Rev. Mat. Iberoam. 41 (2025), no. 3, 1057–1080
DOI 10.4171/RMI/1516

Higher Jacobian ideals, contact equivalence and motivic zeta functions

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Abstract. We show basic properties of higher Jacobian matrices and higher Jacobian ideals for functions and apply it to obtain two main results concerning singularities of functions. Firstly, we prove that a higher Nash blowup algebra is invariant under contact equivalences, which was recently conjectured by Hussain, Ma, Yau and Zuo. Secondly, we obtain an analogue of a result on motivic nearby cycles by Bussi, Joyce and Meinhardt.

1. Introduction

1.1. Higher Jacobian ideals and higher Nash blowups

Jacobian matrices and Jacobian ideals are fundamental objects in the study of singularities of varieties, as well as singularities of morphisms. There are higher versions of these notions, higher Jacobian matrices and higher Jacobian ideals. Duarte introduced them for hypersurface varieties in [12], motivated by the study of higher Nash blowups. The case of more general varieties was later treated in [2, 4]. The aim of the present article is to show basic properties of higher Jacobian matrices and ideals for functions/morphisms, and to apply them to show two results concerning singularities of functions, that is, a recent conjecture on higher Nash blowup algebras by Hussain, Ma, Yau and Zuo [16], as well as an analogue of a result on motivic nearby cycles by Bussi, Joyce and Meinhardt [6].

Let k be a perfect field. By a k-variety we mean a separated and reduced scheme of finite type and of pure dimension over k. For a k-variety S, by S-variety we mean a variety X together with a morphism $X \to S$ such that every irreducible component of X is dominant over some irreducible component of S. For a non-negative integer n and an S-variety X, we have the coherent $\mathcal{O}_X$-module $\mathcal{P}^n_{X/S}$ and $\Omega^{(n)}_{X/S}$, that are called the sheaf of principal parts of order n and the sheaf of Kähler differentials of order n, respectively. These sheaves are closely related to the higher Nash blowup. This blowup was studied, for example, in [2, 4, 8, 11–14, 20, 22, 23] mainly from the viewpoint of the desingularization problem. For a k-variety X, its n-th Nash blowup, denoted by $\mathbf{Nash}_n(X)$, can be defined

Mathematics Subject Classification 2020: 32S05 (primary); 13N05, 13N15, 14B05, 14G10 (secondary).
Keywords: higher Nash blowups, higher Jacobian matrices, higher Jacobian ideals, singularity, contact equivalence, motivic zeta function, motivic nearby cycle.

as the blowup $\mathrm{Bl}_{\mathcal{P}^n_{X/k}}(X) = \mathrm{Bl}_{\Omega^{(n)}_{X/k}}(X)$ associated to $\mathcal{P}^n_{X/k}$ or $\Omega^{(n)}_{X/k}$. Basic properties of blowups associated to general coherent sheaves were studied previously in [19, 21]. It is natural to consider a generalization of this blowup from k-varieties to S-varieties. We may define the n-th Nash blowup $\mathbf{Nash}_n(X/S)$ of X over S to be $\mathrm{Bl}_{\mathcal{P}^n_{X/S}}(X) = \mathrm{Bl}_{\Omega^{(n)}_{X/S}}(X)$. We may speculate that this blowup would shed new light on the study of singularities of morphisms $X \to S$ or singularities of the corresponding foliations on X. The study of higher Nash blowups of S-varieties itself is not the main subject of this paper, but it motivates the study of the sheaves $\mathcal{P}^n_{X/k}$ and $\Omega^{(n)}_{X/k}$.

Our principal interest is in the case where $S = \mathbb{A}^1_k$ and X is smooth. Focusing on the even more specialized situation $X = \mathbb{A}^d_k$ is also important, as the local study of the given function $f: X \to \mathbb{A}^1_k$ is reduced to this situation. Our first result, Proposition 2.3, is an analogue of a result proved in [2, 4, 12]. Namely, we observe that for a morphism $f: \mathbb{A}^d_k \to \mathbb{A}^1_k$, there exists a free presentation

$$\mathcal{O}_{\mathbb{A}^d_k}^{\binom{d-1+n}{d}} \to \mathcal{O}_{\mathbb{A}^d_k}^{\binom{d+n}{d}-1} \to \Omega^{(n)}_{\mathbb{A}^d_k/\mathbb{A}^1_k} \to 0$$

such that the left map is given by a higher Jacobian matrix $\mathrm{Jac}_n(f)$ defined in [2] (for the single function case), which has higher derivatives of f and zeroes as entries. For more information to the reader, there is another version of higher Jacobian matrix associated to a morphism of affine spaces (see Section 1.1 in [7]. Restricting the above exact sequence to the hypersurface defined by f refines an earlier result in [12], and is similar to the exact sequence independently obtained in Corollary 2.27 of [4]. This free presentation shows that the n-th Jacobian ideal sheaf $\mathcal{J}_n(f)$ is generated by the maximal minors of $\mathrm{Jac}_n(f)$. Combining this with a result of Villamayor [21] also gives that for a function $f: X \to \mathbb{A}^1_k$ on a smooth variety, the n-Nash blowup $\mathbf{Nash}_n(X/\mathbb{A}^1_k)$ coincides with the blowup of X with respect to the ideal $\mathcal{J}_n(f)$; this is an analogue of Duarte's result for hypersurfaces, see Proposition 4.11 in [12].

1.2. Higher Nash blowup local algebras

In [16], Hussain, Ma, Yau and Zuo use the definition of higher Jacobian matrices introduced earlier in [12] and work with the ideal of $\mathbb{C}[x]$ generated by all maximal minors of such a higher Jacobian matrix of a polynomial (to compare different versions of higher Jacobian matrices, see Remark 2.14).

Consider the ring $\mathbb{C}\{x\} = \mathbb{C}\{x_1, \dots, x_d\}$ of complex analytic functions on a neighborhood of $\mathbf{0}$ in $\mathbb{C}^d$. We study a version extended to $\mathbb{C}\{x\}$ of a problem in [16]. Let f be in $\mathbb{C}\{x\}$ such that $f(\mathbf{0}) = 0$. The n-th Nash blowup local algebra of f is defined as follows: $\mathcal{T}_n(f) := \mathbb{C}\{x\}/\langle f, \mathcal{J}_n(f)\rangle$. Clearly, $\mathcal{T}_1(f)$ is the Tjurina algebra. When f is a polynomial in $\mathbb{C}[x]$, the n-th Nash blowup local algebra $\mathcal{T}_n(f)$ and the derivation Lie algebra $\mathrm{Der}(\mathcal{T}_n(f))$ are main objectives studied in [16]. In terms of Conjecture 1.5 in [16], it was expected that if $f \in \mathbb{C}[x]$ defines an isolated singularity at $\mathbf{0}$, then $\mathcal{T}_n(f)$ is a contact invariant (this was checked in [16] with $n = d = 2$). We go further as Theorem 2.26 below when checking the conjecture with arbitrary $n, d \in \mathbb{N}^*$; especially, we allow f and g to be in $\mathbb{C}\{x\}$ and do not require the isolatedness of singularities. For the concept of contact equivalence at a singular point, see Definition 2.25.

Theorem (Theorem 2.26). *Let f and g be in $\mathbb{C}\{x\}$ with $f(\mathbf{0}) = g(\mathbf{0}) = 0$. If f is contact equivalent to g at $\mathbf{0}$, then $\mathcal{T}_n(f)$ is isomorphic to $\mathcal{T}_n(g)$ as $\mathbb{C}$-algebras for any $n \in \mathbb{N}^*$.*

1.3. Motivic zeta functions

Assume that the field k has characteristic zero. Let X be a smooth k-variety of dimension d. Let f be a non-constant regular function on X, and let f also denote the corresponding element in $\mathcal{O}_X(X)$. As mentioned above, to such an f, we associate the $\mathcal{O}_X$-module $\Omega_f^{(n)}$, the Nash blowup $\mathbf{Nash}_n(f)$, and the sheaf of ideals $\mathcal{J}_n(f) \subseteq \mathcal{O}_X$. Now, we associate to f the motivic zeta function $Z_f(T)$ and nearby cycles $\mathcal{S}_f$. It would be meaningful to explore the relation of $\mathcal{J}_n(f)$ to $Z_f(T)$ and $\mathcal{S}_f$.

The theory of motivic zeta functions is one of the profound applications of motivic integration to the study of singularities. Indeed, several singularity invariants such as Hodge–Euler characteristic and Hodge spectrum can be recovered from motivic zeta functions via appropriate Hodge realizations (see, e.g., [9,10]). Since the motivic zeta function relates directly to the monodromy conjecture, it has been widely taken care by several geometers and singularity theorists. For instance, Denef–Loeser (cf. [9, 10]) give it an explicit description using log-resolution, hence they can prove its rationality and list a candidate set of poles concerning the log-resolution numerical data, which is very important for any approach to the conjecture. Moreover, the motivic zeta function and the motivic nearby cycles are also important in mathematical physics, especially in the study of motivic Donalson–Thomas invariants theory for noncommutative Calabi–Yau threefolds (cf. [6]).

For any integer $m \geq 1$, let $K_0^{\mu_m}(\mathrm{Var}_S)$ denote the μ_m-equivariant Grothendieck ring of S-varieties endowed with a good μ_m-action. Localizing $K_0^{\mu_m}(\mathrm{Var}_S)$ with respect to the class $\mathbb{L}$ of the trivial line bundle over S, one obtains $\mathcal{M}_S^{\mu_m}$, and taking inductive limit of $\mathcal{M}_S^{\mu_m}$ with respect to m gives the ring $\mathcal{M}_S^{\hat{\mu}}$ (see Section 3.1). Write

$$Z_f(T) = \sum_{m \geq 1} [\mathcal{X}_m(f)]\, \mathbb{L}^{-dm} T^m,$$

where $\mathcal{X}_m(f)$ is the m-th iterated contact locus of f defined in Section 3.1. The following theorem is also a main result of this article.

Theorem (Theorem 3.5). *Let f and g be non-constant regular functions on X with the same scheme-theoretic zero locus X_0. Suppose that $g - f \in \mathcal{J}_2(f)$. If $d = 2$ and X_0 has nodes, suppose additionally that k is quadratically closed. Then, for any integer $m \geq 1$, the identity $[\mathcal{X}_m(f)] = [\mathcal{X}_m(g)]$ holds in $\mathcal{M}_{X_0}^{\mu_m}$. As a consequence, $Z_f(T) = Z_g(T)$ and $\mathcal{S}_f = \mathcal{S}_g$.*

The theorem points out that the motivic zeta function and the motivic nearby cycles of f are invariant modulo the second order Jacobian ideal sheaf $\mathcal{J}_2(f)$. Modifying Theorem 3.5 with the hypotheses replaced by $g - f \in \mathcal{J}_1(f)^3$ for non-constant regular functions f and g with the same scheme-theoretic zero locus, we obtain Theorem 3.8. We can consider Theorem 3.8 as a variant of Theorems 3.2 and 3.6 in [6]. However, it should be noted that the definition of motivic nearby cycles in [6] is slightly different from that in [9,10]. Namely, the Denef–Loeser motivic nearby cycles of a regular function f is supported on the zero locus of f (cf. [9, 10]), while the one of Bussi–Joyce–Meinhardt is supported on the critical locus of f (cf. [6]). Notice that the proof in [6] uses Proposition 4.3 of [3] in a crucial way. Our idea in proving Theorems 3.5 and 3.8 may provide new arguments for Theorems 3.2 and 3.6 in [6] using m-separating log-resolution, whose existence is given in [5].

