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Asymptotic expansions for harmonic functions at conical boundary points

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Abstract. We prove three theorems about the asymptotic behavior of solutions u to the homogeneous Dirichlet problem for the Laplace equation at boundary points with tangent cones. First, under very mild hypotheses, we show that the doubling index of u either has a unique finite limit, or goes to infinity; in other words, there is a well-defined order of vanishing. Second, under more quantitative hypotheses, we prove that if the order of vanishing of u is finite at a boundary point 0, then locally $u(x) = |x|^m \psi(x/|x|) + o(|x|^m)$, where $|x|^m \psi(x/|x|)$ is a homogeneous harmonic function on the tangent cone. Finally, we construct a convex domain in three dimensions where such an expansion fails at a boundary point, showing that some quantitative hypotheses are necessary in general. The assumptions in all of the results only involve regularity at a single point, and in particular are much weaker than what is necessary for unique continuation, monotonicity of Almgren's frequency, Carleman estimates, or other related techniques.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open set with $0 \in \partial\Omega$, and consider solutions to the Dirichlet problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap B_2, \\ u = 0 & \text{on } \partial\Omega \cap B_2. \end{cases} \tag{1.1}$$

A basic question here is to understand the asymptotic behavior of u near 0.

If, instead, $0 \in \Omega$ were an interior point, the asymptotic behavior would be clear: as u is analytic, locally it can be decomposed as a leading-order homogeneous harmonic function plus higher order terms. Similar expansion formulas hold in related contexts where analyticity is not available, including second-order or higher order elliptic equations with C^∞, Lipschitz, or Hölder coefficients; see [3, 4, 9].

Below, we will use the term *asymptotic expansion* loosely for representations of the form $u = v + w$, where v has homogeneity m (or more generally, growth at least $g(|x|)$

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for some nondecreasing $g \geq 0$) and $w = o(|x|^m)$ (or more generally, $w = o(g(|x|))$ for the same modulus g), possibly with more quantitative control over w.

At boundary points (so $0 \in \partial\Omega$), Felli and Ferrero proved in [5] that if Ω is a $C^{1,\alpha}$ perturbation of a regular cone, then suitable rescalings of the solution (i.e., blow-ups) converge to a non-trivial homogeneous harmonic function on that cone; this gives an expansion of u to leading order. In [14], Kenig and Zhao show that when $\partial\Omega \in C^{1,\mathrm{Dini}}$, the expansion formula $u(x) = P_N(x) + O(|x|^N \int_0^{|x|} \tilde{\omega})$ holds, where $\tilde{\omega}$ is a modified modulus of continuity for the unit normal and P_N is a degree N homogeneous harmonic polynomial on $\mathbb{R}^d_+$ that vanishes on the boundary (choosing coordinates so that $\partial\mathbb{R}^d_+$ is tangent to $\partial\Omega$ at 0).

In these results, Ω is required to have certain smoothness in a neighborhood of 0. Here, we aim to discuss rougher domains under only one-point conditions, i.e., conditions which do not imply any smoothness except possibly at the single point 0 itself. It is clear that at least some assumptions are necessary at 0 to have any hope for solutions to (1.1) to have asymptotic expansions. Indeed, assuming Ω is regular for the Dirichlet problem, consider a Green's function G for Ω with some fixed pole: if $G(x) = |x|^m \psi(x/|x|) + o(|x|^m)$, then $\Omega = \{G(x) > 0\}$ is tangent to the cone $\Gamma = \{x : \psi(x/|x|) > 0\}$ at the origin. With this in mind, we begin with the following definitions.

Definition 1.1. Given an open cone Γ, the point $0 \in \partial\Omega$ is called *conical with cone* Γ if, in Hausdorff distance,

$$r^{-1} \operatorname{dist}(\partial\Omega \cap B_r, \partial\Gamma \cap B_r) \to 0.$$

Here, a *cone* Γ is a set invariant under dilation, i.e., $r\Gamma = \Gamma$ for all $r > 0$.

For an open cone Γ, let

$$(0 <)\ \lambda_{1,\Gamma} \leq \lambda_{2,\Gamma} \leq \cdots \leq \lambda_{k,\Gamma} \leq \cdots$$

be the sequence of Dirichlet eigenvalues (counting multiplicity) of the Laplace–Beltrami operator on the cross-section $\Gamma \cap B_1$. Also, let $\psi_{k,\Gamma}$ be the associated eigenfunction, and let

$$m_{k,\Gamma} = \frac{-(d-2) + \sqrt{(d-2)^2 + 4\lambda_{k,\Gamma}}}{2} \quad (> 0) \tag{1.2}$$

be the characteristic constant. Then all homogeneous harmonic function on Γ vanishing on $\partial\Gamma$ can be written as $|x|^{m_{k,\Gamma}} \psi_{\lambda_{k,\Gamma}}(x/|x|)$. Below, the dependence on Γ will be omitted when there is no ambiguity. It is worth mentioning that in Definition 1.1, Γ is allowed to be $\mathbb{R}^d_+$, so in particular every boundary point of a C^1 domain is conical.

If one wants to find an asymptotic expansion for u at 0, the first step is to identify the homogeneity of the leading-order term. On cones, convex domains, or sufficiently regular perturbations of them, the Almgren frequency gives a way to read off this homogeneity (this will be discussed below), but in this more general configuration, it is unclear that the frequency is even approximately monotone. Instead, taking zero extensions of solutions to (1.1) outside Ω, define the doubling index

$$N_u(r) := \frac{\fint_{\partial B_r} |u|^2}{\fint_{\partial B_{r/4}} |u|^2}. \tag{1.3}$$

We say that u is *asymptotically homogeneous* if the limit of $N_u(r)$ at $r = 0$ exists in the extended real number sense ($\lim_{r\searrow 0} N_u(r) \in [0,\infty]$). The doubling index is a rough measure of homogeneity, and the existence of this limit means that there is a unique leading-order homogeneity for u in this rough sense. Our first theorem states that u is asymptotically homogeneous if 0 is a conical point, under a smoothness assumption on Γ.

Assumption 1.2. The open cone Γ is graphical, in the sense that $\Gamma = \{(x', x_n) : x_n > g(x')\}$ for some choice of coordinates and function $g\colon \mathbb{R}^{d-1} \to \mathbb{R}$, and g is Lipschitz.

Theorem 1.3. *If $0 \in \partial\Omega$ is conical with cone Γ satisfying Assumption 1.2, then u is asymptotically homogeneous and, moreover,*

$$\lim_{r\searrow 0} N_u(r) = 4^{2m} \quad \textit{for some } m \in \{m_{\lambda_{k,\Gamma}}\}_{k=1}^{\infty} \cup \{+\infty\},$$

where $m_{\lambda_{k,\Gamma}}$ are the characteristic constants of Γ defined in (1.2).

Note that Theorem 1.3 does not exclude the possibility that u vanishes to infinite order near 0, i.e., that the strong unique continuation property (SUCP) fails. In the literature, (SUCP) is known to hold when Ω is regular enough. See Remark 1.7 below for discussion. Assumption 1.2 can be considerably relaxed, for example to a uniform Lebesgue density condition on Γ^c, but we do not attempt maximal generality here: in fact, this theorem is interesting even when $\Gamma = \mathbb{R}^d_+$.

Theorem 1.3 suggests that one might consider Almgren blow-ups of u, the rescaled functions

$$u_r(y) = \frac{u(ry)}{(\fint_{\partial B_r\cap\Omega} |u|^2)^{1/2}}, \tag{1.4}$$

to attempt to find the leading order term in an asymptotic expansion for u, even when the Almgren frequency is unavailable. Indeed, the boundedness of $N_u(r)$ is enough to guarantee the compactness of $\{u_r\}_{r\in(0,1)}$ in L^2. Hence, along subsequences $r_k \to 0$, u_r converges. Moreover, $\lim_{r\searrow 0} N_u(r) = 4^{2m} < \infty$ guarantees that the blow-up limit has to be a homogeneous harmonic function with homogeneity m. When $d = 2$, one can further obtain the uniqueness of the blow-up limit simply due to the fact that the eigenvalues $\lambda_{k,\Gamma}$ are all simple.

When $d \geq 3$, however, it turns out we require additional assumptions. First, the following essentially says that $\partial\Omega$ is $C^{1,\alpha}$ at 0 only, but with an arbitrary tangent object.

Definition 1.4 (α-conical). Given an open cone Γ, we say 0 is *α-conical with cone* Γ if there exists $\alpha > 0$ such that

$$\limsup_{r\to 0} r^{-(1+\alpha)} \operatorname{dist}(\partial\Omega \cap B_r, \partial\Gamma \cap B_r) < \infty. \tag{1.5}$$

We will also need to assume some smoothness of the limit cone Γ.

Assumption 1.5. The open cone Γ is graphical in the sense of Assumption 1.2 with the graph g being either $C^{1,\text{Dini}}$ or semiconvex on S^{n-1}.

Recall that a function g is called semiconvex if there exists a constant $C > 0$ such that, locally, $g(x+y) - 2g(x) + g(x-y) \geq -C|y|^2$. This is equivalent to say that $\Gamma \cap \partial B_1$ satisfies a uniform exterior ball condition, cf. Theorem 3.9 in [18].

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^d$ with $0 \in \partial\Omega$ and $d \geq 3$. If 0 is α-conical with cone Γ satisfying Assumption 1.5, then for any non-trivial solution u to (1.1), either $\lim_{r\to 0} N_r(u) = +\infty$, or there exist a Laplacian eigenfunction $\psi_{\lambda_{N,\Gamma}}$ on Γ, $C \neq 0$, and $\alpha_0 \in (0,1)$ such that*

$$u(x) = C|x|^{m_{N,\Gamma}} \psi_{\lambda_{N,\Gamma}}(x/|x|) + w(x), \quad \text{where } |w(x)| \leq C|x|^{m_{N,\Gamma}+\alpha_0\alpha}. \tag{1.6}$$

Recall that the characteristic constant is defined in (1.2). It is easy to see that (1.6) implies $(\fint_{\partial B_r \cap \Omega} |u|^2)^{1/2} \approx r^{m_{N,\Gamma}}$ and $u_r \to C|x|^{m_{N,\Gamma}}\psi_{N,\Gamma}$ in L^2. That is, the Almgren blow-up of u is also unique. However, the opposite is not true: (SUCP) or even the validity of Almgren's monotonicity formula combined with the uniqueness of blow-ups do not imply an expansion formula in the format of (1.6). This can be seen from the harmonic function $\mathrm{Re}(x+iy)/\log(x+iy)$ on $\mathbb{R}^2$.

In [14], an expansion formula was proved when $\partial\Omega \in C^{1,\mathrm{Dini}}$. It is interesting to ask whether our α-conical condition can be relaxed to a Dini rate.

Remark 1.7. For $\partial\Omega \in C^{1,\mathrm{Dini}}$ (see [1]) or convex (see [2]), it is known that (SUCP) holds at every boundary point. For $\partial\Omega \in C^1$, combining our Theorem 1.3 and Lemma 4.1 in [20], it can be shown that (SUCP) holds at almost every boundary point. See also [21] for discussion on quasiconvex Lipschitz domains. At a point where (SUCP) holds, any non-trivial solution must vanish to at most finite order, and in particular $N_r(u)$ is bounded. On the other hand, the assumptions in Theorem 1.6 are weaker than any known criterion for (SUCP) even if the cone Γ is a half-space, as far as we are aware. That is, assuming (SUCP), the expansion formula only requires the regularity of $\partial\Omega$ at one point.

Remark 1.8. We expect similar results as Theorem 1.6 hold for operators with scaling subcritical coefficients and lower order terms, i.e.,

$$Lu = D_i(a_{ij}D_ju + \widetilde{W}_i u) + W_i D_i u + Vu,$$

with $a_{ij} \in C^{\varepsilon}$, $\widetilde{W}_i, W_i \in L^{d+\varepsilon}_{\mathrm{loc}}$ and $V \in L^{d/2+\varepsilon}_{\mathrm{loc}}$. See [4] for an interior version which works for higher order elliptic equations with subcritical lower order terms.

One may naturally ask whether the extra convergence rate condition in (1.5) is necessary. We construct a *convex* domain $\Omega \subset \mathbb{R}^3$ for which $0 \in \partial\Omega$ is conical with cone $\mathbb{R}^3_+$, but for which no expansion (1.6) exists for some u:

Theorem 1.9. *There exist a convex domain $\Omega \subset \mathbb{R}^3$ with $0 \in \partial\Omega$ being conical with tangent cone $\mathbb{R}^3_+ = \{(x,y,z) : z > 0\}$, a solution u to (1.1), and a sequence $r_k \to 0$, such that*

$$\frac{u(r_{2k+1}\cdot)}{(\fint_{\partial B_{r_{2k+1}}} |u|^2)^{1/2}} \to 4\sqrt{2/\pi}\, xz \quad \text{and} \quad \frac{u(r_{2k}\cdot)}{(\fint_{\partial B_{r_{2k}}} |u|^2)^{1/2}} \to 4\sqrt{2/\pi}\, yz.$$

The point here is that $r^{-1}(\Omega \cap B_r) \to B_1^+$ slowly, with cross-sections $\partial B_r \cap \Omega$ resembling ellipses with oscillating eccentricity. Then a suitably chosen u can be made to have traces $u|_{\partial B_r}$ "rotate" between two second eigenfunctions of the Lapacian on $\partial B_r \cap \mathbb{R}^3_+$, as r decreases.

Note that when Ω is convex, every point $x_0 \in \partial\Omega$ is conical. Indeed, $(\Omega - x_0)/r$ always converges *monotonically* to a cone Γ_{x_0}. Moreover, Almgren's frequency is monotone on convex domains ([2]), so

$$N_u(r) \searrow N_u(0) = 4^{2m_{N,\Gamma_{x_0}}} \quad \text{for some } N \in \mathbb{N}, \text{ as } r \searrow 0.$$

So the conclusion of Theorem 1.3 for convex Ω holds, and in the following stronger form: $\lim_{r\searrow 0} N_u(r) < \infty$ and (SUCP) is valid. Therefore, the example of Theorem 1.9 shows that to have an asymptotic expansion in the weakest possible sense (uniqueness of limits for the Almgren rescalings u_r), it is not sufficient to have monotonicity of the frequency, or (SUCP), or even monotonicity in the convergence of Ω/r to its tangent cone; some sufficiently summable rate of convergence appears to be needed.

