J. Eur. Math. Soc. 27, 2467–2476 (2025) DOI 10.4171/JEMS/1416

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Geometry of constant mean curvature surfaces in $\mathbb{R}^3$

Received September 13, 2021

Abstract. The crowning achievement of this paper is the proof that round spheres are the only complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with nonzero constant mean curvature. Fundamental to this proof are new results including the existence of intrinsic curvature and radius estimates for compact disks embedded in $\mathbb{R}^3$ with nonzero constant mean curvature. We also prove curvature estimates for compact annuli embedded in $\mathbb{R}^3$ with nonzero constant mean curvature and apply them to obtain deep results on the global geometry of complete surfaces of finite topology embedded in $\mathbb{R}^3$ with constant mean curvature.

Keywords: minimal surface, constant mean curvature, curvature estimates.

1. Introduction

A longstanding problem in classical surface theory is to classify the complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with constant mean curvature. In the case the surface is simply-connected and compact, this classification follows by work of either Hopf [15] in 1951 or of Alexandrov [1] in 1956, who gave different proofs that a round sphere is the only possibility.

In this paper we will prove that a complete, embedded simply-connected surface in $\mathbb{R}^3$ with nonzero constant mean curvature must be compact, which by Hopf's or Alexandrov's theorem gives the next classification result.

Theorem 1.1. *Complete, simply-connected surfaces embedded in $\mathbb{R}^3$ with nonzero constant mean curvature are round spheres.*

Theorem 1.1, together with results of Colding and Minicozzi [11] and Meeks and Rosenberg [23] that show that the complete, simply-connected minimal surfaces embed-

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Mathematics Subject Classification 2020: 53A10 (primary); 49Q05, 53C42 (secondary).

ded in $\mathbb{R}^3$ are planes and helicoids, finishes the classification of complete simply-connected surfaces embedded in $\mathbb{R}^3$ with constant mean curvature.

In Section 2 we explain how the results in [26,28,30,31] lead to the following intrinsic radius and curvature estimates for embedded disks in $\mathbb{R}^3$ with nonzero constant mean curvature, where the *radius* of a compact Riemannian surface with boundary is the maximum intrinsic distance of points in the surface to its boundary.

Theorem 1.2 (Radius estimates). *There exists an $\mathcal{R} \geq \pi$ such that any compact disk embedded in $\mathbb{R}^3$ of constant mean curvature $H > 0$ has radius less than $\mathcal{R}/H$.*

Theorem 1.3 (Curvature estimates). *Given $\delta, \mathcal{H} > 0$, there exists $K(\delta, \mathcal{H}) \geq \sqrt{2}\,\mathcal{H}$ such that any compact disk M embedded in $\mathbb{R}^3$ with constant mean curvature $H \geq \mathcal{H}$ satisfies*

$$\sup_{\{p \in M \mid d_M(p, \partial M) \geq \delta\}} |A_M| \leq K(\delta, \mathcal{H}),$$

where $|A_M|$ is the norm of the second fundamental form and d_M is the intrinsic distance function of M.

The radius estimate in Theorem 1.2 implies that a complete, simply-connected surface embedded in $\mathbb{R}^3$ with nonzero constant mean curvature is compact. In this way Theorem 1.1 follows from Theorem 1.2 and Hopf's or Alexandrov's theorem.

We wish to emphasize to the reader that the curvature estimates for embedded constant mean curvature disks given in Theorem 1.3 depend only on the *lower* positive bound $\mathcal{H}$ for their mean curvature. Other important examples of curvature estimates for constant mean curvature surfaces, assuming certain geometric conditions, can be found in the literature; see for instance [3, 4, 6, 10, 11, 38–41, 45, 46].

Our investigation here is inspired by the pioneering work of Colding and Minicozzi in the minimal case [7–10]; however, in the constant positive mean curvature setting our work leads to the existence of radius and curvature estimates. Since the plane and the helicoid are complete simply-connected minimal surfaces properly embedded in $\mathbb{R}^3$, a radius estimate does not hold in the minimal case. Moreover, rescalings of a helicoid give rise to a sequence of embedded minimal disks with arbitrarily large norms of their second fundamental forms at points that can be arbitrarily far from their boundary curves; therefore in the minimal setting, curvature estimates do not hold either.

For clarity of exposition, we will call an oriented surface M immersed in $\mathbb{R}^3$ an *H-surface* if it is *embedded*, *connected* and it has *positive constant mean curvature H*. We will call an H-surface an *H-disk* if the H-surface is homeomorphic to a closed disk in the Euclidean plane.

