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MHD equilibria with nonconstant pressure in nondegenerate toroidal domains

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Abstract. We prove the existence of piecewise smooth MHD equilibria in three-dimensional toroidal domains of \mathbb{R}^3 where the pressure is constant on the boundary but not in the interior. The pressure is piecewise constant and the plasma current exhibits an arbitrary number of current sheets. We also establish the existence of free boundary steady states surrounded by vacuum with an external surface current. The toroidal domains where these equilibria are shown to exist need not be small perturbations of an axisymmetric domain, and in fact they can have any knotted topology. The building blocks we use in our construction are analytic toroidal domains satisfying a certain nondegeneracy condition, which roughly states that there exists a force-free field that is ergodic on the surface of the domain. The proof involves three main ingredients: a gluing construction of piecewise smooth MHD equilibria, a Hamilton–Jacobi equation on the two-dimensional torus that can be understood as a nonlinear deformation of the cohomological equation (so the nondegeneracy assumption plays a major role in the corresponding analysis), and a new KAM theorem tailored for the study of divergence-free fields in three dimensions whose Poincaré map cannot be computed explicitly.

Keywords. Magnetohydrodynamics, steady states, weak solutions, toroidal domain, cohomological equation, KAM theory

1. Introduction

The magnetohydrodynamics (MHD) equilibria in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ are often described by the solutions to the system of equations

$$B \times \operatorname{curl} B + \nabla P = 0 \quad \text{in } \Omega, \quad (1.1a)$$

$$\operatorname{div} B = 0 \quad \text{in } \Omega, \quad (1.1b)$$

$$B \cdot N = 0 \quad \text{on } \partial\Omega. \quad (1.1c)$$

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Here the vector field $B(x)$ is the steady magnetic field of a perfectly conducting plasma, the scalar function $P(x)$ is the plasma pressure and $N(x)$ denotes the outer unit normal at a boundary point $x \in \partial\Omega$. It is also customary to assume that P is constant on the boundary $\partial\Omega$, so we can set

$$P = 0 \quad \text{on } \partial\Omega. \quad (1.1d)$$

Equations (1.1), which we will refer to as the *fixed boundary problem*, model the equilibrium configurations of a plasma confined in the fixed magnetic domain Ω . Because of this connection, the most interesting case is when Ω is a toroidal domain, that is, when the boundary of the domain is diffeomorphic to a torus. It is worth noting that these equations also describe stationary solutions to the 3D Euler equations in fluid mechanics, with B playing the role of the velocity field of the fluid and P being the negative of the Bernoulli function (see e.g. [1]).

In this paper we will be concerned with the problem of finding toroidal domains that admit MHD equilibria whose pressure is constant on the boundary but not in the interior. This problem can be traced back at least to the work of H. Grad in the 1960s, who conjectured [19] that no smooth solutions fibring the domain with toroidal surfaces exist unless the domain is axially symmetric. An important somewhat related result, due to Bruno and Laurence [8] in the 1990s, is the existence of weak solutions with non-constant stepped-pressure in nonsymmetric toroidal domains that are small perturbations of an axisymmetric solid torus. A very illuminating numerical implementation of this model suggesting the existence of stepped-pressure equilibria in toroidal domains far from axisymmetric was developed in [22,23]. See also [10] for the construction of smooth solutions with nonconstant pressure, but subject to a small force, on toroidal domains that are close to being axisymmetric. However, Grad's influential conjecture remains wide open. A comprehensive recent account of the subject can be found in [20,21].

It is important to remark that in the context of plasma physics, the pressure profile P (usually described as a function of the magnetic flux) and the rotational transform profile (which is essentially the ratio of the components of the frequency vector of the magnetic field on each invariant toroidal surface) are prescribed *a priori*, as inputs of the computations. The relevance of these prescriptions lies in the fact that they yield uniqueness in the solutions of equations (1.1), if they exist. In contrast, in this work we will be interested in the existence of stepped-pressure MHD equilibria in domains very far from symmetric, without imposing any profile, and then the pressure and the rotational transform profiles show up *a posteriori*. The same comment applies to the free boundary problem that is stated below.

Another related equation that appears in plasma physics, particularly concerning the design of plasma confinement devices for nuclear power generation, describes a free boundary steady state surrounded by vacuum with an external current J_{ext} . In terms of the interior and exterior magnetic fields, B and B_{ext} , this system reads in this case

$$B \times \text{curl } B + \nabla P = 0 \quad \text{in } \Omega, \quad (1.2a)$$

$$\text{div } B = 0 \quad \text{in } \Omega, \quad (1.2b)$$

$$\operatorname{div} B_{\text{ext}} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \quad (1.2c)$$

$$\operatorname{curl} B_{\text{ext}} = J_{\text{ext}} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \quad (1.2d)$$

$$(B - B_{\text{ext}}) \cdot N = 0 \quad \text{on } \partial\Omega, \quad (1.2e)$$

$$|B|^2 - |B_{\text{ext}}|^2 = 0 \quad \text{on } \partial\Omega, \quad (1.2f)$$

$$B_{\text{ext}} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (1.2g)$$

For simplicity, we have assumed that $P = 0$ on $\partial\Omega$. The jump condition (1.2f) uses the fact that the (hydrodynamic) pressure in the vacuum is simply $\frac{1}{2}|B_{\text{ext}}|^2$. In this work we shall assume that the external current is a current sheet, i.e.,

$$J_{\text{ext}} = J \, dS \quad (1.2h)$$

where dS is the surface measure on the boundary $\partial\Omega'$ of a certain domain Ω' enclosing $\overline{\Omega}$ and $J : \partial\Omega' \rightarrow \mathbb{R}^3$ is a tangent vector field on the surface. We want to stress that in applications in the context of plasma confinement (stellarators), the external current J_{ext} is generated by a certain number of current coils, but the techniques we use in this work do not allow us to deal with this case (see Remark 1.4). We will additionally impose the tangency condition

$$B \cdot N = 0 \quad \text{on } \partial\Omega \quad (1.2i)$$

and refer to the system of equations (1.2) as the *free boundary problem*.

We observe that a related, although different, free boundary problem for MHD equilibria has been considered in the literature; see e.g. [3, 29] and references therein. In these works, the problem consists in determining the shape of an axisymmetric plasma region, which is surrounded by vacuum and contained in a given axisymmetric shell. It is assumed that the shell is a perfect conductor, so no external current is introduced in the vacuum region; moreover, the vacuum field is assumed to be tangent to the boundaries of the shell and of the plasma, and the continuity of the hydrodynamic pressure is also imposed (in order to have a weak solution of the MHD equations across the plasma boundary). In contrast, in the free boundary problem that we consider above, the vacuum vessel is not a perfect conductor, so this leads to considering MHD equilibria on a fixed plasma region Ω (the fixed boundary problem), and looking for an external current J_{ext} in the vacuum whose corresponding magnetic field B_{ext} forces the plasma to be in equilibrium on the whole space (the reason for the boundary conditions (1.2e) and (1.2f)).

The main result of this article is the existence of piecewise smooth MHD equilibria with nonconstant stepped-pressure in a wide range of toroidal domains, which can be very different from an axisymmetric domain. The same philosophy works, with only minor modifications, for the fixed and free boundary problems. The equilibria we construct are not C^1 , like those of Bruno and Laurence for almost-axisymmetric domains, and in fact they feature singular current sheets (cf. Remark 2.4). The toroidal domains we consider can even be knotted in an arbitrarily complicated fashion. Specifically, the result applies to any toroidal domain with an analytic boundary satisfying a certain nondegeneracy

assumption, which enables us to employ KAM-theoretic techniques in a certain step of the proof.

1.1. *Nondegenerate toroidal domains*

To define the nondegeneracy condition that our toroidal domains must satisfy, we need to introduce some notation. We recall that by toroidal domain we mean that the domain is diffeomorphic to a solid torus. Firstly, recall that Hodge theory ensures the existence of a unique (modulo a multiplicative constant) solution to the boundary problem

$$\operatorname{curl} h = 0 \quad \text{in } \Omega, \quad \operatorname{div} h = 0 \quad \text{in } \Omega, \quad h \cdot N = 0 \quad \text{on } \partial\Omega$$

on a toroidal domain $\Omega \subset \mathbb{R}^3$. We refer to h as the *harmonic field* of Ω .

Beltrami fields are solutions to the equation

$$\operatorname{curl} B = \lambda B \quad \text{in } \Omega, \quad B \cdot N = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

for some nonzero real constant λ . The space of Beltrami fields on a toroidal domain Ω is infinite-dimensional. Specifically, as the curl defines a self-adjoint operator with dense domain in the space $L^2_{\operatorname{div},h}(\Omega)$ of square-integrable divergence-free fields that are L^2 -orthogonal to the harmonic field h , it is standard that there is an orthogonal basis of $L^2_{\operatorname{div},h}(\Omega)$ consisting of Beltrami fields with zero harmonic part, i.e., solutions $\{B^n\}_{n=1}^\infty$ to equation (1.3) for certain nonzero constants $\{\lambda^n\}_{n=1}^\infty$ that satisfy the additional constraint

$$\int_{\Omega} B^n \cdot h \, dx = 0.$$

Note that $|\lambda^n| \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, for all $\lambda \in \mathbb{R} \setminus \{\lambda^n\}_{n=1}^\infty$, there is a unique Beltrami field with parameter λ and prescribed harmonic part, that is, a unique solution to the boundary problem (1.3) subject to the additional constraint

$$\int_{\Omega} B \cdot h \, dx = 1.$$

All along this paper we shall use the term *eigenvalue* when referring to the constant proportionality factor λ of a Beltrami field, even taking into account that, strictly speaking, λ is an eigenvalue of curl only when it belongs to the aforementioned discrete set of points $\{\lambda^n\}_{n=1}^\infty$.

In both cases (that is, when B has a prescribed harmonic part and when $B = B^n$ for some n), the trace on the boundary of the Beltrami field is a smooth vector field tangent to the (embedded, nonsymmetric, possibly knotted) toroidal surface $\partial\Omega$. In any case, $\partial\Omega$ is an invariant torus of B . Now recall that this invariant torus is *Diophantine* with frequency vector $\omega \in \mathbb{R}^2$ if there exist global coordinates $\varphi : \partial\Omega \rightarrow \mathbb{T}^2$, with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, such that the restriction of the field B to $\partial\Omega$ reads in these coordinates as

$$B|_{\partial\Omega} = \omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2}, \tag{1.4}$$

and ω is a Diophantine vector. This means that there exist constants $\gamma > 0$ and $\tau > 1$ such that

$$|k \cdot \omega| \geq \gamma |k|^{-\tau} \quad (1.5)$$

for any vector with integral components $k \in \mathbb{Z}^2 \setminus \{0\}$. It is well known that the Diophantine property (possibly with different γ) of the frequency vector ω is independent of the choice of coordinates φ . We recall that $\{\partial_{\varphi_1}, \partial_{\varphi_2}\}$ stands for the holonomic (or coordinate) basis associated to the coordinate system $\varphi : \partial\Omega \rightarrow \mathbb{T}^2$; as usual, the notation ∂_{φ_j} means that the coordinate basis vector field can be identified with the partial derivative operator (under the usual interpretation of vector fields as differential operators acting on functions).

In this paper, we will say that a toroidal domain $\Omega \subset \mathbb{R}^3$ is *nondegenerate of type I or II* if there is a Beltrami field on Ω for which the boundary $\partial\Omega$ is a Diophantine invariant torus and if the determinant of certain 2×2 constant matrices is not zero (type I) or not equal to a certain constant depending on the Beltrami field (type II). To streamline the introduction, the expression of these matrices (which are the averages on \mathbb{T}^2 of matrix-valued quantities involving the specific Beltrami field, the associated Diophantine frequency vector and the linearizing coordinates φ) is relegated to Definitions 3.1 and 4.1 in the main text. To get some intuition about the meaning of this condition, recall that Beltrami fields appear in the plasma physics literature as force-free fields with constant factor, so the nondegeneracy condition can be heuristically understood as the existence of a generic force-free field on the domain whose rotational transform is irrational on the boundary. For concreteness, we shall refer to a Beltrami field with this property as a *nondegenerate Beltrami field of type I or II*.

Some observations are in order. Firstly, since there are infinitely many curl eigenfields that do not necessarily vanish on the toroidal surface $\partial\Omega$, and since the set of Diophantine vectors has full measure, it is natural to conjecture that a “generic” toroidal domain should be nondegenerate, in this sense. However, genericity questions for vector-valued eigenvalue problems are remarkably hard [15, 30] and we have not been able to rigorously establish this claim. A particular case that one can study in detail is the class of thin toroidal domains, which one can understand as thickenings of an arbitrary closed curve in space. There one has a good understanding on the structure of harmonic fields, which can be used to rigorously show that thin toroidal domains are indeed nondegenerate. Details are given in Proposition 7.1. A concrete consequence of this fact is that there certainly are nondegenerate toroidal domains of any knot type, which are obviously not small perturbations of an axisymmetric domain. The boundary can be chosen to be smooth, and even analytic.

1.2. Statement of the main results

We are now ready to present our main result about the existence of MHD equilibria in nondegenerate toroidal domains.

In the context of the fixed boundary problem, the result can be stated as follows:

Theorem 1.1 (Fixed boundary MHD equilibria). *Let $\Omega_1 \subset \mathbb{R}^3$ be a nondegenerate toroidal domain of type I with analytic boundary, and let B_1 be a nondegenerate Beltrami field of type I with eigenvalue λ_1 on Ω_1 , in the sense of Definition 3.1. Then, for any $N > 1$ and almost all $(\lambda_2, \dots, \lambda_N) \in \mathbb{R}^{N-1}$ with $\lambda_j \neq \lambda_k$, the following statements hold:*

- (i) *There exists a collection $\{\Omega_k\}_{k=2}^N$ of “nested” nondegenerate toroidal domains of type I with analytic boundary, all of them diffeomorphic to Ω_1 , with the property that $\overline{\Omega_{k-1}} \subset \Omega_k$ for all $2 \leq k \leq N$ (see Figure 1).*

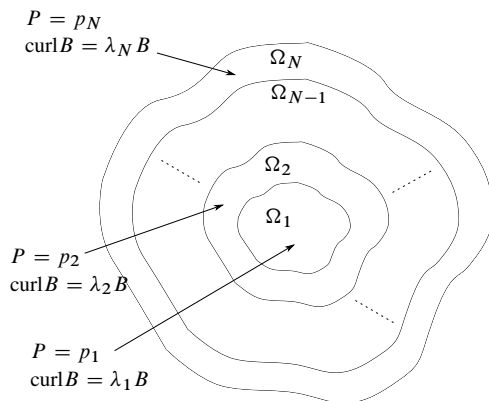


Fig. 1. A cross section of the nested toroidal domains.

- (ii) *There is a piecewise smooth MHD equilibrium (B, P) in the fixed domain Ω_N which satisfies equations (1.1) in the weak sense. In particular, $P + \frac{1}{2}|B|^2$ is continuous across the surfaces $\partial\Omega_k$ for $1 \leq k \leq N - 1$.*
- (iii) *For each $1 \leq k \leq N$, the magnetic field and the pressure satisfy*

$$\text{curl } B = \lambda_k B \quad \text{and} \quad P = p_k$$

in $\Omega_k \setminus \overline{\Omega_{k-1}}$. Here $\{p_k\}_{k=1}^{N-1}$ are distinct nonzero constants, $p_N := 0$ and we have set $\Omega_0 := \emptyset$. Furthermore, $B = B_1$ in Ω_1 .

Remark 1.2. The full measure set of values $(\lambda_2, \dots, \lambda_N)$ for which the theorem holds is described as the whole \mathbb{R}^{N-1} minus a finite set of hyperplanes; the values of λ_j that must be avoided are given by equation (3.5), and depend on the nondegenerate Beltrami field B_1 and the pressure jumps in a very nontrivial way.

