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The holonomy inverse problem

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Abstract. Let (M, g) be a smooth Anosov Riemannian manifold and $\mathcal{C}^\#$ the set of its primitive closed geodesics. Given a Hermitian vector bundle \mathcal{E} equipped with a unitary connection $\nabla^\mathcal{E}$, we define $\mathcal{T}^\#(\mathcal{E}, \nabla^\mathcal{E})$ as the sequence of traces of holonomies of $\nabla^\mathcal{E}$ along elements of $\mathcal{C}^\#$. This descends to a homomorphism on the additive moduli space \mathbb{A} of connections up to gauge $\mathcal{T}^\# : (\mathbb{A}, \oplus) \rightarrow \ell^\infty(\mathcal{C}^\#)$, which we call the *primitive trace map*. It is the restriction of the well-known *Wilson loop* operator to primitive closed geodesics.

The main theorem of this paper shows that the primitive trace map $\mathcal{T}^\#$ is locally injective near generic points of \mathbb{A} when $\dim(M) \geq 3$. We obtain global results in some particular cases: flat bundles, direct sums of line bundles, and general bundles in negative curvature under a spectral assumption which is satisfied in particular for connections with small curvature. As a consequence of the main theorem, we also derive a spectral rigidity result for the connection Laplacian.

The proofs are based on two new ingredients: a Livšic-type theorem in hyperbolic dynamical systems showing that the cohomology class of a unitary cocycle is determined by its trace along closed primitive orbits, and a theorem relating the local geometry of \mathbb{A} to the Pollicott–Ruelle resonance near zero of a certain natural transport operator.

Keywords. Inverse problem, holonomy, hyperbolic dynamical systems, Pollicott–Ruelle resonances, microlocal analysis

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1. Introduction

1.1. Primitive trace map, local injectivity

Let (M, g) be a smooth closed Riemannian Anosov manifold such as a manifold of negative sectional curvature [2]. Recall that this means that there exists a continuous flow-invariant splitting of the tangent bundle to the unit tangent bundle $\mathcal{M} := SM$:

$$T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u,$$

such that

$$\begin{aligned} \forall t \geq 0, \forall v \in E_s, \quad |d\varphi_t(v)| &\leq C e^{-t\theta} |v|, \\ \forall t \leq 0, \forall v \in E_u, \quad |d\varphi_t(v)| &\leq C e^{-|t|\theta} |v|, \end{aligned} \tag{1.1}$$

where $(\varphi_t)_{t \in \mathbb{R}}$ is the geodesic flow on \mathcal{M} generated by the vector field X , and the constants $C, \theta > 0$ are uniform and the metric $|\cdot|$ is arbitrary.

Let $\mathcal{E} \rightarrow M$ be a smooth Hermitian vector bundle. We denote by $\mathbb{A}_{\mathcal{E}}$ the affine space of smooth unitary connections on \mathcal{E} , and $\mathbb{A}_{\mathcal{E}}$ the moduli space of connections up to gauge equivalence: a point $\alpha \in \mathbb{A}_{\mathcal{E}}$ is an orbit $\alpha := \{p^* \nabla^{\mathcal{E}} \mid p \in C^\infty(M, U(\mathcal{E}))\}$ of gauge-equivalent connections, where $\nabla^{\mathcal{E}} \in \alpha$ is arbitrary and $p^* \nabla^{\mathcal{E}}(\bullet) := p^{-1} \nabla^{\mathcal{E}}(p \bullet)$ is the pullback connection. We let $\mathcal{C} = \{c_1, c_2, \dots\}$ be the set of free homotopy classes of loops on M , which is known to be in one-to-one correspondence with closed geodesics [57]. More precisely, given $c \in \mathcal{C}$, there exists a unique closed geodesic $\gamma_g(c) \subset M$ in the class c . It will be important to set apart *primitive* and *non-primitive* homotopy classes (resp. closed geodesics): a free loop is said to be primitive if it cannot be homotoped to a certain power (≥ 2) of another free loop. The set of primitive classes defines a subset $\mathcal{C}^\# = \{c_1^\#, c_2^\#, \dots\} \subset \mathcal{C}$.

Given a class $\alpha \in \mathbb{A}_{\mathcal{E}}$, a unitary connection $\nabla^{\mathcal{E}} \in \alpha$ and an arbitrary point $x_{c^\#} \in \gamma_g(c^\#)$ (for some $c^\# \in \mathcal{C}^\#$), the parallel transport $\text{Hol}_{\nabla^{\mathcal{E}}}(c^\#) \in U(\mathcal{E}_{x_{c^\#}})$, starting at $x_{c^\#}$, with respect to $\nabla^{\mathcal{E}}$ and along $\gamma_g(c^\#)$ depends on the choice of a representative $\nabla^{\mathcal{E}} \in \alpha$ since two gauge-equivalent connections have conjugate holonomies. However, the trace does not depend on the choice of $\nabla^{\mathcal{E}} \in \alpha$, and therefore the *primitive trace map*

$$\mathcal{T}^\# : \mathbb{A}_{\mathcal{E}} \ni \alpha \mapsto (\text{Tr}(\text{Hol}_{\nabla^{\mathcal{E}}}(c_1^\#)), \text{Tr}(\text{Hol}_{\nabla^{\mathcal{E}}}(c_2^\#)), \dots) \in \ell^\infty(\mathcal{C}^\#) \tag{1.2}$$

is well-defined. Observe that the data of the primitive trace map is rather weak information: in particular, it is *not* (a priori) equivalent to the data of the conjugacy class of the holonomy along each closed geodesic (and the latter is the same as the non-primitive trace map, where one considers *all* closed geodesics). One of the main results of this paper is the following:

Theorem 1.1. *Let (M, g) be a smooth Anosov Riemannian manifold of dimension ≥ 3 and let $\mathcal{E} \rightarrow M$ be a smooth Hermitian vector bundle. Let $\alpha_0 \in \mathbb{A}_{\mathcal{E}}$ be a generic point. Then the primitive trace map is locally injective near α_0 .*

By *local injectivity*, we mean the following: there exists $N \in \mathbb{N}$ (independent of α_0) such that \mathcal{T}^\sharp is locally injective in the C^N -quotient topology on $\mathbb{A}_\mathcal{E}$. In other words, for any element $\nabla_0^\mathcal{E} \in \alpha_0$, there exists $\varepsilon > 0$ such that the following holds; if $\nabla_{1,2}^\mathcal{E}$ are two smooth unitary connections such that $\|p_i^* \nabla_i^\mathcal{E} - \nabla_0^\mathcal{E}\|_{C^N} < \varepsilon$ for some $p_i \in C^\infty(M, U(\mathcal{E}))$, and $\mathcal{T}^\sharp(\nabla_1^\mathcal{E}) = \mathcal{T}^\sharp(\nabla_2^\mathcal{E})$, then $\nabla_1^\mathcal{E}$ and $\nabla_2^\mathcal{E}$ are gauge-equivalent.

We say that a point α is *generic* if it enjoys the following two features:

- (A) α is *opaque*. By definition (see [15, Section 5]), this means that for all $\nabla^\mathcal{E} \in \alpha$, the parallel transport map along geodesics does not preserve any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$ (i.e. \mathcal{F} is preserved by parallel transport along geodesics if and only if $\mathcal{F} = \{0\}$ or $\mathcal{F} = \mathcal{E}$). This was proved to be equivalent to the fact that the Pollicott–Ruelle resonance at $z = 0$ of the operator $\mathbf{X} := \pi^* \nabla_X^{\text{End}}$ has multiplicity equal to 1, with resonant space $\mathbb{C} \cdot \mathbb{1}_\mathcal{E}$ (here $\pi : SM \rightarrow M$ is the projection and ∇^{End} is the induced connection on the endomorphism bundle; see Section 2.2 for further details).
- (B) α has *solenoidally injective generalized X-ray transform* $\Pi_1^{\text{End}(\mathcal{E})}$ on twisted 1-forms with values in $\text{End}(\mathcal{E})$. This last assumption is less easy to describe in simple geometric terms: roughly speaking, the X-ray transform is an operator of integration of symmetric m -tensors along closed geodesics. For vector-valued symmetric m -tensors, this might not be well-defined, and one needs a more general (hence more abstract) definition involving the residue at $z = 0$ of the meromorphic extension of the family $\mathbb{C} \ni z \mapsto (-\mathbf{X} - z)^{-1}$ (see Section 2.4).

It was shown in previous articles [14, 15] that in dimension $n \geq 3$, properties (A) and (B) are satisfied on an open dense subset $\omega \subset \mathbb{A}_\mathcal{E}$ with respect to the C^N -quotient topology.¹ When the reference connection α only has property (A) (this is the case for the product connection on the trivial bundle for instance), we are able to show a *weak local injectivity* result; see Theorem 5.1.

We note that the gauge class of a connection is uniquely determined by the holonomies along *all* closed loops [5, 58] and that in mathematical physics our primitive trace map \mathcal{T}^\sharp is known as the *Wilson loop operator* [6, 34, 68, 101]. In stark contrast, our Theorem 1.1 says that the *restriction to closed geodesics* of this operator already determines (locally) the gauge class of the connection.

¹More precisely, there exists $N \in \mathbb{N}$ and a subset $\Omega \subset \mathcal{A}_\mathcal{E}$ of the (affine) Fréchet space of smooth affine connections on \mathcal{E} such that $\omega = \pi_\mathcal{E}(\Omega)$ (where $\pi_\mathcal{E} : \mathcal{A}_\mathcal{E} \rightarrow \mathbb{A}_\mathcal{E}$ is the projection) and

- Ω is invariant by the action of the gauge group: $p^* \Omega = \Omega$ for all $p \in C^\infty(M, U(\mathcal{E}))$;
- Ω is open: for all $\nabla_0^\mathcal{E} \in \Omega$, there exists $\varepsilon > 0$ such that if $\nabla^\mathcal{E} \in \mathcal{A}_\mathcal{E}$ and $\|\nabla^\mathcal{E} - \nabla_0^\mathcal{E}\|_{C^N} < \varepsilon$, then $\nabla^\mathcal{E} \in \Omega$;
- Ω is dense: for all $\nabla_0^\mathcal{E} \in \mathcal{A}_\mathcal{E}$ and all $\varepsilon > 0$, there exists $\nabla^\mathcal{E} \in \Omega$ such that $\|\nabla^\mathcal{E} - \nabla_0^\mathcal{E}\|_{C^N} < \varepsilon$;
- connections in Ω satisfy properties (A) and (B).

1.2. *Global injectivity*

We now mention some global injectivity results. We let $\mathbb{A}_r := \bigsqcup_{\mathcal{E}_r \in \text{Vect}_r(M)} \mathbb{A}_{\mathcal{E}_r}$, where the disjoint union is taken over all Hermitian vector bundles $\mathcal{E}_r \in \text{Vect}_r(M)$ of rank r over M up to isomorphism, and we set

$$\mathbb{A} := \bigsqcup_{r \geq 0} \mathbb{A}_r;$$

we also let $\text{Vect}(M) = \bigsqcup_{r \geq 0} \text{Vect}_r(M)$ be the space of all topological vector bundles up to isomorphism. A point $x \in \mathbb{A}$ corresponds to a pair $([\mathcal{E}], \alpha)$, where $[\mathcal{E}] \in \text{Vect}(M)$ is an equivalence class of Hermitian vector bundles and α a class of gauge-equivalent unitary connections.²

The space \mathbb{A} has a natural monoid structure given by the \oplus -operator of taking direct sums (both for the vector bundle part and the connection part). The primitive trace map can then be seen as a *global* (monoid) homomorphism

$$\mathcal{T}^\# : \mathbb{A} \rightarrow \ell^\infty(\mathcal{C}^\#), \tag{1.3}$$

where $\ell^\infty(\mathcal{C}^\#)$ is endowed with the obvious additive structure. We actually conjecture that the “generic” assumption of Theorem 1.1 is unnecessary and that the primitive trace map (1.3) should be globally injective if $\dim(M) \geq 3$ and $\dim(M)$ is odd. Let us discuss a few partial results supporting this conjecture:

- (1) In Section 5.2.1, we show that the primitive trace map is injective when restricted to *direct sums of line bundles* when $\dim(M) \geq 3$; see Theorem 5.8. Note that it was proved by Paternain [77] that the primitive trace map restricted to line bundles, $\mathcal{T}_1^\# : \mathbb{A}_1 \rightarrow \ell^\infty(\mathcal{C}^\#)$, is injective when $\dim(M) \geq 3$.
- (2) In Section 5.2.2, we show that the primitive trace map $\mathcal{T}^\#$ restricted to *flat* connections is globally injective; see Proposition 5.9.
- (3) In Section 5.2.3, we also obtain a global result in negative curvature under an extra *spectral condition*; see Proposition 5.14. This condition is generic (see Appendix A) and is also satisfied by connections with *small curvature*, i.e. with curvature controlled by a constant depending only on the dimension and an upper bound on the sectional curvature of (M, g) ; see Lemma 5.13.
- (4) In Section 5.2.4, as a consequence of Corollary 1.4 below, we show that the primitive trace map $\mathcal{T}^\#([\mathcal{E}], \alpha)$ allows one to recover the isomorphism class $\pi^*[\mathcal{E}]$. In particular, if $\dim(M)$ is odd, this suffices to recover $[\mathcal{E}]$; see Proposition 5.15.

²Note that if two smooth Hermitian vector bundles \mathcal{E}_1 and \mathcal{E}_2 are isomorphic as topological vector bundles (i.e. there exists an invertible $p \in C^\infty(M, \text{Hom}(\mathcal{E}_1, \mathcal{E}_2))$), then they are also isomorphic as Hermitian vector bundles, that is, p can be taken unitary; the choice of Hermitian structure is therefore irrelevant.

Theorem 1.1 is inspired by earlier work on the subject; see [45,70,77–79] for instance. Nevertheless, it goes beyond the aforementioned literature thanks to an *exact Livšic co-cycle theorem* (Theorem 1.3), explained in Section 1.4. It also belongs to a more general family of *geometric inverse results* which has become a very active field of research in the past twenty years, both on closed manifolds and on manifolds with boundary; see [40,80,84,93,95,98] among other references.

Theorem 1.1 can also be compared to a similar problem called the *marked length spectrum* (MLS) rigidity conjecture, also known as the Burns–Katok [11] conjecture. The latter asserts that if (M, g) is Anosov, then the marked length spectrum

$$L_g : \mathcal{C} \rightarrow \mathbb{R}_+, \quad L_g(c) := \ell_g(\gamma_g(c)) \tag{1.4}$$

(where $\ell_g(\gamma)$ denotes the Riemannian length of the curve $\gamma \subset M$ computed with respect to the metric g), namely the length of all closed geodesics marked by the free homotopy classes of M , should determine the metric up to isometry. Despite some partial answers [7, 19, 44, 48, 54, 74], this conjecture is still widely open. Recently, Guillarmou and the second author proved a local version of the Burns–Katok conjecture [44] using techniques from microlocal analysis and the theory of Pollicott–Ruelle resonances.

1.3. Inverse spectral problem

The *length spectrum* of the Riemannian manifold (M, g) is the collection of lengths of closed geodesics *counted with multiplicities*. It is said to be *simple* if all closed geodesics have distinct lengths, and this is known to be a generic condition (with respect to the metric, even in the non-Anosov case; see [1, 3]). Given $\nabla^\varepsilon \in \alpha$, one can form the *connection Laplacian* $\Delta_{\nabla^\varepsilon} := (\nabla^\varepsilon)^* \nabla^\varepsilon$ (also known as the Bochner Laplacian), which is a differential operator of order 2, non-negative, formally self-adjoint and elliptic, acting on $C^\infty(M, \mathcal{E})$. While $\Delta_{\nabla^\varepsilon}$ depends on the choice of representative ∇^ε in the class α , its spectrum does not and there is a well-defined *spectrum map*

$$\mathcal{S} : \mathbb{A}_\mathcal{E} \ni \alpha \mapsto \text{spec}(\Delta_\alpha), \tag{1.5}$$

where $\text{spec}(\Delta_\alpha) = \{0 \leq \lambda_0(\alpha) \leq \lambda_1(\alpha) \leq \dots\}$ is the spectrum counted with multiplicities. Note that more generally, the spectrum map (1.5) can be defined on the whole moduli space \mathbb{A} (just as the primitive trace map (1.2)). The trace formula of Duistermaat–Guillemin [26, 42] applied to Δ_α reads (when the length spectrum is simple)

$$\lim_{t \rightarrow \ell(\gamma_g(c))} (t - \ell(\gamma_g(c))) \sum_{j \geq 0} e^{-i\sqrt{\lambda_j(\alpha)}t} = \frac{\ell(\gamma_g(c^\sharp)) \text{Tr}(\text{Hol}_{\nabla^\varepsilon}(c))}{2\pi |\det(\mathbb{1} - P_{\gamma_g(c)})|^{1/2}}, \tag{1.6}$$

where $\sharp : \mathcal{C} \rightarrow \mathcal{C}^\sharp$ is the operator giving the primitive orbit associated to an orbit; P_γ is the Poincaré map associated to the orbit γ and $\ell(\gamma)$ its length. Theorem 1.1 therefore has the following straightforward consequence:

Corollary 1.2. *Let (M, g) be a smooth Anosov Riemannian manifold of dimension ≥ 3 with simple length spectrum.*

- Let $\mathcal{E} \rightarrow M$ be a smooth Hermitian vector bundle and $\alpha_0 \in \mathbb{A}_{\mathcal{E}}$ be a generic point. Then the spectrum map \mathcal{S} is locally injective near α_0 .
- The spectrum map \mathcal{S} is also globally injective when restricted to cases (1)–(4) of Section 1.2.

This corollary follows easily from Theorem 1.1 by observing that under the simple length spectrum assumption, the primitive trace map can be recovered from the equality (1.6). Corollary 1.2 is analogous to the Guillemin–Kazhdan [46, 47] rigidity result in which a potential $q \in C^\infty(M)$ is recovered from the knowledge of the spectrum of $-\Delta_g + q$ (see also [20, 82]). As far as the connection Laplacian is concerned, it seems that Corollary 1.2 is the first positive result in this direction. Counter-examples were constructed by Kuwabara [60] using the Sunada method [96] but on coverings of a given Riemannian manifolds; hence the simple length spectrum condition is clearly violated. Up to our knowledge, it is also the first positive general result in an inverse spectral problem on a closed manifold of dimension > 1 with *infinite* gauge group.

This gives hope that similar methods could be used in the classical problem of recovering the isometry class of a metric from the spectrum of its Laplace–Beltrami operator *locally* (similarly to a conjecture of Sarnak for planar domains [89]). Such a result was already obtained in a neighbourhood of negatively-curved locally symmetric spaces by Sharafutdinov [91]. See also [20] for the weaker deformational spectral rigidity results or [24, 51] for recent results in the plane.

1.4. Exact Livšić cocycle theorem

The main ingredient in the proof of Theorem 1.1 is the following Livšić-type result in hyperbolic dynamical systems, which may be of independent interest. It shows that the cohomology class of a unitary cocycle over a transitive Anosov flow is determined by its trace along primitive periodic orbits. We phrase it in a somewhat more general context where we allow non-trivial vector bundles.

Theorem 1.3. *Let \mathcal{M} be a smooth manifold endowed with a smooth transitive Anosov flow $(\varphi_t)_{t \in \mathbb{R}}$. For $i \in \{1, 2\}$, let $\mathcal{E}_i \rightarrow \mathcal{M}$ be a Hermitian vector bundle equipped with a unitary connection $\nabla^{\mathcal{E}_i}$, and denote by $C_i(x, t) : (\mathcal{E}_i)_x \rightarrow (\mathcal{E}_i)_{\varphi_t(x)}$ the parallel transport along the flow lines with respect to $\nabla^{\mathcal{E}_i}$. If the connections have trace-equivalent holonomies in the sense that for all primitive periodic orbits γ , one has*

$$\text{Tr}(C_1(x_\gamma, \ell(\gamma))) = \text{Tr}(C_2(x_\gamma, \ell(\gamma))), \tag{1.7}$$

where $x_\gamma \in \gamma$ is arbitrary and $\ell(\gamma)$ is the period of γ , then there exists $p \in C^\infty(\mathcal{M}, \text{U}(\mathcal{E}_2, \mathcal{E}_1))$ such that for all $x \in \mathcal{M}$ and $t \in \mathbb{R}$,

$$C_1(x, t) = p(\varphi_t x) C_2(x, t) p(x)^{-1}. \tag{1.8}$$

In the language of dynamical systems, note that every unitary cocycle is given by parallel transport along some unitary connection and (1.8) says that the cocycles induced by

parallel transport are *cohomologous*. In particular, in the case of the trivial principal bundle $U(r) \times \mathcal{M} \rightarrow \mathcal{M}$ our theorem can be restated just in terms of $U(r)$ -cocycles. Note that the bundles \mathcal{E}_1 and \mathcal{E}_2 could be a priori distinct (and have different ranks) but Theorem 1.3 shows that they are actually isomorphic:

Corollary 1.4. *Let $\mathcal{E}_1, \mathcal{E}_2 \rightarrow \mathcal{M}$ be two Hermitian vector bundles equipped with respective unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. If the traces of the holonomy maps agree as in (1.7), then \mathcal{E}_1 and \mathcal{E}_2 are isomorphic.*

Theorem 1.3 has other geometric consequences which are further detailed in Section 3.1. Livšic-type theorems have a long history in hyperbolic dynamical systems going back to the seminal paper of Livšic [67] and appear in various settings. They were both developed in the Abelian case, i.e. for functions (see [22, 36, 38, 67, 69] for instance) and in the cocycle case.

Surprisingly, we could not locate any result such as Theorem 1.3 in the literature. The closest works (in the discrete-time case) are those of Parry [75] and Schmidt [90] which mainly inspired the proof of Theorem 1.3. Nevertheless, when considering compact Lie groups, Parry’s and Schmidt’s results seem to be weaker as they need to assume that the conjugacy classes of the cocycles agree (and not only the traces) and that a certain additional cocycle is transitive in order to derive the same conclusion. The literature is mostly concerned with the discrete-time case, namely hyperbolic diffeomorphisms: in that case, a lot of articles are devoted to studying cocycles with values in a non-compact Lie group (and sometimes satisfying a “slow-growth” assumption); see [4, 23, 53, 87, 88]. One can also wonder if Theorem 1.3 could be proved in the non-unitary setting. Other articles such as [71–73, 86, 100] seem to have been concerned with regularity assumptions on the map p , namely bootstrapping its regularity under some weak a priori assumption (such as measurability only). Let us also point out at this stage that some regularity issues will appear while proving Theorem 1.3 but they will be bypassed by the use of a recent regularity statement [39, Theorem 4.1] in hyperbolic dynamics.

1.5. Strategy of proof

We now briefly discuss the strategy of proof for Theorem 1.1. Fix a generic unitary connection $\nabla_0^\mathcal{E}$ (satisfying assumptions (A) and (B)) and pick two nearby connections $\nabla_{1,2}^\mathcal{E}$ such that

$$\mathcal{T}^\#(\nabla_1^\mathcal{E}) = \mathcal{T}^\#(\nabla_2^\mathcal{E}). \tag{1.9}$$

Our aim is to prove that there exists an isometry $p \in C^\infty(M, U(\mathcal{E}))$ such that $\nabla_2^\mathcal{E} = p^* \nabla_1^\mathcal{E} = p^{-1} \nabla_1^\mathcal{E} (p \bullet)$. Equivalently, this is the same as having

$$\nabla^{\text{Hom}(\nabla_2^\mathcal{E}, \nabla_1^\mathcal{E})} p = 0, \tag{1.10}$$

where $\nabla^{\text{Hom}(\nabla_2^\mathcal{E}, \nabla_1^\mathcal{E})}$ is the *mixed connection* (see Section 2.2.1), the natural connection induced by $\nabla_{1,2}^\mathcal{E}$ on the endomorphism bundle $\text{End}(\mathcal{E})$ over M .

- (1) *Non-Abelian Livšic theory:* The first step is to lift the connections to the unit tangent bundle $\pi : SM \rightarrow M$, namely, to consider the pullback bundle $\pi^*\mathcal{E} \rightarrow SM$ and the two pullback connections $\pi^*\nabla_{1,2}^\mathcal{E}$. Taking the parallel transport with respect to the connection $\pi^*\nabla_i^\mathcal{E}$ along a geodesic flow line $(\varphi_s(x, v))_{s \in [0,t]}$ (for $(x, v) \in SM$) yields a natural cocycle map $C_i((x, v), t) : \mathcal{E}_x \rightarrow \mathcal{E}_{\pi(\varphi_t(x,v))}$ as in Section 1.4. It is then straightforward to verify that the equality of the traces (1.9) translates into the fact that the cocycles C_1 and C_2 are trace-equivalent in the sense of (1.7). As a consequence, Theorem 1.3 implies the existence of $p \in C^\infty(SM, U(\pi^*\mathcal{E}))$ such that

$$C_1((x, v), t) = p(\varphi_t(x, v))C_2((x, v), t)p(x, v)^{-1}, \quad \forall (x, v) \in SM, \forall t \in \mathbb{R}.$$

Differentiating the previous equality with respect to time t and taking $t = 0$, one finds that this is actually equivalent to

$$(\pi^*\nabla)_X^{\text{Hom}(\pi^*\nabla_2^\mathcal{E}, \pi^*\nabla_1^\mathcal{E})} p = 0, \tag{1.11}$$

where X is the geodesic vector field. This relation is similar to (1.10) but it holds on the unit tangent bundle SM and not on the base manifold M . The main difficulty is now to show that (1.11) implies (1.10); equivalently, this is the same as showing that the isometry p does not depend on the velocity variable v in SM (only on the base variable x).

- (2) *Convexity of the leading resonance:* The idea is to use the well-established theory of Pollicott–Ruelle resonances – which allows one to define a natural spectrum for Anosov flows (that is, flows satisfying (1.1)), and more generally for transport operators over Anosov flows – and to translate (1.11) into the fact that the first order differential operator $\mathbf{X} := (\pi^*\nabla)_X^{\text{Hom}(\pi^*\nabla_2^\mathcal{E}, \pi^*\nabla_1^\mathcal{E})}$, acting on $C^\infty(SM, \pi^*\text{End}(\mathcal{E}))$, has a Pollicott–Ruelle resonance at $z = 0$. More precisely, we can write $\nabla_i^\mathcal{E} = \nabla_0^\mathcal{E} + A_i$, where $A_i \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ is small, with values in skew-Hermitian endomorphisms. Then, under the generic assumptions **(A)** and **(B)**, the operator \mathbf{X} admits a simple leading resonance λ_{A_1, A_2} which is real and non-positive. Moreover, the generic assumption also ensures that for $A_1 = A_2 = 0$, the leading resonant space is spanned by the identity section $\mathbf{1}_\mathcal{E} \in C^\infty(SM, \pi^*\text{End}(\mathcal{E}))$. The key idea is to show by a convexity argument that $|\lambda_{A_1, A_2}|$ controls *quantitatively* the distance between the connections $\nabla_1^\mathcal{E}$ and $\nabla_2^\mathcal{E}$ in the moduli space $\mathbb{A}_\mathcal{E}$. In particular, (1.11) means that the two connections satisfy $\lambda_{A_1, A_2} = 0$, and thus the connections must be gauge-equivalent.

As pointed out by the referee, the idea to use the strict convexity of the dominant Pollicott–Ruelle resonance is reminiscent of the work of Katsuda–Sunada [56] and Pollicott [85]. The main difference here is that our moduli space of unitary connections $\mathbb{A}_\mathcal{E}$ is infinite-dimensional and the quantitative strict convexity of λ_{A_1, A_2} is given in the end by some elliptic theory.

1.6. Organization of the paper

The paper is divided into three parts:

- First of all, we prove in Section 3 the *exact Livšic cocycle* Theorem 1.3 for general Anosov flows showing that the cohomology class of a unitary cocycle is determined by its trace along closed orbits. The proof is based on the introduction of a new tool which we call *Parry's free monoid*, denoted by \mathbf{G} , and is formally generated by orbits homoclinic to a fixed closed orbit. We show that any unitary connection induces a unitary representation of the monoid \mathbf{G} and that trace-equivalent connections have the same character; we can then apply tools from representation theory.
- In subsequent sections, we develop a microlocal framework, based on the theory of Pollicott–Ruelle resonances. We define a notion of *generalized X-ray transform with values in a vector bundle*, which is mainly inspired by [36, 38, 44]. In Section 4, we relate the geometry of the moduli space of gauge-equivalent connections to the leading Pollicott–Ruelle resonance of a certain natural operator, the mixed connection.
- Eventually, the main results such as Theorem 1.1 are proved in Section 5, where we also deduce the global properties of \mathcal{T}^\sharp involving line bundles, flat bundles, negatively curved base manifolds, and the topology of bundles.

Some technical preliminaries are provided in Section 2.