Similarly, it follows that the Dini condition in [14] cannot be replaced by even very strong geometric assumptions like convexity. In the recent work [15], counterexamples are constructed of barely non-$C^{1,\mathrm{Dini}}$ domains admitting solutions to the Dirichlet problem with large singular sets, but in those examples u still has unique Almgren blow-ups.

It is worth emphasizing that in both Theorems 1.6 and 1.3, we only assume one-point conditions at 0. Compared to earlier results in [5, 14], we do not need any smoothness condition on $\partial\Omega$ or its normal direction $\mathbf{n}$ in a neighborhood. We hope the methodology here could be useful when discussing asymptotic and unique continuation properties of harmonic functions on rough domains.

The paper is organized as follows. In Section 2, we prove Theorem 1.3. After collecting some preliminary facts about Green's functions on cones in Section 3, we provide the proof of Theorem 1.6 in Section 4. Finally, in Section 5, we discuss the uniqueness of Almgren blow-ups on $\Omega \subset \mathbb{R}^2$, and construct the example in Theorem 1.9.

2. Asymptotic homogeneity at a conical point

In this section, we prove Theorem 1.3. The key idea is to combine a compactness argument motivated by [16,17] and a rigidity result. Besides the usual doubling index $N_u(r)$ defined in (1.3), the following version using averages over full balls rather than spheres will also be useful:

$$\widetilde{N}_u(r) := \frac{\fint_{B_r} |u|^2}{\fint_{B_{r/2}} |u|^2}.$$

If there is no ambiguity, we suppress the subscript: $N = N_u$ and $\widetilde{N} = \widetilde{N}_u$. It is worth noting that if u is harmonic, u^2 is subharmonic, and so from the mean value property both $N, \widetilde{N} \geq 1$. The following lemma shows that N and $\widetilde{N}$ are comparable at adjacent scales.

Lemma 2.1. *Let v be a harmonic function on $B_1 \subset \mathbb{R}^d$, $d \geq 2$, with $v \not\equiv 0$ on any neighborhood of 0. Then, for some $C = C(d)$,*

$$\widetilde{N}_v(r) \leq C N_v(r) \quad \textit{and} \quad N_v(s) \leq C \prod_{j=0}^{3} \widetilde{N}_v(2^{1-j} r), \quad \forall r \in (0, 1/2) \textit{ and } s \in (r/2, r).$$

Proof. For the first inequality, by definition,

$$\widetilde{N}_v(r) = \frac{\fint_{B_r} |v|^2}{\fint_{B_{r/2}} |v|^2} \leq C \frac{\fint_{B_r} |v|^2}{\fint_{B_{r/2}\setminus B_{r/4}} |v|^2}. \tag{2.1}$$

By the mean value property of v^2, which is subharmonic, we obtain

$$\text{RHS of (2.1)} \leq C \frac{\fint_{\partial B_r} |v|^2}{\fint_{\partial B_{r/4}} |v|^2} = C N_v(r).$$

The second inequality can be proved similarly:

$$N_v(s) = \frac{\fint_{\partial B_s} |v|^2}{\fint_{\partial B_{s/4}} |v|^2} \le C \frac{\fint_{B_{2r}\setminus B_r} |v|^2}{\fint_{B_{r/8}} |v|^2} \le C \frac{\fint_{B_{2r}} |v|^2}{\fint_{B_{r/8}} |v|^2} = C \prod_{j=0}^{3} \widetilde{N}_v(2^{1-j} r). \qquad \blacksquare$$

The rigidity result is given as follows.

Lemma 2.2. *Let $\Gamma \subset \mathbb{R}^d$, $d \ge 2$, be a cone with vertex at 0 satisfying Assumption 1.2, and let v be a non-trivial solution to*

$$\begin{cases} \Delta v = 0 & \text{in } \Gamma \cap B_2, \\ v = 0 & \text{on } \partial\Gamma \cap B_2. \end{cases}$$

Then both N_v and $\widetilde{N}_v$ are non-decreasing for $r \in (0, 2)$. Moreover, if either $N_v(t) = N_v(s)$ or $\widetilde{N}_v(t) = \widetilde{N}_v(s)$ for some $t > s$, then u is homogeneous of degree $m_{j,\Gamma}$ with a characteristic constant defined in (1.2). In particular, $N_v \equiv 16^{m_{j,\Gamma}}$ and $\widetilde{N}_v \equiv 4^{m_{j,\Gamma}}$.

The proof of Lemma 2.2 is standard, by computing the derivatives of the (generalized) Almgren frequency functions. See Appendix B. The rest of the section is devoted to the proof of Theorem 1.3. From now on, let Γ be the tangent cone of Ω at 0, and let m_j be the characteristic constant defined in (1.2).

2.1. Step 1

We prove that $\liminf_{r\to 0} N_u(r) < \infty$ implies (SUCP). More precisely, we show

$$\limsup_{r\to 0} N_u(r) \le C(\liminf_{r\to 0} N_u(r))^4. \tag{2.2}$$

For this, we first prove the following:

Claim. *For any number $\mu \notin \{m_j\}_j$, there exists $r_0 = r_0(d, \mu, \Omega)$, such that $\widetilde{N}_u(r) \le 2^{2\mu}$ implies $\widetilde{N}_u(r/2) \le 2^{2\mu}$ for all $r \in (0, r_0)$.*

Proof of the Claim. We argue by contradiction. Suppose the contrary, that there exist solutions $u_k \in H^1$ to (1.1) and $r_k \to 0$, such that $\widetilde{N}_{u_k}(r_k) \le 2^{2\mu}$, $\widetilde{N}_{u_k}(r_k/2) > 2^{2\mu}$. Let

$$\tilde{u}_k(y) := \frac{u_k(r_k y)}{\left(\fint_{B_{r_k}} |u_k|^2\right)^{1/2}}.$$

Then we have

$$\Delta\tilde{u}_k = 0 \ \text{ in } r_k^{-1}(\Omega \cap B_{r_k}), \quad \tilde{u}_k = 0 \ \text{ on } B_1 \setminus r_k^{-1}(\Omega \cap B_{r_k}), \quad \text{and} \quad \fint_{B_1} |\tilde{u}_k|^2 = 1.$$

By the Caccioppoli inequality and the Sobolev embeddings, for all $\varepsilon \in (0, 1)$,

$$\tilde{u}_k \to u_\infty \quad \text{weakly in } L^2(B_1), H^1(B_{1-\varepsilon}), \text{ and strongly in } L^2(B_{1-\varepsilon}), \tag{2.3}$$

passing to a subsequence. Since Ω is conical at 0, we claim that

$$\Delta u_\infty = 0 \ \text{ in } B_1 \cap \Gamma \quad \text{and} \quad u_\infty = 0 \ \text{ on } \partial\Gamma \cap B_1. \tag{2.4}$$

To see that u_∞ is harmonic, take any test function $\varphi \in C_c^\infty(B_1 \cap \Gamma)$. From Definition 1.1, for sufficiently large k, we have $\operatorname{supp}(\varphi) \subset r_k^{-1}(\Omega \cap B_{r_k})$. Combining with (2.3), we obtain $\int \nabla \tilde{u}_k \cdot \nabla \varphi = 0$. Passing $k \to \infty$ and noting that φ is chosen arbitrarily, we obtain $\Delta u_\infty = 0$ in $B_1 \cap \Gamma$. For the boundary condition, we first note that $u_\infty = 0$ a.e. on $B_1 \setminus \bar{\Gamma}$ by recalling the $L^2(B_{1-\varepsilon})$ strong convergence. The desired zero boundary value now follows from the fact that Γ is Lipschitz.

Note that from (2.3), we have

$$\fint_{B_1} |u_\infty|^2 \le \liminf_k \fint_{B_1} |\tilde{u}_k|^2 = 1 \quad \text{and} \quad \fint_{B_r} |u_\infty|^2 = \lim_k \fint_{B_r} |\tilde{u}_k|^2, \quad \forall r < 1,$$

In particular, $\fint_{B_{1/2}} |u_\infty|^2 \ge 2^{2\mu} > 0$, so $u_\infty \not\equiv 0$ on Γ. Moreover,

$$\widetilde{N}_{u_\infty}(1) \le \liminf_k \widetilde{N}_{\tilde{u}_k}(1) \le 2^{2\mu} \quad \text{and} \quad \widetilde{N}_{u_\infty}(1/2) = \lim_k \widetilde{N}_{\tilde{u}_k}(1/2) \ge 2^{2\mu}.$$

From Lemma 2.2, this implies $\widetilde{N}_{u_\infty}(r) \equiv 4^m$ for some $m \in \{m_j\}_j$, and in particular, $\mu = m$. But we have assumed $\mu \notin \{m_j\}$, which is a contradiction. Hence, the claim is proved. ∎

Using the Claim, now we prove (2.2). Let $N_\infty = \liminf N_u(r)$. Since $N_\infty < \infty$, we can find a sequence of $r_k \to 0$ such that $N_u(r_k) \le 2N_\infty$ for each k. Using Lemma 2.1, we can deduce that $\widetilde{N}_u(r_k) \le CN_u(r_k) \le 2CN_\infty$. Pick a small $\varepsilon \ge 0$ such that $\mu := \log_4(2CN_\infty + \varepsilon) \notin \{m_j\}_j$, then fix an $r_k < r_0$ with r_0 given in the claim with such μ. Now, applying the claim iteratively, we obtain that for all $j \ge 0$, $\widetilde{N}_u(2^{-j} r_k) \le 2CN_\infty + \varepsilon$. Finally, for all sufficiently small r, we can find some j such that $r \in (2^{-j-2} r_k, 2^{-j-1} r_k)$. Using the second inequality in Lemma 2.1, we obtain $N_u(r) \le C \prod_{i=0}^3 \widetilde{N}_u(2^{-i} r_k) \le C(2CN_\infty + \varepsilon)^4 \le CN_\infty^4$, noting that ε can be chosen arbitrarily close to zero.

2.2. Step 2

We now are in a position to perform a more precise version of the argument in Step 1, this time using N in place of $\widetilde{N}$. Step 1 is used to improve compactness for the less well-behaved quantity N.

Lemma 2.3. *Suppose* $\liminf_{r\to 0} N_u(r) < \infty$. *Then for any* $\mu \notin \{m_j\}$, *there exists some* $r_0 = r_0(d, \mu, \Omega, u)$ *such that* $N_u(r) \le 4^{2\mu}$ *implies* $N_u(\tau r) \le 4^{2\mu}$ *for any* $r < r_0$ *and any* $\tau \in [1/16, 1/4]$.

Proof. We prove this by contradiction. Suppose the contrary, that there exist sequences $r_k \to 0$ and $\tau_k \in [1/16, 1/4]$ such that

$$N_u(r_k) \le 4^{2\mu} \quad \text{and} \quad N_u(\tau_k r_k) > 4^{2\mu}. \tag{2.5}$$

Recall from Step 1 that, for all large enough k,

$$N_u(4r_k) \le 2 \limsup_{r\to 0} N_u(r) \le 2C(\liminf_{r\to 0} N_u(r))^4 = 2CM^4, \tag{2.6}$$

where we denote $M := \liminf_{r\to 0} N_u(r)$. Let

$$u_k(y) := \frac{u(r_k y)}{\left(\fint_{\partial B_{r_k/4}} |u|^2\right)^{1/2}}.$$

Then

$$\fint_{\partial B_{1/4}} |u_k|^2 = 1 \quad \text{and} \quad \begin{cases} \Delta u_k = 0 & \text{in } (\Omega/r_k) \cap B_4, \\ u_k = 0 & \text{on } B_4 \setminus (\Omega/r_k). \end{cases}$$

Combining with (2.5) and $\Delta(u_k)^2 \geq 0$, we obtain $\fint_{\partial B_1} |u_k|^2 \leq 4^{2\mu} \fint_{\partial B_{1/4}} |u_k|^2 = 4^{2\mu}$. Hence, also noting (2.6), we reach

$$\fint_{B_4} |u_k|^2 \leq \fint_{\partial B_4} |u_k|^2 \leq 2CM^4 \fint_{\partial B_1} |u_k|^2 \leq 2CM^4 4^{2\mu}. \tag{2.7}$$

From the Caccioppoli inequality and the Sobolev embedding, passing to a subsequence, for all $\varepsilon \in (0, 4)$,

$$u_k \to u_\infty \quad \text{weakly in } L^2(B_4), H^1(B_{4-\varepsilon}), \text{ strongly in } L^2(B_{4-\varepsilon}).$$

Passing to further subsequences, we can also obtain that $\tau_k \to \tau_\infty \in [1/16, 1/4]$ and

$$u_k \to u_\infty \quad \text{strongly in } L^2(\partial B_{\tau_\infty}), L^2(\partial B_{\tau_\infty/4}), L^2(\partial B_1), L^2(\partial B_{1/4}). \tag{2.8}$$

Hence,

$$N_{u_\infty}(1) = \frac{\fint_{\partial B_1} |u_\infty|^2}{\fint_{\partial B_{1/4}} |u_\infty|^2} = \lim_{k\to\infty} \frac{\fint_{\partial B_1} |u_k|^2}{\fint_{\partial B_{1/4}} |u_k|^2} = \lim_{k\to\infty} N_u(r_k) \leq 4^{2\mu}. \tag{2.9}$$

Here, in the last inequality we used (2.5). Next, we show

$$\fint_{\partial B_{\tau_k}} |u_k|^2 \to \fint_{\partial B_{\tau_\infty}} |u_\infty|^2 \quad \text{and} \quad \fint_{\partial B_{\tau_k/4}} |u_k|^2 \to \fint_{\partial B_{\tau_\infty/4}} |u_\infty|^2. \tag{2.10}$$