The next corollary is an immediate consequence of Theorem 1.3.

Corollary 1.4. *If M is a complete H-surface with positive injectivity radius r_0, then*

$$\sup_M |A_M| \leq K(r_0, H).$$

Since there exists an $\varepsilon > 0$ such that for any $C > 0$, every complete immersed surface Σ in $\mathbb{R}^3$ with $\sup_\Sigma |A_\Sigma| < C$ has injectivity radius greater than ε/C, Corollary 1.4

implies that a necessary and sufficient condition for an H-surface to have bounded norm of the second fundamental form is that it has positive injectivity radius.

Corollary 1.5. *A complete H-surface has positive injectivity radius if and only if it has bounded norm of the second fundamental form.*

As complete H-surfaces of bounded norm of the second fundamental form are properly embedded in $\mathbb{R}^3$ by [24, Theorem 6.1], Corollary 1.4 implies the next result.

Corollary 1.6. *A complete H-surface with positive injectivity radius is properly embedded in $\mathbb{R}^3$.*

In Section 3 we obtain curvature estimates for H-surfaces that are annuli; these estimates are analogous to the curvature estimates in Theorem 1.3 for H-disks but necessarily must also depend on the flux (see Definitions 3.2 and 3.3) of a given annulus. We then apply these new curvature estimates to prove Theorem 1.7 below on the properness of complete H-surfaces of finite topology.

Theorem 1.7. *A complete H-surface with smooth compact boundary* (*possibly empty*) *and finite topology has bounded norm of the second fundamental form and is properly embedded in $\mathbb{R}^3$.*

Earlier, as the main result in [11], Colding and Minicozzi proved the similar theorem that complete minimal surfaces of finite topology embedded in $\mathbb{R}^3$ are proper, thereby solving the classical Calabi–Yau problem in the minimal setting.

Theorem 1.7 shows that certain classical results for H-surfaces hold when the hypothesis of "properly embedded" is replaced by the weaker hypothesis of "complete and embedded." For instance, in the seminal paper [17], Korevaar, Kusner and Solomon proved that the ends of a properly embedded H-surface of finite topology in $\mathbb{R}^3$ are asymptotic to the ends of surfaces of revolution defined by Delaunay [14] in 1841, and that if such a surface has exactly two ends, then it must be a Delaunay surface. Earlier Meeks [19] proved that a properly embedded H-surface of finite topology in $\mathbb{R}^3$ cannot have exactly one end. In particular, this last result together with Theorem 1.7 gives a generalization of Theorem 1.1; namely, a complete H-surface of finite topology cannot have exactly one end, and so, if the H-surface is simply-connected, then it must be compact.

The theory developed in this article also provides key tools for understanding the geometry of H-disks in a Riemannian three-manifold, especially when the manifold is locally homogeneous. These generalizations and applications are work in progress [33]. See [27] for applications of the present paper to obtain area estimates for closed H-surfaces of fixed genus embedded in a flat 3-torus; see [12, 13, 37] for examples that demonstrate that Theorem 1.7 does not hold in the hyperbolic 3-space $\mathbb{H}^3$ when $H \in [0, 1)$ and in the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ when $H \in [0, 1/2)$. In contrast to these aforementioned nonproperness results, we will show in [35] that complete H-surfaces of finite topology in complete hyperbolic three-manifolds must be proper whenever $H \geq 1$.

2. The intrinsic curvature and radius estimates

In [28] we proved the following extrinsic curvature and radius estimates for compact disks embedded in $\mathbb{R}^3$ with constant mean curvature.

Theorem 2.1 (Extrinsic curvature estimates). *Given $\delta, \mathcal{H} > 0$, there exists a constant $K_0(\delta, \mathcal{H})$ such that for any H-disk $\mathcal{D}$ with $H \geq \mathcal{H}$,*

$$\sup_{\{p\in\mathcal{D}\,|\,d_{\mathbb{R}^3}(p,\partial\mathcal{D})\geq\delta\}} |A_{\mathcal{D}}| \leq K_0(\delta, \mathcal{H}).$$

Theorem 2.2 (Extrinsic radius estimates). *There exists a constant $\mathcal{R}_0 > 0$ such that any H-disk $\mathcal{D}$ has extrinsic radius less than $\mathcal{R}_0/H$. In other words, for any point $p \in \mathcal{D}$,*

$$d_{\mathbb{R}^3}(p, \partial\mathcal{D}) < \mathcal{R}_0/H.$$

Thus, Theorems 1.2 and 1.3 are immediate consequences of a chord-arc type result from [26], namely Theorem 2.4 below, and Theorems 2.2 and 2.1.