1.7. Perspectives

We intend to pursue this work in different directions:

- Since the first release of the present article on arXiv (May 2021), the notion of *Parry's free monoid* introduced in Section 3.3 has proved to be extremely powerful. In particular, it was used in the companion papers [17, 65] in order to show that the frame flow of nearly 0.25-pinched negatively-curved Riemannian manifolds is ergodic, thus almost answering a long-standing conjecture of Brin [10, Conjecture 2.6].
- Furthermore, in [13, Theorem 4.5] we were strikingly able to show *global* injectivity of the primitive trace map under a suitable low-rank assumption, by exhibiting a relation with real algebraic geometry.
- Moreover, in [16, Theorem 1.1] we showed a stability estimate version of Theorem 1.1, namely that the distance between connections up to gauge equivalence is controlled (at least locally) by the distance between their images under the primitive trace map \mathcal{T}^\sharp .
- Eventually, the arguments developed in Section 3 mainly rely on the use of homoclinic orbits; to these orbits, we will associate a notion of *length* which is well-defined as an element of $\mathbb{R}/T_\star\mathbb{Z}$, for some real number $T_\star > 0$. We believe that, similarly to the set of periodic orbits where one defines the *Ruelle zeta function*

$$\zeta(s) := \prod_{\gamma^\sharp \in \mathcal{G}^\sharp} (1 - e^{-s\ell(\gamma^\sharp)}),$$

where the product is taken over all primitive periodic orbits $\gamma^\# \in \mathcal{G}^\#$ (and $\ell(\gamma^\#)$ denotes the orbit period) and one shows that this extends meromorphically from $\{\text{Re } s \gg 0\}$ to \mathbb{C} (see [27, 35]), one could also define a complex function for homoclinic orbits by means of a Poincaré series (rather than a product). It should be a consequence of [99, Theorem 4.15] that this function extends meromorphically to \mathbb{C} . It might then be interesting to compute its value at 0; the latter might be independent of the choice of representatives for the length of homoclinic orbits (two representatives differ by mT_\star for some $m \in \mathbb{Z}$) and could be (at least in some particular cases) an interesting topological invariant, as is the case for the Ruelle zeta function on surfaces (see [28]).

2. Setting up the tools

2.1. Microlocal calculus and functional analysis

Let \mathcal{M} be a smooth closed manifold. Given a smooth vector bundle $\mathcal{E} \rightarrow \mathcal{M}$, we denote by $\Psi^m(\mathcal{M}, \mathcal{E})$ the space of pseudodifferential operators of order m acting on \mathcal{E} . When \mathcal{E} is the trivial line bundle, such an operator $P \in \Psi(\mathcal{M})$ can be written (up to a smoothing remainder) in local coordinates as

$$Pf(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi \cdot (x-y)} p(x, \xi) f(y) dy d\xi, \tag{2.1}$$

where f is compactly supported in the local patch and $p \in S^m(T^*\mathbb{R}^n)$ is a *symbol*, i.e. it satisfies the following estimates in local coordinates:

$$\sup_{|\alpha'| \leq \alpha, |\beta'| \leq \beta} \sup_{(x, \xi) \in T^*\mathbb{R}^n} \langle \xi \rangle^{m-|\alpha'|} |\partial_{\xi}^{\alpha'} \partial_x^{\beta'} p(x, \xi)| < \infty \tag{2.2}$$

for all $\alpha, \beta \in \mathbb{N}^n$, with $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. When \mathcal{E} is not the trivial line bundle, the symbol p is an $\text{End}(\mathcal{E})$ -valued symbol, which in local coordinates and local trivializations is identified with a matrix function. Given $p \in S^m(T^*\mathcal{M})$, one can define a (non-canonical) *quantization procedure* $\text{Op} : S^m(T^*\mathcal{M}) \rightarrow \Psi^m(\mathcal{M})$ thanks to (2.1) in coordinate patches. This also works more generally with a vector bundle $\mathcal{E} \rightarrow \mathcal{M}$ and one has a quantization map $\text{Op} : S^m(T^*\mathcal{M}, \text{End}(\mathcal{E})) \rightarrow \Psi^m(\mathcal{M}, \mathcal{E})$ (note that the symbol is then a section of the pullback bundle $\text{End}(\mathcal{E}) \rightarrow T^*\mathcal{M}$ satisfying the bounds (2.2) in local coordinates and local trivializations). There is a well-defined (partial) inverse map

$$\sigma_{\text{princ}} : \Psi(\mathcal{M}, \mathcal{E}) \rightarrow S^m(T^*\mathcal{M}, \text{End}(\mathcal{E})) / S^{m-1}(T^*\mathcal{M}, \text{End}(\mathcal{E}))$$

called the *principal symbol* and satisfying $\sigma_{\text{princ}}(\text{Op}(p)) = [p]$ (the equivalence class as an element of $S^m(T^*\mathcal{M}, \text{End}(\mathcal{E})) / S^{m-1}(T^*\mathcal{M}, \text{End}(\mathcal{E}))$).

We denote by $H^s(\mathcal{M}, \mathcal{E})$ the space of Sobolev sections of order $s \in \mathbb{R}$ and by $C_*^s(\mathcal{M}, \mathcal{E})$ the space of Hölder–Zygmund sections of order $s \in \mathbb{R}$. It is well-known that for $s \in \mathbb{R}_+ \setminus \mathbb{N}$, C_*^s coincides with the space C^s of Hölder-continuous section of order s . Recall that C_*^s is an algebra as long as $s > 0$ and H^s is an algebra for $s > n/2$. If $P \in \Psi^m(\mathcal{M}, \mathcal{E})$ is a pseudodifferential operator of order $m \in \mathbb{R}$, then for $X = H, C_*, P : X^{s+m}(\mathcal{M}, \mathcal{E}) \rightarrow X^s(\mathcal{M}, \mathcal{E})$, is bounded. We refer to [92, 97] for further details.

2.2. Connections on vector bundles

We refer the reader to [25, Chapter 2] for the background on connections on vector bundles.

2.2.1. *Mixed connection on the homomorphism bundle.* In this subsection, we consider two Hermitian vector bundles $\mathcal{E}_1, \mathcal{E}_2 \rightarrow \mathcal{M}$ equipped with respective unitary connections $\nabla_1 = \nabla^{\mathcal{E}_1}$ and $\nabla_2 = \nabla^{\mathcal{E}_2}$ which can be written in some local patch $U \subset \mathbb{R}^n$ of coordinates and in local trivializations of the bundles as $\nabla^{\mathcal{E}_i} = d + \Gamma_i$ for some $\Gamma_i \in C^\infty(U, T^*U \otimes \text{End}_{\text{sk}}(\mathcal{E}_i))$. Let $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ be the vector bundle of homomorphisms from \mathcal{E}_1 to \mathcal{E}_2 , endowed with the natural Hermitian structure.

Definition 2.1. We define the (unitary) *homomorphism connection*, or *mixed connection*, $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$ on $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$, induced by $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$, by the Leibniz property:

$$\forall u \in C^\infty(\mathcal{M}, \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)), \forall s \in C^\infty(\mathcal{M}, \mathcal{E}_1),$$

$$\nabla^{\mathcal{E}_2}(us) = (\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}u) \cdot s + u \cdot (\nabla^{\mathcal{E}_1}s).$$

Equivalently, it is straightforward to check that this is the canonical tensor product connection induced on $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \cong \mathcal{E}_2 \otimes \mathcal{E}_1^*$ and that in local coordinates and local trivializations we have

$$\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}u := du + \Gamma_2(\bullet)u - u\Gamma_1(\bullet). \tag{2.3}$$

Note that this definition does not require the bundles to have the same rank; we stress that the mixed connection $\nabla^{\text{Hom}(\nabla_1, \nabla_2)}$ depends on the choice of connections ∇_1 and ∇_2 . When $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ and $\nabla_1^{\mathcal{E}} = \nabla_2^{\mathcal{E}} = \nabla^{\mathcal{E}}$ we will write $\nabla^{\text{End}(\nabla^{\mathcal{E}})} = \nabla^{\text{Hom}(\nabla^{\mathcal{E}}, \nabla^{\mathcal{E}})}$ for the induced *endomorphism* connection on $\text{End}(\mathcal{E})$. When clear from the context, we will also write $\nabla^{\text{End}(\mathcal{E})}$ for the endomorphism connection induced by $\nabla^{\mathcal{E}}$. The homomorphism and endomorphism connections will play a central role in the upcoming sections.

Given a flow $(\varphi_t)_{t \in \mathbb{R}}$, we will denote by

$$P(x, t) : \text{Hom}(\mathcal{E}_{1_x}, \mathcal{E}_{2_x}) \rightarrow \text{Hom}(\mathcal{E}_{1_{\varphi_t(x)}}, \mathcal{E}_{2_{\varphi_t(x)}})$$

the parallel transport with respect to the mixed connection along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$. This parallel transport has a clear geometric meaning: in fact, for $u \in \text{Hom}(\mathcal{E}_{1_x}, \mathcal{E}_{2_x})$, we have

$$P(x, t)u = C_2(x, t)uC_1(x, t)^{-1}, \tag{2.4}$$

where $C_i(x, t) : \mathcal{E}_{ix} \rightarrow \mathcal{E}_{i_{\varphi_t(x)}}$ is the parallel transport with respect to $\nabla^{\mathcal{E}_i}$ along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$.

Recall that the curvature tensor $F_\nabla \in C^\infty(\mathcal{M}, \Lambda^2(T^*\mathcal{M}) \otimes \text{End}(\mathcal{E}))$ of $\nabla = \nabla^{\mathcal{E}}$ is defined, for any vector fields X, Y on \mathcal{M} and sections S of \mathcal{E} , as

$$F_\nabla(X, Y)S = \nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{[X, Y]}S.$$

Then a quick computation using (2.3) reveals that

$$F_{\nabla^{\text{Hom}(\nabla_1, \nabla_2)}} u = F_{\nabla_2} \cdot u - u \cdot F_{\nabla_1}. \tag{2.5}$$

Eventually, using the Leibniz property, we observe that if $p_i \in C^\infty(\mathcal{M}, \text{U}(\mathcal{E}'_i, \mathcal{E}_i))$ is an isomorphism, then

$$\begin{aligned} \nabla^{\text{Hom}(\nabla_1, p_2^* \nabla_2)} u &= (p_2)^{-1} \nabla^{\text{Hom}(\nabla_1, \nabla_2)} (p_2 u), & \forall u \in C^\infty(\mathcal{M}, \text{Hom}(\mathcal{E}_1, \mathcal{E}'_2)), \\ \nabla^{\text{Hom}(p_1^* \nabla_1, \nabla_2)} u &= \nabla^{\text{Hom}(\nabla_1, \nabla_2)} (u p_1^{-1}) \cdot p_1, & \forall u \in C^\infty(\mathcal{M}, \text{Hom}(\mathcal{E}'_1, \mathcal{E}_2)). \end{aligned} \tag{2.6}$$

2.2.2. Ambrose–Singer formula. Recall that the celebrated Ambrose–Singer formula (see e.g. [59, Theorem 8.1]) determines the tangent space at the identity of the holonomy group with respect to an arbitrary connection, in terms of its curvature tensor. Here we give an integral version of this fact. We start with a Hermitian vector bundle \mathcal{E} over the Riemannian manifold (\mathcal{M}, g) . Equip \mathcal{E} with a unitary connection $\nabla = \nabla^\mathcal{E}$.

Consider a smooth homotopy $\Gamma : [0, 1]^2 \rightarrow \mathcal{M}$ such that $\Gamma(0, 0) = p$. The “vertical” map $C_\uparrow(s, t) : \mathcal{E}_p \rightarrow \mathcal{E}_{\Gamma(s,t)}$ is obtained by parallel transport with respect to ∇ from \mathcal{E}_p to $\mathcal{E}_{\Gamma(0,1)}$, then $\mathcal{E}_{\Gamma(0,1)}$ to $\mathcal{E}_{\Gamma(s,1)}$ and $\mathcal{E}_{\Gamma(s,1)}$ to $\mathcal{E}_{\Gamma(s,t)}$, along $\Gamma(0, \bullet)$, $\Gamma(\bullet, 1)$ and $\Gamma(s, \bullet)$, respectively. Next, define the “horizontal” map $C_\rightarrow(s, t) : \mathcal{E}_p \rightarrow \mathcal{E}_{\Gamma(s,t)}$ by parallel transport with respect to ∇ from \mathcal{E}_p to $\mathcal{E}_{\Gamma(s,0)}$ and $\mathcal{E}_{\Gamma(s,0)}$ to $\mathcal{E}_{\Gamma(s,t)}$, along $\Gamma(\bullet, 0)$ and $\Gamma(s, \bullet)$, respectively. For a better understanding, see Figure 1.

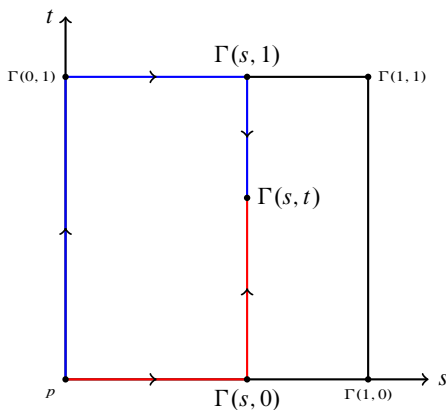


Fig. 1. The homotopy Γ in Lemma 2.2 with the corresponding points in \mathcal{M} : the parallel transport maps C_\uparrow (vertical) and C_\rightarrow (horizontal) are taken, respectively, along blue and red trajectories.

We are ready to prove the formula:

Lemma 2.2. *The following formula holds:*

$$C_\uparrow^{-1}(1, 1)C_\rightarrow(1, 1) - \mathbb{1}_{\mathcal{E}_p} = \int_0^1 \int_0^1 C_\uparrow(s, t)^{-1} F_\nabla(\partial_t, \partial_s) C_\rightarrow(s, t) dt ds. \tag{2.7}$$

Proof. Let $w_1, w_2 \in \mathcal{E}_p$; formally, we will identify the connection ∇ with its pullback $\Gamma^*\nabla$ on the pullback bundle $\Gamma^*\mathcal{E}$ over $[0, 1]^2$, as well as the curvature F_∇ with Γ^*F_∇ . Then we have the following chain of equalities:

$$\begin{aligned} &\langle w_1, (C_\uparrow(1, 1)^{-1}C_\rightarrow(1, 1) - \mathbb{1}_{\mathcal{E}_p})w_2 \rangle \\ &= \langle C_\uparrow(1, 1)w_1, C_\rightarrow(1, 1)w_2 \rangle - \langle C_\uparrow(0, 1)w_1, C_\rightarrow(0, 1)w_2 \rangle \\ &= \int_0^1 \partial_s \langle C_\uparrow(s, 1)w_1, C_\rightarrow(s, 1)w_2 \rangle ds \\ &= \int_0^1 \langle C_\uparrow(s, 1)w_1, \nabla_{\partial_s} C_\rightarrow(s, 1)w_2 \rangle ds \\ &= \int_0^1 \left[\int_0^1 (\partial_t \langle C_\uparrow(s, t)w_1, \nabla_{\partial_s} C_\rightarrow(s, t)w_2 \rangle + \langle C_\uparrow(s, 0)w_1, \nabla_{\partial_s} C_\rightarrow(s, 0)w_2 \rangle) dt \right] ds \\ &= \int_0^1 \int_0^1 \langle C_\uparrow(s, t)w_1, \nabla_{\partial_t} \nabla_{\partial_s} C_\rightarrow(s, t)w_2 \rangle ds dt \\ &= \int_0^1 \int_0^1 \langle w_1, C_\uparrow(s, t)^{-1} \underbrace{(\nabla_{\partial_t} \nabla_{\partial_s} - \nabla_{\partial_s} \nabla_{\partial_t})}_{=F_\nabla(\partial_t, \partial_s)} C_\rightarrow(s, t)w_2 \rangle ds dt, \end{aligned}$$

as the Lie bracket $[\partial_s, \partial_t]$ is 0 and we have used the unitarity of ∇ throughout. This completes the proof, since w_1 and w_2 were arbitrary. ■

We have two applications in mind for this lemma: one if γ is in a neighbourhood of p and we use the radial homotopy via geodesics emanating from p , and the second one for the “thin rectangle” obtained by shadowing a piece of the flow orbit; see Lemma 3.14.

2.3. Fourier analysis in the fibres

In this subsection, we recall some elements of Fourier analysis in the fibres and refer to [45–47, 81, 83] for further details.

2.3.1. Analysis on the trivial line bundle. Let (M, g) be a smooth Riemannian manifold of arbitrary dimension $n \geq 2$. The unit tangent bundle is endowed with the natural Sasaki metric and we let $\pi : SM \rightarrow M$ be the projection on the base. There is a canonical splitting of the tangent bundle to SM :

$$T(SM) = \mathbb{H} \oplus \mathbb{V} \oplus \mathbb{R}X,$$

where X is the geodesic vector field, $\mathbb{V} := \ker d\pi$ is the vertical space and \mathbb{H} is the horizontal space; the latter can be defined as the orthogonal to $\mathbb{V} \oplus \mathbb{R}X$ with respect to the Sasaki metric (see [76, Chapter 1]). Any vector $Z \in T(SM)$ can be decomposed as

$$Z = \alpha(Z)X + Z_\mathbb{H} + Z_\mathbb{V},$$

where α is the Liouville 1-form, $Z_{\mathbb{H}} \in \mathbb{H}$ and $Z_{\mathbb{V}} \in \mathbb{V}$. If $f \in C^\infty(SM)$, its gradient computed with respect to the Sasaki metric can be written as

$$\nabla_{\text{Sas}} f = (Xf)X + \nabla_{\mathbb{H}} f + \nabla_{\mathbb{V}} f,$$

where $\nabla_{\mathbb{H}} f \in \mathbb{H}$ is the horizontal gradient, $\nabla_{\mathbb{V}} f \in \mathbb{V}$ is the vertical gradient. We also let $\mathcal{N} \rightarrow SM$ be the *normal bundle* whose fibre over each $(x, v) \in SM$ is $(\mathbb{R} \cdot v)^\perp$. The bundles \mathbb{H} and \mathbb{V} may be naturally identified with \mathcal{N} (see [76, Section 1]).

For every $x \in M$, the sphere $S_x M = \{v \in T_x M \mid |v|_x^2 = 1\} \subset SM$ endowed with the Sasaki metric is isometric to the canonical sphere $(\mathbb{S}^{n-1}, g_{\text{can}})$. We denote by $\Delta_{\mathbb{V}}$ the vertical Laplacian obtained for $f \in C^\infty(SM)$ as $\Delta_{\mathbb{V}} f(x, v) = \Delta_{g_{\text{can}}}(f|_{S_x M})(v)$, where $\Delta_{g_{\text{can}}}$ is the spherical Laplacian. For $m \geq 0$, we denote by Ω_m the (finite-dimensional) vector space of spherical harmonics of degree m for the spherical Laplacian $\Delta_{g_{\text{can}}}$; they are defined as the elements of $\ker(\Delta_{g_{\text{can}}} + m(m + n - 2))$. We will use the convention that $\Omega_m = \{0\}$ if $m < 0$. If $f \in C^\infty(SM)$, it can be decomposed as $f = \sum_{m \geq 0} f_m$, where $f_m \in C^\infty(M, \Omega_m)$ is the L^2 -orthogonal projection of f onto the spherical harmonics of degree m .

There is a one-to-one correspondence between trace-free symmetric tensors of degree m and spherical harmonics of degree m . More precisely, the map

$$\pi_m^* : C^\infty(M, \otimes_S^m T^* M|_{0\text{-Tr}}) \rightarrow C^\infty(M, \Omega_m),$$

given by $\pi_m^* f(x, v) = f_x(v, \dots, v)$ is an isomorphism. Here, the index 0-Tr denotes the space of trace-free symmetric tensors, that is, tensors f such that if $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ denotes a local orthonormal frame of TM then

$$\text{Tr}(f) := \sum_{i=1}^n f(\mathbf{e}_i, \mathbf{e}_i, \cdot, \dots, \cdot) = 0.$$

We will denote by $\pi_{m*} : C^\infty(M, \Omega_m) \rightarrow C^\infty(M, \otimes_S^m T^* M|_{0\text{-Tr}})$ the adjoint of this map. More generally, the mapping

$$\pi_m^* : C^\infty(M, \otimes_S^m T^* M) \rightarrow \bigoplus_{k \geq 0} C^\infty(M, \Omega_{m-2k}) \tag{2.8}$$

is an isomorphism. We refer to [15, Section 2] for further details.

The geodesic vector field acts as

$$X : C^\infty(M, \Omega_m) \rightarrow C^\infty(M, \Omega_{m-1}) \oplus C^\infty(M, \Omega_{m+1})$$

(see [47, 83]). We define X_+ as the L^2 -orthogonal projection of X on the higher modes Ω_{m+1} : if $u \in C^\infty(M, \Omega_m)$, then $X_+ u := (Xu)_{m+1}$, and X_- as the L^2 -orthogonal projection of X on the lower modes Ω_{m-1} . For $m \geq 0$, the operator $X_+ : C^\infty(M, \Omega_m) \rightarrow C^\infty(M, \Omega_{m+1})$ is elliptic and thus has a finite-dimensional kernel (see [21]). The operator $X_- : C^\infty(M, \Omega_m) \rightarrow C^\infty(M, \Omega_{m-1})$ is of divergence type. The elements in the kernel of X_+ are called *conformal Killing tensors* (CKTs), associated to the trivial line bundle.

For $m = 0$, the kernel of X_+ on $C^\infty(M, \Omega_0)$ always contains the constant functions. We define *non-trivial CKTs* to be those elements in $\ker X_+$ which are not constant functions on SM . The kernel of X_+ is invariant under changing the metric by a conformal factor (see [45, Section 3.6]). It is known (see [83]) that there are no non-trivial CKTs in negative curvature and for Anosov surfaces but the question remains open for general Anosov manifolds. We provide an affirmative answer to this question *generically* as a byproduct of our work [14].

2.3.2. Twisted Fourier analysis. We now consider a Hermitian vector bundle with a unitary connection $(\mathcal{E}, \nabla^\mathcal{E})$ over (M, g) and define the operator $\mathbf{X} := (\pi^*\nabla^\mathcal{E})_X$ acting on $C^\infty(SM, \pi^*\mathcal{E})$, where $\pi : SM \rightarrow M$ is the projection. For the sake of simplicity, we will drop the π^* in the following. If $f \in C^\infty(SM, \mathcal{E})$, then $\nabla^\mathcal{E} f \in C^\infty(SM, T^*(SM) \otimes \mathcal{E})$ and we can write

$$\nabla^\mathcal{E} f = (\mathbf{X}f, \nabla_{\mathbb{H}}f, \nabla_{\mathbb{V}}f),$$

where $\nabla_{\mathbb{H}}f \in C^\infty(SM, \mathbb{H}^* \otimes \mathcal{E})$ and $\nabla_{\mathbb{V}}f \in C^\infty(SM, \mathbb{V}^* \otimes \mathcal{E})$. For future reference, we introduce a bundle endomorphism map R on $\mathcal{N} \otimes \mathcal{E}$, derived from the Riemann curvature tensor via the formula $R(x, v)(w \otimes e) = (R_x(w, v)v) \otimes e$.

If (e_1, \dots, e_r) is a local orthonormal frame of \mathcal{E} , then we define the *vertical Laplacian* as

$$\Delta_{\mathbb{V}}^\mathcal{E} \left(\sum_{k=1}^r u_k e_k \right) := \sum_{k=1}^r (\Delta_{\mathbb{V}} u_k) e_k.$$

Any section $f \in C^\infty(SM, \mathcal{E})$ can be decomposed as $f = \sum_{m \geq 0} f_m$, where $f_m \in \ker(\Delta_{\mathbb{V}}^\mathcal{E} + m(m+n-2))$, and we define

$$C^\infty(M, \Omega_m \otimes \mathcal{E}) := \ker(\Delta_{\mathbb{V}}^\mathcal{E} + m(m+n-2)) \cap C^\infty(SM, \mathcal{E}).$$

Here again, the operator

$$\mathbf{X} : C^\infty(M, \Omega_m \otimes \mathcal{E}) \rightarrow C^\infty(M, \Omega_{m-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{m+1} \otimes \mathcal{E})$$

can be split into $\mathbf{X} = \mathbf{X}_+ + \mathbf{X}_-$. For every $m \geq 0$, the operator \mathbf{X}_+ is elliptic and has finite-dimensional kernel, whereas \mathbf{X}_- is of divergence type. The kernel of \mathbf{X}_+ is invariant under a conformal change of the metric (see [45, Section 3.6]) and elements in its kernel are called *twisted conformal Killing tensors* (twisted CKTs). For simplicity we will often drop the word *twisted* and refer to the latter as CKTs. There are examples of vector bundles with CKTs on manifolds of arbitrary dimension. We proved in a companion paper [15] that the non-existence of CKTs is a generic condition, regardless of the curvature of the manifold (generic with respect to the connection, i.e. there is a residual set of the space of all unitary connections with regularity C^k , $k \geq 2$, which has no CKTs).

It is also known [45] that in negative curvature, there are always a *finite number of degrees* with CKTs (and their number can be estimated thanks to a lower bound on the curvature of the manifold and the curvature of the vector bundle). In other words,

$\ker \mathbf{X}_+|_{C^\infty(SM, \mathcal{E})}$ is finite-dimensional. The proof relies on an energy identity called the Pestov identity. This is also known for Anosov surfaces since any Anosov surface is conformally equivalent to a negatively-curved surface and CKTs are conformally invariant. Nevertheless, to the best of our knowledge, it is still an open question whether for Anosov manifolds of dimension $n \geq 3$, there are at most finitely many CKTs.

2.4. Twisted symmetric tensors

Given a section $u \in C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$, the connection $\nabla^\mathcal{E}$ produces an element $\nabla^\mathcal{E}u \in C^\infty(M, T^*M \otimes (\otimes_S^m T^*M) \otimes \mathcal{E})$. In coordinates, if (e_1, \dots, e_r) is a local orthonormal frame for \mathcal{E} and $\nabla^\mathcal{E} = d + \Gamma$, for some 1-form Γ with values in skew-Hermitian matrices, such that $\nabla^\mathcal{E}e_k = \sum_{i=1}^n \sum_{l=1}^r \Gamma_{ik}^l dx_i \otimes e_l$, we have

$$\begin{aligned} \nabla^\mathcal{E} \left(\sum_{k=1}^r u_k \otimes e_k \right) &= \sum_{k=1}^r (\nabla u_k \otimes e_k + u_k \otimes \nabla^\mathcal{E}e_k) \\ &= \sum_{k=1}^r \left(\nabla u_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k u_l \otimes dx_i \right) \otimes e_k, \end{aligned} \tag{2.9}$$

where $u_k \in C^\infty(M, \otimes_S^m T^*M)$ and ∇ is the Levi-Civita connection. The symmetrization operator $\mathcal{S}^\mathcal{E} : C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E}) \rightarrow C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$ is defined by

$$\mathcal{S}^\mathcal{E} \left(\sum_{k=1}^r u_k \otimes e_k \right) = \sum_{k=1}^r \mathcal{S}(u_k) \otimes e_k,$$

where $u_k \in C^\infty(M, \otimes_S^m T^*M)$ and in coordinates, writing $u_k = \sum_{i_1, \dots, i_m=1}^n u_{i_1 \dots i_m}^{(k)} dx_{i_1} \otimes \dots \otimes dx_{i_m}$, we have

$$\mathcal{S}(dx_{i_1} \otimes \dots \otimes dx_{i_m}) = \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} dx_{\pi(i_1)} \otimes \dots \otimes dx_{\pi(i_m)},$$

where \mathfrak{S}_m denotes the group of permutations of order m . For the sake of simplicity, we will write \mathcal{S} instead of $\mathcal{S}^\mathcal{E}$. We can symmetrize (2.9) to produce an element $D^\mathcal{E} := \mathcal{S} \nabla^\mathcal{E}u \in C^\infty(M, \otimes_S^{m+1} T^*M \otimes \mathcal{E})$ given in coordinates by

$$D^\mathcal{E} \left(\sum_{k=1}^r u_k \otimes e_k \right) = \sum_{k=1}^r \left(Du_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k u_l \otimes dx_i \right) \otimes e_k, \tag{2.10}$$

where $D := \mathcal{S} \nabla$ is the usual symmetric derivative of symmetric tensors.³ Elements of the form $Du \in C^\infty(M, \otimes_S^{m+1} T^*M)$ are called *potential tensors*. By comparison, we will call elements of the form $D^\mathcal{E}f \in C^\infty(M, \otimes_S^{m+1} T^*M \otimes \mathcal{E})$ *twisted potential tensors*.

³Beware of the notation: $\nabla^\mathcal{E}$ is for the connection, $D^\mathcal{E}$ for the symmetric derivative of tensors and $\nabla^{\text{End}(\mathcal{E})}$ is the connection induced by $\nabla^\mathcal{E}$ on the endomorphism bundle.

The operator $D^\mathcal{E}$ is a first order differential operator and its expression can be read off from (2.10):

$$\begin{aligned} \sigma_{\text{princ}}(D^\mathcal{E})(x, \xi) \cdot \left(\sum_{k=1}^r u_k(x) \otimes e_k(x) \right) &= \sum_{k=1}^r (\sigma_{\text{princ}}(D)(x, \xi) \cdot u_k(x)) \otimes e_k(x) \\ &= i \sum_{k=1}^r \sigma(\xi \otimes u_k(x)) \otimes e_k(x), \end{aligned}$$

where $e_k(x) \in \mathcal{E}_x, u_k(x) \in \otimes_S^m T_x^* M$ and the basis $(e_1(x), \dots, e_r(x))$ is assumed to be orthonormal for the metric h on \mathcal{E} . One can check that this is an injective map, which means that $D^\mathcal{E}$ is a left-elliptic operator and can be inverted on the left modulo a smoothing remainder. Its kernel is finite-dimensional and consists of smooth elements.

