned}$$

This settles (D.2). ∎

Proof of (D.4). Observe that (D.3) implies that $|x - y| \approx |y - z|$.

Since for $\alpha > 0$ the function $z_2 \mapsto |z - z_2|^{\alpha-1}$ is integrable with antiderivative $\approx |z - z_2|^{\alpha}$, we have

$$\begin{aligned} |x - y|^{-1} \int_{(x,y)} |z - z_2|^{\alpha-1} &\lesssim |x - y|^{-1} (|z - x|^{\alpha} + |z - y|^{\alpha}) \\ &= |x - y|^{-1} |z - x|^{\alpha} + |x - y|^{-1} |z - y|^{\alpha} \\ &\lesssim |z - x|^{\alpha-1} + |x - y|^{\alpha-1} \\ &\lesssim |z - x|^{\alpha-1} + |z - x|^{\alpha-1}. \end{aligned}$$

In the last step we have used the fact that $\alpha \in (0, 1)$ (i.e. $\alpha - 1 < 0$).

Also observe that

$$|x-y|^{-1}\int_{(x,y)} |z-x|^{\alpha-1} dz_2 \lesssim |z-x|^{\alpha-1}.$$

This settles (D.4). ∎

Proof of (D.6). In this case observe that (D.5) implies $|x-y| \approx |z-x|$ so $(\alpha < 1)$

$$\frac{1}{|x-y|}\int_{(x,y)} |z-x|^{\alpha-1} \approx |x-y|^{\alpha-1} \lesssim |y-z|^{\alpha-1}.$$

For the remainder we argue as for (D.4) to obtain

$$\begin{aligned} |x-y|^{-1}\int_{(x,y)} |z-z_2|^{\alpha-1} &\lesssim |x-y|^{-1}(|z-x|^{\alpha}+|z-y|^{\alpha}) \\ &= |x-y|^{-1}|z-x|^{\alpha}+|x-y|^{-1}|z-y|^{\alpha} \\ &\approx |z-x|^{\alpha-1}+|z-y|^{\alpha-1} \lesssim |z-y|^{\alpha-1}. \end{aligned}$$

This settles (D.6). ∎

For the upcoming statement, we need the notion of the uncentered Hardy–Littlewood maximal function, which is given by

$$\mathcal{M}f(x) = \sup_{B_r(x)\ni y} \frac{1}{|B_r(y)|}\int_{B_r(y)} |f(z)|\,dz.$$

Let us recall the following proposition first.

Proposition D.3 ([69, Proposition 6.6]). *For any* $\alpha \in [0,1]$,

$$|u(x)-u(y)| \lesssim |x-y|^{\alpha}\big(\mathcal{M}(-\Delta)^{\alpha/2}u(x)+\mathcal{M}(-\Delta)^{\alpha/2}u(y)\big).$$

This implies

$$\begin{aligned} &\fint_{x\rhd y} |u(z_1)-u(x)|\,dz_1 \lesssim |x-y|^{\alpha}\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2}u(x), \\ &\fint_{x\rhd y} |u(z_1)-u(y)|\,dz_1 \lesssim |x-y|^{\alpha}\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2}u(y). \end{aligned}$$

We now develop an adapted version of [68, Proposition 6.3].

Lemma D.4. *Let*

$$\begin{aligned} G(x,y,z) &:= |u(y)+u(x)-2u(z)|\fint_{x\rhd y} \big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2, \\ H(x,y,z) &:= \fint_{x\rhd y} |u(z_1)-u(x)|\,dz_1\fint_{x\rhd y} \big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2, \end{aligned}$$

for some $1/\mu \in (0,1)$. *Then for any* $\alpha \in (0,1)$ *and* $\varepsilon \in (0, 1-\alpha)$ *such that* $\varepsilon < 1/\mu - \alpha/2$, $G(x,y,z)$ *and* $H(x,y,z)$ *are, up to a constant, bounded from above by*

$$|x-y|^{\alpha+\varepsilon}\big(\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x) + \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(y) + \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(z)\big) \cdot k_{1/\mu-\alpha/2-\varepsilon,1/\mu}(x,y,z),$$