Likewise, for the free boundary problem, our main result can be stated as follows. In the statement, since B is a Beltrami field, it solves equation (1.2a) in Ω for any constant P , whose value is inessential and can be set to be 0. Since the plasma pressure of the vacuum is 0 as well, this yields free boundary force-free equilibria.

Theorem 1.3 (Free boundary force free equilibria). *Let $\Omega \subset \mathbb{R}^3$ be a nondegenerate toroidal domain of type II with analytic boundary, and let B be a nondegenerate Beltrami field of type II with eigenvalue λ on Ω , in the sense of Definition 4.1. Then there exists an external magnetic field B_{ext} and a current sheet of the form $J_{\text{ext}} = J dS$, where dS is the surface measure on the boundary of an analytic domain Ω' that is diffeomorphic to Ω and encloses $\overline{\Omega}$, and where J is an analytic tangent vector field on $\partial\Omega'$, such that $(B, B_{\text{ext}}, J_{\text{ext}}, \Omega)$ is a solution of the free boundary problem (1.2) with $P = 0$ in Ω .*

Remark 1.4. The external surface current J_{ext} can be approximated (in the sense of distributions) by a current \tilde{J}_{ext} whose support consists of finitely many closed curves on $\partial\Omega'$; see e.g. [18]. Then \tilde{J}_{ext} represents the current produced by a number of external coils (and hence it cannot be longer represented as $J dS$ for some smooth tangent vector field on $\partial\Omega'$). However, $(B, \tilde{B}_{\text{ext}}, \tilde{J}_{\text{ext}}, \Omega)$ is not a solution of the free boundary problem (1.2) because the boundary terms $\tilde{B}_{\text{ext}} \cdot N$ and $|\tilde{B}_{\text{ext}}|^2 - |B|^2$ on $\partial\Omega$ are small but not necessarily zero.

Remark 1.5. A minor modification of the proof of Theorem 1.3 using Theorem 5.1 allows us to construct free boundary MHD equilibria with constant pressure $P = c \neq 0$ in Ω provided that $|c|$ is small enough. In this case, the boundary condition (1.2f) is replaced by the identity $|B_{\text{ext}}|^2 - |B|^2 = 2c$ on $\partial\Omega$, and Ω has to be a nondegenerate toroidal domain of both types I and II. Since the pressure of the vacuum region is 0 and $c \neq 0$, this modified construction yields stepped-pressure free boundary equilibria in \mathbb{R}^3 .

A consequence of these theorems and of the above discussion about nondegenerate domains is the existence of piecewise smooth MHD equilibria with nonconstant pressure and (fixed or free) toroidal boundaries of any knot type. A precise statement is given in Corollary 7.3. For the benefit of the reader, in Section 2 we recall the definition of weak solutions to the system (1.1a)–(1.1c), which is required to make sense of MHD equilibria that are only piecewise smooth.

It is worth mentioning that a minor modification of the proof of Theorem 1.1 permits us to prove the existence of Lipschitz continuous force-free fields with a nonconstant proportionality factor on toroidal domains of complicated geometry. Details are provided in Theorem 8.1. This is interesting because, in a certain precise sense, smooth force-free fields with a nonconstant factor are rare, as discussed in [17, 27].

1.3. Strategy of the proof

The strategy of the proofs of Theorems 1.1 and 1.3 is similar, so let us focus on the former. The basic idea behind Theorem 1.1 is motivated by the work of Bruno and Laurence [8] on MHD equilibria on small perturbations of an axisymmetric toroidal domain. The perturbative construction they use in their proof, however, hinges strongly on having approximately axisymmetric solutions, where one can obtain very precise information about the solutions and their trajectories, and cannot be extended to toroidal domains that are not approximately symmetric.

To explain the gist of our approach, let us stick to the simplest case, $N = 2$. The case of an arbitrary $N \geq 2$ is obtained by repeating the process $N - 1$ times. Our initial observation (Lemma 2.1) is that, if we have two Beltrami fields B_1, B_2 defined on two disjoint domains $\Omega_1, \Omega'_2 := \Omega_2 \setminus \overline{\Omega_1}$, with $\overline{\Omega_1} \subset \Omega_2$, one can define a piecewise smooth MHD equilibrium on the domain Ω_2 , with a certain piecewise constant pressure function P , provided that the difference $|B_1|^2 - |B_2|^2$ is constant on $\partial\Omega_1$.

We start by choosing B_1 as a nondegenerate Beltrami field in the toroidal domain Ω_1 , so that the analytic surface $\partial\Omega_1$ is a Diophantine invariant torus of B_1 . To construct a Beltrami field B_2 in an exterior neighborhood of $\partial\Omega_1$, we use a version of the Cauchy–Kovalevskaya theorem for the curl operator [14] (see also Appendix B) with a Cauchy datum given by an analytic vector field tangent to $\partial\Omega_1$. A key aspect of this theorem is that one can only grant the existence of a local solution to the equation provided that the Cauchy datum satisfies an additional constraint. When one takes this constraint into account, showing that $|B_1|^2 - |B_2|^2$ is constant on $\partial\Omega_1$ becomes equivalent to proving the existence of an analytic solution to a certain nonlinear Hamilton–Jacobi equation on \mathbb{T}^2 .

The key difficulty of the problem is that, as the toroidal domains we consider are far from axisymmetric, we cannot extract from the equations enough information about the trajectories of the vector fields. The first manifestation of this difficulty is that we have not found a way of effectively using trajectories to analyze the aforementioned Hamilton–Jacobi equation. Instead, we have shown that one can exploit the fact that the restriction $B_1|_{\partial\Omega_1}$ is conjugate to a Diophantine rotation, in order to regard the equation as a nonlinear perturbation of the cohomological equation which appears in KAM theory. With this approach, we eventually establish the existence of analytic solutions by means of a Newton quadratic scheme (Theorem 5.1).

The next step is to show that the resulting field B_2 does in fact have an invariant torus enclosing a toroidal domain $\Omega_2 \supset \overline{\Omega_1}$, which permits us to make sense of the basic geometric configuration used to construct the MHD equilibrium. To this end, we prove that $\partial\Omega_1$ is a twist (in the KAM-theoretic sense) invariant torus of B_2 , so that it is accumulated by a set of Diophantine analytic invariant tori. However, the key difficulty is that we cannot compute a good approximation for the trajectories of B_2 . This means that we do not have enough information to apply the existing KAM theorems for divergence-free fields (see e.g. [16, 24–26]), which are based on studying the Poincaré map of the field on a transverse section.

To solve this problem, we establish a KAM theorem for divergence-free vector fields in \mathbb{R}^3 with two key features that make it rather different from other KAM theorems in the same context [6, 7, 24]. First, it applies to vector fields which need not be approximately integrable or in Birkhoff normal form. Secondly, the twist condition is written solely in terms of the vector field and of the approximate invariant torus. An additional advantage is that the formulas take a particularly simple form when the field is Beltrami. Recall that a KAM theorem for perturbations of integrable volume-preserving diffeomorphisms was obtained in [9, 31, 32].

1.4. Organization of the paper

After recalling the definition of weak MHD equilibria, in Section 2 we prove a lemma ensuring that one can construct piecewise smooth MHD equilibria by gluing two Beltrami fields defined on nonintersecting domains with a common boundary component, provided that the boundary traces of these Beltrami fields satisfy a certain constraint. The main arguments of the proofs of Theorems 1.1 and 1.3 are presented in Sections 3 and 4. For clarity of exposition, however, the two essential technical points of proof (which are of independent interest) are relegated to Sections 5 and 6. Specifically, in Section 5 we solve, using a cohomological equation, the Hamilton–Jacobi equation associated with the constraint that we came across in Section 2. Also, in Section 6 we prove our new KAM theorem for divergence-free vector fields in \mathbb{R}^3 . Section 7 is devoted to rigorously proving that thin toroidal domains of any topology are generically nondegenerate (of type I and II). The existence result for Lipschitz-continuous force-free fields with a nonconstant factor is presented in Section 8. The paper concludes with two technical appendices. In the first appendix we show that Beltrami fields are analytic up to the boundary if the domain is analytic, and in the second we record certain results for Beltrami fields that we proved in [13, 14, 16] and which are relevant for the problem under consideration.

2. Construction of weak MHD equilibria from Beltrami fields

In this section we introduce the definition of a weak MHD equilibrium. We say that a pair (B, P) of class, say, $L^2(\Omega)$ is a *weak solution to the stationary MHD equations* in Ω if

$$\int_{\Omega} [(B \otimes B) \cdot \nabla w - (P + \frac{1}{2}|B|^2) \operatorname{div} w] dx = 0 \quad \text{and} \quad \int_{\Omega} B \cdot \nabla \phi dx = 0$$

for any vector field $w \in C_c^1(\Omega)$ and any scalar function $\phi \in C^1(\Omega)$. Of course, if B and P are continuously differentiable, this is equivalent to saying that they satisfy Equations (1.1a)–(1.1c) in Ω . As usual, the subscript c means that we take functions with compact support.

Lemma 2.1. *Let $\{\Omega_k\}_{k=1}^N$ be $N \geq 2$ bounded domains in \mathbb{R}^3 with smooth connected boundaries. Assume that these domains are nested in the sense that $\overline{\Omega_{k-1}} \subset \Omega_k$ for all $1 \leq k \leq N$, with $\Omega_0 := \emptyset$. With $\Omega'_k := \Omega_k \setminus \overline{\Omega_{k-1}}$, suppose furthermore that the vector field B_k satisfies the equation $\operatorname{curl} B_k = \lambda_k B_k$ in Ω'_k for some nonzero constant λ_k . Assume that B_k is tangent to the boundary of Ω'_k and that*

$$|B_{k+1}|^2 - |B_k|^2 = 2c_k \quad \text{on } \partial\Omega_k \tag{2.1}$$

for all $1 \leq k \leq N - 1$, where c_k are constants. Then

$$B(x) := \sum_{k=1}^N B_k(x) 1_{\Omega'_k}(x)$$

is a piecewise smooth MHD equilibrium on Ω_N with piecewise constant pressure

$$P(x) := c_0 - \sum_{k=1}^{N-1} c_k 1_{\Omega_N \setminus \overline{\Omega'_k}}(x).$$

Here c_0 is any real constant (in particular, it can be chosen so that $P(x) = 0$ if $x \in \partial\Omega_N$).

Remark 2.2. As B_k is a Beltrami field defined in smooth domains and tangent to the boundary, it is standard [12] that B_k is smooth up to the boundary. Therefore, the constraint (2.1) makes sense pointwise. A related analytic regularity result up to the boundary, which will be needed later on, is proved in Appendix A.

Remark 2.3. The result and the proof remain valid when $\lambda_k = 0$ if the corresponding vector field B_k is additionally assumed to be divergence-free in Ω'_k .

Proof of Lemma 2.1. To keep the formulas as simple as possible, we will prove the result for $N = 2$; the general case is analogous. We start by noticing that, for all $\phi \in C^1(\Omega_2)$,

$$\begin{aligned} \int_{\Omega_2} B \cdot \nabla \phi \, dx &= \int_{\Omega_1} B_1 \cdot \nabla \phi \, dx + \int_{\Omega'_2} B_2 \cdot \nabla \phi \, dx \\ &= \int_{\partial\Omega_1} \phi B_1 \cdot N \, dS + \int_{\partial(\Omega'_2)} \phi B_2 \cdot N \, dS = 0, \end{aligned}$$

where we have used the fact that $\operatorname{div} B_1 = \operatorname{div} B_2 = 0$ in their respective domains and $B_1 \cdot N = 0$ on $\partial\Omega_1$ and $B_2 \cdot N = 0$ on $\partial\Omega'_2$. Hence $\operatorname{div} B = 0$ in the sense of distributions.

Let us now take an arbitrary vector field $w \in C_c^1(\Omega_2)$. We can write

$$\begin{aligned} I &:= \int_{\Omega_2} [(B \otimes B) \cdot \nabla w - (P + \frac{1}{2}|B|^2) \operatorname{div} w] \, dx \\ &= \int_{\Omega_1} [(B_1 \otimes B_1) \cdot \nabla w - (c_0 + \frac{1}{2}|B_1|^2) \operatorname{div} w] \, dx \\ &\quad + \int_{\Omega'_2} [(B_2 \otimes B_2) \cdot \nabla w - (c_0 - c_1 + \frac{1}{2}|B_2|^2) \operatorname{div} w] \, dx =: I_1 + I_2. \end{aligned}$$

Integrating by parts, and using the fact that $B_1 \cdot N = 0$ on $\partial\Omega_1$, we easily obtain

$$\begin{aligned} I_1 &= \int_{\Omega_1} [(B_1 \otimes B_1) \cdot \nabla w + \frac{1}{2} \nabla |B_1|^2 \cdot w] \, dx - \int_{\partial\Omega_1} (c_0 + \frac{1}{2}|B_1|^2) w \cdot N \, dS \\ &= - \int_{\Omega_1} [\operatorname{div}(B_1 \otimes B_1) - \frac{1}{2} \nabla |B_1|^2] \cdot w \, dx - \int_{\partial\Omega_1} (c_0 + \frac{1}{2}|B_1|^2) w \cdot N \, dS \\ &= - \int_{\partial\Omega_1} (c_0 + \frac{1}{2}|B_1|^2) w \cdot N \, dS. \end{aligned}$$

To pass to the last equation we have used the well known identity for Beltrami fields

$$\operatorname{div}(B_1 \otimes B_1) = \frac{1}{2} \nabla |B_1|^2.$$

Analogously, using the same identity for B_2 on Ω'_2 and the fact that $B_2 \cdot N = 0$ on the boundary, we can compute the term I_2 . Notice that by the connectedness of the boundaries of Ω_1 and Ω_2 we have $\partial\Omega'_2 = \partial\Omega_1 \cup \partial\Omega_2$, and the outward pointing normal vector of Ω'_2 on $\partial\Omega_1$ is $-N$. We thus obtain

$$\begin{aligned} I_2 &= \int_{\Omega'_2} [(B_2 \otimes B_2) \cdot \nabla w + \frac{1}{2} \nabla |B_2|^2 \cdot w] dx + \int_{\partial\Omega_1} (c_0 - c_1 + \frac{1}{2} |B_2|^2) w \cdot N dS \\ &= \int_{\partial\Omega_1} (c_0 - c_1 + \frac{1}{2} |B_2|^2) w \cdot N dS. \end{aligned}$$

The surface integral is taken only on $\partial\Omega_1$ because $w = 0$ on $\partial\Omega_2$.

Putting together these computations and using the boundary condition (2.1), we finally conclude that

$$I = \int_{\partial\Omega_1} [\frac{1}{2} (|B_2|^2 - |B_1|^2) - c_1] w \cdot N dS = 0$$

for all $w \in C_c^1(\Omega_2)$. It then follows that (B, P) is a weak solution of the MHD equations in Ω_2 , as claimed. ■

Remark 2.4. It is easy to check that the plasma current $J := \text{curl } B$ of the solution constructed in Lemma 2.1 is the vector-valued distribution

$$J = \sum_{k=1}^N \lambda_k B_k 1_{\Omega'_k} + \sum_{k=1}^{N-1} (B_k - B_{k+1}) \times N_k dS_k,$$

where dS_k and N_k are the area measure and outer unit normal on $\partial\Omega_k$. The current sheet terms appearing in this formula are a consequence of the discontinuity of the magnetic field across the surfaces $\partial\Omega_k$.

3. Fixed boundary equilibria: Proof of Theorem 1.1

In this section we show how to implement the strategy discussed in the Introduction to prove the first main result of this paper, modulo some technical results that will be presented in later sections.

