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Infinite torsion in Griffiths groups

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Abstract. We show that there are smooth complex projective varieties with infinite 2-torsion in their third Griffiths groups. It follows that the torsion subgroup of Griffiths groups is in general not finitely generated, thereby solving a problem of Schoen from 1992.

Keywords: Griffiths group, algebraic cycles, unramified cohomology, Enriques surfaces.

1. Introduction

The Griffiths group $\text{Griff}^i(X)$ of a smooth complex projective variety X is the group of homologically trivial codimension i cycles modulo algebraic equivalence. This is a countable abelian group which is a basic invariant of X . However, detecting whether a given homologically trivial cycle is nontrivial in the Griffiths group is a subtle problem. For instance, the isomorphism type of the abelian group $\text{Griff}^i(X)$ is not known in any nontrivial example.

Griffiths [15] used his transcendental Abel–Jacobi maps to construct the first example of a smooth complex projective variety with nontrivial Griffiths group. Clemens [7] combined Griffiths’ approach with a degeneration argument to show that in fact $\text{Griff}^i(X) \otimes \mathbb{Q}$ may be an infinite-dimensional \mathbb{Q} -vector space for any $i \geq 2$. This showed that Griffiths groups are in general not finitely generated modulo torsion.

Improving earlier results of Schoen [35] and Rosenschon–Srinivas [32], Totaro [41] showed that $\text{Griff}^i(X)/\ell$ may for $i \geq 2$ be infinite for any prime ℓ . These results rely on Griffiths’ method, a theorem of Bloch–Esnault [3], and ideas from Nori’s proof [28] of Clemens’ theorem.

Schoen [33] used Griffiths’ method to show that Griffiths groups may contain nontrivial torsion. The first nontrivial torsion classes with trivial transcendental Abel–Jacobi invariants have been constructed by Totaro [40] via a topological method; nontorsion classes with that property had earlier been constructed by Nori [29].

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1.1. Main result

This article shows that the theory of refined unramified cohomology developed in [38] furnishes a new method to detect nontriviality of classes in the Griffiths group. As a concrete application, we prove the following.

Theorem 1.1. *Let JC be the Jacobian of a very general quartic curve $C \subset \mathbb{P}_{\mathbb{C}}^2$. Then for any very general Enriques surface X over \mathbb{C} , $\text{Griff}^3(X \times JC)$ has infinite 2-torsion.*

As an immediate corollary, we obtain the following result

Corollary 1.2. *The torsion subgroup of Griffiths groups of smooth complex projective varieties is in general not finitely generated.*

The above results solve a problem of Schoen [33], who writes in the introduction:

Although our experience is that torsion in the Griffiths groups of varieties over \mathbb{C} is difficult to find, we have no compelling evidence that it is always finite or usually zero.

Schoen's problem and Theorem 1.1 above should be compared to a theorem of Merkurjev–Suslin [24], who showed that for any integer n , the n -torsion subgroup of Griff^2 is finite.

The analogue of Corollary 1.2 is trivial for Chow groups, as any elliptic curve E has infinite torsion in $\text{CH}_0(E)$. However, even for Chow groups, the problem becomes interesting if we restrict to ℓ -torsion for a given prime ℓ . It has been shown in [32, 34, 41] that these groups may be infinite, but all the torsion cycles involved are algebraically equivalent to zero.

The infinitely many different 2-torsion classes in $\text{Griff}^3(X \times JC)$ from Theorem 1.1 are given by exterior products $K_X \times z$, where $K_X \in \text{Pic } X$ is the unique 2-torsion class on the Enriques surface X and where z is the Ceresa cycle $C - C^-$ on JC (see [6]), or a pullback of the Ceresa cycle by one of infinitely many isogenies. We will show that infinitely many of the cycles $K_X \times z$ are linearly independent modulo 2. To this end we will use Totaro's result [41], who showed that infinitely many of the classes z are linearly independent modulo 2 in $\text{Griff}^2(JC)$. Totaro used among other ingredients the fact that the Ceresa cycle has nontrivial Abel–Jacobi invariant (see [17]). In contrast, $K_X \times z$ has trivial Abel–Jacobi invariant.

While it is natural to consider cycles of the form $K_X \times z$ as above, our method of proving that they are nontrivial (resp. linearly independent) is new. The main idea is that [38] allows one to use cohomological tools (most notably Poincaré duality) that are a priori not available within the framework of algebraic cycles.

Even though Theorem 1.1 concentrates on 2-torsion in the Griffiths group, our method is flexible and large parts of this paper work for ℓ -torsion for arbitrary primes ℓ . In particular, it is conceivable that our method will allow one to find infinite ℓ -torsion for other primes ℓ by replacing Enriques surfaces by suitable surfaces X with ℓ -torsion in $H^2(X, \mathbb{Z})$.

1.2. An injectivity theorem

Let Y be an arbitrary smooth complex projective variety and let $A^i(Y) := \text{CH}^i(Y)/\sim_{\text{alg}}$ be the Chow group modulo algebraic equivalence. Theorem 1.1 will be deduced from the following, which is the main result of this paper.

Theorem 1.3. *Let Y be a smooth complex projective variety and let X be an Enriques surface that is very general with respect to Y . Then the exterior product map*

$$A^i(Y)/2 \rightarrow A^{i+1}(X \times Y)[2], \quad [z] \mapsto [K_X \times z],$$

is injective. In fact, $[K_X \times z]$ is not divisible by 2 in $A^{i+1}(X \times Y)$ unless $[z] = 0 \in A^i(Y)/2$.

The condition on X means that it lies outside a countable union of proper closed subsets (which may depend on X) of the moduli space of Enriques surfaces.

We let $\mathcal{T}^{i+1}(X \times Y) \subset \text{Griff}^{i+1}(X \times Y)_{\text{tors}}$ denote the kernel of the transcendental Abel–Jacobi map (see (2.10) below) and define

$$E_2^i(Y) := \ker(\text{cl}_Y^i : A^i(Y)/2 \rightarrow H^{2i}(Y, \mathbb{Z}/2)).$$

Corollary 1.4. *In the notation of Theorem 1.3, there is a canonical injection*

$$E_2^i(Y) \hookrightarrow \mathcal{T}^{i+1}(X \times Y)[2], \quad [z] \mapsto [K_X \times z].$$

By [38] (see Theorem 2.7 below), there is a canonical extension

$$0 \rightarrow \text{Griff}^i(Y)/2 \rightarrow E_2^i(Y) \rightarrow Z^i(Y)[2]/H^{2i}(Y, \mathbb{Z})[2] \rightarrow 0, \tag{1.1}$$

where $Z^i(Y)[2]$ denotes the 2-torsion subgroup of $\text{coker}(\text{cl}_Y^i : \text{CH}^i(Y) \rightarrow H^{2i}(Y, \mathbb{Z}))$. Corollary 1.4 thus shows that there are two sources for nontrivial 2-torsion classes with trivial transcendental Abel–Jacobi invariants in the Griffiths group of $X \times Y$: one coming from $\text{Griff}^i(Y)/2$ and one stemming from $Z^i(Y)[2]/H^{2i}(Y, \mathbb{Z})[2]$. While the former may be infinite [13, 41], the latter is always a finite group. One source for nontrivial elements in $Z^i(Y)[2]/H^{2i}(Y, \mathbb{Z})[2]$ are nonalgebraic non-torsion Hodge classes α on Y such that 2α is algebraic. Several examples with that property are known (see e.g. [12, 30]) and we will discuss those applications in Corollaries 9.1–9.3 below. For instance, Corollary 1.4 and [30] lead to the first known example of a smooth complex projective variety X with a rational decomposition of the diagonal for which $\mathcal{T}^3(X) \neq 0$; see Corollaries 9.1 and 9.2 below.

In [39], Soulé and Voisin constructed nondivisible torsion classes in the Griffiths group of products $X \times Y$, where $H^2(X, \mathbb{Z})_{\text{tors}} \neq 0$ and Y is a carefully chosen hypersurface in \mathbb{P}^4 so that the integral Hodge conjecture fails for Y by Kollár’s argument [1]. Their argument relies on degenerations of Y . Instead, we will degenerate X , while Y may be arbitrary and the contribution of $\text{Griff}^i(Y)/2$ will be taken into account in a precise way.

1.3. Degenerations of Enriques surfaces

In addition to the theory of refined unramified cohomology, Theorem 1.3 relies on the following geometric input.

Theorem 1.5. *There is a regular flat projective scheme $\mathcal{X} \rightarrow \text{Spec } R$ over a discrete valuation ring R whose residue field is an algebraically closed field κ of characteristic zero, such that*

- (1) *the geometric generic fibre $X_{\bar{\eta}}$ is an Enriques surface;*
- (2) *the special fibre $X_0 = \mathcal{X} \times \kappa$ is a union of ruled surfaces;*
- (3) *the restriction map $\text{Br}(\mathcal{X})[2] \rightarrow \text{Br}(X_{\bar{\eta}})[2]$ is surjective.*

The geometric meaning of the theorem is as follows. By [11], the unique nonzero class in $\text{Br}(X_{\bar{\eta}}) \simeq \mathbb{Z}/2$ corresponds to a smooth (i.e. unramified) conic bundle $P \rightarrow X_{\bar{\eta}}$ and the above theorem implies that this conic bundle extends to a smooth conic bundle $\mathcal{P} \rightarrow \mathcal{X}$. That is, while the Enriques surface breaks up into ruled components, the conics in the fibration $P \rightarrow X_{\bar{\eta}}$ remain smooth and do not break up into unions of two lines.

The fact that degenerations as above exist was a surprise to the author. To explain one subtle aspect, note that we may assume that the dvr R in Theorem 1.5 is complete. Item (1) together with the proper base change theorem can then be used to show $\text{Br}(\mathcal{X})_{\text{tors}} \simeq \text{Br}(X_0)_{\text{tors}}$ (see Proposition 7.2 and Remark 7.3 below) and so $\text{Br}(X_0)[2] \neq 0$ by item (3). On the other hand, each component X_{0i} of X_0 is ruled by item (2) and so $\text{Br}(X_{0i}) = 0$ for all i .

For us, the crucial consequence of the above theorem will be as follows:

Corollary 1.6. *In the notation of Theorem 1.5, there is a class $\alpha \in \text{Br}(\mathcal{X})[2]$ such that $\alpha|_{X_{\bar{\eta}}}$ generates $\text{Br}(X_{\bar{\eta}}) \simeq \mathbb{Z}/2$ and for any component X_{0i} of the special fibre, $\alpha|_{X_{0i}} = 0 \in \text{Br}(X_{0i})$.*

1.4. Outline of the argument

Let X and Y be as in Theorem 1.3. The cohomological analogue of the exterior product map in Theorem 1.3 is the cup product map

$$H^2(X, \mathbb{Z}/2) \otimes H^{2i}(Y, \mathbb{Z}/2) \rightarrow H^{2i+2}(X \times Y, \mathbb{Z}/2), \quad \alpha \otimes \beta \mapsto p^*\alpha \cup q^*\beta,$$

where $p : X \times Y \rightarrow X$ and $q : X \times Y \rightarrow Y$ denote the natural projections. If $\alpha \in H^2(X, \mathbb{Z}/2)$ is nonzero, Poincaré duality yields a class $\bar{\alpha} \in H^2(X, \mathbb{Z}/2)$ with $\alpha \cup \bar{\alpha} = \text{cl}_X^2(pt)$ and we find

$$q_*(p^*\bar{\alpha} \cup p^*\alpha \cup q^*\beta) = q_*(p^*\text{cl}_X^2(pt) \cup q^*\beta) = \beta \in H^{2i}(Y, \mathbb{Z}/2).$$

This classical argument shows that $p^*\alpha \cup q^*\beta \neq 0$ as long as α and β are nonzero.

We aim to use this approach to prove Theorem 1.3. The obvious obstacle is that $A^1(X)/2$ does not satisfy Poincaré duality (in fact, $K_X \cdot D = 0$ for any divisor D on X).

The crucial input which allows us to circumvent this problem is [38], which shows that Chow groups modulo algebraic equivalence can be computed by refined unramified cohomology. An important observation here is that the Poincaré dual $\bar{\alpha} \in H^2(X, \mathbb{Z}/2)$ of the class $\alpha := [K_X] \in H^2(X, \mathbb{Z}/2)$ generates the second unramified cohomology of X :

$$\text{Br}(X) \simeq H_{\text{nr}}^2(X, \mathbb{Z}/2) = \mathbb{Z}/2[\bar{\alpha}].$$

Hence, algebraic cycles modulo algebraic equivalence as well as the Poincaré dual of K_X live in the world of refined unramified cohomology and so it is natural to try to work there.