where

$$k_{s,\gamma}(x,y,z) := \min\{|x-z|^{s-1}, |y-z|^{s-1}\} \tag{D.7}$$
$$+ \left(\frac{|y-z|}{|x-y|}\right)^{\gamma-s} |y-z|^{s-1} \chi_{\{|y-z| \lesssim \min\{|x-y|,|x-z|\}\}} \tag{D.8}$$
$$+ \left(\frac{|x-z|}{|x-y|}\right)^{\gamma-s} |x-z|^{s-1} \chi_{\{|x-z| \lesssim \min\{|x-y|,|y-z|\}\}}. \tag{D.9}$$

Proof. In the case of $|x-y| \lesssim \min\{|x-z|, |y-z|\}$, we estimate by Proposition D.3:

$$\begin{aligned}|u(y)+u(x)-2u(z)| &\le |u(x)-u(y)| + 2|u(y)-u(z)|\\ &\lesssim |x-y|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(x) + \mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)\\ &\quad + |y-z|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(y) + \mathcal{M}(-\Delta)^{\alpha/4}u(z)\big),\end{aligned}$$

and

$$\fint_{x \rhd y} |u(z_1)-u(x)|\, dz_1 \lesssim |x-y|^{\alpha/2}\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x).$$

Therefore, by (D.2) we get, for $\gamma_1 = \alpha/2+\varepsilon$ and $\gamma_2 = \alpha+\varepsilon \in [0,1]$,

$$\begin{aligned}&G(x,y,z)\\ &\lesssim |x-y|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(x) + \mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1} - |z-x|^{1/\mu-1}\big|\, dz_2\\ &\quad + |y-z|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(y) + \mathcal{M}(-\Delta)^{\alpha/4}u(z)\big)\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1} - |z-x|^{1/\mu-1}\big|\, dz_2\\ &\lesssim \big(\mathcal{M}(-\Delta)^{\alpha/4}u(x) + \mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)|x-y|^{\alpha/2+\gamma_1}|y-z|^{1/\mu-\gamma_1-1}\\ &\quad + \big(\mathcal{M}(-\Delta)^{\alpha/4}u(y) + \mathcal{M}(-\Delta)^{\alpha/4}u(z)\big)|y-z|^{\alpha/2+1/\mu-\gamma_2-1}|x-y|^{\gamma_2},\end{aligned}$$

and

$$\begin{aligned}H(x,y,z) &\lesssim |x-y|^{\alpha/2}\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1} - |z-x|^{1/\mu-1}\big|\, dz_2\\ &\lesssim \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)|x-y|^{\alpha+\varepsilon}|y-z|^{1/\mu-\alpha/2-\varepsilon-1}.\end{aligned}$$

In the case of $|y-z| \lesssim \min\{|x-z|, |x-y|\}$, we start with the same estimates as in the previous case, but apply (D.6) for $\gamma_1 = \alpha/2+\varepsilon$ and $\gamma_2 = \alpha+\varepsilon \in [0,1]$ to get

$$
\begin{aligned}
&G(x,y,z)\\
&\lesssim |x-y|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\\
&+|y-z|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(y)+\mathcal{M}(-\Delta)^{\alpha/4}u(z)\big)\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\\
&\lesssim \big(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)|x-y|^{\alpha/2}|y-z|^{1/\mu-1}\\
&\quad+\big(\mathcal{M}(-\Delta)^{\alpha/4}u(y)+\mathcal{M}(-\Delta)^{\alpha/4}u(z)\big)|y-z|^{\alpha/2}|y-z|^{1/\mu-1}\\
&\lesssim \big(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)|x-y|^{\alpha/2+\gamma_1}|y-z|^{1/\mu-\gamma_1-1}\left(\frac{|y-z|}{|x-y|}\right)^{\gamma_1}\\
&+\big(\mathcal{M}(-\Delta)^{\alpha/4}u(y)+\mathcal{M}(-\Delta)^{\alpha/4}u(z)\big)|x-y|^{\gamma_2}|y-z|^{\alpha/2+1/\mu-\gamma_2-1}\left(\frac{|y-z|}{|x-y|}\right)^{\gamma_1+(\gamma_2-\gamma_1)},
\end{aligned}
$$