Before, we introduce, for $m \in \mathbb{N}$, the operator

$$\pi_m^* : C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}) \rightarrow C^\infty(SM, \pi^* \mathcal{E}),$$

defined by

$$\pi_m^* \left(\sum_{k=1}^r u_k \otimes e_k \right)(x, v) := \sum_{k=1}^r (u_k)_x(v, \dots, v) e_k(x).$$

Similarly to (2.8), the following mappings are isomorphisms (see [15, Section 2]):

$$\begin{aligned} \pi_m^* : C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E}) &\rightarrow \bigoplus_{k \geq 0} C^\infty(M, \Omega_{m-2k} \otimes \mathcal{E}), \\ \pi_m^* : C^\infty(M, \otimes_S^m T^* M|_{0-\text{Tr}} \otimes \mathcal{E}) &\rightarrow C^\infty(M, \Omega_m \otimes \mathcal{E}). \end{aligned}$$

We recall the notation $(\pi^* \nabla^\mathcal{E})_X := \mathbf{X}$. The following remarkable commutation property holds (see [15, Section 2]):

$$\forall m \in \mathbb{Z}_{\geq 0}, \quad \pi_{m+1}^* D^\mathcal{E} = \mathbf{X} \pi_m^*. \tag{2.11}$$

The vector bundle $\otimes_S^m T^* M \otimes \mathcal{E}$ is naturally endowed with a canonical fibrewise metric induced by the metrics g and h which allows us to define a natural L^2 scalar product. The L^2 formal adjoint $(D^\mathcal{E})^*$ of $D^\mathcal{E}$ is of divergence type (in the sense that its principal symbol is surjective for every $(x, \xi) \in T^* M \setminus \{0\}$; see [15, Definition 3.1] for further details). We call the elements in its kernel *twisted solenoidal tensors*.

By ellipticity of $D^\mathcal{E}$, for any twisted m -tensor f there exist unique $p \in (\ker D^\mathcal{E})^\perp \cap C^\infty(M, \otimes_S^{m-1} T^* M \otimes \mathcal{E})$ and $h \in C^\infty(M, \otimes_S^m T^* M \otimes \mathcal{E})$ such that

$$f = D^\mathcal{E} p + h, \quad (D^\mathcal{E})^* h = 0. \tag{2.12}$$

This decomposition bears resemblance to the Hodge decomposition of differential forms; we also note that (2.12) could be extended to other regularities. We define $\pi_{\ker(D^\mathcal{E})^*} f := h$ as the L^2 -orthogonal projection on twisted solenoidal tensors. This can be expressed as

$$\pi_{\ker(D^\mathcal{E})^*} = \mathbb{1} - D^\mathcal{E} [(D^\mathcal{E})^* D^\mathcal{E}]^{-1} (D^\mathcal{E})^*, \tag{2.13}$$

where $[(D^\mathcal{E})^* D^\mathcal{E}]^{-1}$ is the resolvent of the operator $(D^\mathcal{E})^* D^\mathcal{E}$ (defined as follows: $[(D^\mathcal{E})^* D^\mathcal{E}]^{-1} = 0$ on $\ker (D^\mathcal{E})^* D^\mathcal{E}$, and on the L^2 -orthogonal complement of this kernel it is given by the genuine inverse of $(D^\mathcal{E})^* D^\mathcal{E}$, well-defined by Fredholm theory of elliptic operators).

2.5. Pollicott–Ruelle resonances

We explain the link between the widely studied notion of Pollicott–Ruelle resonances (see for instance [12, 27, 29–31, 37, 66]) and the notion of (twisted) conformal Killing tensors introduced in the previous subsection. We also refer to [15] for an extensive discussion of this.

2.5.1. Definition of resolvents. Let \mathcal{M} be a smooth manifold endowed with a vector field $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ generating an Anosov flow in the sense of (1.1). Throughout this subsection, we will always assume that the flow is volume-preserving. It will be important to consider the dual decomposition to (1.1),

$$T^*(\mathcal{M}) = \mathbb{R}E_0^* \oplus E_s^* \oplus E_u^*,$$

where $E_0^*(E_s \oplus E_u) = 0$, $E_s^*(E_s \oplus \mathbb{R}X) = 0$, $E_u^*(E_u \oplus \mathbb{R}X) = 0$. As before, we consider a vector bundle $\mathcal{E} \rightarrow \mathcal{M}$ equipped with a unitary connection $\nabla^\mathcal{E}$ and set $\mathbf{X} := \nabla_X^\mathcal{E}$. Since X preserves a smooth measure $d\mu$ and $\nabla^\mathcal{E}$ is unitary, the operator \mathbf{X} is skew-adjoint on $L^2(\mathcal{M}, \mathcal{E}; d\mu)$, with dense domain

$$\mathcal{D}_{L^2} := \{u \in L^2(\mathcal{M}, \mathcal{E}; d\mu) \mid \mathbf{X}u \in L^2(\mathcal{M}, \mathcal{E}; d\mu)\}. \tag{2.14}$$

Its L^2 -spectrum consists of the absolutely continuous spectrum $i\mathbb{R}$ and of embedded eigenvalues. We introduce the resolvents

$$\begin{aligned} \mathbf{R}_+(z) &:= (-\mathbf{X} - z)^{-1} = - \int_0^\infty e^{-tz} e^{-t\mathbf{X}} dt, \\ \mathbf{R}_-(z) &:= (\mathbf{X} - z)^{-1} = - \int_{-\infty}^0 e^{zt} e^{-t\mathbf{X}} dt, \end{aligned} \tag{2.15}$$

initially defined for $\operatorname{Re} z > 0$. (Let us stress the conventions used here: $-\mathbf{X}$ is associated to the positive resolvent $\mathbf{R}_+(z)$ whereas \mathbf{X} is associated to the negative one, $\mathbf{R}_-(z)$.) Here $e^{-t\mathbf{X}}$ denotes the propagator of \mathbf{X} , that is, the parallel transport by $\nabla^\mathcal{E}$ along the flow lines of X . If $\mathbf{X} = X$ is simply the vector field acting on functions (i.e. \mathcal{E} is the trivial line bundle), then $e^{-tX} f(x) = f(\varphi_{-t}(x))$ is simply the composition with the flow.

There exists a family \mathcal{H}_\pm^s of Hilbert spaces called *anisotropic Sobolev spaces*, indexed by $s > 0$, such that the resolvents can be meromorphically extended to the whole complex plane by making \mathbf{X} act on \mathcal{H}_\pm^s . The poles of the resolvents are called the *Pollicott–Ruelle resonances* and have been widely studied in [12, 27, 29–31, 37, 66]. Note that the resonances and the resonant states associated to them are intrinsic to the flow and do not depend on any choice of construction of the anisotropic Sobolev spaces. More precisely,

there exists a constant $c > 0$ such that $\mathbf{R}_\pm(z) \in \mathcal{L}(\mathcal{H}_\pm^s)$ are meromorphic in $\{\operatorname{Re} z > -cs\}$. For $\mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$), the space \mathcal{H}_+^s (resp. \mathcal{H}_-^s) consists of distributions which are microlocally H^s in a neighborhood of E_s^* (resp. microlocally H^s in a neighborhood of E_u^*) and microlocally H^{-s} in a neighborhood of E_u^* (resp. microlocally H^{-s} in a neighborhood of E_s^* ; see [27, 30]). These spaces also satisfy $(\mathcal{H}_+^s)' = \mathcal{H}_-^s$ (where one identifies the spaces using the L^2 -pairing).

From now on, we will assume that s is fixed and small enough, and set $\mathcal{H}_\pm := \mathcal{H}_\pm^s$. We have

$$H^s \subset \mathcal{H}_\pm \subset H^{-s}, \tag{2.16}$$

and there is a certain strip $\{\operatorname{Re} z > -\varepsilon_{\text{strip}}\}$ (for some $\varepsilon_{\text{strip}} > 0$) on which $z \mapsto \mathbf{R}_\pm(z) \in \mathcal{L}(\mathcal{H}_\pm)$ is meromorphic (and the same holds for small perturbations of \mathbf{X}).

These resolvents satisfy the following equalities on \mathcal{H}_\pm , for z not a resonance:

$$\mathbf{R}_\pm(z)(\mp \mathbf{X} - z) = (\mp \mathbf{X} - z)\mathbf{R}_\pm(z) = \mathbf{1}_\mathcal{E}. \tag{2.17}$$

Given $z \in \mathbb{C}$ which not a resonance, we have

$$\mathbf{R}_+(z)^* = \mathbf{R}_-(\bar{z}), \tag{2.18}$$

which is understood in the following way: given $f_1, f_2 \in C^\infty(\mathcal{M}, \mathcal{E})$, we have

$$\langle \mathbf{R}_+(z)f_1, f_2 \rangle_{L^2} = \langle f_1, \mathbf{R}_-(\bar{z})f_2 \rangle_{L^2}.$$

We will always use this convention for the definition of the adjoint.

Since the connections are unitary and the flow preserves a smooth measure, the propagators $e^{-t\mathbf{X}}$ preserve the norm in $L^2(\mathcal{M}, \mathcal{E}; d\mu)$. As a consequence, the integrals in (2.15) converge when $\operatorname{Re} z > 0$ and thus we obtain the following statement that we record for future purposes:

$$\text{the resonance spectrum of } \pm \mathbf{X} \text{ is contained in } \{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0\}. \tag{2.19}$$

A point $z_0 \in \mathbb{C}$ is a resonance for $-\mathbf{X}$ (resp. \mathbf{X}), i.e. is a pole of $z \mapsto \mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$) if and only if there exists a non-zero $u \in \mathcal{H}_+^s$ (resp. \mathcal{H}_-^s) for some $s > 0$ such that $-\mathbf{X}u = z_0u$ (resp. $\mathbf{X}u = z_0u$). If γ is a small counterclockwise oriented circle around z_0 , then the spectral projector onto the resonant states is

$$\Pi_{z_0}^\pm = -\frac{1}{2\pi i} \int_\gamma \mathbf{R}_\pm(z) dz = \frac{1}{2\pi i} \int_\gamma (z \pm \mathbf{X})^{-1} dz,$$

where we abuse notation by writing $-(\mathbf{X} + z)^{-1}$ (resp. $(\mathbf{X} - z)^{-1}$) for the meromorphic extension of $\mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$).

2.5.2. *Resonances at $z = 0$.* By the above, in a neighborhood of $z = 0$ we can write the following Laurent expansion (beware of the conventions):

$$\mathbf{R}_+(z) = -\mathbf{R}_0^+ - \frac{\Pi_0^+}{z} + \mathcal{O}(z)$$

(or in other words, using our abuse of notation, $(\mathbf{X} + z)^{-1} = \mathbf{R}_0^+ + \Pi_0^+/z + \mathcal{O}(z)$ and

$$\mathbf{R}_-(z) = -\mathbf{R}_0^- - \frac{\Pi_0^-}{z} + \mathcal{O}(z)$$

(or in other words, $(z - \mathbf{X})^{-1} = \mathbf{R}_0^- + \Pi_0^-/z + \mathcal{O}(z)$). As a consequence, these equalities define the two operators \mathbf{R}_0^\pm as the holomorphic part (at $z = 0$) of the resolvents $-\mathbf{R}_\pm(z)$. We introduce

$$\Pi := \mathbf{R}_0^+ + \mathbf{R}_0^- \tag{2.20}$$

Lemma 2.3. *We have $(\mathbf{R}_0^+)^* = \mathbf{R}_0^-$ and $(\Pi_0^+)^* = \Pi_0^- = \Pi_0^+$. Thus Π is formally self-adjoint. Moreover, it is non-negative in the sense that for all $f \in C^\infty(\mathcal{M}, \mathcal{E})$, $\langle \Pi f, f \rangle_{L^2} = \langle f, \Pi f \rangle_{L^2} \geq 0$. Also, $\langle \Pi f, f \rangle = 0$ if and only if $\Pi f = 0$ if and only if $f = \mathbf{X}u + v$ for some $u \in C^\infty(\mathcal{M}, \mathcal{E})$ and $v \in \ker \mathbf{X}|_{\mathcal{H}_\pm}$.*

Proof. See [15, Lemma 5.1]. ■

We also record for clarity the following identities:

$$\begin{aligned} \Pi_0^\pm \mathbf{R}_0^\pm &= \mathbf{R}_0^\pm \Pi_0^\pm = 0, \quad \Pi_0^\pm \mathbf{R}_0^\mp = \mathbf{R}_0^\mp \Pi_0^\pm = 0, \\ \mathbf{X} \Pi_0^\pm &= \Pi_0^\pm \mathbf{X} = 0, \quad \mathbf{X} \mathbf{R}_0^\pm = \mathbf{R}_0^\pm \mathbf{X} = \mathbb{1} - \Pi_0^\pm, \quad -\mathbf{X} \mathbf{R}_0^\mp = -\mathbf{R}_0^\mp \mathbf{X} = \mathbb{1} - \Pi_0^\mp. \end{aligned} \tag{2.21}$$

2.5.3. Perturbation theory of resonances. We will need to apply the framework of Pollicott–Ruelle resonances to connections with finite regularity. Let $\nabla_0^\mathcal{E}$ be an arbitrary unitary connection of regularity C_*^s (with $s > 1$) on a smooth Hermitian vector bundle $\mathcal{E} \rightarrow \mathcal{M}$ and define the first order differential operator $\mathbf{X}_0 := (\nabla_0)_{\mathcal{X}}^\mathcal{E}$ acting on sections of \mathcal{E} .

Lemma 2.4. *There exists a constant $C > 0$, depending only on the vector field X , and anisotropic Sobolev spaces \mathcal{H}_\pm , such that the resolvents $z \mapsto \mathbf{R}_\pm(z) = (\mp \mathbf{X}_0 - z)^{-1} \in \mathcal{L}(\mathcal{H}_\pm)$ admit a meromorphic extension from $\{\operatorname{Re} z \gg 0\}$ to $\{\operatorname{Re} z \geq -Cs\}$.*

For a proof, we refer to the article by Guedes Bonthonneau–de Poyferré–Guillarmou [41, Theorem 3] (we note however that less precise statements were obtained by microlocal methods also by Dyatlov–Zworski, see for instance [27, Remark (i) on page 4]). It is also immediate to extend the perturbation theory of Pollicott–Ruelle resonances of Bonthonneau [8, Corollary 1.2] to finite regularity (in fact, our case is easier to handle because the perturbations we consider are by order zero terms):

Lemma 2.5. *Let $C > 0$ and \mathcal{H}_\pm be as in Lemma 2.4. Let $z_0 \in \mathbb{C}$ with $\operatorname{Re} z_0 > -Cs$ be a Pollicott–Ruelle resonance of $-\mathbf{X}_0$ and γ be a small contour around z_0 enclosing no other resonances of $-\mathbf{X}_0$. Then there exist $\varepsilon > 0$ and $s_0 \gg 1$ such that for any $s > s_0$, and any connection $\nabla^\mathcal{E} = \nabla_0^\mathcal{E} + A$ for some $A \in C_*^s(\mathcal{M}, T^*\mathcal{M} \otimes \operatorname{End}(\mathcal{E}))$ such that $\|A\|_{C_*^s} < \varepsilon$, the projector*

$$\Pi_A := \frac{1}{2\pi i} \oint_\gamma (-\mathbf{X} - z)^{-1} dz$$

is well-defined, and the map $A \mapsto \Pi_A \in \mathcal{L}(\mathcal{H}_+)$ is C^∞ -regular with locally constant rank; here $\mathbf{X}_A := \mathbf{X}_0 + V_A = \nabla_X^\xi$ and $V_A := A(X)$. Moreover, the map $A \mapsto \lambda_A$ associating to A the sum of the resonances of $-\mathbf{X}_A$ enclosed by γ (with multiplicities) is smooth near $A = 0$.

Proof. We sketch the proof for the convenience of the reader. Note that for any $s > s_1$, the multiplication map

$$C_*^s(\mathcal{M}, \text{End}(\mathcal{E})) \times \mathcal{H}_\pm \ni (V, u) \mapsto Vu \in \mathcal{H}_\pm \tag{2.22}$$

is continuous; this follows from [41, Section 2].

Note that $V_A = A(X) \in C_*^s(\mathcal{M}, \text{End}(\mathcal{E}))$. Then for any z in a small neighbourhood of γ ,

$$\mathbf{X}_0 + V_A + z = (\mathbf{X}_0 + z)(\mathbb{1} + (\mathbf{X}_0 + z)^{-1}V_A)$$

and so by (2.22), the map $\|(\mathbf{X}_0 + z)^{-1}V\|_{\mathcal{H}_+ \rightarrow \mathcal{H}_+}$ has norm smaller than 1 if ε is small enough, so the operator $\mathbb{1} + (\mathbf{X}_0 + z)^{-1}V_A$ is invertible on the domain $\mathcal{D}_+ = \{u \in \mathcal{H}_+ \mid \mathbf{X}_0 u \in \mathcal{H}_+\}$ (note that $\mathbf{X}_0 V_A = [\mathbf{X}_0, V_A] + V_A \mathbf{X}_0$ where $[\mathbf{X}_0, V_A] \in C_*^{s-1}(\mathcal{M}, \text{End}(\mathcal{E}))$, so the domain \mathcal{D}_+ is invariant under multiplication with V when $s > s_1 + 1$). This implies that $\mathbf{X}_0 + V_A + z : \mathcal{D}_+ \rightarrow \mathcal{H}_+$ is invertible for z in a neighbourhood of γ , with inverse bounded by some uniform constant.

Therefore Π_A is well-defined and we may differentiate to get, in the direction of some $B \in C_*^s(\mathcal{M}, T^*\mathcal{M} \otimes \text{End}(\mathcal{E}))$,

$$D_B \Pi_A = \frac{1}{2\pi i} \oint_\gamma (-\mathbf{X}_A - z)^{-1} V_B (-\mathbf{X}_A - z)^{-1} dz.$$

It follows from (2.22) that $\|D_B \Pi_A\|_{\mathcal{H}_+ \rightarrow \mathcal{H}_+} \leq C_1 \|V_B\|_{\mathcal{H}_+ \rightarrow \mathcal{H}_+} \leq C_2 \|B\|_{C_*^s}$ for some uniform constants $C_1, C_2 > 0$. This shows that $A \mapsto \Pi_A \in \mathcal{L}(\mathcal{H}_+)$ is C^1 , and iterating this argument shows that in fact this map is smooth.

To show that the rank of Π_A is locally constant we refer to [8, Section 4] (see also [18, Section 6]). Finally, consider a basis $(u_i)_{i=1}^r \subset \mathcal{H}_+$ of $-\mathbf{X}_0$ of (generalized) resonant states, where r is the rank of Π_0 . Then the map $A \mapsto (\Pi_A u_i)_{i=1}^r \in \mathcal{H}_+$ is smooth and so for small enough ε , the sequence $(\Pi_A u_i)_{i=1}^r$ is a basis of the range of Π_A (of generalized resonant states of $-\mathbf{X}_A$). The map $-\mathbf{X}_A$ acts on the range of Π_A by definition and so since λ_A equals the trace of $-\mathbf{X}_A$ in the constructed smooth basis, this gives the smoothness of λ_A . ■

2.6. Generalized X-ray transform

The discussion is carried out here in the closed case, but could also be generalized to the case of a manifold with boundary. We introduce the operator

$$\Pi := \mathbf{R}_0^+ + \mathbf{R}_0^-,$$

where \mathbf{R}_0^+ (resp. \mathbf{R}_0^-) denotes the holomorphic part at 0 of $-\mathbf{R}_+(z)$ (resp. $-\mathbf{R}_-(z)$) and Π_0^+ is the L^2 -orthogonal projection on the (smooth) resonant states at 0. Such an operator

was first introduced in the non-twisted case by Guillarmou [38]. The operator $\Pi + \Pi_0^+$ is the derivative of the (total) L^2 -spectral measure at 0 of the skew-adjoint operator \mathbf{X} .

Definition 2.6 (Generalized X-ray transform of twisted symmetric tensors). We define the *generalized X-ray transform* of twisted symmetric tensors as the operator

$$\Pi_m := \pi_{m*}(\Pi + \Pi_0^+)\pi_m^*$$

In what follows, we will mostly use this operator with $m = 1$. In this case, the operator Π_1 takes a 1-form with values in some bundle \mathcal{E} , pulls it back onto the unit tangent bundle to a spherical harmonic of degree 1 twisted by some bundle (π_1^* -operator), then “averages” this spherical harmonic along the geodesic flow lines ($(\Pi + \Pi_0^+)$ -operator) and then selects the first spherical harmonic of this distribution in order to give a twisted 1-form on the base manifold M (π_{1*} -operator). We remark that when we want to emphasize the dependence of Π_m on a connection $\nabla^\mathcal{E}$, we will write $\Pi_m^{\nabla^\mathcal{E}}$ (this will appear in particular in Section 4.3).

Remark 2.7. This also allows us to define a generalized (twisted) X-ray transform Π_m for an arbitrary unitary connection $\nabla^\mathcal{E}$ on \mathcal{E} . Indeed, it is not clear a priori whether one sticks to the usual definition of the X-ray transform that one can find a “natural” candidate for the X-ray transform on twisted tensors. For instance, one could consider the map

$$\mathcal{C} \ni \gamma \mapsto I_m^{\nabla^\mathcal{E}} f(\gamma) := \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} (e^{-t\mathbf{X}} f)(x_\gamma, v_\gamma) dt \in \mathcal{E}_{x_\gamma},$$

where $\gamma \in \mathcal{C}$ is a closed geodesic and $(x_\gamma, v_\gamma) \in \gamma$. However, this definition does depend on the choice of a base point $(x_\gamma, v_\gamma) \in \gamma$ and it would no longer be true that $I_m^{\nabla^\mathcal{E}}(D^\mathcal{E} p)(\gamma) = 0$ unless the connection is transparent.

By (2.11) and (2.21), we have the equalities

$$(D^\mathcal{E})^* \Pi_m = 0 = \Pi_m D^\mathcal{E}, \tag{2.23}$$

showing that Π_m maps the set of twisted solenoidal tensors to itself. We say that the generalized X-ray transform is *solenoidally injective* (*s-injective*) on m -tensors if for all $u \in C^\infty(SM, \mathcal{E})$ and $f \in C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$,

$$\mathbf{X}u = \pi_m^* f \implies \exists p \in C^\infty(M, \otimes_S^{m-1} T^*M \otimes \mathcal{E}) \text{ such that } f = D^\mathcal{E} p. \tag{2.24}$$

Lemma 2.8. *The generalized X-ray transform is s-injective on m-tensors if and only if Π_m is injective on solenoidal tensors (if this holds, we say Π_m is s-injective).*

Proof. Assume that $\Pi_m f = 0$ and f is a twisted solenoidal m -tensor. Then

$$\langle \Pi_m f, f \rangle_{L^2} = \langle \Pi \pi_m^* f, \pi_m^* f \rangle_{L^2} + \langle \Pi_0^+ \pi_m^* f, \pi_m^* f \rangle_{L^2} = 0.$$

Both terms on the right hand side are non-negative by Lemma 2.3, hence both vanish, and the same lemma implies that $\Pi \pi_m^* f = 0$ and $\Pi_0^+ \pi_m^* f = 0$. Thus $\mathbf{X}u = \pi_m^* f$ for some

smooth u , so by the s -injectivity of the generalized X-ray transform, f is potential, which implies $f = 0$.

The other direction is obvious by (2.21). ■

Next, we show Π_m enjoys good analytical properties:

Lemma 2.9. *The operator $\Pi_m : C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E}) \rightarrow C^\infty(M, \otimes_S^m T^*M \otimes \mathcal{E})$ has the following properties:*

- (1) Π_m is a pseudodifferential operator of order -1 ,
- (2) Π_m is formally self-adjoint and elliptic on twisted solenoidal tensors (its Fredholm index is thus equal to 0 and its kernel and cokernel are finite-dimensional),
- (3) If Π_m is s -injective, then the following stability estimates hold:

$$\forall s \in \mathbb{R}, \forall f \in H^s(M, \otimes_S^m T^*M \otimes \mathcal{E}), \quad \|\pi_{\ker(D^\mathcal{E})^*} f\|_{H^s} \leq C_s \|\Pi_m f\|_{H^{s+1}},$$

for some $C_s > 0$, and for some $C > 0$,

$$\forall f \in H^{-1/2}(M, \otimes_S^m T^*M \otimes \mathcal{E}), \quad \langle \Pi_m f, f \rangle_{L^2} \geq C \|\pi_{\ker(D^\mathcal{E})^*} f\|_{H^{-1/2}}^2.$$

In particular, these estimates hold if (M, g) has negative curvature and $\nabla^\mathcal{E}$ has no twisted CKTs.

Point (3) is a quantitative improvement of the following statement: $\Pi_m f = 0, f \in \ker(D^\mathcal{E})^* \Rightarrow f = 0$, i.e. it provides a stability estimate for the X-ray transform (see Lemma 2.8 for the relation between Π_m and the X-ray transform).

Proof of Lemma 2.9. The first two points follow from a rather straightforward adaptation of the proof of [64, Theorem 2.5.1] (see also [38] for the original arguments); we omit it. It remains to prove the third point.

The first estimate follows from (2), the elliptic estimate and the fact that Π_m is s -injective. The last estimate in the non-twisted case follows from [43, Lemma 2.1] (or [64, Theorem 2.5.6]) and subsequent remarks; the twisted case follows by minor adaptations.

If (M, g) has negative curvature and $\nabla^\mathcal{E}$ has no twisted CKTs, using Lemma 2.8 and [45, Sections 4, 5] we find that Π_m is s -injective, proving the claim. ■

3. Exact Livšic cocycle theory

We phrase this section in a very general context which is that of a transitive Anosov flow on a smooth manifold. It is independent of the rest of the article.

3.1. Statement of the results

3.1.1. A weak exact Livšic cocycle theorem. Let \mathcal{M} be a smooth closed manifold endowed with a flow $(\varphi_t)_{t \in \mathbb{R}}$ with infinitesimal generator $X \in C^\infty(\mathcal{M}, T\mathcal{M})$. We assume

that the flow is Anosov in the sense of (1.1) and that it is *transitive*, that is, it admits a dense orbit.⁴ We denote by \mathcal{G} the set of all periodic orbits for the flow and by $\mathcal{G}^\#$ the set of all *primitive* orbits, which cannot be written as a shorter orbit to some positive power ≥ 2 .

Let $(\mathcal{E}, \nabla^\mathcal{E})$ be a smooth Hermitian vector bundle of rank r equipped with a unitary connection $\nabla^\mathcal{E}$. We will denote by

$$C(x, t) : \mathcal{E}_x \rightarrow \mathcal{E}_{\varphi_t(x)}$$

the parallel transport along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$ with respect to the connection $\nabla^\mathcal{E}$. In the more general setting, we may consider two Hermitian vector bundles $\mathcal{E}_1, \mathcal{E}_2 \rightarrow \mathcal{M}$, equipped with unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. Recall that if $\nabla^{\mathcal{E}_2} = p^* \nabla^{\mathcal{E}_1}$ for some unitary map $p \in C^\infty(\mathcal{M}, \text{U}(\mathcal{E}_2, \mathcal{E}_1))$,⁵ i.e. the connections are gauge-equivalent, then parallel transport along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$ satisfies the commutation relation

$$C_1(x, t) = p(\varphi_t x) C_2(x, t) p(x)^{-1}.$$

We say that such cocycles are *cohomologous*. In particular, given a closed orbit $\gamma = (\varphi_t x_0)_{t \in [0, T]}$ of the flow, one has

$$C_1(x_0, T) = p(x_0) C_2(x_0, T) p(x_0)^{-1},$$

i.e. the parallel transport maps are conjugate.

Definition 3.1. We say that the connections $\nabla^{\mathcal{E}_1, 2}$ have *trace-equivalent holonomies* if for all primitive closed orbits $\gamma \in \mathcal{G}^\#$, we have

$$\text{Tr}(C_1(x_\gamma, \ell(\gamma))) = \text{Tr}(C_2(x_\gamma, \ell(\gamma))), \tag{3.1}$$

where $x_\gamma \in \gamma$ is arbitrary and $\ell(\gamma)$ is the period of γ .

This condition could be a priori obtained with $\text{rank}(\mathcal{E}_1) \neq \text{rank}(\mathcal{E}_2)$. We shall see that this cannot be the case. The following result is one of the main theorems of this paper. It seems to improve known results of Livšic cocycle theory (in particular [75, 90]); see Section 1.4 for a more extensive discussion.

Theorem 3.2. *Assume \mathcal{M} is endowed with a smooth transitive Anosov flow. Let $\mathcal{E}_1, \mathcal{E}_2 \rightarrow \mathcal{M}$ be two Hermitian vector bundles equipped with respective unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. If the connections have trace-equivalent holonomies in the sense of Definition 3.1, then there exists $p \in C^\infty(\mathcal{M}, \text{U}(\mathcal{E}_2, \mathcal{E}_1))$ such that for all $x \in \mathcal{M}$ and $t \in \mathbb{R}$,*

$$C_1(x, t) = p(\varphi_t x) C_2(x, t) p(x)^{-1}, \tag{3.2}$$

i.e. the cocycles induced by parallel transport are cohomologous. Moreover, \mathcal{E}_1 and \mathcal{E}_2 are isomorphic.

In order to prove the injectivity Theorem 1.1, we will apply Theorem 3.2 with $\mathcal{M} = SM$, the geodesic flow, and the pullback bundle $\pi^* \mathcal{E}$ equipped with two pullback

⁴Note that there are examples of non-transitive Anosov flows [33].