Following this approach, it is straightforward to prove injectivity of the exterior product map in Theorem 1.3 on the level of cycles (not modulo algebraic equivalence) purely in terms of cohomology. While that statement is of course trivial, passing to algebraic equivalence will introduce an error term of the form

$$q_*(p^*\bar{\alpha} \cup \iota_*\xi) \in H^{2i-1}(V, \mathbb{Z}/2), \quad \text{where } \xi \in \bigoplus_{w \in (X \times Y)^{(i)}} H^1(w, \mathbb{Z}/2). \quad (1.2)$$

Here $V \subset Y$ is some open subset whose complement $R = Y \setminus V$ has codimension at least $i - 1$. The main technical difficulty in the proof of Theorem 1.3 is to show that (1.2) vanishes (possibly up to a class that extends to all of Y).

At this point we use specialization maps for refined unramified cohomology that we construct in Section 4 below and which serve as a replacement for Fulton’s specialization maps on Chow groups [14]. These maps are inspired by well-known specialization maps in Galois cohomology; see also Remark 4.8 below.

Even though we would like to show that (1.2) vanishes for X and Y smooth, we will be able to reduce the problem to the situation where X splits up into many components as in Theorem 1.5. This may be surprising, as usually one cannot prove the vanishing of an invariant by showing that it vanishes after specialization, but we will be able to put ourselves in a situation where that argument actually works.

At this final step, the geometry of Enriques surfaces via Theorem 1.5 comes in, as it is exactly the kind of degeneration needed to ensure that $\bar{\alpha}$ vanishes on each component of the special fibre of the degeneration and we will show in Lemma 5.1 that this implies that the specialization of (1.2) vanishes. This concludes the proof of Theorem 1.3 up to the proof of Theorem 1.5. The latter relies in turn on an analysis of flower pot degenerations of Enriques surfaces, constructed by Persson and Horikawa [20, 27, 31]; cf. Theorem 7.1 below.

2. Preliminaries

2.1. Conventions

For an abelian group G , we denote by $G[\ell^r]$ the subgroup of ℓ^r -torsion elements. Whenever G and H are abelian groups such that there is a canonical map $H \rightarrow G$ (and there

is no reason to confuse this map with a different map), we write G/H as shorthand for $\text{coker}(H \rightarrow G)$.

All schemes are separated. An *algebraic scheme* X is a scheme of finite type over a field. Its Chow group of codimension i cycles modulo rational equivalence is denoted by $\text{CH}^i(X)$; the quotient of $\text{CH}^i(X)$ modulo algebraic equivalence is denoted by $A^i(X)$.

A *variety* is an integral scheme of finite type over a field. For an equi-dimensional algebraic scheme X , we denote by $X^{(i)}$ the set of all codimension i points of X . For a scheme X over a field k , we write for any field extension K of k the scheme given by extension of scalars as $X_K := X \times_k K$. A *very general point* of a scheme over \mathbb{C} is a closed point outside a countable union of proper closed subsets.

If R is an integral local ring with residue field κ and fraction field K , with algebraic closure \bar{K} , then for any flat R -scheme $\mathcal{X} \rightarrow \text{Spec } R$, we write $X_0 := \mathcal{X} \times_R \kappa$ (resp. $X_{\bar{0}} := \mathcal{X} \times_R \bar{\kappa}$) for the special (resp. geometric special) fibre and $X_\eta := \mathcal{X} \times_R K$ (resp. $X_{\bar{\eta}} := \mathcal{X} \times_R \bar{K}$) for the generic (resp. geometric generic) fibre.

An irreducible flat R -scheme $\mathcal{X} \rightarrow \text{Spec } R$ over a discrete valuation ring R is called *strictly semistable* if \mathcal{X} is regular, the generic fibre X_η is smooth and the special fibre X_0 is a geometrically reduced simple normal crossing divisor on \mathcal{X} , i.e. the components of X_0 are smooth and the scheme-theoretic intersection of r different components of X_0 is either empty or smooth and equi-dimensional of codimension r in \mathcal{X} .

2.2. Cohomology

For a scheme X and a prime ℓ invertible on X we write

$$H^i(X, \mu_{\ell^r}^{\otimes n}) := H^i(X_{\text{ét}}, \mu_{\ell^r}^{\otimes n}) \quad \text{and} \quad H^i(X, \mathbb{Z}/\ell^r(n)) := H^i_{\text{cont}}(X_{\text{ét}}, \mathbb{Z}/\ell^r(n)), \quad (2.1)$$

where H^i_{cont} denotes Jannsen’s continuous étale cohomology (see [21]); sometimes we also write $H^i(X, \mathbb{Z}/\ell^r(n))$ in place of $H^i(X, \mu_{\ell^r}^{\otimes n})$. These groups are functorial with respect to pullbacks along arbitrary morphisms.

For a ring A , we write $H^i(A, \mu_{\ell^r}^{\otimes n}) := H^i(\text{Spec } A, \mu_{\ell^r}^{\otimes n})$. If $A = K$ is a field, then these groups coincide with the Galois cohomology of the absolute Galois group of K .

For a scheme X and a point $x \in X$, we write

$$H^i(x, \mu_{\ell^r}^{\otimes n}) := \varinjlim_{\emptyset \neq V_x \subset \overline{\{x\}}} H^i(V_x, \mu_{\ell^r}^{\otimes n}),$$

where V_x runs through all open dense subsets of the closure of x . By [25, p. 88, III.1.16], this direct limit coincides with the étale cohomology of $\text{Spec } \kappa(x)$ and hence with the Galois cohomology of the field $\kappa(x)$.

Remark 2.1. If X is a smooth equi-dimensional algebraic scheme over a field k , then the groups in (2.1) agree with the Borel–Moore cohomology groups used in [38] (see [38, Lemma 6.5 and Proposition 6.6]).

Lemma 2.2. *Let $f : X \rightarrow Y$ be a proper morphism between equi-dimensional smooth algebraic schemes over a field k . Let ℓ be a prime invertible in k and let $c := \dim Y - \dim X$. Then for $A = \mathbb{Z}/\ell^r$ or $A = \mathbb{Z}_\ell$, there are pushforward maps*

$$f_* : H^{i-2c}(X, A(n-c)) \rightarrow H^i(Y, A(n))$$

that are compatible with respect to open immersions.

Proof. This follows from Remark 2.1 and [38, (P1) in Proposition 6.6]. ■

Lemma 2.3. *Let X be a smooth equi-dimensional algebraic scheme over a field k and let $Z \subset X$ be a smooth closed subscheme of pure codimension c and with complement U . Then for $A = \mathbb{Z}/\ell^r$ or $A = \mathbb{Z}_\ell$, there is a long exact Gysin sequence*

$$H^i(X, A(n)) \rightarrow H^i(U, A(n)) \xrightarrow{\partial} H^{i+1-2c}(Z, A(n-c)) \xrightarrow{\iota_*} H^{i+1}(X, A(n)), \quad (2.2)$$

where ι_* is the pushforward map from Lemma 2.2. This sequence is functorial with respect to pullbacks along open immersions as well as pushforwards along proper morphisms $f : X' \rightarrow X$ such that X' and $Z' = f^{-1}(Z)$ are smooth and equi-dimensional.

Proof. This follows from Remark 2.1 and [38, (P2) in Proposition 6.6]. ■

The Gysin sequence is compatible with cup products in the following sense. If $\alpha \in H^j(X, A(m))$, then cup product with α induces a commutative diagram

$$\begin{CD} H^i(X, A(n)) @>>> H^i(U, A(n)) @>\partial>> H^{i+1-2c}(Z, A(n-c)) \\ @V\cup\alpha VV @V\cup\alpha|_U VV @V\cup\alpha|_Z VV \\ H^{i+j}(X, A(n+m)) @>>> H^{i+j}(U, A(n+m)) @>\partial>> H^{i+j+1-2c}(Z, A(n+m-c)) \end{CD} \quad (2.3)$$

This fact is well-known; the case $A = \mathbb{Z}/\ell^r$ (which is enough for the purpose of this paper) is for instance spelled out in [37, Lemma 2.4].

Remark 2.4. For $A = \mathbb{Z}/\ell^r$, $H^i(X, A(n))$ commutes with filtered inverse limits of schemes with affine transition maps (see [25, p. 88, III.1.16]). Using this, it follows that Lemma 2.3 and (2.2) remain true when X is a regular scheme of finite type over a local ring $R = \mathcal{O}_{B,0}$ of a smooth k -variety B at a closed point $0 \in B$ and $Z \subset X$ is a smooth equi-dimensional k -variety contained in the special fibre of $X \rightarrow \text{Spec } R$.

2.3. Classical unramified cohomology and Brauer groups

Let X be an integral regular scheme and let $x \in X^{(1)}$ be a codimension 1 point such that ℓ is invertible in the residue field $\kappa(x)$. Then there is a residue map in Galois cohomology

$$\partial_x : H^i(k(X), \mu_{\ell^r}^{\otimes n}) \rightarrow H^{i-1}(\kappa(x), \mu_{\ell^r}^{\otimes n-1}), \quad (2.4)$$

where $k(X)$ denotes the residue field of the generic point of X . If ℓ is invertible in each residue field of X , then the unramified cohomology of X may be defined as follows (see [8, Theorem 4.1.1 (a)]):

$$H_{\text{nr}}^i(X, \mu_{\ell^r}^{\otimes n}) := \{\alpha \in H^i(k(X), \mu_{\ell^r}^{\otimes n}) \mid \partial_x \alpha = 0 \forall x \in X^{(1)}\}.$$

For a scheme X , we let $\text{Br}(X) := H^2(X_{\text{ét}}, \mathbb{G}_m)$. For any prime ℓ that is invertible on X , the Kummer sequence yields an isomorphism

$$\text{coker}(c_1 : \text{Pic}(X) \rightarrow H^2(X_{\text{ét}}, \mu_{\ell^r})) \simeq \text{Br}(X)[\ell^r]. \tag{2.5}$$

If X is a regular variety, then $\text{Br}(X)[\ell^r] \simeq H_{\text{nr}}^2(X, \mu_{\ell^r})$ (see [8, Proposition 4.2.3]).

2.4. Refined unramified cohomology

In this paper we use refined unramified cohomology only for smooth equi-dimensional algebraic schemes. In this case the cohomology theory in (2.1) agrees with Borel–Moore cohomology used in [38] (see Remark 2.1). For an equi-dimensional algebraic scheme X , we denote by F_*X the increasing filtration given by

$$F_0X \subset F_1X \subset \dots \subset F_{\dim X}X = X, \quad \text{where } F_jX := \{x \in X \mid \text{codim}_X(x) \leq j\}.$$

Fix a prime ℓ invertible on X and let $A = \mathbb{Z}/\ell^r$ or $A = \mathbb{Z}_\ell$. If X is smooth (or more generally, if F_jX is contained in a smooth open subset of X), we define

$$H^i(F_jX, A(n)) := \varinjlim_{F_jX \subset U \subset X} H^i(U, A(n)),$$

where U runs through all open subsets of X that contain F_jX , and where the cohomology functor is (continuous) étale cohomology from (2.1).

Lemma 2.5. *Let $f : X \rightarrow Y$ be a morphism between smooth equi-dimensional algebraic schemes. If f is flat, then there is a pullback map*

$$f^* : H^i(F_jY, A(n)) \rightarrow H^i(F_jX, A(n)).$$

If f is proper of relative codimension $c := \dim Y - \dim X$, there is a pushforward map

$$f_* : H^i(F_jX, A(n)) \rightarrow H^{i+2c}(F_{j+c}Y, A(n+c)).$$

Proof. Let $V \subset Y$ be an open subset with $F_jY \subset V$. The complement $W = Y \setminus V$ has codimension at least $j + 1$. If f is flat, then $f^{-1}(W) \subset X$ has codimension at least $j + 1$ as well. Hence there is a well-defined pullback map $H^i(V, A(n)) \rightarrow H^i(F_jX, A(n))$. This map is compatible with restrictions to smaller open subsets $V' \subset V$ with $F_jY \subset V'$ and hence induces the pullback f^* stated in the lemma.