2. Higher Jacobian ideals

2.1. Higher Nash blowups of morphisms

In Sections 2.1 and 2.2, k is an arbitrary perfect field.

We first recall the construction of Oneto–Zatini [19] on the Nash blowup associated to a coherent sheaf. Let X be a reduced Noetherian scheme, and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module which is locally free of constant rank r on an open dense subscheme U of X. Consider the functor $\mathcal{G}$ from the category of X-schemes to the category of sets which sends each X-scheme Y to the set of $\mathcal{O}_Y$-modules that are locally free quotient of rank r of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Then $\mathcal{G}$ is contravariant and represented by an X-scheme $\mathbf{Grass}_r\, \mathcal{M}$. It is a fact that $\mathbf{Grass}_r\, \mathcal{M} \times_X U$ is isomorphic to U.

Definition 2.1 (Oneto–Zatini). The closure of $\mathbf{Grass}_r\, \mathcal{M} \times_X U$ in $\mathbf{Grass}_r\, \mathcal{M}$ is called the *blowup of X at $\mathcal{M}$*, and is denoted by $\mathrm{Bl}_{\mathcal{M}}(X)$.

The natural morphism $\pi_{\mathcal{M}} \colon \mathrm{Bl}_{\mathcal{M}}(X) \to X$ is birational as X is reduced, and it is projective as $\mathcal{M}$ is coherent. Note that $(\pi_{\mathcal{M}}^* \mathcal{M})/\mathrm{Tor}(\pi_{\mathcal{M}}^* \mathcal{M})$ is locally free. As shown in [19], $\mathrm{Bl}_{\mathcal{M}}(X)$ satisfies the following universal property: if $h \colon Y \to X$ is a modification such that $(h^*\mathcal{M})/\mathrm{Tor}(h^*\mathcal{M})$ is locally free, then there exists a unique morphism $\phi \colon Y \to \mathrm{Bl}_{\mathcal{M}}(X)$ such that $\pi_{\mathcal{M}} \circ \phi = h$.

Let S be a reduced Noetherian scheme, and let $f \colon X \to S$ be a morphism of schemes. The diagonal morphism

$$\Delta = \Delta_f : X \to X \times_S X$$

of f is locally a closed immersion, so $\Delta(X)$ is closed in an open subset V of $X \times_S X$. Let $\mathcal{I} = \mathcal{I}_f$ be the ideal sheaf defining $\Delta(X)$ in V, i.e., $\mathcal{I}$ is the kernel of

$$\Delta^\# : \mathcal{O}_{X \times_S X} \to \Delta_* \mathcal{O}_X.$$

For any $n \in \mathbb{N}^*$, we define

$$\mathcal{P}_f^n := \mathcal{P}_{X/S}^n := \mathcal{O}_{X \times_S X}/\mathcal{I}^{n+1} \quad \text{and} \quad \Omega_f^{(n)} := \Omega_{X/S}^{(n)} := \mathcal{I}/\mathcal{I}^{n+1}.$$

As explained in Sections 16.3 and 16.4 of [15], $\mathcal{P}_f^n$ (hence $\Omega_f^{(n)}$) depends functorially on f, and the construction is local, i.e., for any open subscheme U of X we have $\mathcal{P}_{f|U}^n = \mathcal{P}_f^n|_U$ and $\Omega_{f|U}^{(n)} = \Omega_f^{(n)}|_U$. We regard $\mathcal{P}_f^n$ (respectively, $\Omega_f^{(n)}$) as an $\mathcal{O}_X$-algebra (respectively, an $\mathcal{O}_X$-module) using the first projection $X \times_S X \to X$. The map $\mathcal{O}_X \to \mathcal{P}_f^n$ induced by the first projection $X \times_S X \to X$ gives a splitting of the exact sequence

$$0 \to \Omega_f^{(n)} \to \mathcal{P}_f^n \to \mathcal{O}_X \to 0.$$

In particular, $\mathcal{P}_f^n \cong \mathcal{O}_X \oplus \Omega_f^{(n)}$.

Definition 2.2. The $\mathcal{O}_X$-modules $\mathcal{P}^n_f$ and $\Omega^{(n)}_f$ are called the *sheaf of principal parts of order* n of f and the *module of Kähler differentials of order* n of f, respectively.

Consider morphisms of separated schemes $f: X \to S$ and $g: S \to B$, and the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{Id}} & X & \xrightarrow{f} & S \\ {\scriptstyle\Delta_f}\downarrow & & \downarrow{\scriptstyle\Delta_{g\circ f}} & & \downarrow{\scriptstyle\Delta_g} \\ X \times_S X & \xrightarrow{\ell} & X \times_B X & \xrightarrow{f\times f} & S \times_B S \end{array}$$

Similarly as in equation (16.4.3.3) of [15], the left and right squares give rise to the canonical morphisms of $\mathcal{O}_X$-modules

$$\Theta_n : \Omega^{(n)}_{g\circ f} \to \Omega^{(n)}_f \quad \text{and} \quad \Psi'_n : f^*\Omega^{(n)}_g = \Omega^{(n)}_g \otimes_{\mathcal{O}_S} \mathcal{O}_X \to \Omega^{(n)}_{g\circ f},$$

respectively. In particular, Θ_n is induced by $\ell^*(\mathcal{I}^n_{g\circ f}) \subseteq \mathcal{I}^n_f$ for every $n \in \mathbb{N}$ due to the left diagram. The right square gives an inclusion of coherent sheaves of ideals $\mathcal{K} := (f \times f)^*(\mathcal{I}_g) \subseteq \mathcal{I}_{g\circ f}$. From the proof of Proposition 16.4.18 in [15], Θ_n is exactly the canonical projection

$$\Omega^{(n)}_{g\circ f} \to \Omega^{(n)}_{g\circ f} / \big((\mathcal{I}^{n+1}_{g\circ f} + \mathcal{K})/\mathcal{I}^{n+1}_{g\circ f}\big) \cong \Omega^{(n)}_f.$$

On the other hand, the tensor product over $\mathcal{O}_{S\times_B S}$ of $\mathcal{I}_g \to \Omega^{(n)}_g$ and $\mathcal{O}_{X\times_B X} \to \mathcal{O}_X$ gives rise to a natural morphism $\mathcal{K} \to f^*\Omega^{(n)}_g$. The composition of this natural morphism with Ψ'_n is denoted by Ψ''_n. Let Ψ_n be the homomorphism $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f} \to \Omega^{(n)}_{g\circ f}$ induced by Ψ''_n. Namely, Ψ_n is the composition of

$$\Psi''_n \otimes \mathrm{Id} : \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f} \to \Omega^{(n)}_{g\circ f} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f}$$

with the product homomorphism

$$\Omega^{(n)}_{g\circ f} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f} \to \Omega^{(n)}_{g\circ f}.$$

Then $\mathrm{Im}(\Psi_n)$ is nothing but the sheaf of ideals generated by $\Psi'_n(\mathcal{K}) = \mathrm{Im}(\Psi''_n)$.

Proposition 2.3. *With the previous notation,* Θ_n *is surjective and* $\ker(\Theta_n) = \mathrm{Im}(\Psi_n)$. *As a consequence, there is an exact sequence of sheaves of* $\mathcal{O}_X$*-modules*

$$\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^{n-1}_{g\circ f} \xrightarrow{\Psi_n} \Omega^{(n)}_{g\circ f} \xrightarrow{\Theta_n} \Omega^{(n)}_f \longrightarrow 0.$$

Proof. From Proposition 16.4.18 in [15], the image of $\mathcal{K}$ in $\Omega^{(n)}_{g\circ f}$ generates the kernel of Θ_n as a $\mathcal{P}^n_{g\circ f}$-submodule. Namely, the natural morphism

$$\Psi_n : \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f} \to \Omega^{(n)}_{g\circ f}$$

is surjective. We need to show that this factors through

$$\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f} \twoheadrightarrow \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^{n-1}_{g\circ f}.$$

To do so, we first observe that $\mathcal{K} \subseteq \mathcal{I}_{g\circ f}$. If a is a section of $\mathcal{K}$ and if b is a section of $\mathcal{I}^n_{g\circ f}/\mathcal{I}^{n+1}_{g\circ f}$, then the section $a \otimes b$ of $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{P}^n_{g\circ f}$ maps to the section $ab = 0$ of $\Omega^{(n)}_{g\circ f} = \mathcal{I}_{g\circ f}/\mathcal{I}^{n+1}_{g\circ f}$. Thus, we get the desired factorization. ∎

Lemma 2.4. *Let $f\colon X \to S$ and $g\colon S \to B$ be morphisms of separated schemes. Suppose that f is étale. Then, $f^*\mathcal{P}^n_g \cong \mathcal{P}^n_{g\circ f}$ and $f^*\Omega^{(n)}_g \cong \Omega^{(n)}_{g\circ f}$.*

Proof. Let $\Delta_g(S)^{(n)}$ denote the n-th infinitesimal neighborhood of $\Delta_g(S)$, that is, the closed subscheme of $S \times_B S$ defined by $\mathcal{I}^{n+1}_g$. Then, $\mathcal{P}^n_g$ is identified with the structure sheaf of $\Delta_g(S)^{(n)}$ and $\Omega^{(n)}_g$ is identified with the defining ideal sheaf of $\Delta_g(S)$ as a closed subscheme of $\Delta_g(S)^{(n)}$. We show the lemma by using these identifications.

Consider the natural morphisms

$$\Delta_{g\circ f}(X) \hookrightarrow (f \times f)^{-1}(\Delta_g(S)) \to \Delta_g(S).$$

Since the composite map and the right map are étale, so is the left map. Since the left map is also a closed immersion, it is an open and closed immersion from Theorem 17.9.1 in [15]. This shows that the natural maps

$$\Delta_{g\circ f}(X)^{(n)} \hookrightarrow (f \times f)^{-1}(\Delta_g(S))^{(n)} \to \Delta_g(S)^{(n)}$$

are also étale. Here the superscript (n) again means the n-th infinitesimal neighborhood. We now regard $\Delta_{g\circ f}(X)^{(n)}$ and $\Delta_g(S)^{(n)}$ as an X-scheme and an S-scheme using the first projections $X \times_B X \to X$ and $S \times_B S \to S$ respectively. Then, the étale morphism $\Delta_{g\circ f}(X)^{(n)} \to \Delta_g(S)^{(n)}$ is compatible with the morphism $X \to S$, and hence factors as

$$\Delta_{g\circ f}(X)^{(n)} \to \Delta_g(S)^{(n)} \times_S X \to \Delta_g(S)^{(n)}.$$

We conclude that $\Delta_{g\circ f}(X)^{(n)} \to \Delta_g(S)^{(n)} \times_S X$ is étale. Since

$$X = (\Delta_{g\circ f}(X)^{(n)})_{\mathrm{red}} \to (\Delta_g(S)^{(n)} \times_S X)_{\mathrm{red}} = X$$

is an isomorphism, $\Delta_{g\circ f}(X)^{(n)} \to \Delta_g(S)^{(n)} \times_S X$ is an isomorphism again from Theorem 17.9.1 in [15]. The last two isomorphisms are translated to the desired isomorphisms of sheaves via identifications mentioned at the beginning of the proof. ∎

Notation 2.5. Let $x = (x_1, \ldots, x_d)$ be an ordered family of n variables, and let f be in $k[x]$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ be in $\mathbb{N}^d$. From now on, we use the following notation:

(i) $\alpha! = \alpha_1! \cdots \alpha_d!$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$;

(ii) $\alpha \geq \beta$ (equivalently, $\beta \leq \alpha$) if $\alpha_i \geq \beta_i$ for all $1 \leq i \leq d$;
$\alpha > \beta$ (equivalently, $\beta < \alpha$) if $\alpha \geq \beta$ but $\alpha \neq \beta$;

(iii) $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, and

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}};$$

(iv) For k of positive characteristic, we will consider $\partial^\gamma f(x)/\gamma!$ just as a formal symbol which stands for the coefficient of y^γ in the y-variable polynomial $f(x+y)-f(x)$ (see [18]).