3.1. Nondegenerate toroidal domains of type I

Let us begin by defining the class of toroidal domains that we consider in Theorem 1.1.

Definition 3.1. A toroidal domain $\Omega \subset \mathbb{R}^3$ is *nondegenerate of type I* if there exists a Beltrami field B on Ω such that the following conditions hold:

- (i) $\partial\Omega$ is a Diophantine invariant torus of the field.
- (ii) The constant 2×2 matrix

$$M := \int_{\mathbb{T}^2} G(\varphi)^{-1} \begin{pmatrix} 1 - \omega_1 \partial_{\varphi_1} \mathcal{R} & -\omega_2 \partial_{\varphi_1} \mathcal{R} \\ -\omega_1 \partial_{\varphi_2} \mathcal{R} & 1 - \omega_2 \partial_{\varphi_2} \mathcal{R} \end{pmatrix} d\varphi \tag{3.1}$$

is invertible. Here ω is the frequency vector of B on $\partial\Omega$, φ are the linearizing coordinates in equation (1.4), G is the metric matrix (or first fundamental form) of the surface $\partial\Omega$ in the coordinates φ , and $\mathcal{R}(\varphi)$ is the unique zero mean solution to the equation on the torus

$$\omega_1 \partial_{\varphi_1} \mathcal{R} + \omega_2 \partial_{\varphi_2} \mathcal{R} = \varkappa - |X|^2,$$

where $X := B|_{\partial\Omega}$ and the constant \varkappa is chosen so that $\int_{\mathbb{T}^2} (\varkappa - |X|^2) d\varphi = 0$. By rescaling the Beltrami field, we can henceforth assume that the above constant is $\varkappa = 1$ (which has already been considered when writing equation (3.1)).

The Beltrami field and the analytic toroidal domain satisfying the nondegeneracy assumption are B_1 and Ω_1 , and the corresponding eigenvalue is λ_1 . Clearly, the nondegeneracy of B_1 is invariant under rescaling the field by a nonzero constant.

Here and in what follows,

$$[f]_{\mathbb{T}^2} := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f d\varphi$$

denotes the average of a function on \mathbb{T}^2 (in φ -coordinates) and we set $\omega^\perp := (\omega_2, -\omega_1)$ for each two-component vector $\omega = (\omega_1, \omega_2)$.

3.2. Construction of the first layer

As $\partial\Omega_1$ is an invariant torus of B_1 with Diophantine frequency vector $\omega^{(1)}$, it is standard [11] that one can parametrize the invariant torus by an embedding $K_1 : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ satisfying the equation

$$L_{\omega^{(1)}} K_1 = B_1 \circ K_1,$$

where $L_{\omega^{(1)}} K_1 := DK_1 \omega^{(1)}$ is the pointwise derivative of K_1 in the direction of $\omega^{(1)}$. In this picture, of course, the invariant torus is the image $\partial\Omega_1 = K_1(\mathbb{T}^2)$. Let us emphasize from the beginning that parametrizing the invariant tori by embeddings is essential for the KAM theorem that we will prove in Section 6 (Theorem 6.2), which will play a key role in this proof later on.

Since the boundary of the domain Ω_1 is analytic, Theorem A.1 implies that B_1 is analytic up to the boundary. A theorem of Herman [32, 33] then ensures that the linearizing parametrization K_1 is analytic, or in other words, there are analytic coordinates $\varphi : \partial\Omega_1 \rightarrow \mathbb{T}^2$ in which $X_1 := B_1|_{\partial\Omega_1}$ takes the form

$$X_1 = G_{K_1}^{-1} DK_1^\top B_1 \circ K_1 = \omega_1^{(1)} \partial_{\varphi_1} + \omega_2^{(1)} \partial_{\varphi_2}.$$

Here $G_{K_1} := DK_1^\top DK_1$ is the matrix representation of the pullback of the Euclidean metric to the surface $\partial\Omega_1$ obtained using the embedding K_1 , so it is a positive definite symmetric (nonconstant) 2×2 matrix.

Theorem 5.1 implies that, for any constant c_2 that is small enough in absolute value, there is an analytic vector field X_2 on $\partial\Omega_1$ such that:

- (i) The 1-form that is dual to X_2 with respect to the metric on $\partial\Omega_1$ induced by the Euclidean metric in \mathbb{R}^3 is closed.
- (ii) The pointwise norm of X_2 , computed with the induced metric on $\partial\Omega_1$, satisfies the equation

$$|X_2|^2 = (1 + b_2)|X_1|^2 + c_2$$

for some constants with $|b_2| \leq C|c_2|$.

- (iii) The vector field X_2 depends continuously on the parameter c_2 , in the C^r -topology of vector fields for any r , and is Diophantine with frequency vector

$$\omega^{(2)} := (1 + c_2)^{1/2}\omega^{(1)}. \tag{3.2}$$

In particular, one can write

$$X_2 = X_1 + \mathcal{O}(c_2) = \omega_1^{(1)}\partial_{\varphi_1} + \omega_2^{(1)}\partial_{\varphi_2} + \mathcal{O}(c_2);$$

here and in what follows, $\mathcal{O}(c_2)$ stands for a quantity (which may vary from line to line) whose C^r norm is bounded by $C|c_2|$, for any fixed integer r . We remark that the fact that the frequency vectors $\omega^{(2)}$ and $\omega^{(1)}$ are proportional (see (3.2)), is a consequence of the proof of Theorem 5.1. It is not a necessary condition, but it is convenient to write all the estimates of the proof of Lemma 5.4 in a simple way, and to keep the same Diophantine constants.

Now we consider the Cauchy problem

$$\text{curl } B'_2 = \lambda_2 B'_2, \quad B'_2|_{\partial\Omega_1} = X_2,$$

for some nonzero constant $\lambda_2 \neq \lambda_1$ that we will fix later. Since $\partial\Omega_1$ and X_2 are analytic, and the 1-form dual to X_2 is closed, Theorem B.1 (which is a sort of Cauchy–Kovalevskaya theorem for the curl operator) implies that there exists a unique analytic solution to this Cauchy problem in a neighborhood of $\partial\Omega_1$. Eventually, we will only be interested in the behavior of the solution outside Ω_1 .

By construction, $\partial\Omega_1$ is a Diophantine invariant torus of the vector field B'_2 . We claim that it is twist (in the KAM sense; see Definition 6.1) for almost all choices of the constant λ_2 . This implies that $\partial\Omega_1$ is accumulated by a set of Diophantine invariant tori of B'_2 contained in $\mathbb{R}^3 \setminus \overline{\Omega_1}$. Since B'_2 is analytic, these tori are analytic as well. This follows from Corollary 6.3 to the KAM theorem for divergence-free fields that we shall prove in Section 6.

Let us denote by $K_2 : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ an embedding that is a linearizing parametrization of the invariant torus $\partial\Omega_1$ with frequency vector $\omega^{(2)}$ of the vector field B'_2 . Then we can introduce coordinates (which we still denote by φ) such that $X_2 = B'_2|_{\partial\Omega_1}$ becomes the Diophantine linear field

$$X_2 = G_{K_2}^{-1}DK_2^\top B'_2 \circ K_2 = \omega_1^{(2)}\partial_{\varphi_1} + \omega_2^{(2)}\partial_{\varphi_2}.$$

In general, K_2 is different from the parametrization K_1 that linearizes X_1 , but it follows

from the previous discussion that both parametrizations differ by a higher order correction, i.e.,

$$K_2 = K_1 + \mathcal{O}(c_2). \tag{3.3}$$

Since B'_2 satisfies the equation $\text{curl } B'_2 = \lambda_2 B'_2$, an easy computation shows that the following identity holds:

Lemma 3.2. $DB_2'^\top + DB'_2 = 2DB_2'^\top + \lambda_2 B'_2 \times$, where DB'_2 is the Jacobian matrix of B'_2 and \times denotes the vector product, both computed in Cartesian coordinates.

Proof. The proof is straightforward. Indeed,

$$DB_2'^\top + DB'_2 = 2DB_2'^\top + (DB'_2 - DB_2'^\top) = 2DB_2'^\top + \text{curl } B'_2 \times,$$

so the claim follows from the equation $\text{curl } B'_2 = \lambda_2 B'_2$. ■

To invoke Corollary 6.3, we must check the twist condition (see Definition 6.1). This involves computing a two-component vector field (or a 2×1 matrix) appearing in (6.3), which in this case takes the form

$$A_2 = -\frac{G_{K_2}^{-1}}{|n_2|^2} [2DK_2^\top DB_2'^\top n_2 + \lambda_2 DK_2^\top (B'_2 \times n_2)].$$

Here $DB_2'^\top$ and B'_2 are evaluated at $K_2(\varphi)$ and the normal vector

$$n_2(\varphi) := \partial_{\varphi_1} K_2(\varphi) \times \partial_{\varphi_2} K_2(\varphi)$$

is defined in terms of K_2 as in Definition 3.1. Observe that DK_2 is a 3×2 matrix.

Since the vector field $B'_2 \times n_2$ is tangent to $\partial\Omega_1$ and perpendicular to B'_2 , we infer that there is a nonvanishing vector (given by a 2×1 matrix) α_2 on $\partial\Omega_1$ such that

$$B'_2 \times n_2 = DK_2 \alpha_2.$$

Therefore,

$$DK_2^\top (B'_2 \times n_2) = (DK_2^\top DK_2) \alpha_2 = G_{K_2} \alpha_2.$$

The matrix A_2 then takes the form

$$\begin{aligned} A_2 &= -\frac{2G_{K_2}^{-1}}{|n_2|^2} DK_2^\top DB_2'^\top n_2 - \frac{\lambda_2}{|n_2|^2} \alpha_2 \\ &= \frac{2G_{K_2}^{-1}}{|n_2|^2} DK_2^\top L_{\omega^{(2)}} n_2 - \frac{\lambda_2}{|n_2|^2} \alpha_2 =: A_2^{(1)} + A_2^{(2)}, \end{aligned}$$

where we have used Lemma 6.4 to pass to the second equality. It is clear from this expression that the vector $A_2(\varphi)$ only depends on the way the torus $\partial\Omega_1$ is embedded in \mathbb{R}^3 , on the Diophantine vector $\omega^{(2)}$, on the parametrization K_2 linearizing X_2 , and on the eigenvalue λ_2 .

According to Definition 6.1, the invariant torus $\partial\Omega_1$ of B'_2 is *twist* if the twist constant

$$\begin{aligned} T_2 &:= ([A_2^{(1)}]_{\mathbb{T}^2} + [A_2^{(2)}]_{\mathbb{T}^2}) \cdot (\omega^{(2)})^\perp \\ &= [A_2^{(1)}]_{\mathbb{T}^2} \cdot (\omega^{(2)})^\perp - \lambda_2 |\omega^{(2)}| \left[\frac{F_2}{|n_2|^2} \right]_{\mathbb{T}^2} \end{aligned} \tag{3.4}$$

is nonzero. Here $F_2(\varphi)$ is the function defined as the projection of the field α_2 in the $(\omega^{(2)})^\perp$ direction, i.e.,

$$F_2 := \alpha_2 \cdot \frac{(\omega^{(2)})^\perp}{|\omega^{(2)}|}.$$

This function is nonvanishing because the field $DK_2 \alpha_2$ is perpendicular to $B'_2|_{\partial\Omega_1}$, so α_2 and $\omega^{(2)}$ cannot be proportional at some point of \mathbb{T}^2 .

Arguing in the same way, we obtain an analogous expression for the twist constant T_1 of the invariant torus $\partial\Omega_1$ for the vector field B_1 . As $\partial\Omega_1$ is an invariant torus for B_1 and B'_2 , and the corresponding parametrizations K_1 and K_2 and Diophantine vectors $\omega^{(1)}$ and $\omega^{(2)}$ differ just by an error of order c_2 by (3.2)–(3.3), we conclude that

$$\begin{aligned} T_2 &= T_1 - \lambda_2 \left(|\omega^{(1)}| \left[\frac{F_1}{|n_1|^2} \right]_{\mathbb{T}^2} + O(c_2) \right) + \lambda_1 |\omega^{(1)}| \left[\frac{F_1}{|n_1|^2} \right]_{\mathbb{T}^2} + O(c_2) \\ &=: T_1 - a\lambda_2 + b, \end{aligned}$$

where $n_1 := \partial_{\varphi_1} K_1 \times \partial_{\varphi_2} K_1$ and F_1 is also nonvanishing. The constants a, b are therefore nonzero if $|c_2|$ is small enough, so $T_2 \neq 0$ provided that

$$\lambda_2 \neq \frac{b + T_1}{a} = 1 + \frac{T_1}{|\omega^{(1)}| \left[\frac{F_1}{|n_1|^2} \right]_{\mathbb{T}^2}} + O(c_2). \tag{3.5}$$

This shows that $\partial\Omega_1$ is a twist invariant torus of B'_2 for almost all choices of λ_2 .

Hence, we can take a Diophantine analytic invariant torus Σ_2 of the vector field B'_2 , lying outside $\overline{\Omega_1}$, which is η -close to $\partial\Omega_1$. By this we mean that, for any fixed r , there is a diffeomorphism Ψ_1 of \mathbb{R}^3 which maps $\partial\Omega_1$ into Σ_2 and which is close to the identity because $\|\Psi_1 - \text{id}\|_{C^r} < \eta$. The invariant torus Σ_2 is then the boundary of a toroidal domain $\Omega_2 \supset \overline{\Omega_1}$. It is easy to check that the matrix M_2 in (3.1), associated to the vector field $B'_2|_{\partial\Omega_2}$, is related to the matrix M_1 of $B_1|_{\partial\Omega_1}$ as

$$M_2 = M_1 + \mathcal{O}(\eta + |c_2|).$$

As M_1 is invertible and $\partial\Omega_1$ is accumulated by Diophantine invariant tori of B'_2 , we can take η (and $|c_2|$) small enough so that M_2 is invertible too. We then conclude that Ω_2 is a nondegenerate toroidal domain of type I.

3.3. Conclusion of the proof

As Ω_2 is another nondegenerate toroidal domain of type I, we can repeat the argument to construct a vector field B'_3 in a neighborhood of $\partial\Omega_2$ that solves the equation

$$\text{curl } B'_3 = \lambda_3 B'_3, \quad B'_3|_{\partial\Omega_2} = X_3,$$

for some constant $\lambda_3 \neq \lambda_2$, and the Cauchy datum X_3 satisfies

$$|X_3|^2 = (1 + b_3)|\tilde{X}_2|^2 + c_3$$

with arbitrarily small constants c_3 and $b_3 = \mathcal{O}(c_3)$. Here $\tilde{X}_2 := B'_2|_{\partial\Omega_2}$. Again, one can pick c_3 and λ_3 so that $\partial\Omega_2$ is a twist Diophantine invariant torus of B'_3 . Therefore there is an analytic Diophantine invariant torus of B'_3 , which is the boundary of another nondegenerate toroidal domain of type I $\Omega_3 \supset \overline{\Omega_2}$.

This process can be iterated $N - 1$ times to obtain a family $\{\Omega_k\}_{k=1}^N$ of (analytic) nested tubes, different constants λ_k , small constants c_k, b_k and vector fields B'_k satisfying $\text{curl } B'_k = \lambda_k B'_k$ in $\Omega'_k := \Omega_k \setminus \overline{\Omega_{k-1}}$ for all $2 \leq k \leq N$.