Assume now that f is proper and let $U \subset X$ be an open subset with $F_jX \subset U$ and complement $Z = X \setminus U$. Then $f(Z) \subset Y$ has codimension at least $j + c + 1$. Moreover,

Lemma 2.2 induces a pushforward map

$$f_* : H^i(X \setminus f^{-1}(f(Z)), A(n)) \rightarrow H^{i+2c}(Y \setminus f(Z), A(n+c)).$$

Pre- and postcomposing this with the canonical restriction maps, we get a pushforward map

$$f_* : H^i(U, A(n)) \rightarrow H^{i+2c}(F_{j+c}Y, A(n+c)).$$

By the compatibility of pushforwards with restrictions along open immersions (see Lemma 2.2), this map is compatible with respect to restriction from U to a smaller open subset $U' \subset X$ with $F_j X \subset U' \subset U$. This implies the existence of f_* as claimed in the lemma. ■

We define the j -th refined unramified cohomology of X with values in $A(n)$ by

$$H^i_{j,\text{nr}}(X, A(n)) := \text{im}(H^i(F_{j+1}X, A(n)) \rightarrow H^i(F_j X, A(n))).$$

As indicated above, these groups coincide with the refined unramified cohomology groups from [38] defined via Borel–Moore cohomology (see Remark 2.1).

Taking direct limits over (2.2) twice, we get the following (see [38, Lemma 5.8]).

Lemma 2.6. *Let X be a smooth equi-dimensional algebraic scheme over k . For any $j, n \in \mathbb{Z}$, there is a long exact sequence*

$$\begin{aligned} \dots \rightarrow H^i(F_j X, A(n)) \rightarrow H^i(F_{j-1} X, A(n)) \xrightarrow{\partial} \bigoplus_{x \in X^{(j)}} H^{i+1-2j}(x, A(n-j)) \\ \xrightarrow{\iota_*} H^{i+1}(F_j X, A(n)) \rightarrow \dots, \end{aligned}$$

where ι_* (resp. ∂) is induced by the pushforward (resp. residue) map from (2.2).

The above lemma implies $H^i_{0,\text{nr}}(X, \mu_{\ell^r}^{\otimes n}) = H^i_{\text{nr}}(X, \mu_{\ell^r}^{\otimes n})$, and, by [38, Corollary 5.10],

$$H^i(X, A(n)) \simeq H^i(F_j X, A(n)) \quad \text{for all } j \geq \lceil i/2 \rceil. \tag{2.6}$$

We define a decreasing filtration F^* on $H^i(F_j X, A(n))$ by

$$F^m H^i(F_j X, A(n)) := \text{im}(H^i(F_m X, A(n)) \rightarrow H^i(F_j X, A(n)))$$

for $m \geq j$. In particular, $F^{j+1} H^i(F_j X, A(n)) = H^i_{j,\text{nr}}(X, A(n))$. We also define (see [38, Definition 5.4]) a decreasing filtration G^* on $H^i(F_j X, \mu_{\ell^r}^{\otimes n})$ by

$$\alpha \in G^m H^i(F_j X, \mu_{\ell^r}^{\otimes n}) \iff \delta(\alpha) \in F^m H^{i+1}(F_j X, \mathbb{Z}_{\ell}(n)),$$

where δ denotes the Bockstein map associated to $0 \rightarrow \mathbb{Z}_{\ell}(n) \rightarrow \mathbb{Z}_{\ell}(n) \rightarrow \mu_{\ell^r}^{\otimes n} \rightarrow 0$. Moreover,

$$G^m H^i_{j,\text{nr}}(X, \mu_{\ell^r}^{\otimes n}) := \text{im}(G^m H^i(F_{j+1} X, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(F_j X, \mu_{\ell^r}^{\otimes n})).$$

Related to δ , there is also the Bockstein map $\tilde{\delta}$ that is associated to the short exact sequence $0 \rightarrow \mu_{\ell^r}^{\otimes n} \rightarrow \mu_{\ell^{2r}}^{\otimes n} \rightarrow \mu_{\ell^r}^{\otimes n} \rightarrow 0$. The Bockstein $\tilde{\delta}$ is compatible with δ in the sense that $\tilde{\delta}(\alpha)$ is the reduction modulo ℓ^r of $\delta(\alpha)$. The key feature of $\tilde{\delta}$ is the derivation property (see e.g. [19, p. 304]), which yields

$$\tilde{\delta}(\alpha \cup \beta) = \tilde{\delta}(\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup \tilde{\delta}(\beta). \tag{2.7}$$

In analogy to G^* , we have the filtration \tilde{G}^* on $H^i(F_j X, \mu_{\ell^r}^{\otimes n})$, given by

$$\alpha \in \tilde{G}^m H^i(F_j X, \mu_{\ell^r}^{\otimes n}) \iff \tilde{\delta}(\alpha) \in F^m H^{i+1}(F_j X, \mu_{\ell^r}^{\otimes n})$$

(see [38, Definition 7.24]). Moreover,

$$\tilde{G}^m H_{j,\text{nr}}^i(X, \mu_{\ell^r}^{\otimes n}) := \text{im}(\tilde{G}^m H^i(F_{j+1} X, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(F_j X, \mu_{\ell^r}^{\otimes n})).$$

The compatibility of δ and $\tilde{\delta}$ implies directly

$$G^m H^i(F_j X, \mu_{\ell^r}^{\otimes n}) \subset \tilde{G}^m H^i(F_j X, \mu_{\ell^r}^{\otimes n}). \tag{2.8}$$

2.5. Cycle groups in terms of refined unramified cohomology

In [38, Section 7], several cycle groups are computed in terms of refined unramified cohomology. The set-up in [38] works for arbitrary separated schemes of finite type over a field; the corresponding cohomology functor should be taken as Borel–Moore pro-étale cohomology (see [38, Section 4]). We will only use refined unramified cohomology for smooth varieties over a field, in which case we may by [38, Lemma 6.5 and Proposition 6.6] take the cohomology groups from (2.1). It follows that the results of [38, Sections 5 and 7] hold true in our setting. We will repeatedly make use of this fact in what follows and in particular freely use the results from [38, Section 7]. We also note that over algebraically closed fields, [38, Proposition 6.6 and Lemma 7.5] imply that the group $A^i(X)_{\mathbb{Z}_\ell}$ from [38, Definition 7.2] agrees with $A^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, where $A^i(X)$ denotes the Chow group of algebraic cycles modulo algebraic equivalence.

For instance, if we define

$$E_{\ell^r}^i(X) := \ker(A^i(X)/\ell^r \rightarrow H^{2i}(X, \mu_{\ell^r}^{\otimes i})), \tag{2.9}$$

where the morphism is given by the reduction modulo ℓ^r of the cycle class map, then [38, Corollary 7.12 and Lemma 7.13] imply the following.

Theorem 2.7. *Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime that is invertible in k . Then there is a canonical isomorphism*

$$E_{\ell^r}^i(X) \simeq H_{i-2,\text{nr}}^{2i-1}(X, \mu_{\ell^r}^{\otimes i})/H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})$$

and a canonical extension

$$0 \rightarrow \text{Griff}^i(X)/\ell^r \rightarrow E_{\ell^r}^i(X) \rightarrow Z^i(X)[\ell^r]/H^{2i}(X, \mathbb{Z}_\ell(i))[\ell^r] \rightarrow 0,$$

where $Z^i(X)[\ell^r] := \text{coker}(\text{cl}_X^i : \text{CH}^i(X)_{\mathbb{Z}_\ell} \rightarrow H^{2i}(X, \mathbb{Z}_\ell(i)))[\ell^r]$.

2.6. Indivisible torsion classes with trivial transcendental Abel–Jacobi invariant

Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime invertible in k . By [38, Section 7.5], there is a transcendental Abel–Jacobi map on torsion cycles

$$\lambda_{\text{tr}}^i : \text{Griff}^i(X)[\ell^\infty] \rightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i)), \tag{2.10}$$

where N^* denotes the coniveau filtration. If X is projective, then the above map agrees with Bloch’s map and if in addition $k = \mathbb{C}$, then it agrees with Griffiths’ [15] transcendental Abel–Jacobi map on torsion cycles (see [38, Proposition 8.5]). We write

$$\mathcal{T}^i(X)[\ell^r] := \ker(\lambda_{\text{tr}}^i)[\ell^r] \subset \text{Griff}^i(X)[\ell^r].$$

Theorem 2.8. *Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime invertible in k . Then there are subgroups $\mathcal{T}_0^i(X)[\ell^r] \subset \mathcal{T}^i(X)[\ell^r]$ with $\mathcal{T}^i(X)[\ell^\infty] = \bigcup_r \mathcal{T}_0^i(X)[\ell^r]$ and canonical isomorphisms*

$$\mathcal{T}_0^i(X)[\ell^r] \simeq \frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})} \simeq \frac{H_{i-3, \text{nr}}^{2i-2}(X, \mu_{\ell^r}^{\otimes i})}{G^i H_{i-3, \text{nr}}^{2i-2}(X, \mu_{\ell^r}^{\otimes i})}.$$

Moreover, the kernel of the canonical surjection

$$\mathcal{T}_0^i(X)[\ell^r] \simeq \frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})} \twoheadrightarrow \frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{\tilde{G}^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}$$

is given by all classes in $\mathcal{T}_0^i(X)[\ell^r]$ that are ℓ^r -divisible in $A^i(X)$.

Proof. This follows from [38, Corollary 7.23 and Proposition 7.25]. ■

3. Product maps

Here and in what follows, all tensor products will be over \mathbb{Z} if not mentioned otherwise.

Lemma 3.1. *Let k be an algebraically closed field and let ℓ be a prime invertible in k . Let X and Y be smooth varieties over k . Then there is a well-defined linear map*

$$\Lambda : \tilde{\delta}(H^1(X, \mu_{\ell^r})) \otimes \frac{H_{i-2, \text{nr}}^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})} \rightarrow \frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{\tilde{G}^{i+1} H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})},$$

which on elementary tensors is given by

$$\Lambda([\tilde{\delta}(\alpha)] \otimes [\beta]) := [p^* \alpha \cup q^* \beta],$$

where $p : X \times Y \rightarrow X$ and $q : X \times Y \rightarrow Y$ denote the natural projections and where $\alpha \in H^1(X, \mu_{\ell^r})$ and $\beta \in H^{2i-1}(F_{i-1}Y, \mu_{\ell^r}^{\otimes i})$.

Proof. Recall $G^m \subset \tilde{G}^m$ from (2.8). Well-definedness in β follows therefore from (2.8) together with the isomorphism

$$\frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{G^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})} \simeq \frac{H_{i-2, nr}^{2i}(X \times Y, \mu_{\ell^r}^{\otimes i+1})}{G^{i+1}H_{i-2, nr}^{2i}(X \times Y, \mu_{\ell^r}^{\otimes i+1})}$$

from Theorem 2.8. To prove well-definedness in α , assume that $\tilde{\delta}(\alpha) = 0$. Then by the derivation property of the Bockstein $\tilde{\delta}$ (see (2.7)) together with its functoriality, we find

$$\tilde{\delta}(p^*\alpha \cup q^*\beta) = -p^*\alpha \cup q^*\tilde{\delta}(\beta).$$

By [38, Corollary 7.9], $\tilde{\delta}(\beta)$ extends to Y and so $p^*\alpha \cup q^*\beta \in \tilde{G}^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})$. This shows that Λ is well-defined, which concludes the proof of the lemma. ■

Lemma 3.2. *Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime invertible in k . Then there is a canonical isomorphism*

$$\tilde{\delta}(H^1(X, \mu_{\ell^r})) \simeq A^1(X)[\ell^r]/\ell^r A^1(X)[\ell^{2r}].$$

Proof. Since algebraic and homological equivalence coincides for divisors on smooth varieties,

$$A^1(X)[\ell^r] \simeq H^2(X, \mathbb{Z}_{\ell}(1))[\ell^r] \simeq \delta(H^1(X, \mu_{\ell^r})),$$

where we use the fact that any class in $H^2(X, \mathbb{Z}_{\ell}(1))[\ell^r] \simeq \delta(H^1(X, \mu_{\ell^r}))$ is algebraic as it comes from a μ_{ℓ^r} -torsor and hence from a line bundle. The lemma follows therefore from the compatibility of δ and $\tilde{\delta}$. ■

Let X and Y be smooth varieties over an algebraically closed field k and let ℓ be a prime invertible in k . The natural exterior product map on cycles yields a map

$$\frac{A^1(X)[\ell^r]}{\ell^r A^1(X)[\ell^{2r}]} \otimes A^i(Y)/\ell^r \rightarrow \frac{A^{i+1}(X \times Y)[\ell^r]}{\ell^r A^{i+1}(X \times Y)[\ell^{2r}]}, \quad [z_1] \otimes [z_2] \mapsto [z_1 \times z_2]. \quad (3.1)$$

To compare that map with Λ from Lemma 3.1, we consider the diagram

$$\begin{CD} \frac{A^1(X)[\ell^r]}{\ell^r A^1(X)[\ell^{2r}]} \otimes A^i(Y)/\ell^r @>(3.1)>> \frac{A^{i+1}(X \times Y)[\ell^r]}{\ell^r A^{i+1}(X \times Y)[\ell^{2r}]} \\ @VV\uparrow V @VV\uparrow V \\ \tilde{\delta}(H^1(X, \mu_{\ell^r})) \otimes \frac{H_{i-2, nr}^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})} @>\Lambda>> \frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{\tilde{G}^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})} \end{CD} \quad (3.2)$$

where the vertical injection on the left comes from Lemma 3.2 and Theorem 2.7, while the vertical injection on the right stems from Theorem 2.8.