For the first limit, we estimate

$$\left| \fint_{\partial B_{\tau_k}} |u_k|^2 - \fint_{\partial B_{\tau_\infty}} |u_k|^2 \right| = \left| \fint_{\partial B_1} (|u_k(\tau_k x)|^2 - |u_k(\tau_\infty x)|^2)\, d\sigma_x \right| \tag{2.11}$$

$$= \left| \fint_{\partial B_1} \int_{\tau_\infty}^{\tau_k} \frac{d}{dr} |u_k(rx)|^2 \, dr \, d\sigma_x \right| \leq C \|u_k\|_{L^\infty(B_{1/4})} \left| \fint_{\partial B_1} \int_{\tau_\infty}^{\tau_k} |\nabla u_k(rx)| \, dr \, d\sigma_x \right|.$$

From the mean value property for u_k^2 (which is subharmonic) and the inequality (2.7), we have $\|u_k\|_{L^\infty(B_{1/4})} \leq C_\mu$. Hence,

$$\begin{aligned} \text{RHS of (2.11)} &\leq C \left| \fint_{\partial B_1} \int_{\tau_\infty}^{\tau_k} |\nabla u_k(rx)| \, dr \, d\sigma_x \right| \\ &\leq C \left| \int_{\tau_\infty}^{\tau_k} \fint_{\partial B_r} |\nabla u_k| \, d\sigma \, dr \right| \leq C \|\nabla u_k\|_{L^2(B_{1/4})} \sqrt{|\tau_k - \tau_\infty|} \to 0. \end{aligned}$$

The last step used that ∇u_k is uniformly bounded in $L^2(B_{1/4})$. Combining with (2.8), we have

$$\begin{aligned} &\left| \fint_{\partial B_{\tau_k}} |u_k|^2 - \fint_{\partial B_{\tau_\infty}} |u_\infty|^2 \right| \\ &\qquad \leq \left| \fint_{\partial B_{\tau_k}} |u_k|^2 - \fint_{\partial B_{\tau_\infty}} |u_k|^2 \right| + \left| \fint_{\partial B_{\tau_\infty}} |u_k|^2 - \fint_{\partial B_{\tau_\infty}} |u_\infty|^2 \right| \to 0. \end{aligned}$$

This proves the first convergence in (2.10). The proof for the second convergence is almost identical. From (2.10) and (2.5),

$$N_{u_\infty}(\tau_\infty) = \frac{\fint_{\partial B_{\tau_\infty}} |u_\infty|^2}{\fint_{\partial B_{\tau_\infty/4}} |u_\infty|^2} = \lim_{k\to\infty} \frac{\fint_{\partial B_{\tau_k}} |u_k|^2}{\fint_{\partial B_{\tau_k/4}} |u_k|^2} = \lim_{k\to\infty} N_u(\tau_k r_k) \geq 4^{2\mu}. \tag{2.12}$$

As before, u_∞ satisfies (2.4). From (2.9), (2.12), and the rigidity in Lemma 2.2, we must have $N_{u_\infty} \equiv 16^m$ for some $m \in \{m_j\}$. Hence, $\mu = m$, a contradiction. ■

2.3. Step 3: Conclusion of the proof of Theorem 1.3

We may as well assume that $\liminf N_u(r) < +\infty$. Recall that $N_u(r) \geq 1$ for all r, and let $m := 2^{-1} \log_4(\liminf_{r\to 0} N_u(r))$. We have $m \in [0, \infty)$.

Now, we find a sequence of positive numbers $\varepsilon_k \to 0$ such that $m + \varepsilon_k \notin \{m_j\}$. For each k, we further find a small enough r_k with $N_u(r_k) < m + \varepsilon_k$ and $r_k < r_0(d, m + \varepsilon_k, \Omega, u)$ (where r_0 is given in Lemma 2.3). Applying Lemma 2.3 iteratively, we have that $\sup_{r \leq r_k/4} N_u(r) \leq 4^{2(m+\varepsilon_k)}$, and so in particular $\limsup_{r\to 0} N_u(r) \leq 4^{2(m+\varepsilon_k)}$. Sending $k \to \infty$,

$$\limsup_{r\to 0} N_u(r) \leq 4^{2m} = \liminf_{r\to 0} N_u(r).$$

This implies, passing to a subsequence, $u_r = u(r\cdot)/(\fint_{\partial B_r} |u|^2)^{1/2}$ converges to a non-trivial, homogeneous harmonic function on $B_1 \cap \Gamma$, with the homogeneity m. This implies that m must be one of the characteristic constants defined in (1.2).

3. Eigenvalues, eigenfunctions, and Green's functions on cones

Before proving Theorem 1.6. we make some preparatory remarks concerning the Green's function on the limit cone. Let $\Gamma \subset \mathbb{R}^d$ be a cone with vertex at the origin and let $\Sigma = \Gamma \cap \partial B_1$ be its spherical cross-section.

3.1. Eigenvalues and eigenfunctions of spherical cross-sections

Let $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq \cdots$ be the Dirichlet eigenvalues (counting multiplicity) of the spherical cross-section Σ and let $\{\psi_k\}_{k=1}^\infty$ be a corresponding basis of eigenfunctions, orthonormal in L^2. We have the following properties.

Lemma 3.1. *For each $q > (d-1)/4$, there exists $C = C(q, d, \Sigma) > 0$, independent of k, such that*

$$\|\psi_k\|_{L^\infty(\Sigma)} \leq C\lambda_k^q.$$

Proof. Applying the local maximum principle to $v = |\psi_k|$, which is a weak subsolution of $-\Delta_{S^{n-1}} v \leq \lambda_k v$, in charts (see, e.g., Theorem 8.17 in [7]),

$$\|\psi_k\|_{L^\infty(\Sigma)} \leq C[\|\lambda_k \psi_k\|_{L^{2q}} + \|\psi_k\|_{L^2}]$$

noting $2q > (d-1)/2$. Then

$$\|\lambda_k \psi_k\|_{L^{2q}} \leq \lambda_k \|\psi_k\|_{L^\infty}^{(q-1)/q} \|\psi_k\|_{L^2}^{1/q} \leq \varepsilon \|\psi_k\|_{L^\infty} + C_\varepsilon \lambda_k^q,$$

using that $\|\psi_k\|_{L^2} = 1$. Choosing ε small and reabsorbing the first term gives

$$\|\psi_k\|_{L^\infty(\Sigma)} \le C\lambda_k^q. \qquad \blacksquare$$

Lemma 3.2. *For some $C = C(d)$, $\lambda_k \ge \frac{1}{C} k^{2/(d-1)}$.*

Proof. Since $\Sigma \subset \partial B_1$, we have $\lambda_k \ge \lambda_k(S^{d-1})$, where $\lambda_k(S^{d-1})$ is the kth eigenvalue of Laplacian operator on a $(d-1)$-dimensional unit sphere. We know that $\{\lambda_k(S^{d-1})\}_k$ contains exactly one zero and $\binom{d+j-1}{d-1} - \binom{d+j-3}{d-1} \approx j^{d-2}$ copies of $j(j+d-2)$, for $j = 1, 2, \ldots$ Hence, counting all eigenvalues up to the size $j(j+d-2) \approx j^2$, we reach

$$\lambda_k \ge \frac{1}{C}\left(\left(\frac{k}{C}\right)^{1/(d-1)} - 1\right)^2 \ge \frac{1}{C} k^{2/(d-1)}. \qquad \blacksquare$$

As a direct corollary of Lemma 3.2,

$$\sum_{j=1}^{\infty} \mu^{\sqrt{\lambda_k}} < \infty, \quad \forall \mu \in (0,1). \tag{3.1}$$

3.2. Green's function on a cone and orthogonal expansions

From standard elliptic regularity theory, the Green's function $G(x, y)$ exists on an arbitrary Lipschitz cone Γ. More precisely, for every $x \in \Gamma \cap B_1$, $f, g_i \in L^\infty(\Gamma \cap B_1)$, and $h \in C_c^0(\partial\Gamma \cap B_1)$, the unique continuous weak solution to

$$\begin{cases} \Delta u = f + \partial_i g_i & \text{in } \Gamma \cap B_1, \\ u = h & \text{on } \partial(\Gamma \cap B_1), \\ u \to 0 & \text{as } |x| \to \infty \end{cases}$$

can be represented by

$$u(x) = \int_\Gamma G(x,y) f(y)\, dy - \int_\Gamma \frac{\partial}{\partial y_i} G(x,y) g_i(y)\, dy + \int_{\partial\Gamma} h(y) \frac{\partial}{\partial \mathbf{n_y}} G(x,y)\, d\sigma_y.$$

See, for instance, Theorem 1.1 in [8] and [13]. See also [12] for discussions on unbounded domains. The following properties are standard: symmetry $G(x,y) = G(y,x)$; scaling $G(\lambda x, \lambda y) = \lambda^{2-d} G(x,y)$; and a pointwise bound

$$G(x,y) \le C \frac{1}{|x-y|^{d-2}}. \tag{3.2}$$

Furthermore, we have the following derivative bounds when Γ is regular enough.

Lemma 3.3. *Suppose that Γ satisfies Assumption 1.5. Then for $x, y \in \Gamma$, $x \ne y$, we have*

$$|\nabla_y G(x,y)| \le C\,(|y|^{-1} + |x-y|^{-1})\,|x-y|^{2-d}, \tag{3.3}$$

$$|\nabla_y G(x,y)| \le C\delta(x)\,(|x|^{-1} + |x-y|^{-1})\,(|y|^{-1} + |x-y|^{-1})|x-y|^{2-d}, \tag{3.4}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Gamma)$.

The proof is standard, and it is based on scaling, the point-wise bound (3.2), and a local Lipschitz estimate coming from the smoothness of Σ. For completeness, we provide a proof in Appendix A. Throughout the rest of the paper, we denote

$$K_i(x,y) := \frac{\partial G(x,y)}{\partial y_i} \quad \text{for } x, y \in \Gamma, x \neq y,$$
$$k(x,y) := \frac{\partial G(x,y)}{\partial \mathbf{n}_y} \quad \text{for } x \in \Gamma, y \in \partial\Gamma \setminus \{0\}.$$

From Lemma 3.3, we know that $K_i(\cdot, y), k(\cdot, y) \in L^\infty_{\text{loc}}(B_{|y|})$ are harmonic functions, which have orthogonal expansions

$$K_i(x,y) = \sum_{j=1}^\infty b_i^{(j)}(y)\,|x|^{m_j}\,\psi_j(x/|x|) \quad \text{and} \quad K(x,y) = \sum_{j=1}^\infty b^{(j)}(y)\,|x|^{m_j}\,\psi_j(x/|x|),$$

where

$$b_i^{(j)}(y) = \frac{\int_{\partial B_{2|y|/3}\cap\Gamma} K_i(z,y)|z|^{m_j}\,\psi_j(z/|z|)\,dz}{\int_{\partial B_{2|y|/3}\cap\Gamma}(|z|^{m_j}\,\psi_j(z/|z|))^2\,dz},$$
$$b^{(j)}(y) = \frac{\int_{\partial B_{2|y|/3}\cap\Gamma} k(z,y)|z|^{m_j}\,\psi_j(z/|z|)\,dz}{\int_{\partial B_{2|y|/3}\cap\Gamma}(|z|^{m_j}\,\psi_j(z/|z|))^2\,dz}.$$

Here, $2/3$ could have been any fixed number smaller than 1. By scaling,

$$b_i^{(j)}(y) = |y|^{1-d-m_j}\,b_i^{(j)}(y/|y|) \quad \text{and} \quad b^{(j)}(y) = |y|^{1-d-m_j}\,b^{(j)}(y/|y|). \tag{3.5}$$

Denote the partial sums as

$$K_i^{(N)}(x,y) = \sum_{j\le N} b_i^{(j)}(y/|y|)|y|^{1-d-m_j}\,|x|^{m_j}\,\psi_j(x/|x|),$$
$$k^{(N)}(x,y) = \sum_{j\le N} b^{(j)}(y/|y|)|y|^{1-d-m_j}\,|x|^{m_j}\,\psi_j(x/|x|). \tag{3.6}$$

We have the following estimates.

Lemma 3.4. *Suppose that Γ satisfies Assumption 1.5 and that $b_i^{(j)}$ and $b^{(j)}$ are defined as above. Then for some $C = C(d,\Sigma)$,*

$$|b_i^{(j)}(y/|y|)| + |b^{(j)}(y/|y|)| \le C\,(\lambda_j)^{d/4}\,(3/2)^{m_j}, \quad j = 1, 2, \ldots. \tag{3.7}$$

Furthermore, we have remainder estimates: for some $C = C(d,\Sigma,N)$,

$$|K_i(x,y) - K_i^{(N)}(x,y)| \le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}, \quad \forall y \in \Gamma \setminus \{0\}, x \in \Gamma \cap B_{|y|/2},$$
$$|k(x,y) - k^{(N)}(x,y)| \le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}, \quad \forall y \in \partial\Gamma \setminus \{0\}, x \in \Gamma \cap B_{|y|/2}. \tag{3.8}$$

Here, (3.7) is not sharp in general, but as it is enough for later use, we do not pursue more precise bounds.