To construct a weak solution (B, P) of the MHD equations in the toroidal domain Ω_N , we set

$$B(x) := B_1(x) 1_{\Omega_1}(x) + \sum_{k=2}^N B'_k(x) 1_{\Omega'_k}(x) \prod_{j=2}^k (1 + b_j)^{-1/2},$$

$$P(x) := p_1 1_{\Omega_1}(x) + \sum_{k=2}^N p_k 1_{\Omega'_k}(x).$$

The constant p_1 is arbitrary, and the constants p_k are defined in terms of c_j, b_j as

$$p_k := p_1 - \frac{1}{2} \sum_{l=2}^k \prod_{j=2}^l (1 + b_j)^{-1} c_l,$$

Note that, generically, $p_k \neq p_j$ if $k \neq j$. A straightforward application of Lemma 2.1 shows that (B, P) is a piecewise smooth MHD equilibrium with all the properties stated in Theorem 1.1.

Remark 3.3. In view of potential applications of our method of proof in the context of plasma physics, we want to highlight that the toroidal surfaces $\partial\Omega_2, \dots, \partial\Omega_N$ we construct in the proof of Theorem 1.1 are not too distorted from one another, and their separation is small. Quantifying such distortions is feasible (but not trivial) using quantitative versions of the Cauchy–Kovalevskaya and KAM theorems (see also Remark 4.2).

4. Free boundary equilibria: Proof of Theorem 1.3

We first introduce the class of toroidal domains that we consider in Theorem 1.3.

Definition 4.1. A toroidal domain $\Omega \subset \mathbb{R}^3$ is *nondegenerate of type II* if there exists a Beltrami field B on Ω (with eigenvalue λ) such that the following conditions hold:

- (i) $\partial\Omega$ is an invariant torus of the field with Diophantine frequency vector ω .
- (ii) The *twist constant* T (see Definition 6.1) satisfies

$$T + \lambda \left[\frac{\alpha \cdot \omega^\perp}{|n|^2} \right]_{\mathbb{T}^2} \neq 0,$$

where $\alpha \cdot \omega^\perp \equiv \alpha_1 \omega_2 - \alpha_2 \omega_1$, $K : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ is the linearizing embedding of $\partial\Omega$ in coordinates φ (so, in particular, $\partial\Omega = K(\mathbb{T}^2)$), $n := \partial_{\varphi_1} K \times \partial_{\varphi_2} K$ is a normal vector and the \mathbb{R}^2 -valued function $\alpha(\varphi)$ is defined as

$$(B \circ K) \times n =: DK \alpha.$$

The nondegeneracy of B is invariant under rescaling the field by a nonzero constant.

The proof of Theorem 1.3 follows the same strategy as the proof of Theorem 1.1, although it is easier because it does not make use of the Hamilton–Jacobi equation we study in Section 5. Therefore, to make the presentation easier, we will use the same notation as in the previous section without further mention.

As the analytic toroidal domain is nondegenerate in the sense of Definition 4.1, there exists a Beltrami field B in Ω satisfying the equation

$$\text{curl } B = \lambda B$$

for some nonzero constant λ and the boundary condition $B \cdot N = 0$, such that $\partial\Omega$ is a Diophantine invariant torus of B with frequency vector ω and its twist constant satisfies

$$T + \lambda \left[\frac{\alpha \cdot \omega^\perp}{|n|^2} \right]_{\mathbb{T}^2} \neq 0. \tag{4.1}$$

We focus on the special case of a globally force-free equilibrium with $P = 0$, whose magnetic field B is a Beltrami field. In Remark 1.5 we explain how to modify this construction to obtain more general equilibria. It is then clear that $(B, B_{\text{ext}}, \Omega)$ are a solution to the equations for a free boundary MHD equilibrium with external current J_{ext} if the external magnetic field and the external current satisfy

$$\text{curl } B_{\text{ext}} = J_{\text{ext}} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{4.2a}$$

$$\text{div } B_{\text{ext}} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{4.2b}$$

$$B_{\text{ext}} \cdot N = 0 \quad \text{on } \partial\Omega, \tag{4.2c}$$

$$|B_{\text{ext}}|^2 - |B|^2 = 0 \quad \text{on } \partial\Omega. \tag{4.2d}$$

To ensure that J_{ext} is a current sheet, we also aim to construct a toroidal domain $\Omega' \supset \overline{\Omega}$ and a tangent vector field J on $\partial\Omega'$ such that

$$J_{\text{ext}} = J \, dS, \tag{4.2e}$$

$$B_{\text{ext}} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega'}, \tag{4.2f}$$

where dS is the surface measure on $\partial\Omega'$. Note that the tangent vector field J must be divergence-free with respect to the induced metric on $\partial\Omega'$ because (4.2a) implies that, in the sense of distributions,

$$0 = \text{div } J_{\text{ext}} = (\text{div}_{\partial\Omega'} J) \, dS.$$

Thus proving Theorem 1.3 boils down to constructing a domain Ω' and an analytic divergence-free tangent vector field J on $\partial\Omega'$ such that the solution to the exterior div-curl problem (4.2a)–(4.2c) on $\mathbb{R}^3 \setminus \overline{\Omega}$ satisfies the conditions (4.2d)–(4.2f).

To construct solutions to this overdetermined system, we follow the same philosophy as in Section 3. Let $X := B|_{\partial\Omega}$ be the restriction of the Beltrami field B on the boundary of the domain Ω . Observe that X is analytic in view of Theorem A.1 and the associated 1-form X^b is closed on $\partial\Omega$ by Theorem B.1. Therefore, the Cauchy problem

$$\operatorname{curl} h = 0, \quad \operatorname{div} h = 0, \quad h|_{\partial\Omega} = X,$$

has a unique analytic solution in a small tubular neighborhood U of $\partial\Omega$ as a consequence of Theorem B.1. By construction, $h \cdot N = 0$ and $|h|^2 = |B|^2$ on $\partial\Omega$. Lemma 2.1 then ensures that the field

$$B \mathbf{1}_\Omega + h \mathbf{1}_{U'}$$

with $U' := U \setminus \overline{\Omega}$ is a weak solution to the stationary MHD equations in the toroidal domain $\Omega_2 := U \cup \Omega$.

Proceeding just as in Section 3.2, the twist constant T_h of the invariant torus $\partial\Omega$ of the harmonic field h can be readily shown to be

$$T_h = T + \lambda|\omega| \left[\frac{F}{|n|^2} \right]_{\mathbb{T}^2} = T + \lambda \left[\frac{\alpha \cdot \omega^\perp}{|n|^2} \right]_{\mathbb{T}^2},$$

where T is the twist constant of B (see Definition 6.1). This is simply (3.4), where we have set $\lambda_2 = 0$ because the field h is harmonic and $c_2 = 0$ because $|h|^2 = |B|^2$ on $\partial\Omega$.

The nondegeneracy assumption of type II, i.e., (4.1), ensures that $T_h \neq 0$. Thus Corollary 6.3 implies that $\partial\Omega$ is accumulated (in both components of its complement) by analytic Diophantine invariant tori of h . We can therefore choose an analytic domain $\Omega' \supset \overline{\Omega}$ whose boundary is one of these invariant tori.

To conclude, let us now define the vector field

$$B_{\text{ext}}(x) := h(x) \mathbf{1}_{\Omega' \setminus \overline{\Omega}}(x)$$

for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. As h is divergence free in $\Omega' \setminus \overline{\Omega}$ and tangent to $\partial\Omega$ and $\partial\Omega'$, an elementary computation shows that $\operatorname{div} B_{\text{ext}} = 0$ in the sense of distributions. Furthermore, the corresponding current is readily computed using that

$$\int_{\mathbb{R}^3 \setminus \overline{\Omega}} B_{\text{ext}} \cdot \operatorname{curl} v \, dx = \int_{\partial\Omega'} v \cdot J \, dS$$

for any $v \in C_c^\infty(\mathbb{R}^3 \setminus \overline{\Omega}, \mathbb{R}^3)$, where

$$J := h \times N'$$

and N' is the outer unit normal on $\partial\Omega'$. Therefore, $(B_{\text{ext}}, J, \Omega')$ satisfies the system (4.2), so Theorem 1.3 follows.

Remark 4.2. Quantitative versions of the Cauchy–Kovalevskaya theorem (Theorem B.1) and the KAM theorem (Theorem 6.2) provide an estimate for the separation that we can obtain between the current sheet $\partial\Omega'$ and the domain Ω in terms of the Diophantine constants of the frequency vector ω and of the analyticity radii and the analytic norms of the different objects that appear in the construction (namely, the Beltrami field $B|_{\partial\Omega}$ and the linearizing embedding K of the invariant torus $\partial\Omega$). However, this is a hard job that will be the object of future contributions.

5. Solving a Hamilton–Jacobi problem via the cohomological equation

Let Ω be an analytic nondegenerate toroidal domain of type I in \mathbb{R}^3 . By definition, there exists a Beltrami field B in Ω that satisfies the equation

$$\text{curl } B = \lambda B$$

for some constant λ , $\partial\Omega$ is a Diophantine invariant torus of B , and the corresponding matrix M defined in (3.1) is invertible. Arguing as at the beginning of Section 3.2 we infer that B is analytic up to the boundary and there are analytic coordinates $\varphi : \partial\Omega \rightarrow \mathbb{T}^2$ such that

$$Y := B|_{\partial\Omega} = \omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2},$$

where $\omega \in \mathbb{R}^2$ is a Diophantine frequency vector. With a slight abuse of notation, in this section we will use the same name for a quantity on $\partial\Omega$ (a function or a vector field) and for its expression in these coordinates.

In this section, if Z_1, Z_2 are two vector fields on \mathbb{T}^2 , $|Z_1|$ and $Z_1 \cdot Z_2$ denote the norm of Z_1 and the scalar product of Z_1 and Z_2 , respectively, computed with respect to the metric on $\partial\Omega$ induced by the Euclidean metric, which we write in the coordinates φ . As before, $[f]_{\mathbb{T}^2}$ will denote the mean of a function f in φ -coordinates, i.e.,

$$[f]_{\mathbb{T}^2} := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f(\varphi) \, d\varphi$$

with $d\varphi := d\varphi_1 \, d\varphi_2$.

The main result of this section is the following theorem. This result is used in the proof of Theorem 1.1 to construct Cauchy data which satisfy the constraint equation (2.1) appearing in the Cauchy–Kovalevskaya theorem for the curl operator.

Theorem 5.1. *Let Ω and Y be as above. Then, for any constant c with small enough absolute value, there is a nonnegative constant b and an analytic vector field X on $\partial\Omega$ of the form*

$$X = (1 + c)^{1/2} Y + \nabla H + a_1 \nabla \varphi_1 + a_2 \nabla \varphi_2,$$

where ∇ denotes the gradient operator on $\partial\Omega$ associated with the induced metric, such that

- (i) $|X|^2 = (1 + b)|Y|^2 + c$,
- (ii) X is analytically conjugate to a linear field with Diophantine frequency vector $\omega' = (1 + c)^{1/2} \omega$,

(iii) for any fixed r , the scalar analytic function H on $\partial\Omega$ and the constants b, a_1, a_2 are bounded as

$$\|H\|_{C^r(\partial\Omega)} + |b| + |a_1| + |a_2| \leq C|c|.$$

Moreover, X depends continuously on the parameter c in the C^r topology of vector fields, for any fixed r .

Remark 5.2. We are working in the analytic category because we need analytic solutions to apply the Cauchy–Kovalevskaya theorem in the proof of Theorem 1.1. In this category, we can prove Theorem 5.1 using a quadratic Newton scheme. Using instead a Nash–Moser iteration, one can prove a completely analogous result in the C^r setting, for large enough r .

Remark 5.3. The expression of X guarantees that the dual 1-form of X (computed with respect to the metric induced from \mathbb{R}^3), which we denote by X^b , is closed: $dX^b = 0$. This condition is also required to apply the Cauchy–Kovalevskaya theorem. We also observe that the introduction of nonzero constants a_1, a_2 in the expression of X is important to prove Lemma 5.4 below, where they are crucial to ensure the solvability of certain cohomological equations that appear in the Newton’s iterative scheme.

Proof of Theorem 5.1. As in Section 3, we can rescale B so that

$$[|Y|^2]_{\mathbb{T}^2} = 1. \tag{5.1}$$

In coordinates φ , condition (i) for the vector field X is equivalent to picking the analytic function $H : \mathbb{T}^2 \rightarrow \mathbb{R}$ and the constants $a = (a_1, a_2)$ so that the equation

$$2(1 + c)^{1/2}L_\omega H + |\nabla H|^2 + 2\nabla H \cdot (a\nabla\varphi) + f = b|Y|^2 + c(1 - |Y|^2) \tag{5.2}$$

is satisfied. Here $L_\omega \equiv \omega_1\partial_{\varphi_1} + \omega_2\partial_{\varphi_2}$ is the derivative in the ω direction and we have set

$$\begin{aligned} f &:= (a\nabla\varphi)^2 + 2(1 + c)^{1/2}(a\omega), \\ b &:= [|\nabla H|^2 + 2\nabla H \cdot (a\nabla\varphi) + f]_{\mathbb{T}^2}. \end{aligned}$$

Throughout this proof, we use the shorthand notation

$$\begin{aligned} a\nabla\varphi &:= a_1\nabla\varphi_1 + a_2\nabla\varphi_2, \\ a\omega &:= a_1\omega_1 + a_2\omega_2. \end{aligned}$$

To study the existence of solutions to (5.2) it is convenient to define the nonlinear operator

$$T_c(H, a) := 2(1 + c)^{1/2}L_\omega H + |\nabla H|^2 + 2\nabla H \cdot (a\nabla\varphi) + f - b|Y|^2 - c(1 - |Y|^2).$$

The definition of the constant b ensures that $[T_c(H, a)]_{\mathbb{T}^2} = 0$.

To study the operator T_c we will make use of the Banach space $\dot{\mathcal{H}}_\rho$ of holomorphic functions H on the complex strip

$$\Delta(\rho) := \{\varphi : \operatorname{Re} \varphi \in \mathbb{T}^2, |\operatorname{Im} \varphi| < \rho\} \tag{5.3}$$

that have zero mean on $\{\operatorname{Im} \varphi = 0\}$ (i.e., $[H]_{\mathbb{T}^2} = 0$). This space is endowed with the supremum norm $\|H\|_\rho := \sup_{\varphi \in \Delta(\rho)} |H(\varphi)|$. We will also denote by $\dot{\mathcal{H}}_\rho$ an analogous space of vector or matrix-valued functions. With some abuse of notation, we still use the notation φ for the complexification of the toroidal coordinates φ , and the same name for a function (vector or matrix) on \mathbb{T}^2 and its complexification on $\Delta(\rho)$. Since the induced metric on $\partial\Omega$ is analytic and Y is also an analytic vector field, it then follows that there is some $\rho_0 > 0$ such that T_c defines a map

$$T_c : \dot{\mathcal{H}}_\rho \rightarrow \dot{\mathcal{H}}_\rho$$

for all $\rho < \rho_0$.

To solve the equation

$$T_c(H, a) = 0, \tag{5.4}$$

we will crucially use the additional requirement that X is analytically conjugate to the linear field $(1 + c)^{1/2}\omega$. More precisely, let us consider the equation

$$\Phi^* \left((1 + c)^{1/2}\omega + \nabla H + a\nabla\varphi \right) - (1 + c)^{1/2}\omega = 0, \tag{5.5}$$

where $\Phi(\varphi) := \varphi + v(\varphi)$ is a diffeomorphism of \mathbb{T}^2 and $v : \mathbb{T}^2 \rightarrow \mathbb{R}^2$. We denote the LHS of this equation by $R_c(H, v, a)$.

Our goal is to find analytic solutions (H, v, a) to (5.4) and (5.5) when $|c|$ is small enough. Notice that (5.5) automatically guarantees that condition (ii) is satisfied.