Example 2.6. Let $x=(x_1,\dots,x_d)$, $x'=(x'_1,\dots,x'_d)$ and $R=k[x]$. Let f be in R, which defines a morphism $f\colon X=\operatorname{Spec} R\to \operatorname{Spec} k[t]$. Let $S=\operatorname{Spec} k[t]=\operatorname{Spec} k[t']$ and $B=\operatorname{Spec} k$. Then we have

$$X\times_k X=\operatorname{Spec} k[x,x'],\quad S\times_k S=\operatorname{Spec} k[t,t'],$$

and

$$\mathcal{I}_{g\circ f}=\langle x'_1-x_1,\dots,x'_d-x_d\rangle,\quad \mathcal{I}_g=\langle t-t'\rangle,\quad \mathcal{K}=\langle f(x')-f(x)\rangle.$$

We obtain furthermore that $\mathcal{P}^{n-1}_{g\circ f}\cong R^{\binom{d+n-1}{d}}$, and that its canonical R-basis is

$$\{[(x'-x)^\beta]\mid 0\le|\beta|\le n-1\};$$

similarly, $\Omega^{(n)}_{g\circ f}\cong R^{\binom{d+n}{d}-1}$, and its canonical R-basis is $\{[(x'-x)^\alpha]\mid 1\le|\alpha|\le n\}$. A system of generators of the R-module $\mathcal{K}\otimes_R\mathcal{P}^{n-1}_{g\circ f}$ is formed by the following vectors:

$$(f(x')-f(x))\otimes[(x'-x)^\beta],\quad 0\le|\beta|\le n-1,$$

because, thanks to Notation 2.5,

$$f(x')-f(x)=\sum_{\gamma>0}\frac{\partial^\gamma f(x)}{\gamma!}(x'-x)^\gamma,$$

we have for $0\le|\beta|\le n-1$,

$$\begin{aligned}\Psi_n\big((f(x')-f(x))\otimes[(x'-x)^\beta]\big)&=\Big[(x'-x)^\beta\sum_{\gamma>0}\frac{\partial^\gamma f(x)}{\gamma!}(x'-x)^\gamma\Big]\\&=\sum_{1\le|\alpha|\le n,\,\alpha>\beta}\frac{\partial^{\alpha-\beta}f(x)}{(\alpha-\beta)!}[(x'-x)^\alpha].\end{aligned}$$

Using the exact sequence in Proposition 2.3, we have that

$$\Big\{\sum_{1\le|\alpha|\le n,\,\alpha>\beta}\frac{\partial^{\alpha-\beta}f(x)}{(\alpha-\beta)!}[(x'-x)^\alpha]\ \Big|\ 0\le|\beta|\le n-1\Big\}$$

is a system of generators of $\ker(\Theta_n)$ as an R-module.

Similarly as in Corollary 16.4.22 of [15], we can show that if f is of finite type, then $\Omega^{(n)}_f$ is a quasi-coherent sheaf on X and an $\mathcal{O}_X$-module of finite type. As S is Noetherian and f is of finite type, this implies that f is of finite presentation, hence $\Omega^{(n)}_f$ is an $\mathcal{O}_X$-module of finite presentation (this can be also seen explicitly due to Proposition 2.3).

If f is a morphism of varieties, we denote by $\operatorname{Crit}(f)$ its critical locus. To define a class of morphisms for which we will define higher Nash blowups, we consider the following condition for a morphism $f\colon X\to S$ of k-varieties.

Condition 2.7. For every irreducible component X' of X, the closure of $f(X')$ is an irreducible component of S, and $X \setminus \mathrm{Crit}(f)$ is dense in X.

The relative dimension of a morphism $f: X \to S$ satisfying Condition 2.7 is defined to be $\dim X - \dim S$. Note that if k is of characteristic zero, X is smooth and $S = \mathbb{A}^1_k$; then the above condition is satisfied if and only if $f|_{X'}$ is non-constant for every irreducible component X' of X.

Lemma 2.8. *Let $f: X \to S$ be a morphism satisfying Condition 2.7 of relative dimension e. Then $\Omega_f^{(n)}$ is a coherent $\mathcal{O}_X$-module locally free of constant rank $r = \binom{e+n}{e} - 1$ on an open dense subset of X.*

Proof. Let $U \subset X$ be the smooth locus of f, which is an open dense subset of X by Condition 2.7. From Definition 16.10.1 and Proposition 17.12.4 in [15], $\mathcal{P}^n_f|_U = \mathcal{O}_U \oplus \Omega_f^{(n)}|_U$ is locally free and has the same rank as $\bigoplus_{i=0}^n S^i_{\mathcal{O}_U} \Omega^1_f|_U$ everywhere on U, where $S^i_{\mathcal{O}_U} \Omega^1_f|_U$ denotes the i-th symmetric power of $\Omega^1_f|_U$. The lemma now follows from the fact that $\Omega^1_f|_U$ has constant rank e. ■

Definition 2.9. Let $f: X \to S$ be a morphism of k-varieties satisfying Condition 2.7. The *n-th Nash blowup of f* is defined to be the blowup of X at $\Omega_f^{(n)}$, $\mathrm{Bl}_{\Omega_f^{(n)}}(X)$, and denoted by $\mathbf{Nash}_n(f)$, or by $\mathbf{Nash}_n(X/S)$ when the morphism $f: X \to S$ is fixed.

Consider the particular case where f is $X \to \operatorname{Spec} k$. Assume the k-dimension of X is d. Then $\mathbf{Nash}_n(X/\operatorname{Spec} k)$ is nothing but the n-th Nash blowup of X, denoted by $\mathbf{Nash}_n(X)$, in the sense of Yasuda [22]. Denote by $\Delta(X)^{(n)}$ the n-th infinitesimal neighborhood of $\Delta(X)$, and by pr_1 the restricted first projection $\Delta(X)^{(n)} \to X$. According to [22], $\mathbf{Nash}_n(X)$ is the irreducible component dominating X of the relative Hilbert scheme $\mathbf{Hilb}_{\binom{d+n}{d}}(\mathrm{pr}_1)$ for a constant Hilbert polynomial $\binom{d+n}{d}$. Since the moduli schemes $\mathbf{Hilb}_{\binom{d+n}{d}}(\mathrm{pr}_1)$ and $\mathbf{Grass}_{\binom{d+n}{d}}(\mathcal{O}_X \oplus \Omega_f^{(n)})$ present equivalent functors, we have

$$\mathbf{Nash}_n(X) \cong \mathrm{Bl}_{\mathcal{O}_X \oplus \Omega_f^{(n)}}(X) \cong \mathrm{Bl}_{\Omega_f^{(n)}}(X) = \mathbf{Nash}_n(X/\operatorname{Spec} k),$$

see Proposition 1.8 and Corollary 1.9 in [22].

Let $f: X \to S$ be a morphism satisfying Condition 2.7 of relative dimension e, and let $\mathcal{Q}(X)$ be the sheaf of total quotient rings of X. Let $n \in \mathbb{N}^*$ and $r = \binom{e+n}{e} - 1$. Let ψ be the composition of the canonical morphism

$$\bigwedge^r \Omega_f^{(n)} \to \bigwedge^r \Omega_f^{(n)} \otimes_{\mathcal{O}_X} \mathcal{Q}(X)$$

and a fixed isomorphism

$$\bigwedge^r \Omega_f^{(n)} \otimes_{\mathcal{O}_X} \mathcal{Q}(X) \to \mathcal{Q}(X). \tag{2.1}$$

Then $\operatorname{Im} \psi$ is a coherent fractional ideal of $\mathcal{Q}(X)$, locally free of rank 1 on an open dense subscheme U of X. In general, there are several isomorphisms as (2.1), so the identification of $\bigwedge^r \Omega_f^{(n)} \otimes_{\mathcal{O}_X} \mathcal{Q}(X)$ and $\mathcal{Q}(X)$ is not canonical, thus $\operatorname{Im} \psi$ is well defined up to isomorphism.

Proposition 2.10. *Let $f\colon X \to S$ be a morphism satisfying Condition 2.7 of relative dimension e, and let $n \in \mathbb{N}^*$ and $r = \binom{e+n}{e} - 1$. There is an isomorphism of k-varieties*

$$\mathbf{Nash}_n(f) \cong \mathrm{Bl}_{\bigwedge^r \Omega_f^{(n)}}(X).$$

Moreover, $\mathbf{Nash}_n(f)$ *is isomorphic to the blowup of X with respect to the fractional ideal* $\operatorname{Im}\psi$.

Proof. This is a direct application of Oneto–Zatini's result of Theorem 3.1 in [19] to the k-variety X and the sheaf $\mathcal{M} = \Omega_f^{(n)}$. ∎

2.2. Higher Jacobian ideals of regular functions

Let X be a scheme, and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. For each non-negative integer i, the *i-th Fitting ideal sheaf of $\mathcal{M}$*, denoted by $\mathbf{Fitt}_i(\mathcal{M})$, is defined as follows. We take an open covering $X = \bigcup_\lambda U_\lambda$ such that for each λ, $\mathcal{M}|_{U_\lambda}$ admits a free presentation:

$$\mathcal{O}_{U_\lambda}^{N'} \xrightarrow{\psi_\lambda} \mathcal{O}_{U_\lambda}^{N} \xrightarrow{\theta} \mathcal{M}|_{U_\lambda} \longrightarrow 0.$$

For each λ, we define $\mathbf{Fitt}_i(\mathcal{M})|_{U_\lambda} \subseteq \mathcal{O}_{U_\lambda}$ to be the ideal sheaf generated by $(N-i) \times (N-i)$-minors of the matrix representing ψ_λ. We see that these ideal sheaves glue together to give an ideal sheaf on the entire scheme X; we define $\mathbf{Fitt}_i(\mathcal{M})$ to be this ideal sheaf. We denote by $\mathcal{K}$ the ideal sheaf glued from the kernel of θ varying λ.

Remark 2.11. Similarly, we can define Fitting ideals $\mathbf{Fitt}_i(\mathcal{M})$ of a coherent sheaf $\mathcal{M}$ on a complex analytic space X.

We discuss more on the local setting. Let U be an affine open subscheme of X, $M = \mathcal{M}(U)$, $R = \mathcal{O}_X(U)$ and $K = \mathcal{K}(U) \subseteq R^{N'}$. Let $\{e_1, \dots, e_N\}$ be the canonical basis of R^N, and $\{v_1, \dots, v_N\}$ a system of generators of M. Let $\theta\colon R^N \to M$ be the R-homomorphism defined by $\theta(e_i) = v_i$, $1 \le i \le N$. Let $\{(a_{1j}, \dots, a_{Nj}) \mid 1 \le j \le N'\}$ be a system of generators of K, and let

$$A := (a_{ij})_{1 \le i \le N, 1 \le j \le N'}.$$

Then A is called a *relation matrix of M with respect to* $\{v_1, \dots, v_N\}$. Then $\mathrm{Fitt}_i(M) := \mathbf{Fitt}_i(\mathcal{M})(U)$ is generated by all $((N-i) \times (N-i))$-minors of A, which is independent of the choice of A and $\{v_1, \dots, v_N\}$ (cf. Lemmas D.1 and D.2 in [17]).