Proof. We only prove for K_i and $b_i^{(j)}$ as the computation for k and $b^{(j)}$ is almost identical. For (3.7), by the scaling property (3.5) and $\|\psi_j\|_{L^2} = 1$,

$$\begin{aligned}|b_i^{(j)}(y/|y|)| &= (3/2)^{-1+d+m_j}\,|b_i^{(j)}(3y/(2|y|))|\\ &= (3/2)^{-1+d+m_j}\left|\int_{\Gamma\cap\partial B_1} K_i(w, 3y/(2|y|))\,\psi_j(w)\,dw\right|\\ &\le (3/2)^{-1+d+m_j}\,C\,(1/2)^{1-d}\,\|\psi_j\|_{L^\infty} \le C\,\lambda_j^{d/4}\,(3/2)^{m_j}.\end{aligned}$$

Here we have also used the point-wise bound (3.3) and Lemma 3.1 with $q = d/4$. Next, we prove (3.8):

$$\begin{aligned}|K_i(x,y) - K_i^{(N)}(x,y)| &= \left|\sum_{j=N+1}^{\infty} b_i^{(j)}(y/|y|)\,|y|^{1-d-m_j}\,|x|^{m_j}\,\psi_j(x/|x|)\right|\\ &\le \frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}\sum_{j=N+1}^{\infty}|b_i^{(j)}(y/|y|)|\,(|x|/|y|)^{m_j-m_{N+1}}\,\|\psi_j\|_{L^\infty}\\ &\le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}\sum_{j=N+1}^{\infty}(3/2)^{m_j}\,(1/2)^{m_j-m_{N+1}}\,\lambda_j^{d/2}.\end{aligned}$$

In the last inequality, we used (3.7), $|x| < |y|/2$, and Lemma 3.1. Hence,

$$\begin{aligned}|K_i(x,y) - K_i^{(N)}(x,y)| &\le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}\sum_{j=N+1}^{\infty}(3/4)^{m_j}\,\lambda_j^{d/2}\\ &\le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}\sum_{j=N+1}^{\infty}(3/4)^{m_j-(d/2)\log_{4/3}(\lambda_j)}\\ &\le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}\sum_{j=N+1}^{\infty}(3/4)^{\sqrt{\lambda_j}/2}\\ &\le C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}\sum_{j=1}^{\infty}(3/4)^{\sqrt{\lambda_j}/2} = C\,\frac{|x|^{m_{N+1}}}{|y|^{m_{N+1}+d-1}}.\end{aligned}$$

Here we have also used (1.2) and (3.1). The lemma is proved. ∎

4. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. From Assumption 1.5, we first fix coordinates $x = (x', x_d)$ such that the tangent cone Γ at the origin can be locally represented by $\{x_d > \Psi(x')\}$ for some 1-homogeneous function Ψ. By (1.5), we know that for sufficiently large C, locally,

$$\mathcal{C} := \{(x', x_d) : x_d > \Psi(x') + C|x'|^{1+\alpha}\} \subset \Omega. \tag{4.1}$$

The following De Giorgi-type estimate plays a key role in our proof.

Lemma 4.1. *Assume Ω together with its tangent cone at the origin Γ satisfy assumptions of Theorem 1.6, and let $\mathscr{C}$ be defined as above. Let $u \in H^1_{\mathrm{loc}}$ satisfy*

$$\partial_i(a_{ij}\partial_j u) = 0 \ \text{ in } \Omega \cap B_2 \quad \text{and} \quad u = 0 \ \text{ on } \partial\Omega \cap B_2, \tag{4.2}$$

with bounded and measurable coefficients $a_{ij} = a_{ij}(x)$ satisfying, for some $\lambda, \Lambda > 0$,

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \text{and} \quad |a_{ij}| \leq \Lambda. \tag{4.3}$$

Then there exists a constant $\alpha_0 \in (0,1)$, such that for all small enough $r > 0$,

$$\sup_{B_r \cap (\Omega \setminus \mathscr{C})} |u| \leq C r^{\alpha\alpha_0} \sup_{B_{2r}\cap\Omega} |u|.$$

Proof. Fix a point $x \in B_r \cap (\Omega \setminus \mathscr{C})$. From the definition of $\mathscr{C}$ and (1.5), there exists a constant $C_1 > 0$, such that for all sufficiently small r, we have

$$\mathrm{dist}(\partial\Omega \cap B_r, \partial\mathscr{C} \cap B_r) \leq C_1 r^{1+\alpha}$$

and

$$|B_s(x) \cap \Omega^c| \geq c_0|B_s|, \quad \forall s \in (10C_1 r^{1+\alpha}, r/2). \tag{4.4}$$

Now, from $u = 0$ on $\partial\Omega$, $B_{10C_1r^{1+\alpha}}(x) \cap \partial\Omega \neq \emptyset$, and the De Giorgi improvement of oscillation lemma, we have that, for some $\alpha_0 \in (0,1)$,

$$|u(x) - 0| \leq \mathrm{osc}_{\Omega\cap B_{10C_1r^{1+\alpha}}(x)}(u) \leq C\left(\frac{10C_1r^{1+\alpha}}{r/2}\right)^{\alpha_0} \sup_{B_{r/2}(x)} |u| \leq C r^{\alpha\alpha_0} \sup_{B_{2r}} |u|.$$

See for instance Theorem 8.27 in [7]. Note that in the proof of Theorem 8.27 in [7], the exterior cone condition can be replaced by the exterior measure condition. Moreover, here we only need the estimate up to the scale $C_1r^{1+\alpha}$ instead of zero, and at these scales, the exterior measure condition is true due to (4.4). ∎

Proposition 4.2. *Let Ω and Γ satisfy assumptions of Theorem 1.6, and in addition, for some small $R > 0$, $\Gamma \cap B_R \subset \Omega \cap B_R$. Let $m_N < m_{N+1}$ be two distinct characteristic constants of Γ. Suppose that $u \in H^1_{\mathrm{loc}}$ satisfies (4.2) with coefficients satisfying (4.3) and*

$$|a_{ij}(x) - \delta_{ij}| = O(|x|^\beta), \quad \text{as } x \to 0 \tag{4.5}$$

for some $\beta > 0$. Then, if $u(x) = O(|x|^\mu)$ for some $\mu \in [m_N, m_{N+1})$, we have

$$\begin{aligned} &u(x) = C|x|^{m_N}\psi_N\left(\frac{x}{|x|}\right) + w(x) \quad \text{in } \Gamma, \\ &\text{with } \left(\fint_{B_r\cap\Gamma} |w|^p\right)^{1/p} = O(r^{\min\{\mu+\alpha_0\min\{\alpha,\beta\}, m_{N+1}\}}), \end{aligned} \tag{4.6}$$

where $p > 2d/(d-2)$ and $\alpha_0 \in (0,1)$ are constants depending only on $(\Omega, d, \lambda, \Lambda)$. Furthermore, when $\mu \in (m_N, m_{N+1})$, we must have $C \equiv 0$.

Assuming Proposition 4.2, we prove Theorem 1.6.

Proof of Theorem 1.6. From Theorem 1.3, $\lim_{r\to 0} N_u(r) \in (0,\infty]$ exists. Clearly, we only need to consider the case of $\lim_{r\to 0} N_u < \infty$. We claim that there exists a characteristic constant m_N such that

$$|u(x)| = O(|x|^{m_N}) \quad \text{and} \quad |u(x)| \neq O(|x|^{m_{N+1}}). \tag{4.7}$$

Indeed, if $N_u(r) \le M$, we have $⨍_{B_{2^{-k}}} u^2 \ge cM^{-k}$ from Lemma 2.1, and so

$$\|u\|^2_{L^\infty(B_r)} \ge ⨍_{B_r} u^2 \ge cr^\beta$$

for some $\beta > 0$. This shows that $|u| \neq O(|x|^\mu)$ for some $\mu < \infty$. We still need to show $u = O(|x|^{m_1})$, which can be done by constructing a barrier function. By (1.5), we know that for sufficiently large C, locally,

$$\Omega \subset \mathcal{C}_1 := \{(x', x_d) : x_d > \Psi(x') - C|x'|^{1+\alpha}\}.$$

Now for sufficiently small $R > 0$, let $u_1 \in H^1_{\mathrm{loc}}$ be the solution to

$$\Delta u_1 = 0 \text{ in } \mathcal{C}_1 \cap B_R, \quad u_1 = 0 \text{ on } \partial\mathcal{C}_1 \cap B_R, \quad u_1 = 1 \text{ on } \mathcal{C}_1 \cap \partial B_R,$$

where the boundary condtions are understood in the sense of non-tangential limit. Such solution exists since $\mathcal{C}_1 \cap B_R$ is a Lipschitz domain. Now, $\mathcal{C}_1$ is a "$C^{1,\alpha}$ perturbation" of the cone Γ, which verifies the assumptions in Theorem 1.1 of [5]. Hence, for some nontrivial homogeneous harmonic function P_{m_1} of degree m_1, we have $(0 <)u_1 = P_{m_1}(x) + o(|x|^{m_1})$, noting that m_1 is the characteristic constant associated with the leading eigenvalue of the tangent cone Γ. Now, since $\Omega \cap B_R \subset \mathcal{C}_1 \cap B_R$ by choosing R small enough, by comparison, we have $|u| \le u_1 = O(|x|^{m_1})$. Combining these, we have proved (4.7).

Now we prove (4.6). Take a $C^{1,\alpha}$ change of variables

$$(\tilde{x}', \tilde{x}_d) = \Phi(x', x_d) = (x', x_d - C|x'|^{1+\alpha}),$$

where C is the number given in (4.1). It is easy to see that $\widetilde{\Omega} := \Phi(\Omega)$ is still α-conical with the tangent cone $\widetilde{\Gamma} := \Phi(\mathcal{C})$, which satisfies Assumption 1.5. More importantly, now the tangent cone $\widetilde{\Gamma}$ is locally contained in $\widetilde{\Omega}$.

In the new coordinates,

$$\tilde{u} := u \circ \Phi^{-1}$$

satisfies (4.2) locally on $\widetilde{\Omega}$, with coefficients verifying all the conditions in Proposition 4.2. Moreover, we still have $\tilde{u} = O(|\tilde{x}|^{m_N})$ since Φ is locally a diffeomorphism. Applying Proposition 4.2 with $\mu = m_N$ to $\tilde{u}$ on $\widetilde{\Omega}$, we obtain a homogeneous harmonic function $P(\tilde{x}) = C_1|\tilde{x}|^{m_N}\psi_N(\tilde{x}/|\tilde{x}|)$ on $\widetilde{\Gamma}$, such that

$$\left(⨍_{\widetilde{\Gamma}\cap B_r} |\tilde{u}(\tilde{x}) - P(\tilde{x})|^p \, d\tilde{x}\right)^{1/p} \le Cr^{\min\{m_N+\alpha_0\alpha,\, m_{N+1}\}}. \tag{4.8}$$

We first show the leading term $P \not\equiv 0$, i.e., $C_1 \neq 0$. Suppose the contrary, that is, $C_1 = 0$. Again from the fact that Φ is locally a diffeomorphism, by Lemma 4.1, we have

$$\|\tilde{u}\|_{L^\infty(B_r\cap(\widetilde{\Omega}\setminus\widetilde{\Gamma}))} \le C\|u\|_{L^\infty(B_{Cr}\cap(\Omega\setminus\mathcal{C}))} \le Cr^{\alpha\alpha_0} \sup_{B_{2Cr}} |u| \le Cr^{m_N+\alpha\alpha_0}.$$

Combining this, (4.8), and a local maximum principle, we reach $|u(x)| = O(|x|^{\mu_1})$, where $\mu_1 := \min\{m_N + \alpha_0\alpha, m_{N+1}\}$. If $\mu_1 = m_{N+1}$, this is a contradiction. Otherwise, repeating the above procedure, but now applying Proposition 4.2 with $\mu = \mu_1$ instead of m_N, we can further improve the vanishing order of u, and in finitely many steps reach $|u| = O(|x|^{m_{N+1}})$, which is again a contradiction. Hence, $C_1 \neq 0$.

Set

$$w(x) = u(x) - P(x),$$

where u and P are extended by zero outside Ω and Γ, respectively. We are left to show

$$\|w\|_{L^\infty(B_r)} \leq C r^{\min\{m_N+\alpha_0\alpha, m_{N+1}\}}.$$

Note that both u and P satisfy assumptions of Lemma 4.1 (for P, we take $\Omega = \Gamma$). Hence,

$$\sup_{B_r\setminus\mathcal{C}} |w| \leq \sup_{B_r\setminus\mathcal{C}} (|u| + |P|) \leq C r^{m_N+\alpha\alpha_0}.$$

Here, we also used $u = 0$ on $B_r \setminus \Omega$ and $P = 0$ on $B_r \setminus \Gamma$. To bound w on $B_r \cap \mathcal{C}$, we transform (4.8) back to x-coordinates and apply the triangle inequality. This gives

$$\begin{aligned}
&\Big(\fint_{\mathcal{C}\cap B_r} |u(x) - P(x)|^p \, dx\Big)^{1/p} \\
&\leq \Big(\fint_{\mathcal{C}\cap B_r} |u(x) - P\circ\Phi(x)|^p \, dx\Big)^{1/p} + \Big(\fint_{\mathcal{C}\cap B_r} |P(x) - P\circ\Phi(x)|^p \, dx\Big)^{1/p} \\
&\leq C\Big(\fint_{\tilde{\Gamma}\cap B_{Cr}} |u\circ\Phi^{-1} - P|^p\Big)^{1/p} + \sup_{B_r\cap\mathcal{C}} (|x - \Phi(x)||\nabla\Phi|) \leq C r^{\min\{m_N+\alpha_0\alpha, m_{N+1}\}}.
\end{aligned}$$

Here, we also used $|x - \Phi(x)| \leq C|x|^{1+\alpha}$ and $|\nabla P| = O(|x|^{m_N-1})$. Now, note that

$$\Delta w = \Delta(u - P) = 0 \text{ in } B_1 \cap \mathcal{C}, \quad |w| = |u - P| \leq C|x|^{m_N+\alpha\alpha_0} \text{ on } B_1 \cap \partial\mathcal{C}.$$

We apply a local maximum principle to $(w - Cr^{m_N+\alpha\alpha_0})_+$ on $B_{2r} \cap \mathcal{C}$, which is subharmonic in $B_{2r} \cap \mathcal{C}$ and vanishes on $B_{2r} \cap \partial\mathcal{C}$ by choosing C large enough, to obtain

$$\|(w - Cr^{m_N+\alpha\alpha_0})_+\|_{L^\infty(B_r\cap\mathcal{C})} \leq Cr^{d/2}\|(w - Cr^{m_N+\alpha\alpha_0})_+\|_{L^2(B_r\cap\mathcal{C})} \leq Cr^{m_N+\alpha\alpha_0}.$$

Similarly, we can bound $\|(-w - Cr^{m_N+\alpha\alpha_0})_+\|_{L^\infty(B_r\cap\mathcal{C})}$, and hence, as desired, $\|w\|_{L^\infty(B_r\cap\mathcal{C})} \leq Cr^{m_N+\alpha\alpha_0}$. ∎

The rest of Section 4 will be devoted to the proof of Proposition 4.2.