To solve this equation, we apply Lemma 5.4 to the approximate solution

$$(H_0, v_0, a_0) := (0, 0, 0).$$

We shall now use the notation of this lemma without further notice. As $T_c(H_0, a_0) = -c(1 - |Y|^2)$ and $R_c(H_0, v_0, a_0) = 0$, it is clear that

$$E_0 = \|1 - |Y|^2\|_\rho |c|.$$

It is obvious that there is $c_0 > 0$ small enough such that the assumption (5.8) holds for all $|c| < c_0$ (of course, the smallness assumption on v_0 is also satisfied because $\|v_0\|_\rho = 0$). It remains to check the generic condition on the matrix $M^{(0)}$ (see (5.15)). Since $\Phi_0 = \operatorname{id}$, an easy computation shows that the columns of the 2×2 matrix $M^{(0)}$ are given by the vectors

$$[\nabla\varphi_i - \omega_i \nabla L_\omega^{-1}(1 - |Y|^2)]_{\mathbb{T}^2}$$

with $i = 1, 2$. In terms of the positive definite symmetric matrix G describing the metric on $\partial\Omega$ in the φ -coordinates, it is immediate to check that (5.15) is equivalent to

$$\det \left[G(\varphi)^{-1} \cdot \begin{pmatrix} 1 - \omega_1 \partial_{\varphi_1} \mathcal{R} & -\omega_2 \partial_{\varphi_1} \mathcal{R} \\ -\omega_1 \partial_{\varphi_2} \mathcal{R} & 1 - \omega_2 \partial_{\varphi_2} \mathcal{R} \end{pmatrix} \right]_{\mathbb{T}^2} \neq 0$$

where $\mathcal{R}(\varphi)$ is the unique zero mean solution to the equation

$$\omega_1 \partial_{\varphi_1} \mathcal{R} + \omega_2 \partial_{\varphi_2} \mathcal{R} = 1 - |Y|^2.$$

This condition is immediately satisfied, by Definition 3.1, if B is a nondegenerate Beltrami field of type I on the toroidal domain Ω .

For any $\rho' \in (0, \rho)$, we can then conclude from Lemma 5.4 that there exists a unique triple $(H, v, a) \in \mathcal{H}_{\rho'} \times \mathcal{H}_{\rho'} \times \mathbb{R}^2$ in a neighborhood of $(0, 0, 0)$ such that (5.4) and (5.5) hold provided that $|c|$ is small enough. It is clear that (H, v, a) depends continuously on c .

The bound (iii) then follows from estimate (5.9) below, the usual Cauchy estimate

$$\|H\|_{C^r(\mathbb{T}^2)} + |a| \leq C_r \|H\|_{\rho'} + |a| \leq C|c|,$$

and the obvious bound

$$|b| \leq C(|a| + |a|^2 + |a| \|H\|_{C^1(\mathbb{T}^2)} + \|H\|_{C^1(\mathbb{T}^2)}^2) \leq C|c|$$

for $|c| < 1$. ■

5.1. Existence of solutions of the Hamilton–Jacobi equation

In this section we prove the basic lemma used to establish the existence of analytic solutions to (5.4)–(5.5). To this end, note that (5.5) reads

$$R_c(H, v, a) := (1 + c)^{1/2} L_\omega v - \nabla H \circ (\text{id} + v) - (a \cdot \nabla \varphi) \circ (\text{id} + v) = 0. \quad (5.6)$$

Here and in what follows, when the operator L_ω acts on vector- or matrix-valued functions, its action is understood componentwise.

To solve the system

$$T_c(H, a) = 0, \quad R_c(H, v, a) = 0, \quad (5.7)$$

we shall use Newton’s quadratic scheme and the Rüssmann estimates for analytic solutions to cohomological equations [28]. We recall that the constants $\gamma > 0$ and $\tau > 1$ in the proof appear in the definition of the Diophantine vector ω , and that one can assume $\gamma \leq 1$ without any loss of generality.

Lemma 5.4. *Take $c \in [-1/2, 1/2]$ and consider a triple $(H_0, v_0, a_0) \in (\mathcal{H}_\rho, \mathcal{H}_\rho, \mathbb{R}^2)$. For any $\rho' \in (0, \rho)$, if $\|v_0\|_\rho$ and*

$$E_0 := \|T_c(H_0, a_0)\|_\rho + \|R_c(H_0, v_0, a_0)\|_\rho \quad (5.8)$$

are smaller than a certain constant $\varepsilon_0 > 0$ that depends on ρ' but not on c , and if the approximate solution (H_0, a_0, v_0) satisfies the generic assumption given by (5.15) below, then there exists a unique solution $(H, v, a) \in \mathcal{H}_{\rho'} \times \mathcal{H}_{\rho'} \times \mathbb{R}^2$ to (5.4) and (5.6) (or equivalently (5.7)) satisfying

$$\|H - H_0\|_{\rho'} + \|v - v_0\|_{\rho'} + |a - a_0| < CE_0. \quad (5.9)$$

Proof. To set a quadratic Newton iteration, we introduce corrections (ξ, η, α) to the approximate solution so that

$$(H_1, v_1, a_1) := (H_0, v_0, a_0) + (\xi, \eta, \alpha)$$

is a solution to the equations modulo a quadratic error, which is bounded by CE_0^2 (precise estimates will be shown later). We also take a constant b_0 ensuring that $[T_c(H_0, a_0)]_{\mathbb{T}^2} = 0$, and introduce the correction

$$b_1 := b_0 + \beta,$$

where the constant β will be fixed later.

Setting $E_H^0 := T_c(H_0, a_0)$ and $E_v^0 := R_c(H_0, v_0, a_0)$, we then obtain (ξ, η, α) as solutions to the linearized equations

$$2X_0 \cdot \nabla \xi + 2\nabla H_0 \cdot (\alpha \nabla \varphi) + 2(a_0 \nabla \varphi) \cdot (\alpha \nabla \varphi) + 2(1+c)^{1/2}(\alpha \omega) - \beta |Y|^2 = -E_H^0 \quad (5.10)$$

and

$$(1+c)^{1/2}L_\omega \eta - (D(\nabla H_0) \circ (\text{id} + v_0))\eta - (\alpha \nabla \varphi) \circ (\text{id} + v_0) - (D(a_0 \nabla \varphi) \circ (\text{id} + v_0))\eta = -E_v^0 + \nabla \xi \circ (\text{id} + v_0). \quad (5.11)$$

In (5.10), the vector field X_0 is defined as

$$X_0 := (1+c)^{1/2}\omega + \nabla H_0 + a_0 \nabla \varphi,$$

and the constant β will be chosen later to ensure the solvability of the equation (that is, so that a certain zero mean condition holds). In (5.11), the symbol D is used to denote the Jacobian matrix of a vector field.

Taking the pullback of (5.10) with the diffeomorphism $\Phi_0 := \text{id} + v_0$, defining the function $\hat{\xi} := \xi \circ \Phi_0$, and using

$$\Phi_0^* X_0 = (1+c)^{1/2}\omega - (I + Dv_0)^{-1} E_v^0,$$

we can rewrite (5.10) as

$$2(1+c)^{1/2}L_\omega \hat{\xi} - 2((I + Dv_0)^{-1} E_v^0) \hat{\xi} = \Phi_0^* (-E_H^0 + \beta |Y|^2 - 2\nabla H_0 \cdot (\alpha \nabla \varphi) - 2(a_0 \nabla \varphi) \cdot (\alpha \nabla \varphi) - 2(1+c)^{1/2}(\alpha \omega)).$$

In this equation, I denotes the 2×2 identity matrix. We also observe that if $\|v_0\|_\rho$ is small enough, the matrix $I + Dv_0$ is invertible. The second summand, which denotes the action of the vector field $-2(I + Dv_0)^{-1} E_v^0$ (understood as a first order differential operator) on the function $\hat{\xi}$, is in fact a quadratic term; precise estimates will be given below. Thus,

we can drop this term and consider the following equation:

$$\begin{aligned}
 & 2(1+c)^{1/2}L_\omega\hat{\xi} \\
 & = \Phi_0^*(-E_H^0 + \beta|Y|^2 - 2\nabla H_0 \cdot (\alpha\nabla\varphi) - 2(a_0\nabla\varphi) \cdot (\alpha\nabla\varphi) - 2(1+c)^{1/2}(\alpha\omega)). \quad (5.12)
 \end{aligned}$$

Following Zehnder [34], to study (5.11) we define a new function $\tilde{\eta}$ via

$$\eta =: (I + Dv_0)\tilde{\eta}.$$

Computing the Jacobian matrix of the equation that defines E_v^0 , one obtains the identity

$$\begin{aligned}
 DE_v^0 & = (1+c)^{1/2}L_\omega(Dv_0) - (D(\nabla H_0) \circ (\text{id} + v_0))(I + Dv_0) \\
 & \quad - (D(a_0\nabla\varphi) \circ (\text{id} + v_0))(I + Dv_0).
 \end{aligned}$$

Plugging this expression into (5.11), and dropping the term $(DE_v^0)\tilde{\eta}$, which is quadratic, we can write

$$L_\omega\tilde{\eta} = \frac{1}{(1+c)^{1/2}}(I + Dv)^{-1}((\alpha\nabla\varphi + \nabla\xi) \circ (\text{id} + v_0) - E_v^0). \quad (5.13)$$

Summarizing, we have changed the original linearized system of equations to the equivalent linear cohomological equations (5.12) and (5.13). If we choose the constant β in (5.12) so that

$$[\Phi_0^*(-E_H^0 + \beta|Y|^2 - 2\nabla H_0 \cdot (\alpha\nabla\varphi) - 2(a_0\nabla\varphi) \cdot (\alpha\nabla\varphi) - 2(1+c)^{1/2}(\alpha\omega))]_{\mathbb{T}^2} = 0,$$

then (5.12) admits a unique zero-mean solution ξ , depending on the constant vector α , which is of the form

$$\xi = \xi^E + (\xi_1^H + \xi_1^a + \xi_1^\omega)\alpha_1 + (\xi_2^H + \xi_2^a + \xi_2^\omega)\alpha_2.$$

Here

$$\begin{aligned}
 \xi^E & := \Phi_{0*}L_\omega^{-1}\Phi_0^*\left(\frac{\beta_0|Y|^2 - E_H^0}{2(1+c)^{1/2}}\right), \\
 \xi_i^H & := \Phi_{0*}L_\omega^{-1}\Phi_0^*\left(\frac{\beta_1^{(i)}|Y|^2 - \nabla H_0 \cdot \nabla\varphi_i}{(1+c)^{1/2}}\right), \\
 \xi_i^a & := \Phi_{0*}L_\omega^{-1}\Phi_0^*\left(\frac{\beta_2^{(i)}|Y|^2 - (a_0\nabla\varphi) \cdot \nabla\varphi_i}{(1+c)^{1/2}}\right), \\
 \xi_i^\omega & := \Phi_{0*}L_\omega^{-1}\Phi_0^*(\beta_3^{(i)}|Y|^2 - \omega_i),
 \end{aligned}$$

and the constants $\beta_0, \beta_1^{(i)}, \beta_2^{(i)}, \beta_3^{(i)}$ (with $i = 1, 2$) guarantee that all the above functions of the form $\Phi_0^*(\dots)$ have zero mean. This ensures that the action of the operator L_ω^{-1} (mapping functions of zero mean to functions of zero mean) is well defined. Note, in particular,

$$\beta = \beta_0 + (\beta_1^{(1)} + \beta_2^{(1)} + \beta_3^{(1)})\alpha_1 + (\beta_1^{(2)} + \beta_2^{(2)} + \beta_3^{(2)})\alpha_2.$$

Next, let us plug the expression for ξ in (5.13) and consider the 2×2 matrix-valued function $M^{(0)}$ whose columns are the vector fields

$$\nabla\varphi_i + \nabla(\xi_i^H + \xi_i^a + \xi_i^\omega)$$

with $i = 1, 2$. The solvability of (5.13) then follows if and only if one can pick a vector $\alpha \in \mathbb{R}^2$ such that

$$[(I + Dv_0)^{-1}M^{(0)} \circ (\text{id} + v_0)]_{\mathbb{T}^2}\alpha = [(I + Dv_0)^{-1}(E_v^0 - \nabla\xi^E \circ (\text{id} + v_0))]_{\mathbb{T}^2}. \tag{5.14}$$

This linear equation has a solution if and only if the matrix $M^{(0)}$ satisfies the invertibility condition

$$\det[(I + Dv_0)^{-1}M^{(0)} \circ (\text{id} + v_0)]_{\mathbb{T}^2} \neq 0. \tag{5.15}$$

This is the generic assumption appearing in the statement of the lemma. Note that equation (5.15) only depends on (H_0, v_0, a_0) , on the vector field Y and on the domain Ω .

We have thus proved that, fixing the constants α and β as above, there is a unique solution $(\xi, \tilde{\eta})$ to the linearized equations (5.12)–(5.13). Now let us estimate the analytic norms of these solutions to show that this scheme is indeed quadratic, and that it can be iterated because the norms of the corrected approximate solutions are uniformly bounded. For this we use Rüssmann estimates [28]: if $A \in \dot{\mathcal{H}}_\rho$ and ω is a Diophantine vector, for each $\delta > 0$ the cohomological equation $L_\omega B = A$ has a unique solution $B \in \dot{\mathcal{H}}_{\rho-\delta}$ that satisfies

$$\|B\|_{\rho-\delta} \leq C\gamma^{-1}\delta^{-\tau}\|A\|_\rho.$$

The constant C is independent of δ .

In what follows, let us fix a small constant $0 < \delta < 1$, and $\rho > \delta$ (which will measure the loss of analytic band in the iteration) and assume that $E_0 < \varepsilon_0$ (see (5.8)), where ε_0 satisfies the smallness condition

$$\varepsilon_0 \ll \gamma^6\delta^{9+6\tau}. \tag{5.16}$$

By assumption, $\|v_0\|_\rho < \varepsilon_0$, and hence (5.16) implies that $\|v_0\|_\rho \ll \delta$, which guarantees, by the Cauchy estimate, that $I + Dv_0$ is close to the unit matrix.

A straightforward computation using Rüssmann estimates and the Cauchy estimate for derivatives then implies

$$\begin{aligned} \|\xi^E\|_{\rho-\delta} &\leq C\gamma^{-1}\delta^{-\tau}\|E_H^0\|_\rho, & \|\xi_i^a\|_{\rho-\delta} &\leq C\gamma^{-1}\delta^{-\tau}|a_0|, \\ \|\xi_i^H\|_{\rho-2\delta} &\leq C\gamma^{-1}\delta^{-\tau-1}\|H_0\|_\rho, & \|\xi_i^\omega\|_{\rho-\delta} &\leq C\gamma^{-1}\delta^{-\tau}. \end{aligned}$$

Here we have used the fact that the condition $\|v_0\|_\rho \ll \delta$ ensures that the diffeomorphism Φ_0 is close to the identity, and hence the constant C can be taken independent of v_0 . The constant does depend on Y and Ω , though.

Solving (5.14) for α , one then concludes that

$$\begin{aligned} |\alpha| &\leq C(\|E_v^0\|_\rho + \gamma^{-1}\delta^{-1-\tau}\|E_H^0\|_\rho) \leq C\gamma^{-1}\delta^{-1-\tau}E_0, \\ \|\xi\|_{\rho-2\delta} &\leq C(\gamma^{-1}\delta^{-\tau}\|E_H^0\|_\rho + \gamma^{-2}\delta^{-2-2\tau}E_0) \leq C\gamma^{-2}\delta^{-2-2\tau}E_0. \end{aligned}$$

Now, solving (5.13) for $\tilde{\eta}$, and using again $\|v_0\|_\rho \ll \delta$, we readily estimate

$$\|\eta\|_{\rho-4\delta} \leq C\gamma^{-3}\delta^{-3-3\tau}E_0.$$

Analogously, the constant β in (5.12) satisfies

$$|\beta| \leq C(\|E_H^0\|_\rho + \delta^{-1}|\alpha|) \leq C\gamma^{-1}\delta^{-2-2\tau}E_0.$$

In these bounds, the constant C only depends on $\|H_0\|_\rho$, $|a_0|$, Y and Ω .