Let X be a smooth algebraic k-variety of dimension d, and let $f\colon X \to \mathbb{A}_k^1$ be a non-constant regular function. Then $\Omega_f^{(n)}$ is a coherent $\mathcal{O}_X$-module locally free of constant rank $r = \binom{d-1+n}{d-1} - 1$ on an open subset of X (cf. Lemma 2.8).

Definition 2.12. The r-th Fitting ideal $\mathbf{Fitt}_r(\Omega_f^{(n)})$ of the $\mathcal{O}_X$-module $\Omega_f^{(n)}$ is called the *n-th Jacobian ideal of f*, and is denoted by $\mathcal{J}_n(f)$.

Let us give an explicit description of $\mathcal{J}_n(f)$ in the affine case $X = \operatorname{Spec} R$ with $R = k[x] = k[x_1, \dots, x_d]$. This case is also enough for results and applications we will mention.

To an $f \in R$ and an $n \in \mathbb{N}^*$, we associate a matrix described as follows:

$$\mathrm{Jac}_n(f) := (r_{\beta,\alpha})_{0\le|\beta|\le n-1, 1\le|\alpha|\le n},$$

where the ordering for row and column indices is graded lexicographical,

$$r_{\beta,\alpha} = r_{\beta,\alpha}(f) := \begin{cases} 0 & \text{if } \alpha_i < \beta_i \text{ for some } 1 \le i \le d, \\ 0 & \text{if } \alpha = \beta, \\ \frac{\partial^{\alpha-\beta} f}{(\alpha-\beta)!} & \text{if } \alpha > \beta, \end{cases} \tag{2.2}$$

using Notation 2.5. Clearly, $\mathrm{Jac}_n(f)$ is a matrix of type $\binom{d-1+n}{d} \times \left(\binom{d+n}{d} - 1\right)$ with entries in R.

Definition 2.13. For $f \in R$, the matrix $\mathrm{Jac}_n(f)$ is called the *Jacobian matrix of order n of f*, or the *n-th Jacobian matrix of f*.

Remark 2.14. There are a few versions of higher Jacobian matrix which are slightly different from one another. Our definition above follows the one adopted in [2]. Another version considered in [12, 16] differs in that the diagonal entries $r_{\alpha,\alpha}$ are f instead of 0. These two versions coincide modulo f. The one in [4], which the authors call the *Jacobi–Taylor matrix*, has one extra column by allowing $|\alpha| = 0$.

Let us consider the ring homomorphism $k[t] \to R$ which maps t to f. Let I be the kernel of the diagonal homomorphism $R \otimes_{k[t]} R \to R$, which is an R-module via the homomorphism $R \to R \otimes_{k[t]} R$ given by $\xi \mapsto \xi \otimes 1$. By Lemma 2.8, $\Omega_f^{(n)} = I/I^{n+1}$ is an R-module of generic rank $r = \binom{d-1+n}{d-1} - 1$. For $1 \le i \le d$, put $\delta x_i := 1 \otimes x_i - x_i \otimes 1$, and for $\alpha = (\alpha_1, \dots, \alpha_d)$, put $(\delta x)^\alpha := \prod_{i=1}^d (\delta x_i)^{\alpha_i}$. Then $I = \langle \delta x_1, \dots, \delta x_d \rangle$ and $\{[(\delta x)^\alpha] \mid 1 \le |\alpha| \le n\}$ is a system of generators of the R-module $\Omega_f^{(n)}$. Let $\{e_\alpha \mid 1 \le |\alpha| \le n\}$ be the canonical R-basis of $R^{\binom{d+n}{d}-1}$, using graded lexicographical ordering for indices α in $\mathbb{N}^d$. Consider the homomorphism of R-modules

$$\theta : R^{\binom{d+n}{d}-1} \to \Omega_f^{(n)}$$

defined by

$$\theta(e_\alpha) = [(\delta x)^\alpha].$$

Proposition 2.15. *With the previous notation, for $f \in R$, the transpose of $\mathrm{Jac}_n(f)$ is a relation matrix of $\Omega_f^{(n)}$ with respect to $\{[(\delta x)^\alpha] \mid 1 \le |\alpha| \le n\}$. As a consequence, $\mathcal{J}_n(f)$ is the ideal of R generated by all the maximal minors of the matrix $\mathrm{Jac}_n(f)$.*

Proof. Consider the homomorphism Θ_n defined in Section 2.1 for the case $f \colon \mathbb{A}_k^d \to \mathbb{A}_k^1$ and $g \colon \mathbb{A}_k^1 \to \operatorname{Spec} k$. Via the natural isomorphism $\Omega_{g\circ f}^{(n)} \xrightarrow{\cong} R^{\binom{d+n}{d}-1}$ mapping $[(x'-x)^\alpha]$ to e_α, Θ_n is nothing but θ. Also, via this isomorphism, it is computed in Example 2.6 that

$$\left\{ \sum_{1\le|\alpha|\le n,\, \alpha>\beta} \frac{\partial^{\alpha-\beta} f(x)}{(\alpha-\beta)!} e_\alpha \,\middle|\, 0 \le |\beta| \le n-1 \right\} = \left\{ \sum_{1\le|\alpha|\le n} r_{\beta,\alpha} e_\alpha \,\middle|\, 0 \le |\beta| \le n-1 \right\}$$

is a system of generators of $\ker(\theta)$ as an R-module. This proves that the transpose of $\mathrm{Jac}_n(f)$ is a relation matrix of $\Omega_f^{(n)}$ with respect to $\{[(\delta x)^\alpha] \mid 1 \leq |\alpha| \leq n\}$. ∎

Example 2.16. For $f \in R$, we have

$$\mathrm{Jac}_2(f) = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_d} & \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d} & \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} & \cdots \\ 0 & 0 & \cdots & 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_d} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \frac{\partial f}{\partial x_1} & \cdots & 0 & \frac{\partial f}{\partial x_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{\partial f}{\partial x_1} & 0 & \cdots \end{pmatrix}.$$

More particularly, for $f(x_1, x_2) = x_1^3 - x_2^2$, we have

$$\mathrm{Jac}_2(f) = \begin{pmatrix} 3x_1^2 & -2x_2 & 3x_1 & 0 & -1 \\ 0 & 0 & 3x_1^2 & -2x_2 & 0 \\ 0 & 0 & 0 & 3x_1^2 & -2x_2 \end{pmatrix}$$

and

$$\mathcal{J}_2(f) = \langle x_1^6,\ x_1^4 x_2,\ x_1^2 x_2^2,\ x_2^3,\ 4x_1 x_2^2 - 3x_1^4 \rangle.$$

Remark 2.17. We have a more direct way to show that the row vectors of $\mathrm{Jac}_n(f)$ are elements of the kernel of θ. Indeed, let r_β denote the β-row of $\mathrm{Jac}_n(f)$, namely

$$r_\beta = (r_{\beta,\alpha})_{1 \leq |\alpha| \leq n} = \sum_{1 \leq |\alpha| \leq n} r_{\beta,\alpha}\, e_\alpha \ \in R^{\binom{d+n}{d}-1}.$$

Then, we have

$$\begin{aligned} \theta(r_\beta) &= \sum_{1 \leq |\alpha| \leq n,\, \alpha > \beta} \frac{\partial^{\alpha-\beta} f(x)}{(\alpha-\beta)!}[(\delta x)^\alpha] \\ &= \Big[(\delta x)^\beta \sum_{1 \leq |\alpha| \leq n,\, \alpha > \beta} \Big(\frac{\partial^{\alpha-\beta} f(x)}{(\alpha-\beta)!} \otimes 1\Big) \cdot (\delta x)^{\alpha-\beta}\Big] \\ &= \Big[(\delta x)^\beta \sum_{1 \leq |\gamma| \leq n-|\beta|,\, \gamma > 0} \frac{\partial^\gamma f(x \otimes 1)}{\gamma!} \cdot (\delta x)^\gamma\Big]. \end{aligned}$$

Note that $[(\delta x)^{\beta+\gamma}] = 0$ in $\Omega_f^{(n)}$ for $|\beta| + |\gamma| > n$. Thus, using Taylor's expansion,

$$\begin{aligned} \theta(r_\beta) &= \Big[(\delta x)^\beta \sum_{\gamma > 0} \frac{\partial^\gamma f(x \otimes 1)}{\gamma!} \cdot (\delta x)^\gamma\Big] \\ &= \Big[(\delta x)^\beta \sum_{\gamma \geq 0} \frac{\partial^\gamma f(x \otimes 1)}{\gamma!} \cdot (\delta x)^\gamma - (\delta x)^\beta f(x \otimes 1)\Big] \\ &= \big[(\delta x)^\beta \big(f(1 \otimes x) - f(x \otimes 1)\big)\big] = \big[(\delta x)^\beta \big(1 \otimes f - f \otimes 1\big)\big] = 0. \end{aligned}$$

Furthermore, the rank of $\mathrm{Jac}_n(f)$ is $\binom{d-1+n}{d}$, which is equal to $\binom{d+n}{d} - 1 - r$. This is only enough to show that $\mathcal{J}_n(f)$ is isomorphic as a fractional ideal to the ideal of R generated by all the maximal minors of the matrix $\mathrm{Jac}_n(f)$, which is weaker than the statement of Proposition 2.15.

Proposition 2.18. *Let f be a non-constant regular function on a smooth d-dimensional k-variety X, and let $n \in \mathbb{N}^*$. Then $\mathbf{Nash}_n(f)$ is isomorphic to the blowup of X with respect to $\mathcal{J}_n(f)$.*

Proof. By Proposition 2.15, as well as by Proposition 2.5 and Corollary 2.6 in [21], $\mathcal{J}_n(f)$ and $\mathrm{Im}\,\psi$ mentioned in Proposition 2.10 are isomorphic as fractional ideals. Thus $\mathbf{Nash}_n(f)$ is isomorphic to the blowup of X with respect to $\mathcal{J}_n(f)$. ∎

Proposition 2.19. *Let f be a regular function on a smooth d-dimensional k-variety X. Let $\varphi\colon X' \to X$ be an étale morphism. Then, we have $\varphi^{-1}\mathcal{J}_n(f) = \mathcal{J}_n(f \circ \varphi)$. Similarly for the case where f is a holomorphic function on a complex manifold X and $\varphi\colon X' \to X$ is a local isomorphism of complex manifolds.*

Proof. From Lemma 2.4, we have $\varphi^*\Omega_f^{(n)} = \Omega_{f\circ\varphi}^{(n)}$. From a basic property of Fitting ideals, we have

$$\mathbf{Fitt}_r(\varphi^*\mathcal{M}) = \varphi^{-1}\,\mathbf{Fitt}_r(\mathcal{M})$$

for any coherent sheaf $\mathcal{M}$ on X. Similarly for the holomorphic setting. ∎

Proposition 2.20. *We have an inclusion $\mathcal{J}_n(f) \subseteq \mathcal{J}_1(f)^{\binom{d-2+n}{d-1}}$. In particular, $\mathcal{J}_n(f) \subseteq \mathcal{J}_1(f)^3$, if either*

(i) *$d \geq 3$ and $n \geq 2$, or*

(ii) *$d = 2$ and $n \geq 3$.*

Proof. If $|\beta| = n-1$, then the β-row of the matrix $\mathrm{Jac}_n(f)$ has only zeroes and first derivatives $\partial f/\partial x_i$ as its entries. Indeed, if α is given by $\alpha_i = \beta_i + 1$ and $\alpha_j = \beta_j$ $(i \neq j)$ for some i, then the (β,α)-entry of the matrix is $\partial f/\partial x_i$. If α is not of this form, then the (β,α)-entry is zero. The number of β's with $|\beta| = n-1$ is $\binom{d-2+n}{d-1}$, and hence there are $\binom{d-2+n}{d-1}$ rows as above in the matrix. It follows that every maximal minor of $\mathrm{Jac}_n(f)$ belongs to $\mathcal{J}_1(f)^{\binom{d-2+n}{d-1}}$, which shows the first assertion of the proposition.