and

$$
\begin{aligned}
H(x,y,z)&\lesssim |x-y|^{\alpha/2}\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\\
&\lesssim \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)|x-y|^{\alpha/2}|y-z|^{1/\mu-1}\\
&=\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)|x-y|^{\alpha+\varepsilon}|y-z|^{1/\mu-\alpha/2-\varepsilon-1}\left(\frac{|y-z|}{|x-y|}\right)^{\alpha/2+\varepsilon}.
\end{aligned}
$$

Observe that $\frac{|y-z|}{|x-y|}\le 1$.

For the last case of $|x-z|\lesssim \min\{|y-z|,|x-y|\}$, by Proposition D.3 we get

$$
\begin{aligned}
|u(y)+u(x)-2u(z)|\le{}& |u(x)-u(y)|+2|u(x)-u(z)|\\
&\lesssim |x-y|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y)\big)\\
&\quad+|x-z|^{\alpha/2}\big(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(z)\big),
\end{aligned}
$$

and hence, by (D.4) for $\gamma_1=\alpha/2+\varepsilon$ and $\gamma_2=\alpha+\varepsilon\in[0,1]$,

$$
\begin{aligned}
&G(x,y,z)\\
&\lesssim |x-y|^{\alpha/2}(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y))\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\\
&+|x-z|^{\alpha/2}(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(z))\fint_{x\rhd y}\big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\\
&\lesssim (\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y))|x-y|^{\alpha/2}|x-z|^{1/\mu-1}\\
&\quad+(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(z))|x-z|^{\alpha/2}|x-z|^{1/\mu-1}\\
&\lesssim (\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(y))|x-y|^{\alpha/2+\gamma_1}|x-z|^{1/\mu-\gamma_1-1}\left(\frac{|x-z|}{|x-y|}\right)^{\gamma_1}\\
&+(\mathcal{M}(-\Delta)^{\alpha/4}u(x)+\mathcal{M}(-\Delta)^{\alpha/4}u(z))|x-y|^{\gamma_2}|x-z|^{\alpha/2+1/\mu-\gamma_2-1}\left(\frac{|x-z|}{|x-y|}\right)^{\gamma_1+(\gamma_2-\gamma_1)},
\end{aligned}
$$

and

$$\begin{aligned} H(x,y,z) &\lesssim |x-y|^{\alpha/2}\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x) \fint_{x\triangleright y} \big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2 \\ &\lesssim \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)|x-y|^{\alpha/2}|x-z|^{1/\mu-1} \\ &= \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x)|x-y|^{\alpha+\varepsilon}|x-z|^{1/\mu-\alpha/2-\varepsilon-1}\Big(\frac{|x-z|}{|x-y|}\Big)^{\alpha/2+\varepsilon}. \end{aligned}$$

Eventually, by Lebesgue's differentiation theorem, for a.e. $x\in\mathbb{R}$,

$$\mathcal{M}(-\Delta)^{\alpha/4}u(x) \lesssim \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/4}u(x). \qquad \blacksquare$$

Furthermore, we need a result inspired by [68, Proposition 6.4].

Proposition D.5. *Let $F,G,H:\mathbb{R}^n\to\mathbb{R}_+$, $\alpha\in(0,n)$, $s,\beta\in(0,1)$, $s+\alpha<\beta$, and consider*

$$I := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (F(x)+F(y))(G(x)+G(y)+G(z))|x-y|^{\alpha-n}H(z) \cdot k_{s,\beta}(x,y,z)\,dx\,dy\,dz,$$

where $k_{s,\beta}(x,y,z)$ is of the form (D.7), (D.8), *or* (D.9). *Then*

$$\begin{aligned} I \le &\int_{\mathbb{R}^n} GH\mathcal{I}_{s+\alpha}F + \int_{\mathbb{R}^n} FG\mathcal{I}_{s+\alpha}H + \int_{\mathbb{R}^n} F\mathcal{I}_\alpha G\cdot\mathcal{I}_s H \\ &+ \int_{\mathbb{R}^n} G\mathcal{I}_\alpha F\cdot\mathcal{I}_s H. \end{aligned}$$