Definition 2.3. Given a point p on a compact surface $\Sigma \subset \mathbb{R}^3$, $\Sigma(p, R)$ denotes the closure of the component of $\Sigma \cap \mathbb{B}(p, R)$ passing through p, where $\mathbb{B}(p, R)$ denotes the extrinsic open ball in $\mathbb{R}^3$ of radius R centered at p.

Theorem 2.4 (Weak chord arc property). *There exists a $\delta_1 \in (0, 1/2)$ such that the following holds. Let Σ be an H-disk in $\mathbb{R}^3$. Then for all closed intrinsic balls $\overline{B}_\Sigma(x, R)$ in $\Sigma - \partial\Sigma$,*

(1) *$\Sigma(x, \delta_1 R)$ is a disk with piecewise smooth boundary $\partial\Sigma(x, \delta_1 R) \subset \partial\mathbb{B}(x, \delta_1 R)$;*

(2) *$\Sigma(x, \delta_1 R) \subset B_\Sigma(x, R/2)$.*

We begin by applying Theorem 2.4 to prove the intrinsic radius estimate.

Proof of Theorem 1.2. Without loss of generality, fix $H = 1$. Arguing by contradiction, if the radius estimates were false, then for each $n \in \mathbb{N}$, there would exist a 1-disk Σ_n containing an intrinsic closed ball $\overline{B}_\Sigma(\vec{0}, n) \subset \Sigma_n \setminus \partial\Sigma_n$. Theorem 2.4 implies that $B_\Sigma(\vec{0}, n) \subset \Sigma_n$ contains a 1-disk centered at $\vec{0}$ of extrinsic radius $\delta_1 n$. For n large enough, the existence of such a 1-disk contradicts the extrinsic radius estimate and completes the proof of Theorem 1.2. ∎

We next prove the intrinsic curvature estimate.

Proof of Theorem 1.3. Let $\varepsilon = \delta_1\delta$, where $\delta_1 \in (0, 1/2)$ is given in Theorem 2.4, and let $K(\delta, \mathcal{H}) := K_0(\varepsilon, \mathcal{H})$, where $K_0(\varepsilon, \mathcal{H})$ is given in Theorem 2.1. Let $\mathcal{D}$ be an H-disk with $H \geq \mathcal{H}$ and let $p \in \mathcal{D}$ be a point with $d_{\mathcal{D}}(p, \partial\mathcal{D}) \geq \delta$. By Theorem 2.4, the closure of the component E of $\mathcal{D} \cap \mathbb{B}(p, \varepsilon)$ containing p is an H-disk in the interior of $\mathcal{D}$ with $\partial E \subset \partial\mathbb{B}(p, \varepsilon)$. By Theorem 2.1,

$$|A_E|(p) \leq K_0(\varepsilon, \mathcal{H}) = K(\delta, \mathcal{H}).$$

This completes the proof of Theorem 1.3. ∎

3. Curvature estimates for H-annuli and properness of H-surfaces with finite topology

A classical conjecture in the global theory of minimal surfaces, first stated by Calabi in 1965 [5] and later revisited by Yau [47, 48], is the following:

Conjecture 3.1 (Calabi–Yau conjecture). *There do not exist complete immersed minimal surfaces in a bounded domain in* $\mathbb{R}^3$.

Based on earlier work of Jorge and Xavier [16], Nadirashvili [36] proved the existence of a complete, bounded, immersed minimal surface in $\mathbb{R}^3$, thereby disproving the above conjecture. In contrast to these results, Colding and Minicozzi proved in [11] that complete, finite topology minimal surfaces *embedded* in $\mathbb{R}^3$ are proper. Thus, the Calabi–Yau conjecture holds in the classical setting of complete, embedded, finite topology minimal surfaces.

In this section we will apply Proposition 3.4 below to obtain Theorem 1.7, a result that generalizes the properness result of Colding and Minicozzi for embedded minimal surfaces of finite topology to the setting of H-surfaces.

Recall the definition of flux of a 1-cycle in an H-surface; see for instance [17, 18, 42] for further discussion of this invariant, specifically [42, Section 4] for the next definition.

Definition 3.2. Let γ be a piecewise-smooth 1-cycle in an H-surface M. The *flux vector* of γ is $\int_\gamma (H\gamma + \xi) \times \dot{\gamma}$, where ξ is the unit normal to M along γ and γ is parameterized by arc length.