In order to show that this scheme can be iterated, let us check that the norms of the corrected approximate solution (H_1, v_1, a_1) and the constant b_1 remain uniformly bounded. Indeed, if $\|H_0\|_\rho + |a_0| + |b_0| < \varkappa_0$ for some positive constant \varkappa_0 , we can easily derive

$$\|H_1\|_{\rho-2\delta} + |a_1| + |b_1| \leq \|H_0\|_\rho + |a_0| + |b_0| + C\gamma^{-2}\delta^{-2-2\tau}E_0 < \varkappa_0,$$

where in the last inequality we have used the assumptions (5.8) and (5.16). Analogously, if E_0 is small enough (see (5.16)),

$$\begin{aligned} \|v_1\|_{\rho-4\delta} &\leq \|v_0\|_\rho + C\gamma^{-3}\delta^{-4-3\tau}E_0 \ll \delta, \\ \|(I + Dv_1) - (I + Dv_0)\|_{\rho-5\delta} &= \|D\eta\|_{\rho-5\delta} \leq C\gamma^{-3}\delta^{-4-3\tau}E_0 \ll 1. \end{aligned}$$

The generic assumption (5.15) is then satisfied in the iteration because

$$\|M^{(1)} - M^{(0)}\|_{\rho-5\delta} \leq C\gamma^{-3}\delta^{-4-3\tau}E_0 \ll 1.$$

Therefore, if δ is small enough,

$$\begin{aligned} &|\det [(I + Dv_1)^{-1}M^{(1)} \circ (\text{id} + v_1)]_{\mathbb{T}^2}| \\ &\geq |\det [(I + Dv_0)^{-1}M^{(0)} \circ (\text{id} + v_0)]_{\mathbb{T}^2}| - C\delta > 0 \end{aligned}$$

because $M^{(0)}$ satisfies the invertibility condition by hypothesis.

To complete the proof of the lemma, we have to check that the new errors $E_H^1 := T_c(H_1, a_1)$ and $E_v^1 := R_c(H_1, v_1, a_1)$ are quadratic with respect to the errors E_H^0 and E_v^0 . This follows from the fact that

$$\begin{aligned} E_H^1 &= (\nabla\xi)^2 + 2\nabla\xi \cdot (\alpha\nabla\varphi) + (\alpha\nabla\varphi)^2 - 2((I + Dv_0)^{-1}E_v^0)\xi \circ \Phi_0, \\ E_v^1 &= (D^2(\nabla H_0 + a_0\nabla\varphi) \circ \Phi_0)\eta \otimes \eta \\ &\quad + (D(\nabla\xi - \alpha\nabla\varphi) \circ \Phi_0)\eta + (DE_v^0(I + Dv_0)^{-1})\eta. \end{aligned}$$

A straightforward computation using the previous estimates shows

$$\|E_H^1\|_{\rho-3\delta} \leq C\gamma^{-4}\delta^{-6-4\tau}E_0^2, \quad \|E_v^1\|_{\rho-4\delta} \leq C\gamma^{-6}\delta^{-9-6\tau}E_0^2,$$

so the scheme is indeed quadratic. In particular,

$$E_1 := \|E_H^1\|_{\rho-3\delta} + \|E_v^1\|_{\rho-4\delta} \leq C\gamma^{-6}\delta^{-9-6\tau}E_0^2,$$

so the new error E_1 is smaller than ε_0 because of the smallness condition (5.16).

It is now standard that the scheme can therefore be iterated to yield a unique solution $(H, v, a) \in \mathcal{H}_{\rho'} \times \dot{\mathcal{H}}_{\rho'} \times \mathbb{R}^2$ to (5.4) and (5.6) that is bounded as in (5.9) with $\rho' := \rho - 8\delta$. This is an easy consequence of the previous estimates and the following well known lemma:

Lemma 5.5. *Let $\{E_n\}_{n=0}^\infty$ be a sequence of positive real numbers such that*

$$E_{n+1} \leq C \gamma^{-a} \delta_n^{-b-a\tau} E_n^2$$

for some constant $C > 0$, positive reals a, b , and small constant δ_0 , with $\delta_{n+1} := \delta_n/2$. Then $E_n \rightarrow 0$ as $n \rightarrow \infty$ provided that $E_0 \ll \gamma^a \delta^{b+a\tau}$.

Lemma 5.4 is thus proven. ■

6. A KAM theorem adapted to divergence-free vector fields

In this section we prove a KAM theorem that applies to divergence-free vector fields in \mathbb{R}^3 that are not necessarily close to integrable ones or in Birkhoff normal form. Another key technical advantage of this result is that the twist condition is written in terms of the (approximately) invariant torus and the vector field itself, so one does not need any fine information about the trajectories of the vector field. The proof follows the parametrization method as presented in [11] in the context of Hamiltonian systems.

It is convenient to study the reducibility properties of the invariant tori using embeddings of $\mathbb{T}^2 \rightarrow \mathbb{R}^3$, denoting by φ the natural coordinates on the torus. As before, in this section D denotes the Jacobian matrix of a vector-valued function and $|\cdot|$ stands for its norm computed with the induced metric.

6.1. Statement of results

Let B be an analytic divergence-free vector field in \mathbb{R}^3 and let $\omega \in \mathbb{R}^2$ be a frequency vector satisfying the Diophantine condition (1.5). Let $K : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ be an analytic embedding, which parametrizes a torus $\mathcal{T} := K(\mathbb{T}^2) \subset \mathbb{R}^3$ that we will eventually assume to be approximately invariant under B with frequency ω . The associated error, or defect, of invariance is measured using the function

$$E(\varphi) := L_\omega K(\varphi) - B(K(\varphi)), \tag{6.1}$$

where, as before, $L_\omega K(\varphi) := DK(\varphi)\omega$ is the derivative of K in the direction of ω . We still denote by

$$G_K(\varphi) := DK(\varphi)^\top DK(\varphi) \tag{6.2}$$

the matrix representation of the pullback to \mathbb{T}^2 of the Euclidean metric. It is obviously a nonsingular 2×2 matrix because K is an embedding. As K and B are analytic, they can be analytically continued to a complex strip that we call $\Delta(\rho)$ of the form (5.3).

An important ingredient of a KAM theorem is its twist condition, which in this case we define in terms of the embedding K and the vector field B as follows:

Definition 6.1. The embedding $K : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ is *twist* for the vector field B if the twist constant

$$T := [A_1]_{\mathbb{T}^2} \omega_2 - [A_2]_{\mathbb{T}^2} \omega_1 \equiv [A]_{\mathbb{T}^2} \cdot \omega^\perp$$

is nonzero. Here the 2×1 matrix A is defined as

$$A(\varphi) := -\frac{G_K(\varphi)^{-1} DK(\varphi)^\top [DB(K(\varphi))^\top + DB(K(\varphi))]n(\varphi)}{|n(\varphi)|^2} \tag{6.3}$$

and n is the vector field normal to the torus \mathcal{T} given by

$$n(\varphi) := \partial_{\varphi_1} K(\varphi) \times \partial_{\varphi_2} K(\varphi). \tag{6.4}$$

The main result of this section, which is of interest in itself, is the following KAM theorem. We recall that two tori are said to be η -close if there is a diffeomorphism Ψ of \mathbb{R}^3 mapping one into the other and such that $\|\Psi - \text{id}\|_{C^r} < \eta$, where $r \geq 4$ is any fixed integer.

Theorem 6.2. *Let B , ω , ρ and K be as above. Assume that the embedding K is twist with respect to the vector field B . If the invariance error $\|E\|_\rho$ is small enough, then there is a constant λ_* and an analytic Diophantine invariant torus \mathcal{T}_* of B with frequency vector $\omega_* := (1 + \lambda_*)\omega$. Furthermore, $|\lambda_*| \leq C\|E\|_\rho$ and the torus \mathcal{T}_* is $(C\|E\|_\rho)$ -close to $\mathcal{T} := K(\mathbb{T}^2)$.*

The following corollary, which is a standard consequence of Theorem 6.2, was employed in the proof of Theorem 1.1.

Corollary 6.3. *Assume that the (analytic, divergence-free) vector field B has an invariant torus \mathcal{T} with Diophantine frequency vector ω . If \mathcal{T} is twist (in the sense of Definition 6.1), then it is accumulated (in both connected components of the complement $\mathbb{R}^3 \setminus \mathcal{T}$) by a set of Diophantine analytic invariant tori of B .*

Proof. Simply apply Theorem 6.2 to the triple (B, ω', K) , with a Diophantine frequency vector $\omega' \neq \omega$ which is very close to ω and has the same Diophantine constants (γ, τ) . This ensures the invariance error is as small as one wishes. The accumulation property of the statement follows from the fact that the set of Diophantine numbers with fixed (γ, τ) has positive measure in any neighborhood of ω . The invariant torus \mathcal{T}' with frequency vector $(1 + \lambda')\omega'$ that one obtains with Theorem 6.2 lies in the exterior component of $\mathbb{R}^3 \setminus \mathcal{T}$ or in the interior one depending on whether ω'_2/ω'_1 is smaller or greater than ω_2/ω_1 (because of the twist condition). ■

To prove Theorem 6.2 we will iteratively correct the embedding and the frequency vector by means of the Newton method. Denoting the corrected quantities by

$$\bar{K}(\varphi) := K(\varphi) + \Delta_K(\varphi), \quad \bar{\omega} := \omega + \delta_\omega,$$

one is led to choose $\Delta_K(\varphi)$ as a solution of the linearized equation

$$\mathcal{R}(\Delta_K(\varphi)) := L_\omega \Delta_K(\varphi) - DB(K(\varphi))\Delta_K(\varphi) = -E(\varphi) - DK(\varphi)\delta_\omega. \tag{6.5}$$

Our next goal is to analyze this equation, which will involve developing a geometric setting in which the analytic properties of the equation are laid bare. The most efficient way to do this is by first considering the (trivial) case where \mathcal{T} is an actual invariant torus of B . This case will be considered in the next subsection. Subsequently, in Section 6.3 we will refine this approach to deal with the approximately invariant case, which will enable us to prove Theorem 6.2.

6.2. Geometric study of the invariant case

In this subsection we shall assume that \mathcal{T} is an invariant torus of the (divergence-free) vector field B with frequency vector $\omega \in \mathbb{R}^2$, which we parametrize by the map $K : \mathbb{T}^2 \rightarrow \mathbb{R}^3$. The invariance equation reads

$$L_\omega K(\varphi) = B(K(\varphi)). \tag{6.6}$$

The columns of the matrix DK are obviously a basis of the tangent space $T_{K(\varphi)}\mathcal{T}$. With $n(\varphi)$ being the normal vector (6.4), the key geometric observation is that the frame

$$\left(DK(\varphi), \frac{n(\varphi)}{|n(\varphi)|^2} \right) \tag{6.7}$$

greatly simplifies the analysis of the operator \mathcal{R} . More precisely, in this frame, the linear operator \mathcal{R} has a triangular structure that reduces the study of equation (6.5) to that of two cohomological equations with constant coefficients.

To prove this, we start by computing the action of the operator \mathcal{R} on the frame (6.7). Firstly, by taking derivatives of both sides of (6.6), we obtain

$$L_\omega DK(\varphi) = DB(K(\varphi))DK(\varphi),$$

which implies that $\mathcal{R}(DK) = 0$. To compute $\mathcal{R}(n(\varphi)/|n(\varphi)|^2)$, we use the following lemma:

Lemma 6.4. *If equation (6.6) is satisfied, then*

$$L_\omega n(\varphi) = -DB(K(\varphi))^\top n(\varphi).$$

Proof. A direct computation yields

$$\begin{aligned} L_\omega n(\varphi) &= L_\omega(\partial_{\varphi_1} K(\varphi)) \times \partial_{\varphi_2} K(\varphi) + \partial_{\varphi_1} K(\varphi) \times L_\omega(\partial_{\varphi_2} K(\varphi)) \\ &= (DB(K(\varphi))\partial_{\varphi_1} K(\varphi) \times \partial_{\varphi_2} K(\varphi) + \partial_{\varphi_1} K(\varphi) \times (DB(K(\varphi))\partial_{\varphi_2} K(\varphi)). \end{aligned}$$

Then the elementary identity

$$(Uv_1) \times v_2 + v_1 \times (Uv_2) + U^\top(v_1 \times v_2) = 0$$

holds if U is a traceless 3×3 matrix and v_1, v_2 are 3×1 vectors. If we take U to be $DB(K(\varphi))$, and note that DB has zero trace because B is divergence-free, the lemma follows. ■

We are ready to compute the action of the linear operator \mathcal{R} on the normal vector:

$$\begin{aligned} \mathcal{R}\left(\frac{n(\varphi)}{|n(\varphi)|^2}\right) &= L_\omega\left(\frac{n(\varphi)}{|n(\varphi)|^2}\right) - \frac{DB(K(\varphi))n(\varphi)}{|n(\varphi)|^2} \\ &= \frac{L_\omega n(\varphi)}{|n(\varphi)|^2} + n(\varphi)L_\omega[|n(\varphi)|^2]^{-1} - \frac{DB(K(\varphi))n(\varphi)}{|n(\varphi)|^2} \\ &= \frac{L_\omega n(\varphi) - DB(K(\varphi))n(\varphi)}{|n(\varphi)|^2} - \frac{n(\varphi)(L_\omega n(\varphi)^\top n(\varphi) + n(\varphi)^\top L_\omega n(\varphi))}{|n(\varphi)|^4} \\ &= -\frac{[DB(K(\varphi))^\top + DB(K(\varphi))]n(\varphi)}{|n(\varphi)|^2} \\ &\quad + \frac{n(\varphi)^\top [DB(K(\varphi))^\top + DB(K(\varphi))]n(\varphi)}{|n(\varphi)|^4}n(\varphi). \end{aligned} \tag{6.8}$$

Here we have used Lemma 6.4 to pass to the last line. It is straightforward to check that this expression can be written in the frame (6.7) as

$$\mathcal{R}\left(\frac{n(\varphi)}{|n(\varphi)|^2}\right) = DK(\varphi)A(\varphi), \tag{6.9}$$

where the 2×1 matrix $A(\varphi)$ is

$$A(\varphi) = -\frac{G_K(\varphi)^{-1}DK(\varphi)^\top [DB(K(\varphi))^\top + DB(K(\varphi))]n(\varphi)}{|n(\varphi)|^2}. \tag{6.10}$$

Observe $A(\varphi)$ is just as in (6.3).

6.3. Proof of Theorem 6.2

Our goal now is to use the frame introduced in the previous section to analyze equation (6.5), which determines the small corrections Δ_K and δ_ω . The problem ultimately boils down to studying the inverse of the operator \mathcal{R} .