4.1. A representation by Green's function

Let $\eta \in C_c^\infty(B_R)$ be a usual cut-off function with $\eta = 1$ on $B_{R/2}$. Let

$$v := -\int_\Gamma K_i(x, y) f_i(y) \, dy + \int_{\partial\Gamma} k(x, y) h(y) \, d\sigma_y \; := -\mathrm{I} + \mathrm{II},$$

where $f_i = \eta(\delta_{ij} - a_{ij})\,\partial_{x_j} u$, $h = u\eta$, and K_i and k are the kernels defined in Section 3.2. Now, $u - v$ satisfies

$$\begin{cases} \Delta(u-v) = 0 & \text{in } \Gamma \cap B_{R/2}, \\ u - v = 0 & \text{on } \partial\Gamma \cap B_{R/2}. \end{cases}$$

Hence, we have the full series expansion

$$u - v = \sum_{j=1}^{\infty} C_j\,|x|^{m_j}\,\psi_j(x/|x|) = \sum_{j=1}^{N} C_j\,|x|^{m_j}\,\psi_j(x/|x|) + O(|x|^{m_{N+1}}).$$

Next, we expand v. First, from Lemma 4.1, we have

$$\sup_{B_r \cap \partial\Gamma} |h| = O(r^{\mu+\alpha\alpha_0}). \tag{4.9}$$

By (4.5),

$$|f_i| = \eta|\delta_{ij} - a_{ij}|\,|\partial_{x_j} u| \le C\,|x|^{\beta}\,|\nabla u|. \tag{4.10}$$

4.2. Estimating the term I

Recall the definition for $K_i^{(N)}$ in (3.6). We split

$$\begin{aligned} \mathrm{I} = &\sum_{i=1}^{d} \int_{\Gamma} K_i^{(N)}(x,y) f_i(y)\,dy - \sum_{i=1}^{d} \int_{\Gamma \cap B_{2|x|}} K_i^{(N)}(x,y) f_i(y)\,dy \\ &+ \sum_{i=1}^{d} \int_{\Gamma \cap B_{2|x|}^c} (K_i - K_i^{(N)})(x,y) f_i(y)\,dy + \sum_{i=1}^{d} \int_{\Gamma \cap B_{2|x|}} K_i(x,y) f_i(y)\,dy \\ &:= \mathrm{I}_1 - \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4. \end{aligned}$$

In the following,we prove that I_1 is a finite combination of homogeneous harmonic functions with degree up to m_N, and the rest three terms I_2, I_3 and I_4 are of higher order.

Convergence and expansion of I_1.

For each $j = 1, \ldots, N$ and $i = 1, \ldots, d$, from (3.7) and (4.10),

$$|b_i^{(j)}(y/|y|)|\,|y|^{1-d-m_j}\,|f(y)| \le C\,|y|^{1-d-m_j+\beta}\,|\nabla u(y)|,$$

where C is a constant that could depend on m_j, λ_j.

Hence,

$$\begin{aligned} &\int_{\Gamma \cap B_R} \left| b_i^{(j)}\left(\frac{y}{|y|}\right)\right| |y|^{1-d-m_j} |f|\,dy \\ &\qquad \le C \sum_{l=0}^{\infty} (2^{-l}R)^{1-d-m_j+\beta}\,(2^{-l}R)^d \fint_{\Gamma \cap (B_{2^{-l}R} \setminus B_{2^{-l-1}R})} |\nabla u| \\ &\qquad \le C \sum_{l=0}^{\infty} (2^{-l}R)^{1-m_j+\beta} \left(\fint_{\Gamma \cap (B_{2^{-l}R} \setminus B_{2^{-l-1}R})} |\nabla u|^2 \right)^{1/2}. \end{aligned} \tag{4.11}$$

By the Caccioppoli inequality, $u = O(|x|^\mu)$, and $m_j \le \mu$, we can further compute

$$\begin{aligned}
\text{RHS of (4.11)} &\le C \sum_{l=0}^{\infty} (2^{-l} R)^{1-m_j+\beta}\, (2^{-l} R)^{-1}\, \|u\|_{L^\infty(\Gamma\cap B_{2^{-l+1}R})} \\
&\le C \sum_{l=0}^{\infty} (2^{-l} R)^{-m_j+\beta}\, (2^{-l+1} R)^{\mu} \\
&\le C\, R^{\mu-m_j+\beta} \sum_{l=0}^{\infty} 2^{-l\beta}\, 2^{-l(\mu-m_j)} \qquad (4.12)\\
&\le C\, R^{m_N-m_j+\varepsilon+\beta} \sum_{l=0}^{\infty} 2^{-l\beta} \le C\, R^{\mu-m_j+\beta}. \qquad (4.13)
\end{aligned}$$

This proves that the integrands in I_1 are in L^1. Hence,

$$\begin{aligned}
\mathrm{I}_1 &= \sum_{i=1}^{d}\sum_{j=1}^{N} \int_\Gamma b_i^{(j)}\Big(\frac{y}{|y|}\Big)\, |y|^{1-d-m_j}\, |x|^{m_j}\, \psi_j\Big(\frac{x}{|x|}\Big)\, f_i(y)\, dy \\
&= \sum_{j=1}^{N} |x|^{m_j}\, \psi_j\Big(\frac{x}{|x|}\Big)\Big(\sum_{i=1}^{d} \int_\Gamma b_i^{(j)}\Big(\frac{y}{|y|}\Big)\, |y|^{1-d-m_j}\, f_i(y)\, dy\Big)
\end{aligned}$$

is a combination of homogeneous harmonic functions of degree up to m_N.

Smallness of I_2.

From Lemma 3.1 and (4.11)–(4.13) with R replaced by $2|x|$, we have

$$\begin{aligned}
|\mathrm{I}_2| &\le C \sum_{j=1}^{N} |x|^{m_j}\, |\psi_j| \int_{\Gamma\cap B_{2|x|}} |b_i^{(j)}|(y/|y|)\, |y|^{1-d-m_j}\, |f|\, dy \\
&\le C \sum_{j=1}^{N} |x|^{m_j}\, \lambda_j^{d/4}\, (2|x|)^{\mu-m_j+\beta} \le C|x|^{\mu+\beta}.
\end{aligned}$$

Smallness of I_3.

By (3.8) and (4.10),

$$\begin{aligned}
|\mathrm{I}_3| &\le C|x|^{m_{N+1}} \int_{\Gamma\cap\{2|x|\le|y|<R\}} |y|^{1-d-m_{N+1}+\beta}\, |Du(y)|\, dy \\
&\le C|x|^{m_{N+1}} \sum_{l=1}^{\log_2(R/|x|)} (2^l|x|)^{1-d-m_{N+1}+\beta}\, (2^l|x|)^d \fint_{\Gamma\cap(B_{2^{l+1}|x|}\setminus B_{2^l|x|})} |\nabla u(y)|\, dy \\
&\le C|x|^{m_{N+1}} \sum_{l=1}^{\log_2(R/|x|)} (2^l|x|)^{1-m_{N+1}+\beta} \Big(\fint_{\Gamma\cap(B_{2^{l+1}|x|}\setminus B_{2^l|x|})} |\nabla u(y)|^2\, dy\Big)^{1/2}. \qquad (4.14)
\end{aligned}$$

Again by the Caccioppoli inequality and $u = O(|x|^{\mu})$, we obtain

$$\text{RHS of (4.14)} \le C|x|^{m_{N+1}} \sum_{l=1}^{\log_2(R/|x|)} (2^l|x|)^{1-m_{N+1}+\beta}(2^l|x|)^{-1}\|u\|_{L^\infty(\Gamma\cap(B_{2^{l+1}|x|}\setminus B_{2^l|x|}))}$$
$$\le C|x|^{m_{N+1}} \sum_{l=1}^{\log_2(R/|x|)} (2^l|x|)^{-m_{N+1}+\beta}(2^l|x|)^{\mu} \le C|x|^{\mu+\beta} \sum_{l=1}^{\log_2(R/|x|)} 2^{l(\mu-m_{N+1}+\beta)}$$
$$\le C|x|^{\mu+\beta}\left(2^{\log_2(R/|x|)(\mu-m_{N+1}+\beta)}+1\right) = C\,R^{\mu-m_{N+1}+\beta}|x|^{m_{N+1}} + C|x|^{\mu+\beta}.$$

Smallness of I_4.

Note that the kernel $K_i(x,y)$ has two singular points for $y \in B_{2|x|}$: at $y = x$ and $y = 0$. This motivates us to split

$$\mathrm{I}_4 = \int_{\Gamma\cap(B_{2|x|}(x)\setminus B_{|x|/10}(x))} K_i(x,y)f_i(y)\,dy + \int_{\Gamma\cap B_{|x|/10}(x)} K_i(x,y)f_i(y)\,dy$$
$$=: \mathrm{I}_{41} + \mathrm{I}_{42}.$$

For I_{41}, noting (3.3), (4.10), and $|x| \approx |y| \approx |x-y|$ for all $y \in B_{2|x|}(x) \setminus B_{|x|/10}(x)$,

$$|\mathrm{I}_{41}| \le C\int_{\Gamma\cap(B_{2|x|}(x)\setminus B_{|x|/10}(x))} (|x-y|^{-1}+|y|^{-1})\,|x-y|^{2-d}\,|y|^{\beta}\,|\nabla u(y)|\,dy$$
$$\le C|x|^{1-d+\beta}\int_{B_{2|x|}(x)} |\nabla u(y)|\,dy \le C|x|^{1-d+\beta}\,|x|^d\left(\fint_{B_{2|x|}(x)} |\nabla u|^2\right)^{1/2}$$
$$\le C|x|^{1-d+\beta}\,|x|^d\,|x|^{-1}\,\|u\|_{L^\infty B_{4|x|}(x)} \le C|x|^{\beta+\mu}.$$

Here, in the last line, we also used the Caccioppoli inequality and $u = O(|x|^{\mu})$.

For I_{42}, by (3.3), (4.10), and $|y| \approx |x| \ge 10|x-y|$,

$$|\mathrm{I}_{42}| \le C\int_{\Gamma\cap B_{|x|/10}(x)} \frac{1}{|x-y|^{d-1}}\,|y|^{\beta}\,|\nabla u(y)|\,dy$$
$$\le C|x|^{\beta}\int_{\Gamma\cap B_{2|x|}} \frac{1}{|x-y|^{d-1}}\,|\nabla u(y)|\,dy.$$

Hence, for $|x| \le r$,

$$|\mathrm{I}_{42}| \le Cr^{\beta}\left|\int_{\Gamma\cap B_{2|x|}} \frac{1}{|x-y|^{d-1}}\,|\nabla u(y)|\,dy\right|$$
$$= Cr^{\beta}\left|\int_{\Gamma\cap B_{3r}(x)\cap B_{2r}(0)} \frac{1}{|x-y|^{d-1}}\,|\nabla u(y)|\,dy\right|$$
$$\le Cr^{\beta}\left|\left((|z|^{-d+1}1_{B_{3r}}) * (|\nabla u|1_{B_{2r}\cap\Gamma})\right)(x)\right|.$$

By Young's inequality for convolution, a reverse Hölder inequality for ∇u (cf. [6]), the Caccioppoli inequality, and $u = O(|x|^{\mu})$, we obtain that for some $q < d/(d-1)$ to be fixed later, a small $\varepsilon > 0$ coming from the reverse Hölder inequality, and p satisfying $1 + 1/p = 1/q + 1/(2+\varepsilon)$,

$$\begin{aligned}\|\mathrm{I}_{42}\|_{L^p(B_r\cap\Gamma)} &\le C\,r^{\beta}\,\|1_{B_{3r}}\,|z|^{-d+1}\|_{L^q}\,\||\nabla u|\,1_{B_{2r}\cap\Gamma}\|_{L^{2+\varepsilon}}\\ &\le C\,r^{\beta}\,r^{-d+1+d/q}r^{d/(2+\varepsilon)}\Big(\fint_{B_{4r}\cap\Omega}|\nabla u|^2\Big)^{1/2}\\ &\le C\,r^{\beta}\,r^{-d+1+d/q}\,r^{d/(2+\varepsilon)}\,r^{-1}\,\|u\|_{L^\infty(\Gamma\cap B_{8r})} \le C\,r^{-d+d/q+d/(2+\varepsilon)}\,r^{\mu+\beta}.\end{aligned}$$

That is,

$$\Big(\fint_{\Gamma\cap B_r}|\mathrm{I}_{42}|^p\Big)^{1/p} \le C\,r^{\mu+\beta}.$$

Finally, we can make $p > 2d/(d-2)$ by choosing q to be sufficiently close to $d/(d-1)$.