The flux is a homological invariant and we say that M has *zero flux* if the flux vector of any 1-cycle in M is zero; in particular, since the first homology group of a disk is zero, the flux of an H-disk is zero.

Definition 3.3. Let E be an H-annulus. The *flux* $F(E)$ of E is the length of the flux vector of either generator of the first homology group of E.

The next proposition implies that given a compact 1-annulus with a fixed positive (or zero) flux, and given $\delta > 0$, the injectivity radius function on this annulus is bounded away from zero at points of distance greater than δ from its boundary.

Proposition 3.4. *Given $\rho > 0$ and $\delta \in (0, 1)$ there exists a positive constant $I_0(\rho, \delta)$ such that if E is a compact 1-annulus with $F(E) \geq \rho$ or with $F(E) = 0$, then*

$$\inf_{\{p \in E \mid d_E(p, \partial E) \geq \delta\}} I_E \geq I_0(\rho, \delta),$$

where $I_E \colon E \to [0, \infty)$ is the injectivity radius function of E.

Proof. Arguing by contradiction, suppose there exist a $\rho > 0$ and a sequence $E(n)$ of compact 1-annuli satisfying $F(E(n)) \geq \rho > 0$ or $F(E(n)) = 0$, with injectivity radius functions $I_n \colon E(n) \to [0, \infty)$ and points $p(n)$ in $\{q \in E(n) \mid d_{E(n)}(q, \partial E(n)) \geq \delta\}$ with

$$I_n(p(n)) \leq 1/n.$$

We next use the fact that the injectivity radius function on a complete Riemannian manifold with boundary is continuous. For each $p(n)$ defined above consider a point $q(n) \in \overline{B}_{E(n)}(p(n), \delta/2)$ where the following positive continuous function attains its maximum value:

$$f\colon \overline{B}_{E(n)}(p(n), \delta/2) \to (0, \infty),$$
$$f(x) = \frac{d_{E(n)}(x, \partial \overline{B}_{E(n)}(p(n), \delta/2))}{I_n(x)}.$$

Let $r(n) = \frac{1}{2} d_{E(n)}(q(n), \partial \overline{B}_{E(n)}(p(n), \delta/2))$ and note that

$$\frac{\delta/2}{I_n(q(n))} \geq \frac{2r(n)}{I_n(q(n))} = f(q(n)) \geq f(p(n)) \geq n\delta/2.$$

Moreover, if $x \in \overline{B}_{E(n)}(q(n), r(n))$, then by the triangle inequality,

$$\frac{r(n)}{I_n(x)} \leq \frac{d_{E(n)}(x, \partial \overline{B}_{E(n)}(p(n), \delta/2))}{I_n(x)} = f(x) \leq f(q(n)) = \frac{2r(n)}{I_n(q(n))}.$$

Therefore, for n large the H_n-surfaces

$$M(n) = \frac{1}{I_n(q(n))}[\overline{B}_{E(n)}(q(n), r(n)) - q(n)],$$

with $H_n = I_n(q(n))$, satisfy the following conditions:

- $I_{M(n)}(\vec{0}) = 1$;
- $d_{M(n)}(\vec{0}, \partial M(n)) \geq n\delta/4$;
- $I_{M(n)}(x) \geq 1/2$ for any $x \in \overline{B}_{M(n)}(\vec{0}, n\delta/4)$.

By [26, Theorem 3.2], for any $k \in \mathbb{N}$, there exists an $n(k) \in \mathbb{N}$ such that the closure of the component $\Delta(n(k))$ of $M(n(k)) \cap \mathbb{B}(k)$ containing the origin is a compact $H_{n(k)}$-surface with boundary in $\partial\mathbb{B}(k)$ and the injectivity radius function of $\Delta(n(k))$ restricted to points in $\Delta(n(k)) \cap \mathbb{B}(k - 1/2)$ is at least $1/2$. By [29, Theorem 1.3], for k sufficiently large, $\Delta(n(k))$ contains a simple closed curve $\Gamma(n(k))$ with the length of its nonzero flux vector bounded from above by some constant $C > 0$. Since the curves $\Gamma(n(k))$ are rescalings of simple closed curves $\widetilde{\Gamma}(n(k)) \subset E(n(k))$, the $\widetilde{\Gamma}(n(k))$ are simple closed curves with nonzero flux. Hence these simple closed curves are generators of the first homology group of the annuli $E(n(k))$. This immediately gives a contradiction in the case that $F(E(n(k))) = 0$. If $F(E(n(k))) \geq \rho > 0$, we have

$$\begin{aligned} C \geq |F(\Gamma(n(k)))| &= \left| F\left(\frac{1}{I_{n(k)}(q(n(k)))} \widetilde{\Gamma}(n(k)) \right) \right| = \frac{|F(\widetilde{\Gamma}(n(k)))|}{I_{n(k)}(q(n(k)))} \\ &= \frac{F(E(n(k)))}{I_{n(k)}(q(n(k)))} \geq \frac{\rho}{I_{n(k)}(q(n(k)))} \geq \rho n(k). \end{aligned}$$