Lemma 3.3. *The diagram (3.2) commutes.*

Proof. Let $\alpha \in H^1(X, \mu_{\ell^r})$ and $[\beta] \in H_{i-2, \text{nr}}^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})/H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})$ with representative $\beta \in H^{2i-1}(F_{i-1}Y, \mu_{\ell^r}^{\otimes i})$. We claim first that

$$\delta(\beta) \in \text{im}(H^{2i}(Y, \mathbb{Z}_\ell(i)) \rightarrow H^{2i}(F_{i-1}Y, \mathbb{Z}_\ell(i))). \tag{3.3}$$

By (2.6), it suffices to show that

$$\partial(\delta(\beta)) \in \bigoplus_{y \in Y^{(i)}} H^1(y, \mathbb{Z}_\ell(0))$$

vanishes, which holds because the right hand side is torsion-free, while $\delta(\beta)$ is torsion.

Mapping $\alpha \otimes [\beta]$ to the upper right corner in (3.2) via (3.1), we get the element

$$z_1 \times [\partial(\beta)] \in A^{i+1}(X \times Y)[\ell^r]/\ell^r A^{i+1}(X \times Y)[\ell^{2r}],$$

where $z_1 \in A^1(X)[\ell^r]$ is the unique class with $\text{cl}_X^1(z_1) = \delta(\alpha) \in H^2(X, \mathbb{Z}_\ell(1))$ and where

$$\partial(\beta) \in \bigoplus_{y \in Y^{(i)}} [y]\mathbb{Z}_\ell.$$

On the other hand, the image of $\alpha \otimes [\beta]$ in $\frac{A^{i+1}(X \times Y)[\ell^r]}{\ell^r A^{i+1}(X \times Y)[\ell^{2r}]}$ via Λ is represented by

$$\partial\gamma \in \bigoplus_{z \in (X \times Y)^{(i+1)}} [z]\mathbb{Z}/\ell^r,$$

where $\gamma \in H^{2i+1}(F_i(X \times Y), \mu_{\ell^r}^{\otimes i+1})$ is a lift of $\tilde{\delta}(p^*\alpha \cup q^*\beta)$. The derivation property of $\tilde{\delta}$ (see (2.7)) and functoriality of the Bockstein $\tilde{\delta}$ yield

$$\tilde{\delta}(p^*\alpha \cup q^*\beta) = p^*(\tilde{\delta}(\alpha)) \cup q^*\beta - p^*\alpha \cup q^*(\tilde{\delta}(\beta)) \in H^{2i+1}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1}).$$

By (3.3), $\delta(\beta)$ and hence the reduction $\tilde{\delta}(\beta)$ extends to a class on Y . Since α is a global class on X , we find that for any lift $\gamma' \in H^{2i+1}(F_i(X \times Y), \mu_{\ell^r}^{\otimes i+1})$ of $p^*(\tilde{\delta}(\alpha)) \cup q^*\beta$,

$$[\partial\gamma] = [\partial\gamma'] \in A^{i+1}(X \times Y)/\ell^r.$$

Since $\tilde{\delta}(\alpha)$ is an algebraic class, it is supported on some divisor $D \subset X$. Since β is defined away from a codimension i subset $Z \subset Y$, $p^*(\tilde{\delta}(\alpha)) \cup q^*\beta$ is defined away from the codimension $i + 1$ subset $D \times Z$ and so we may pick a lift γ' as above in such a way that

$$\partial\gamma' = \partial p^*(\tilde{\delta}(\alpha)) \cup q^*\beta = z_1 \times \partial\beta \in \bigoplus_{z \in (X \times Y)^{(i+1)}} [z]\mathbb{Z}/\ell^r.$$

This concludes the proof of the lemma. ■

4. Smooth specialization of refined unramified cohomology

Let κ be a field and let $R = \mathcal{O}_{B,0}$ be the local ring of a smooth pointed curve $(B, 0)$ over κ , where $0 \in B$ is a κ -rational point. Let $K = \text{Frac } R$ and let ℓ be a prime invertible in κ . The main result of this section is as follows.

Proposition 4.1. *Let R be as above, fix a uniformizer $\pi \in R$, and let $\mathcal{X} \rightarrow \text{Spec } R$ be a smooth morphism with equi-dimensional fibres $X_0 = \mathcal{X} \times_R \kappa$ and $X_\eta = \mathcal{X} \times_R K$. There are linear specialization maps*

$$sp : H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(F_j X_0, \mu_{\ell^r}^{\otimes n}) \tag{4.1}$$

with the following properties:

- (0) sp does not depend on π if κ is algebraically closed;
- (1) sp respects the filtration F^* ;
- (2) if $\mathcal{U} \subset \mathcal{X}$ is an open subset with $F_{j+1} X_\eta \subset U_\eta$ and $F_{j+1} X_0 \subset U_0$, then any class $\alpha \in H^i(\mathcal{U}, \mu_{\ell^r}^{\otimes n})$ with restriction $[\alpha|_{U_\eta}] \in H^i_{j,\text{nr}}(X_\eta, \mu_{\ell^r}^{\otimes n})$ satisfies $sp([\alpha|_{U_\eta}]) = [\alpha|_{U_0}]$;
- (3) the pushforwards from Lemma 2.2 commute with specialization: if $\mathcal{Y} \rightarrow \text{Spec } R$ is a smooth morphism with equi-dimensional fibres Y_η and Y_0 , and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a proper R -morphism of pure relative dimension, then $(f|_{X_0})_* \circ sp = sp \circ (f|_{X_\eta})_*$;
- (4) sp induces a specialization map between geometric fibres $X_{\bar{\eta}} := X_\eta \times_K \bar{K}$ and $X_{\bar{0}} := X_0 \times_\kappa \bar{\kappa}$:

$$\bar{sp} : H^i(F_j X_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(F_j X_{\bar{0}}, \mu_{\ell^r}^{\otimes n}).$$

The map \bar{sp} does not depend on the choice of π and respects the filtration F^* .

Corollary 4.2. *In the above notation, there are well-defined specialization maps*

$$sp : H^i_{j,\text{nr}}(X_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i_{j,\text{nr}}(X_0, \mu_{\ell^r}^{\otimes n}) \quad \text{and} \quad \bar{sp} : H^i_{j,\text{nr}}(X_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i_{j,\text{nr}}(X_{\bar{0}}, \mu_{\ell^r}^{\otimes n}).$$

Remark 4.3. In the above corollary, sp may depend on the choice of the uniformizer $\pi \in R$, while \bar{sp} is independent of that choice.

4.1. Construction: Part 1

Let $\mathcal{U} \rightarrow \text{Spec } R$ be a smooth morphism with equi-dimensional fibres $U_0 = \mathcal{U} \times_R \kappa$ and $U_\eta = \mathcal{U} \times_R K$. By Remark 2.4, (2.2) yields an exact sequence

$$H^i(\mathcal{U}, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_\eta, \mu_{\ell^r}^{\otimes n}) \xrightarrow{\partial} H^{i-1}(U_0, \mu_{\ell^r}^{\otimes n-1}) \rightarrow H^{i+1}(\mathcal{U}, \mu_{\ell^r}^{\otimes n}). \tag{4.2}$$

We fix a uniformizer $\pi \in R$. Then π gives rise to a class in $H^1(K, \mu_{\ell^r}) \simeq K^*/(K^*)^{\ell^r}$ and so we get a class $(\pi) \in H^1(U_\eta, \mu_{\ell^r})$ via pullback. We then define

$$sp : H^i(U_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_0, \mu_{\ell^r}^{\otimes n}), \quad \alpha \mapsto -\partial((\pi) \cup \alpha),$$

where ∂ is the residue map in (4.2). (If the residue field κ is not algebraically closed, then the class (π) and hence the map sp may depend on the choice of π ; cf. proof of Proposition 4.1 below.)

Lemma 4.4. *If $\alpha \in H^i(U_\eta, \mu_{\ell^r}^{\otimes n})$ extends to a class $\tilde{\alpha} \in H^i(\mathcal{U}, \mu_{\ell^r}^{\otimes n})$, then $sp(\alpha) = \tilde{\alpha}|_{U_0}$.*

Proof. By (2.3), $-\partial((\pi) \cup \alpha) = -(\partial(\pi)) \cup \tilde{\alpha}|_{U_0}$ and so the result follows from the fact that $\partial(\pi) = -1 \in H^0(U_0, \mu_{\ell^r}^{\otimes 0}) = \mathbb{Z}/\ell^r$; see e.g. [38, (P6) in Proposition 6.6]. ■

Lemma 4.5. *In the above notation,*

$$sp(\ker(H^i(U_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n})) \subset \ker(H^i(U_0, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_{\bar{0}}, \mu_{\ell^r}^{\otimes n})).$$

Proof. Replacing κ by the algebraic closure $\bar{\kappa}$, we may assume that κ is algebraically closed. Then K has cohomological dimension 1 and so the Hochschild–Serre spectral sequence [25, p. 105, III.2.20] yields an exact sequence

$$0 \rightarrow H^1(K, H^{i-1}(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n})) \rightarrow H^i(U_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n}).$$

For any class $\beta \in H^j(U_\eta, \mu_{\ell^r}^{\otimes m})$, cup product with β is compatible with the above spectral sequence. The action on the E_2 -page is induced by the action of β on the coefficients and hence only depends on the image of β in $H^j(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes m})$. This description shows that cup product with the class $(\pi) \in H^1(K, \mu_{\ell^r})$ acts trivially on the E_2 -page of the above spectral sequence. It thus follows from the above short exact sequence that for any class $\alpha \in \ker(H^i(U_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n}))$, we have

$$(\pi) \cup \alpha = 0 \in H^{i+1}(U_\eta, \mu_{\ell^r}^{\otimes n+1}),$$

and so $sp(\alpha) = 0$. This proves the lemma. ■

4.2. *Construction: Part 2*

Let R and π be as above and let $\mathcal{X} \rightarrow \text{Spec } R$ be a smooth morphism with equi-dimensional fibres. For any $j \geq i$, we define a specialization map

$$sp : H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(F_j X_0, \mu_{\ell^r}^{\otimes n})$$

as follows. Let $\alpha \in H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$. Then there is a closed subset $Z_\eta \subset X_\eta$ of codimension $> j$ such that $\alpha = [\alpha_{U_\eta}]$ is represented by a class

$$\alpha_{U_\eta} \in H^i(U_\eta, \mu_{\ell^r}^{\otimes n}),$$

where $U_\eta = X_\eta \setminus Z_\eta$. The closure $\mathcal{Z} \subset \mathcal{X}$ of Z_η is automatically flat over R and so the special fibre Z_0 has codimension $> j$ in X_0 . Let $\mathcal{U} := \mathcal{X} \setminus \mathcal{Z}$ with special fibre $U_0 = X_0 \setminus Z_0$. By Section 4.1, we get a class $\alpha_{U_0} := sp(\alpha_{U_\eta}) \in H^i(U_0, \mu_{\ell^r}^{\otimes n})$ and define

$$sp(\alpha) = [\alpha_{U_0}] \in H^i(F_j X_0, \mu_{\ell^r}^{\otimes n}).$$

Functoriality of the Gysin sequence with respect to open immersions (see Lemma 2.3) immediately shows that this definition is well-defined.