Taking derivatives in (6.1), we obtain

$$L_\omega DK(\varphi) = DB(K(\varphi))DK(\varphi) + DE(\varphi). \tag{6.11}$$

Arguing as in the proof of Lemma 6.4 but using this equation instead of (6.6), we prove the following:

Lemma 6.5. *The normal vector $n(\varphi)$ satisfies the equation*

$$L_\omega n(\varphi) = -DB(K(\varphi))^\top n(\varphi) + \partial_{\varphi_1} E(\varphi) \times \partial_{\varphi_2} K(\varphi) + \partial_{\varphi_1} K(\varphi) \times \partial_{\varphi_2} E(\varphi).$$

This allows us to compute the quantity $\mathcal{R}(n(\varphi)/|n(\varphi)|^2)$ as in (6.8):

$$\mathcal{R}\left(\frac{n(\varphi)}{|n(\varphi)|^2}\right) = DK(\varphi)(A(\varphi) + A'(\varphi)) + \frac{n(\varphi)}{|n(\varphi)|^2}b(\varphi), \tag{6.12}$$

Here $A(\varphi)$ is the vector (6.10) and

$$A'(\varphi) := \frac{G_K(\varphi)^{-1} DK(\varphi)^\top (\partial_{\varphi_1} E(\varphi) \times \partial_{\varphi_2} K(\varphi) + \partial_{\varphi_1} K(\varphi) \times \partial_{\varphi_2} E(\varphi))}{|n(\varphi)|^2}, \quad (6.13)$$

$$b(\varphi) := -\frac{(\partial_{\varphi_1} E(\varphi) \times \partial_{\varphi_2} K(\varphi) + \partial_{\varphi_1} K(\varphi) \times \partial_{\varphi_2} E(\varphi))^\top n(\varphi)}{|n(\varphi)|^2}. \quad (6.14)$$

The following lemma shows how to get an approximate solution of (6.5), modulo a quadratic error, using solutions to a pair of cohomological equations:

Lemma 6.6. *Suppose that the functions ξ_1, ξ_2 on \mathbb{T}^2 satisfy the cohomological equations*

$$L_\omega \xi_1(\varphi) + A(\varphi) \xi_2(\varphi) = -G_K(\varphi)^{-1} DK(\varphi)^\top E(\varphi) - \delta_\omega, \quad (6.15)$$

$$L_\omega \xi_2(\varphi) = -n(\varphi)^\top E(\varphi). \quad (6.16)$$

Then

$$\Delta_K(\varphi) := DK(\varphi) \xi_1(\varphi) + \frac{n(\varphi)}{|n(\varphi)|^2} \xi_2(\varphi) \quad (6.17)$$

solves equation (6.5) modulo a quadratic error. More precisely,

$$\mathcal{R}(\Delta_K(\varphi)) + E(\varphi) + DK(\varphi) \delta_\omega = DK(\varphi) E_1(\varphi) + \frac{n(\varphi)}{|n(\varphi)|^2} E_2(\varphi) \quad (6.18)$$

with

$$E_1(\varphi) := G_K(\varphi)^{-1} DK(\varphi)^\top DE(\varphi) \xi_1(\varphi) + A'(\varphi) \xi_2(\varphi), \quad (6.19)$$

$$E_2(\varphi) := n(\varphi)^\top DE(\varphi) \xi_1(\varphi) + b(\varphi) \xi_2(\varphi). \quad (6.20)$$

Remark 6.7. We observe that, at least formally, the quantities $E_1(\varphi)$ and $E_2(\varphi)$ are quadratic in $E(\varphi)$. This is obvious except, perhaps, for the presence of the constant δ_ω in (6.15). The fact that this constant is of order $[E]_{\mathbb{T}^2}$ is a consequence of the solvability condition that the average of (6.15) must be zero (see the discussion after (6.21) below).

Proof. First we compute

$$\begin{aligned} \mathcal{R}(\Delta_K(\varphi)) &= \mathcal{R}(DK(\varphi)) \xi_1(\varphi) + DK(\varphi) L_\omega \xi_1 + \mathcal{R}\left(\frac{n(\varphi)}{|n(\varphi)|^2}\right) \xi_2(\varphi) + \frac{n(\varphi) L_\omega \xi_2}{|n(\varphi)|^2} \\ &= DE(\varphi) \xi_1(\varphi) + DK(\varphi) (L_\omega \xi_1(\varphi) + (A(\varphi) + A'(\varphi)) \xi_2(\varphi)) \\ &\quad + \frac{n(\varphi) (L_\omega \xi_2(\varphi) + b(\varphi) \xi_2(\varphi))}{|n(\varphi)|^2}. \end{aligned}$$

Next, we plug this expression in (6.18) and read off the errors $E_1(\varphi)$ and $E_2(\varphi)$:

$$E_1(\varphi) = G_K(\varphi)^{-1} DK(\varphi)^\top DE(\varphi) \xi_1(\varphi) + L_\omega \xi_1(\varphi) + (A(\varphi) + A'(\varphi)) \xi_2 \\ + G_K(\varphi)^{-1} DK(\varphi)^\top E(\varphi) + \delta_\omega,$$

$$E_2(\varphi) = n(\varphi)^\top DE(\varphi) \xi_1(\varphi) + b(\varphi) \xi_2(\varphi) + L_\omega \xi_2(\varphi) + n(\varphi)^\top E(\varphi).$$

Then expressions (6.19) and (6.20) follow from (6.15) and (6.16). ■

To solve the cohomological equations (6.15) and (6.16), we need to ensure that the RHS of these equations have zero mean. In the case of (6.16), a simple computation shows that

$$\begin{aligned} [n^\top E]_{\mathbb{T}^2} &= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} n(\varphi)^\top E(\varphi) \, d\varphi \\ &= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} n(\varphi)^\top [L_\omega K(\varphi) - B(K(\varphi))] \, d\varphi \\ &= -\frac{1}{4\pi^2} \int_{\mathbb{T}^2} n(\varphi)^\top B(K(\varphi)) \, d\varphi, \end{aligned}$$

where we have used $n(\varphi)^\top L_\omega K(\varphi) = 0$. The last integral can be equivalently written as

$$\int_{\mathbb{T}^2} n(\varphi)^\top B(K(\varphi)) \, d\varphi = \int_{\mathcal{T}} N^\top B \, dS = \int_{\Omega} \operatorname{div} B \, dx = 0,$$

where $dS := |n| \, d\varphi$ is the induced area form on \mathcal{T} , $N := n/|n|$ is the unit normal and Ω is the domain in \mathbb{R}^3 bounded by \mathcal{T} . To obtain the last equality, we have also used the fact that B is divergence-free. We thus conclude that $[n^\top E]_{\mathbb{T}^2} = 0$, so there exists a unique zero mean solution $\tilde{\xi}_2$ to (6.16). The general solution to this equation is

$$\xi_2(\varphi) = \tilde{\xi}_2(\varphi) + [\xi_2]_{\mathbb{T}^2}$$

with any constant $[\xi_2]_{\mathbb{T}^2} \in \mathbb{R}$.

Plugging the expression for ξ_2 in (6.15), we find that the average of the RHS of this equation is

$$[A\tilde{\xi}_2]_{\mathbb{T}^2} + [A]_{\mathbb{T}^2} [\xi_2]_{\mathbb{T}^2} + [G_K^{-1} DK^\top E]_{\mathbb{T}^2} + \delta_\omega. \tag{6.21}$$

We now make the additional assumption that δ_ω and ω are collinear, so

$$\delta_\omega := \Lambda \omega$$

for some $\Lambda \in \mathbb{R}$. This is crucial to preserve the Diophantine properties of the frequency vector. The twist condition in Definition 6.1 then ensures that there is a unique pair $([\xi_2]_{\mathbb{T}^2}, \Lambda) \in \mathbb{R}^2$ for which the quantity (6.21) is 0. It is immediate to check that the solution $([\xi_2]_{\mathbb{T}^2}, \Lambda)$ is of order $[E]_{\mathbb{T}^2}$.

We thus obtain an expression for the corrected embedding $\bar{K}(\varphi) := K(\varphi) + \Delta_K(\varphi)$. The invariance error associated with the corrected embedding can be easily computed:

$$\begin{aligned} \bar{E}(\varphi) &:= L_{\bar{\omega}} \bar{K}(\varphi) - B(\bar{K}(\varphi)) \\ &= (\mathcal{R}(\Delta_K(\varphi)) + E(\varphi) + DK(\varphi)\delta_\omega) \\ &\quad + (B(K(\varphi)) + DB(K(\varphi))\Delta_K(\varphi) - B(K(\varphi) + \Delta_K(\varphi))) + \mathcal{O}(E^2) \\ &= \left(DK(\varphi)E_1(\varphi) + \frac{n(\varphi)}{|n(\varphi)|^2} E_2(\varphi) \right) + \mathcal{O}(E^2). \end{aligned}$$

Therefore, the new error is quadratic in $E(\varphi)$ by Lemma 6.6.

In view of the above estimates, it is now well known that, if a certain smallness assumption is satisfied, the associated quadratic scheme starting with $(K_0, \Lambda_0) := (K, 1) \in \dot{\mathcal{H}}_\rho \times \mathbb{R}^+$ converges to some $(K_*, \Lambda_*) \in \dot{\mathcal{H}}_\rho \times \mathbb{R}^+$. The embedded torus $\mathcal{T}_* := K_*(\mathbb{T}^2)$ is then a Diophantine invariant torus of B with frequency vector $\omega_* := \Lambda_*\omega$. The details are standard (see e.g. [11]), and go essentially as in Section 5.1, so we will just sketch the argument.

Rüssmann estimates for the solutions ξ_1, ξ_2 of the cohomological equations (6.15)–(6.16) ensure that, for any small $\delta > 0$,

$$\|\xi_2\|_{\rho-\delta} \leq C\gamma^{-1}\delta^{-\tau}\|E\|_\rho, \quad \|\xi_1\|_{\rho-2\delta} \leq C\gamma^{-2}\delta^{-2\tau}\|E\|_\rho.$$

The Cauchy estimate then permits us to estimate B, b, E_1, E_2 and \bar{E} ; in particular,

$$\|\bar{E}\|_{\rho-2\delta} \leq C\gamma^{-4}\delta^{-4\tau}\|E\|_\rho^2.$$

These are the estimates for each step of the Newton method, which goes as follows:

- (i) Initialize the scheme with $(K_0, \Lambda_0) := (K, 1)$.
- (ii) Given (K_n, Λ_n) , set $\omega_n := \Lambda_n\omega$ and compute the invariance error $E_n := L_{\omega_n}K_n - B \circ K_n$.
- (iii) Construct the adapted frame by computing DK_n and n_n .
- (iv) Compute the new 2×1 matrix A_n as in (6.10).
- (v) Solve the cohomological equations (6.15) and (6.16), thus obtaining $\xi_{1,n}$ and $\xi_{2,n}$. The new constant vector $\Lambda_{n+1}\omega_n$ is then obtained from (6.21).
- (vi) Compute Δ_{K_n} from (6.17) and set $K_{n+1} := K_n + \Delta_{K_n}$.
- (vii) Repeat the iteration step with $\delta_{n+1} := \delta_n/2$.

Lemma 5.5 ensures the existence of an invariant torus of B given by

$$(K_*, \omega_*) := \lim_{n \rightarrow \infty} (K_n, \Lambda_n\omega) \in \dot{\mathcal{H}}_{\rho-4\delta} \times \mathbb{R}^+$$

provided that $\|E_0\|_\rho \ll \gamma^4\delta^{4\tau}$, so the theorem follows.

7. Thin toroidal domains are generically nondegenerate

As discussed in the introduction, it is reasonable to expect that a “generic” toroidal domain satisfies both nondegeneracy conditions (see Definitions 3.1 and 4.1), but proving generic results for vectorial problems is often extremely hard. Our objective in this section is to prove an analog of this result in the class of thin toroidal domains, where one can analyze the harmonic field (and other Beltrami fields) in detail.

Given a closed smooth curve $\gamma : \mathbb{T} \rightarrow \mathbb{R}^3$, let us denote by $\Omega(\gamma, \varepsilon)$ the toroidal domain (or tube) of thickness ε defined by this curve:

$$\Omega(\gamma, \varepsilon) := \{x \in \mathbb{R}^3 : \text{dist}(x, \gamma(\mathbb{T})) < \varepsilon\}. \tag{7.1}$$

The main result of this section can then be stated as follows. When we say that the result holds for almost all small enough ε , it means that there exists some $\varepsilon_0 > 0$ and a subset $Z \subset [0, \varepsilon_0]$ of measure zero such that the result holds for all $\varepsilon \in (0, \varepsilon_0] \setminus Z$.

Proposition 7.1. *For a generic curve γ (in the sense of a dense subset in the C^r topology, for any $r \geq 3$), the toroidal domain $\Omega(\gamma, \varepsilon)$ is analytic and nondegenerate of type I and II for almost all small enough $\varepsilon > 0$.*

Proof. Theorem B.2 implies that, for a generic curve γ , the toroidal domain $\Omega_1 := \Omega(\gamma, \varepsilon)$ admits a Beltrami field B satisfying $\text{curl } B = \lambda_1 B$, for some constant $\lambda_1 = O(\varepsilon^3)$, and $\partial\Omega_1$ is an analytic Diophantine invariant torus of B . This immediately establishes property (i) in Definitions 3.1 and 4.1.

Let us next prove condition (ii) in Definition 3.1. In the following computations we shall use the formulas and results obtained in [16] without further mention. Consider the coordinates $(\alpha, r, \theta) : \Omega_1 \rightarrow \mathbb{T} \times (0, 1) \times \mathbb{T}$ that parametrize Ω_1 (we are assuming, without any loss of generality, that the length of the core curve γ is 2π); in particular, $\partial\Omega_1$ corresponds to the surface $\{r = 1\}$. In these coordinates, $Y := B|_{r=1}$ takes the form

$$Y = \partial_\alpha - \tau(\alpha)\partial_\theta + \mathcal{O}(\varepsilon),$$

where τ is the torsion of γ , and $\mathcal{O}(\varepsilon)$ stands for a quantity whose C^m norm is bounded by $C|\varepsilon|^m$, for any fixed integer m . Using the expression of the metric on $\partial\Omega_1$ induced from \mathbb{R}^3 , in coordinates (α, θ) , we can readily compute $|Y|^2$, which leads to

$$|Y|^2 = 1 + \mathcal{O}(\varepsilon). \tag{7.2}$$

Moreover, we can compute the linearizing coordinates (φ_1, φ_2) in terms of (α, θ) as

$$\varphi_1 = \alpha + \mathcal{O}(\varepsilon), \quad \varphi_2 = \theta + \int_0^\alpha \tau(s) ds - [\tau]_{\mathbb{T}^1} \alpha + \mathcal{O}(\varepsilon),$$

and hence Y takes the following form in these coordinates:

$$Y = \partial_{\varphi_1} - [\tau]_{\mathbb{T}^1} \partial_{\varphi_2} + \mathcal{O}(\varepsilon).$$

This implies that the frequency vector of Y (which is Diophantine) is given by

$$\omega_1 = 1 + \mathcal{O}(\varepsilon), \quad \omega_2 = -[\tau]_{\mathbb{T}^1} + \mathcal{O}(\varepsilon).$$

Accordingly, we infer from (7.2) that the function \mathcal{R} in Definition 3.1 solves the cohomological equation

$$(1 + \mathcal{O}(\varepsilon))\partial_{\varphi_1} \mathcal{R} + (-[\tau]_{\mathbb{T}^1} + \mathcal{O}(\varepsilon))\partial_{\varphi_2} \mathcal{R} = (1 + \mathcal{O}(\varepsilon)) - |Y|^2 = \mathcal{O}(\varepsilon).$$

It then follows that $\mathcal{R} = \mathcal{O}(\varepsilon)$.

Putting together all these computations, we obtain a matrix M (see (3.1)) of the form

$$M = \int_{\mathbb{T}^2} G_\varepsilon^{-1} \cdot \begin{pmatrix} 1 + \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & 1 + \mathcal{O}(\varepsilon) \end{pmatrix} d\varphi,$$

where G_ε is the matrix of the metric (which depends on ε) in φ -coordinates. It is obvious that M is invertible because the (inverse) metric matrix G_ε^{-1} is positive definite, which shows that Ω_1 is nondegenerate of type I. In fact, [16, Theorem 7.8] ensures that the twist constant T of the invariant torus $\partial\Omega_1$ of B_1 (see Definition 6.1) is

$$T = c_\gamma \varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{7.3}$$

where c_γ is a certain explicit constant that depends on the curve γ through its curvature and torsion but not on ε or λ . This constant is nonzero for a generic curve γ [16, Lemma 7.9].