To show the second assertion, we first note that if $r > s$, then

$$\binom{r}{s} \geq \binom{r-1}{s-1} \geq \cdots \geq \binom{r-s+1}{1} = r-s+1.$$

If $d \geq 3$ and $n \geq 2$, then

$$\binom{d-2+n}{d-1} = \binom{d-3+n}{d-2} + \binom{d-3+n}{d-1} \geq n + (n-1) = 2n-1 \geq 3.$$

If $d = 2$ and $n \geq 3$, then

$$\binom{d-2+n}{d-1} = n \geq 3.$$ ∎

2.3. Higher Jacobian ideals of complex analytic functions

In this subsection, we consider the ring $\mathbb{C}\{x\} = \mathbb{C}\{x_1, \dots, x_d\}$ and $f \in \mathbb{C}\{x\}$ with $f(\mathbf{0}) = 0$, where $\mathbf{0}$ is the origin of $\mathbb{C}^d$. Denote by $V = V(f)$ or $(V, \mathbf{0})$ the germ of the complex hypersurface singularity at $\mathbf{0}$ defined by f. For a complex analytic function f, we also define $\mathrm{Jac}_n(f)$ and $\mathcal{J}_n(f)$ similarly as in Section 2.2.

Lemma 2.21. *Let f be in $\mathbb{C}\{x\}$ with $f(\mathbf{0}) = 0$. Let φ be an automorphism of $\mathbb{C}\{x\}$. Then for $n \in \mathbb{N}^*$, we have the equality $\varphi(\mathcal{J}_n(f)) = \mathcal{J}_n(\varphi(f))$ of ideals of $\mathbb{C}\{x\}$.*

Proof. This lemma is a particular case of the second part of Proposition 2.19. ∎

Recall that $f(x) = \sum_\alpha a_\alpha x^\alpha \in \mathbb{C}[x]$ is a *weighted homogeneous* polynomial with respect to weights $\deg x_i = w_i \in \mathbb{Z}$ $(1 \le i \le d)$ if $\alpha_1 w_1 + \cdots + \alpha_d w_d$ is a constant $N \in \mathbb{N}^*$ for every α with $a_\alpha \neq 0$. One calls N the *weight* or the *degree* of f.

Corollary 2.22. *Let f be a weighted homogeneous polynomial in $\mathbb{C}[x]$ with respect to positive weights, and let u be in $\mathbb{C}\{x\}$ with $u(\mathbf{0}) \neq 0$. Then for every $n \in \mathbb{N}^*$, we have the equality $\mathcal{J}_n(f) = \mathcal{J}_n(uf)$ of ideals of $\mathbb{C}\{x\}$.*

Proof. Let $w_1, \dots, w_d \in \mathbb{N}^*$ be the weights of f, and let N be the degree of f. Consider the automorphism φ of $\mathbb{C}\{x\}$ defined by

$$x \mapsto (u^{w_1} x_1, \dots, u^{w_d} x_d).$$

Then we have

$$\varphi(f) = f(u^{w_1} x_1, \dots, u^{w_d} x_d) = u^N f.$$

For every $n \in \mathbb{N}^*$, let us compare the n-th Jacobian matrix of f in [1] and our $\mathrm{Jac}_n(f)$. They are the same at almost all but (β, β)-entries, for $\beta \in \mathbb{N}^d$ and $0 \le |\beta| \le n-1$. The (β, β)-entries in [1] are f, while our (β, β)-entries are 0. By Theorem 1.6 in [1] and its proof, the n-th Jacobian ideal admits a system of generators such that each generator g is a weighted homogeneous polynomial of weights $w_1, \dots, w_d$, hence so does our $\mathcal{J}_n(f)$. As above, we have $\varphi(g) = u^N g$, which implies that $\varphi(\mathcal{J}_n(f)) = \mathcal{J}_n(f)$ because u is a unit. Now, the corollary follows from Lemma 2.21. ∎

Remark 2.23. Let f be in $\mathbb{C}[x]$ such that $f(\mathbf{0}) = 0$. We are going to describe the stalk $(\Omega_f^{(n)})_{\mathbf{0}}$ of the sheaf $\Omega_f^{(n)}$ at the origin of $\mathbb{C}^d$. Consider the ring homomorphism

$$\mathbb{C}\{t\} \to \mathbb{C}\{x\}, \quad t \mapsto f.$$

Let I' be the kernel of the diagonal homomorphism $\mathbb{C}\{x\} \otimes_{\mathbb{C}\{t\}} \mathbb{C}\{x\} \to \mathbb{C}\{x\}$, which is an $\mathbb{C}\{x\}$-module via the map $\mathbb{C}\{x\} \to \mathbb{C}\{x\} \otimes_{\mathbb{C}\{t\}} \mathbb{C}\{x\}$ sending ξ to $\xi \otimes 1$. We can prove that I'/I'^{n+1} is a $\mathbb{C}\{x\}$-module of generic rank $r = \binom{d-1+n}{d-1} - 1$. Then, similarly as in Corollary 16.4.16 of [15], we have $(\Omega_f^{(n)})_{\mathbf{0}} \cong I'/I'^{n+1}$ as $\mathbb{C}\{x\}$-modules.

Lemma 2.24. *Let f and u be in $\mathbb{C}\{x\}$ such that $f(\mathbf{0}) = 0$ and $u(\mathbf{0}) \neq 0$. Then for any $n \in \mathbb{N}^*$, $\langle f, \mathcal{J}_n(f) \rangle_{\mathbb{C}\{x\}} = \langle f, \mathcal{J}_n(uf) \rangle_{\mathbb{C}\{x\}}$.*

Proof. First, we use the convention that $\partial^{\alpha-\beta} = 0$ if there is an i such that $\alpha_i < \beta_i$. For each injection

$$\iota : \{\beta \mid 0 \le |\beta| \le n-1\} \to \{\alpha \mid 1 \le |\alpha| \le n\},$$

we define

$$M_\iota^f := \big(r_{\beta,\iota(\beta')}(f)\big)_{\beta,\beta'}.$$

Up to permutation of columns, this is equal to the maximal square submatrix of $\mathrm{Jac}_n(f)$ corresponding to the image of ι. Let

$$T := \left(\frac{\partial^{\beta'-\beta} u}{(\beta'-\beta)!}\right)_{0\le|\beta|,|\beta'|\le n-1} \quad \text{and} \quad N_\iota := T M_\iota^f.$$

We see that T is an upper triangular matrix with every diagonal entry equal to u. Indeed, if $\beta > \beta'$ for the graded lexicographic order, then for some i, $\beta_i > \beta'_i$ and hence the (β, β')-entry is zero. It follows that

$$\det(N_\iota) = u^{\binom{d-1+n}{d}} \det(M_\iota^f).$$

A direct computation shows that the (β, β')-entry of N_ι is

$$\sum_{0\le|\beta''|\le n-1} \frac{\partial^{\beta''-\beta} u}{(\beta''-\beta)!} \frac{\partial^{\iota(\beta')-\beta''} f}{(\iota(\beta')-\beta'')!},$$

with the convention that $\partial^{(0,\dots,0)} f = 0$. By the general Leibniz rule, this is equal to

$$r_{\beta,\iota(\beta')}(uf).$$

Thus $M_\iota^{uf} = N_\iota$ and $\det(M_\iota^{uf}) = \det(N_\iota) = u^{\binom{d-1+n}{d}} \det(M_\iota^f)$. Because the ideals $\mathcal{J}_n(f)$ and $\mathcal{J}_n(uf)$ are generated by $\{\det(M_\iota^f)\}_\iota$ and $\{\det(M_\iota^{uf})\}_\iota$ respectively, and because of the previous convention, the lemma follows. ∎

Definition 2.25. Let f and g be in $\mathbb{C}\{x\}$. Then f is said to be *contact equivalent* to g (at $\mathbf{0}$) if there exist an automorphism φ of $\mathbb{C}\{x\}$ and a unit u in $\mathbb{C}\{x\}$ such that $g = u \cdot \varphi(f)$.

Theorem 2.26. *Let f and g be in $\mathbb{C}\{x\}$ with $f(\mathbf{0}) = g(\mathbf{0}) = 0$. If f is contact equivalent to g at $\mathbf{0}$, then $\mathcal{T}_n(f)$ is isomorphic to $\mathcal{T}_n(g)$ as $\mathbb{C}$-algebras for any $n \in \mathbb{N}^*$.*

Proof. By the hypothesis, there exists an automorphism φ of $\mathbb{C}\{x\}$ and a unit u in $\mathbb{C}\{x\}$ such that $g = u \cdot \varphi(f)$. Take any $n \in \mathbb{N}^*$. It follows that

$$\begin{aligned}
\mathcal{T}_n(g) = \mathbb{C}\{x\}/\langle g, \mathcal{J}_n(g)\rangle &= \mathbb{C}\{x\}/\langle \varphi(f), \mathcal{J}_n(u\cdot\varphi(f))\rangle \\
&= \mathbb{C}\{x\}/\langle \varphi(f), \mathcal{J}_n(\varphi(f))\rangle \quad \text{(by Lemma 2.24)} \\
&= \mathbb{C}\{x\}/\langle \varphi(f), \varphi(\mathcal{J}_n(f))\rangle \quad \text{(by Lemma 2.21)} \\
&= \mathbb{C}\{x\}/\varphi(\langle f, \mathcal{J}_n(f)\rangle).
\end{aligned}$$

Since φ is an automorphism φ of $\mathbb{C}\{x\}$, the well defined map

$$\mathcal{T}_n(f) = \mathbb{C}\{x\}/\langle f, \mathcal{J}_n(f)\rangle \to \mathbb{C}\{x\}/\varphi(\langle f, \mathcal{J}_n(f)\rangle),$$

sending $h + \langle f, \mathcal{J}_n(f)\rangle$ to $\varphi(h) + \varphi(\langle f, \mathcal{J}_n(f)\rangle)$, is an isomorphism of $\mathbb{C}$-algebras. ∎

Corollary 2.27 ([16], Conjecture 1.5). *Let f and g be in $\mathbb{C}[x]$ such that $V(f)$ and $V(g)$ have an isolated singularity at $\mathbf{0}$. If f is contact equivalent to g at $\mathbf{0}$, then $\mathcal{T}_n(f)$ is isomorphic to $\mathcal{T}_n(g)$ as $\mathbb{C}$-algebras for any $n \in \mathbb{N}^*$.*

We remark that Conjecture 1.5 of [16] was proved by its authors for $d = n = 2$, see Theorem A in [16]. Furthermore, Theorem 2.26 is much stronger than this conjecture since Theorem 2.26 accepts f and g to be in $\mathbb{C}\{x\}$, and it does not need the condition that f and g have an isolated singularity at $\mathbf{0}$.

3. Motivic zeta functions of regular functions

In this section, k is a field of characteristic zero.

3.1. Motivic zeta functions

By a *k-prevariety*, we mean a separated and reduced scheme of finite type over k. (We introduce this term for a temporal use, restricted to this and the next paragraphs, in order to mean a "variety not necessarily of pure dimension".) For an integer $m \geq 1$, we denote by μ_m the group scheme of n-th roots of unity $\mathrm{Spec}(k[\tau]/(\tau^m - 1))$. These schemes together with the mappings $\mu_{ml} \to \mu_m$ given by $\xi \mapsto \xi^l$ form a projective system, whose limit is denoted by $\hat{\mu}$. A *good* μ_m-action on a k-prevariety X is a group action of μ_m on X such that every orbit is contained in an affine open subset, a *good* $\hat{\mu}$-action is an action of $\hat{\mu}$ factoring via a good μ_m-action.