Lemma 4.6. *The specialization map*

$$sp : H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(F_j X_0, \mu_{\ell^r}^{\otimes n}), \quad \alpha \mapsto [\alpha_{U_0}],$$

defined above is compatible with the filtration F^ .*

Proof. Let $m \geq j$ and assume in the above notation that $\alpha \in F^m H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$. This means that we may choose $\alpha_{U_\eta} \in H^i(U_\eta, \mu_{\ell^r}^{\otimes n})$ in such a way that Z_η actually has codimension $> m$ in X_η . But then the above construction immediately shows that $sp(\alpha)$ lifts to a class in $H^i(F_m X_0, \mu_{\ell^r}^{\otimes n})$ and hence lies in $F^m H^i(F_j X_0, \mu_{\ell^r}^{\otimes n})$. ■

Lemma 4.7. *Let $\mathcal{Y} \rightarrow \text{Spec } R$ be another smooth morphism with equi-dimensional fibres Y_η and Y_0 and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a proper R -morphism of pure relative codimension $c := \dim X_\eta - \dim Y_\eta$. Then the following diagram commutes:*

$$\begin{CD} H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n}) @>sp>> H^i(F_j X_0, \mu_{\ell^r}^{\otimes n}) \\ @Vf_*VV @VVf_*V \\ H^{i+2c}(F_{j+c} Y_\eta, \mu_{\ell^r}^{\otimes n+c}) @>sp>> H^{i+2c}(F_{j+c} Y_0, \mu_{\ell^r}^{\otimes n+c}) \end{CD}$$

Proof. This follows directly from the projection formula and the fact that the Gysin sequence is functorial with respect to proper pushforwards (see Lemma 2.3). ■

4.3. Proofs of Proposition 4.1 and Corollary 4.2

Proof of Proposition 4.1. The existence of the specialization map together with item (1) follows from the construction in Section 4.2 and Lemma 4.6. Items (2) and (3) follow from Lemmas 4.4 and 4.7, respectively. By construction, sp depends on the class $(\pi) \in H^1(K, \mu_{\ell^r})$ of π . On the other hand, the map will not change if we replace R by its completion, and so we may from now on assume that R is complete. If κ is algebraically closed, then $H^1(\text{Spec } R, \mu_{\ell^r}) \simeq H^1(\kappa, \mu_{\ell^r}) = 0$ by [25, Corollary VI.2.7] and so $\partial : H^1(K, \mu_{\ell^r}) \rightarrow H^0(\kappa, \mathbb{Z}/\ell^r) = \mathbb{Z}/\ell^r$ is an isomorphism, which implies that in this case the class $(\pi) \in H^1(K, \mu_{\ell^r})$ is independent of π . Hence, sp does not depend on the choice of π if κ is algebraically closed, as claimed in item (0).

To prove the existence of the specialization map on geometric fibres in item (4), up to replacing κ by its algebraic closure we may assume that κ is algebraically closed. It follows from what we have said above that sp is in this situation independent of the choice of π . To prove (4), it thus suffices by Lemmas 4.5 and 4.6 to show that, up to a finite base change, any class $[\alpha] \in H^i(F_j X_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n})$ comes from $H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$. To prove this, let $U_{\bar{\eta}} \subset X_{\bar{\eta}}$ be an open subset with $F_j X_{\bar{\eta}} \subset U_{\bar{\eta}}$ and let $\alpha \in H^i(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n})$. Taking the Galois closure of the complement of $U_{\bar{\eta}} \subset X_{\bar{\eta}}$, up to shrinking $U_{\bar{\eta}}$ we may assume that $U_{\bar{\eta}} = U_\eta \times \bar{K}$ for some open subset $U_\eta \subset X_\eta$ with $F_j X_\eta \subset U_\eta$. Since $H^i(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n})$ is a finite group, up to a finite base change we may also assume that the Galois group $G = \text{Gal}(\bar{K}/K)$ acts trivially on α . Since κ is algebraically closed, K has cohomological dimension 1. The Hochschild–Serre spectral sequence [25, p. 105, III.2.20] thus shows that α lies in the image of $H^i(U_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(U_{\bar{\eta}}, \mu_{\ell^r}^{\otimes n})$ as we want. This concludes the proof of the proposition. ■

Proof of Corollary 4.2. Since $H_{j,\text{nr}}^i(X, A(n)) = F^{j+1} H^i(F_j X, A(n))$, the corollary follows from Proposition 4.1. ■

Remark 4.8. The above specialization maps yield as a special case (where $j = 0$) maps

$$sp : H^i(K(X_\eta), \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(\kappa(X_0), \mu_{\ell^r}^{\otimes n}) \tag{4.3}$$

that are well-known from Galois cohomology. On the other hand, the specialization maps between refined unramified cohomology from Corollary 4.2 seem new even in the case $j = 0$, where the groups in question coincide with traditional unramified cohomology. In fact, the situation is somewhat subtle: Unramified classes may in proper flat families specialize via (4.3) to ramified classes and this was the main technique to prove nontriviality of certain unramified classes in [36, Section 6]. The main point is that the families $\mathcal{X} \rightarrow \text{Spec } R$ considered in *loc. cit.* are flat but not smooth and our results here show that the ramification has to lie on the singular locus of \mathcal{X} over R .

5. A vanishing result

Let κ be an algebraically closed field. We assume for simplicity that κ has characteristic zero. Let R be the local ring of a smooth pointed curve $(B, 0)$ over κ with fraction field $K = \text{Frac } R$. Let \mathcal{X}, \mathcal{Y} , and \mathcal{W} be flat R -schemes with equi-dimensional fibres and let

$$p : \mathcal{W} \rightarrow \mathcal{X} \quad \text{and} \quad q : \mathcal{W} \rightarrow \mathcal{Y}$$

be R -morphisms with $c := \text{codim}_{\mathcal{Y}}(q(\mathcal{W}))$. Assume that

- (1) \mathcal{X} is regular, \mathcal{W} is integral and normal, $\mathcal{Y} \rightarrow \text{Spec } R$ is smooth;
- (2) q is proper and generically finite onto its image $q(\mathcal{W}) \subset \mathcal{Y}$;
- (3) p is dominant.

Since κ has characteristic zero, the generic fibre W_η of \mathcal{W} is generically smooth. Lemma 2.5 thus yields pushforward maps

$$q_* : H^i(F_0 W_\eta, \mu_{\ell^r}^{\otimes n}) \rightarrow H^{i+2c}(F_c Y_\eta, \mu_{\ell^r}^{\otimes n+c}),$$

with c as above.

Lemma 5.1. *In the above notation, let ℓ be a prime and let $\alpha \in H_{\text{nr}}^i(\mathcal{X}, \mu_{\ell^r}^{\otimes n})$ be an unramified class on \mathcal{X} whose restriction to the generic point of any component of the special fibre X_0 vanishes. Then for any $\xi \in H^j(K(W_\eta), \mu_{\ell^r}^{\otimes m})$, the class*

$$q_*(p^* \alpha \cup \xi) \in H^{i+j+2c}(F_c Y_\eta, \mu_{\ell^r}^{\otimes n+m+c})$$

lies in the kernel of $sp : H^{i+j+2c}(F_c Y_\eta, \mu_{\ell^r}^{\otimes n+m+c}) \rightarrow H^{i+j+2c}(F_c Y_0, \mu_{\ell^r}^{\otimes n+m+c})$ from Section 4.2.

Proof. Since κ has characteristic zero and \mathcal{W} extends to a normal κ -variety over some neighbourhood of $0 \in B$, up to shrinking \mathcal{W} we may assume that $\mathcal{W} \rightarrow \text{Spec } R$ is smooth. By linearity of q_* , up to shrinking \mathcal{W} we may assume that W_0 is irreducible.

By Lemma 4.7, it then suffices to show that

$$sp(p^* \alpha \cup \xi) = 0 \in H^{i+j}(F_0 W_0, \mu_{\ell^r}^{\otimes n+m}).$$

This vanishing in turn follows from the fact that the residue map

$$\partial : H^{i+j+1}(F_0 W_\eta, \mu_{\ell^r}^{\otimes n+m+1}) \rightarrow H^{i+j}(\kappa(W_0), \mu_{\ell^r}^{\otimes n+m})$$

factorizes through the cohomology of the completion $\widehat{\mathcal{O}}_{\mathcal{W}, W_0}$ of the local ring of \mathcal{W} at the generic point of W_0 together with the claim that

$$p^* \alpha = 0 \in H^i(\widehat{\mathcal{O}}_{\mathcal{W}, W_0}, \mu_{\ell^r}^{\otimes n}).$$

To prove this last claim, note that the restriction map

$$H^i(\widehat{\mathcal{O}}_{\mathcal{W}, W_0}, \mu_{\ell^r}^{\otimes n}) \rightarrow H^i(\kappa(W_0), \mu_{\ell^r}^{\otimes n})$$

is injective (see [25, Corollary VI.2.7]) and $p^* \alpha$ lies in the kernel of the above map because α vanishes on any component of the special fibre \mathcal{X} by assumption. (This last step uses the fact that p is dominant by assumption (3).) This concludes the proof of the lemma. ■

6. An injectivity theorem

Theorem 6.1. *Let κ be an algebraically closed field of characteristic zero and let R be the local ring of a smooth pointed curve $(B, 0)$ over κ . Let k be an algebraic closure of $\text{Frac } R$ and let ℓ be a prime. Assume that there is a proper strictly semistable R -scheme $\mathcal{X} \rightarrow \text{Spec } R$ with connected fibres of relative dimension 2 such that the following holds, where $X := \mathcal{X} \times_R k$ denotes the geometric generic fibre:*

- (C1) *the restriction map $H_{\text{nr}}^2(\mathcal{X}, \mu_{\ell^r}) \rightarrow H_{\text{nr}}^2(X, \mu_{\ell^r})$ is surjective;*
- (C2) *for each component X_{0i} of X_0 , the restriction map $H_{\text{nr}}^2(\mathcal{X}, \mu_{\ell^r}) \rightarrow H_{\text{nr}}^2(X_{0i}, \mu_{\ell^r})$ is zero.*

Then for any smooth projective variety Y_κ over κ with base change $Y = Y_\kappa \times_\kappa k$, and for any free \mathbb{Z}/ℓ^r -module $M \subset \tilde{\delta}(H^1(X, \mu_{\ell^r}))$, the following composition is injective:

$$M \otimes \frac{H_{i-2, \text{nr}}^{2i-1}(Y_\kappa, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y_\kappa, \mu_{\ell^r}^{\otimes i})} \rightarrow M \otimes \frac{H_{i-2, \text{nr}}^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})} \xrightarrow{\Lambda} \frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{\tilde{G}^{i+1} H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})},$$

where Λ is the map from Lemma 3.1.

Proof. Let $\alpha_1, \dots, \alpha_n \in H^1(X, \mu_{\ell^r})$, such that the free \mathbb{Z}/ℓ^r -module M is given by

$$M = \bigoplus_{j=1}^n \tilde{\delta}(\alpha_j) \mathbb{Z}/\ell^r \mathbb{Z} \subset \tilde{\delta}(H^1(X, \mu_{\ell^r})). \tag{6.1}$$

Let further

$$\beta_{\kappa,j} \in H^{2i-1}(F_{i-1}Y_{\kappa}, \mu_{\ell^r}^{\otimes i}) \quad \text{for } j = 1, \dots, n \tag{6.2}$$

and denote the image of $\beta_{\kappa,j}$ in $H^{2i-1}(F_{i-1}Y, \mu_{\ell^r}^{\otimes i})$ by β_j . For a contradiction, we assume that

$$\sum_{j=1}^n \tilde{\delta}(\alpha_j) \otimes [\beta_{\kappa,j}] \neq 0 \in M \otimes \frac{H_{i-2, \text{nr}}^{2i-1}(Y_{\kappa}, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y_{\kappa}, \mu_{\ell^r}^{\otimes i})}, \tag{6.3}$$

and

$$\Lambda\left(\sum_{j=1}^n \tilde{\delta}(\alpha_j) \otimes [\beta_j]\right) = \sum_{j=1}^n [p^*\alpha_j \cup \beta_j] = 0 \in \frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{\tilde{G}^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}.$$

By the derivation property for $\tilde{\delta}$ (see (2.7)), the latter is equivalent to

$$\sum_{j=1}^n p^*(\tilde{\delta}\alpha_j) \cup q^*\beta_j - \sum_{j=1}^n p^*(\alpha_j) \cup q^*(\tilde{\delta}\beta_j) \in F^{i+1}H^{2i+1}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1}), \tag{6.4}$$

which by (2.6) means that the above class in $H^{2i+1}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})$ extends to a class on $X \times Y$. The theorem will be proven if we derive a contradiction from (6.3) and (6.4).

Since X is a smooth proper connected surface over an algebraically closed field,

$$H^4(X, \mu_{\ell^r}^{\otimes 2}) = \mathbb{Z}/\ell^r \cdot \text{cl}_X^2(pt)$$

where $\text{cl}_X^2(pt)$ denotes the cycle class of a closed point on X .

Step 1. For any $j_0 \in \{1, \dots, n\}$, there is a class $\overline{\tilde{\delta}(\alpha_{j_0})} \in H^2(X, \mu_{\ell^r})$ with

$$\overline{\tilde{\delta}(\alpha_{j_0})} \cup \tilde{\delta}(\alpha_j) = \begin{cases} \text{cl}_X^2(pt) & \text{if } j = j_0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the above property does not change if we add to $\overline{\tilde{\delta}(\alpha_{j_0})}$ a class that lifts to $H^2(X, \mathbb{Z}_{\ell}(1))$.