Lemma D.6. *Let $G,H:\mathbb{R}\to\mathbb{R}_+$, $\alpha,\beta\in(0,1)$ such that $\beta<\alpha<1/\mu$, and*

$$\begin{aligned} I := &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} G(x)\fint_{x\triangleright y}|u(z_1)-u(y)|^2\,dz_1\,H(z) \\ &\cdot \fint_{x\triangleright y}\big||z-z_2|^{1/\mu-1}-|z-x|^{1/\mu-1}\big|\,dz_2\,|x-y|^{-2+\alpha(q-2)}\,dx\,dy\,dz. \end{aligned}$$

Then

$$\begin{aligned} I \lesssim &\int_{\mathbb{R}} \mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1}G\ \mathcal{M}\big(\mathcal{M}(-\Delta)^{\alpha/2}u\,\mathcal{M}(-\Delta)^{\beta/2}u\big)\mathcal{I}_{1/\mu-\varepsilon}H \\ &+ \int_{\mathbb{R}} G\mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1}\mathcal{M}\big(\mathcal{M}(-\Delta)^{\alpha/2}u\,\mathcal{M}(-\Delta)^{\beta/2}u\big)\mathcal{I}_{1/\mu-\varepsilon}H \\ &+ \int_{\mathbb{R}} \mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1}G\ \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2}u\ \mathcal{M}\mathcal{M}(-\Delta)^{\beta/2}u\ \mathcal{I}_{1/\mu-\varepsilon}H \\ &+ \int_{\mathbb{R}} G\mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1}\big(\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2}u\ \mathcal{M}\mathcal{M}(-\Delta)^{\beta/2}u\big)\mathcal{I}_{1/\mu-\varepsilon}H \end{aligned}$$

for any admissible $\varepsilon\in(0,1)$ with $\alpha(q-1)+\beta-1<\varepsilon<1/\mu$.

Proof. We begin by treating the term $\fint_{x \triangleright y} |u(z_1) - u(y)|^2 \, dz_1$ by Proposition D.3:

$$\begin{aligned}
&\fint_{x \triangleright y} |u(z_1) - u(y)|^2 \, dz_1 \\
&\lesssim \fint_{x \triangleright y} |z_1 - y|^\alpha (\mathcal{M}(-\Delta)^{\alpha/2} u(z_1) + \mathcal{M}(-\Delta)^{\alpha/2} u(y)) |z_1 - y|^\beta \\
&\qquad\qquad \cdot \big(\mathcal{M}(-\Delta)^{\beta/2} u(z_1) + \mathcal{M}(-\Delta)^{\beta/2} u(y)\big) \, dz_1 \\
&\leq |x - y|^{\alpha+\beta} \Big(\fint_{x \triangleright y} \mathcal{M}(-\Delta)^{\alpha/2} u(z_1) \mathcal{M}(-\Delta)^{\beta/2} u(z_1) \, dz_1 \\
&\quad + \fint_{x \triangleright y} \mathcal{M}(-\Delta)^{\alpha/2} u(z_1) \, dz_1 \mathcal{M}(-\Delta)^{\beta/2} u(y) \\
&\quad + \mathcal{M}(-\Delta)^{\alpha/2} u(y) \fint_{x \triangleright y} \mathcal{M}(-\Delta)^{\beta/2} u(z_1) \, dz_1 + \mathcal{M}(-\Delta)^{\alpha/2} u(y) \mathcal{M}(-\Delta)^{\beta/2} u(y) \Big) \\
&\lesssim |x - y|^{\alpha+\beta} \\
&\quad \cdot \big(\mathcal{M}(\mathcal{M}(-\Delta)^{\alpha/2} u \mathcal{M}(-\Delta)^{\beta/2} u)(y) + \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2} u(y) \mathcal{M}\mathcal{M}(-\Delta)^{\beta/2} u(y) \big).
\end{aligned}$$