Let S be a k-prevariety, with trivial μ_m-action. The *μ_m-equivariant Grothendieck group* $K_0^{\mu_m}(\mathrm{Var}_S)$ is the quotient of the free abelian group generated by the μ_m-equivariant isomorphism classes $[X \to S, \sigma]$, where σ is a good action of μ_m on X, considered as an automorphism of X over S, by the subgroup generated by

$$[X \to S, \sigma] - [Y \to S, \sigma|_Y] - [X \setminus Y \to S, \sigma|_{X \setminus Y}]$$

for any invariant Zariski closed subset Y of X, and by

$$[X \times_k \mathbb{A}_k^n \to S, \sigma] - [X \times_k \mathbb{A}_k^n \to S, \sigma']$$

when σ and σ' are cartesian products of μ_m-actions that coincide on X and are linear on $\mathbb{A}_k^n$. There is a natural structure of a commutative ring with unity on $K_0^{\mu_m}(\mathrm{Var}_S)$ due to fiber product over S, with μ_m-action on the fiber product induced from the diagonal one. Let $\mathbb{L}$ be the class of the trivial line bundle over S. We define

$$\mathcal{M}_S^{\mu_m} := K_0^{\mu_m}(\mathrm{Var}_S)[\mathbb{L}^{-1}], \quad K_0^{\hat{\mu}}(\mathrm{Var}_S) := \varinjlim K_0^{\mu_m}(\mathrm{Var}_S), \quad \text{and}$$
$$\mathcal{M}_S^{\hat{\mu}} := \varinjlim \mathcal{M}_S^{\mu_m} = K_0^{\hat{\mu}}(\mathrm{Var}_S)[\mathbb{L}^{-1}].$$

Forgetting actions recovers the (classical) Grothendieck ring $\mathcal{M}_S$.

To a k-variety X and $m \in \mathbb{N}$ corresponds the k-scheme $\mathcal{L}_m(X)$ that represents the functor

$$K \mapsto \mathrm{Mor}_{k-\mathrm{schemes}}(\mathrm{Spec}(K[t]/\langle t^{m+1}\rangle K[t]), X)$$

from the category of k-algebras to the category of sets. For integers $l \geq m \geq 0$, the truncation map $k[t]/\langle t^{l+1}\rangle \to k[t]/\langle t^{m+1}\rangle$ induces a morphism of k-schemes $\pi_m^l \colon \mathcal{L}_l(X) \to$

$\mathcal{L}_m(X)$. If X is smooth of dimension d, π^l_m is a locally trivial fibration with fiber $\mathbb{A}_k^{(l-m)d}$. Let $\mathcal{L}(X)$ be the projective limit of $\mathcal{L}_m(X)$ in the category of k-schemes, and let π_m be the natural morphism $\mathcal{L}(X) \to \mathcal{L}_m(X)$.

Let X be a smooth k-variety of dimension d, and let $f: X \to \mathbb{A}^1_k$ be a non-constant function with the scheme-theoretic zero locus X_0. For any integer $m \geq 1$, put

$$\mathcal{X}_m(f) := \{\psi \in \mathcal{L}_m(X) \mid f(\psi) = t^m \mod t^{m+1}\}$$

which is a k-variety endowed with the good μ_m-action as below, for $\xi \in \mu_m$,

$$\xi \cdot \psi(t) = \psi(\xi t).$$

This together with the natural morphism $\mathcal{X}_m(f) \to X_0$ yields an element $[\mathcal{X}_m(f)]$ in $\mathcal{M}_{X_0}^{\mu_m}$.

Definition 3.1. The formal power series $Z_f(T) = \sum_{m\geq 1}[\mathcal{X}_m(f)]\mathbb{L}^{-dm}T^m$ is called the *motivic zeta function* of f.

Let $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ be the $\mathcal{M}_{X_0}^{\hat{\mu}}$-submodule of $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ generated by 1 and by finite products of elements of the form

$$\frac{\mathbb{L}^p T^q}{1-\mathbb{L}^p T^q} := \sum_{m\geq 1} (\mathbb{L}^p T^q)^m, \quad (p,q) \in \mathbb{Z} \times \mathbb{N}^*.$$

An element in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ is called a *rational series*. According to [9], Section 4, there is a unique $\mathcal{M}_{X_0}^{\hat{\mu}}$-linear morphism $\lim_{T\to\infty} : \mathcal{M}_{X_0}^{\hat{\mu}}[[T]]_{\mathrm{sr}} \to \mathcal{M}_{X_0}^{\hat{\mu}}$ with $\lim_{T\to\infty} \frac{\mathbb{L}^p T^q}{1-\mathbb{L}^p T^q} = -1$ for all $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$.

By a *log-resolution of* (X, X_0), we mean a proper birational morphism $h: Y \to X$ such that Y is smooth, h is an isomorphism over $(X \setminus X_0) \cup (X_0)_{\mathrm{sm}}$, and $h^{-1}(X_0)$ has simple normal crossing support. Consider a log-resolution $h: Y \to X$ of (X, X_0). Let E_i, $i \in J$, be all the irreducible components of $h^{-1}(X_0)$. Assume that

$$\operatorname{div}(h^* f) = \sum_{i\in J} N_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i\in J}(\nu_i - 1)E_i,$$

where N_i and ν_i are positive integers. For $I \subseteq J$, put $E_I = \bigcap_{i\in I} E_i$ and $E_I^\circ = E_I \setminus \bigcup_{j\notin I} E_j$. If $I = \{i\}$, we write E_i° instead of $E_{\{i\}}^\circ$. Let U be an affine Zariski open subset of Y such that $U \cap E_I^\circ \neq \emptyset$, $U \cap E_I^\circ$ is closed in U, $U \cap \bigcup_{j\notin I} E_j = \emptyset$, and such that, on U,

$$h^* f(y) = u(y) \prod_{i\in I} y_i^{N_i},$$

where for each i, $y_i = 0$ is a local equation on the chart (U, y) defining E_i, and u is a morphism $U \to \mathbb{G}_m$. Let N_I be the greatest common divisor of $(N_i)_{i\in I}$. Denef–Loeser in [10] construct an unramified Galois covering $\pi_I: \widetilde{E}_I^\circ \to E_I^\circ$, with Galois group μ_{N_I} given over $U \cap E_I^\circ$ by

$$\widetilde{E}_I^\circ|_{U\cap E_I^\circ} := \{(z,y) \in \mathbb{A}^1_k \times (U \cap E_I^\circ) \mid z^{N_I} = u(y)^{-1}\} \to U \cap E_I^\circ$$
$$(z,y) \mapsto y.$$

Choose a covering of $Y \setminus \bigcup_{j \notin I} E_j$ by affine Zariski open subvarieties U. Then the varieties $\widetilde{E}_I^\circ|_{U \cap E_I^\circ}$ are naturally glued together into a unramified Galois covering $\widetilde{E}_I^\circ$, which is endowed with a natural μ_{N_I}-action. The μ_{N_I}-equivariant morphism $\widetilde{E}_I^\circ \to E_I^\circ \to X_0$ determines a class $[\widetilde{E}_I^\circ]$ in $\mathcal{M}_{X_0}^{\mu_{N_I}}$.

Theorem 3.2 (Denef–Loeser [10]). *Using a log-resolution h of (X, X_0), and the previous notation,*

$$[\mathcal{X}_m(f)] = \mathbb{L}^{md} \sum_{\emptyset \neq I \subseteq J} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}_I^\circ] \Bigg(\sum_{\substack{k_i \geq 1, i \in I \\ \sum_{i \in I} k_i N_i = m}} \mathbb{L}^{-\sum_{i \in I} k_i \nu_i} \Bigg)$$

and

$$Z_f(T) = \sum_{\emptyset \neq I \subseteq J} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}_I^\circ] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

In particular, $Z_f(T)$ is a rational series.

Definition 3.3. The motivic quantity

$$\mathcal{S}_f := -\lim_{T \to \infty} Z_f(T) = \sum_{\emptyset \neq I \subseteq J} (1-\mathbb{L})^{|I|-1} [\widetilde{E}_I^\circ] \in \mathcal{M}_{X_0}^{\hat{\mu}}$$

is called the *motivic nearby cycle* of f.

Definition 3.4. Keeping the notation above, for $m \in \mathbb{N}^*$, we say that a log-resolution h is called *m-separating* if $N_i + N_j > m$ whenever $E_i \cap E_j \neq \emptyset$.

We notice that an m-separating log-resolution always exists. This was proved over $\mathbb{C}$, with a slight different definition of log-resolutions, in Lemma 2.9 of [5]; they do not assume that a log-resolution $h\colon Y \to X$ needs to be an isomorphism over $(X \setminus X_0) \cup (X_0)_{\mathrm{sm}}$. However, the same proof shows that the result holds over any field of characteristic zero with our definition of log-resolutions. If h is an m-separating log-resolution of (X, X_0), it follows from Theorem 3.2 that

$$[\mathcal{X}_m(f)] = \mathbb{L}^{md} \sum_{N_i \mid m} [\widetilde{E}_i^\circ] \, \mathbb{L}^{-m\nu_i/N_i}. \tag{3.1}$$

3.2. Motivic zeta function is invariant modulo second order Jacobian ideal

Let X be a smooth k-variety of dimension $d \geq 2$. If f is a non-constant regular function on X, we also denote by f the corresponding element in $\mathcal{O}_X(X)$. Fix an algebraic closure $\bar{k}$ of k. In what follows, we say that a $\bar{k}$-point x of X_0 is a *node* if its complete local ring $\widehat{\mathcal{O}}_{(X_0)_{\bar{k}}, x}$ is isomorphic to $\bar{k}[[x_1, x_2]]/\langle x_1^2 + x_2^2 \rangle$. We say that X_0 *has nodes* if it has $\bar{k}$-points which are nodes.

Theorem 3.5. *Let f and g be non-constant regular functions on X with the same scheme-theoretic zero locus X_0. Suppose that $g - f \in \mathcal{J}_2(f)$. If $d = 2$ and X_0 has nodes, suppose additionally that k is quadratically closed. Then, for any integer $m \geq 1$, the identity $[\mathcal{X}_m(f)] = [\mathcal{X}_m(g)]$ holds in $\mathcal{M}_{X_0}^{\mu_m}$. As a consequence, $Z_f(T) = Z_g(T)$ and $\mathcal{S}_f = \mathcal{S}_g$.*

Proof. We first outline the structure of the proof. We will discuss two cases separately; the case where $d = 2$ and X_0 has a node, i.e., when $f(x)$ is written as $x_1^2 + x_2^2 +$ (terms of higher degree) for local coordinates x_1 and x_2, see case (iv) below; and the remaining case (cases (i), (ii) and (iii) below). In the latter case, we can show that $g - f$ has higher order than f at any point in question. Using this fact, we can show $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ for every i, where $\widetilde{E}_i^\circ(f)$ and $\widetilde{E}_i^\circ(g)$ are $\widetilde{E}_i^\circ$ defined using f and g, respectively, and using some m-separating log-resolution of (X, X_0). The equality $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ gives the desired conclusion. In the former case, we use a log-resolution h of (X, X_0) which is m-separating for exceptional divisors coming from singular points in X_0 that are not nodes (i.e., for such exceptional divisors $N_i(f) + N_j(f) > m$ if $E_i \cap E_j \neq \emptyset$); similarly as above, we can show $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$; also, for a node of X_0, an explicit computation shows that $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ if E_i is an exceptional prime divisor contracting to the point. Again, this is enough to obtain the desired conclusion.