Proof. Since X is a smooth proper surface over the algebraically closed field k , Poincaré duality implies that the cup product pairing on $H^2(X, \mu_{\ell^r})$ is perfect (see e.g. [26, Theorem 24.1]). This implies the existence of the classes $\overline{\tilde{\delta}(\alpha_{j_0})}$, because $\tilde{\delta}(\alpha_j)$ are \mathbb{Z}/ℓ^r -linearly independent for $j = 1, \dots, n$ by (6.1).

By the compatibility of δ and $\tilde{\delta}$, the class $\tilde{\delta}(\alpha_j)$ is the reduction modulo ℓ^r of a torsion class in $H^2(X, \mathbb{Z}_{\ell}(1))$. Since any class in $H^2(X, \mathbb{Z}_{\ell}(1))$ has trivial cup product pairing with a torsion class in $H^2(X, \mathbb{Z}_{\ell}(1))$, we find that the properties in question do not change if we add to $\overline{\tilde{\delta}(\alpha_{j_0})}$ a class that lifts to $H^2(X, \mathbb{Z}_{\ell}(1))$. This concludes Step 1. ■

By (6.2), there is an open subset $U_\kappa \subset Y_\kappa$ with $\text{codim}_{Y_\kappa}(Y_\kappa \setminus U_\kappa) > i - 1$ and such that for all $j = 1, \dots, n$,

$$\beta_{\kappa,j} \in H^{2i-1}(U_\kappa, \mu_{\ell r}^{\otimes i}) \quad \text{and} \quad \beta_j \in H^{2i-1}(U, \mu_{\ell r}^{\otimes i}),$$

where $U = U_\kappa \times k$, and where by slight abuse of notation we do not distinguish between the above classes on the open subsets U_κ and U and their restrictions to $F_{i-1}Y_\kappa$ and $F_{i-1}Y$, respectively.

Assumption (6.4) together with Lemma 2.6 implies that there is a class

$$\xi \in \bigoplus_{w \in (X \times U)^{(i)}} H^1(w, \mu_{\ell r})$$

such that

$$\gamma := \sum_{j=1}^n p^*(\tilde{\delta}\alpha_j) \cup q^*\beta_j - \sum_{j=1}^n p^*(\alpha_j) \cup q^*(\tilde{\delta}\beta_j) + \iota_*\xi \in H^{2i+1}(F_i(X \times U), \mu_{\ell r}^{\otimes i+1}) \tag{6.5}$$

extends to a class on $X \times Y$. (Note that the above class lies on $F_i(X \times U)$, not on $F_i(X \times Y)$.)

Step 2. *The map*

$$q_* : H^{2i+3}(F_i(X \times U), \mu_{\ell r}^{\otimes i+2}) \rightarrow H^{2i-1}(F_{i-2}U, \mu_{\ell r}^{\otimes i})$$

from Lemma 2.5 satisfies, for any $j_0 \in \{1, \dots, n\}$,

$$q_*(p^*(\overline{\tilde{\delta}(\alpha_{j_0})}) \cup \gamma) = \beta_{j_0} + q_*(p^*(\overline{\tilde{\delta}(\alpha_{j_0})}) \cup \iota_*\xi) \in H^{2i-1}(F_{i-2}U, \mu_{\ell r}^{\otimes i}). \tag{6.6}$$

Proof. This is a consequence of (6.5), the computation in Step 1 and the fact that

$$q_*(p^*(\overline{\tilde{\delta}(\alpha_{j_0})}) \cup \alpha_j) \cup q^*(\tilde{\delta}\beta_j) = 0,$$

which follows from the projection formula, because $q_*(p^*(\overline{\tilde{\delta}(\alpha_{j_0})}) \cup \alpha_j) = 0$ for degree reasons. This concludes Step 2. ■

Recall that $U = U_\kappa \times_\kappa k$ and consider the smooth R -scheme $\mathcal{U} := U_\kappa \times_\kappa R$. Since κ is algebraically closed, applying item (4) of Proposition 4.1 to this family, we get a specialization map

$$\bar{p} : H^{2i-1}(F_{i-2}U, \mu_{\ell r}^{\otimes i}) \rightarrow H^{2i-1}(F_{i-2}U_\kappa, \mu_{\ell r}^{\otimes i}).$$

Step 3. *We have*

$$\bar{p}(q_*(p^*(\overline{\tilde{\delta}(\alpha_{j_0})}) \cup \iota_*\xi)) = 0 \in H^{2i-1}(F_{i-2}U_\kappa, \mu_{\ell r}^{\otimes i}).$$

Proof. By linearity, we may assume that $\xi \in H^1(\kappa(w), \mu_{\ell r})$ for some $w \in (X \times U)^{(i)}$. Since X is a surface, $q(w) \in U$ has codimension at least $i - 2$. If $q(w)$ has codimension greater than $i - 2$, then

$$q_*(p^*(\overline{\tilde{\delta}(\alpha_{j_0})}) \cup \iota_*\xi) = 0 \in H^{2i-1}(F_{i-2}U, \mu_{\ell r}^{\otimes i})$$

by Lemma 2.6 and we are done. Hence, we may assume $q(w) \in U^{(i-2)}$.

The pullback of $\overline{\delta(\alpha_{j_0})}$ via p^* factorizes through the restriction

$$\overline{\delta(\alpha_{j_0})}|_{p(w)} \in H^2(\kappa(p(w)), \mu_{\ell^r}).$$

If $p(w)$ is not the generic point of X , then the latter group vanishes for dimensional reasons and we are done. Hence, we may assume that $p(w)$ is the generic point of X .

Recall that R is the local ring of a smooth pointed curve $(B, 0)$ over κ . We wish to perform a base change corresponding to a ring map $R \rightarrow R'$, where R' is the local ring of a smooth pointed curve $(B', 0')$ over κ and $B' \rightarrow B$ is a finite morphism that maps $0'$ to 0 . When we perform such a base change, the model \mathcal{X} becomes singular, but it follows from [18, Proposition 2.2] that $\mathcal{X} \times_R R'$ can be made into a strictly semistable R' -scheme $\mathcal{X}' \rightarrow \text{Spec } R'$ by repeatedly blowing up all non-Cartier components of the special fibre. The exceptional divisors introduced in these blowups are ruled surfaces over the algebraically closed field κ and so they have trivial second unramified cohomology. For this reason, assumptions (C1) and (C2) remain true after such a base change. We may thus in what follows freely apply base changes as above.

Up to a base change as described above, we may assume that the point w is defined over $K = \text{Frac}(R)$. We may then consider the normalization \mathcal{W} of the closure of w in $\mathcal{X} \times_R \mathcal{U}$. The projections of $\mathcal{X} \times_R \mathcal{U}$ to the two factors yield natural maps

$$p|_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{X} \quad \text{and} \quad q|_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{U}.$$

By the above reduction step, $p|_{\mathcal{W}}$ is dominant and $q|_{\mathcal{W}}$ is generically finite onto its image. It follows that (1)–(3) from Section 5 are satisfied.

Since $H^1(\kappa(w), \mu_{\ell^r}) \simeq \kappa(w)^*/(\kappa(w)^*)^{\ell^r}$, up to a finite base change as above we may assume that ξ extends to a class on the generic point of \mathcal{W} . By assumption (C1), $\overline{\delta(\alpha_{j_0})}$ lifts to a class in $H_{\text{nr}}^2(\mathcal{X}, \mu_{\ell^r})$ whose restriction to each component of the special fibre of $\mathcal{X} \rightarrow \text{Spec } R$ vanishes by (C2). The vanishing claimed in Step 3 is therefore a consequence of Lemma 5.1. ■

By assumption, β_j is the image of $\beta_{\kappa, j}$ via the natural map

$$H^{2i-1}(F_{i-2}U_{\kappa}, \mu_{\ell^r}^{\otimes i}) \rightarrow H^{2i-1}(F_{i-2}U, \mu_{\ell^r}^{\otimes i}),$$

where we recall $U = U_{\kappa} \times_{\kappa} k$. The pullback of $\beta_{j, \kappa}$ to the product $\mathcal{U} := U_{\kappa} \times_{\kappa} \text{Spec } R$ is thus a class that restricts to β_j on the geometric generic fibre and to $\beta_{j, \kappa}$ on the special fibre. Proposition 4.1 (2) thus implies

$$\overline{s\bar{p}}(\beta_j) = \beta_{\kappa, j} \in H^{2i-1}(F_{i-2}U_{\kappa}, \mu_{\ell^r}^{\otimes i})$$

for all j . Step 3 together with (6.6) in Step 2 thus imply

$$\overline{s\bar{p}}(q_*(p^*(\overline{\delta(\alpha_{j_0})}) \cup \gamma)) = \overline{s\bar{p}}(\beta_{j_0}) = \beta_{\kappa, j_0} \in H^{2i-1}(F_{i-2}U_{\kappa}, \mu_{\ell^r}^{\otimes i}). \tag{6.7}$$

Since $H^2(X, \mu_{\ell^r}) \rightarrow H_{\text{nr}}^2(X, \mu_{\ell^r})$ is surjective by (2.6), $p^*(\overline{\delta(\alpha_{j_0})})$ extends to $X \times Y$. The same holds for γ by assumption. Hence,

$$q_*(p^*(\overline{\delta(\alpha_{j_0})}) \cup \gamma) \in H^{2i-1}(F_{i-2}U, \mu_{\ell^r}^{\otimes i})$$

extends to a class on Y . Note that $F_{i-2}Y = F_{i-2}U$ and $F_{i-2}Y_\kappa = F_{i-2}U_\kappa$. Item (2) (resp. (4)) in Proposition 4.1 thus implies that the left hand side of (6.7) extends to a class on Y_κ and so

$$\beta_{\kappa, j_0} \in H^{2i-1}(F_{i-2}U_\kappa, \mu_{\ell^r}^{\otimes i}) = H^{2i-1}(F_{i-2}Y_\kappa, \mu_{\ell^r}^{\otimes i})$$

extends to a class in Y_κ . This holds for all $j_0 = 1, \dots, n$ and so the class in (6.3) vanishes, which contradicts our assumptions. This concludes the proof of the theorem. ■

Remark 6.2. The proof of Theorem 6.1 shows that conditions (C1) and (C2) can be slightly weakened as follows. The surjectivity in (C1) is only needed to lift the classes $\overline{\delta}(\alpha_j)$ from Step 1 to \mathcal{X} . These classes may by Step 2 be modified by the image of integral classes and so (C1) may be weakened to only ask that the composition

$$H_{nr}^2(\mathcal{X}, \mu_{\ell^r}) \rightarrow H_{nr}^2(X, \mu_{\ell^r}) \rightarrow H_{nr}^2(X, \mu_{\ell^r})/H_{nr}^2(X, \mathbb{Z}_\ell(1))$$

is surjective, or even weaker, that the images of $\overline{\delta}(\alpha_j)$ in $H_{nr}^2(X, \mu_{\ell^r})/H_{nr}^2(X, \mathbb{Z}_\ell(1))$ are contained in the image of the above map.

Condition (C2) is only used to ensure that the lifts of the classes $\overline{\delta}(\alpha_{j_0})$ to \mathcal{X} from Step 1 restrict trivially to the components of the special fibre of \mathcal{X} and it would be enough to replace (C2) by this more precise condition.

Recall $E_{\ell^r}^i(Y) \subset A^i(Y)/\ell^r$ from (2.9). The exterior product map (3.1) induces a map

$$\times : A^1(X)[\ell^r] \otimes E_{\ell^r}^i(Y) \rightarrow A^{i+1}(X \times Y)[\ell^r], \quad [z_1] \otimes [z_2] \mapsto [z_1 \times z_2].$$

Corollary 6.3. *In the notation of Theorem 6.1, the kernel of the natural composition*

$$A^1(X)[\ell] \otimes E_\ell^i(Y_\kappa) \rightarrow A^1(X)[\ell] \otimes E_\ell^i(Y) \xrightarrow{\times} A^{i+1}(X \times Y)[\ell] \rightarrow A^{i+1}(X \times Y)/\ell$$

is given by $(\ell \cdot A^1(X)[\ell^2]) \otimes E_\ell^i(Y_\kappa)$.