⁵Here, we denote by $\text{U}(\mathcal{E}_2, \mathcal{E}_1) \rightarrow \mathcal{M}$ the bundle of unitary maps $\mathcal{E}_2 \rightarrow \mathcal{E}_1$. Of course, it may be empty if the bundles are not isomorphic.

connections $\pi^*\nabla_1, \pi^*\nabla_2$. However, as we shall see in Section 5, Theorem 3.2 will not directly imply Theorem 1.1: indeed, after differentiating (3.2) at $t = 0$, it only gives the existence of a map $p \in C^\infty(SM, U(\pi^*\mathcal{E}))$ such that

$$\pi^*\nabla_X^{\text{Hom}(\nabla_2, \nabla_1)} p = 0.$$

We will then have to prove that $p = \pi^*p_0$ for some unitary isomorphism $p_0 \in C^\infty(M, U(\mathcal{E}))$ on the base such that $\nabla^{\text{Hom}(\nabla_2, \nabla_1)} p_0 = 0$ (this is equivalent to the connections being gauge-equivalent, as follows directly from Definition 2.1).

Remark 3.3. The simplest example in which the automorphism p does not descend to the base can be constructed as follows (originally in [77]). If (Σ, g_Σ) is a Riemannian surface of negative curvature, then along any closed geodesic γ the parallel transport with respect to the Levi-Civita connection ∇^{LC} is the identity (as P_γ fixes $\dot{\gamma}$ and hence also its normal $\dot{\gamma}^\perp$). Thus $\pi^*\nabla^{\text{LC}}$ on the pullback $\pi^*\mathcal{K}$ of the canonical bundle $\mathcal{K} = (T_\mathbb{C}^*\Sigma)^{0,1}$ and the trivial connection on $S\Sigma \times \mathbb{C}$ have trace-equivalent holonomies. By Theorem 3.2, there is $p \in C^\infty(\mathcal{M}, U(\pi^*\mathcal{K}, \mathbb{C}))$ such that $\pi^*\nabla_X^{\text{Hom}(\nabla^{\text{LC}}, d)} p = 0$, but clearly p is not of degree zero as the bundle \mathcal{K} is not topologically trivial. In fact, p can be chosen to be of degree 1 and similar examples exist in higher dimensions (see [13]).

As we shall see in the proof, for any given $L > 0$, it suffices to assume that the trace-equivalent holonomy condition (1.2) holds for all primitive periodic orbits of length $\geq L$ in order to get the conclusion of the theorem. Surprisingly, the rather weak condition (1.2) implies in particular that the bundles are isomorphic as stated in Corollary 1.4 and the trace of the holonomy of unitary connections along closed orbits should allow one in practice to classify vector bundles over manifolds carrying Anosov flows. Even more surprisingly, the ranks of \mathcal{E}_1 and \mathcal{E}_2 might be a priori different and Theorem 3.2 actually shows that the ranks have to coincide.

The idea relies on a key notion which we call *Parry’s free monoid*, whose introduction goes back to Parry [75]. This free monoid \mathbf{G} corresponds (at least formally) to the free monoid generated by the set of homoclinic orbits to a given periodic orbit of a point x_\star (see Section 3.2.1 for a definition) and we shall see that a connection induces a unitary representation $\rho : \mathbf{G} \rightarrow U(\mathcal{E}_{x_\star})$ (it is not canonical but we shall see that its important properties are). Geometric properties of the connection can be read off from this representation; see Theorem 3.6 below. Moreover, tools from representation theory can be applied and this is how we will eventually prove Theorem 3.2.

3.1.2. Opaque and transparent connections. Theorem 3.2 has an interesting straightforward corollary. Recall that a unitary connection is said to be *transparent* if the holonomy along all closed orbits is trivial.

Corollary 3.4. *Assume \mathcal{M} is endowed with a smooth transitive Anosov flow. Let $\mathcal{E} \rightarrow \mathcal{M}$ be a Hermitian vector bundle of rank r equipped with a unitary connection $\nabla^\mathcal{E}$. If the connection is transparent, then \mathcal{E} is trivial and trivialized by a smooth orthonormal family $e_1, \dots, e_r \in C^\infty(\mathcal{M}, \mathcal{E})$ such that $\nabla_X^\mathcal{E} e_i = 0$.*

In order to prove the previous corollary, it suffices to apply Theorem 3.2 with $\mathcal{E}_1 = \mathcal{E}$ equipped with $\nabla^{\mathcal{E}}$ and $\mathcal{E}_2 = \mathbb{C}^r \times M$ equipped with the trivial flat connection. Then $C_2(x, t) = \mathbb{1}$ and (e_1, \dots, e_n) is obtained as the image by p of the canonical basis of \mathbb{C}^n . This corollary seems to be folklore but nowhere written down. It is stated in [78, Proposition 9.2] under the extra assumption that $\mathcal{E} \oplus \mathcal{E}^*$ is trivial.

The notion “opposite” to transparent connections is that of *opaque* connections, which do not preserve any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$ by parallel transport along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$. It was shown in [15, Section 5] that the opacity of a connection is equivalent to the fact that

$$\ker \nabla_X^{\text{End}}|_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))} = \mathbb{C} \cdot \mathbb{1}_{\mathcal{E}}.$$

Also note that when X is volume-preserving, this corresponds to the Pollicott–Ruelle (co)resonant states at 0 associated to the operator ∇_X^{End} . We shall also connect this notion with the representation $\rho : \mathbf{G} \rightarrow \text{U}(\mathcal{E}_{x_\star})$ of the free monoid:

Proposition 3.5. *The following statements are equivalent:*

- (1) *the connection $\nabla^{\mathcal{E}}$ is opaque;*
- (2) $\ker \nabla_X^{\text{End}}|_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))} = \mathbb{C} \cdot \mathbb{1}_{\mathcal{E}};$
- (3) *the representation $\rho : \mathbf{G} \rightarrow \text{U}(\mathcal{E}_{x_\star})$ is irreducible.*

3.1.3. Kernel of the endomorphism connection. The previous proposition actually follows from a more general statement which we now describe. The representation $\rho : \mathbf{G} \rightarrow \text{U}(\mathcal{E}_{x_\star})$ gives rise to an orthogonal splitting

$$\mathcal{E}_{x_\star} = \bigoplus_{i=1}^K \mathcal{E}_i^{\oplus n_i},$$

where $\mathcal{E}_i \subset \mathcal{E}_{x_\star}$ and $n_i \geq 1$; each factor \mathcal{E}_i is \mathbf{G} -invariant and the induced representation on each factor is irreducible; furthermore, for $i \neq j$, the induced representations on \mathcal{E}_i and \mathcal{E}_j are not isomorphic. Let $\mathbb{C}[\mathbf{G}]$ be the formal algebra generated by \mathbf{G} over \mathbb{C} and let

$$\mathbf{R} := \rho(\mathbb{C}[\mathbf{G}]).$$

By Burnside’s Theorem (see [63, Corollary 3.3] for instance), one has

$$\mathbf{R} = \bigoplus_{i=1}^K \Delta_{n_i} \text{End}(\mathcal{E}_i),$$

where

$$\Delta_{n_i} u = u \oplus \dots \oplus u \quad (n_i \text{ times})$$

for $u \in \text{End}(\mathcal{E}_i)$. We introduce the *commutant* \mathbf{R}' of \mathbf{R} by

$$\mathbf{R}' := \{u \in \text{End}(\mathcal{E}_{x_\star}) \mid \forall v \in \mathbf{R}, uv = vu\}.$$

We then have

Theorem 3.6. *There exists a natural isomorphism*

$$\Phi : \mathbf{R}' \rightarrow \ker \nabla_X^{\text{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))}.$$

In particular, these spaces have the same dimension:

$$\dim(\ker \nabla_X^{\text{End}(\mathcal{E})} |_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))}) = \dim(\mathbf{R}^I) = \sum_{i=1}^K n_i^2.$$

3.1.4. *Invariant sections.* To conclude this subsection, we now investigate the existence of smooth *invariant* sections of the bundle $\mathcal{E} \rightarrow \mathcal{M}$, that is, elements of $\ker \nabla_X^\mathcal{E} |_{C^\infty(\mathcal{M}, \mathcal{E})}$. First of all, observe that if $u \in C^\infty(\mathcal{M}, \mathcal{E})$ is an invariant section, then $u_\star := u(x_\star)$ is invariant by the \mathbf{G} -action. The converse is also true:

Lemma 3.7. *Assume that there exists $u_\star \in \mathcal{E}_{x_\star}$ such that $\rho(g)u_\star = u_\star$ for all $g \in \mathbf{G}$. Then there exists (a unique) $u \in C^\infty(\mathcal{M}, \mathcal{E})$ such that $u(x_\star) = u_\star$ and $\nabla_X^\mathcal{E}u = 0$.*

Such an approach turns out to be useful when trying to understand a sort of *weak version* of Livšic theory, such as the following: if $\mathcal{E} \rightarrow \mathcal{M}$ is a vector bundle equipped with a unitary connection $\nabla^\mathcal{E}$ and for each periodic orbit $\gamma \in \mathcal{G}$, there exists a section $u_\gamma \in C^\infty(\gamma, \mathcal{E}|_\gamma)$ such that $\nabla_X^\mathcal{E}u_\gamma = 0$, then one can wonder whether this implies the existence of a global invariant smooth section $u \in C^\infty(\mathcal{M}, \mathcal{E})$. It turns out that the answer depends on the rank of \mathcal{E} :

Lemma 3.8. *Assume that $\text{rank}(\mathcal{E}) \leq 2$ and that for all periodic orbits $\gamma \in \mathcal{G}$, there exists $u_\gamma \in C^\infty(\gamma, \mathcal{E}|_\gamma)$ such that $\nabla_X^\mathcal{E}u_\gamma = 0$. Then there exists $u \in C^\infty(\mathcal{M}, \mathcal{E})$ such that $\nabla_X^\mathcal{E}u = 0$.*

We shall see that the proof of this lemma is purely representation-theoretic and completely avoids the need to understand dynamics and the distribution of periodic orbits. We leave as an exercise the fact that Lemma 3.8 does not hold when $\text{rank}(\mathcal{E}) \geq 3$. A simple counter-example can be built using the following argument: any matrix in $\text{SO}(3)$ preserves an axis; hence, taking any $\text{SO}(3)$ -connection on a *real* vector bundle of rank 3 and then complexifying the bundle, one gets a vector bundle and a connection satisfying the assumptions of Lemma 3.8; it then suffices to produce an $\text{SO}(3)$ -connection without any invariant sections.

We believe that other links between properties of the representation ρ and the geometry and/or dynamics of the parallel transport along the flow lines could be discovered. To conclude, let us also mention that all the results are presented here for complex vector bundles; most of them could be naturally restated for real vector bundles modulo obvious modifications in the statements.

3.2. Dynamical preliminaries on Anosov flows

3.2.1. *Shadowing lemma and homoclinic orbits.* Fix an arbitrary Riemannian metric g on \mathcal{M} . As usual, we define the *local strong (un)stable manifolds* as

$$\begin{aligned} W_\delta^s(x) &:= \{y \in \mathcal{M} \mid \forall t \geq 0, d(\varphi_t y, \varphi_t x) < \delta, d(\varphi_t x, \varphi_t y) \rightarrow_{t \rightarrow +\infty} 0\}, \\ W_\delta^u(x) &:= \{y \in \mathcal{M} \mid \forall t \leq 0, d(\varphi_t y, \varphi_t x) < \delta, d(\varphi_t x, \varphi_t y) \rightarrow_{t \rightarrow -\infty} 0\}, \end{aligned}$$

where $\delta > 0$ is chosen small enough. For $\delta = \infty$, we obtain the sets $W^{s,u}(x)$ which are the strong stable/unstable manifolds of x . We also set $W_{loc}^{s,u}(x) := W_{\delta_0}^{s,u}(x)$ for some fixed $\delta_0 > 0$ small enough. The *local weak (un)stable manifolds* $W_{\delta}^{ws,wu}(x)$ are the sets of points $y \in B(x, \delta)$ such that there exists $t \in \mathbb{R}$ with $|t| < \delta$ and $\varphi_t y \in W_{loc}^{s,u}(x)$. The following lemma is known as the *local product structure* (see [32, Theorem 5.1.1] for more details):

Lemma 3.9. *There exist $\varepsilon_0, \delta_0 > 0$ small enough such that for all $x, y \in \mathcal{M}$ such that $d(x, y) < \varepsilon_0$, the intersection $W_{\delta_0}^{wu}(x) \cap W_{\delta_0}^s(y)$ is a single point $\{z\}$. We write $z := \llbracket x, y \rrbracket$.*

The main tool we will use to construct suitable *homoclinic orbits* is the following classical shadowing property of Anosov flows for which we refer to [55, Corollary 18.1.8], [32, Theorem 5.3.2] and [32, Proposition 6.2.4]. For the sake of simplicity, we now write $\gamma = [xy]$ if γ is an orbit segment of the flow with endpoints x and y .

Theorem 3.10. *There exist $\varepsilon_0, \theta, C > 0$ with the following property. Consider $\varepsilon < \varepsilon_0$, and a finite or infinite sequence of orbit segments $\gamma_i = [x_i y_i]$ of length T_i greater than 1 such that for any n , $d(y_n, x_{n+1}) \leq \varepsilon$. Then there exists a genuine orbit γ and times τ_i such that γ restricted to $[\tau_i, \tau_i + T_i]$ shadows γ_i up to $C\varepsilon$. More precisely, for all $t \in [0, T_i]$, one has*

$$d(\gamma(\tau_i + t), \gamma_i(t)) \leq C\varepsilon e^{-\theta \min(t, T_i - t)}. \tag{3.3}$$

Moreover, $|\tau_{i+1} - (\tau_i + T_i)| \leq C\varepsilon$. Finally, if the sequence of orbit segments γ_i is periodic, then the orbit γ is periodic.

It is instructive for the reader to have Figure 2(a) in mind, where the upper curve corresponds to the orbit γ approximating the segment given by the lower curve. Let us also make the following important comment. In the theorem, one can also allow the first orbit segment γ_i to be infinite on the left, and the last orbit segment γ_j to be infinite on the right. In this case, (3.3) should be replaced by: assuming that γ_i is defined on $(-\infty, 0]$ and γ_j on $[0, +\infty)$, we would get for some $\tilde{\tau}_{i+1}$ within $C\varepsilon$ of τ_{i+1} , and all $t \geq 0$,

$$d(\gamma(\tilde{\tau}_{i+1} - t), \gamma_i(-t)) \leq C\varepsilon e^{-\theta t}, \quad d(\gamma(\tau_j + t), \gamma_j(t)) \leq C\varepsilon e^{-\theta t}.$$

Fix an arbitrary periodic point $x_\star \in \mathcal{M}$ of period T_\star and denote by γ_\star its primitive orbit.

Definition 3.11 (Homoclinic orbits). A point $p \in \mathcal{M}$ is said to be *homoclinic* to x_\star if $p \in W^{ws}(x_\star) \cap W^{wu}(x_\star)$ (in other words, $d(\varphi_{t+t_0^\pm} p, \varphi_t x_\star) \rightarrow_{t \rightarrow \pm\infty} 0$ for some $t_0^\pm \in \mathbb{R}$). We say that an orbit γ is homoclinic to x_\star if it contains a point $p \in \gamma$ that is homoclinic to x_\star , and we denote by $\mathcal{H} \subset \mathcal{M}$ the set of orbits homoclinic to x_\star .

Note that by hyperbolicity, the convergence of the point p to x_\star is exponentially fast. More precisely, let γ be the orbit of p and let $\mathbb{R} \ni t \mapsto \gamma(t)$ be the flow parametrization of γ . Then there exist uniform constants $C, \theta > 0$ (independent of γ) and $A_\pm \in \mathbb{R}$

(depending on γ) such that

$$d(\gamma(A_{\pm} \pm nT_{\star}), x_{\star}) \leq C e^{-\theta n}. \tag{3.4}$$

The points $\gamma(A_{\pm})$ correspond to an arbitrary choice of points in $W_{\delta_0}^{s,u}(x_{\star}) \cap \gamma$ (for an arbitrary $\delta_0 > 0$ small enough). Homoclinic orbits have infinite length (except the orbit of x_{\star} itself) but it will be convenient to introduce a notion of *length* T_{γ} which we define to be $T_{\gamma} := A_+ - A_-$ (note that this is a highly non-canonical definition). We define the *trunk* to be the central segment $\gamma([A_-, A_+])$. In other words, the length of γ is equal to the length of its trunk. We also define the points $x_n^{\pm} := \gamma(A_{\pm} \pm nT_{\star})$. Note that another choice of values A'_{\pm} has to differ from A_{\pm} by mT_{\star} for some $m \in \mathbb{Z}$. Homoclinic orbits will play a key role, as we shall see in due course.

Lemma 3.12. *Assume that the flow is transitive. Then the set \mathcal{W} of points belonging to a homoclinic orbit in \mathcal{H} is dense in \mathcal{M} .*

Proof. This is a straightforward consequence of the shadowing Theorem 3.10: one concatenates a long segment S of a transitive orbit with γ_{\star} , i.e. one applies Theorem 3.10 with $\dots \gamma_{\star} \gamma_{\star} S \gamma_{\star} \gamma_{\star} \dots$. ■

Remark 3.13. In the particular case of an Anosov geodesic flow on the unit tangent bundle, one can check that \mathcal{H} is in one-to-one correspondence with $\pi_1(M)/\langle \tilde{\gamma}_{\star} \rangle$, where $\tilde{\gamma}_{\star} \in \pi_1(M)$ is any element such that the conjugacy class of $\tilde{\gamma}_{\star}$ in $\pi_1(M)$ corresponds⁶ to the free homotopy class $c \in \mathcal{C}$ whose unique geodesic representative is γ_{\star} .

3.2.2. *Applications of the Ambrose–Singer formula.* Consider a Hermitian vector bundle \mathcal{E} over (\mathcal{M}, g) equipped with a unitary connection $\nabla = \nabla^{\mathcal{E}}$. If $x, y \in \mathcal{M}$ are at a distance less than the injectivity radius of \mathcal{M} , denote by $C_{x \rightarrow y} : \mathcal{E}_x \rightarrow \mathcal{E}_y$ the parallel transport with respect to $\nabla^{\mathcal{E}}$ along the shortest geodesic from x to y , by $C(x, t) : \mathcal{E}_x \rightarrow \mathcal{E}_{\varphi_t x}$ the parallel transport along the flow and by C_{γ} the parallel transport along a curve γ . For $U \in C^{\infty}(\mathcal{M}, \text{End}(\mathcal{E}))$, we define, for all $x \in \mathcal{M}$,

$$\|U\|_x := \text{Tr}(U^*(x)U(x))^{1/2},$$

and $\|U\|_{L^{\infty}} := \sup_{x \in \mathcal{M}} \|U\|_x$. In particular, if $U \in C^{\infty}(\mathcal{M}, \text{U}(\mathcal{E}))$ is unitary, then $\|U\|_{L^{\infty}} = \sqrt{\text{rank}(\mathcal{E})}$. We record the following consequences of Lemma 2.2:

Lemma 3.14. *The following consequences of the Ambrose–Singer formula hold:*

- (1) *Assume we are in the setting of Theorem 3.10: for some $C, \varepsilon, T > 0$, let $x, p \in \mathcal{M}$ satisfy $d(\varphi_t x, \varphi_t p) \leq C \varepsilon e^{-\theta \min(t, T-t)}$ for all $t \in [0, T]$. Then for any $0 \leq T_1 \leq T$,*

$$\|C(\varphi_{T_1} x, -T_1) C_{\varphi_{T_1} p \rightarrow \varphi_{T_1} x} C(p, T_1) C_{x \rightarrow p} - \mathbb{1}_{\mathcal{E}_x}\|_x \leq \frac{c_0 C \varepsilon}{\theta} \|F_{\nabla}\|_{C^0},$$

where $c_0 = c_0(X, g) > 0$ depends only on the vector field X and the metric.

⁶Recall that the set \mathcal{C} of free homotopy classes is in one-to-one correspondence with the conjugacy classes of $\pi_1(M)$ [49, Chapter 1].

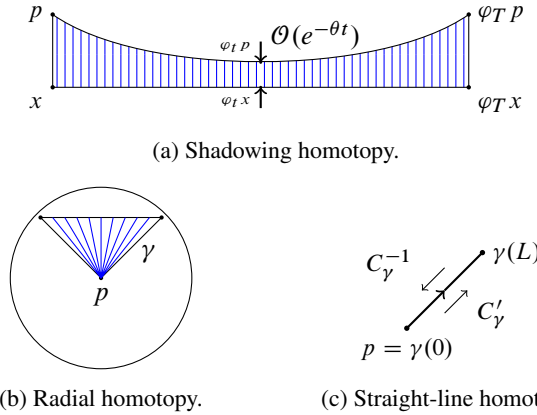


Fig. 2. Presentation of the geometries considered in Lemma 3.14.

(2) Assume $\gamma \subset B(p, \iota/2)$ is a closed piecewise smooth curve at p of length L , where ι denotes the injectivity radius of (\mathcal{M}, g) . Then for some $C = C(g) > 0$ depending on the metric,

$$\|C_\gamma - \text{Id}_p\|_p \leq CL \cdot \sup_{y \in \gamma} d(p, y) \cdot \|F_\nabla\|_{C^0}.$$

(3) Let $\gamma : [0, L] \rightarrow M$ be a unit speed curve based at p , and ∇' be a second unitary connection on \mathcal{E} , whose parallel transport along γ we denote by C'_γ . Then

$$\|C_\gamma^{-1} C'_\gamma - \text{Id}_p\|_p \leq L \|\nabla - \nabla'\|_{C^0}.$$

The geometries appearing in (1)–(3) are depicted in Figure 2 (a)–(c), respectively.

Proof. We first prove (1). For C, ε small enough, for all $t \in [0, T]$ we denote by τ_t the unit speed shortest geodesic, of length $\ell(t)$, from $\varphi_t x$ to $\varphi_t p$. Define a smooth homotopy $\Gamma : [0, 1]^2 \rightarrow \mathcal{M}$ by setting

$$\Gamma(s, t) := \tau_{tT_1}(s\ell(t)),$$

and note that by assumption $\ell(t) \leq C\varepsilon e^{-\theta \min(t, T-t)}$. We apply Lemma 2.2 to the homotopy Γ to obtain, after a rescaling of parameters s and t ,

$$\begin{aligned} & C(\varphi_{T_1} x, -T_1) C_{\varphi_{T_1} p \rightarrow \varphi_{T_1} x} C(p, T_1) C_{x \rightarrow p} - \mathbb{1}_{\mathcal{E}_x} \\ &= \int_0^{T_1} \int_0^{\ell(t)} C_\uparrow^{-1}(s, t) F_\nabla(\partial_t \tau_t(s), \partial_s \tau_t(s)) C_\rightarrow(s, t) ds dt. \end{aligned}$$

Here we recall that C_\uparrow and C_\rightarrow are parallel transport maps obtained by parallel transport along curves as in Figure 1. Since C_\uparrow and C_\rightarrow are isometries, and since by compactness $|\partial_t \tau_t(s)| \leq D$ for some $0 < D = D(X, g)$, we have

$$\begin{aligned} & \|C(\varphi_{T_1} x, -T_1) C_{\varphi_{T_1} p \rightarrow \varphi_{T_1} x} C(p, T_1) C_{x \rightarrow p} - \mathbb{1}_{\mathcal{E}_x}\|_x \\ & \leq CD\varepsilon \|F_\nabla\|_{C^0} \int_0^T e^{-\theta \min(t, T-t)} dt \leq \frac{2DC}{\theta} \varepsilon \|F_\nabla\|_{C^0}. \end{aligned}$$

For (2), we may assume by approximation that γ is smooth. Then taking the homotopy

$$\Gamma(s, t) = \exp_x(t \exp_x^{-1}(\gamma(sL))),$$

and applying Lemma 2.2, we obtain, by a rescaling of s and writing $\tilde{\Gamma}(s, t) = \Gamma(s/L, t)$,

$$C_\gamma - \text{Id}_x = \int_0^L \int_0^1 C_1(s, t)^{-1} F_\nabla(\partial_s \tilde{\Gamma}, \partial_t \tilde{\Gamma}) C_2(s, t) dt ds.$$

The estimate now follows by using $\|\partial_t \tilde{\Gamma}\| \leq Cd(x, \gamma(s))$, where we introduce the positive constant $C = \sup_{x \in M} \sup_{|y|_{g_x} < l/2} \|d \exp_x(y)\|_{T_x M \rightarrow T_{\exp_x(y)}}$.

For (3), denote by C_t, C'_t the parallel transports along $\gamma|_{[0,t]}$ with the connections ∇, ∇' , respectively. Then it is straightforward that

$$\partial_t(C_t^{-1}C'_t) = C_t^{-1}(\nabla - \nabla')(\dot{\gamma}(t))C'_t,$$

so

$$C_\gamma^{-1}C'_\gamma - \text{Id}_p = \int_0^L C_t^{-1}(\nabla - \nabla')(\dot{\gamma}(t))C'_t dt.$$

The required estimate follows. ■

We also have the following result to which we will refer to as the *spiral lemma*:

Lemma 3.15. *Let $x_\star \in \mathcal{M}$ be a periodic point of period T_\star and let $x_0 \in W_{\text{loc}}^s(x_\star)$. Define $x_n := \varphi_{nT_\star} x_0$ and write $q_n := C(x_\star, nT_\star)^{-1}C_{x_n \rightarrow x_\star}C(x_0, nT_\star)C_{x_\star \rightarrow x_0}$. Then*

$$\rho(x_0) := \lim_{n \rightarrow \infty} q_n \in \text{U}(\mathcal{E}_{x_\star})$$

exists. Moreover, there exist uniform constants $C, \theta > 0$ such that

$$|q_n - \rho(x_0)| \leq Ce^{-\theta n}.$$

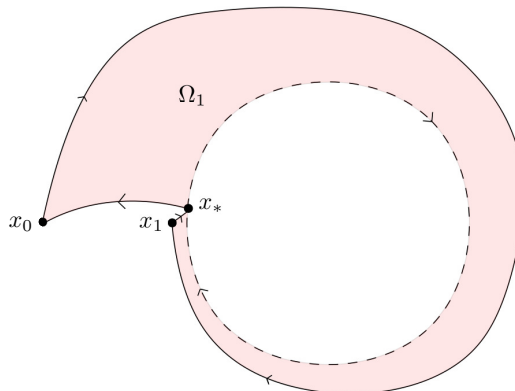


Fig. 3. The spiral lemma: the set Ω_1 corresponds to the area over which the integral in the Ambrose–Singer formula is computed for $n = 1$.

Proof. Apply the Ambrose–Singer formula as in the first item of the previous lemma (same notations as in the previous proof):

$$q_n - \mathbb{1}_{\mathcal{E}_{x_\star}} = \int_0^{nT_\star} \int_0^{\ell(t)} C_{\uparrow}^{-1}(s, t) F_{\nabla}(\partial_t \tau_t(s), \partial_s \tau_t(s)) C_{\rightarrow}(s, t) ds dt,$$

where τ_t is the unit speed shortest geodesic of length $\ell(t)$ from $\varphi_t x_0$ to $\varphi_t x_\star$. Observe that this integral converges absolutely because (see (3.4))

$$\begin{aligned} \int_0^{nT_\star} \left\| \int_0^{\ell(t)} C_{\uparrow}^{-1}(s, t) F_{\nabla}(\partial_t \tau_t(s), \partial_s \tau_t(s)) C_{\rightarrow}(s, t) ds \right\| dt \\ \leq \int_0^{nT_\star} C \|F_{\nabla}\|_{C^0} e^{-\theta t} dt < \infty, \end{aligned}$$

and thus the limit exists. Moreover, it is clear that the convergence is exponential. ■

3.3. Proof of the exact Livšić cocycle theorem

3.3.1. *Parry’s free monoid.* As we shall see, Parry’s free monoid is the key notion to understand the holonomy of unitary connections. Whereas flat connections up to gauge equivalence correspond to representations of the fundamental group up to conjugacy, in the setting of hyperbolic dynamics, we will show that *arbitrary* connections up to cocycle equivalence correspond to representations of Parry’s free monoid. Recall from Section 3.2.1 that $x_\star \in \mathcal{M}$ is a periodic point of period T_\star . Let \mathbf{G} be the free monoid generated by \mathcal{H} (homoclinic orbits to x_\star), that is, the formal set of words

$$\mathbf{G} := \{\gamma_1^{m_1} \dots \gamma_k^{m_k} \mid k \in \mathbb{N}, m_1, \dots, m_k \in \mathbb{N}_0, \gamma_1, \dots, \gamma_k \in \mathcal{H}\},$$

endowed with the obvious monoid structure. The empty word corresponds to the identity element denoted by $\mathbf{1}_{\mathbf{G}}$. Note the periodic orbit corresponding to x_\star also belongs to the set of homoclinic orbits. We call \mathbf{G} *Parry’s free monoid* as the idea (although not written like this) was first introduced in his work [75] (see also [90] for a related approach). The main result of this subsection is the following:

Proposition 3.16. *Let $\nabla^\mathcal{E}$ be a unitary connection on the Hermitian vector bundle $\mathcal{E} \rightarrow \mathcal{M}$. Then $\nabla^\mathcal{E}$ induces a representation*

$$\rho : \mathbf{G} \rightarrow \mathrm{U}(\mathcal{E}_{x_\star}).$$

Formally, this proposition could have also been stated as a definition.

Proof. Since \mathbf{G} is a free monoid, it suffices to define ρ on the set of generators of \mathbf{G} , namely for all homoclinic orbits $\gamma \in \mathcal{H}$. For the neutral element we set $\rho(\mathbf{1}_{\mathbf{G}}) = \mathbb{1}_{\mathcal{E}_{x_\star}}$. For the periodic orbit γ_\star of x_\star , we set $\rho(\gamma_\star) := C(x_\star, T_\star)$.