We first consider the cases

$$|x - y| \lesssim \min\{|x - z|, |y - z|\} \quad \text{and} \quad |y - z| \lesssim \min\{|x - z|, |x - y|\}$$

and observe by (D.2) and (D.6) that

$$\fint_{x \triangleright y} \big| |z - z_2|^{1/\mu - 1} - |z - x|^{1/\mu - 1} \big| \, dz_2 \lesssim |x - y|^\varepsilon |y - z|^{1/\mu - \varepsilon - 1}$$

for an admissible $\varepsilon > 0$. Therefore, we get

$$\begin{aligned}
&\iiint_{\mathbb{R}^3} G(x) \; \mathcal{M}\big(\mathcal{M}(-\Delta)^{\alpha/2} u \mathcal{M}(-\Delta)^{\beta/2} u\big)(y) \; H(z) |x - y|^{-2+\alpha(q-1)+\beta+\varepsilon} \\
&\qquad\qquad \cdot |y - z|^{1/\mu - \varepsilon - 1} \, dx \, dy \, dz \\
&+ \iiint_{\mathbb{R}^3} G(x) \; \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2} u(y) \; \mathcal{M}\mathcal{M}(-\Delta)^{\beta/2} u(y) \; H(z) |x - y|^{-2+\alpha(q-1)+\beta+\varepsilon} \\
&\qquad\qquad \cdot |y - z|^{1/\mu - \varepsilon - 1} \, dx \, dy \, dz \\
&\approx \int_{\mathbb{R}} \mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1} G(y) \mathcal{M}(\mathcal{M}(-\Delta)^{\alpha/2} u \mathcal{M}(-\Delta)^{\beta/2} u)(y) \mathcal{I}_{1/\mu - \varepsilon} H(y) \, dy \\
&\quad + \int_{\mathbb{R}} \mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1} G(y) \, \mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2} u(y) \, \mathcal{M}\mathcal{M}(-\Delta)^{\beta/2} u(y) \mathcal{I}_{1/\mu - \varepsilon} H(y) \, dy.
\end{aligned}$$

For the case

$$|x - z| \lesssim \min\{|y - z|, |x - y|\},$$

by (D.4) we have

$$\fint_{x \triangleright y} \big| |z - z_2|^{1/\mu - 1} - |z - x|^{1/\mu - 1} \big| \, dz_2 \lesssim |x - y|^\varepsilon |x - z|^{1/\mu - \varepsilon - 1}$$

for an admissible $\varepsilon > 0$, which implies

$$\iiint_{\mathbb{R}^3} G(x)\mathcal{M}\big(\mathcal{M}(-\Delta)^{\alpha/2}u\,\mathcal{M}(-\Delta)^{\beta/2}u\big)(y)H(z)|x-y|^{-2+\alpha(q-1)+\beta+\varepsilon} \cdot |x-z|^{1/\mu-\varepsilon-1}\,dx\,dy\,dz$$
$$+ \iiint_{\mathbb{R}^3} G(x)\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2}u(y)\,\mathcal{M}\mathcal{M}(-\Delta)^{\beta/2}u(y)H(z)|x-y|^{-2+\alpha(q-1)+\beta+\varepsilon} \cdot |x-z|^{1/\mu-\varepsilon-1}\,dx\,dy\,dz$$
$$\approx \int_{\mathbb{R}} G(x)\mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1}\mathcal{M}\big(\mathcal{M}(-\Delta)^{\alpha/2}u\,\mathcal{M}(-\Delta)^{\beta/2}u\big)(x)\mathcal{I}_{1/\mu-\varepsilon}H(x)\,dy$$
$$+ \int_{\mathbb{R}} G(x)\mathcal{I}_{\alpha(q-1)+\beta+\varepsilon-1}\big(\mathcal{M}\mathcal{M}(-\Delta)^{\alpha/2}u(x)\mathcal{M}\mathcal{M}(-\Delta)^{\beta/2}u(x)\big)\mathcal{I}_{1/\mu-\varepsilon}H(x)\,dy. \quad \blacksquare$$

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