Proof. It is clear that any class in $(\ell \cdot A^1(X)[\ell^2]) \otimes E_\ell^i(Y_\kappa)$ maps to zero in the group $A^{i+1}(X \times Y)/\ell$. The converse implication follows by the commutative diagram in (3.2) (see Lemma 3.3) from Theorem 6.1, applied to $M = \tilde{\delta}(H^1(X, \mu_\ell))$. ■

7. Flower pot degenerations of Enriques surfaces

Let k be an algebraically closed field of characteristic zero and let $R = k[[t]]$. Following previous work of Kulikov and Perrson, Morrison classified all semistable degenerations $\mathcal{X} \rightarrow \text{Spec } R$ of Enriques surfaces over R in [27, Corollary 6.2]. In contrast to Kulikov’s original claim in [22], it is not true that up to birational equivalence we can always assume that $K_{\mathcal{X}}$ is 2-torsion. In fact, there are three additional types of degenerations (called (ib), (iib), and (iiib) in [27, Corollary 6.2]) that do not admit an étale $2 : 1$ cover by a Kulikov degeneration of K3 surfaces. The simplest example of such an exceptional degeneration of Enriques surfaces is given by type (ib), called flower pot degenerations

(see [31, Proposition 3.3.1 (3)]). The central fiber X_0 of a flower pot degeneration is constructed as follows. One starts with a rational surface S_0 which contains a disjoint union of smooth rational (-4) -curves C_1, \dots, C_r . For each $i = 1, \dots, r$, one attaches to S_0 a chain of Hirzebruch surfaces F_4 by identifying $C_i \subset S_0$ with a suitable section of the first Hirzebruch surface in the chain. Finally, for each attached chain of Hirzebruch surfaces, one attaches a \mathbb{P}^2 as an end-component which is glued to the last F_4 along a smooth conic. On the level of K3 covers, a flower pot degeneration corresponds to the degeneration of a K3 surface T with a fixed point free involution ι to a nodal K3 surface T_0 and an involution ι_0 which fixes exactly the nodes of T_0 .

We will need the following evident properties for a flower pot degeneration $\mathcal{X} \rightarrow \text{Spec } R$:

- if we perform a base change $t \rightarrow t^m$ and resolve the resulting family by repeatedly blowing up all non-Cartier components of the special fibre, then the new semistable degeneration of Enriques surfaces is again a flower pot degeneration;
- each component of the central fibre X_0 is rational and the dual complex of X_0 is a tree (i.e. a finite connected undirected acyclic graph).

Explicit examples of flower pot degenerations of Enriques surfaces have been constructed by Horikawa [20, Section 10.2] and by Persson [31, Appendix 2]. The dual complex of the central fibre is in both constructions given by a straight line, i.e. the components of the special fibre form a chain of rational surfaces.

Theorem 7.1. *Let κ be an algebraically closed field of characteristic zero and let R be the local ring at a closed point of a smooth curve over κ . Let $\mathcal{X} \rightarrow \text{Spec } R$ be a strictly semistable degeneration of Enriques surfaces such that the special fibre X_0 is a flower pot as in [27, type (ib), Corollary 6.2]. Then up to a base change, the restriction map*

$$H_{\text{nr}}^2(\mathcal{X}, \mu_2) \rightarrow H_{\text{nr}}^2(X_{\bar{\eta}}, \mu_2)$$

to the geometric generic fibre $X_{\bar{\eta}}$ is surjective.

Proof. Up to a base change (followed by resolving the resulting family by blowing up all non-Cartier components of the special fibre), we may assume that the monodromy action on $H^2(X_{\bar{\eta}}, \mu_2)$ is trivial. The Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X_{\bar{\eta}}, \mu_2)) \implies H^{p+q}(X_{\bar{\eta}}, \mu_2)$$

degenerates at E_2 , because the absolute Galois group G of $\text{Frac } R$ has cohomological dimension 1. Hence we get an exact sequence

$$H^2(X_{\bar{\eta}}, \mu_2) \rightarrow H^2(X_{\bar{\eta}}, \mu_2) \xrightarrow{d_2} H^2(G, H^1(X_{\bar{\eta}}, \mu_2)).$$

Since G has cohomological dimension 1, we find that $H^2(G, H^1(X_{\bar{\eta}}, \mu_2)) = 0$ and so $H^2(X_{\bar{\eta}}, \mu_2) \rightarrow H^2(X_{\bar{\eta}}, \mu_2)$ is surjective. Hence there is a class $\alpha \in H^2(X_{\bar{\eta}}, \mu_2)$ whose image in $H^2(X_{\bar{\eta}}, \mu_2)$ generates

$$H_{\text{nr}}^2(X_{\bar{\eta}}, \mu_2) \simeq \text{coker}(\text{cl}_{X_{\bar{\eta}}}^1 : \text{CH}^1(X_{\bar{\eta}}) \rightarrow H^2(X_{\bar{\eta}}, \mu_2)) \simeq \mathbb{Z}/2.$$

To prove the theorem, it suffices to show that α is unramified on \mathcal{X} (as the latter implies that α lifts to a class on $H^2(\mathcal{X}, \mu_2)$; see Lemma 2.3 and Remark 2.4). To prove the latter, let X_{0i} for $i \in I$ denote the components of X_0 . There is a natural residue map

$$\partial : H^2(X_\eta, \mu_2) \rightarrow \bigoplus_{i \in I} H^1(k(X_{0i}), \mathbb{Z}/2)$$

and we let

$$\gamma_i := \partial\alpha \in H^1(k(X_{0i}), \mathbb{Z}/2).$$

Up to a ramified 2 : 1 base change (followed by resolving the resulting family as above), by the commutative diagram in [9, p. 148], we may assume that $\gamma_i = 0$ for all components that have not been introduced by resolving the singularities introduced by the 2 : 1 base change. Since the dual graph of X_0 is a tree, we conclude that whenever X_{0i} meets X_{0j} with $i \neq j$, then either γ_i or γ_j is zero. This implies that there is a subset $J \subset I$ such that

$$X'_0 := \bigcup_{j \in J} X_{0j} \subset X_0$$

is smooth and α extends to an unramified class on $X \setminus X'_0$. Since α has degree 2, it actually extends to an honest class

$$\alpha' \in H^2(\mathcal{X} \setminus X'_0, \mu_2)$$

(see (2.6)). Since X'_0 is smooth, the residue of α' satisfies

$$\partial\alpha' \in H^1(X'_0, \mathbb{Z}/2).$$

Since each component of X_0 is rational, the same holds for the components of X'_0 and so $H^1(X'_0, \mu_2) = 0$. Hence, $\partial\alpha' = 0$ and we find that α' extends to a class in $H^2(\mathcal{X}, \mu_2)$. This concludes the proof of the theorem. ■

Proof of Theorem 1.5. By [20, Section 10.2] or [31, Appendix 2], there is a strictly semistable degeneration $\mathcal{X} \rightarrow \text{Spec } R$ of Enriques surfaces such that the special fibre X_0 is a flower pot as in [27, type (ib), Corollary 6.2] and where R is the local ring at a closed point of a smooth curve over an algebraically closed field of characteristic zero. Then $H^2_{\text{nr}}(\mathcal{X}, \mu_{\ell^r}) \simeq \text{Br}(\mathcal{X})[\ell^r]$ and $H^2_{\text{nr}}(X_\eta, \mu_{\ell^r}) \simeq \text{Br}(X_\eta)[\ell^r]$ by [8, Proposition 4.2.3] and Remark 2.4. Theorem 1.5 follows therefore from Theorem 7.1. ■

We end this section with the following result which reveals some subtlety of the geometry of degenerations as in Theorem 1.5.

Proposition 7.2. *Let $\mathcal{X} \rightarrow \text{Spec } R$ be as in Theorem 1.5. Then $\text{Br}(X_0)[2] \neq 0$ while $\text{Br}(X_{0i}) = 0$ for each component X_{0i} of X_0 .*

Proof. Since each X_{0i} is a smooth projective ruled surface over an algebraically closed field, $\text{Br}(X_{0i}) = 0$ is clear and it suffices to prove $\text{Br}(X_0)[2] \neq 0$. Replacing R by its completion, we may assume that R is complete and so there is a noncanonical isomorphism $R \simeq \kappa[[t]]$ because the residue field κ has characteristic zero.

By the proper base change theorem [25, Corollary 2.7], we get canonical isomorphisms

$$H^2(\mathcal{X}, \mu_{2^r}) \simeq H^2(X_0, \mu_{2^r}) \quad \text{and} \quad H^2(\mathcal{X}, \mathbb{Z}_2(1)) \simeq H^2(X_0, \mathbb{Z}_2(1)), \tag{7.1}$$

where the latter uses [21, (0.2)]. Let $X_\eta = \mathcal{X} \times_R K$ where $K = \text{Frac } R \simeq \kappa((t))$. The Gysin sequence yields an exact sequence

$$\bigoplus_i \mathbb{Z}_2[X_{0i}] \rightarrow H^2(\mathcal{X}, \mathbb{Z}_2(1)) \rightarrow H^2(X_\eta, \mathbb{Z}_2(1)), \tag{7.2}$$

where X_{0i} denote the components of the special fibre X_0 and the first arrow is given by $\sum a_i X_{0i} \mapsto \sum a_i c_1(\mathcal{O}_{\mathcal{X}}(X_{0i}))$. Note that $G := \text{Gal}(\bar{K}/K)$ has cohomological dimension 1 because $R \simeq \kappa[[t]]$ with κ algebraically closed. Moreover, Kummer theory shows

$$H^1(G, \mu_{2^r}) \simeq \kappa((t))^*/(\kappa((t))^*)^{2^r}.$$

This is a finite group (isomorphic to $\mathbb{Z}/2^r$) and so $R^1 \lim H^1(G, \mu_{2^r}) = 0$. Hence, $H_{\text{cont}}^2(G, \mathbb{Z}_2(1)) \simeq \lim H^2(G, \mu_{2^r}) = 0$ by [21, (1.6)] because G has cohomological dimension 1. Moreover, $H^1(X_{\bar{\eta}}, \mathbb{Z}_2(1)) = 0$ because $X_{\bar{\eta}}$ is an Enriques surface. The Hochschild–Serre spectral sequence for continuous étale cohomology (see [21, (0.3)]) thus yields an isomorphism

$$H^2(X_\eta, \mathbb{Z}_2(1)) \simeq H^2(X_{\bar{\eta}}, \mathbb{Z}_2(1))^G.$$

Since $H^1(G, \mathbb{G}_m) = H^2(G, \mathbb{G}_m) = 0$, the Hochschild–Serre spectral sequence yields similarly an isomorphism $\text{Pic}(X_\eta) \simeq \text{Pic}(X_{\bar{\eta}})^G$. Since $X_{\bar{\eta}}$ is an Enriques surface, $H^2(X_{\bar{\eta}}, \mathbb{Z}_2(1)) \simeq \text{Pic}(X_{\bar{\eta}}) \otimes \mathbb{Z}_2$ is algebraic. We thus conclude from the above isomorphism that

$$H^2(X_\eta, \mathbb{Z}_2(1)) \simeq \text{Pic}(X_\eta) \otimes \mathbb{Z}_2.$$

Taking closures of divisors on X_η thus shows via (7.2) that $H^2(\mathcal{X}, \mathbb{Z}_2(1))$ is algebraic. Since algebraic classes restrict to algebraic classes, the second isomorphism in (7.1) shows similarly that $H^2(X_0, \mathbb{Z}_2(1))$ is algebraic. Hence, (2.5) implies

$$\begin{aligned} \text{Br}(\mathcal{X})[2^r] &\simeq H^2(\mathcal{X}, \mu_{2^r})/H^2(\mathcal{X}, \mathbb{Z}_2(1)), \\ \text{Br}(X_0)[2^r] &\simeq H^2(X_0, \mu_{2^r})/H^2(X_0, \mathbb{Z}_2(1)). \end{aligned} \tag{7.3}$$

We thus conclude from (7.1) that there are canonical isomorphisms

$$\text{Br}(\mathcal{X})[2^r] \simeq \text{Br}(X_0)[2^r]$$

for all $r \geq 0$. By Theorem 1.5 (3), $\text{Br}(\mathcal{X})[2] \neq 0$ and so $\text{Br}(X_0)[2] \neq 0$, which proves the proposition. ■

Remark 7.3. The above proof shows more generally that if R in Theorem 1.5 is complete, then the natural restriction map $\text{Br}(\mathcal{X})_{\text{tors}} \rightarrow \text{Br}(X_0)_{\text{tors}}$ is an isomorphism.

Remark 7.4. One can use Proposition 7.2 to check that several degenerations of Enriques surfaces as in [27, Corollary 6.2] (e.g. those of type (iia)) do not satisfy the conclusion of Theorem 1.5.