To conclude, let us establish property (ii) in Definition 4.1. Straightforward computations using the formulas for thin tubes derived in [16] show that

$$|\alpha \cdot \omega^\perp| \leq C, \quad |n| \geq C\varepsilon,$$

for some ε -independent constant C . Here α is the \mathbb{R}^2 -valued function introduced in Definition 4.1, and not an angular coordinate. One can then use (7.3) to see that

$$T + \lambda \left[\frac{\alpha \cdot \omega^\perp}{|n|^2} \right]_{\mathbb{T}^2} = c_\gamma \varepsilon^2 + \mathcal{O}\left(\frac{\lambda}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon^3).$$

Therefore,

$$T + \lambda \left[\frac{\alpha \cdot \omega^\perp}{|n|^2} \right]_{\mathbb{T}^2} \neq 0$$

provided that the nonzero constant λ satisfies $|\lambda| < C|c_\gamma|\varepsilon^4$ for a certain positive constant C independent of ε . ■

Remark 7.2. The nondegenerate Beltrami field B in the proof of Proposition 7.1 has eigenvalue

$$\lambda_1 = \mathcal{O}(\varepsilon^3).$$

In particular, the ratio between the L^2 norms of the plasma current $\|\text{curl } B\|_{L^2}$ and the magnetic field $\|B\|_{L^2}$, in the toroidal domain $\Omega(\gamma, \varepsilon)$, is small. Since the Beltrami equation is linear, we can safely normalize the magnetic field so that $\|B\|_{L^2} = 1$, thus implying that the plasma current is small, which is a desirable feature for stellarator design [4].

As any toroidal domain is isotopic to a thin tube, we infer that there are MHD equilibria of the kind described in Theorems 1.1 and 1.3 that have arbitrary topology:

Corollary 7.3. *There exist piecewise smooth MHD equilibria with fixed or free toroidal boundaries of arbitrary topology. More precisely, given any toroidal domain $\Omega_0 \subset \mathbb{R}^3$, one can assume that the domains Ω_k of Theorem 1.1 and Ω, Ω' of Theorem 1.3 are diffeomorphic to Ω_0 .*

Proof. It is an immediate consequence of Theorems 1.1 and 1.3 and of the fact that, by Proposition 7.1, a generic thin tube is nondegenerate of type I and II. ■

8. Lipschitz continuous force-free fields with nonconstant factor

A *force-free field* in a domain Ω is a vector field B that satisfies the equations

$$\operatorname{curl} B = fB \quad \text{in } \Omega, \quad \operatorname{div} B = 0 \quad \text{in } \Omega, \quad B \cdot N = 0 \quad \text{on } \partial\Omega$$

for some scalar function f . In the context of hydrodynamics, these fields are called Beltrami fields with nonconstant factor [17, 27].

It is obvious that a force-free field satisfies the MHD equations in Ω with $P = 0$. This is used to define force-free fields of low regularity. Specifically, one says that a vector field $B \in L^2(\Omega)$ is *force-free* if

$$\int_{\Omega} [(B \otimes B) \cdot \nabla w - \frac{1}{2}|B|^2 \operatorname{div} w] dx = 0 \quad \text{and} \quad \int_{\Omega} B \cdot \nabla \phi dx = 0$$

for any vector field $w \in C_c^1(\Omega)$ and any scalar function $\phi \in C^1(\Omega)$.

The strategy of the proof of Theorem 1.1 can be readily adapted to show the existence of Lipschitz-continuous force-free fields with nonconstant factor on toroidal domains of any topology. More precisely, one can prove the following:

Theorem 8.1. *Let B_1 be a nondegenerate Beltrami field of type I with eigenvalue λ_1 on an analytic toroidal domain Ω_1 . For any $N \geq 2$ and almost all distinct constants $\{\lambda_k\}_{k=2}^N$, there exists a family $\{\Omega_k\}_{k=1}^N$ of nested analytic toroidal domains as in Theorem 1.1 and a Lipschitz continuous vector field B satisfying the equation*

$$\operatorname{curl} B = fB, \quad \operatorname{div} B = 0$$

in Ω , where the factor

$$f := \sum_{k=1}^N \lambda_k 1_{\Omega_k \setminus \overline{\Omega_{k-1}}}$$

is not constant in Ω_N .

Proof. We use the same construction as in the proof of Theorem 1.1, but we consider the particular case where $c_k := 0$ for all $k = 2, \dots, N$. This obviously implies that $b_k = 0$ as well. The effect of this choice of constants is that the vector field B_k in a neighborhood of $\partial\Omega_{k-1}$ is constructed using the Cauchy–Kovalevskaya theorem with Cauchy datum given by $B_{k-1}|_{\partial\Omega_{k-1}}$, so $\partial\Omega_{k-1}$ is no longer a discontinuity surface of the magnetic field, and that we do not need to use Theorem 5.1.

For almost all choices of the constant λ_k , the Diophantine invariant torus $\partial\Omega_{k-1}$ one obtains is twist. This ensures the existence of a family of nested toroidal domains $\{\Omega_k\}_{k=1}^N$ as above and of a weak solution (B, P) to the MHD equations with constant pressure $P = 0$. In each set $\Omega_k \setminus \overline{\Omega_{k-1}}$, B satisfies the equation $\operatorname{curl} B = \lambda_k B$ and is analytic up to the boundary; in particular, B is Lipschitz continuous on Ω_N . The plasma current $\operatorname{curl} B$ is given by Remark 2.4, and the singular terms supported on the toroidal surfaces vanish because B is continuous. This formula shows that B is in fact a force-free field with piecewise constant factor f as above. The theorem is thus proven. ■

Remark 8.2. The assumption that the matrix M is invertible that appears in the definition of nondegenerate toroidal domains of type I is not used to prove Theorem 8.1 because here we do not need Theorem 5.1. Thus, the theorem holds under the slightly weaker assumption on the domain Ω_1 that it admits a Beltrami field for which the boundary is a Diophantine invariant torus.

Remark 8.3. In view of Proposition 7.1, Theorem 8.1 implies the existence of Lipschitz-continuous force-free fields with nonconstant factor on thin tubes of any topology.

Appendix A. Analyticity up to the boundary of Beltrami fields

In this appendix we show that, if Ω is a bounded, analytic domain of \mathbb{R}^3 and B is a Beltrami field on Ω that is tangent to the boundary of the domain, then B is analytic up to the boundary. More generally, we shall prove the following:

Theorem A.1. *Let $B \in H^1(\Omega)$ be a divergence-free field satisfying*

$$\text{curl } B - \lambda B = F \quad \text{in } \Omega, \quad B \cdot N = 0 \quad \text{on } \partial\Omega$$

for some constant λ . If $F \in C^\omega(\overline{\Omega})$, then B is analytic up to the boundary of the domain.

The motivation for this result is that solutions to elliptic boundary value problems on real analytic domains and with analytic coefficients are analytic up to the boundary. While curl is not an elliptic operator, intuitively speaking it is not far from being one provided that we restrict our attention to divergence-free fields.

Perhaps the most efficient way to make this intuition precise, thereby proving Theorem A.1, is via a Dirac-type operator acting on differential forms of mixed degrees. Specifically, let

$$\Lambda^\bullet := \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3$$

be the exterior algebra of \mathbb{R}^3 , where Λ^p is the $\binom{3}{p}$ -dimensional vector space of p -forms. Λ^\bullet is then a real vector space of dimension 8.

Given a differential form $\psi \in H^1(\Omega, \Lambda^\bullet)$, we now define a Dirac-type operator as

$$\mathcal{D}\psi := d\psi + d^*\psi,$$

where d and d^* respectively denote the exterior derivative and the codifferential. \mathcal{D} is an elliptic first order differential operator satisfying $\mathcal{D}^2 = -\Delta$ (minus the Hodge Laplacian). The principal symbol of \mathcal{D} is the map $\sigma_1(\mathcal{D}) : \mathbb{R}^3 \rightarrow \mathcal{L}(\Lambda^\bullet)$ given by $\sigma_1(\mathcal{D})(\xi) = i\gamma(\xi)$, where

$$\gamma(\xi)\Psi := \xi \wedge \Psi + \star(\xi \wedge \star\Psi),$$

the wedge denotes the exterior product and \star is the Hodge star operator. Note that

$$\gamma(\xi)^2 = -|\xi|^2 I.$$

Proof of Theorem A.1. For each point $x \in \partial\Omega$ and for $p = 1, 2$, let

$$\begin{aligned} \Lambda_{x,\text{tan}}^p &:= \{\Psi \in \Lambda^p : i_{N(x)}\Psi = 0\}, \\ \Lambda_{x,\text{nor}}^1 &:= \{\Psi \in \Lambda^1 : i_\xi\Psi = 0 \text{ for all } \xi \in T_x\partial\Omega\}, \\ \Lambda_{x,\text{nor}}^2 &:= \{\Psi \in \Lambda^2 : i_\xi\Psi = 0 \text{ for some } \xi \in T_x\partial\Omega\}, \end{aligned}$$

denote the vector spaces (of dimension $3 - p$ and p , respectively) of p -forms that are tangent and normal to the boundary at the point x . Here and in what follows,

$$T_x\partial\Omega := \{\xi \in \mathbb{R}^3 : \xi \cdot N(x) = 0\}$$

is the tangent space to the boundary at the point x and $i_\xi\Psi$ denotes the interior product of the vector ξ with the form Ψ . For each $x \in \partial\Omega$, we now define the 4-dimensional vector subspaces

$$E'_x := \Lambda_{x,\text{tan}}^1 \oplus \Lambda_{x,\text{nor}}^2, \quad E''_x := \Lambda^0 \oplus \Lambda_{x,\text{nor}}^1 \oplus \Lambda_{x,\text{tan}}^2 \oplus \Lambda^3.$$

Obviously $\Lambda^\bullet = E'_x \oplus E''_x$ for any $x \in \partial\Omega$ and these subspaces are orthogonal. Let us denote the orthogonal projector $\Lambda^\bullet \rightarrow E''_x$ by P''_x and introduce the trace-type operator

$$(P''\psi)(x) := P''_x\psi(x) \in H^{1/2}(\partial\Omega, \Lambda^\bullet).$$

Then P'' maps each differential form $\psi \in H^1(\Omega, \Lambda^\bullet)$ to the projection of its boundary trace to the space of $H^{1/2}$ sections of the vector bundle E'' over the boundary whose fiber at x is E''_x . Equivalently, one can understand P'' as the action of a zeroth order pseudodifferential operator (whose principal symbol is an orthogonal projector) on the usual boundary trace $\psi|_{\partial\Omega} \in H^{1/2}(\partial\Omega, \Lambda^\bullet)$.

We are interested in the boundary value problem for the Dirac operator defined as

$$\mathcal{D}\psi - \lambda \star \psi = G \quad \text{in } \Omega, \tag{A.1a}$$

$$P''\psi = 0 \quad \text{on } \Omega. \tag{A.1b}$$

The key observation now is that, for each $x \in \partial\Omega$ and each $\xi \in T_x^*\partial\Omega$, the operator

$$a(x, \xi) := \gamma(N^b(x))\gamma(\xi) \in \mathcal{L}(\Lambda^\bullet)$$

maps E'_x onto E''_x and vice versa. It is then well known [2, Corollary 3.18] that (A.1) defines an elliptic boundary value problem. For the benefit of the reader, let us mention that this is a straightforward consequence of the fact that the principal symbol of the Calderón projector associated to the boundary value problem (A.1) is $[|\xi|I - ia(x, \xi)]/(2|\xi|)$. Since (A.1) is an elliptic boundary value problem, the theory of Boutet de Monvel [5] ensures that ψ is analytic up to the boundary if $G \in C^\omega(\bar{\Omega}, \Lambda^\bullet)$.

This implies that the vector field B is analytic up to the boundary. In order to see this, let ψ be the differential 1-form dual to B , which we can regard as an element of $H^1(\Omega, \Lambda^\bullet)$. If $G \in C^\omega(\bar{\Omega}, \Lambda^\bullet)$ denotes the 1-form dual to F , the equations $\text{div } B = 0$ and $\text{curl } B - \lambda B = F$ read, in terms of the associated forms,

$$d^*\psi = 0, \quad \star d\psi - \lambda\psi = G.$$

As $\star \star$ is the identity, we infer that ψ satisfies equation (A.1a) in Ω . Furthermore, the condition $B \cdot N = 0$ amounts to saying that the differential form ψ (which is of pure degree 1) satisfies $i_N \psi = 0$. This ensures that ψ satisfies the boundary condition (A.1b). The analytic regularity result for solutions to the elliptic problem (A.1) then implies that $B \in C^\omega(\bar{\Omega})$, as claimed. ■

Appendix B. Beltrami fields, Cauchy–Kovalevskaya and thin tubes

In this appendix we review some previous results about Beltrami fields. The first result is an analog of the Cauchy–Kovalevskaya theorem for the curl operator which was proved in [14] and which plays an important role in the proof of our main theorem. The basic idea is that even though the curl operator does not admit noncharacteristic surfaces, a variant of the Cauchy–Kovalevskaya theorem holds provided that the Cauchy datum satisfies a certain necessary constraint. In the statement, the reader should note that the second equation, $\operatorname{div} B = 0$, is a consequence of the first when $\lambda \neq 0$, so in that case it can be omitted.

Theorem B.1. *Let Σ be an embedded and oriented analytic surface of \mathbb{R}^3 . For any real constant λ , the Cauchy problem*

$$\operatorname{curl} B = \lambda B, \quad \operatorname{div} B = 0, \quad B|_\Sigma = X, \tag{B.1}$$

with X an analytic field tangent to Σ , has a unique solution in a neighborhood of Σ if and only if

$$d(j_\Sigma^* X^b) = 0, \tag{B.2}$$

where X^b is the metric dual 1-form of X , and $j_\Sigma : \Sigma \rightarrow \mathbb{R}^3$ is the embedding of Σ in \mathbb{R}^3 .

The second result concerns Beltrami fields on thin toroidal domains $\Omega(\gamma, \varepsilon)$, as defined in (7.1). The following theorem, which was proved in [13, 16], shows that in a generic thin tube $\Omega(\gamma, \varepsilon)$ there exist Beltrami fields that are tangent to the boundary and exhibit some nice “almost-integrability” properties.

Theorem B.2. *Let γ be a generic closed curve, where by “generic” we mean an open and dense set, in the C^3 topology, of smooth closed curves in \mathbb{R}^3 . For almost every small enough $\varepsilon > 0$ and almost every nonzero constant $\lambda_1 = \mathcal{O}(\varepsilon^3)$, the boundary of $\Omega(\gamma, \varepsilon)$ is a Diophantine invariant torus of a Beltrami field B , which satisfies the equation $\operatorname{curl} B = \lambda_1 B$ in $\Omega(\gamma, \varepsilon)$. Moreover, the boundary $\partial\Omega(\varepsilon, \gamma)$ can be chosen to be analytic for a dense set of curves.*

Remark B.3. In fact, the toroidal domain $\Omega(\gamma, \varepsilon)$ is covered by a set of Diophantine invariant tori \mathcal{J} of almost full measure in the sense that, for any $\delta > 0$,

$$\frac{|\mathcal{J}|}{|\Omega(\gamma, \varepsilon)|} > 1 - \delta$$

provided that ε is small enough.

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