4.3. Estimating the term II

Similar to the treatment of I, we split

$$\begin{aligned}\mathrm{II} &= \int_{\partial\Gamma} k^{(N)}(x,y)\,h(y)\,d\sigma_y - \int_{\partial\Gamma\cap B_{2|x|}} k^{(N)}(x,y)\,h(y)\,d\sigma_y\\ &\quad+ \int_{\partial\Gamma\cap B_{2|x|^c}} (k(x,y)-k^{(N)}(x,y))\,h(y)\,d\sigma_y + \int_{\partial\Gamma\cap B_{2|x|}} k(x,y)\,h(y)\,d\sigma_y\\ &=: \mathrm{II}_1 - \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4,\end{aligned}$$

and further,

$$\begin{aligned}\mathrm{II}_4 &= \int_{\partial\Gamma\cap(B_{2|x|}(x)\setminus B_{|x|/10}(x))} k(x,y)\,h(y)\,d\sigma_y + \int_{\partial\Gamma\cap B_{|x|/10}(x)} k(x,y)\,h(y)\,d\sigma_y\\ &=: \mathrm{II}_{41} + \mathrm{II}_{42}.\end{aligned}$$

The estimates for II_1, II_2, II_3, and II_{41} are very similar to those of the corresponding terms in I. Actually, the estimates here are simpler since a point-wise bound (4.9) for h is available, instead of merely an L^2 bound for Du from Caccioppoli's inequality. For II_1, formally,

$$\begin{aligned}\mathrm{II}_1 &= \int_{\partial\Gamma}\sum_{j\le N} b^{(j)}\Big(\frac{y}{|y|}\Big)\,|y|^{1-d-m_j}\,|x|m_j\,\psi_j\Big(\frac{x}{|x|}\Big)\,h(y)\,d\sigma_y\\ &= \sum_{j\le N}|x|^{m_j}\,\psi_j\Big(\frac{x}{|x|}\Big)\int_{\partial\Gamma} b^{(j)}\Big(\frac{y}{|y|}\Big)\,|y|^{1-d-m_j}\,h(y)\,d\sigma_y,\end{aligned}$$

which is a combination of homogeneous solutions on Γ with homogeneity at most m_N. Such formal computation is rigorous since all integrands are in L^1, which we check below. By (3.7) and (4.9),

$$\begin{aligned}\int_{\partial\Gamma}|b^{(j)}|\,|y|^{1-d-m_j}\,|h|\,d\sigma_y &\le C\int_{\partial\Gamma\cap B_R}|y|^{1-d-m_j}\,|y|^{\mu+\alpha\alpha_0}\,d\sigma_y\\ &\le C\int_0^R r^{1-d-m_j}\,r^{\mu+\alpha\alpha_0}\,r^{d-2}\,dr = C\,R^{\mu-m_j+\alpha\alpha_0}.\end{aligned}$$

As explained when estimating I_2, similar computation yields $|\mathrm{II}_2| \le C|x|^{\mu+\alpha\alpha_0}$. For II_3, we use (3.8) and (4.9) to obtain

$$\begin{aligned}|\mathrm{II}_3| &\le C|x|^{m_{N+1}} \int_{\partial\Gamma\cap B^c_{2|x|}} |y|^{1-d-m_{N+1}}|y|^{\mu+\alpha\alpha_0}\,d\sigma_y \\ &\le C|x|^{m_{N+1}} \int_{2|x|}^{R} r^{\mu-m_{N+1}+\alpha\alpha_0-1}\,dr \le C|x|^{\mu+\alpha\alpha_0} + C|x|^{m_{N+1}}.\end{aligned}$$

For II_{41}, from (3.3), (4.9), and the fact that $|x| \approx |y| \approx |x-y|$ for any $y \in B_{2|x|}(x) \setminus B_{|x|/10}(x))$, we obtain

$$\begin{aligned}|\mathrm{II}_{41}| &\le C \int_{\partial\Gamma\cap(B_{2|x|}(x)\setminus B_{|x|/10}(x))} (|y|^{-1} + |x-y|^{-1})\,|x-y|^{2-d}\,|y|^{\mu+\alpha\alpha_0}\,d\sigma_y \\ &\le C|x|^{\mu+\alpha\alpha_0}.\end{aligned}$$

Finally, we estimate II_{42}, which is different from estimating I_{42}. Here, using (3.4) instead of (3.3), and noting (4.9) and the fact that $|y| \approx |x| \ge 10|x-y|$ for $y \in B_{|x|/10}(x)$, we obtain

$$\begin{aligned}|\mathrm{II}_{42}| &\le C \int_{\partial\Gamma\cap B_{|x|/10}(x)} \frac{\delta(x)}{|x-y|^d}\,|y|^{\mu+\alpha\alpha_0}\,d\sigma_y \\ &\approx C|x|^{\mu+\alpha\alpha_0} \int_{\partial\Gamma\cap B_{|x|/10}(x)} \frac{\delta(x)}{|x-y|^d}\,d\sigma_y.\end{aligned}$$

Let $x^* \in \partial\Gamma$ be the point with $|x-x^*| = \delta(x)$. Since $y \in \partial\Gamma$, by definition and the triangle inequality,

$$|x-y| \ge \delta(x)\ (= \mathrm{dist}(x,\partial\Gamma)) \quad \text{and} \quad |x^*-y| \le |x-y| + \delta(x) \le 2|x-y|.$$

Hence,

$$\begin{aligned}|\mathrm{II}_{42}| &\le C|x|^{\mu+\alpha\alpha_0} \int_{\partial\Gamma\cap B_{|x|/10}(x)} \frac{\delta(x)}{\delta(x)^d + |x^*-y|^d}\,d\sigma_y \\ &\le C|x|^{\mu+\alpha\alpha_0} \int_{\mathbb{R}^{d-1}} \frac{\delta(x)}{\delta(x)^d + |y'|^d}\,dy' = C|x|^{\mu+\alpha\alpha_0} \int_{\mathbb{R}^{d-1}} \frac{1}{1+|y'|^d}\,dy' \le C|x|^{\mu+\alpha\alpha_0}.\end{aligned}$$

4.4. Concluding the proof of Proposition 4.2

With all above, we have constructed

$$P = \sum_{j=1}^{N} C_j |x|^{m_j} \psi_j(x/|x|)$$

such that

$$\left(\fint_{\Gamma\cap B_r} |u-P|^p\right)^{1/p} \le C r^{\min\{\mu+\alpha_0\min\{\alpha,\beta\},m_{N+1}\}}.$$

When $\mu \in (m_N, m_{N+1})$, clearly we must have $P = 0$ in order to match the vanishing order. When $\mu = m_N$, i.e., $u(x) = O(|x|^{m_N})$, similarly P cannot have any term with homogeneity lower than m_N. Hence, P is homogeneous of degree m_N. Choosing ψ_N from the eigenspace properly, we have $P(\tilde{x}) = C|\tilde{x}|^{m_N}\psi_N(\tilde{x}/|\tilde{x}|)$. This finishes the proof of Proposition 4.2.

5. Uniqueness and non-uniqueness of blow-ups

As mentioned in the introduction, Theorem 1.3 implies the existence of Almgren blow-ups (along subsequences) for any non-trivial harmonic functions vanishing locally on the boundary near a conical point. When $d = 2$, such limit is also unique.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ and let $0 \in \partial\Omega$ be a conical point with a tangent cone Γ. Suppose that u is a non-trivial harmonic function vanishing locally on $\partial\Omega$ near 0, with $u = O(|x|^N)$ for some $N > 0$. Then there exists a homogeneous harmonic function P on Γ, vanishing on $\partial\Gamma$, such that for the function u_r defined in (1.4), we have $u_r \to P$ weakly in $H^1(B_1)$ and strongly in $C^{0,\alpha_0}(B_1)$, where α_0 is a constant depending only Γ.*

As before, in the statement we do not distinguish u and P with their zero extensions.

We give a sketch of the proof. First, as explained in the introduction, all subsequence limits of u_r have to be L^2-normalized on ∂B_1, lying in an eigenspace of λ_N determined by $\lim_{r\to 0} N_u(r)$. Now from the fact that the eigenvalue λ_N is simple, there exists an L^2-normalized eigenfunction ψ_N, such that all possible Almgren blow-ups along subsequences have to be either $+\psi_N$ or $-\psi_N$. Noting that the full blow-up sequence u_r varies continuously with respect to r in $L^2(\partial B_1)$, the limit has to be unique.

When the dimension is three or more, higher eigenvalues need not to be simple, so the argument fails. This suggests that the Almgren blow-up sequences might "rotate" within these eigenspaces, leading to non-unique limits. In this section, we confirm that this actually can happen by constructing the example promised in Theorem 1.9.

5.1. Setup

We construct Ω by intersecting cones. For $k = 1, 2, \dots$, let $\alpha_k, \beta_k \in (0, \pi/2)$ be numbers satisfying

$$\beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \cdots \to \pi/2. \tag{5.1}$$

The values of α_k, β_k are not important. For instance, here we can fix $\alpha_k = \pi/2 - 2^{-(2k+2)}$ and $\beta_k = \pi/2 - 2^{-(2k+1)}$. Let

$$\Gamma_k = \left\{(x, y, z) : z > \sqrt{\left(\frac{x}{\tan\beta_k}\right)^2 + \left(\frac{y}{\tan\alpha_k}\right)^2}\right\}$$

be an elliptic cone with opening angles $2\alpha_k$ and $2\beta_k$. For a sequence of numbers with $\{\delta_k\}_{k=1}^{\infty} \subset (0,1)$, $\delta_k \downarrow 0$, to be chosen later, we set

$$\Omega = \bigcap_{k=1}^{\infty}(O_k\Gamma_k - \delta_k),$$

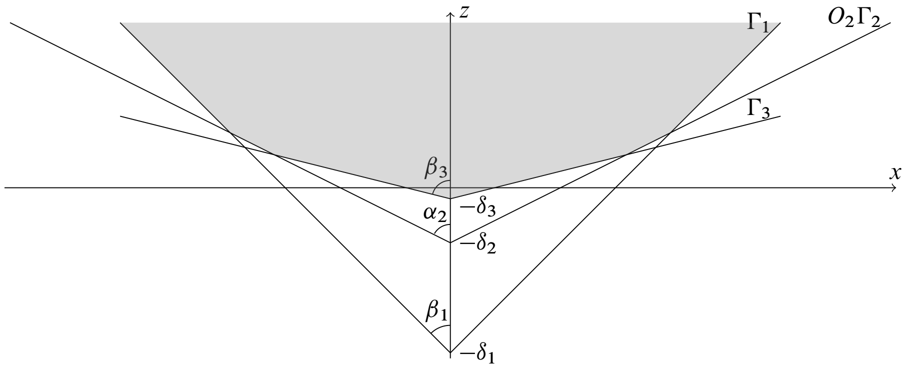
where

$$O_k = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k \text{ odd}, \quad \text{and} \quad O_k = O = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k \text{ even.}$$

Here and throughout this section, we use the abbreviation

$$\Gamma_k - \delta_k = \Gamma_k - \delta_k(0,0,1) = \Gamma_k - \delta_k e_z$$

for cones shifted along the z axis. Clearly, this Ω is convex with $0 \in \partial\Omega$, and is symmetric with respect to reflections about the x and y axes.



Let $\lambda_{x,k}$ and $\lambda_{y,k}$ be the first Dirichlet eigenvalues of the bisected cross-sections $\Gamma_k \cap \partial B_1 \cap \{x > 0\}$ and $\Gamma_k \cap \partial B_1 \cap \{y > 0\}$, respectively. Due to the eccentricity of the elliptical region $\Gamma_k \cap \partial B_1$, we have $\lambda_{x,k} < \lambda_{y,k}$. In the next section, we sketch a proof of this fact via an elementary perturbation argument from a spherical cap. A more detailed exposition (in the case of ellipses in the plane, but the approach is the same) can be found in [11].

5.2. The spectrum of perturbations of spherical caps

Let $E_0 \subset S^{d-1} \subset \mathbb{R}^d$ be a spherical cap of the form $E_0 = \{x : |x| = 1, x_d > s\}$ for a fixed $s \in (-1,1)$. We parametrize $S^{d-1} \setminus \{(0,\dots,0,1)\}$ by $(\psi,\theta) \in [0,\pi) \times S^{d-2} = T_{(1,0,\dots,0)}S^{d-1}$ via the exponential map, and write g for the round metric. Let $\phi_t : S^{d-1} \times [0,T) \to S^{d-1}$ be a family of diffeomorphisms smooth in both parameters and with $\phi_0(x) = x$. Set $E_t = \phi_t(E_0)$.

Now consider the Dirichlet eigenvalues $\{\lambda_k(E_t)\}_{k=1}^{\infty}$ of these domains, i.e., the non-decreasing sequences of numbers for which

$$\begin{cases} -\Delta_{S^{d-1}} v_k = \lambda_k(E_t) v_k & \text{on } E_t, \\ v_k = 0 & \text{on } \partial E_t. \end{cases}$$

At $t = 0$, $\lambda_k(E_0)$ have a straightforward structure which may be verified by separation of variables: $\lambda_1(E_0)$ is simple, then $\lambda_2(E_0) = \cdots = \lambda_d(E_0) < \lambda_{d+1}(E_0)$ with an orthonormal (in $L^2(E_0)$) basis of $(d-1)$ eigenfunctions $\{v_i\}_{i=1}^{d-1}$.

The eigenvalues $\{\lambda_k(E_t)\}_{k=2}^d$ of E_t form, for t small, a union of C^1 curves of the following form: let

$$m_{ij} = -\int_{\partial E_0} g(\nabla v_i, \nabla v_j)\, g(V, \nu)\, dA,$$

where A is the surface measure on ∂E_0 and v_i are the basis of second eigenfunctions, $V = \partial_t \phi_t|_{t=0}$ (this is a vector field), and ν is the outward unit normal vector to E_0. This is an $(d-1)$-dimensional symmetric matrix, with eigenvalues $\mu_1 \leq \cdots \leq \mu_{d-1}$. Then

$$\lambda_k(E_t) = \lambda_2(E_0) + t\mu_k + O(t^2).$$

This formula can be found in [19] in the case of subsets $\mathbb{R}^d$, but remains valid over any Riemannian manifold by the same argument after a direct computation of the variation of the Dirichlet and volume integrals ([10] carries out such computations). As a consequence, if the numbers μ_k are all distinct, then there is a $t_0 > 0$ such that for $t \in (0, t_0)$ the eigenvalues $\lambda_2(E_t), \ldots, \lambda_n(E_t)$ are simple.