Let $\gamma \in \mathcal{H}$ (and $\gamma \neq \gamma_\star$) and consider a parametrization $\mathbb{R} \ni t \mapsto \gamma(t)$. Following the notations of Section 3.2.1, we let $x_n^\pm := \gamma(A_\pm \pm nT_\star)$, $x_n^+ = \varphi_{T_n}(x_n^-)$ for some

$T_n = A_+ - A_- + 2nT_*$, where $T_\gamma := A_+ - A_-$ (length of the trunk), and the points $(x_n^\pm)_{n \in \mathbb{N}}$ converge exponentially fast to x_* as $n \rightarrow \infty$. As we shall see, there is a small technical issue coming from the fact that $C(x_*, T_*)$ is not trivial and this can be overcome by considering a subsequence $k_n \rightarrow \infty$ such that⁷

$$C(x_*, T_*)^{k_n} \rightarrow \mathbb{1}_{\mathcal{E}_{x_*}}, \quad n \rightarrow \infty. \tag{3.5}$$

For $n, m \in \mathbb{N}$, we define $\rho_{m,n}(\gamma) \in U(\mathcal{E}_{x_*})$ as follows:

$$\rho_{m,n}(\gamma) := C_{x_{k_m}^+ \rightarrow x_*} C(x_0^+, k_m T_*) C(x_0^-, T_\gamma) C(x_{k_n}^-, k_n T_*) C_{x_* \rightarrow x_{k_n}^-}, \tag{3.6}$$

and we will write $\rho_n(\gamma) := \rho_{n,n}(\gamma)$.

Lemma 3.17. *There exists $\rho(\gamma) \in U(\mathcal{E}_{x_*})$ such that*

$$\rho_{m,n}(\gamma) \xrightarrow{n,m \rightarrow \infty} \rho(\gamma),$$

and $\rho(\gamma)$ does not depend on the way the limit in n, m is taken.

Proof. We have by construction:

$$\begin{aligned} \rho_{m,n}(\gamma) &= C_{x_{k_m}^+ \rightarrow x_*} C(x_0^+, k_m T_*) C(x_0^-, T_\gamma) C(x_{k_n}^-, k_n T_*) C_{x_* \rightarrow x_{k_n}^-} \\ &= [C_{x_{k_m}^+ \rightarrow x_*} C(x_0^+, k_m T_*) C_{x_* \rightarrow x_0^+} C(x_*, k_m T_*)^{-1}] \\ &\quad \times C(x_*, k_m T_*) C_{x_0^+ \rightarrow x_*} C(x_0^-, T_\gamma) C_{x_* \rightarrow x_0^-} C(x_*, k_n T_*) \\ &\quad \times [C(x_*, k_n T_*)^{-1} C_{x_0^- \rightarrow x_*} C(x_{k_n}^-, k_n T_*) C_{x_* \rightarrow x_{k_n}^-}], \end{aligned} \tag{3.7}$$

where T_γ is independent of n (trunk of γ). For the middle term we have, by (3.5),

$$\begin{aligned} C(x_*, k_m T_*) C_{x_0^+ \rightarrow x_*} C(x_0^-, T_\gamma) C_{x_* \rightarrow x_0^-} C(x_*, k_n T_*) \\ = C_{x_0^+ \rightarrow x_*} C(x_0^-, T_\gamma) C_{x_* \rightarrow x_0^-} + o(1), \end{aligned}$$

as n, m go to ∞ . Moreover, the convergence of the terms in brackets follows from the spiral lemma 3.15 (the convergence is exponentially fast). ■

This concludes the proof of Proposition 3.16. ■

Remark 3.18. For $\gamma \in \mathcal{H}$, (3.7) shows that $\rho(\gamma)$ does not depend on the choice of subsequence $(k_n)_{n \in \mathbb{N}}$ as long as $C(x_*, T_*)^{k_n} \rightarrow \mathbb{1}$. However, $\rho(\gamma)$ does depend on the choice of trunk $[x_0^- x_0^+]$ for γ and another choice of trunk produces a $\rho'(\gamma)$ which differs from $\rho(\gamma)$:

$$\rho'(\gamma) = C(x_*, T_*)^{m_L(\gamma)} \rho(\gamma) C(x_*, T_*)^{m_R(\gamma)}, \tag{3.8}$$

where $m_L(\gamma), m_R(\gamma) \in \mathbb{Z}$.

⁷For any compact metric group G , if $g \in G$, there exists a sequence $k_n \in \mathbb{N}$ such that $g^{k_n} \rightarrow_{n \rightarrow \infty} \mathbf{1}_G$.

3.3.2. *Conjugate representations.* We introduce the submonoid $\mathbf{G}^* := \mathbf{G} \setminus \{\gamma_\star^k \mid k \geq 1\}$, that is, \mathbf{G} minus the powers of γ_\star . Recall that the character of a representation ρ is defined by $\chi_\rho(\bullet) := \text{Tr}(\rho(\bullet))$. This subsection is devoted to proving the following:

Proposition 3.19. *Let $\nabla^{\mathcal{E}_{1,2}}$ be two unitary connections on the Hermitian vector bundles $\mathcal{E}_1, \mathcal{E}_2 \rightarrow \mathcal{M}$. Assume that the connections have trace-equivalent holonomies in the sense of Definition 3.1. Then the induced representations $\rho_{1,2} : \mathbf{G}^* \rightarrow \text{U}(\mathcal{E}_{1,2x_\star})$ have the same character. In particular, this implies that they are isomorphic, i.e. there exists $p_\star \in \text{U}(\mathcal{E}_{2x_\star}, \mathcal{E}_{1x_\star})$ such that*

$$\forall \gamma \in \mathbf{G}, \quad \rho_1(\gamma) = p_\star \rho_2(\gamma) p_\star^{-1}. \tag{3.9}$$

Proof. Following Lemma 3.17, we consider a subsequence $(k_n)_{n \in \mathbb{N}}$ such that $C_{1,2}(x_\star, T_\star)^{k_n} \rightarrow 1$. Once we know that the representations have the same character, the conclusion is a straightforward consequence of a general fact of representation theory [63, Corollary 3.8]. For the sake of simplicity, we take $\gamma = \gamma_1 \cdot \gamma_2$, where $\gamma_{1,2} \in \mathcal{H}$ (and both $\gamma_{1,2}$ cannot be equal to γ_\star at the same time since the word γ is in \mathbf{G}^*) but the generalization to longer words is straightforward as we shall see and words of length 1 are also handled similarly (one does not even need to concatenate orbits in this case). The empty word (corresponding to the identity element in \mathbf{G}^*) will also be dealt with separately. This proposition is based on the shadowing theorem 3.10 and the fact that one can concatenate orbits. But we will have to be careful to produce periodic orbits which are primitive.

By Lemma 3.17 we have

$$\rho_1(\gamma) = \rho_1(\gamma_1)\rho_1(\gamma_2) = \rho_{1;n,N}(\gamma_1)\rho_{1;n,n}(\gamma_2) + o(1), \quad n \rightarrow \infty,$$

where we use $\rho_{i;m,n}$ to denote the expression in (3.6) with respect to $\nabla^{\mathcal{E}_i}$ for $i = 1, 2$. The term $N = N(n) \geq n$ will ensure that a certain orbit is primitive as we shall see below. Let $x_n^\pm(i)$ be the points on the orbit γ_i that are exponentially close to x_\star , given by Section 3.2.1. Consider the concatenation of the orbits $S := [x_{k_N}^-(1)x_{k_n}^+(1)] \cup [x_{k_n}^-(2)x_{k_n}^+(2)]$. Note that the starting points and endpoints of these segments are at distance at most $\mathcal{O}(e^{-\theta k_n})$. Thus by the shadowing theorem 3.10, there exists a genuine periodic orbit $\tilde{\gamma}_n$ and a point $y_n \in \tilde{\gamma}_n$ (of period T'_n) which $\mathcal{O}(e^{-\theta k_n})$ -shadows the concatenation S (here, if we have a longer word of length k , it suffices to apply the shadowing Theorem 3.10 with k segments).

We claim that $\tilde{\gamma}_n$ is primitive for all N large enough. Indeed, observe that $\tilde{\gamma}_n$ can be decomposed into the following six subsegments as in Figure 4.

Moreover, the total length of $\tilde{\gamma}_n$ is

$$T'_n = T_{\gamma_2} + 2k_n T_\star + T_{\gamma_1} + (k_N + k_n)T_\star + \mathcal{O}(e^{-\theta k_n}).$$

Take $x \in \gamma_1 \cup \gamma_2$ with $x \notin \gamma_\star$, and consider a small $\varepsilon > 0$ such that $d(x, \gamma_\star) > 3\varepsilon$. Let n be large enough so that for all $m \geq n$ the tail $[x_m^-(1)x_n^-(1)]$ is in $B(\gamma_\star, \varepsilon)$, $\tilde{\gamma}_n$ satisfies $d(\tilde{\gamma}_n, x) < \varepsilon$ and finally the shadowing factor of Theorem 3.10 satisfies $\mathcal{O}(e^{-\theta k_n}) < \varepsilon$.

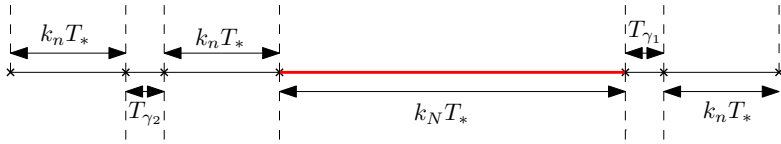


Fig. 4. The orbit $\tilde{\gamma}_n$ is made up of six segments: in the first segment (of length $k_n T_*$), it shadows the first portion $[x_{k_n}^-(2)x_0^-(2)]$ which wraps around γ_* ; in the second (of length T_{γ_2}), it shadows the trunk $[x_0^-(2)x_0^+(2)]$, in the third (of length $k_n T_*$), it shadows the last portion $[x_0^+(2)x_{k_n}^+(2)]$ which also wraps around γ_* ; then this process is repeated but for the second orbit γ_1 .

Pick $N \geq n$ such that $(k_N - k_n)T_* > T'_n/2$. We argue by contradiction and assume that $\tilde{\gamma}_n = \gamma_0^k$ for some $k \geq 2$ and $\gamma_0 \in \mathcal{G}^\#$, a primitive orbit.

This implies that there is a copy of γ_0 in the central red segment of Figure 4 which $\mathcal{O}(e^{-\theta k_n})$ -shadows the orbit of $x_N^-(1)$ and this forces $\tilde{\gamma}_n \subset B(\gamma_*, 2\varepsilon)$. Thus $d(\tilde{\gamma}_n, x) > \varepsilon$, which is a contradiction.

By the first and second items of Lemma 3.14, we have

$$\rho_{1;n,N}(\gamma_1)\rho_{1;n,n}(\gamma_2) = C_{1,y_n \rightarrow x_*} C_1(y_n, T'_n) C_{1,y_n \rightarrow x_*}^{-1} + \mathcal{O}(e^{-\theta k_n}).$$

By assumption, we have $\text{Tr}(C_1(y_n, T'_n)) = \text{Tr}(C_2(y_n, T'_n))$. This yields

$$\begin{aligned} \text{Tr}(\rho_1(\gamma)) &= \text{Tr}(C_{1,y_n \rightarrow x_*} C_1(y_n, T'_n) C_{1,y_n \rightarrow x_*}^{-1}) + o(1) \\ &= \text{Tr}(C_1(y_n, T'_n)) + o(1) \\ &= \text{Tr}(C_2(y_n, T'_n)) + o(1) = \text{Tr}(\rho_2(\gamma)) + o(1). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we obtain the claimed result about characters for all non-empty words $\gamma \in \mathbf{G}^*$.

It remains to deal with the empty word. For that, take any $\gamma \in \mathbf{G}^*$, and consider a subsequence $n_i \in \mathbb{N}$ such that $\rho_1(\gamma)^{n_i} \rightarrow \mathbb{1}_{\mathcal{E}_{1,x_*}}$ and $\rho_2(\gamma)^{n_i} \rightarrow \mathbb{1}_{\mathcal{E}_{2,x_*}}$. Then

$$\text{Tr}(\rho_1(\gamma)^{n_i}) = \text{Tr}(\rho_2(\gamma)^{n_i}),$$

and taking the limit as $i \rightarrow \infty$ gives

$$\text{Tr}(\rho_1(\mathbf{1}_{\mathbf{G}^*})) = \text{rank}(\mathcal{E}_1) = \text{rank}(\mathcal{E}_2) = \text{Tr}(\rho_2(\mathbf{1}_{\mathbf{G}^*})).$$

By [63, Corollary 3.8], there is a p_* satisfying (3.9) for $\gamma \in \mathbf{G}^*$.

It is now straightforward to show (3.9) for all $\gamma \in \mathbf{G}$. Applying (3.9) with $\gamma_*\gamma \in \mathbf{G}^*$, where $\gamma \in \mathcal{H} \setminus \{\gamma_*\}$ is arbitrary, we get

$$\rho_1(\gamma_*\gamma) = \rho_1(\gamma_*)\rho_1(\gamma) = p_*\rho_2(\gamma_*\gamma)p_*^{-1} = p_*\rho_2(\gamma_*)p_*^{-1}p_*\rho_2(\gamma)p_*^{-1}.$$

Since $\rho_1(\gamma) = p_*\rho_2(\gamma)p_*^{-1}$ (because $\gamma \in \mathbf{G}^*$), we find that $\rho_1(\gamma_*) = p_*\rho_2(\gamma_*)p_*^{-1}$, that is, $C_1(x_*, T_*) = p_*C_2(x_*, T_*)p_*^{-1}$ or equivalently $P(x_*, T_*)p_* = p_*$ (where P denotes the parallel transport along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$ with respect to the mixed connection $\nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_2}, \nabla^{\mathcal{E}_1})}$, as in (2.4)), concluding the proof. ■

Remark 3.20. Although the representations $\rho_{1,2}$ depend on choices (namely on the choice of trunk for each homoclinic orbit $\gamma \in \mathcal{H}$), the map $p_\star \in U(\mathcal{E}_{x_\star})$ does not. Indeed, taking two other representations $\rho'_{1,2}$ (for some other choices of trunks), by (3.8) one gets

$$\begin{aligned} \rho'_1(\gamma) &= C_1(x_\star, T_\star)^{m_L(\gamma)} \rho_1(\gamma) C_1(x_\star, T_\star)^{m_R(\gamma)} \\ &= C_1(x_\star, T_\star)^{m_L(\gamma)} p_\star \rho_2(\gamma) p_\star^{-1} C_1(x_\star, T_\star)^{m_R(\gamma)} \\ &= C_1(x_\star, T_\star)^{m_L(\gamma)} p_\star C_2(x_\star, T_\star)^{-m_L(\gamma)} \rho'_2(\gamma) C_2(x_\star, T_\star)^{-m_R(\gamma)} p_\star^{-1} C_1(x_\star, T_\star)^{m_R(\gamma)} \\ &= (P(x_\star, m_L(\gamma) T_\star) p_\star) \rho'_2(\gamma) (P(x_\star, m_R(\gamma) T_\star) p_\star)^{-1} = p_\star \rho'_2(\gamma) p_\star^{-1}, \end{aligned}$$

since $P(x_\star, T_\star) p_\star = p_\star$, that is, p_\star also conjugates the representations $\rho'_{1,2}$. Note that the map p_\star given by [63, Corollary 3.8] is generally not unique. Nevertheless, if the representation is irreducible, it is unique modulo the trivial S^1 -action.

3.3.3. *Proof of Theorem 3.2.* Let \mathcal{W} be the set of all points belonging to homoclinic orbits in \mathcal{H} . By Lemma 3.12, \mathcal{W} is dense in \mathcal{M} and we are going to define the map p (which will conjugate the cocycles) on \mathcal{W} and then show that p is Lipschitz-continuous on \mathcal{W} so that it extends naturally to \mathcal{M} . The map p is defined as the parallel transport of p_\star with respect to the mixed connection.

By assumptions, $C_i(x_\star, T_\star)^{k_n} \rightarrow \mathbb{1}_{\mathcal{E}_x}$, and thus $P(x_\star, T_\star)^{k_n} \rightarrow \mathbb{1}_{\text{Hom}(\mathcal{E}_{2x_\star}, \mathcal{E}_{1x_\star})}$ (where $\mathbb{1}_{\text{Hom}(\mathcal{E}_{2x_\star}, \mathcal{E}_{1x_\star})}(q) = q$ for $q \in \text{Hom}(\mathcal{E}_{2x_\star}, \mathcal{E}_{1x_\star})$). Consider a point $x \in \gamma$, where $\gamma \in \mathcal{H}$ is a homoclinic orbit, and also consider a parametrization of γ as in Section 3.2.1. For $n \in \mathbb{N}$ large enough, consider the point $x_n^- \in \gamma$ (which is exponentially close to x_\star) and write $x = \varphi_{T_n^-}(x_n^-)$ for some $T_n^- > 0$. Define

$$p_n^-(x) := P(x_{k_n}^-, T_{k_n}^-) P_{x_\star \rightarrow x_{k_n}^-} p_\star \in U(\mathcal{E}_{2x}, \mathcal{E}_{1x}).$$

Lemma 3.21. *Fix $\gamma \in \mathcal{H}$. Then for all $x \in \gamma$, there exists $p_-(x) \in U(\mathcal{E}_{2x}, \mathcal{E}_{1x})$ such that $p_n^-(x) \xrightarrow{n \rightarrow \infty} p_-(x)$. There exists $C > 0$ such that $|p_n^-(x) - p_-(x)| \leq C/n$. Moreover, $\nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_2}, \nabla^{\mathcal{E}_1})} p_- = 0$ on γ .*

In particular, this shows that p_- is smooth when restricted to γ as $\nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_2}, \nabla^{\mathcal{E}_1})}$ is elliptic on γ .

Proof of Lemma 3.21. By construction, the differential equation is clearly satisfied if the limit exists. Moreover, for some time T_0 (independent of n , $T_{k_n}^- = T_0 + k_n T_\star$) we have

$$\begin{aligned} p_n^-(x) &= P(x_{k_n}^-, T_{k_n}^-) P_{x_\star \rightarrow x_{k_n}^-} p_\star = P(x_0^-, T_0) P(x_{k_n}^-, k_n T_\star) P_{x_\star \rightarrow x_{k_n}^-} p_\star \\ &= P(x_0^-, T_0) P_{x_\star \rightarrow x_0^-} P(x_\star, T_\star)^{k_n} [P(x_\star, T_\star)^{-k_n} P_{x_0^- \rightarrow x_\star} P(x_{k_n}^-, k_n T_\star) P_{x_\star \rightarrow x_{k_n}^-} p_\star]. \end{aligned}$$

By assumption, the term outside the brackets converges as $n \rightarrow \infty$ and the term in brackets converges by the spiral lemma 3.15. ■

We now claim the following:

Lemma 3.22. *There exists a uniform constant $C > 0$ such that the following holds. Assume that x and z belong to two homoclinic orbits in \mathcal{H} and $z \in W_{\text{loc}}^u(x)$. Then*

$$\|P_{x \rightarrow z} p_-(x) - p_-(z)\| \leq Cd(x, z).$$

Proof. By the previous proofs, the point x is associated to points x_n^- on the homoclinic orbit and we will use the same notations for the point z associated to the points z_n^- . There is here a slight subtlety coming from the fact that the parametrizations of the homoclinic orbits γ were chosen in a non-canonical way (via a choice of A_{\pm}). In particular, it is not true that the flow lines of $z_{k_n}^-$ and $x_{k_n}^-$ shadow each other; in other words, we might not have $T_{k_n}^-(z) = T_{k_n}^-(x)$ but we rather have $T_{k_n}^-(z) = T_{k_n}^-(x) + mT_{\star}$ for some $m \in \mathbb{Z}$ depending on both x and z .

We have

$$\begin{aligned} \|P_{x \rightarrow z} p_-(x) - p_-(z)\| &= \|P_{x \rightarrow z} p_n^-(x) - p_n^-(z)\| + o(1) \\ &= \|P_{x \rightarrow z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_{\star} \rightarrow x_{k_n}^-} p_{\star} - P(z_{k_n}^-, T_{k_n}^-(z)) P_{x_{\star} \rightarrow z_{k_n}^-} p_{\star}\| + o(1) \\ &\leq C \|P_{z_{k_n}^- \rightarrow x_{\star}} P(z_{k_n}^-, T_{k_n}^-(z))^{-1} P_{x \rightarrow z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_{\star} \rightarrow x_{k_n}^-} p_{\star} - p_{\star}\| + o(1) \\ &\leq C \|P_{z_{k_n}^- \rightarrow x_{\star}} P(z_{k_n}^-, mT_{\star})^{-1} P_{x_{\star} \rightarrow z_{k_n}^-} \\ &\quad \times [P_{z_{k_n}^-} P(z_{k_n}^-, mT_{\star})^{-1} P_{x \rightarrow z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_{\star} \rightarrow x_{k_n}^-}] p_{\star} - p_{\star}\| \\ &\quad + o(1). \end{aligned}$$

Applying Lemma 3.14 (1), we have

$$\begin{aligned} \|P_{z_{k_n}^-} P(z_{k_n}^-, mT_{\star})^{-1} P_{x \rightarrow z} P(x_{k_n}^-, T_{k_n}^-(x)) P_{x_{\star} \rightarrow x_{k_n}^-} - \mathbb{1}_{\text{End}(\mathcal{E}_{x_{\star}})}\| \\ \leq Cd(x, z), \end{aligned}$$

where the constant $C > 0$ is uniform in n . Moreover, observe that

$$\lim_{n \rightarrow \infty} P_{z_{k_n}^-} P(z_{k_n}^-, mT_{\star})^{-1} P_{x_{\star} \rightarrow z_{k_n}^-} = P(x_{\star}, mT_{\star})^{-1}.$$

Hence

$$\|P_{x \rightarrow z} p_-(x) - p_-(z)\| \leq C (\|P(x_{\star}, mT_{\star})^{-1} p_{\star} - p_{\star}\| + d(x, z) + o(1)).$$

Since $P(x_{\star}, T_{\star}) p_{\star} = p_{\star}$ the first term on the right hand side vanishes. Letting $n \rightarrow \infty$, we obtain the announced result. ■

Note that we could have done the same construction “in the future” by considering instead

$$p_+(x) = \lim_{n \rightarrow \infty} P(x, T_{k_n}^+)^{-1} P_{x_{\star} \rightarrow x_{k_n}^+} p_{\star} \in U(\mathcal{E}_{2x}, \mathcal{E}_{1x}),$$

where $x_n^+ := \varphi_{T_n^+}(x)$ is exponentially closed to x_{\star} as in Section 3.2.1. A statement similar to Lemma 3.22 holds with the unstable manifold replaced by the stable one. We have

Lemma 3.23. For all $x \in \mathcal{W}$, $p_-(x) = p_+(x)$.

Proof. This follows from Proposition 3.19. Indeed,

$$\begin{aligned} \|p_-(x) - p_+(x)\| &= \|P(x_{k_n}^-, T_{k_n}^-)P_{x_\star \rightarrow x_{k_n}^-} p_\star - P(x, T_{k_n}^+)^{-1}P_{x_\star \rightarrow x_{k_n}^+} p_\star\| + o(1) \\ &\leq C \|P_{x_{k_n}^+ \rightarrow x_\star} P(x, T_{k_n}^+)P(x_{k_n}^-, T_{k_n}^-)P_{x_\star \rightarrow x_{k_n}^-} p_\star - p_\star\| + o(1) \\ &\leq C \|P_{x_{k_n}^+ \rightarrow x_\star} P(x_{k_n}^-, T_{k_n}^-)P_{x_\star \rightarrow x_{k_n}^-} p_\star - p_\star\| + o(1), \end{aligned}$$

where $T_n := T_n^- + T_n^+$. Observe that

$$\begin{aligned} P_{x_{k_n}^+ \rightarrow x_\star} P(x_{k_n}^-, T_{k_n}^-)P_{x_\star \rightarrow x_{k_n}^-} p_\star &= C_{1, x_{k_n}^+ \rightarrow x_\star} C_1(x_{k_n}^-, T_{k_n}^-)C_{1, x_\star \rightarrow x_{k_n}^-} p_\star (C_{2, x_{k_n}^+ \rightarrow x_\star} C_2(x_{k_n}^-, T_{k_n}^-)C_{2, x_\star \rightarrow x_{k_n}^-})^{-1} \\ &= \rho_{1,n}(\gamma) p_\star \rho_{2,n}(\gamma)^{-1} = \rho_1(\gamma) p_\star \rho_2(\gamma)^{-1} + o(1) = p_\star + o(1), \end{aligned}$$

by Proposition 3.19. Hence $\|p_-(x) - p_+(x)\| = o(1)$, that is, $p_-(x) = p_+(x)$. ■

We can now prove the following lemma:

Lemma 3.24. The map p_- is Lipschitz-continuous.

Proof. Consider $x, y \in \mathcal{W}$ close enough. Let $z := \llbracket x, y \rrbracket \in W_{\text{loc}}^{wu}(x) \cap W_{\text{loc}}^s(y)$ and define τ such that $\varphi_\tau(z) \in W_{\text{loc}}^u(x)$. Note that $|\tau| \leq Cd(x, y)$ for some uniform constant $C > 0$; also observe that the point z is homoclinic to the periodic orbit x_\star . We have

$$\begin{aligned} \|p_-(x) - P_{y \rightarrow x} p_-(y)\| &\leq \|p_-(x) - P_{z \rightarrow x} p_-(z)\| + \|p_-(z) - p_+(z)\| + \|P_{z \rightarrow x} p_+(z) - P_{y \rightarrow x} p_+(y)\| \\ &\quad + \|p_+(y) - p_-(y)\| \\ &\leq \|p_-(x) - P_{z \rightarrow x} p_-(z)\| + \|P_{x \rightarrow y} P_{z \rightarrow x} p_+(z) - p_+(y)\| \\ &\leq \|p_-(x) - P_{\varphi_\tau(z) \rightarrow x} p_-(\varphi_\tau(z))\| + \|P_{\varphi_\tau(z) \rightarrow x} p_-(\varphi_\tau(z)) - P_{z \rightarrow x} p_-(z)\| \\ &\quad + \|P_{x \rightarrow y} P_{z \rightarrow x} p_+(z) - p_+(y)\|, \end{aligned}$$

where the terms disappear between the third and fourth line by Lemma 3.23. By Lemma 3.22, the first term is controlled by

$$\|p_-(x) - P_{\varphi_\tau(z) \rightarrow x} p_-(\varphi_\tau(z))\| \leq Cd(x, \varphi_\tau(z)) \leq Cd(x, y).$$

As for the second term, using Lemma 3.14(2), we have

$$\begin{aligned} \|P_{\varphi_\tau(z) \rightarrow x} p_-(\varphi_\tau(z)) - P_{z \rightarrow x} p_-(z)\| &= \|P_{x \rightarrow z} P_{\varphi_\tau(z) \rightarrow x} P(z, \tau) p_-(z) - p_-(z)\| \\ &\leq Cd(x, y). \end{aligned}$$

Eventually, the last term $\|P_{x \rightarrow y} P_{z \rightarrow x} p_+(z) - p_+(y)\|$ is controlled similarly to the first term by applying Lemma 3.22 (but with the stable manifold instead of unstable). ■

As \mathcal{W} is dense, p_- extends to a Lipschitz-continuous map on \mathcal{M} which satisfies the equation $\nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_2}, \nabla^{\mathcal{E}_1})} p_- = 0$, and by [39, Theorem 4.1], this implies that p_- is smooth. This concludes the proof of Theorem 3.2.

3.3.4. Proof of the geometric properties.

Proof of Proposition 3.5. The equivalence between (1) and (2) can be found in [15, Section 5]. If $\mathcal{F} \subset \mathcal{E}$ is a non-trivial subbundle that is invariant by parallel transport along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$, it is clear that ρ will leave the space \mathcal{F}_{x_\star} invariant and thus is not irreducible. Conversely, if ρ is not irreducible, then there exists a non-trivial $\mathcal{F}_{x_\star} \subset \mathcal{E}_{x_\star}$ preserved by ρ . Let $\pi_\star : \mathcal{E}_{x_\star} \rightarrow \mathcal{F}_{x_\star}$ be the orthogonal projection. For x on a homoclinic orbit, define $\pi(x) : \mathcal{E}_x \rightarrow \mathcal{E}_x$ similarly to p_- in Lemma 3.21 by parallel transport of the section π_\star with respect to the connection $\nabla^{\text{End}(\mathcal{E})}$. Following the previous proofs (we only use $\rho\pi_\star = \pi_\star\rho$), one shows that π extends to a Lipschitz-continuous section on homoclinic orbits which satisfies $\pi^2 = \pi$ and $\nabla_X^{\text{End}(\mathcal{E})} \pi = 0$. By [39, Theorem 4.1], π extends to a smooth section, i.e. $\pi \in C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))$. Moreover, $\pi(x_\star) = \pi_\star$, hence π is the projection onto a non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$. ■

Proof of Theorem 3.6. The linear map $\Phi : \mathbf{R}' \rightarrow \ker \nabla_X^{\text{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))}$ is defined in the following way. Consider $u_\star \in \mathbf{R}'$ and define, as in Lemma 3.21, for x on a homoclinic orbit, $u_-(x)$ as the parallel transport of u_\star from x_\star to x along the orbit (with respect to the endomorphism connection $\nabla^{\text{End}(\mathcal{E})}$). Similarly, one can define $u_+(x)$ by parallel transport from the future. The fact that $u_\star \in \mathbf{R}'$ is then used in the following observation (see Lemma 3.23):

$$\|u_-(x) - u_+(x)\| = \|\rho(\gamma)u_\star\rho(\gamma)^{-1} - u_\star\| = 0.$$

(Note that $\rho(\gamma)u_\star\rho(\gamma)^{-1}$ corresponds formally to the parallel transport of u_\star with respect to $\nabla^{\text{End}(\mathcal{E})}$ from x_\star to x_\star along the homoclinic orbit γ .) Hence, by Lemma 3.24, u_- is Lipschitz-continuous and satisfies $\nabla_X^{\text{End}(\mathcal{E})} u_- = 0$. By [39, Theorem 4.1], it is smooth and we set $u_- := \Phi(u_\star) \in \ker \nabla_X^{\text{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))}$.