8. Proofs of Theorems 1.1 and 1.3

Proof of Theorem 1.3. Let Y be a smooth complex projective variety and let X be an Enriques surface that is very general with respect to Y . We aim to show that the exterior product map

$$A^i(Y)/2 \rightarrow A^{i+1}(X \times Y)/2, \quad [z] \mapsto [K_X \times z],$$

is injective. Using the Künneth decomposition for cohomology with $\mathbb{Z}/2$ -coefficients together with the fact that K_X has nontrivial cycle class in $H^2(X, \mu_2)$, we see that $[K_X \times z] = 0$ implies that $[z]$ has trivial cycle class in $H^{2i}(Y, \mu_2^{\otimes i})$. The problem thus reduces to the statement that

$$E_2^i(Y) \rightarrow A^{i+1}(X \times Y)/2, \quad [z] \mapsto [K_X \times z],$$

is injective, where $E_2^i(Y) = \ker(\text{cl}_Y^i : A^i(Y)/2 \rightarrow H^{2i}(Y, \mu_2^{\otimes i}))$ from (2.9).

By a straightforward specialization argument and because the moduli space of complex Enriques surfaces is irreducible, it suffices to prove the result for some smooth projective Enriques surfaces X over \mathbb{C} .

Since Chow groups modulo algebraic equivalence are countable, there is a countable algebraically closed field $\kappa \subset \mathbb{C}$ such that $Y = Y_\kappa \times \mathbb{C}$ for some smooth projective variety Y_κ over κ and such that the natural map

$$E_2^i(Y_\kappa) \rightarrow E_2^i(Y)$$

is surjective, hence an isomorphism because κ and \mathbb{C} are algebraically closed.

Let $R := \kappa[[t]]$. By Theorem 1.5, up to enlarging κ we may assume that there is a regular flat proper scheme $\mathcal{X} \rightarrow \text{Spec } R$ whose generic fibre is a smooth Enriques surface and such that

- there is a class $\alpha \in H_{\text{nr}}^2(\mathcal{X}, \mu_2^{\otimes 2})$ whose pullback to the geometric generic fibre is the unique nonzero Brauer class of the Enriques surface $X_{\bar{\eta}}$;
- the restriction of α to each component of the special fibre is trivial.

Let k be an algebraic closure of the fraction field of R . Since κ is countable, we may assume that $k \subset \mathbb{C}$.

It follows from Corollary 6.3 that

$$E_2^i(Y_\kappa) \rightarrow A^{i+1}(X_{\bar{\eta}} \times Y_k)/2, \quad [z] \mapsto [K_X \times z],$$

is injective. Since $k \subset \mathbb{C}$, we may consider the base change $X := X_{\bar{\eta}} \times_k \mathbb{C}$. Since $k \subset \mathbb{C}$ is an extension of algebraically closed fields, a well-known and straightforward specialization argument shows that

$$A^{i+1}(X_{\bar{\eta}} \times Y_k)/2 \rightarrow A^{i+1}(X \times Y)/2$$

is injective. Altogether, we thus see that

$$E_2^i(Y) \simeq E_2^i(Y_\kappa) \rightarrow A^{i+1}(X \times Y)/2, \quad [z] \mapsto [K_X \times z],$$

is injective. Here the Enriques surface X is somewhat special, but as noted above, this

also shows that the map in question is injective for a very general Enriques surface in place of X . This concludes the proof. ■

Proof of Corollary 1.4. Note that $K_X \in A^1(X)$ is 2-torsion and the Jacobian variety of X is trivial. It follows that the Deligne cycle class (see e.g. [42]) of K_X is 2-torsion. Hence, for $z \in A^i(Y)$ with $[z] \in E_2^i(Y) = \ker(\text{cl}_Y^i : A^i(Y)/2 \rightarrow H^{2i}(Y, \mu_2^{\otimes i}))$, the class $[K_X \times z] \in A^{i+1}(X \times Y)$ has trivial Deligne cycle class, because the Deligne cycle class of K_X is 2-torsion, while that of z is divisible by 2. It follows that the exterior product map

$$E_2^i(Y) \rightarrow A^{i+1}(X \times Y), \quad [z] \mapsto [K_X \times z],$$

lands in the subspace of classes of the Griffiths group with trivial Abel–Jacobi invariant. Since the image of the above map is clearly 2-torsion, we find that its image is contained in

$$\mathcal{T}^{i+1}(X \times Y)[2] \subset \text{Griff}^{i+1}(X \times Y) \subset A^{i+1}(X \times Y).$$

Corollary 1.4 follows therefore from Theorem 1.3. ■

Proof of Theorem 1.1. Let $C \subset \mathbb{P}_{\mathbb{C}}^2$ be a very general quartic curve. Since the group of algebraically trivial cycles modulo rational equivalence is divisible, we have $\text{CH}^2(Y)/2 \simeq A^2(Y)/2$. Hence, [41] shows that $\text{Griff}^2(JC)/2$ is infinite. Theorem 1.3 implies that for a very general Enriques surface X , the map

$$\text{Griff}^2(JC)/2 \rightarrow \text{Griff}^3(X \times JC)[2], \quad [z] \mapsto [K_X \times z],$$

is injective. Hence, $\text{Griff}^3(X \times JC)$ has infinite 2-torsion, as we want. ■

Remark 8.1. Replacing [41] in the above proof by [13], we see that Theorem 1.1 remains true if JC is replaced by the product of three very general elliptic curves.

Corollary 8.2. *For any $n \geq 5$ and any $3 \leq i \leq n - 2$, there is a smooth complex projective n -fold X with infinite torsion (in fact 2-torsion) in $\text{Griff}^i(X)$.*

Proof. For $n \geq 5$, the projective bundle formula shows that the smooth complex projective variety $X \times JC \times \mathbb{P}^{n-5}$, where C and X are as in the proof of Theorem 1.1, has infinite 2-torsion in Griff^i for all $3 \leq i \leq n - 2$. This proves the corollary. ■

Remark 8.3. It remains open whether $\text{Griff}^i(X)$ may have infinite torsion for $i = 2$ or $n - 1$; but recall that the n -torsion subgroup of $\text{Griff}^2(X)$ is finite for any $n \geq 1$ (see [24, Section 18]).

9. Further applications

In [23, 42], Voisin and Ma showed that for any smooth complex projective variety X whose Chow group of zero-cycles is supported on a threefold, there is a short exact sequence

$$0 \rightarrow \left(\frac{H^5(X, \mathbb{Z})}{N^2 H^5(X, \mathbb{Z})} \right)_{\text{tors}} \rightarrow H_{\text{nr}}^4(X, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathcal{T}^3(X) \rightarrow 0,$$

where $N^*H^5(X, \mathbb{Z})$ denotes the coniveau filtration and where $\mathcal{T}^3(X) \subset \text{Griff}^3(X)$ denotes the subgroup of torsion classes with trivial transcendental Abel–Jacobi invariants. All three terms in the above sequence are birational invariants of X . In [42, Section 4], Voisin considered the question whether it can happen that the first term vanishes, while the unramified cohomology group in the middle is nonzero. In *loc. cit.*, Voisin showed that the generalized Hodge conjecture implies the existence of such examples; by the generalized Bloch conjecture, her examples should actually satisfy $\text{CH}_0(X) = \mathbb{Z}$. By a result of Ottem and Suzuki [30], Theorem 1.3 answers Voisin’s question unconditionally.

Corollary 9.1. *For any $n \geq 5$, there is a smooth complex projective n -fold X with $\text{CH}_0(X) \simeq \mathbb{Z}$ such that*

$$N^2H^5(X, \mathbb{Z}) = H^5(X, \mathbb{Z}) \quad \text{and} \quad H_{\text{nr}}^4(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathcal{T}^3(X) = \text{Griff}^3(X)_{\text{tors}} \neq 0.$$

Proof. By [30], there is a smooth complex projective threefold Y , given as a pencil of Enriques surfaces, such that $\text{CH}_0(Y) \simeq \mathbb{Z}$, $H^*(Y, \mathbb{Z})$ is torsion-free and Y admits a non-algebraic Hodge class $\alpha \in H^{2,2}(Y, \mathbb{Z})$ such that 2α is algebraic. It thus follows from Theorem 1.3 that for any very general Enriques surface X , the product

$$Z := X \times Y \times \mathbb{P}^{n-5}$$

contains a nonzero 2-torsion class in $\mathcal{T}^3(Z)$. Moreover, $\text{CH}_0(Z) \simeq \mathbb{Z}$, because X has trivial Chow group of zero-cycles. Since Y has torsion-free cohomology, the Künneth formula applies and shows that $H^5(Z, \mathbb{Z})$ decomposes as

$$(H^3(X, \mathbb{Z}) \otimes H^2(Y \times \mathbb{P}^{n-5}, \mathbb{Z})) \oplus (H^2(X, \mathbb{Z}) \otimes H^3(Y \times \mathbb{P}^{n-5}, \mathbb{Z})) \oplus H^5(Y \times \mathbb{P}^{n-5}, \mathbb{Z}),$$

where we have used $b_1(Y) = 0$, since $\text{CH}_0(Z) \simeq \mathbb{Z}$ (see [5]). Since CH_0 of X and Y are supported on a point, the positive degree integral cohomology of X and Y is contained in N^1 (see [10, Proposition 3.3]). Using this, the above decomposition (together with the Künneth decomposition for $Y \times \mathbb{P}^{n-5}$) easily shows that $N^2H^5(Z, \mathbb{Z}) = H^5(Z, \mathbb{Z})$. Since $\text{CH}_0(Z) \simeq \mathbb{Z}$, this implies by [42, Corollary 0.3] that $H_{\text{nr}}^4(Z, \mathbb{Q}/\mathbb{Z}) \simeq \mathcal{T}^3(Z)$. Moreover, $N^2H^5(Z, \mathbb{Z}) = H^5(Z, \mathbb{Z})$ implies that the intermediate Jacobian $J^5(Z)$ of Z is generated by the images of $J^1(\widetilde{W})$, where $W \subset Z$ runs through all subvarieties of codimension 2 and \widetilde{W} denotes a resolution of W . In particular, the transcendental intermediate Jacobian $J_{\text{tr}}^5(Z)$ vanishes, and so $\mathcal{T}^3(Z) = \text{Griff}^3(Z)_{\text{tors}}$. This concludes the proof of the corollary. ■

The condition $\text{CH}_0(X) \simeq \mathbb{Z}$ means that X admits a rational decomposition of the diagonal. By [5, Theorem 1 (ii)],

$$\text{CH}_0(X) \simeq \mathbb{Z} \implies \text{Griff}^2(X) = 0.$$

This implication fails for Griff^3 , because we may blow up varieties with $\text{CH}_0 = \mathbb{Z}$ along a smooth subvariety with nontrivial Griff^2 . We are however not aware of any other construction that would yield varieties with small Chow groups of zero-cycles but nontrivial

Griffiths groups. For instance, to the best of our knowledge, it has not been known whether varieties with a rational decomposition of the diagonal always admit a birational model with trivial Griffiths groups. The following consequence of the above corollary solves that problem.

Corollary 9.2. *For any $n \geq 5$, there is a smooth complex projective n -fold X such that any smooth complex projective variety X' that is birational to X satisfies*

$$\mathrm{CH}_0(X') \simeq \mathbb{Z} \quad \text{and} \quad \mathrm{Griff}^3(X') \neq 0.$$

Proof. The result follows directly from Corollary 9.1 and the fact that $\mathcal{T}^3(Z)$ is a birational invariant of smooth complex projective varieties (see [42, Lemma 2.2]). ■

Corollary 9.3. *Let J_C be the Jacobian of a smooth quartic $C \subset \mathbb{P}_{\mathbb{C}}^2$ defined over \mathbb{Q} , with good reduction at 2 and with associated Kummer variety $Y = \widetilde{J_C}/\pm$. Then for any very general Enriques surface X over \mathbb{C} ,*

$$\mathrm{Griff}^3(X \times Y)[2] \neq 0 \quad \text{and} \quad \mathrm{Griff}^3(X \times Y)/2 \neq 0.$$

Proof of Corollary 9.3. Let C be a smooth plane curve of degree 4 defined over \mathbb{Q} and with good reduction at 2. Then J_C has good reduction at 2, and [12, Corollary 2.13] implies that the Kummer variety $Y = \widetilde{J_C}/\pm$ associated to J_C is a smooth complex projective variety with $H_{\mathrm{nr}}^3(Y, \mu_2) \neq 0$. As noted in *loc. cit.* the Chow group of Y is supported on a surface (see [5, Section 4, Example (1)]) and the integral cohomology of Y is torsion-free, so that [10] implies that there is a nonalgebraic nontorsion Hodge class $\alpha \in H^{2,2}(Y, \mathbb{Z})$ such that 2α is algebraic (cf. [12, Corollary 3.3]). For any very general complex projective Enriques surface X , Theorem 1.3 thus implies that $\mathrm{Griff}^3(X \times Y)[2] \neq 0$ and $\mathrm{Griff}^3(X \times Y)/2 \neq 0$, as we want. ■

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