When $d = 3$, the eigenfunctions v_1 and v_2, by separation of variables, are easily seen to be of the form $v_1(\psi, \theta) = q(\psi)\cos(\theta)$ and $v_2(\psi, \theta) = q(\psi)\sin(\theta)$ (after a rotation), for a smooth function q which is positive on $[0, \arccos s)$ and vanishes at $\psi = \arccos s$. This gives the explicit formula

$$m = -(q')^2 \int \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} g(V, \nu)\, dA.$$

Then $\mu_1 = \mu_2$ if and only if this matrix is a multiple of the identity, or equivalently if

$$\int_{\partial E_0} \sin(2\theta)\, g(V, \nu)\, dA = 0 \quad \text{and} \quad \int_{\partial E_0} \cos(2\theta)\, g(V, \nu)\, dA = 0.$$

If $\phi_t(\phi, \theta, t) = (\phi(1 + h(\theta, t)), \theta)$ where $h(\theta, 0) = 0$, h is even and π-periodic in θ, and has $\partial_t h(\cdot, 0)$ strictly decreasing on $[0, \pi/2]$, then the second integral is positive and E_t has $\lambda_2(E_t) < \lambda_3(E_t)$ for $t \in (0, t_0)$. Moreover, the domains E_t are symmetric across the planes $\theta = 0$ and $\theta = \pi/2$, so odd reflections of the first eigenfunctions of the bisected domains $E_t \cap \{\theta \in (0, \pi)\}$, $E_t \cap \{\theta \in (-\pi/2, \pi/2)\}$ give eigenfunctions on E_t. As all the eigenvalues are continuous in t, these must be the second and third eigenfunctions: in particular, $\lambda_1(E_t \cap \{\theta \in (0, \pi)\}) \neq \lambda_1(E_t \cap \{\theta \in (-\pi/2, \pi/2)\})$. A more careful examination of the matrix m shows that in fact $\lambda_1(E_t \cap \{\theta \in (0, \pi)\})$ is the larger of the two.

It is then easy to see that, given E_0 a hemisphere, one may construct a ϕ_t of this form for which each E_t is the cross-section of an elliptic cone with opening angles $\alpha = \pi/2 - t$ and $\beta = \pi/2 - 2t$ (along the $\theta = 0$ and $\theta = \pi/2$ axes, respectively). We conclude that $\lambda_{x,k} < \lambda_{y,k}$ for the cones Γ_k above so long as $\beta_k - \alpha_k$ is small enough.

5.3. Back to the example

The discussion in Section 5.2 shows that the second eigenvalue of the hemisphere $\partial B_1 \cap \{(x, y, z) : z > 0\}$, which equals 6 with a multiplicity of 2, splits into simple second and third eigenvalues on every $\Gamma_k \cap \partial B_1$. Furthermore, from $\lambda_{x,k} < \lambda_{y,k}$, we have the same order for their characteristic constants (see (1.2)), i.e., $m_{x,k} < m_{y,k}$. Recall that by separation of variables, $m_{x,k}$ and $m_{y,k}$ are equal to the homogeneities of the corresponding extended harmonic functions on Γ_k.

Now find harmonic functions u_1 and u_2 solving the Dirichlet problems

$$\begin{cases} \Delta u_1 = 0, & \Omega \cap B_1 \cap \{x > 0\}, \\ u_1 = 0, & (\partial\Omega \cup \{x = 0\}) \cap B_1, \\ u_1 = 1, & \partial B_1 \cap (\Omega \cap \{x > 0\}), \end{cases} \quad \text{and} \quad \begin{cases} \Delta u_2 = 0, & \Omega \cap B_1 \cap \{y > 0\}, \\ u_2 = 0, & (\partial\Omega \cup \{y = 0\}) \cap B_1, \\ u_2 = 1, & \partial B_1 \cap (\Omega \cap \{y > 0\}). \end{cases}$$

These are in $W^{1,2}(B_t \cap \Omega)$ for $t < 1$. Take odd extensions of u_1 and u_2 with respect to $x = 0$ and $y = 0$, respectively, and then extend both by zero outside Ω. Still denote the resulting functions by u_1 and u_2. It is not difficult to see that

$$N_{u_1}(r), N_{u_2}(r) \downarrow 4^{2\times 2} = 256, \quad \text{as } r \downarrow 0,$$

and

$$\frac{u_1(r\cdot)}{(⨍_{\partial B_r} |u_1|^2)^{1/2}} \to 4\sqrt{\frac{2}{\pi}}\, xz, \quad \frac{u_2(r\cdot)}{(⨍_{\partial B_r} |u_2|^2)^{1/2}} \to 4\sqrt{\frac{2}{\pi}}\, yz, \quad \text{as } r \downarrow 0,$$

which are L^2-normalized eigenfunctions associated with the second eigenvalue $\lambda_2 = 6$ on the hemisphere $\partial B_1 \cap \{(x, y, z) : z > 0\}$. In the following, we show that the desired "rotation" occurs for $u = u_1 + u_2$ by choosing δ_k properly. We consider auxiliary domains Ω_k and $\widehat{\Omega}_k$ and auxiliary functions $u_1^{(k)}$ and $u_2^{(k)}$. Here,

$$\Omega_k := \bigcap_{j \le k} (O_j \Gamma_j - \delta_j) \cap (O_{k+1}\Gamma_{k+1}) \quad \text{and} \quad \widehat{\Omega}_k := \bigcap_{j \le k} (O_j \Gamma_j - \delta_j).$$

It is not difficult to see that

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_k \subset \cdots \subset \Omega \subset \cdots \subset \widehat{\Omega}_k \subset \cdots \subset \widehat{\Omega}_2 \subset \widehat{\Omega}_1. \tag{5.2}$$

The functions $u_1^{(k)}$ and $u_2^{(k)}$ are solutions to

$$\begin{cases} \Delta u_1^{(k)} = 0, & \Omega_{k-1} \cap B_1 \cap \{x > 0\}, \\ u_1^{(k)} = 0, & (\partial\Omega_{k-1} \cup \{x = 0\}) \cap B_1, \\ u_1^{(k)} = 1, & \partial B_1 \cap (\Omega_{k-1} \cap \{x > 0\}), \end{cases} \quad \begin{cases} \Delta u_2^{(k)} = 0, & \widehat{\Omega}_k \cap B_1 \cap \{y > 0\}, \\ u_2^{(k)} = 0, & (\partial\widehat{\Omega}_k \cup \{y = 0\}) \cap B_1, \\ u_2^{(k)} = 1, & \partial B_1 \cap (\widehat{\Omega}_k \cap \{y > 0\}), \end{cases} \tag{5.3}$$

if k is odd, or

$$\begin{cases} \Delta u_1^{(k)} = 0, & \widehat{\Omega}_k \cap B_1 \cap \{x > 0\}, \\ u_1^{(k)} = 0, & (\partial\widehat{\Omega}_k \cup \{x = 0\}) \cap B_1, \\ u_1^{(k)} = 1, & \partial B_1 \cap (\widehat{\Omega}_k \cap \{x > 0\}), \end{cases} \quad \begin{cases} \Delta u_2^{(k)} = 0, & \Omega_{k-1} \cap B_1 \cap \{y > 0\}, \\ u_2^{(k)} = 0, & (\partial\Omega_{k-1} \cup \{y = 0\}) \cap B_1, \\ u_2^{(k)} = 1, & \partial B_1 \cap (\Omega_{k-1} \cap \{y > 0\}), \end{cases} \tag{5.4}$$

if k is even.

Like for u_1 and u_2, in the following we first take odd extensions of $u_1^{(k)}$ and $u_2^{(k)}$ with respect to $x = 0$ and $y = 0$, respectively, and then extend both by zero outside. Still denote the resulting functions by $u_1^{(k)}$ and $u_2^{(k)}$.

Below, we attempt to choose a sequence of δ_k decreasing to zero, such that, due to the alternation of $m_{x,k}$ and $m_{y,k}$,

$$\frac{\fint_{\partial B_{\delta_k}} |u_1^{(k)}|^2}{\fint_{\partial B_{\delta_k}} |u_2^{(k)}|^2} > k \quad \text{if } k \text{ is odd, and} \quad \frac{\fint_{\partial B_{\delta_k}} |u_1^{(k)}|^2}{\fint_{\partial B_{\delta_k}} |u_2^{(k)}|^2} < \frac{1}{k} \quad \text{if } k \text{ is even.} \tag{5.5}$$

In Section 5.5, we show that (5.5) together with a comparison, which comes from (5.2), imply that the desired "rotation" occurs.

5.4. Choosing δ_k

Starting point $k = 1$.

Recall that $u_1^{(1)}$ and $u_2^{(1)}$ solve (5.3) with $\Omega_0 = \Gamma_1$ and $\widehat{\Omega}_1 = \Gamma_1 - \delta_1$. Denote $\psi_{x,1}$ to be the (odd extension with respect to $x = 0$ of) L^2-normalized leading Dirichlet eigenfunction on $\partial B_1 \cap \Gamma_1 \cap \{x > 0\}$. From the orthogonality properties of the spherical harmonics, it is easy to see that

$$\Big(\fint_{\partial B_r \cap \Gamma_1} |u_1^{(1)}|^2\Big)^{1/2} \geq r^{m_{x,1}} \Big(\fint_{\partial B_1 \cap \Gamma_1} u_1^{(1)}\, \psi_{x,1}\Big)^{1/2} \geq C^{-1} r^{m_{x,1}}, \quad \forall r < 1. \tag{5.6}$$

For $u_2^{(1)}$, by Almgren's monotonicity formula on the convex domain (actually, cone) $\widehat{\Omega}_1$, centered at $-\delta_1 e_z \in \partial\widehat{\Omega}_1$, and the fact that the lowest non-orthogonal mode of $u_2^{(1)}$ is $\lambda_{y,1}$, we have

$$\frac{\fint_{\partial B_r(-\delta_1 e_z)} |u_2^{(1)}|^2}{\fint_{\partial B_{r/4}(-\delta_1 e_z)} |u_2^{(1)}|^2} = N_{u_2^{(1)}}(r) \geq N_{u_2^{(1)}}(0) = 4^{2m_{y,1}}, \quad \forall r \in (0, 1 - \delta_1). \tag{5.7}$$

Here and also later in this section, we abuse the notation $N_{u_2^{(1)}}(r)$, which includes a shift of the center. Iterating (5.7), we obtain, for all $r < 3/4$ and $\delta_1 < 1/4$ (away from 1 is enough),

$$\Big(\fint_{\partial B_r(-\delta_1 e_z)} |u_2^{(1)}|^2\Big)^{1/2} \leq C r^{m_{y,1}} \Big(\fint_{\partial B_{3/4}(-\delta_1 e_z)} |u_2^{(1)}|^2\Big)^{1/2} \leq C r^{m_{y,1}}.$$

By the subharmonicity of $|u_2^{(1)}|^2$ and the mean value property, for $r \in (\delta_1/2, 3/16)$ and $\delta_1 < 1/4$,

$$\begin{aligned}\Big(\fint_{\partial B_r} |u_2^{(1)}|^2\Big)^{1/2} &\leq C\Big(\fint_{B_{2r}} |u_2^{(1)}|^2\Big)^{1/2} \leq C\Big(\fint_{B_{4r}(-\delta_1 e_z)} |u_2^{(1)}|^2\Big)^{1/2} \\ &\leq C\Big(\fint_{\partial B_{4r}(-\delta_1 e_z)} |u_2^{(1)}|^2\Big)^{1/2} \leq C r^{m_{y,1}}.\end{aligned} \tag{5.8}$$

Here, C is a constant independent of r and δ_1.

Combining (5.6)–(5.8) with $r = \delta_1$,

$$\frac{⨍_{\partial B_{\delta_1}} |u_1^{(1)}|^2}{⨍_{\partial B_{\delta_1}} |u_2^{(1)}|^2} \geq C^{-1} \delta_1^{2(m_{x,1}-m_{y,1})}, \quad \forall \delta_1 < 1/4.$$

Finally, noting that $m_{x,1} < m_{y,1}$, we are able to choose δ_1 small enough, such that $C^{-1}\delta_1^{2(m_{x,1}-m_{y,1})} > 1$.

Given $\{\delta_j\}_{j\leq k-1}$, *choose* δ_k.

By switching the roles of x and y, without loss of generality, we can always assume that k is an even number.

Recall that $u_1^{(k)}$ and $u_2^{(k)}$ solve (5.4). The estimate of $u_1^{(k)}$ is similar to that of $u_2^{(1)}$, noting that $\widehat{\Omega}_k$ is still convex. By Almgren's monotonicity formula centered at $-\delta_k e_z \in \partial\widehat{\Omega}_k$,

$$\frac{⨍_{\partial B_r(-\delta_k e_z)} |u_1^{(k)}|^2}{⨍_{\partial B_{r/4}(-\delta_k e_z)} |u_1^{(k)}|^2} = N_{u_1^{(k)}}(r) \geq N_{u_1^{(k)}}(0) = 4^{2m_{y,k}}, \quad \forall r < 1-\delta_k.$$

Here, in the last step, we used the fact that the lowest non-orthogonal mode of $u_1^{(k)}$ is $\lambda_{y,k}$, noting that the cone $O_k\Gamma_k$ was rotated by 90 degree in (x, y). Hence, following the proof of (5.8), we obtain, whenever $\delta_k < \delta_1 < 1/4$,

$$\left(⨍_{\partial B_r} |u_1^{(k)}|^2\right)^{1/2} \leq C r^{m_{y,k}}, \quad \forall r \in (\delta_k/2, 3/16). \tag{5.9}$$

Compared to $u_1^{(1)}$, the estimate of $u_2^{(k)}$ requires some extra work, since now Ω_{k-1} is not exactly a cone. Define

$$R_k := \sup\{r : B_r \cap \partial\Omega_{k-1} \subset \partial(O_k\Gamma_k)\}.$$

From the monotonicity of the cones coming from (5.1), we have $R_k > 0$. Clearly, R_k depends on $\{\delta_j\}$ up to $j \leq k-1$. As in the proof of (5.6), denote $\psi_{x,k}$ to be the (odd extension of) L^2-normalized leading Dirichlet eigenfunction on $\partial B_1 \cap \Gamma_k \cap \{x > 0\}$. Note that O_k is a 90 degree rotation in (x, y) and the symmetry, the projection of $u_2^{(k)}$ onto $\psi_{x,k}$, is non-trivial, from which

$$\begin{aligned} \left(⨍_{\partial B_r} |u_2^{(k)}|^2\right)^{1/2} &= \left(⨍_{\partial B_r \cap O_k\Gamma_k} |u_2^{(k)}|^2\right)^{1/2} \\ &\geq \left(\frac{r}{R_k}\right)^{m_{x,k}} \left(⨍_{\partial B_{R_k} \cap O_k\Gamma_k} u_2^{(k)}\psi_{x,k}\right)^{1/2} \geq C\left(\frac{r}{R_l}\right)^{m_{x,k}}, \quad \forall r < R_l, \end{aligned} \tag{5.10}$$

where C is a constant depending on $\{\delta_j\}_{j\leq k-1}$. Here we used the fact that

$$\partial B_r \cap \partial\Omega_{l-1} \subset \partial O_l\Gamma_l, \quad \forall r < R_l,$$

coming from our definition of R_l.