These inequalities lead to a contradiction for $n(k) > C/\rho$, which completes the proof of the proposition. ∎

An immediate consequence of Proposition 3.4 and the intrinsic curvature estimates for H-disks is the following result.

Corollary 3.5. *Given $\rho > 0$ and $\delta \in (0,1)$ there exists a positive constant $A_0(\rho,\delta)$ such that if E is a compact 1-annulus with $F(E) \geq \rho$ or with $F(E) = 0$, then*

$$\sup_{\{p\in E \mid d_E(p,\partial E)\geq\delta\}} |A_E| \leq A_0(\rho,\delta).$$

When M has finite topology, the flux of each of its finitely many annular ends is either zero or bounded away from zero by a fixed positive number. Thus, Proposition 3.4 implies that the injectivity radius function of M is positive, and so the norm of its second fundamental form is bounded by Theorem 1.3. The next corollary is a consequence of this last property and the fact that a complete nonflat H-surface of bounded norm of the second fundamental form is properly embedded in $\mathbb{R}^3$; see [24, Theorem 6.1] or [25, Corollary 2.5 (1)] for this properness result.

Corollary 3.6. *A complete surface M with finite topology embedded in $\mathbb{R}^3$ with nonzero constant mean curvature has bounded norm of the second fundamental form and is properly embedded in $\mathbb{R}^3$.*

Remark 3.7. With slight modifications, the proof of the above corollary generalizes to the case where the H-surface M above is allowed to have smooth compact boundary. Thus, Theorem 1.7 holds as well.

4. Generalizations and open problems

Corollary 3.6 motivates Conjecture 4.1 below concerning the properness and at most cubical area growth estimates for complete H-surfaces M embedded in $\mathbb{R}^3$ with finite genus and constant mean curvature.

Conjecture 4.1. A complete connected surface M of finite genus embedded in $\mathbb{R}^3$ with constant mean curvature has at most cubical area growth in the sense that such an M has area less than CR^3 in ambient balls of radius $R \geq 1$ for some C depending on M. In particular, every such surface is properly embedded in $\mathbb{R}^3$.

Conjecture 4.1 holds for complete, nonflat H-surfaces embedded in $\mathbb{R}^3$ with a countable number of ends and finite genus. In the case of minimal surfaces this cubical volume growth result follows from the properness of such minimal surfaces (by Meeks, Perez and Ros [22]), because properly embedded minimal surfaces in $\mathbb{R}^3$ of finite genus have bounded norm of the second fundamental form (by Meeks, Perez and Ros [20]) and because connected, properly embedded minimal surfaces in $\mathbb{R}^3$ with bounded norm of the second fundamental form have at most cubical volume growth (by Meeks and Rosenberg [24]). In the case of 1-surfaces this cubical area growth result follows from [29], where we prove that 1-surfaces embedded in $\mathbb{R}^3$ with a countable number of ends and finite genus are proper and, secondly, any properly embedded 1-surface in $\mathbb{R}^3$ with a uniform estimate on its genus in ambient balls of any fixed radius have at most cubical area growth. These last results in [29] are based on a general structure theorem for compact 1-surfaces in $\mathbb{R}^3$ with boundary curves on the boundary of balls in $\mathbb{R}^3$, and the proof of this general structure theorem depends on the main results mentioned in the introduction of the present paper.

Many of the results in this paper can be generalized to the Riemannian three-manifold setting. In [33], we will prove the existence of intrinsic curvature estimates for $(H > 0)$-disks in complete locally homogeneous three-manifolds, a result that generalizes Theorem 1.3. As in the present paper, the existence of these universal curvature estimates plays an essential role in understanding global properties of H-surfaces of locally bounded genus in Riemannian three-manifolds. For other important applications see our forthcoming papers [32–35].

Funding. This material is based upon work for the NSF under Award No. DMS-1309236. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

The second author was partially supported by EPSRC grant no. EP/M024512/1

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