Also observe that this construction is done by using parallel transport with respect to the unitary connection $\nabla^{\text{End}(\mathcal{E})}$. As a consequence, if $u_\star, u'_\star \in \mathbf{R}$ are orthogonal (i.e. $\text{Tr}(u_\star^* u'_\star) = 0$), then $\Phi(u_\star)$ and $\Phi(u'_\star)$ are also pointwise orthogonal. This proves that Φ is injective.

It now remains to show the surjectivity of Φ . Let $u \in \ker \nabla_X^{\text{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \text{End}(\mathcal{E}))}$. Following [15, Section 5], we can write $u = u_R + iu_I$, where $u_R^* = u_R, u_I^* = u_I$ and $\nabla_X^{\text{End}(\mathcal{E})} u_R = \nabla_X^{\text{End}(\mathcal{E})} u_I = 0$. By [15, Lemma 5.6], we can then further decompose $u_R = \sum_{i=1}^p \lambda_i \pi_{\mathcal{F}_i}$ (and the same for u_I), where $\lambda_i \in \mathbb{R}, p \in \mathbb{N}$ and $\mathcal{F}_i \subset \mathcal{E}$ is a maximally invariant subbundle of \mathcal{E} (i.e. it does not contain any non-trivial subbundle that is invariant under parallel transport along the flow lines of $(\varphi_t)_{t \in \mathbb{R}}$ with respect to $\nabla^{\mathcal{E}}$), and $\pi_{\mathcal{F}_i}$ is the orthogonal projection onto \mathcal{F}_i . Set $(\pi_{\mathcal{F}_i})_\star := \pi_{\mathcal{F}_i}(x_\star)$. Invariance of \mathcal{F}_i by parallel transport implies that $\rho(\gamma)(\pi_{\mathcal{F}_i})_\star = (\pi_{\mathcal{F}_i})_\star\rho(\gamma)$ for all $\gamma \in \mathbf{G}$, that is, $(\pi_{\mathcal{F}_i})_\star \in \mathbf{R}'$. Moreover,

$\Phi((\pi_{\mathcal{F}_i})_\star) = \pi_{\mathcal{F}_i}$. This proves that both u_R and u_I are in $\text{ran}(\Phi)$. This concludes the proof. ■

It remains to prove the results concerning invariant sections:

Proof of Lemma 3.7. Uniqueness is immediate since $\nabla_X^\mathcal{E}u = 0$ implies that

$$X|u|^2 = \langle \nabla_X^\mathcal{E}u, u \rangle = \langle u, \nabla_X^\mathcal{E}u \rangle = 0,$$

that is, $|u|$ is constant. Now, given u_\star which is \mathbf{G} -invariant, we can define $u_-(x)$ for x on a homoclinic orbit γ by parallel transport of u_\star from x_\star to x along γ with respect to $\nabla^\mathcal{E}$, similarly to Lemma 3.21 and to the proof of Theorem 3.6. We can also define $u_+(x)$ in the same fashion (by parallel transport in the other direction). Then one gets $\|u_-(x) - u_+(x)\| = \|u_\star - \rho(\gamma)u_\star\| = 0$ and the same arguments as before show that u_- extends to a smooth function in the kernel of $\nabla_X^\mathcal{E}$. ■

Proof of Lemma 3.8. This is based on the following:

Lemma 3.25. *Assume that for all periodic orbits $\gamma \in \mathcal{G}$, there exists $u_\gamma \in C^\infty(\gamma, \mathcal{E}|_\gamma)$ such that $\nabla_X^\mathcal{E}u_\gamma = 0$. Then for all $g \in \mathbf{G}$, there exists $u_g \in \mathcal{E}_{x_\star}$ such that $\rho(g)u_g = u_g$.*

Proof. Recall that by the construction of Proposition 3.19, each element $\rho(g) \in \text{U}(\mathcal{E}_{x_\star})$ can be approximated by the holonomy $C_{y_n \rightarrow x_\star} C(y_n, T'_n) C_{x_\star \rightarrow y_n}$ along a sequence of periodic orbits of points y_n converging to x_\star . Now, each $C(y_n, T'_n)$ has 1 as eigenvalue by assumption and taking the limit as $n \rightarrow \infty$, we deduce that 1 is an eigenvalue of $\rho(g)$. ■

As a consequence, we can write for all $g \in \mathbf{G}$, in a fixed orthonormal basis of \mathcal{E}_{x_\star} ,

$$\rho(g) = \alpha_g \begin{pmatrix} 1 & 0 \\ 0 & s(g) \end{pmatrix} \alpha_g^{-1}$$

for some $\alpha_g \in \text{U}(\mathcal{E}_{x_\star})$, where $s(g)$ is an $(r - 1) \times (r - 1)$ matrix. For $\text{rank}(\mathcal{E}) = 1$, the lemma is then a straightforward consequence of Lemma 3.7 since the conjugacy α_g does not appear. For $\text{rank}(\mathcal{E}) = 2$, one has the remarkable property that $s(g)$ is still a representation of \mathbf{G} since $\det \rho(g) = s(g) \in \text{U}(1)$. As a consequence, $\rho : \mathbf{G} \rightarrow \text{U}(\mathcal{E}_{x_\star})$ has the same character as $\rho' : \mathbf{G} \rightarrow \text{U}(\mathcal{E}_{x_\star})$ defined by

$$\rho'(g) := \begin{pmatrix} 1 & 0 \\ 0 & s(g) \end{pmatrix}.$$

By [63, Corollary 3.8], we then conclude that these representations are isomorphic, that is, there exists $p_\star \in \text{U}(\mathcal{E}_{x_\star})$ such that $\rho(g) = p_\star \rho'(g) p_\star^{-1}$. If $u'_\star \in \mathcal{E}_{x_\star}$ denotes the vector fixed by $\rho'(\mathbf{G})$, then $u_\star := p_\star u'_\star$ is fixed by $\rho(\mathbf{G})$. We then conclude by Lemma 3.7. ■

4. Pollicott–Ruelle resonances and local geometry on the moduli space of connections

This section is devoted to the study of the moduli space of connections, from the point of view of Pollicott–Ruelle resonances. We will first deal with the opaque case and then

outline the main distinctions with the non-opaque case. We consider a Hermitian vector bundle $(\mathcal{E}, \nabla^\mathcal{E})$ endowed with a unitary connection over the Anosov Riemannian manifold (M, g) . Recall the notation of Section 2.4: we write $\mathbf{X} = (\pi^* \nabla^\mathcal{E})_X$, $\mathbf{R}_\pm(z) = (\pm \mathbf{X} + z)^{-1}$ for its resolvent and \mathbf{R}_0^\pm , Π_0^\pm for the holomorphic parts and the spectral projector at zero, respectively.

4.1. The Coulomb gauge

We study the geometry of the space of connections (and of the moduli space of gauge-equivalent connections) in a neighbourhood of a given unitary connection $\nabla^\mathcal{E}$ of regularity C_*^s (for $1 < s < \infty$ ⁸) such that $\ker(\nabla^{\text{End}(\mathcal{E})}) = \mathbb{C} \cdot \mathbb{1}_\mathcal{E}$. For the standard differential topology of Banach manifolds, we refer the reader to [62]. We denote by

$$\mathcal{O}_s(\nabla^\mathcal{E}) := \{\nabla^\mathcal{E} + p^{-1} \nabla^{\text{End}(\mathcal{E})} p \mid p \in C_*^{s+1}(M, \text{U}(\mathcal{E})), \|p - \mathbb{1}\|_{C_*^{s+1}} < \delta\}$$

the orbit of gauge-equivalent connections of regularity C_*^s , where $\delta > 0$ is small enough so that $\mathcal{O}_s(\nabla^\mathcal{E})$ is a smooth Banach submanifold. We also define the slice at $\nabla^\mathcal{E}$ by

$$\mathcal{S}_s(\nabla^\mathcal{E}) := \nabla^\mathcal{E} + \ker(\nabla^{\text{End}(\mathcal{E})})^* \cap \{A \in C^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E})) \mid \|A\|_{C_*^s} < \delta\}.$$

Note that \mathbb{S}^1 acts by multiplication freely and properly on $C_*^s(M, \text{U}(\mathcal{E}))$ and hence we may form the quotient Banach manifold, denoted by $C_*^s(M, \text{U}(\mathcal{E}))/\mathbb{S}^1$, which in particular satisfies

$$T_{\mathbb{1}_\mathcal{E}}(C_*^s(M, \text{U}(\mathcal{E}))/\mathbb{S}^1) = C_*^s(M, \text{End}_{\text{sk}}(\mathcal{E}))/(\mathbb{R} \cdot (i\mathbb{1}_\mathcal{E})), \tag{4.1}$$

where we use the identification of tangent spaces given by the exponential map. Next, observe that the map $O : p \mapsto p^* \nabla^\mathcal{E}$ is injective modulo the multiplication action of \mathbb{S}^1 on $C_*^{s+1}(M, \text{U}(\mathcal{E}))$ and that it is an immersion at $p = \mathbb{1}$ with $d_{\mathbb{1}} O(\Gamma) = \nabla^{\text{End}(\mathcal{E})} \Gamma$. Therefore by (2.12), $\mathcal{O}_s(\nabla^\mathcal{E})$ and \mathcal{S}_s are smooth transverse Banach manifolds with

$$T_{\nabla^\mathcal{E}} \mathcal{O}(\nabla^{\text{End}(\mathcal{E})}) = \text{ran}(\nabla^{\text{End}(\mathcal{E})}), \quad T_{\nabla^{\text{End}(\mathcal{E})}} \mathcal{S}_s = \ker(\nabla^{\text{End}(\mathcal{E})})^*.$$

We will say that a connection $\nabla_2^\mathcal{E}$ is in the *Coulomb gauge* with respect to $\nabla_1^\mathcal{E}$ if $(\nabla_1^{\text{End}(\mathcal{E})})^*(\nabla_2^\mathcal{E} - \nabla_1^\mathcal{E}) = 0$. The following lemma shows that, near $\nabla^\mathcal{E}$, we may always put a pair of connections in the Coulomb gauge with respect to each other. It is a slight generalization of the usual claim (see [25, Proposition 2.3.4]).

Lemma 4.1 (Coulomb gauge). *Let $s > 1$. There exists $\varepsilon = \varepsilon(s, \nabla^\mathcal{E}) > 0$ and a neighbourhood $\mathcal{U} \subset C_*^{s+1}(M, \text{U}(\mathcal{E}))/\mathbb{S}^1$ of $\mathbb{1}_\mathcal{E}$ such that for any $A_i \in C^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ with $\|A_i\|_{C_*^s} < \varepsilon$, after setting $\nabla_i^\mathcal{E} = \nabla^\mathcal{E} + A_i$ for $i = 1, 2$, there exists a unique $p_{A_1, A_2} \in \mathcal{U}$ such that $p_{A_1, A_2}^* \nabla_2^\mathcal{E} - \nabla_1^\mathcal{E} \in \ker(\nabla_1^{\text{End}(\mathcal{E})})^*$. Furthermore, if A_i are smooth,*

⁸It is very likely that the case $s = \infty$ still works. This would require to use the Nash–Moser Theorem.

then p_{A_1, A_2} is smooth. Moreover, the map

$$(C^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E})))^2 \ni (A_1, A_2) \mapsto \phi(A_1, A_2) := p_{A_1, A_2}^* \nabla_2^\mathcal{E} \in \mathcal{S}_s(\nabla_1^\mathcal{E})$$

is smooth. Setting $\phi(A) := \phi(0, A)$, we have

$$d\phi|_{A=0} = \pi_{\ker(\nabla^{\text{End}(\mathcal{E})})^*}.$$

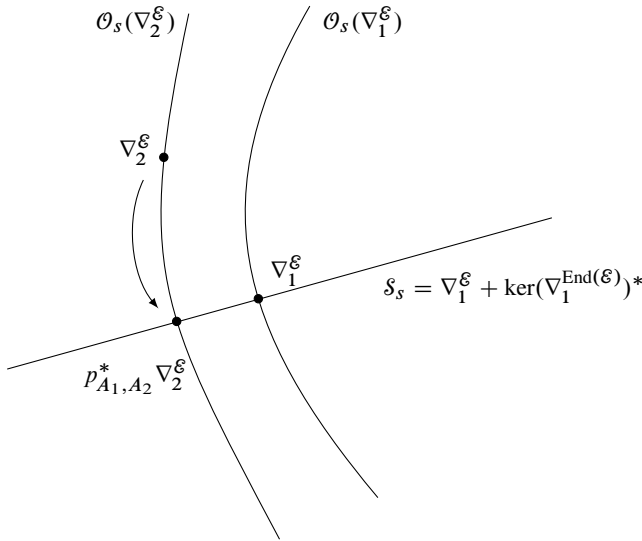


Fig. 5. A schematic representation of Lemma 4.1.

Proof. Note that the exponential map

$$\exp : C_*^{s+1}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{i \mathbb{1}_\mathcal{E}\}^{\perp L^2} \rightarrow C_*^{s+1}(M, \text{U}(\mathcal{E}))/\mathbb{S}^1$$

is well-defined and a local diffeomorphism at zero, so we reduce the claim to finding a neighbourhood $\mathcal{V} \subset C_*^{s+1}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{i \mathbb{1}_\mathcal{E}\}^{\perp L^2}$ of 0 and setting $p = p_{A_1, A_2} = \exp(\chi_{A_1, A_2})$ for $\chi = \chi_{A_1, A_2} \in \mathcal{V}$, that is, $\mathcal{U} = \exp(\mathcal{V})$. Define the functional

$$F : (C_*^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E})))^2 \times C_*^{s+1}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{i \mathbb{1}_\mathcal{E}\}^{\perp L^2} \rightarrow C_*^{s-1}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{i \mathbb{1}_\mathcal{E}\}^{\perp L^2}$$

by

$$F(A_1, A_2, \chi) := (\nabla_1^{\text{End}(\mathcal{E})})^* (\exp(-\chi) \nabla^{\text{End}(\mathcal{E})} \exp(\chi) + \exp(-\chi) A_2 \exp(\chi) - A_1).$$

We see that F is well-defined, i.e. with values in skew-Hermitian endomorphisms, since $\nabla^\mathcal{E}$ is unitary, and integrating by parts we get $\langle F(A_1, A_2, \chi), \mathbb{1}_\mathcal{E} \rangle_{L^2} = 0$; note that F is

smooth in its entries. Next, we compute the partial derivative with respect to the χ variable at $A_1 = A_2 = 0$ and $\chi = 0$:

$$d_\chi F(0, 0, 0)(\Gamma) = \partial_t|_{t=0} F(0, 0, t\Gamma) = (\nabla^{\text{End}(\mathcal{E})})^* \nabla^{\text{End}(\mathcal{E})} \Gamma.$$

This derivative is an isomorphism on

$$\begin{aligned} (\nabla^{\text{End}(\mathcal{E})})^* \nabla^{\text{End}(\mathcal{E})} : C_*^{s+1}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{i \mathbb{1}_{\mathcal{E}}\}^{\perp L^2} \\ \rightarrow C_*^{s-1}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{i \mathbb{1}_{\mathcal{E}}\}^{\perp L^2}, \end{aligned}$$

by the Fredholm property of $(\nabla^{\text{End}(\mathcal{E})})^* \nabla^{\text{End}(\mathcal{E})}$ and since $\ker \nabla^{\text{End}(\mathcal{E})} = \mathbb{C} \cdot \mathbb{1}_{\mathcal{E}}$ by assumption. The first claim then follows by an application of the implicit function theorem for Banach spaces.

The fact that p is smooth if (A_1, A_2) is, is a consequence of elliptic regularity and the fact that C_*^s is an algebra, along with the Coulomb property:

$$(\nabla_1^{\text{End}(\mathcal{E})})^* \nabla_1^{\text{End}(\mathcal{E})} p = (\nabla_1^{\text{End}(\mathcal{E})} p) p^{-1} \bullet \nabla_1^{\text{End}(\mathcal{E})} p + p (\nabla_1^{\text{End}(\mathcal{E})})^* (p^{-1} (A_1 - A_2) p) \in C_*^s$$

implies $p \in C_*^{s+2}$. Bootstrapping we obtain $p_{A_1, A_2} \in C^\infty$. Here \bullet denotes the operation of taking the inner product on the differential form side and multiplication on the endomorphism side.

Eventually, we compute the derivative of $\phi(A)$. Write $p_A := p_{0, A}$ and $\chi_A := \chi_{0, A}$, where χ_A is orthogonal to $i \mathbb{1}_{\mathcal{E}}$ with respect to the L^2 scalar product, so that by definition

$$\phi(A) = \nabla^{\mathcal{E}} + p_A^{-1} \nabla^{\text{End}(\mathcal{E})} p_A + p_A^{-1} A p_A. \tag{4.2}$$

By differentiating the relation $F(A, \chi_A) := F(0, A, \chi_A) = 0$ at $A = 0$, we obtain, for every $\Gamma \in C_*^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$,

$$\begin{aligned} 0 &= d_A F|_{A=0, \chi=0}(\Gamma) + d_\chi F|_{A=0, \chi=0}(d\chi_A|_{A=0}(\Gamma)) \\ &= (\nabla^{\text{End}(\mathcal{E})})^* \Gamma + (\nabla^{\text{End}(\mathcal{E})})^* \nabla^{\text{End}(\mathcal{E})} d\chi_A|_{A=0}(\Gamma), \end{aligned}$$

that is, $d\chi_A|_{A=0}(\Gamma) = -[(\nabla^{\text{End}(\mathcal{E})})^* \nabla^{\text{End}(\mathcal{E})}]^{-1} (\nabla^{\text{End}(\mathcal{E})})^* \Gamma$. Observe that $d p_A|_{A=0} = d\chi_A|_{A=0}$ via the exponential map and by (4.2), we obtain

$$\begin{aligned} d\phi|_{A=0}(\Gamma) &= \nabla^{\text{End}(\mathcal{E})} d\chi_A|_{A=0}(\Gamma) + \Gamma \\ &= \Gamma - \nabla^{\text{End}(\mathcal{E})} [(\nabla^{\text{End}(\mathcal{E})})^* \nabla^{\text{End}(\mathcal{E})}]^{-1} (\nabla^{\text{End}(\mathcal{E})})^* \Gamma. \end{aligned}$$

We then conclude by (2.13). ■

In particular, the proof also shows that the map

$$C_*^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E})) \ni A \mapsto \phi(A) \in \mathcal{S}_s := \mathcal{S}_s(\nabla^{\mathcal{E}})$$

is constant along orbits of gauge-equivalent connections (by construction).

4.2. Resonances at $z = 0$: finer remarks

We recall that ∇^ε is an arbitrary smooth unitary connection on a Hermitian vector bundle $\mathcal{E} \rightarrow M$ and that the differential operator $\mathbf{X} := (\pi^*\nabla^\varepsilon)_X$ is defined over SM .

In the subsequent lemma, we will use the following characterization: $z_0 \in \mathbb{C}$ is a Pollicott–Ruelle resonance of $-\mathbf{X}$ if and only if there exists a non-zero distribution $u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, \mathcal{E})$ such that $-\mathbf{X}u = z_0u$. Here for a closed conic set $\Gamma \subset T^*\mathcal{M}$, we denote by $\mathcal{D}'_\Gamma(\mathcal{M}, \mathcal{E})$ the set of distributional sections u such that the wavefront set satisfies $\text{WF}(u) \subset \Gamma$ (see [52, Chapter 8] for the background on wavefront sets). This characterization follows by the flexibility in the choice of anisotropic spaces (see e.g. [30, Theorem 13] for details).

Lemma 4.2. *The Pollicott–Ruelle resonance spectrum of \mathbf{X} is symmetric with respect to the real axis.*

Proof. If z_0 is a resonance associated to $-\mathbf{X}$, i.e. a pole of $z \mapsto \mathbf{R}_+(z)$, then by (2.18), \bar{z}_0 is a resonance associated to $+\mathbf{X}$, i.e. a pole of $z \mapsto \mathbf{R}_-(z)$. Let $u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, \mathcal{E})$ be a non-zero resonant state such that $-\mathbf{X}u = z_0u$. Let $R : (x, v) \mapsto (x, -v)$ be the antipodal map on SM ; note that the pullback R^* acts on sections of $\pi^*\mathcal{E}$ and that $R^*\pi^*\nabla^\varepsilon = \pi^*\nabla^\varepsilon$ since $\pi \circ R = \pi$. Observe that $R^* : \mathcal{D}'_{E_u^*}(\mathcal{M}, \mathcal{E}) \rightarrow \mathcal{D}'_{E_s^*}(\mathcal{M}, \mathcal{E})$ is an isomorphism, since $R^*X = -X$, so R^* will swap the stable and the unstable bundles. Then $z_0R^*u = -R^*\mathbf{X}u = \mathbf{X}R^*u$ and $R^*u \in \mathcal{D}'_{E_s^*}(\mathcal{M}, \mathcal{E})$. Thus R^*u is a resonant state associated to the resonance z_0 . So both z_0 and \bar{z}_0 are resonances for $+\mathbf{X}$, which completes the proof.⁹ ■

We remark that the preceding lemma also holds in sufficiently high finite regularity by a density argument and the continuity of resonances established in Lemma 2.5.

Consider a contour $\gamma \subset \mathbb{C}$ such that $-\mathbf{X}$ has no resonances other than zero inside or on γ . By continuity of resonances (see Lemma 2.5), there is an $\varepsilon > 0$ such that for all skew-Hermitian 1-forms A with $\|A\|_{C^s_*} < \varepsilon$ the operator $-\mathbf{X}_A := -(\pi^*(\nabla^\varepsilon + A))_X$ has no resonances on γ . Here we need to take s large enough (depending on the dimension), so that the framework of microlocal analysis applies.

In the specific case where $\dim \ker \mathbf{X}|_{\mathcal{H}_+} = 1$, we denote by λ_A the unique resonance of $-\mathbf{X}_A$ enclosed by γ . Note that the map $A \mapsto \lambda_A$ is C^3 -regular for $\varepsilon > 0$ small enough (see Lemma 2.5).

Lemma 4.3. *Assume that $\dim \ker \mathbf{X}|_{\mathcal{H}_+} = 1$. Then $\lambda_A \in \mathbb{R}$ and for $\Gamma \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$,*

$$d\lambda_A|_{A=0} = 0, \quad d^2\lambda_A|_{A=0}(\Gamma, \Gamma) = -\langle \Pi\pi_1^*\Gamma u_0, \pi_1^*\Gamma u_0 \rangle_{L^2},$$

where u_0 is a resonant state associated to $A = 0$ and $\|u_0\|_{L^2} = 1$.

⁹Alternatively, by inspecting the construction of the anisotropic Sobolev space in [30], we see that we may assume $R^* : \mathcal{H}^s_\pm \rightarrow \mathcal{H}^s_\mp$ is an isomorphism, simply by replacing the degree function m in the construction by $\frac{m-R^*m}{2}$, which then implies that $R^*m = -m$.

Proof. By the symmetry property of Lemma 4.2 and continuity of resonances we know $\lambda_A \in \mathbb{R}$. Also, observe that u_0 is either pure odd or pure even with respect to v (i.e. $R^*u_0 = u_0$ or $R^*u_0 = -u_0$) because R^* keeps $\ker \mathbf{X}$ fixed and $\ker \mathbf{X}$ is assumed to be one-dimensional.

For the second claim, it is sufficient to start with the equality $-\mathbf{X}_{\tau\Gamma}u_{\tau\Gamma} = \lambda_{\tau\Gamma}u_{\tau\Gamma}$, where $\Gamma \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$, $\tau \in (-\delta, \delta)$ is small enough so that $\tau \mapsto \lambda_{\tau\Gamma}$ and $\tau \mapsto u_{\tau\Gamma} \in \mathcal{H}_+$ are C^3 , and to compute the derivatives at $\tau = 0$. Observe that $\dot{\mathbf{X}}_0 = \pi_1^*\Gamma$ and $\ddot{\mathbf{X}}_0 = 0$. We obtain $-\dot{\mathbf{X}}_0u_0 - \mathbf{X}_0\dot{u}_0 = \dot{\lambda}_0u_0$ and taking the L^2 scalar product with u_0 , we find $\dot{\lambda}_0 = 0$, using the fact that u_0 is either pure odd or pure even. Thus $\dot{u}_0 - \Pi_0^+\dot{u}_0 = -\mathbf{R}_0^+\pi_1^*\Gamma u_0$. Then, taking the second derivative at $\tau = 0$, we get $-2\pi_1^*\Gamma\dot{u}_0 - \mathbf{X}_0\ddot{u}_0 = \ddot{\lambda}_0u_0$, and taking once again the scalar product with u_0 , we find $\ddot{\lambda}_0 = -2\langle \mathbf{R}_0^+\pi_1^*\Gamma u_0, \pi_1^*\Gamma u_0 \rangle_{L^2}$. It is then sufficient to observe that by symmetry (using $(\mathbf{R}_0^+)^* = \mathbf{R}_0^-$ and $\ddot{\lambda}_0 \in \mathbb{R}$)

$$\langle \mathbf{R}_0^+\pi_1^*\Gamma u_0, \pi_1^*\Gamma u_0 \rangle_{L^2} = \langle \mathbf{R}_0^-\pi_1^*\Gamma u_0, \pi_1^*\Gamma u_0 \rangle_{L^2}.$$

This proves the result. ■

4.3. P-R resonance at 0 of the mixed connection: Opaque case

We now further assume that $\mathbf{X} := (\pi^*\nabla^{\text{End}(\mathcal{E})})_X$ has the resonant space at 0 spanned by $\mathbb{1}_{\mathcal{E}}$. This condition is known as the *opacity* of the connection $\pi^*\nabla^{\mathcal{E}}$. When (M, g) is Anosov, this is known to be a generic condition [15, Theorem 1.6].

As in Section 2.2.1, we assume that $s \gg 1$ (so that standard microlocal analysis and the perturbation lemma 2.5 apply) and we introduce the mixed connection induced by $\nabla_1^{\mathcal{E}} = \nabla^{\mathcal{E}} + A_1$ and $\nabla_2^{\mathcal{E}} = \nabla^{\mathcal{E}} + A_2$, namely

$$\nabla^{\text{Hom}(\nabla_1^{\mathcal{E}}, \nabla_2^{\mathcal{E}})}u = \nabla^{\text{End}(\mathcal{E})}u + A_2u - uA_1,$$

and we set $\mathbf{X}_{A_1, A_2} := (\pi^*\nabla^{\text{Hom}(\nabla_1^{\mathcal{E}}, \nabla_2^{\mathcal{E}})})_X$ and $z \mapsto \mathbf{R}_{\pm}(z, A_i)$ for the resolvents. The operator $\mathbf{X} := \mathbf{X}_{0,0}$ has the resonant space at $z = 0$ spanned by $\mathbb{1}_{\mathcal{E}}$. For $\|A_1\|_{C_*^s}, \|A_2\|_{C_*^s}$ small enough by Lemma 2.5 the map $(A_1, A_2) \mapsto \lambda_{A_1, A_2}$ is C^3 -regular, where we denote by λ_{A_1, A_2} the unique resonance close to 0, namely the unique pole of $\mathbf{R}_{\pm}(z, A_i)$ inside the small contour γ around zero (see Section 4.2). Since $\nabla^{\text{Hom}(\nabla_1^{\mathcal{E}}, \nabla_2^{\mathcal{E}})}$ is unitary, by (2.19) we have $\text{Re } \lambda_{A_1, A_2} \leq 0$, and by Lemmas 4.3 and 2.5 we get $\lambda_{A_1, A_2} \in \mathbb{R}$ (as otherwise the rank of the projector in Lemma 2.5 would be at least 2, contradicting the fact that it is locally constant). In fact, λ_{A_1, A_2} descends to the moduli space: if $p_i^*(\nabla^{\mathcal{E}} + A'_i) = \nabla^{\mathcal{E}} + A_i$ for $\|A'_i\|_{C_*^s}$ small enough, then using (2.6) we get

$$\mathbf{X}_{A'_1, A'_2}u = (p_2)^{-1} \cdot \mathbf{X}_{A_1, A_2}(p_2u(p_1)^{-1}) \cdot p_1, \quad u \in \mathcal{H}_+. \tag{4.3}$$

Here we use the fact that \mathcal{H}_+ is stable under multiplication by C_*^s for s large enough; hence \mathbf{X}_{A_1, A_2} and $\mathbf{X}_{A'_1, A'_2}$ have equal P-R spectra and so $\lambda_{A_1, A_2} = \lambda_{A'_1, A'_2}$.