Combining (5.9)–(5.10) and the fact that $m_{x,l+1} < m_{y,l+1}$, we can choose δ_k small enough such that $\delta_k < R_k$ and

$$\frac{\fint_{B_{\delta_k}} |u_1^{(k)}|^2}{\fint_{\partial B_{\delta_k}} |u_2^{(k)}|^2} \leq C R_k^{-2m_{x,k}} \delta_k^{2(m_{y,k}-m_{x,k})} < \frac{1}{k}.$$

With all above, we have finished our choice of $\{\delta_k\}_{k=1}^{\infty}$.

5.5. Conclusion of the proof of Theorem 1.9

Since $\Omega_{2k} \subset \Omega \subset \widehat{\Omega}_{2k+1}$, by the comparison principle, we have that $|u_1| \geq |u_1^{(2k+1)}|$ and $|u_2| \leq |u_2^{(2k+1)}|$. Hence,

$$\limsup_{r\to\infty} \frac{\fint_{\partial B_r} |u_1|^2}{\fint_{\partial B_r} |u_2|^2} \geq \lim_{k\to\infty} \frac{\fint_{\partial B_{\delta_{2k+1}}} |u_1^{(2k+1)}|^2}{\fint_{\partial B_{\delta_{2k+1}}} |u_2^{(2k+1)}|^2} \geq \lim_{k\to\infty} k = \infty.$$

Similarly,

$$\liminf_{r\to\infty} \frac{\fint_{\partial B_r} |u_1|^2}{\fint_{\partial B_r} |u_2|^2} \leq \lim_{k\to\infty} \frac{\fint_{\partial B_{\delta_{2k}}} |u_1^{(2k)}|^2}{\fint_{\partial B_{\delta_{2k}}} |u_2^{(2k)}|^2} \leq \lim_{k\to\infty} \frac{1}{k} = 0.$$

Then in strong $L^2(B_1)$ topology, $u = u_1 + u_2$ satisfies

$$\lim_{k\to\infty} \frac{u(\delta_{2k+1}\cdot)}{(\fint_{\partial B_{\delta_{2k+1}}} |u|^2)^{1/2}} = \lim_{k\to\infty} \frac{u_1(\delta_{2k+1}\cdot)}{(\fint_{\partial B_{\delta_{2k+1}}} |u_1|^2)^{1/2}} = 4\sqrt{\frac{2}{\pi}}\, xz$$

and

$$\lim_{k\to\infty} \frac{u(\delta_{2k}\cdot)}{(\fint_{\partial B_{\delta_{2k}}} |u|^2)^{1/2}} = \lim_{k\to\infty} \frac{u_2(\delta_{2k}\cdot)}{(\fint_{\partial B_{\delta_{2k}}} |u_2|^2)^{1/2}} = 4\sqrt{\frac{2}{\pi}}\, yz,$$

which are second eigenfunctions on a hemisphere, symmetric with respect to x and y axis, respectively. Hence, u has non-unique blow-up limits near the origin. This finishes the proof of Theorem 1.9.

A. Proof of gradient estimates in Lemma 3.3

The proof of (3.3) is standard. Take $R := \frac{1}{2}\min\{|y|, |x-y|\}$. Now, since $0 \notin B_R(y)$ and $x \notin B_R(y)$, we apply a local Lipschitz estimate for harmonic functions in $B_R(y) \cap \Gamma$ and the point-wise bound (3.2) to obtain

$$|\nabla_y G(x,y)| \leq C R^{-1} \sup_{z\in B_R(y)\cap\Gamma} |G(x,z)| \leq C R^{-1} |x-y|^{2-d},$$

which proves (3.3). Here, Assumption 1.5 implies the same smoothness of $\partial\Gamma \cap B_R$, and hence a local Lipschitz estimate for harmonic functions with zero Dirichlet boundary conditions. When g in Assumption 1.5 is $C^{1,\mathrm{Dini}}$, such estimate is standard. When it is semiconvex, a local Lipschitz estimate can be obtained by constructing a simple barrier coming with the exterior ball.

The proof of (3.4) requires more work. First, by (3.3) and the fact that $G(x,y)=0$ for $y\in\partial\Gamma$, we immediate obtain

$$G(x,y)\le C\,\delta(y)\,(|y|^{-1}+|x-y|^{-1})\,|x-y|^{2-d}.$$

By symmetry, we also have

$$G(x,y)\le C\,\delta(x)\,(|x|^{-1}+|x-y|^{-1})\,|x-y|^{2-d}. \tag{A.1}$$

Now, repeating the argument in the proof of (3.3) but using the new point-wise bound (A.1) instead of (3.2), we reach (3.4). The lemma is proved.

B. Doubling indices and Almgren's frequencies on cones

In this section, we prove Lemma 2.2. Define the (generalized) Almgren frequency functions

$$F(r):=\frac{r\int_{B_r}|\nabla u|^2}{\int_{\partial B_r}|u|^2}\quad\text{and}\quad \tilde F(r):=\frac{\int_{B_r}|\nabla u|^2(r^2-|x|^2)}{\int_{B_r}|u|^2}.$$

By standard computations, we have

$$F'(r)=\frac{2r}{h^2}\left(\Big(\int_{\partial B_r}|u|^2\Big)\Big(\int_{\partial B_r\cap\Gamma}(\nu\cdot\nabla u)^2\Big)-\Big(\int_{\partial B_r}u(\nu\cdot\nabla)u\Big)^2\right) \tag{B.1}$$

and

$$\tilde F'(r)=\frac{4}{r\tilde h^2}\left(\Big(\int_{B_r}|u|^2\Big)\Big(\int_{B_r\cap\Gamma}|x\cdot\nabla u|^2\Big)-\Big(\int_{B_r\cap\Gamma}u(x\cdot\nabla)u\Big)^2\right).$$

See for instance [2]. Note that here all contributions from $\partial\Gamma$ vanish since Γ is a cone. Moreover, by standard elliptic regularity theory, we have $Du|_{\partial\Omega}\in L^2$, which is understood in the sense of non-tangential limit. This guarantees that all the integration by parts in the process are justified. Now we give the proof of Lemma 2.2.

Proof of Lemma 2.2. Noting

$$F=\frac{rD(r)}{h(r)}=\frac{r}{2}\frac{d}{dr}\log h-\frac{d-1}{2}\quad\text{and}\quad \tilde F=\frac{\tilde D(r)}{\tilde h(r)}=r\frac{d}{dr}\log\tilde h-d,$$

we have

$$\begin{aligned}N_u(4^t)&=\frac{\fint_{\partial B_{4^t}}|u|^2}{\fint_{\partial B_{4^{t-1}}}|u|^2}=4^{d-1}\,\frac{h(4^t)}{h(4^{t-1})}=4^{d-1}\,e^{\int_{t-1}^t\frac{d}{ds}\log h(4^s)\,ds}\\&=4^{d-1}\,e^{\log(4)\int_{t-1}^t(r\log(h)')|_{r=4^s}\,ds}=16^{d-1}\,16^{\int_{t-1}^tF(4^s)\,ds}\end{aligned}$$

and

$$\widetilde{N}_u(2^t) = \frac{\fint_{B_{2^t}} |u|^2}{\fint_{B_{2^{t-1}}} |u|^2} = 2^d \, \frac{\tilde{h}(2^t)}{\tilde{h}(2^{t-1})} = 2^d \, e^{\int_{t-1}^t \frac{d}{ds} \log \tilde{h}(2^s)\, ds}$$
$$= 2^d \, e^{\log(2) \int_{t-1}^t (r \log(\tilde{h})')|_{r=2^s}\, ds} = 4^d \, 2^{\int_{t-1}^t \widetilde{F}(2^s)\, ds}.$$

From these, in the following we only prove the monotonicity and rigidity of F and $\widetilde{F}$, since those for N and $\widetilde{N}$ naturally follow. Moreover, we only prove for F as the proof for $\widetilde{F}$ is almost identical. First, from (B.1) and Hölder's inequality, clearly $F' \geq 0$. Now, if $F(t) = F(s)$ for some $t > s$, by the condition for achieving "$=$" in Hölder's inequality, $\partial u/\partial r = C_r u$ for all $r \in (s, t)$. Expanding as spherical harmonics, clearly this can only be true when u is homogeneous in r. Once u is homogeneous in r, we immediately have $F \equiv$ constant and u is a homogeneous harmonic function. ∎

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References

[1] Adolfsson, V. and Escauriaza, L.: $C^{1,\alpha}$ domains and unique continuation at the boundary. *Comm. Pure Appl. Math.* **50** (1997), no. 10, 935–969. Zbl 0899.31004 MR 1466583

[2] Adolfsson, V., Escauriaza, L. and Kenig, C.: Convex domains and unique continuation at the boundary. *Rev. Mat. Iberoamericana* **11** (1995), no. 3, 513–525. Zbl 0840.31005 MR 1363203

[3] Bers, L.: Local behavior of solutions of general linear elliptic equations. *Comm. Pure Appl. Math.* **8** (1955), 473–496. Zbl 0066.08101 MR 0075416

[4] Chen, X.-Y.: Polynomial asymptotics near zero points of solutions of general elliptic equations. *Comm. Partial Differential Equations* **22** (1997), no. 7-8, 1227–1250. Zbl 0886.35013 MR 1466314

[5] Felli, V. and Ferrero, A.: Almgren-type monotonicity methods for the classification of behaviour at corners of solutions to semilinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013), no. 5, 957–1019. Zbl 1296.35065 MR 3109767

[6] Giaquinta, M. and Martinazzi, L.: *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*. Second edition. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) 11, Edizioni della Normale, Pisa, 2012. Zbl 1262.35001 MR 3099262

[7] Gilbarg, D. and Trudinger, N. S.: *Elliptic partial differential equations of second order*. Classics in Mathematics, Springer, Berlin, 2001. Zbl 1042.35002 MR 1814364

[8] Grüter, M. and Widman, K.-O.: The Green function for uniformly elliptic equations. *Manuscripta Math.* **37** (1982), no. 3, 303–342. Zbl 0485.35031 MR 0657523

[9] Han, Q.: Singular sets of solutions to elliptic equations. *Indiana Univ. Math. J.* **43** (1994), no. 3, 983–1002. Zbl 0817.35020 MR 1305956

[10] Henrot, A. and Pierre, M.: *Variation et optimisation de formes*. Math. Appl. (Berlin) 48, Springer, Berlin, 2005. Zbl 1098.49001 MR 2512810

[11] Henry, D.: *Perturbation of the boundary in boundary-value problems of partial differential equations*. London Math. Soc. Lecture Note Ser. 318, Cambridge University Press, Cambridge, 2005. Zbl 1170.35300 MR 2160744

[12] Hofmann, S. and Kim, S.: The Green function estimates for strongly elliptic systems of second order. *Manuscripta Math.* **124** (2007), no. 2, 139–172. Zbl 1130.35042 MR 2341783

[13] Kenig, C. E.: *Harmonic analysis techniques for second order elliptic boundary value problems*. CBMS Reg. Conf. Ser. Math. 83, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994. Zbl 0812.35001 MR 1282720

[14] Kenig, C. and Zhao, Z.: Expansion of harmonic functions near the boundary of Dini domains. *Rev. Mat. Iberoam.* **38** (2022), no. 7, 2117–2152. Zbl 1516.31010 MR 4525136

[15] Kenig, C. and Zhao, Z.: Examples of non-Dini domains with large singular sets. *Adv. Nonlinear Stud.* **23** (2023), no. 1, article no. 20220058, 33 pp. Zbl 1512.35217 MR 4580700

[16] Lin, F. and Shen, Z.: Nodal sets and doubling conditions in elliptic homogenization. *Acta Math. Sin. (Engl. Ser.)* **35** (2019), no. 6, 815–831. Zbl 1421.35074 MR 3952693

[17] Lin, F. and Shen, Z.: Critical sets of solutions of elliptic equations in periodic homogenization. *Comm. Pure Appl. Math.* **77** (2024), no. 7, 3143–3183. Zbl 1540.35029 MR 4764741

[18] Mitrea, D., Mitrea, M. and Yan, L.: Boundary value problems for the Laplacian in convex and semiconvex domains. *J. Funct. Anal.* **258** (2010), no. 8, 2507–2585. Zbl 1201.35095 MR 2593333

[19] Rousselet, B.: Shape design sensitivity of a membrane. *J. Optim. Theory Appl.* **40** (1983), no. 4, 595–623. Zbl 0497.73097 MR 0717179

[20] Tolsa, X.: Unique continuation at the boundary for harmonic functions in C^1 domains and Lipschitz domains with small constant. *Comm. Pure Appl. Math.* **76** (2023), no. 2, 305–336. Zbl 1529.31002 MR 4544799

[21] Zhu, J. and Zhuge, J.: Nodal sets of Dirichlet eigenfunctions in quasiconvex Lipschitz domains. Preprint, arXiv: 2303.02046v1, 2023.

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