We will now move on to a detailed discussion. Let us write $g - f = a$, for some $a \in \mathcal{J}_2(f)$. Since f and g have the same zero locus X_0, we have $a|_{X_0} = 0$. Write locally, on W, $a = a'f$, where W is an open affine subset of X and $a' \in \mathcal{O}_X(W)$. Let p be a non-smooth point of $X_0 \cap W$. Up to étale base change, we may assume that $W = \mathbb{A}_k^d$. Let $N \geq 2$ be the multiplicity of f at p, so that we can write f as

$$f(x) = f_N(x) + (\text{terms of degree} > N),$$

using local coordinates $x = (x_1, \ldots, x_d)$ and a nonzero homogeneous polynomial f_N of degree N. First, we consider the following three cases:

(i) $d \geq 3$,

(ii) $d = 2$ and $N > 2$,

(iii) $d = N = 2$ and $f_2(x_1, x_2) = cx_1^2$, $c \in k^*$ (up to linear change of variables).

We are going to prove that $a'(p) = 0$. Let $\alpha = \alpha(p)$ be a vector in $\mathbb{N}^d$ such that $|\alpha| = N$ and $(\partial^\alpha f)(p) \neq 0$. By the general Leibniz rule, applied to the equality $a = a'f$ on $W = \mathbb{A}_k^d$,

$$\partial^\alpha a = a'\partial^\alpha f + \sum_{\alpha' > 0,\ \alpha' + \beta = \alpha} \frac{\alpha!}{\alpha'!\beta!} (\partial^{\alpha'} a')(\partial^\beta f). \tag{3.2}$$

In the sum $\sum$ on the right-hand side, since $\alpha' > 0$, we have $|\beta| < |\alpha| = N$, hence $(\partial^\beta f)(p) = 0$. If $d \geq 3$, since a is a section of $\mathcal{J}_1(f)^3$ by Proposition 2.20, we can write a in the form

$$a = \sum_{1 \leq i,j,l \leq d} a_{ijl} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_l},$$

where the a_{ijl}'s are sections of $\mathcal{O}_X$. By the general Leibniz rule,

$$\partial^\alpha a = \sum_{1 \leq i,j,l \leq d} \sum_{\alpha' + \beta = \alpha} \frac{\alpha!}{\alpha'!\beta!} (\partial^{\alpha'} a_{ijl}) \Big(\partial^\beta \Big(\frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_l}\Big)\Big).$$

Let e_i be the i-th vector of the canonical basis of $\mathbb{N}^d$, and let $\gamma(i) = \gamma + e_i$ for any $\gamma \in \mathbb{N}^d$. Then, we have

$$\partial^\beta\Big(\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j}\,\frac{\partial f}{\partial x_l}\Big) = \sum_{\beta'+\beta''+\beta'''=\beta} \frac{\beta!}{\beta'!\,\beta''!\,\beta'''!}\,(\partial^{\beta'(i)} f)\,(\partial^{\beta''(j)} f)\,(\partial^{\beta'''(l)} f).$$

If all the three numbers $|\beta'(i)|$, $|\beta''(j)|$ and $|\beta'''(l)|$ are simultaneously $\geq N$, then

$$N + 3 \geq |\beta| + 3 = |\beta'(i)| + |\beta''(j)| + |\beta'''(l)| \geq 3N,$$

so $N \leq 3/2$, which is impossible. Thus, at least one of these numbers is strictly less than N, for instance, $|\beta'(i)| < N$, hence $(\partial^{\beta'(i)} f)(p) = 0$. It follows that

$$\partial^\beta\Big(\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j}\,\frac{\partial f}{\partial x_l}\Big)(p) = 0 \tag{3.3}$$

for every $\beta \leq \alpha$, hence $(\partial^\alpha a)(p) = 0$. Thus, it follows from (3.2) that $a'(p) = 0$.

If $d = 2$, then $\mathcal{J}_2(f)$ has a system of generators consisting of

$$\Big(\frac{\partial f}{\partial x_1}\Big)^3, \quad \Big(\frac{\partial f}{\partial x_1}\Big)^2\frac{\partial f}{\partial x_2}, \quad \frac{\partial f}{\partial x_1}\Big(\frac{\partial f}{\partial x_2}\Big)^2, \quad \Big(\frac{\partial f}{\partial x_2}\Big)^3,$$

and

$$\sigma := \frac{1}{2}\,\frac{\partial^2 f}{\partial x_1^2}\Big(\frac{\partial f}{\partial x_2}\Big)^2 - \frac{\partial^2 f}{\partial x_1\partial x_2}\,\frac{\partial f}{\partial x_1}\,\frac{\partial f}{\partial x_2} + \frac{1}{2}\,\frac{\partial^2 f}{\partial x_2^2}\Big(\frac{\partial f}{\partial x_1}\Big)^2.$$

By (3.3), in order to prove $(\partial^\alpha a)(p) = 0$, hence $a'(p) = 0$, it suffices to prove that

$$(\partial^\beta \sigma)(p) = 0$$

for every $\beta \leq \alpha$. By the general Leibniz rule,

$$\partial^\beta\Big(\frac{\partial f}{\partial x_i}\,\frac{\partial f}{\partial x_j}\,\frac{\partial^2 f}{\partial x_l\partial x_p}\Big) = \sum_{\beta'+\beta''+\beta'''=\beta} \frac{\beta!}{\beta'!\,\beta''!\,\beta'''!}\,(\partial^{\beta'(i)} f)\,(\partial^{\beta''(j)} f)\,(\partial^{\beta'''(l)(p)} f).$$

If $|\beta'(i)| \geq N$, $|\beta''(j)| \geq N$ and $|\beta'''(l)(p)| \geq N$, then $N + 4 \geq |\beta| + 4 \geq 3N$, hence $N \leq 2$. Thus, if $N > 2$, the contradiction argument (see a similar detail as in the case $d \geq 3$) shows that $(\partial^\beta\sigma)(p) = 0$ for every $\beta \leq \alpha$. If $f_2(x_1, x_2) = cx_1^2$, $c \in k^*$, we may choose $\alpha = (2, 0)$. Assume for simplicity that $p = (0, 0)$. Then, we have that $\sigma = 0 +$ (terms of degree ≥ 3), hence $\sigma(0,0) = \frac{\partial\sigma}{\partial x_1}(0,0) = \frac{\partial^2\sigma}{\partial x_1^2}(0,0) = 0$.

Now, we take care of the fourth case:

(iv) $d = N = 2$ and $f_2(x_1, x_2) = x_1^2 + x_2^2$ (up to linear change of variables).

In this case, p is a node of $X_0 \cap W$, and for simplicity we assume $p = (0, 0)$. Then, f and σ have the forms $f(x_1, x_2) = x_1^2 + x_2^2 +$ (terms of degree ≥ 3), $\sigma = 4(x_1^2 + x_2^2) +$ (terms of degree ≥ 3). Since $(0, 0)$ is also a nodal zero of g, we have $g(x_1, x_2) = c(x_1^2 + x_2^2) +$ (terms of degree ≥ 3) for some $c \in k^*$.

To finish the proof, we are going to consider the following two cases.

Case I. *All singular points of X_0 satisfy either* (i)*, or* (ii)*, or* (iii).

Let $h\colon Y \to X$ be an m-separating log-resolution of (X, X_0). We write

$$\operatorname{div}(h^* f) = \sum_{i\in J} N_i(f)E_i \quad \text{and} \quad K_{Y/X} = \sum_{i\in J}(\nu_i - 1)E_i,$$

where E_i, $i \in J$, are irreducible components of $h^{-1}(X_0)$. Consider an arbitrary exceptional divisor E_i. Let U be an open affine subscheme of $Y \setminus \bigcup_{j\neq i} E_j$ with $U \cap E_i^\circ \neq \emptyset$, so that on U we have

$$h^* f = u y_i^{N_i(f)},$$

with $y_i = 0$ being the local equation of E_i, and $u(y) \neq 0$ for every $y \in U \cap E_i^\circ$. Since $g - f = a$, we have $(h^*g)|_U - (h^* f)|_U = (h^*a)|_U$. This implies that, shrinking U if necessary,

$$(h^*g)|_U - (h^* f)|_U \in \langle y_i^{\operatorname{ord}_{E_i}(h^*a)}\rangle \subseteq \mathcal{O}_Y(U), \tag{3.4}$$

from which

$$(h^*g)|_U \in (h^* f)|_U + \langle y_i^{\operatorname{ord}_{E_i}(h^*a)}\rangle.$$

Since $a = a' f$ on W and $a'(p) = 0$ for any singular point p of X_0, we have $\operatorname{ord}_{E_i}(h^*a) > N_i(f)$. Therefore, we deduce that there exists a morphism $v\colon U \to \mathbb{G}_m$ such that

$$(h^*g)|_U = v y_i^{N_i(f)}.$$

This argument works for all exceptional divisors E_i, $i \in J$, hence $h\colon Y \to X$ is also an m-separating log-resolution of (X, X_0) *for* g with the same exceptional divisors E_i's as for f and $N_i(f) = N_i(g) =: N_i$ on each E_i. Thus, by (3.4) we have

$$u(y) - v(y) \in \langle y_i^{\operatorname{ord}_{E_i}(h^*a)-N_i}\rangle, \tag{3.5}$$

hence $u = v$ on $U \cap E_i^\circ$. In fact, proving $u = v$ on $U \cap E_i^\circ$, which we have just attained due to (3.5), is the key step in the proof of Theorem 3.5. Consider the Galois unramified coverings $\widetilde{E}_i^\circ(f)$ and $\widetilde{E}_i^\circ(g)$ described over $U \cap E_i^\circ$ as follows:

$$\begin{aligned}\widetilde{E}_i^\circ(f)|_{U\cap E_i^\circ} &= \{(z, y) \in \mathbb{A}_k^1 \times (U \cap E_i^\circ) \mid z^{N_i} = u(y)^{-1}\},\\ \widetilde{E}_i^\circ(g)|_{U\cap E_i^\circ} &= \{(z, y) \in \mathbb{A}_k^1 \times (U \cap E_i^\circ) \mid z^{N_i} = v(y)^{-1}\}.\end{aligned}$$

Hence $\widetilde{E}_i^\circ(f)|_{U\cap E_i^\circ} = \widetilde{E}_i^\circ(g)|_{U\cap E_i^\circ}$ because $u = v$ on $U \cap E_i^\circ$.

Let U' be another open affine subscheme of $Y \setminus \bigcup_{j\neq i} E_j$ such that $U' \cap E_i^\circ \neq \emptyset$, and on U' we have $h^* f = u' z_i^{N_i}$, with $z_i = 0$ defining $U' \cap E_i$, and u' a unit. Similarly as previous, $h^*g = v' z_i^{N_i}$ on U' for some unit v', and furthermore, $\widetilde{E}_i^\circ(f)|_{U'\cap E_i^\circ} = \widetilde{E}_i^\circ(g)|_{U'\cap E_i^\circ}$ Then, on $U \cap U'$, $z_i = \xi_i y_i$ with ξ_i a unit, hence $u = u'\xi_i^{N_i}$ and $v = v'\xi_i^{N_i}$; thus the map

$$\begin{aligned}&\{(z, y) \in \mathbb{A}_k^1 \times (U \cap U' \cap E_i^\circ) \mid z^{N_i} = u(y)^{-1}\}\\ &\qquad\qquad \to \{(z, y) \in \mathbb{A}_k^1 \times (U \cap U' \cap E_i^\circ) \mid z^{N_i} = u'(y)^{-1}\}\end{aligned}$$

sending (z, y) to $(\xi_i z, y)$ is an isomorphism, and the same observation also holds for v and v'. It follows that

$$\widetilde{E}_i^\circ(f)|_{U\cap U'\cap E_i^\circ} \cong \widetilde{E}_i^\circ(g)|_{U\cap U'\cap E_i^\circ}.$$

hence the gluing yields $\widetilde{E}_i^\circ(f) \cong \widetilde{E}_i^\circ(g)$, hence $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ in $K_0^{\mu_{N_i}}(\mathrm{Var}_{X_0})$.