Next, we need a uniform estimate for the generalized X-ray transform operator $\Pi_1^{\text{End}(\mathcal{E})}$, introduced in Definition 2.6 with $m = 1$ and with respect to the endomorphism connection $\text{End}(\nabla^{\mathcal{E}})$ (see Definition 2.1), in a neighbourhood of $\nabla^{\mathcal{E}}$:

Lemma 4.4. *Assume $\Pi_1^{\text{End}(\mathcal{E})}$ is s -injective. There are constants $\varepsilon, C > 0$ depending only on ∇^ε such that for all skew-Hermitian 1-forms with $\|A\|_{C_*^s} < \varepsilon$,*

$$\forall f \in H^{-1/2}(M, T^*M \otimes \text{End}(\mathcal{E})),$$

$$\langle \Pi_1^{\text{End}(\nabla^\varepsilon + A)} f, f \rangle_{L^2} \geq C \|\pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A)})^*} f\|_{H^{-1/2}}^2.$$

Proof. Observe first that the left hand side of the inequality vanishes on potential tensors by (2.23) and hence it suffices to consider $f \in \ker(\nabla^{\text{End}(\nabla^\varepsilon + A)})^*$. Then

$$\begin{aligned} \langle \Pi_1^{\text{End}(\nabla^\varepsilon + A)} f, f \rangle_{L^2} &= \langle \Pi_1^{\text{End}(\nabla^\varepsilon)} f, f \rangle_{L^2} + \langle (\Pi_1^{\text{End}(\nabla^\varepsilon + A)} - \Pi_1^{\text{End}(\nabla^\varepsilon)}) f, f \rangle_{L^2} \\ &\geq C_0 \|\pi_{\ker(\nabla^{\text{End}(\mathcal{E})})^*} f\|_{H^{-1/2}}^2 - \|\Pi_1^{\text{End}(\nabla^\varepsilon + A)} - \Pi_1^{\text{End}(\nabla^\varepsilon)}\|_{H^{-1/2} \rightarrow H^{1/2}} \|f\|_{H^{-1/2}}^2 \\ &\geq \frac{1}{4} C_0 \|f\|_{H^{-1/2}}^2 - \frac{1}{2} C_0 \|\pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon)})^*} - \pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A)})^*}\|_{H^{-1/2} \rightarrow H^{-1/2}} \|f\|_{H^{-1/2}}^2 \\ &\geq \frac{1}{8} C_0 \|f\|_{H^{-1/2}}^2. \end{aligned}$$

In the second line, we have used Lemma 2.9 (3) with a constant C_0 . In the next line we have used the fact that the map $A \mapsto \Pi_1^{\text{End}(\nabla^\varepsilon + A)} \in \Psi^{-1}$ is continuous; the proof of this fact is analogous to the proof of [43, Proposition 4.1] and we omit it. Thus for $\varepsilon > 0$ small enough, $\|\Pi_1^{\text{End}(\nabla^\varepsilon + A)} - \Pi_1^{\text{End}(\nabla^\varepsilon)}\|_{H^{-1/2} \rightarrow H^{1/2}} \leq C_0/4$. Similarly in the last line we have used the continuity of $A \mapsto \pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A)})^*} \in \mathcal{L}(H^{-1/2})$, which follows by standard microlocal analysis from (2.13): more precisely, this is a consequence of the fact that $(\nabla^{\text{End}(\nabla^\varepsilon)})^* \nabla^{\text{End}(\nabla^\varepsilon)}$ is an isomorphism on $C^\infty(M, \text{End}(\mathcal{E})) \cap (\mathbb{C}\mathbb{1}_\mathcal{E})^\perp$ by the opacity assumption (where the orthogonal is taken with respect to the L^2 scalar product) and by continuity, this holds for all operators $(\nabla^{\text{End}(\nabla^\varepsilon + A)})^* \nabla^{\text{End}(\nabla^\varepsilon + A)}$ as long as $\|A\|_{C_*^s}$ is small enough (for s large enough); hence the operator $[(\nabla^{\text{End}(\nabla^\varepsilon + A)})^* \nabla^{\text{End}(\nabla^\varepsilon + A)}]^{-1}$ in (2.13) is locally uniformly bounded and so by the resolvent formula, it varies continuously with respect to the connection. So again we choose $\varepsilon > 0$ small enough so that

$$\|\pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon)})^*} - \pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A)})^*}\|_{H^{-1/2} \rightarrow H^{-1/2}} \leq C_0/4.$$

The claim follows by setting $C = C_0/8$. ■

Recall that $\lambda_{A_1, A_2} \leq 0$ in the following:

Lemma 4.5. *Assume that the generalized X-ray transform $\Pi_1^{\text{End}(\mathcal{E})}$ defined with respect to the connection $\nabla^{\text{End}(\mathcal{E})}$ is s -injective. For $s \gg 1$ large enough, there exist constants $\varepsilon, C > 0$ such that for all $A_i \in C^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ with $\|A_i\|_{C_*^s} < \varepsilon$ for $i = 1, 2$, we have*

$$0 \leq \|\phi(A_1, A_2) - \nabla_1^\varepsilon\|_{H^{-1/2}(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))}^2 \leq C |\lambda_{A_1, A_2}|.$$

Proof. We introduce two functionals in the vicinity of ∇^ε , for small enough $\varepsilon > 0$:

$$\begin{aligned} F_1, F_2 &: (C_*^{s_0}(M, \text{End}_{\text{sk}}(\mathcal{E})) \cap \{\|A\|_{C_*^{s_0}} < \varepsilon\})^2 \rightarrow \mathbb{R}, \\ F_1(A_1, A_2) &:= \lambda_{A_1, A_2}, \quad F_2(A_1, A_2) := -\|\phi(A_1, A_2) - \nabla_1^\varepsilon\|_{H^{-1/2}}^2. \end{aligned}$$

They are well-defined and restrict to C^3 -regular maps on \mathcal{S}_{s_0} for some $s_0 \gg 1$ large enough by Lemma 4.1 and the discussion above. Moreover, using Lemma 4.3 we have, for all A ,

$$F_1(A, A) = F_2(A, A) = 0 \quad \text{and} \quad dF_1|_{(A,A)} = dF_2|_{(A,A)} = 0.$$

We will compare the second partial derivatives in A_2 at a point (A, A) . Given $\Gamma \in T_{\nabla^\varepsilon} \mathcal{S}_{s_0} \simeq \ker(\nabla^{\text{End}(\mathcal{E})})^*$, by Lemma 4.1 we have

$$d_{A_2}^2 F_2|_{(A,A)}(\Gamma, \Gamma) = -2 \|\pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A))^*} \Gamma\|_{H^{-1/2}}^2. \tag{4.4}$$

By Lemma 4.3,

$$d_{A_2}^2 F_1|_{(A,A)}(\Gamma, \Gamma) = -c^2 \langle \Pi^{\text{End}(\nabla^\varepsilon + A)} \pi_1^* \Gamma \mathbb{1}_\mathcal{E}, \pi_1^* \Gamma \mathbb{1}_\mathcal{E} \rangle_{L^2} = -c^2 \langle \Pi_1^{\text{End}(\nabla^\varepsilon + A)} \Gamma, \Gamma \rangle_{L^2}$$

for some constant $c > 0$, where $\Pi^{\text{End}(\nabla^\varepsilon + A)}$ denotes the Π operator with respect to the endomorphism connection induced by $\nabla^\varepsilon + A$. We use here the fact that the orthogonal projection to the resonant space $\mathbb{C} \mathbb{1}_\mathcal{E}$ of $\mathbf{X}_{A,A}$ at zero vanishes, because $\langle \pi_1^* \Gamma, \mathbb{1}_\mathcal{E} \rangle_{L^2} = 0$ as $\pi_1^* \Gamma$ is odd. For $\varepsilon > 0$ small enough, by Lemma 4.4 we know $\Pi_1^{\text{End}(\nabla^\varepsilon + A)}$ is s -injective and there is a constant $C' = C'(\nabla^\varepsilon) > 0$ such that

$$\begin{aligned} d_{A_2}^2 F_1|_{(A,A)}(\Gamma, \Gamma) &\leq -C' \|\pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A))^*} \Gamma\|_{H^{-1/2}}^2 \\ &= \frac{1}{2} C' d_{A_2}^2 F_2|_{(A,A)}(\Gamma, \Gamma). \end{aligned} \tag{4.5}$$

As a consequence, writing $G(A_2) := F_1(A, A_2) - \frac{1}{4} C' F_2(A, A_2)$, we have $G(A) = 0$, $dG|_{A_2=A} = 0$ and by (4.5), (4.4),

$$d^2 G|_{A_2=A}(\Gamma, \Gamma) \leq \frac{1}{4} C' d_{A_2}^2 F_2|_{(A,A)}(\Gamma, \Gamma) = -\frac{1}{2} C' \|\pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A))^*} \Gamma\|_{H^{-1/2}}^2.$$

If we now Taylor expand the C^3 -map $\mathcal{S}_{s_0} \ni A_2 \mapsto G(A_2)$ at $A_2 = A$, we obtain

$$\begin{aligned} G(A + \Gamma) &= \frac{1}{2} d^2 G|_{A_2=A}(\Gamma, \Gamma) + \mathcal{O}(\|\Gamma\|_{C_*^{s_0}}^3) \\ &\leq -\frac{1}{4} C' \|\Gamma\|_{H^{-1/2}}^2 + \frac{1}{4} C' \|\pi_{\ker(\nabla^{\text{End}(\mathcal{E}))^*} \Gamma - \pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A))^*} \Gamma\|_{H^{-1/2}}^2 + C'' \|\Gamma\|_{C_*^{s_0}}^3 \\ &\leq -\frac{1}{8} C' \|\Gamma\|_{H^{-1/2}}^2 + C'' \|\Gamma\|_{C_*^{s_0}}^3. \end{aligned}$$

In the second line we have introduced a uniform constant $C'' = C''(\nabla^\varepsilon) > 0$ using the C^3 -regular property and $\pi_{\ker(\nabla^{\text{End}(\mathcal{E}))^*} \Gamma = \Gamma$. For the last line, we observe that $A \mapsto \pi_{\ker(\nabla^{\text{End}(\nabla^\varepsilon + A))^*} \Gamma \in \mathcal{L}(H^{-1/2})$ is a continuous map by (2.13) (see the last subsection of Lemma 4.4) and hence the $H^{-1/2} \rightarrow H^{-1/2}$ estimate is arbitrarily small for ε small enough. This estimate holds for all $\|A\|_{C_*^{s_0}}, \|\Gamma\|_{C_*^{s_0}} < \varepsilon/2$.

Choosing $s \gg s_0$ and assuming that $A \in C_*^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ with $\|A\|_{C_*^s} < \varepsilon$, there is a $C'''(\nabla^\varepsilon) > 0$ such that for $\varepsilon > 0$ with $C''' \varepsilon \leq \frac{1}{16} C'$, by interpolation we obtain

$$G(A + \Gamma) \leq -\frac{1}{8} C' \|\Gamma\|_{H^{-1/2}}^2 + \underbrace{C''' \|\Gamma\|_{C_*^s} \|\Gamma\|_{H^{-1/2}}^2}_{\leq C'/16} \leq -\frac{1}{16} C' \|\Gamma\|_{H^{-1/2}}^2 \leq 0.$$

After taking $\varepsilon > 0$ small enough, the statement holds with $C = \frac{1}{2} C'$. ■

We note that the preceding lemma shows that λ_{A_1, A_2} controls the distance in the moduli space between $\nabla^\varepsilon + A_1$ and $\nabla^\varepsilon + A_2$.

Remark 4.6. It was proved in [43] that there exists a metric G on the moduli space of isometry classes (of metrics with negative sectional curvature) which generalizes the usual Weil–Petersson metric on Teichmüller space in the sense that, in the case of a surface, the restriction of G to Teichmüller space is equal to the Weil–Petersson metric. We point out that the operator Π_1 also allows us to define a metric G at a generic point $\alpha_0 \in \mathbb{A}_\varepsilon$, similarly to [43]. Indeed, if $\alpha_0 \in \mathbb{A}_\varepsilon$, taking a representative $\nabla^\varepsilon \in \alpha_0$, one has $T_{\alpha_0} \mathbb{A}_\varepsilon \simeq \ker(\nabla^{\text{End}})^*$ and thus, given $\Gamma \in \ker(\nabla^{\text{End}})^*$, one can consider

$$G_{\alpha_0}(\Gamma, \Gamma) := \langle \Pi_1 \Gamma, \Gamma \rangle_{L^2(M, T^*M \otimes \text{End}(\mathcal{E}))} \geq c \|\Gamma\|_{H^{-1/2}}^2$$

for some constant $c > 0$. Lemma 4.4 shows that the constant c is locally uniform with respect to α_0 .

4.4. *P-R resonance at 0 of the mixed connection: Non-opaque case*

The aim of this subsection is to deal with neighbourhoods of connections that are not necessarily opaque, and only assume $\Pi_1^{\text{End}(\mathcal{E})}$ is injective. In other words, we do not want to assume the resonant space of $-(\pi^* \nabla^{\text{End}(\mathcal{E})})_X$ at zero is spanned by $\mathbb{1}_\varepsilon$ necessarily.

Next, as in Section 4.3, we introduce the mixed connection with respect to $\nabla^\varepsilon + A$ and ∇^ε , denoted by $\nabla^{\text{Hom}(\nabla^\varepsilon + A, \nabla^\varepsilon)}$, and set $\mathbf{X}_A := (\pi^* \nabla^{\text{Hom}(\nabla^\varepsilon + A, \nabla^\varepsilon)})_X$. We assume $s \gg 1$. As before, consider a contour $\gamma \subset \mathbb{C}$ around zero such that $\mathbf{X} := \mathbf{X}_0$ has only the resonance zero enclosed by γ and $\varepsilon > 0$ such that $-\mathbf{X}_A$ has no resonances on γ for all $\|A\|_{C_*^s} < \varepsilon$. We introduce

$$\Pi_A^+ := \frac{1}{2\pi i} \int_\gamma (z + \mathbf{X}_A)^{-1} dz, \quad \lambda_A := \text{Tr}(-\mathbf{X}_A \Pi_A^+).$$

This generalizes the quantity studied in Section 4.2, where it was assumed that the multiplicity of \mathbf{X} at zero is 1. By Lemma 2.5, $A \mapsto \Pi_A^+$ and $A \mapsto \lambda_A$ are C^3 -regular. As in (4.3), the operators $-\mathbf{X}_A$ and $-\mathbf{X}_{A'}$ are unitarily equivalent on \mathcal{H}_+ whenever $\nabla^\varepsilon + A$ and $\nabla^\varepsilon + A'$ are gauge-equivalent; hence $\lambda_A = \lambda_{A'}$ and so λ_A descends to the local moduli space.

Note also that $\text{Re } \lambda_A \leq 0$, since by (2.19) all resonances of $-\mathbf{X}_A$ lie in the half-plane $\{\text{Re } z \leq 0\}$ and this gives us hope that $\text{Re } \lambda_A$ controls the distance between the connections. Assume that the resonant space of $-\mathbf{X}$ at zero is spanned by smooth L^2 -orthonormal resonant states $\{u_i\}_{i=1}^p$. We have the following generalization of Lemma 4.3:

Lemma 4.7. *For $A \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ with $\|A\|_{C_*^s} < \varepsilon$, we have $\lambda_A \in \mathbb{R}$ and the following perturbation formulas hold:*

$$d\lambda_A|_{A=0} = 0, \quad d^2\lambda_A|_{A=0}(\Gamma, \Gamma) = -\sum_{i=1}^p \langle \Pi u_i \pi_1^* \Gamma, u_i \pi_1^* \Gamma \rangle_{L^2}.$$

Proof. The fact that λ_A is real follows from the symmetry of the spectrum of $-\mathbf{X}_A$ shown in Lemma 4.2. The first derivative formula is obvious as $\lambda_A \leq 0$; the second one follows from minor adaptations of [15, Lemma 5.9], where the analogous case of endomorphisms was considered. ■

Next, by a straightforward adaptation of Lemma 4.1, we obtain the existence of $\varepsilon > 0$ such that for all A with $\|A\|_{C^s_*} < \varepsilon$, there is a smooth map $A \mapsto \phi(A) \in \mathcal{D}_s$ that sends $\nabla^\varepsilon + A$ to Coulomb gauge with respect to ∇^ε , that is, $\phi(A) - \nabla^\varepsilon \in \ker(\nabla^{\text{End}(\mathcal{E})})^*$.

Remark 4.8. We cannot get the statement analogous to Lemma 4.1 for parameters (A_1, A_2) , because the range of $F(A_1, A_2, \chi)$ equals $\ker(\nabla_1^{\text{End}(\mathcal{E})})^\perp$ and this is not uniform in A_1 (i.e. $\ker \nabla_1^{\text{End}(\mathcal{E})}$ changes as we move A_1); the space $\mathbb{A}_\mathcal{E}$ is not a smooth manifold at reducible connections.

In the following lemma, we will need to assume that $\pi_{1*}\Pi_0^+ = 0$. Equivalently, this means that the resonant states $u_i \in \ker \mathbf{X}|_{\mathcal{H}_+}$ satisfy $\pi_{1*}u_i = 0$ for $i = 1, \dots, p$, i.e. the degree 1 Fourier modes of all the u_i vanish.

Lemma 4.9. *Assume that the generalized X-ray transform $\Pi_1^{\text{End}(\mathcal{E})}$ defined with respect to the connection $\nabla^{\text{End}(\mathcal{E})}$ is s -injective and additionally that $\pi_{1*}\Pi_0^+ = 0$. For $s \gg 1$ large enough, there exist constants $\varepsilon, C > 0$ such that for all $A \in C^s(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ with $\|A\|_{C^s_*} < \varepsilon$,*

$$0 \leq \|\phi(A) - \nabla^\varepsilon\|_{H^{-1/2}(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}))}^2 \leq C|\lambda_A|.$$

Proof. This is straightforward from the proof of Lemma 4.5. With the same functionals $F_1(A) = \lambda_A$ and $F_2(A) = -\|\phi(A) - \nabla^\varepsilon\|_{H^{-1/2}}^2$, the only slight difference is the computation of $d^2 F_1$. Pick an L^2 -orthonormal basis u_1, \dots, u_p of the resonant space of $-\mathbf{X}$ at zero such that $u_1 = c\mathbb{1}_\mathcal{E}$, where c is a fixed constant. By Lemmas 4.7 and 2.3, we have

$$d^2 F_1|_{A=0}(\Gamma, \Gamma) = -\sum_{i=1}^p \langle \Pi u_i \pi_1^* \Gamma, u_i \pi_1^* \Gamma \rangle_{L^2} \leq -c^2 \langle \Pi_1^{\text{End}(\mathcal{E})} \Gamma, \Gamma \rangle_{L^2}.$$

Note that we have used $\Pi_0^+ \pi_1^* \Gamma = 0$. This follows from the expression for the projector $\Pi_0^+ = \sum_{i=1}^p \langle \bullet, u_i \rangle_{L^2} u_i$ and $\pi_{1*}u_i = 0$ for all i . This suffices to run the proof in the same manner. ■

5. Injectivity of the primitive trace map

We can now prove the main results stated in the introduction.

5.1. The local injectivity result

We now prove the injectivity result of Theorem 1.1.

Proof of Theorem 1.1. We fix a regularity exponent $N \gg 1$ large enough so that the results of Section 4 apply. We fix a smooth unitary connection $\nabla^\mathcal{E}$ on \mathcal{E} and assume that it is *generic*. By mere continuity, the same properties hold for every connection $\nabla^\mathcal{E} + A$ such that $\|A\|_{C_*^N} < \varepsilon$, where $\varepsilon > 0$ is small enough depending on $\nabla^\mathcal{E}$.

Consider two smooth unitary connections $\nabla_i^\mathcal{E} = \nabla^\mathcal{E} + A_i$ such that $\|A_i\|_{C_*^N} < \varepsilon$ for $i = 1, 2$. Assume that $\mathcal{T}^\sharp(\nabla_1^\mathcal{E}) = \mathcal{T}^\sharp(\nabla_2^\mathcal{E})$. By differentiating with respect to time and taking $t = 0$ in (3.2), the exact Livšic cocycle theorem 3.2 yields the existence of a smooth map $p \in C^\infty(SM, U(\mathcal{E}))$ such that

$$\pi^* \nabla_X^{\text{Hom}(\nabla_1^\mathcal{E}, \nabla_2^\mathcal{E})} p = 0,$$

that is, p is a resonant state for the operator \mathbf{X}_{A_1, A_2} associated to the eigenvalue 0. Assumptions (A) and (B) allow us to apply Lemma 4.5. We then obtain

$$\lambda_{A_1, A_2} = 0 \leq -C \|\phi(A_1, A_2) - \nabla_1^\mathcal{E}\|_{H^{-1/2}(M, T^*M \otimes \text{End}(\mathcal{E}))}^2 \leq 0,$$

where $C = C(\nabla^\mathcal{E}) > 0$ only depends on $\nabla^\mathcal{E}$. Hence $\phi(A_1, A_2) = p_{A_1, A_2}^* \nabla_2^\mathcal{E} = \nabla_1^\mathcal{E}$. In other words, the connections are gauge-equivalent. ■

Next, we discuss a version of local injectivity in a neighbourhood of a connection which is non-opaque. We will say a map $f : X \rightarrow Y$ of topological spaces is *weakly locally injective at* $x_0 \in X$ if there exists a neighbourhood $U \ni x_0$ such that $f(x) = f(x_0)$ for $x \in U$ implies $x = x_0$. This notion appears in non-linear inverse problems where the linearization is not continuous [94, Section 2]. We have:

Theorem 5.1. *If $N \gg 1$ and $[\nabla^\mathcal{E}] \in \mathbb{A}_\mathcal{E}$ is such that the generalized X-ray transform $\Pi_1^{\text{End}(\mathcal{E})}$ is s -injective, as well as $\pi_{1*} \ker \pi^* \nabla_X^{\text{End}(\mathcal{E})}|_{C^\infty} = 0$, then the primitive trace map \mathcal{T}^\sharp is weakly locally injective at $[\nabla^\mathcal{E}]$ in the C^N -quotient topology.*

Proof. The proof is analogous to the proof of Theorem 1.1, by using the results of Section 4.4. We omit the details. ■

We shall see below (see Lemma 5.10) that flat connections have an injective generalized X-ray transform $\Pi_1^{\text{End}(\mathcal{E})}$ and satisfy the additional condition that $\ker(\pi^* \nabla^\mathcal{E})_X|_{C^\infty}$ consists of elements of degree zero (but might not be opaque). The previous theorem therefore shows that the primitive trace map is weakly locally injective near such connections. Let us state this as a corollary for the trivial connection, as it partially answers an open question of Paternain [79, p. 33, Question (3)].

Corollary 5.2. *Let $\mathcal{E} = M \times \mathbb{C}^r$ be the trivial Hermitian vector bundle equipped with the trivial flat connection d . Then there exists a neighbourhood $\mathcal{U} \ni [d]$ in $\mathbb{A}_\mathcal{E}$ with C^N -quotient topology such that $[d]$ is the unique gauge class of transparent connections in \mathcal{U} .*

5.2. Global injectivity results

We now detail some cases in which Theorem 1.1 can be upgraded.

5.2.1. *Line bundles.* We let $\mathcal{T}_1^\#$ be the restriction of the total primitive trace map (1.3) to line bundles. The moduli space of all connections on line bundles \mathbb{A}_1 carries a natural Abelian group structure using the tensor product. When restricted to line bundles, the primitive trace map $\mathcal{T}_1^\#$ takes values in $\ell(\mathcal{C}^\#, \mathbb{U}(1))$, the set of sequences indexed by primitive free homotopy classes. We have

Lemma 5.3. *The map $\mathcal{T}_1^\# : \mathbb{A}_1 \rightarrow \ell^\infty(\mathcal{C}^\#, \mathbb{U}(1))$ is a multiplicative group homomorphism.*

Proof. Left as an exercise to the reader. ■

Remark 5.4. There also exists a group homomorphism for higher rank vector bundles by taking the determinant instead of the trace. More precisely, writing \mathbb{A}_r for the set of all unitary connections on all possible Hermitian vector bundles of rank r (up to isomorphism), one can set

$$\det^\# : \mathbb{A} \rightarrow \ell^\infty(\mathcal{C}^\#, \mathbb{U}(1)), \tag{5.1}$$

by taking the determinant of the holonomy along each closed primitive geodesic. This map is also a group homomorphism (where the group structure on $\bigsqcup_{r \geq 0} \mathbb{A}_r$ is also obtained by tensor product). Nevertheless, the determinant map (5.1) cannot be injective as all trivial bundles (of different ranks) have the same image.

We have the following result, mainly due to Paternain [77]:

Proposition 5.5 (Paternain). *Let (M, g) be a smooth Anosov n -manifold. If $n \geq 3$, then the restriction of the primitive trace map to line bundles,*

$$\mathcal{T}_1^\# : \mathbb{A}_1 \rightarrow \ell^\infty(\mathcal{C}^\#), \tag{5.2}$$

is globally injective. Moreover, if $n = 2$ then

$$\ker \mathcal{T}_1^\# = \{([\kappa^{\otimes n}], [\nabla^{\text{LC}\otimes n}]) \mid n \in \mathbb{Z}\},$$

where $\kappa \rightarrow M$ denotes the canonical line bundle and ∇^{LC} is the connection induced on κ by the Levi-Civita connection.

Observe that on surfaces, the trivial line bundle $\mathbb{C} \times M \rightarrow M$ (with the trivial connection) and the canonical line bundle $\kappa \rightarrow M$ (with the Levi-Civita connection) both have trivial holonomy but are not isomorphic. This explains the existence of a non-trivial kernel for $n = 2$. We will need this preliminary lemma:

Lemma 5.6. *Let (M, g) be a smooth closed Riemannian manifold of dimension ≥ 3 and let $\pi : SM \rightarrow M$ be the projection. Let $\mathcal{L}_1 \rightarrow M$ and $\mathcal{L}_2 \rightarrow M$ be two Hermitian line bundles. If $\pi^* \mathcal{L}_1 \simeq \pi^* \mathcal{L}_2$, then $\mathcal{L}_1 \simeq \mathcal{L}_2$.*

Proof. The topology of line bundles is determined by their first Chern class. As a consequence, it suffices to show that $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$. By assumption, we have $c_1(\pi^* \mathcal{L}_1) = \pi^* c_1(\mathcal{L}_1) = c_1(\pi^* \mathcal{L}_2) = \pi^* c_1(\mathcal{L}_2)$ and thus it suffices to show that $\pi^* : H^2(M, \mathbb{Z}) \rightarrow H^2(SM, \mathbb{Z})$ is injective when $\dim(M) \geq 3$. But this is a mere consequence of the Gysin exact sequence [9, Proposition 14.33]. ■

Proof of Proposition 5.5. Assume that $\mathcal{T}_1^\#(\alpha_1) = \mathcal{T}_1^\#(\alpha_2)$, where $\alpha_1 \in \mathbb{A}_{\mathcal{L}_1}$ and $\alpha_2 \in \mathbb{A}_{\mathcal{L}_2}$ are two classes of connections defined on two (classes of) line bundles. By Theorem 3.2, the pullback bundles $\pi^*\mathcal{L}_1$ and $\pi^*\mathcal{L}_2$ are isomorphic, hence $\mathcal{L}_1 \simeq \mathcal{L}_2$ by Lemma 5.6. Up to composing with a first bundle (unitary) isomorphism, we can therefore assume that $\mathcal{L}_1 = \mathcal{L}_2 =: \mathcal{L}$. Let $\nabla_1^\mathcal{L} \in \alpha_1$ and $\nabla_2^\mathcal{L} \in \alpha_2$ be two representatives of these classes. They satisfy $\mathcal{T}^\#(\nabla_1^\mathcal{L}) = \mathcal{T}^\#(\nabla_2^\mathcal{L})$. Combining Theorem 3.2 with [77, Theorem 3.2], the primitive trace map $\mathcal{T}_\mathcal{L}^\#$ is known to be globally injective for connections on the same fixed bundle. Hence $\nabla_1^\mathcal{L}$ and $\nabla_2^\mathcal{L}$ are gauge-equivalent.

For the second claim, let $x = ([\mathcal{L}], \alpha)$. If $\mathcal{T}_1^\#(x) = (1, 1, \dots)$ (i.e. the connection is transparent), then by Theorem 3.2, $\pi^*\mathcal{L} \rightarrow SM$ is trivial. By the Gysin sequence [9, Proposition 14.33], this implies that $c_1(\mathcal{L})$ is divisible by $2g - 2$, where g is the genus of M (see [77, Theorem 3.1]), hence $[\mathcal{L}] = [\kappa^{\otimes n}]$ for some $n \in \mathbb{Z}$. Moreover, the Levi-Civita connection on $\kappa^{\otimes n}$ is transparent and by uniqueness (see [77, Theorem 3.2]), this implies that $\alpha = [\nabla^{\text{LC}\otimes n}]$. ■

Remark 5.7. The target space in (5.2) is actually $\ell^\infty(\mathcal{C}^\#, U(1))$ (sequences indexed by $\mathcal{C}^\#$ and taking values in $U(1)$), which can be seen as a subset of $U(\ell^\infty(\mathcal{C}^\#))$, the group of unitary operators of the Banach space $\ell^\infty(\mathcal{C}^\#)$ (equipped with the sup norm). Then $\mathcal{T}_1^\#$ is a group homomorphism and Proposition 5.5 asserts that

$$\mathcal{T}_1^\# : \mathbb{A}_1 \rightarrow U(\ell^\infty(\mathcal{C}^\#))$$

is a faithful unitary representation of the Abelian group \mathbb{A}_1 .

We end this subsection with a generalization of Proposition 5.5. There is a natural submonoid $\mathbb{A}' \subset \mathbb{A}$ which is obtained by considering sums of line bundles equipped with unitary connections, that is,

$$\mathbb{A}' := \{x_1 \oplus \dots \oplus x_k \mid k \in \mathbb{N}, x_i \in \mathbb{A}_1\}.$$

We then have the following:

Theorem 5.8. *Let (M, g) be a smooth Anosov Riemannian manifold of dimension ≥ 3 . Then the restriction of the primitive trace map to \mathbb{A}' ,*

$$\mathcal{T}^\# : \mathbb{A}' \rightarrow \ell^\infty(\mathcal{C}^\#),$$

is globally injective.

Proof. Let $\mathcal{L} := \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_k$ and $\mathcal{J} = \mathcal{J}_1 \oplus \dots \oplus \mathcal{J}_{k'}$ be two Hermitian vector bundles over M , equipped with the respective connections $\nabla^{\mathcal{L}_1} \oplus \dots \oplus \nabla^{\mathcal{L}_k}$ and $\nabla^{\mathcal{J}_1} \oplus \dots \oplus \nabla^{\mathcal{J}_{k'}}$, and assume that they have the same image by the primitive trace map. Fixing a periodic point (x_\star, v_\star) and applying Proposition 3.19, we see that $k = k'$ and there exist isomorphic representations $\rho_\mathcal{L} : \mathbf{G} \rightarrow U(\pi^*\mathcal{L}_{(x_\star, v_\star)})$ and $\rho_\mathcal{J} : \mathbf{G} \rightarrow U(\pi^*\mathcal{J}_{(x_\star, v_\star)})$, where \mathbf{G} denotes Parry’s free monoid at (x_\star, v_\star) . Since these representations are sums of one-dimensional representations, there is a unitary isomorphism $p_\star : \pi^*\mathcal{L}_{(x_\star, v_\star)} \rightarrow$

$\pi^* \mathcal{J}_{(x_*, v_*)}$ such that for each $i \in \{1, \dots, k\}$, there exists $\sigma(i) \in \{1, \dots, k\}$ such that $p_\star^{(i)} := p_\star |_{\pi^* \mathcal{L}_{i, (x_*, v_*)}}$ is a representation isomorphism

$$p_\star^{(i)} : \pi^* \mathcal{L}_{i, (x_*, v_*)} \rightarrow \pi^* \mathcal{J}_{\sigma(i), (x_*, v_*)}.$$

Now, following the arguments of Lemma 3.21, we parallel-transport $p_\star^{(i)}$ along the homoclinic orbits with respect to the pullback $\pi^* \nabla_X^{\text{Hom}(\nabla^{\mathcal{L}_i}, \nabla^{\mathcal{J}_{\sigma(i)}})}$ of the mixed connection (induced by $\nabla^{\mathcal{L}_i}$ on \mathcal{L}_i and $\nabla^{\mathcal{J}_{\sigma(i)}}$ on $\mathcal{J}_{\sigma(i)}$); the Lipschitz-regularity of the resulting section follows, as in Lemma 3.24, from the fact that $p_\star^{(i)} \rho_{\mathcal{L}_i}(g) = \rho_{\mathcal{J}_{\sigma(i)}}(g) p_\star^{(i)}$ for all $g \in \mathbf{G}$. Using the regularity result of [39], we thus obtain a unitary section

$$p^{(i)} \in C^\infty(SM, \pi^* \text{Hom}(\mathcal{L}_i, \mathcal{J}_{\sigma(i)}))$$

conjugating the parallel transports along geodesic flow lines with respect to the connections $\pi^* \nabla^{\mathcal{L}_i}$ and $\pi^* \nabla^{\mathcal{J}_{\sigma(i)}}$. In particular, the existence of such a $p^{(i)}$ ensures that $\mathcal{T}_1^\#(\mathcal{L}_i, \nabla^{\mathcal{L}_i}) = \mathcal{T}_1^\#(\mathcal{J}_{\sigma(i)}, \nabla^{\mathcal{J}_{\sigma(i)}})$. We then conclude by Proposition 5.5, showing that each pair $(\mathcal{L}_i, \nabla^{\mathcal{L}_i})$ is isomorphic to $(\mathcal{J}_{\sigma(i)}, \nabla^{\mathcal{J}_{\sigma(i)}})$ for $i = 1, \dots, k$. ■

5.2.2. *Flat bundles.* We discuss the particular case of flat vector bundles. It is well-known that the data of a vector bundle equipped with a unitary connection (modulo isomorphism) is equivalent to a unitary representation of the fundamental group (modulo inner automorphisms of the unitary group). More precisely, given $\rho \in \text{Hom}(\pi_1(M), U(r))$, one can associate a Hermitian bundle $\mathcal{E} \rightarrow M$ equipped with a flat unitary connection $\nabla^\mathcal{E}$ by the following process: Let \tilde{M} be the universal cover of M ; consider the trivial bundle $\mathbb{C}^r \times \tilde{M}$ equipped with the flat connection d and define the relation $(x, v) \sim (x', v')$ if and only if $x' = \gamma(x)$, $v' = \rho(\gamma)v$, for some $\gamma \in \pi_1(M)$; then $(\mathcal{E}, \nabla^\mathcal{E})$ is obtained by taking the quotient $\mathbb{C}^r \times \tilde{M} / \sim$. Changing ρ to an isomorphic representation $\rho' = p \cdot \rho \cdot p^{-1}$ (for $p \in U(r)$) changes the connection to a gauge-equivalent connection and this process gives a one-to-one correspondence between the moduli spaces.

For $r \geq 0$, we let

$$\mathcal{M}_r := \text{Hom}(\pi_1(M), U(r)) / \sim$$

be the moduli space of unitary representations of the fundamental group, where two representations are \sim -equivalent whenever they are isomorphic. The space \mathcal{M}_r is called the *character variety*; see [61] for instance. For $r = 0$, it is reduced to a point; for $r = 1$, it is given by $\mathcal{M}_1 = U(1)^{b_1(M)}$, where $b_1(M)$ denotes the first Betti number of M . Given $x \in \mathcal{M}_r$, we let $\Psi(x) = (\mathcal{E}_x, \nabla^{\mathcal{E}_x})$ be the data of a Hermitian vector bundle equipped with a unitary connection (up to gauge equivalence) described by the above process. The primitive trace map $\mathcal{T}^\#$ can then be seen as a map

$$\mathcal{T}^\# : \bigsqcup_{r \geq 0} \mathcal{M}_r \rightarrow \ell^\infty(\mathcal{C}^\#), \quad \mathcal{T}^\#(x) := \mathcal{T}^\#(\nabla^{\mathcal{E}_x}),$$

where the right hand side is understood by (1.2). We then have the following:

Proposition 5.9. *Let (M, g) be an Anosov manifold of dimension ≥ 2 . Then the primitive trace map*

$$\mathcal{T}^\# : \bigsqcup_{r \geq 0} \mathcal{M}_r \rightarrow \ell^\infty(\mathcal{E}^\#)$$

is globally injective. Moreover, given $x_0 = ([\mathcal{E}_0], [\nabla_0^\mathcal{E}]) \in \mathcal{M}_r$, the primitive trace map is weakly locally injective (in the sense of Theorem 5.1) near x_0 in the space $\mathbb{A}_{[\mathcal{E}_0]}$ of all unitary connections on $[\mathcal{E}_0]$.

Proposition 5.9 will be strengthened below when further assuming that (M, g) has negative curvature (see Lemma 5.13): we will show that the primitive trace map is globally injective on connections with *small* curvature. The first part of Proposition 5.9 could be proved by purely algebraic arguments; nevertheless, we provide a proof with dynamical flavour, more in the spirit of the present article. We need a preliminary result:

Lemma 5.10. *Assume (M, g) is Anosov and $\nabla^\mathcal{E}$ is a flat and unitary connection on the Hermitian vector bundle $\mathcal{E} \rightarrow M$. Let $\mathbf{X} := (\pi^* \nabla^\mathcal{E})_X$.*

- *If $\mathbf{X}u = f$ with $f = f_0 + f_1 \in C^\infty(M, (\Omega_0 \oplus \Omega_1) \otimes \mathcal{E})$ and $u \in C^\infty(SM, \pi^* \mathcal{E})$, then $f_0 = 0$ and u is of degree 0.*
- *In particular, smooth invariant sections $u \in \ker \mathbf{X}|_{C^\infty(SM, \pi^* \mathcal{E})}$ are of degree 0.*
- *The operator $\Pi_1^\mathcal{E}$ is s -injective.*

Proof. The proof is based on the twisted Pestov identity for flat connections.

Lemma 5.11 (Twisted Pestov identity). *Let $u \in H^2(SM, \pi^* \mathcal{E})$. Then*

$$\|\nabla_{\nabla}^\mathcal{E} \mathbf{X}u\|_{L^2}^2 = \|\mathbf{X}\nabla_{\nabla}^\mathcal{E} u\|_{L^2}^2 - \langle R\nabla_{\nabla}^\mathcal{E} u, \nabla_{\nabla}^\mathcal{E} u \rangle_{L^2} + (n - 1)\|\mathbf{X}u\|_{L^2}^2.$$

For the notation, see Section 2.3.2; for a proof, we refer to [45, Proposition 3.3]. An important point is that the following inequality holds for Anosov manifolds:

$$\|\mathbf{X}\nabla_{\nabla}^\mathcal{E} u\|_{L^2}^2 - \langle R\nabla_{\nabla}^\mathcal{E} u, \nabla_{\nabla}^\mathcal{E} u \rangle_{L^2} \geq C \|\nabla_{\nabla}^\mathcal{E} u\|_{L^2}^2,$$

where $C > 0$ is independent of u ; see [83, Theorem 7.2] for the case of the trivial line bundle (the generalization to the twisted case is straightforward). We thus obtain

$$\|\nabla_{\nabla}^\mathcal{E} \mathbf{X}u\|_{L^2}^2 \geq C \|\nabla_{\nabla}^\mathcal{E} u\|_{L^2}^2 + (n - 1)\|\mathbf{X}u\|_{L^2}^2. \tag{5.3}$$

By assumption, $\mathbf{X}u = f_0 + f_1 \in C^\infty(M, (\Omega_0 \oplus \Omega_1) \otimes \mathcal{E})$. Observe that this equation can be split into odd/even parts: $\mathbf{X}u_{\text{even}} = f_1$, $\mathbf{X}u_{\text{odd}} = f_0$, and $u_{\text{even,odd}} \in C^\infty(SM, \pi^* \mathcal{E})$ have respective even/odd Fourier components. Applying (5.3) with u_{odd} , we obtain $f_0 = 0$, $\mathbf{X}u_{\text{odd}} = 0$ and $\nabla_{\nabla}^\mathcal{E} u_{\text{odd}} = 0$, that is, u_{odd} is of degree 0 but 0 is even so $u_{\text{odd}} = 0$. As far as u_{even} is concerned, observe that $\nabla_{\nabla}^\mathcal{E} \mathbf{X}u_{\text{even}} = \nabla_{\nabla}^\mathcal{E} f_1$ and

$$\|\nabla_{\nabla}^\mathcal{E} f_1\|_{L^2}^2 = \langle -\Delta_{\nabla}^\mathcal{E} f_1, f_1 \rangle_{L^2} = (n - 1)\|f_1\|_{L^2}^2.$$

Hence, applying the twisted Pestov identity with u_{even} , we obtain

$$0 = \|\mathbf{X}\nabla_{\nabla}^\mathcal{E} u_{\text{even}}\|_{L^2}^2 - \langle R\nabla_{\nabla}^\mathcal{E} u_{\text{even}}, \nabla_{\nabla}^\mathcal{E} u_{\text{even}} \rangle_{L^2} \geq C \|\nabla_{\nabla}^\mathcal{E} u_{\text{even}}\|_{L^2}^2,$$

that is, u_{even} is of degree 0. This proves the first point, and the second point is a direct consequence of the first point.

For the last point, consider the equation $\mathbf{X}u = \pi_1^* f$. By the first point, u is of degree zero, so $u = \pi_0^* u'$ for some $u' \in C^\infty(M, \mathcal{E})$. Hence by (2.11) we get $f = \nabla^\mathcal{E} u'$ and the conclusion follows by Lemma 2.8. ■

Proof of Proposition 5.9. We prove the first part of Proposition 5.9. We assume that $\mathcal{T}^\sharp(x_1) = \mathcal{T}^\sharp(x_2)$, or equivalently $\mathcal{T}^\sharp(\nabla^{\mathcal{E}_{x_1}}) = \mathcal{T}^\sharp(\nabla^{\mathcal{E}_{x_2}})$. The exact Livšic cocycle theorem 3.2 implies that the bundles $\pi^* \mathcal{E}_{x_1}$ and $\pi^* \mathcal{E}_{x_2}$ are isomorphic and yields the existence of a section $p \in C^\infty(SM, U(\pi^* \mathcal{E}_{x_2}, \pi^* \mathcal{E}_{x_1}))$ such that

$$C_{x_1}(x, t) = p(\varphi_t x) C_{x_2}(x, t) p(x)^{-1}$$

for all $x \in \mathcal{M}$ and $t \in \mathbb{R}$, which is equivalent to

$$\pi^* \nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_{x_2}}, \nabla^{\mathcal{E}_{x_1}})} p = 0,$$

where $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_{x_2}}, \nabla^{\mathcal{E}_{x_1}})}$ is the mixed connection induced by $\nabla^{\mathcal{E}_{x_2}}$ and $\nabla^{\mathcal{E}_{x_1}}$ on $\text{Hom}(\mathcal{E}_{x_2}, \mathcal{E}_{x_1})$. Observe that by (2.5), the curvature of $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_{x_2}}, \nabla^{\mathcal{E}_{x_1}})}$ vanishes as both curvatures $F_{\nabla^{\mathcal{E}_{x_1,2}}}$ vanish. Applying Lemma 5.10 with $\mathbf{X} := \pi^* \nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_{x_2}}, \nabla^{\mathcal{E}_{x_1}})}$ acting on the pullback bundle $\pi^* \text{Hom}(\mathcal{E}_{x_2}, \mathcal{E}_{x_1})$, we find that p is of degree 0, which is equivalent to the fact that the connections are gauge-equivalent.

As for the second part of Proposition 5.9, by Theorem 5.1 it is a straightforward consequence of the s -injectivity of $\Pi_1^{\text{End}(\mathcal{E}_0)}$ and the fact that elements of $\ker(\pi^* \nabla^{\text{End}(\mathcal{E}_0)})_X|_{C^\infty}$ are of degree zero, which follows from Lemma 5.10 (2, 3). ■

5.2.3. *Negative sectional curvature.* We now assume further that the Riemannian manifold (M, g) has negative sectional curvature. We introduce the following condition:

Definition 5.12. We say that the pair of connections $(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})$ satisfies the *spectral condition* if the mixed connection $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$ has no non-trivial twisted CKTs.

This condition is symmetric in the pair $(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})$. Observe that by (2.6), the previous condition is invariant under changing one of the two connections to $p^* \nabla^{\mathcal{E}_i}$ for some vector bundle isomorphism p , and thus this condition descends to the moduli space. We then define

$$\mathbf{S} \subset \mathbb{A} \times \mathbb{A}, \tag{5.4}$$

the subspace of all pairs of equivalence classes of connections satisfying the spectral condition. The set \mathbf{S} is open and dense (for the C_*^N -topology, $N \gg 1$), as shown in Appendix A. Moreover, it also contains all pairs of connections with *small curvature*, that is, if

$$\Omega_\varepsilon := \{x = ([\mathcal{E}], [\nabla^\mathcal{E}]) \in \mathbb{A} \mid \|F_{\nabla^\mathcal{E}}\|_{L^\infty(M, \Lambda^2 T^* M \otimes \text{End}(\mathcal{E}))} < \varepsilon\} \subset \mathbb{A},$$

then we have the following:

Lemma 5.13. *Let (M, g) be a negatively-curved Riemannian manifold of dimension ≥ 2 and let $-\kappa < 0$ be an upper bound for the sectional curvature. There exists $\varepsilon(n, \kappa) > 0$ such that*

$$\Omega_{\varepsilon(n, \kappa)} \times \Omega_{\varepsilon(n, \kappa)} \subset \mathbf{S}.$$

One can take $\varepsilon(n, \kappa) = \kappa \sqrt{n - 1} / 4$.

Proof. We start by a preliminary discussion. Given a Hermitian vector bundle $\mathcal{E} \rightarrow M$ with metric $\langle \bullet, \bullet \rangle$, and a unitary connection $\nabla^\mathcal{E}$, we introduce, following [45, Section 3], an operator $\mathcal{F}^\mathcal{E} \in C^\infty(SM, \mathcal{N} \otimes \text{End}_{\text{sk}}(\mathcal{E}))$ (recall that \mathcal{N} is the normal bundle, see Section 2.3.1) defined by the equality

$$\langle \mathcal{F}^\mathcal{E}(x, v)e, w \otimes e' \rangle := \langle F_{\nabla^\mathcal{E}}(v, w)e, e' \rangle, \tag{5.5}$$

where $F_{\nabla^\mathcal{E}}$ is the connection of $\nabla^\mathcal{E}$, and $(x, v) \in SM$, $e, e' \in \mathcal{E}_x$, $w \in \mathcal{N}(x, v)$, and the metric on the left hand side is the natural extension of the metric $\langle \bullet, \bullet \rangle$ on \mathcal{E} to $\mathcal{N} \otimes \mathcal{E}$ by tensoring with the metric g . A straightforward computation shows that

$$\|\mathcal{F}^\mathcal{E}\|_{L^\infty(SM, \mathcal{N} \otimes \text{End}_{\text{sk}}(\mathcal{E}))} \leq \|F_{\nabla^\mathcal{E}}\|_{L^\infty(M, \Lambda^2 T^*M \otimes \text{End}(\mathcal{E}))}. \tag{5.6}$$

Now, let $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$ be two unitary connections, $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$ be the mixed connection and $\mathcal{F}^{\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)}$ be the operator induced by the mixed connection as in (5.5). Observe that by (2.5) and (5.6), we get

$$\|\mathcal{F}^{\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)}\|_{L^\infty} \leq \|F_{\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}}\|_{L^\infty} \leq \|F_{\nabla^{\mathcal{E}_1}}\|_{L^\infty} + \|F_{\nabla^{\mathcal{E}_2}}\|_{L^\infty} < 2\varepsilon(n, \kappa). \tag{5.7}$$

By [45, Theorem 4.5], if $m \geq 1$ satisfies

$$m(m + n - 2) \geq 4 \frac{\|\mathcal{F}^{\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)}\|_{L^\infty}^2}{\kappa^2}, \tag{5.8}$$

then there are no twisted CKTs of degree m (for the connection $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$). Now, the choice of $\varepsilon(n, \kappa) > 0$ combined with (5.7) guarantees that (5.8) is satisfied for any $m \geq 1$. ■

We then have the following statement:

Proposition 5.14. *Let (M, g) be a negatively-curved Riemannian manifold of dimension ≥ 2 . Let $(\alpha, \alpha') \in \mathbf{S}$ be such that $\mathcal{T}^\#(\alpha) = \mathcal{T}^\#(\alpha')$. Then $\alpha = \alpha'$.*

In other words, two connections satisfying the spectral condition and whose images by the primitive trace map are equal, are actually gauge-equivalent.

Proof. Consider two representatives $\nabla^{\mathcal{E}_1} \in \alpha$ and $\nabla^{\mathcal{E}_2} \in \alpha'$. The exact Livšić cocycle theorem 3.2 provides a section $p \in C^\infty(SM, U(\pi^*\mathcal{E}_2, \pi^*\mathcal{E}_1))$ such that

$$\pi^* \nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_2}, \nabla^{\mathcal{E}_1})} p = 0.$$

By assumption, (M, g) has negative curvature and thus p has finite Fourier degree by

[45, Theorem 4.1]. Moreover, since $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_2}, \nabla^{\mathcal{E}_1})}$ has no non-trivial twisted CKTs, p is of degree 0 (see [45, Theorem 5.1]). This shows that the connections are gauge-equivalent. ■

5.2.4. *Topological results.* In this section we prove a global *topological* uniqueness result for the primitive trace map.

Proposition 5.15. *Let (M, g) be an orientable Anosov manifold. If $x_i = ([\mathcal{E}_i], [\nabla^{\mathcal{E}_i}]) \in \mathbb{A}$ for $i = 1, 2$, then $\mathcal{T}^\#(x_1) = \mathcal{T}^\#(x_2)$ implies:*

- *If $\dim(M)$ is odd or more generally the Euler characteristic $\chi(M)$ vanishes, then $\mathcal{E}_1 \simeq \mathcal{E}_2$ as vector bundles.*
- *If $\dim(M) = 2d$ for some $d \in \mathbb{N}$ and $\chi(M) \neq 0$, then*
 - *the Chern classes satisfy $c_i(\mathcal{E}_1) = c_i(\mathcal{E}_2)$ for $i = 1, \dots, d - 1$; also $c_d(\mathcal{E}_1) - c_d(\mathcal{E}_2) \in H^{2d}(M; \mathbb{Z}) \cong \mathbb{Z}$ is a multiple of $\chi(M)$;*
 - *if the even cohomology ring $H^{\text{even}}(M; \mathbb{Z})$ is torsion-free, and the rank of the bundles is less than d or more generally $c_d(\mathcal{E}_1) = c_d(\mathcal{E}_2)$, then \mathcal{E}_1 and \mathcal{E}_2 are stably isomorphic, i.e. there is an $m \geq 0$ such that $\mathcal{E}_1 \oplus \mathbb{C}^m \simeq \mathcal{E}_2 \oplus \mathbb{C}^m$.*

Proof. As a direct consequence of Theorem 3.2, from $\mathcal{T}^\#(x_1) = \mathcal{T}^\#(x_2)$ we deduce that $\pi^*\mathcal{E}_1 \simeq \pi^*\mathcal{E}_2$.

If (M, g) has a vanishing Euler characteristic, there is a non-vanishing vector field $V \in C^\infty(M, TM)$ (see [9, Chapter 11]), which we normalize to unit norm using the metric g and hence see as a section of SM . Then since $\pi \circ V = \text{Id}_M$, we get

$$\mathcal{E}_1 \simeq V^*\pi^*\mathcal{E}_1 \simeq V^*\pi^*\mathcal{E}_2 \simeq \mathcal{E}_2,$$

completing the proof of the first item.

The first point of the second item is immediate after an application of the Gysin exact sequence [9, Proposition 14.33] for the sphere bundle SM . The second point follows from the first one and the fact that the Chern character gives an isomorphism between rational K -theory and even rational cohomology [50, Proposition 4.5]. ■

It is not known to the authors if further results hold about the injectivity of $\pi^* : \text{Vect}(M) \rightarrow \text{Vect}(SM)$ in even dimensions ($\dim(M) \geq 4$).

Appendix A. Generic absence of CKTs for the mixed connection

In this section we assume that (M, g) has negative curvature. For $i = 1, 2$, we let $\nabla^{\mathcal{E}_i}$ be a (smooth) unitary connection on the Hermitian vector bundle $\mathcal{E}_i \rightarrow M$. By [45, Theorem 4.5], there exists $m_0 \gg 1$ (depending on the dimension and the sup norm of the curvatures of $\nabla^{\mathcal{E}_{1,2}}$) such that the mixed connection $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$ has no non-trivial twisted CKTs of degree $m \geq m_0$. This property is stable by any small perturbation of $\nabla^{\mathcal{E}_i}$ in the C^1 -topology (so that the curvature is well-defined). From this, we deduce by standard elliptic theory that if the pair $(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})$ satisfies the spectral condition, then any

small perturbation (in the C^1 -topology) will also satisfy the spectral condition: indeed, absence of twisted CKTs of degree m is equivalent to the invertibility of a natural Laplacian operator acting on $\Omega_m \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ and this is an open property as there are only a priori a finite number of integers $m \leq m_0$ to check; see [15] for further details. This shows that the set \mathbf{S} defined in (5.4) is open.

We now show that it is dense. More precisely, we show the following:

Lemma A.1. *Let $\nabla^{\mathcal{E}_{1,2}}$ be a smooth unitary connection on $\mathcal{E}_{1,2} \rightarrow M$ and assume that the mixed connection $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$ admits non-trivial twisted CKTs of degree $m \geq 1$. Then, for any $\varepsilon > 0$ and $k_0 \gg 1$ large enough, there exists a small perturbation $\nabla^{\mathcal{E}_2} + \Gamma_2$, where $\Gamma_2 \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}_2))$ and $\|\Gamma_2\|_{C^{k_0}} \leq \varepsilon$, such that the mixed connection induced by the pair $(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2} + \Gamma_2)$ has no non-trivial twisted CKTs of degree m .*

Proof. We let $\mathbf{X}^{\Gamma_2} := \pi^* \nabla_X^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2} + \Gamma_2)}$, where Γ_2 is small and, as in Section 2.3.2, we define

$$\mathbf{X}_{\pm}^{\Gamma_2} : C^\infty(M, \Omega_m \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)) \rightarrow C^\infty(M, \Omega_{m \pm 1} \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)),$$

and $\mathbf{X}_{\pm} := \mathbf{X}^{\Gamma_2=0}$. We also let $\Delta^{\Gamma_2} := (\mathbf{X}_+^{\Gamma_2})^* \mathbf{X}_+^{\Gamma_2}$, the Laplacian-type operator acting on sections of $\Omega_m \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$. The existence of twisted CKTs of degree m for the mixed connection $\nabla^{\text{Hom}(\nabla^{\mathcal{E}_1}, \nabla^{\mathcal{E}_2})}$ is equivalent to the existence of a non-trivial kernel for $\Delta^{\Gamma_2=0}$; see [15, Section 4] for further details.

Given $\gamma \subset \mathbb{C}$, a small contour in \mathbb{C} around 0 (containing only the eigenvalue 0 of $\Delta^{\Gamma_2=0}$), we let λ^{Γ_2} be the sum of the eigenvalues of Δ^{Γ_2} inside γ . We see that $C^{k_0} \ni \Gamma_2 \mapsto \lambda^{\Gamma_2}$ is at least C^3 when $k_0 \gg 1$ is chosen large enough. We have $\lambda^{\Gamma_2=0} = 0$ and $d\lambda^{\Gamma_2} = 0$ [15, Section 4]. Moreover, it was shown in [15, Lemma 4.2] that the second derivative is

$$d^2\lambda^{\Gamma_2=0}(A_2, A_2) = \sum_{i=1}^d \|\pi_{\ker \mathbf{X}_-}[d\mathbf{X}^{\Gamma_2=0}(A_2)]_{+u_i}\|_{L^2}^2 = \sum_{i=1}^d \|\pi_{\ker \mathbf{X}_-}(A_2)_{+u_i}\|_{L^2}^2,$$

where $\{u_1, \dots, u_d\}$ is an L^2 -orthonormal basis of $\ker \Delta^{\Gamma_2=0}$ (each u_i is a smooth section of $\Omega_m \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow M$), $A_2 \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}_2))$ and $(A_2)_+$ is the positive part of the operator (see [15, Section 2.2]), and $\pi_{\ker \mathbf{X}_-}$ is the L^2 -orthogonal projection onto $\ker \mathbf{X}_-|_{L^2(M, \Omega_{m+1} \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2))}$. The formula for $d\mathbf{X}^{\Gamma_2=0}(A_2) = A_2$ can be directly read off from (2.3).

It thus suffices to produce a small perturbation such that this second derivative is positive. We can argue by contradiction and assume that for any perturbation, this second derivative vanishes. Following *verbatim* the arguments of [15, Section 4.3], and using the fact that the operator \mathbf{X}_- is of *uniform divergence type* (see [15, Section 3]), this would imply that

$$\langle u_1(x), (A_2)_- w \rangle_x = 0$$

for all $x \in M$, $A_2 \in C^\infty(M, T^*M \otimes \text{End}_{\text{sk}}(\mathcal{E}_2))$, $w \in \Omega_{m+1}(x) \otimes \text{Hom}(\mathcal{E}_1(x), \mathcal{E}_2(x))$. It thus suffices to show, as in [15, Lemma 4.8] that this forces u_1 to be zero, which is a contradiction since $\|u_1\|_{L^2} = 1$.

We now fix an arbitrary point x_0 . We can write $u_1(x_0) = \sum_{i=1}^k p_i \otimes s_i$, where $k = \text{rank}(\text{Hom}(\mathcal{E}_1, \mathcal{E}_2))$, $\{s_1, \dots, s_k\}$ is an orthonormal basis of $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ at x_0 , and $p_i \in \Omega_m(x_0)$. We write $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ for an orthonormal basis of $T_{x_0}M$. Taking $A_2 = i \mathbb{1}_{\mathcal{E}_2} \otimes \mathbf{e}_1^*$, $w = f \otimes s_{i_0}$ where $f \in \Omega_{m+1}(x_0)$ is such that $(v_1)_- f = p_{i_0}$ (here $v_1 = \mathbf{e}_1^*(v)$ and $(v_1)_-$ is the minus operator associated; this operator is surjective by [15, Lemma 2.4]), we get, as in [15, Lemma 4.8],

$$\begin{aligned} \langle u_1(x_0), (A_2)_- w \rangle_{x_0} &= \sum_{i=1}^k \langle p_i \otimes s_i, (v_1)_- f \otimes i \mathbb{1}_{\mathcal{E}_2} \cdot s_{i_0} \rangle \\ &= i \|p_{i_0}\|^2 \|s_{i_0}\|^2 = i \|p_{i_0}\|_{\mathcal{L}^2}^2 = 0. \end{aligned}$$

Hence $u_1 \equiv 0$. This concludes the proof. \blacksquare

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