Since $X_0(f) = X_0(g) = X_0$ and, again, $u = v$ on $U \cap E_i^\circ$ for every appropriate U and i, it follows that if E_j is a strict transform for f, then, up to isomorphism, it is also a strict transform for g, thus $[\widetilde{E}_j^\circ(f)] = [E_j^\circ] = [\widetilde{E}_j^\circ(g)]$.

Since h is an m-separating log-resolution of (X, X_0) common for f and g, the numerical invariants ν_i are common for f and g. Using the formula (3.1) we have

$$[\mathcal{X}_m(f)] = \mathbb{L}^{md}\sum_{N_i|m}[\widetilde{E}_i^\circ(f)]\,\mathbb{L}^{-m\nu_i/N_i} \quad \text{and} \quad [\mathcal{X}_m(g)] = \mathbb{L}^{md}\sum_{N_i|m}[\widetilde{E}_i^\circ(g)]\,\mathbb{L}^{-m\nu_i/N_i}.$$

Therefore, by the above discussions, we get $[\mathcal{X}_m(f)] = [\mathcal{X}_m(g)]$.

Case II. *$d = 2$ and X_0 contains a node, i.e., there is a point $p \in X_0$ satisfying* (iv).

Let $h: Y \to X$ be a log-resolution of (X, X_0) for f. As in the proof of Lemma 2.9 in [5] (applying to the case $d = 2$), given h, we can blow up along intersections $E_l \cap E_s$, and do this procedure many times to attain the property $N_i + N_j > m$ whenever $E_i \cap E_j \neq \emptyset$ and E_i and E_j come from non-nodal singular points (i.e., m-separating for non-nodal singular points). For each node of X_0, we only use once the blowup $(y_1, y_2) \mapsto (y_1 y_2, y_2)$, locally. The resulting m-separating log-resolution for non-nodal singular points is also denoted by h. Using the argument in Case I, if E_i is an exceptional prime divisor coming from a non-nodal singular point of X_0, or if E_i is a strict transform, we have $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ in $K_0^{\mu_{N_i}}(\mathrm{Var}_{X_0})$. It remains to prove that, if E_i is the exceptional prime divisor corresponding to the node p of X_0, then $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ in $K_0^{\mu_{N_i}}(\mathrm{Var}_{X_0})$.

Assume $p = (0, 0)$ again. As explained in case (iv), $f(x_1, x_2) = x_1^2 + x_2^2+$ (terms of degree ≥ 3), and $g(x_1, x_2) = c(x_1^2 + x_2^2) +$ (terms of degree ≥ 3), $c \in k^*$. On an affine Zariski open U around $(0, 0)$, we have

$$h^* f(y_1, y_2) = u y_2^2, \quad u = 1 + y_1^2 + y_2 \cdot (\text{terms of degree} \geq 0)$$

and

$$h^* g(y_1, y_2) = c v y_2^2, \quad v = 1 + y_1^2 + y_2 \cdot (\text{terms of degree} \geq 0).$$

Here, $U \cap E_i$ is defined by $y_2 = 0$ and $N_i(f) = N_i(g) = 2$. Since $u = v = 1 + y_1^2$ on $U \cap E_i$ and $c \neq 0$, the map

$$\{(z, y_1, 0) \in \mathbb{A}_k^1 \times (U \cap E_i^\circ) \mid z^2 = (1 + y_1^2)^{-1}\}$$
$$\to \{(z, y_1, 0) \in \mathbb{A}_k^1 \times (U \cap E_i^\circ) \mid c z^2 = (1 + y_1^2)^{-1}\}$$

sending $(z, y_1, 0)$ to $(z/\sqrt{c}, y_1, 0)$ is an isomorphism. Therefore, by gluing we obtain $\widetilde{E}_i^\circ(f) \cong \widetilde{E}_i^\circ(g)$, hence $[\widetilde{E}_i^\circ(f)] = [\widetilde{E}_i^\circ(g)]$ in $K_0^{\mu_{N_i}}(\mathrm{Var}_{X_0})$. ■

Corollary 3.6. *Let f and g be non-constant regular functions on X with the same scheme-theoretic zero locus X_0. Suppose that d, n, f and g satisfy one of the following conditions*:

- $d \geq 3$, $n \geq 2$, $g - f \in \mathcal{J}_n(f)$,

- $d = 2$, $n \geq 2$, $g - f \in \mathcal{J}_n(f)$, X_0 *has no node,*
- $d = 2$, $n \geq 2$, $g - f \in \mathcal{J}_n(f)$, X_0 *has nodes, k is quadratically closed,*
- $d = 2$, $n \geq 3$, $g - f \in \mathcal{J}_n(f)$.

Then, for any integer $m \geq 1$, the identity $[\mathcal{X}_m(f)] = [\mathcal{X}_m(g)]$ holds in $\mathcal{M}_{X_0}^{\mu_m}$. As a consequence, $Z_f(T) = Z_g(T)$ and $\mathcal{S}_f = \mathcal{S}_g$.

Example 3.7. Consider $f = x_1^2 x_2$ and $g = x_1^2 x_2(1 + x_1^2)$ be in $k[x_1, (1 + x_1^2)^{-1}, x_2]$, which define morphisms $D(1 + x_1^2) \times_k \mathbb{A}_k^1 \to \mathbb{A}_1^1$, where $D(1 + x_1^2)$ is the principal open subset of $\mathbb{A}_k^1$ given by $1 + x_1^2$. Then $\mathcal{J}_2(f) = \langle x_1^3 x_2^3, x_1^4 x_2 \rangle$. We can check that $X_0(f) = X_0(g) =: X_0$ as k-schemes, and that $g - f = x_1^4 x_2 \in \mathcal{J}_2(f)$. By Theorem 3.5, $\mathcal{S}_f = \mathcal{S}_g$ in $\mathcal{M}_{X_0}^{\hat{\mu}}$.

The proof method of Theorem 3.5 can be applied to prove the following result, which is a variant of Theorem 3.2 in [6]. Remark, again, that the definition of motivic nearby cycles in [6] is slightly different from that in [9] and [10], thus Theorem 3.8 is not exactly the same result as that of Theorem 3.2 in [6].

Theorem 3.8. *Let f and g be non-constant regular functions on X with the same scheme-theoretic zero locus X_0. Suppose that $g - f \in \mathcal{J}_1(f)^3$. Then, for any integer $m \geq 1$, the identity $[\mathcal{X}_m(f)] = [\mathcal{X}_m(g)]$ holds in $\mathcal{M}_{X_0}^{\mu_m}$. Consequently, $Z_f(T) = Z_g(T)$ and $\mathcal{S}_f = \mathcal{S}_g$.*

Proof. As in the proof of Theorem 3.5, we can write $g - f = a \in \mathcal{J}_1(f)^3$ with $a|_{X_0} = 0$. On an open affine subset W of X we have $a = a' f$, with $a' \in \mathcal{O}_X(W)$. We may assume $W = \mathbb{A}_k^d$. Let p be a non-smooth point of $X_0 \cap W$. Since

$$a = \sum_{1 \leq i,j,l \leq d} a_{ijl} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_l},$$

with a_{ijl} being sections of $\mathcal{O}_X$, it follows from the general Leibniz rule and (3.2) that $a'(p) = 0$ (see the proof of Theorem 3.5 for details). Consequently, we have $\mathrm{ord}_{E_i}(h^* a) > \mathrm{ord}_{E_i}(h^* f)$ for any log-resolution h of (X, X_0) and any exceptional divisor E_i of h. The rest is completely the same as in Case I of the proof of Theorem 3.5. ∎

Remark 3.9. We observe, from Proposition 2.20 and the proof of Theorems 3.5 and 3.8, the following facts:

(a) if $d = 2$, Theorem 3.5 is stronger than Theorem 3.8;

(b) if $d \geq 3$, Theorem 3.8 is stronger than Theorem 3.5.

Remark 3.10. We now borrow Example 3.5 in [6] to show that Theorem 3.5 will not be true if we replace $\mathcal{J}_2(f)$ by $\mathcal{J}_1(f)$. Further, we show $u \neq v$ in $U \cap E_i^\circ$, where E_i is an exceptional prime divisor. Indeed, consider $X = \mathbb{A}_k^1 \times_k \mathbb{G}_m$, $f(x_1, x_2) = x_1^2$ and $g(x_1, x_2) = x_1^2 x_2$. Then

$$X_0(f) = \mathrm{Spec}(k[x_1, x_2, x_2^{-1}]/\langle x_1^2 \rangle) = \mathrm{Spec}(k[x_1, x_2, x_2^{-1}]/\langle x_1^2 x_2 \rangle) = X_0(g),$$

which will be denoted by X_0. The singular locus of X_0 is the reduced part $(X_0)_{\mathrm{red}}$ of X_0. Since $g = x_2 f$, where $x_2 \colon X \to \mathbb{G}_m$ sending (x_1, x_2) to x_2, the latter means $g - f = (x_2 - 1) f \in \mathcal{J}_1(f) = \langle x_1 \rangle$.

Let h be the blowup of X along X_0. By definition, h is nothing but the identity morphism of X, which has no strict transform, while whose exceptional prime divisor is $E_1 = X_0 = h^{-1}(X_0)$. Using the notation in the proof of Theorem 3.5 we have $U = X$, $u = 1$ and $v = x_2$ on $E_1 = U \cap E_1^\circ$, hence $u \neq v$ in $U \cap E_1^\circ$. Furthermore,

$$\widetilde{E}_1^\circ(f) = \{(z; x_1, x_2) \in \mathbb{A}_k^1 \times_k E_1 \mid z^2 = 1\} = \{(z; 0, x_2) \in \mathbb{A}_k^1 \times_k E_1 \mid z^2 = 1\},$$
$$\widetilde{E}_1^\circ(g) = \{(z; x_1, x_2) \in \mathbb{A}_k^1 \times_k E_1 \mid z^2 = x_2\} = \{(z; 0, x_2) \in \mathbb{A}_k^1 \times_k E_1 \mid z^2 = x_2\},$$

hence $[\widetilde{E}_1^\circ(f)] = (\mathbb{L} - 1)[\mu_2]$ and $[\widetilde{E}_1^\circ(g)] = \mathbb{L} - 1$. By Theorem 3.2, $\mathcal{S}_f = (\mathbb{L} - 1)[\mu_2]$ and $\mathcal{S}_g = \mathbb{L} - 1$.

Acknowledgments. The first author thanks the Department of Mathematics, Graduate School of Science, of Osaka University, for warm hospitality during his visit as a Specially Appointed Researcher supported by the MSJ Tosio Kato Fellowship. He would also like to acknowledge support from the ICTP through the Associates Programme (2020-2025).

Funding. The first author was funded by VinGroup and supported by Vingroup Innovation Foundation (VinIF) under the project code VINIF.2021.DA00030. The second author was supported by JSPS KAKENHI Grant numbers JP21H04994 and JP23H01070.

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Received November 8, 2023; revised June 21, 2024.

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